OWNERSHIP AND CONTROL AND THE FINANCING
AND INVESTMENT POLICIES OF THE FIRM

by

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Except where otherwise indicated this thesis is entirely my own work.
PREFACE

This thesis is concerned with deriving optimal financing and investment policies of the firm. A number of models is developed with differing assumptions relating to the capital market and the firm's motivation. All of these models are dynamic: they explicitly take into account the proposition that some funds can be obtained more quickly than others and they describe the optimum time path of adjustment from a non-optimum position one to an optimum one.

The overall purpose of the thesis is to investigate, at a theoretical level only, the effects of the separation of ownership and control on the investment and financing policies of the firm. Chapter I establishes the motivational assumptions of the models to follow. This is essentially a survey chapter as it is not the purpose of this thesis to contribute to the debate on motivation. Chapter II surveys the main theories and evidence on the determination of share prices and the cost of capital and states the assumptions on these matters to be used in the model. Chapter III establishes notation and continues the statement and discussion of the assumptions. Optimal conditions relating to the firm terminating its operations are also obtained.
The first and simplest model is developed in Chapter IV. It is assumed that the firm wants to maximize the price of its ordinary shares but that the number of shares is fixed and optimal policies relating to ploughback finance and debt finance over time are derived. This model is discussed in considerable detail because the same method is used to obtain solutions to other models. Chapter V analyses the same model excepting that the firm is now able to issue shares. In Chapters IV and V it is assumed that the product and factor market conditions are given. In Chapter VI two things are attempted. Firstly the model developed in Chapter IV is reexamined under the assumption that the product market conditions vary over time. Secondly solutions are obtained for the model derived in Chapter IV under cost of capital assumptions which are different from those used in that chapter.

A growth maximization model is developed in Chapter VII under the assumption that share finance is not available to the firm and this assumption is relaxed in Chapter VIII. The last chapter considers different formulations of the problems which may have been analysed.

My interest in the effects of the separation of ownership and control on the behaviour of the firm was
initially stimulated by writings on this subject by
W.J. Baumol, R. Marris, and O.E. Williamson and I am indebted
to these authors. Their works are referred to later. I am
especially indebted to Professor J.D. Pitchford of the
Australian National University who readily extended assistance
whenever I sought it and to Messrs J. Logan, E. Sieper and
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I am also greatly indebted to Mrs I. Everitt
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Chapter I

MOTIVATIONAL ASSUMPTIONS

1. Managerial Theories of Motivation

One of the most important developments in the theory of the firm since 1958 has been the emergence of theories of motivation of the firm and their use in formal models. Prior to this development, firms were generally assumed to be profit maximizers or share price maximizers and no systematic analysis was given of the decision making process which led to these goals. These motivational assumptions have had their very vocal critics and for a time P.W.S. Andrews'¹ 'satisficing' hypothesis was popular but the difficulties of developing models without assuming that something is being maximized or minimized meant that the hypothesis remained largely void of specific implications for the traditional considerations of the theory of the firm. Relief came when the firm became viewed as an organization in which owners and managers act as separate and possibly conflicting pressure groups. It was then possible to analyse the personal goals of these relatively homogeneous groups and to view the goals of the firm as the outcome of the conflict between these groups.

Theories of the firm based on this approach have become known as 'managerial theories'.

It is not the purpose of this thesis to engage in the debate on motivation. The outcome of the debate, however, is crucial to the problems considered later and a more detailed account of the debate is given in the Appendix to this chapter. For our purpose it will be sufficient here to give only the briefest account of the arguments by the three most important managerial theorists, W.J. Baumol, O.E. Williamson, and R. Marris.

Each writer holds the view that the owners or shareholders of a firm are primarily interested in the financial returns that can be gained from that firm whether stated in terms of share prices, profits or rates of return. Thus, when shareholders exert a dominating influence on the firm the assumptions of profit maximization or share price maximization are appropriate. Shareholders of large firms, however, are typically disunified, disinterested and ignorant of the firm's vast and complex operations and as a group they are not in a position to impose their interests in each major decision. Consequently managers are able to further their interests at the expense of shareholders' interests. This

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is reflected in a lower share price or profit or rate of return, and the limits to which these can be depressed are determined by managers' fear of unified shareholder action against them or fear of a successful takeover bid. In both cases managers may be demoted or sacked or have other conditions imposed which threaten the security of their status.

The top row in Table I shows those personal goals of managers whose achievement is believed to be related to the firm's activities. 'Salary' refers to managers' basic salaries; 'emoluments' refers to such rewards as expense accounts and gifts of shares; the next set of goals refers to the psychological and social rewards managers may receive from their positions in the firm; and 'security' is the retention of managers' status when defined in terms of the previously mentioned goals. The far left hand column shows the goals of management dominated firms. 'Sales' refers to sales revenue; 'growth' is the rate of growth of the book value of the firm's assets; 'staff expenses' and 'emoluments' are self explanatory; 'discretionary profit' is profit in excess of the constraint level and the last item refers to the security constraint in its various forms. A name in a cell indicates that the author believes there exists a positive relation between the two goals. The difference between the writers is more apparent than real since the goals of the firm chosen depend very greatly upon the things the writers want to explain. A much more just explanation is given in the Appendix to this chapter.
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There is nothing in the general account of the relationship between managers and owners given by these writers which will be disputed here. These writers have confirmed that there is a great deal of evidence to support it and it is logically acceptable and analytically useful. We need, however, to be specific on the motivational assumptions that will be made in the models to follow and these will possess a degree of arbitrariness.

An owner will be assumed to be interested in maximizing the market value, or his subjective value of his net worth. If \( W(0) \) is the market value of his net worth, \( P'_i(0) \) is the market value of the \( i \)th investment and \( B(0) \) is the size of his debt, all evaluated at time \( t = 0 \), then he may wish to maximize

\[
W(0) = \sum_{i=1}^{n} P'_i(0) - B(0) \tag{1}
\]

or

\[
W'(0) = \sum_{i=1}^{n} w_i P'_i(0) - B(0) \tag{2}
\]

where there are \( n \) possible investments. Expression (2) is this subjective value of his net worth. If the owner wishes to maximize \( W(0) \) then he accepts the market estimate of future returns from the investment and its evaluation of the uncertainty of the returns. If, however, he is a speculator then his evaluation of the investments will be different from the current market evaluation and \( w_i \) is a weight reflecting
that difference. Obviously (1) is a special case of (2) when \( w_i = 1, i = 1, 2, \ldots, n \).

If \( P_i(0) \) is the market price of the \( i \)th investment and \( I_i(0) \) is the number of units (shares etc) of that investment held by the investor at \( t = 0 \) then

\[
\sum_{i=1}^{n} w_i P_i(0) = \sum_{i=1}^{n} w_i P_i(0) \cdot I_i(0) \quad \ldots (3)
\]

and

\[
\sum_{i=1}^{n} \frac{d}{dt} w_i P_i(t) = \sum_{i=1}^{n} w_i \left[ \frac{d}{dt} I_i(t) \cdot P_i(t) + \frac{d}{dt} P_i(t) \cdot I_i(t) \right] \quad (4)
\]

It is assumed that there are no relationships between the prices. The first term inside the brackets is the increase in the value of \( W'(0) \) which results from new purchases and the second term is the capital gain or loss. Obviously, on this assumption, owners' interests are best served by the managers of the \( i \)th investment if they follow a policy which maximizes \( P_i(t) \). If, however, some prices are related inversely to each other then this may not be so. We will ignore this complication and assume that owner dominated firms attempt to maximize their share prices.

Shareholders are distinct among economic phenomena. They have certain resources available to them in the form of credit worthiness and savings but this 'capital' is like a current factor of production. They invest and disinvest without significant lag. There is no need for them to lay down a plan and follow it for a considerable period of time.
Consequently the models of portfolio behaviour assume that investors want to maximize income from investments in the current period. In the case of firms, however, there are irreversibilities and lags of adjustment which mean that firms must plan over considerable periods of time, and we will treat the optimizing problem of the firm as an exercise in optimal control.

Like Baumol and Marris we will assume that managers are primarily interested in gaining the benefits of size and growth of the firm. Their formulations, however, are entirely static and consequently imply extreme myopia. Since 'maximize the rate of growth' holds for each period the dependence of growth rates at each time is ignored. This may lead to a policy which maximizes growth in this period but prevents further growth in later periods. Even if growth is to be maximized in the 'long run' the time interval must be specified if the objective is to mean anything. We will assume that the managerial group wish to remain in control of the firm forever. If t is time then the planning interval is $t \in [0, \infty)$. If $K(t)$ is the book value of the firm's assets then the simple growth maximization goal becomes

$$\maximize \int_{0}^{\infty} \frac{dK(t)}{dt} \cdot \frac{1}{K(t)} \, dt = \maximize \log [K(\infty) - K(0)] \quad \text{...(5)}$$

Managers, however, will have a preference for growth now rather than later so that a more appropriate goal is

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maximize \( \int_0^\infty \frac{dK(t)}{dt} \frac{1}{K(t)} e^{-\rho t} \, dt, \; \rho > 0 \) \quad \ldots \quad (6)

A goal which has as much justification as (6), however, is

maximize \( \int_0^\infty K(t)e^{-\rho t} \, dt, \; \rho > 0 \) \quad \ldots \quad (7)

and this will be used in the following chapters. It incorporates size as a benefit and the pressure for change now rather than later through the discount factor. One of the constraints in the managerial model will be a minimum dividend and this is used to represent ownership interests.

The purpose of this thesis is the development of a number of models which vary in their motivational assumptions and which retain reasonably realistic assumptions about the supply of finance and some elements of dynamics. The models will be structurally similar and will enable a comparison of results. To put these models in some context, however, we will briefly review the managerial models.

2. Managerial Models

Only the models developed by Baumol, Williamson and Marris will be considered since these are the only models to consider managerial goals in other than a minor way.

Baumol has both a static and 'dynamic' model. In the static model\(^{4}\) he assumes that the firm wants to maximize the level of sales subject to a minimum profit level. It is

\(^{4}\) W.J. Baumol: Business Behavior, Value and Growth.
intuitively clear that the optimum output, under usual assumptions concerning revenue and cost conditions, is either where marginal revenue is zero, or where the minimum profit constraint is binding, whichever is the lower output (see Diagram I). To add rigor to Baumol's diagramatic proof, and to allow us to solve for the shadow price which is implicit in the problem, we will treat it as a non-linear programming problem.

Let \( p(x) \) be the price of the firm's product, \( C(x) \) be the average cost of production, \( x \) be the volume of output, \( \Pi \) be the level of profit and \( k \) be the minimum profit constraint. Then

\[
\Pi = p(x)x - c(x)x
\]  
and the optimizing problem involves

maximizing \( p(x)x \)  
subject to \( \Pi \geq k \)

The Lagrangian is

\[
L = p(x)x + \lambda [p(x)x - c(x)x - k]
\]  
when \( \lambda \geq 0, \lambda[\Pi - k] = 0 \)

The necessary condition of optimization is

\[
\frac{dL}{dx} = 0
\]  
If \( p''(x) < 0 \) and \( c''(x) > 0 \) then the Kuhn-Tucker constraint qualification is met and this necessary condition is also a sufficient condition.
Diagram I

Diagram II

BALANCED GROWTH RATES
Since
\[ \frac{d\pi}{dx} = 0 = p'(x)x + p(x) + \lambda[p'(x)x + p(x) - c'(x)x - c(x)] \ldots (14) \]

\[ [p'(x)x + p(x)](1 + \lambda) = \lambda[c'(x)x + c(x)] \ldots (15) \]

Clearly, if it is optimal for profits to be above the constraint level then

\[ p'(x)x + p(x) = 0 \ldots (16) \]

Since \( \lambda = 0 \) may hold when \( \Pi - k = 0 \), (16) may also hold when the optimum position involves the constraint holding with equality. Thus, when the shadow price of the constraint is zero, so that the constraint is not effectively binding, the optimum output is that where marginal revenue is zero.

It will never be optimal to operate where marginal revenue is negative because this is a lower level of sales than could be achieved by reducing output. However, the constraint may be binding before marginal revenue is zero. In this case

\[ \Pi - k = 0 \text{ and } \lambda > 0 \]

and

\[ p'(x)x + p(x) \left( \frac{1}{\lambda} + 1 \right) = c'(x)x + c(x) \ldots (17) \]

The shadow price can then be evaluated for the value of \( x \) corresponding to the level of profit being equal to the constraint.
If the profit constraint is so high that it is equal to the maximum profit the firm can earn then marginal revenue must equal marginal cost, and the shadow price becomes infinite. This is the case of an owner dominated firm.

Baumol does not solve his 'dynamic' model but believes it to be a truer representation of the way firms behave. Let \( g \) be the rate of growth of revenue, \( I \) be investment as a proportion of capital stock, \( n \) be profit as a proportion of shareholders' equity, and \( D \) be dividends and \( E \) earnings both as a proportion of shareholders' equity. The problem then, is

\[
\text{maximize} \quad g = f \left( I, n \right) \quad \text{...(18)}
\]

\[
\text{subject to} \quad I = \phi \left( n, D \right) + E \quad \text{...(19)}
\]

\[
\begin{align*}
\quad & n = D + E \\
\end{align*} \quad \text{...(20)}
\]

Baumol is content with showing the nature of the problem and the solution will, of course, depend on the particular functions assumed. We will be concerned with this type of problem later.

Williamson's model is chronologically next to Baumol's but in technique it more closely resembles

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6 O.E. Williamson: The Economics of Discretionary Behavior.
T. Scitovsky's\(^7\) famous model than it does either Baumol's or Marris'. The analysis involves a managerial utility function which is assumed to be continuous and differentiable. Where \( U \) is the level of utility, \( S \) is the level of staff costs, \( M \) is the level of managerial emoluments, \( x \) is output, \( \Pi_r \) is reported profit, \( \Pi_0 \) is the minimum profit constraint after tax, \( C \) is the cost of production, \( t \) is the tax rate on profit and \( T \) is tax paid, the mathematical problem involved is

\[
\text{maximize } U = U [S, M, \Pi_r - \Pi_0 - T] \quad \ldots(21)
\]

subject to \( \Pi_r \geq \Pi_0 + T \) \quad \ldots(22)

By ignoring corner solutions, the problem becomes

\[
\text{maximize } U = U [S, M, (1 - t)(R - C - S - M) - \Pi_0] \quad \ldots(23)
\]

and the necessary conditions are

\[
\frac{\partial R}{\partial x} = \frac{\partial C}{\partial x} \quad \ldots(24)
\]

\[
\frac{\partial R}{\partial S} = \frac{-U_1 + (1 - t) U_3}{(1 - t) U_3} \quad \ldots(25)
\]

\[
U_2 = (1 - t) U_3 \quad \ldots(26)
\]

The notation \( U_i \) means the derivative of \( U \) with respect to the \( i \)th argument in the utility function.

The analysis is continued in terms of these marginal utilities. Firstly \( M \) is dropped from (23) and the dependence

of the remaining variables is analysed, and then M is replaced and S is dropped and dependence is observed. Apart from propositions relating to output, no attempt is made to draw out these conclusions in terms of the more conventional considerations of the theory of the firm. With no analysis of the implications for investment policy given, the model must remain static.

This survey does most harm to Marris' account of the managerial firm. His account of firm behaviour is extensive and complex, but the basic relationships in the model are relatively simple and we will concentrate on these. The rate of growth of demand for the firm's products, \( D \), is assumed to increase with the rate of diversification of the firm's products, \( d \), because expansion strategies are less likely to prompt retaliation. It is inversely related to the profit margin, \( m \), because a high level of expenditure on research and development is needed to maintain a high rate of growth. Thus

\[
D = D(m,d) 
\]

\[
\frac{\partial D}{\partial m} < 0, \quad \frac{\partial D}{\partial d} > 0 \quad \ldots (28)
\]

The rate of profit, \( p \), is also a function of \( m \) and \( d \). Obviously, \( p \) increases as \( m \) increases, other things being equal. The profit rate will also increase for a range (from

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8 A much simpler version is given in his article "A Model of the 'Managerial' Enterprise", Quarterly Journal of Economics, May 1963, pp. 185-209, but there is an important difference between the versions in relation to the supply of capital.
zero) of $d$ but the capital-output ratio will increase very rapidly for high values of $d$ and so the profit rate will eventually fall with $d$. Thus

$$\dot{p} = p(m,d) \quad \ldots (29)$$

$$\frac{\partial p}{\partial m} > 0, \frac{\partial p}{\partial d} < 0 \quad \ldots (30)$$

Capital may come from internal and external sources. If leverage is as high as the security constraint will allow, then the rate of growth will be the sum of the rates of growth of accumulated funds and shareholders' contributions. The higher the retention rate and the higher the rate of new issues, the lower will be the value of shares and the greater the threat to management's security. Suppose the retention rate and new issue rate are chosen to meet the security constraints. Then, because $p$ is a function of $m$ and $d$, the rate of growth of capital, $C$, is

$$\dot{C} = C(m,d) \quad \ldots (31)$$

Finally, by adding the assumption that the equilibrium capital-output ratio is a constant, equilibrium requires that

$$\dot{D} = \dot{C} \quad \ldots (32)$$

Thus, in this presentation, there are four unknowns and three equations. The system becomes determinant by adding the condition that the rate of growth must be maximized. In
Diagram II, $m_1 < m_2 < m_3$; the C lines show rates of growth of capital for given $m$ but varying $d$ and D lines show rates of growth in demand for given $m$ but varying $d$. The 'balanced growth rate' line gives solutions to the system (27), (31), (32). Obviously, the optimal rate of diversification is $d^*$. More detailed solutions can be obtained from a system specified in more detail\(^9\) but the basic system and method are those shown here. The situation when owners dominate is represented in this model by the situation where the security constraint is so high that management has no choice on growth policy. The maximum rate of growth in this case will be lower.

Since $D$ is also the rate of growth of sales revenue there are no basic conflicts between the essentials of Marris' and Baumol's models.

This completes the summary of the managerial models. Only Marris has made a thorough attempt at analysing the implications of managerialism, and although he alone included reasonably complete assumptions concerning the supply of finance, there is no element of dynamics in his model. This is where we will attempt to make a contribution. We will begin by analysing a share price maximization model of the firm which takes into account the varying speeds with which finance can be obtained. Firstly it will be assumed that

product and factor market conditions are given and then that they vary over time. Then a size maximization model will be analysed.
Appendix A

SEPARATION OF OWNERSHIP AND CONTROL

1. The Firm as an Organisation

In a classic work, C.I. Barnard\(^1\) distinguishes two types of organizations, the 'formal' and the 'informal'. A 'formal' organization is a set of actions that is consciously co-ordinated. It involves the existence of persons (human or institutional) willing to contribute to the activity of the system, the existence of deliberate communication between the persons concerned, and the existence of common goals. An 'informal' organization possesses the above characteristics, excepting that the goals are not common to the participants.

Firms are formal organizations. They combine the efforts of labourers, suppliers of materials (including land), creditors and shareholders through remuneration. They have highly developed information systems which record, analyse and disperse knowledge relating to material and financial events. The existence of remuneration to participants, a highly developed information system, and a highly ordered decision making structure allows actions to be consciously co-ordinated and for the existence of common goals.

\(^1\) C.I. Barnard: The Functions of the Executive, (Harvard University Press, 1938).
K.J. Arrow\(^2\) has established that 'dictatorship' is necessary to ensure the transitivity of group preferences. 'Dictatorship' means that group preferences correspond to those of a member of the group, regardless of the preferences of others, providing his preferences are complete and transitive. Arrow showed that provided there are at least three states and three people to order them, group ordering by simple majority rating may result in an intransitive ordering, even though each individual is rational in the sense described above. In recent literature the dictator, or more correctly, the person (human or institutional) who has the responsibility of ensuring that group preferences are rational, is called the 'peak co-ordinator'\(^3\) or the 'benevolent dictator'\(^4\). These concepts are similar to the notion of the entrepreneur where he is regarded as an organizer-manager rather than an innovator or risk-taker. He is at the head of the executive structure and considers the preferences of others when exercising discretion, and he may have very little power over group preferences, or he may be a major force. The distinction between a peak co-ordinator and a benevolent dictator lies in this attitude towards


management and non-management preferences. Peak co-ordinators form their notions of group preferences by considering management and non-management preferences alike, while benevolent dictators are concerned only with management goals and considers non-management demands as constraints.

Which is, in general, the more appropriate characterization is not vital to us. Only the fact that firms, in general, have some person to ensure rationality is important.

At any time the goals of the firm will be the result of many and diverse pressures exerted on it. Pressures will come from all of the participants in actions and possibly from those who don't participate directly, such as governments and the press. The importance of the desires of each participant in the forming of the firm's goals depends on the power-relationship between each, which is determined by legal relationships, customs, personal factors, ignorance, and so on. Some participants are potentially powerful but choose to remain inactive because they consider their side payments to be sufficient. The active participants remain so because the rewards they would receive from inactivity are thought to be insufficient.

The two most important groups which can be distinguished are managers and owners. Within each group the members' functions, type of remuneration and power are similar, though they vary greatly from firm to firm. The preferences of the members within each group are similar but they differ between the groups, and intertemporal rationality for the
firm requires that the preferences within each group and the relative power of each group remain the same over time. There are reasons to believe that both of these conditions generally hold.

2. The Development of Managerial Theories

Theories of the firm based on these propositions have become known as 'managerial theories'. They have developed in response to the realization that ownership and management have become separated in large corporations, and that owners have in general become passive. If the latter was not the case then the emergence of the managerial class would have little significance for the traditional considerations of the theory of the firm, which has assumed that only the owners are active. Managers would remain mere agents of the firm's owners. The assumption of profit maximization, or its corporate equivalent of share price maximization, would remain justified. Managerial theorists regard this as a special case among large corporations and it exists when managers' remunerations, in all their forms, are tied directly to the owners' goals. In this case both managers and owners

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are active, but there is no conflict of interest.

Managerial theorists argue that the more usual case is that where the conflict does exist and that managers do not act in the best interests of the shareholders. This idea is not new, but what is new is the detail with which the conflict is drawn and the derivation of models based on this conflict. Williamson, for instance, points out that both Adam Smith and Alfred Marshall had clear statements of the conflict, as did J.M. Keynes and J.R. Hicks later. The empiricists, especially A.A. Berle and G.C. Means, have done the most systematic work in this field until recently, drawing attention to the growth in absolute and relative importance of large corporations, the increasing dispersion of share ownership, the separation of ownership from control and the need to rethink traditional motivational assumptions.

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6 O.E. Williamson: The Economics of Discretionary Behaviour: Managerial Objectives in a Theory of the Firm, pp. 3-5
They presented a number of interesting empirical propositions. Of the 200 largest corporations in the U.S.A. which they studied, 65 percent of the number or 80 percent by assets were controlled by management or by a legal device involving the separation of ownership from control, 23 percent of the number or 14 percent by assets had majority ownership, and 12 percent by number or 6 percent by assets were controlled by a majority of ownership.

Similar evidence is now available for other western countries. E.L. Wheelwright has published the results of a study of the ownership and control of 102 large public companies in Australia. The figures are for 1953 and the following results relate to domestic and foreign owned companies combined: 8 percent of the number or 5 percent by shareholders' funds of the firms had no separation of ownership and control, ownership and control were partially separated for 28 percent of the number or 19 percent by shareholders' funds, 11 percent of the number or 7 percent by shareholders' funds were in a 'marginal area', ownership and control were 'virtually completely separated' for 44 percent of the number or 51 percent by shareholders' funds, and 9 percent of the number or 18 percent by shareholders' funds was unclassified.

The empiricists did not proceed, however, to establish new motivational assumptions which could be

used in the theory of the firm. Meanwhile, dissatisfaction with the traditional motivational assumptions continued and the debate centred, for a time, on P.W.S. Andrews' \(^{13}\) 'satisficing' hypothesis. Although it had much support it was difficult to handle in models and the profit maximization assumption continued to be used. Only T. Scitovsky\(^{14}\) derived a motivational assumption from an examination of the firm's decision-making process, but in this case the decision maker was an owner-manager and the problem was to optimize between profits and leisure. He was not concerned with large firms.

An important article which in some ways heralded the development of managerial theories was written by R.M. Cyert and J.G. March\(^{15}\) and published in 1956. Like many writers before them, they argued that the 'objective of the firm is to attain a satisfactory level of profits'.\(^{16}\) This is so because of the creditor-like position of shareholders (which was not stressed) and because of the planning process used by firms. The essential tool of planning is the firm's budget and this tends to fix expectations and aspirations and the use of rules of thumb militates against maximization. 'So long as the profit level


\(^{16}\) Cyert and March, op. cit., p.47.
and sales continue to be satisfactory, the budgetary decisions are exceptionally dependent on decisions of previous years with shifts tending to reflect the expansionist inclinations of subunits rather than systematic reviews by top management'. Their most important contribution in this article was the introduction of the concept of 'organizational slack'. Because the firm does not maximize profits there exists some 'slack' which 'appears to imply that significant amounts of individual energies potentially utilizable by the organization are, in fact, being directed to the satisfaction of other roles'. The way in which the slack is used is determined by the employees and it continues until some 'shock' prompts its reduction. Cyert and March, however, did not explore employees' goals and how these are reflected in the firm's goals.

Managerial theories derive specific goals for the firm from an analysis of managers' goals and this deficiency prevents their model being considered a managerial model. The three most important managerial models are those developed by W.J. Baumol, O.E. Williamson, and

17 Cyert and March, op. cit., p. 52.
18 Cyert and March, op. cit., p. 53
R. Marris. Baumol's ideas were greatly influenced by his contact with firms as a consulting economist, while Williamson and Marris relied heavily on economic, sociological and organization theory literature. The details of their descriptions of the relationship between owners and managers differ but the essentials are the same. Owners or shareholders are seen as (generally) a mass of disunified individuals, without the ability to understand the operations of the firm and with no more than a passing interest in the firm as such. They behave more like short term creditors than owners. They remain inactive so long as profits, in the case of Baumol and Williamson, or share prices in the case of Marris don't fall much below those of companies with competitive shares and so long as the firm avoids significant adverse publicity, as might result from an important project failing or the discovery of corruption. If these conditions aren't maintained then shareholders may take administrative action which results in policies which aren't in the managers' best interests, or in the rearrangement or sacking of administrative staff. Marris stresses the threat of takeovers, when share prices are low. If share prices are low, then a raiding company may be able to obtain control over the firm's assets for less than their aggregate purchase price.

Insofar as share prices are determined by profits this proposition also holds for low profits. After a takeover, top management in the raided company is often sacked or demoted or rearranged or tightly controlled by the parent company, and the firm's managers would like to avoid any of these. Provided managers avoid the dangers of shareholder action or takeovers, they have a monopoly of administration and are able to manipulate the firm to achieve their own goals.

Baumol says that managers' primary goals are prestige, salary, security of tenure, and a 'quite life'. This leads him to the proposition that the firm's objective is to maximize sales revenue, subject to a minimum profit constraint. He says that prestige among managers relates to sales and not profits and quotes the prestigious 'Young Presidents Organization' of the U.S.A., which requires that a member be a president of a company whose annual sales is in excess of a million dollars. Its profit position is irrelevant. A statistical study is quoted which indicates that managerial salaries are more closely related to sales than to profits. More studies have since become available and they confirm this proposition. The achievement of the last two goals is assisted by belonging to a large organization and meeting the constraints already mentioned. Further, managers are likely to be conservative in profit seeking than owner-managers because a successful gamble in terms of profits is unlikely to result in increased salaries, but a failure could lead to
embarrassment, demotion, or dismissal.

Clearly the constraint level of profits may be so high that managers have no slack and the firm is then a profit maximizing firm. In a dynamic version, Baumol uses the rate of growth of sales as the objective and profit ceases to be a constraint because sales varies directly with profits.

The structure of Williamson's model is more like that of Cyert and March than Baumol's. Managerial dominance is reflected through shifts in costs and the notion of 'expense preference' is crucial. By 'expense preference' Williamson means that 'some types of expenses have positive values attached to them: they are incurred not merely for their contributions to productivity (if any) but, in addition, for the manner in which they enhance the individual and collective objectives of managers.' Williamson chooses increasing salary, security, professional excellence and dominance as managers' main goals, with status, power and prestige as subgoals of dominance. These lead to positive expense preferences for staff expenses, emoluments and profit.

Increases in staff expenses, which correspond roughly to increases in general administrative and selling expenses, are desired because they are likely to lead to an improvement in an existing managers' position in the

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22 W.J. Baumol: 'On the Theory of Expansion of the Firm'
23 O.E. Williamson: 'The Economics of Discretionary Behavior..' p. 33.
administrative hierarchy. Staff is usually recruited for low positions in the hierarchy. As the manager's position improves, his ability to satisfy the above goals improve. 'Emoluments' are those parts of managerial salaries and perquisites which are discretionary and are rents paid to managers for their monopoly of control. Obviously it is in managers' interests to make those as large as possible.

A major difference between Williamson on the one hand and Baumol and Marris on the other, arises in relation to the firm's profit. He distinguishes 'minimum profit' and 'discretionary profit'. 'Minimum profit' is the profit constraint, discussed above, while 'discretionary profit' is profit in excess of this. It is required to provide capital for expansion and because it is a source of managerial utility itself. Unlike Baumol and Marris, Williamson holds that profit is a measure of self-fulfilment and organizational achievement and it is desired by management for its own sake.

After the most exhaustive analysis of managers, Marris concludes that managers' goals are power, prestige, salary and security, and he ties the first three to the rate of growth of the firm and the last to the rate of return on the firm's assets. Thus the objective is the maximization of the firm's rate of growth and the rate of return is a constraint. If the sales-assets ratio is assumed to be constant, this model is very similar to Baumol's dynamic model.
Like Williamson, Marris claims that a major way for a manager to satisfy his goals is to be promoted within the organization. Promotion is usually restricted for a manager in the higher ranks since inter-firm transfers are difficult and the retirement, sacking or death of his superiors is uncommon. His chances of promotion are much greater, however, if the firm is growing. Marris assumes that the rate of change of staff expenses depends on the rate of change of the book value of the firm's assets. In this respect Marris' model is dynamic and uses a more general objective.

Promotion gives managers more power and prestige. Basic salary is the largest single source of income for managers excepting those of giant corporations and salaries are dependent on the number of the managers' subordinates and not on profits. Bonus schemes for managers are also largely scale dependent. The granting of stock options to managers probably encourages them to increase their shareholdings in the corporation and thus encourage them to adopt owners' goals. He believes, however, that managerial shareholdings on the present scale, or a substantially increased one, is unlikely to lead to profit maximizing behaviour.
Chapter II
DETERMINANTS OF SHARE PRICES

1. Theory

In the share-price maximization models the assumptions made about the determinants of the price of ordinary shares are of course crucial, and in this chapter these assumptions, and others relating to the cost of capital, will be discussed.

The price of an ordinary share at any time \( t = 0 \), when the firm is expected to exist for some finite time \( t \in [0,T] \) is assumed to be

\[
P(0) = \int_0^T D(t)e^{-\delta t}dt + \left[ 1 + \frac{N(T)}{S(T)} \right]e^{-\delta T}
\]

when \( D(t) \) is dividend per dollar of an ordinary share; \( \delta \) is the market determined rate of capitalization, or rate of discount, of the returns to shareholders; \( N(T) \) is the book value of accumulated funds at \( T \); \( S(T) \) is the book value of shareholders' contributions on which dividends are paid and on which capital distributions are made, at \( T \); and \( T \) is the date at which the firm is wound up. The first term in expression (1), then, is the present value of dividends "per share" received over the life of the firm. The second term in the expression is the present value of the capital
distribution per share which is made when the firm liquidates. No account is taken in this formulation of the possibility of capital loss or gain when the firm liquidates and this will remain so throughout most of the following discussion. The reason for this, and what happens when this possibility is admitted, is discussed in the next chapter. A more detailed account of expression (1) is also given there.

If the firm exists forever then expression (1) becomes

\[ P(0) = \int_{0}^{\infty} D(t)e^{-\delta t} dt \quad \ldots \quad (2) \]

As \( T \) tends to infinity the last term in (1) tends to zero because \( e^{-\delta T} \) tends to zero and any realistic value of \( \frac{N(T)}{S(T)} \) will not prevent the product of these terms tending to zero.

For a share price maximizing firm the period it remains in existence in its present form must be decided with reference to its objective and cannot be regarded as being determined regardless of market conditions.\(^1\) This is not so, however, for growth maximizing firms because it is essential for the welfare of managers that the firm is not liquidated. Thus it will be assumed that the growth maximizing firms have an infinite life.

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\(^1\) Usual formulations of the share price maximization problem, however, do not give the firm the option of a finite horizon. None of the models referred to in this thesis do so.
At this stage three things need emphasizing about the above formulation of share prices. The first is that the interim benefits received by shareholders are dividends per share and not "earnings rate of return" (profits per share).

The Traditional Theory\(^2\) of the cost of capital assumes that shareholders believe that the firm's managers are acting in their best interests and so it is argued that shareholders should be indifferent between a return in the form of increased dividends or in the form of increased equity in the firm. Thus it is the sum of these which compose the 'return' to shareholders. Two methods\(^3\) capitalizing returns are used. The first is to capitalize profits after deducting interest payments, and the second is to capitalize profits before deducting interest payments. In the latter method debt is deducted from the capitalized value to obtain the market value of the firm's ordinary shares. The market values will be different if the same rates of capitalization are used and the latter method implies that all of the risk of capital loss is borne by the shareholders. The former method of capitalization is by far the most common used. The


Modigliani-Miller theory,\textsuperscript{4} which differs in many respects from the Traditional Theory also uses the earnings as 'returns'.

M.J. Gordon\textsuperscript{5} points out that the practice of using earnings and not dividends necessarily involves double counting if the firm ploughs back any of the profit, because shareholders receive all of the profit in the current year and the returns from investing part of that profit in subsequent years. This point, however, is not necessarily relevant since it is how people behave, and not the logic of their behaviour, that is important.

Lack of research in this field makes any statement on the issue questionable. However, we will hold the position that shareholders regard dividends, and not earnings, as their returns. Certainly earnings are important to shareholders but this is because they are used in the prediction of future dividends. Empirical evidence, which will be referred to later, marginally supports the dividends hypothesis in favour of the earnings hypothesis and the belief that managers try to iron out fluctuations in dividends by adding to and withdrawing from accumulated funds also supports this view.


\textsuperscript{5} M.J. Gordon: The Investment, Financing and Valuation of the Corporation, (Richard D. Irwin, Inc., Illinois, 1962), Ch. 5.
The remaining two things that need discussing at this stage are the supply of debt finance to the firm and the rate of capitalization of future dividends, and in both of these matters attitudes towards the variability of returns are crucial. The usual assumption, and the one adopted here, is that a creditor or a shareholder prefers a less variable stream of income to a more variable one, when the means of both streams are the same. In principle, of course, the variability of a stream of income can be ironed out by arbitrage but the uncertainty of the stream prevents this from being the case in practice. It is also assumed that investments with a high probability of return of capital are preferred to those with a low probability. Thus the returns from investments, whether in the form of the rate of interest or the rate of capitalization must be higher for investments with high variability of returns and low probability of return of capital than in the opposite case.

Theories of the cost of capital have concentrated on the effects of leverage on the rate of interest and the rate of capitalization. Traditional Theory holds that the rate of interest is a constant over a wide range of leverage and then it increases rapidly. If it is assumed that a firm will try to honour its interest debt then it will default only when interest is greater than current profits plus available debt finance and accumulated funds. Thus there will be no default until leverage (interest payment) is large. Thenceafter the likelihood of insolvency becomes very great.
As the difference between interest payable and interest paid increases some creditors will press for a winding up of the firm, and on doing so those creditors with less secure claims on the firm's assets may suffer capital loss. Since fluctuations in operating profits aren't known with certainty, the rate of interest does not become infinite after what is generally believed to be the critical leverage, but it does increase very rapidly.

In Diagrams I, II, III and IV, \( r \) is the average rate of interest payable on the firm's debt, \( \delta \) retains its previous meaning as the rate at which dividends are capitalized, and \( L \) is the measure of leverage. The precise measure of leverage varies between writers although most writers use the ratio of the market value of hands to the market value of the firm's shares. The market value of the firm's shares is, of course, the market value of the shareholders' equity, \([N(t) + S(t)]\) in the firm. Diagram I shows the traditional view. The rate of interest is a constant over a wide range of leverage and then it increases rapidly. The Modigliami-Miller theory does not consider the assumption explicitly but it must assume that the interest rate is a constant over all leverage for its propositions to produce reasonable results. The dotted lines show the case where this is not assumed. Again leverage is measured as a ratio of market values.

The assumption to be used in the following models regarding the supply of debt finance is shown in Diagram IV. It differs in two respects from the traditional assumption.
DIAGRAM I
TRADITIONAL THEORY

DIAGRAM II
MODIGLIANI-MILLER THEORY
The rate of interest is a constant over a wide range of leverage and then it becomes variable. This assumption is that the expected rate of interest is a constant over a wide range of leverage and then it becomes variable. The assumption is that it is constant until investment exceeds some critical point and then it becomes variable. Another way of stating this assumption is that, debt finance is available at a constant rate of interest when available. Debt finance is not available outside that range. Thus, we have a step function assumption about r(L) and δ(L) and we have a step function.
The first is that the measure of leverage is different. In our models leverage is measured as the ratio of the book value of the firm's debt, $B$, to the book value of the shareholders' total equity in the firm, $(N + S)$. Thus

$$L = \frac{B}{N + S} \quad \ldots \quad (3)$$

The rationale behind using leverage as a determinant of the cost of capital remains as stated before, namely the fear of capital loss and its affect on the variability of returns to shareholders and creditors and can be justified along similar lines.

The second respect in which our assumption about the supply of debt finance differs from the Traditional assumption is of little importance. The Traditional assumption is that the average rate of interest is a constant over a wide range of leverage and then it increases rapidly. The assumption to be used here is that it is constant until leverage reaches some critical value, $k$, and then it becomes infinite. Another way of stating this assumption is this, debt finance is available at a constant rate of interest, $r$, for $L \in [0, k]$; debt finance is not available outside that range. Thus

$$0 \leq B \leq k(N + S) \quad \ldots \quad (4)$$

This is a simplifying assumption. If we assume a more gradual slope then solving the following models would require assumptions about $r'(L)$ and $r''(L)$ and we have no detailed knowledge about these.

\[ The \ 'function \ of \ t' \ part \ of \ the \ following \ statements \ has \ been \ deleted. \]
Again, for our purposes, the rate of capitalization function will be simplified. It will be assumed the $\delta(L)$ is a constant. This seems to be preferable to the Traditional assumption that $\delta(L)$ is an increasing function when dividends and not earnings are used as returns because in the former case the firm can draw on accumulated funds to maintain the dividend rate. Strictly the $\delta(L)$ function should be that shown in Diagram III where the level of accumulated funds is higher for $\delta_3(L)$ than $\delta_2(L)$ and higher for $\delta_2(L)$ than $\delta_1(L)$. However the simpler case shown in Diagram IV will be used.

The relationship between $\delta$ and $r$ depends on the particular circumstances surrounding the firm. Normally, of course, it would be expected that $\delta > r$ because both shareholders and creditors are likely to be equally security conscious and the income and capital associated with debt is more secure than that associated with shares. In the case where shareholders are few and strongly associate themselves with the firm, there may be benefits which more than compensate for the lack of security and result in $\delta \leq r$. Thus this possibility will also be considered.

2. Empirical Results

A number of empirical examinations of the determinants of share prices have been conducted and the more relevant ones for our purposes deserve comment. It is not always meaningful to departmentalize results into such headings as 'earnings versus dividends' because the limitations of regression techniques makes an evaluation of conclusions concerning
variables dependent on the other variables in the model, and so we will discuss results by models rather than by variables.

M.J. Gorden, however, has constructed models for the purpose of resolving the earnings versus dividends dispute. In 1959 he published an article in which the model

\[ P = \alpha + \beta D + \gamma Y \] 

where \( P \) was the price of shares at the end of the year, \( D \) was dividend per share and \( Y \) was earnings per share, was fitted to data relating to a group of companies in the U.S.A. for the years 1951 and 1954. The sample was made up of 32 chemical companies, 52 foodstuff companies, 34 steel companies and 46 machine tool companies. He found that dividends were statistically significant in seven of the eight samples and that earnings were statistically significant in six of the eight samples. In addition, the standard errors of \( \beta \) were in general smaller in relation to \( \beta \) than the standard errors of \( \gamma \) were in relation to \( \gamma \). The coefficients of determination of all samples were above 0.73. Content that the test favoured the dividend hypothesis he then fitted

\[ P = \alpha + \beta D + \gamma (Y - D) \] 

to the same sample, where \( (Y - D) \) was used as an index of the expected growth in dividends. The coefficients of determination, the estimates of \( \gamma \) and their standard errors

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remained unchanged but the estimates of $\beta$ changed and the standard errors decreased. Current dividends become a stronger determinant when expected dividends are entered separately and this adds more weight to the dividend hypothesis.

In an article published in 1960, Gordon uses the same sample to test the importance of dividends, an index of the rate of growth of dividends, an index of its size, and index of uncertainty of earnings, and leverage as determinants of share prices. Dividends were statistically significant in all eight samples, and the coefficient was close to the expected value in six of the eight samples. The growth coefficients were significant in seven out of eight samples, but the coefficients were very small. Size was significant in only three of the eight samples (at the five percent level of significance). The uncertainty index had the right sign in all but one sample but it was significant in only three of the eight samples. The results concerning leverage are interesting. The coefficient was negative in four samples and positive in four samples, and it was statistically significant in only three samples. The coefficients of determination were above 0.74 for all samples.

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The poor performance of the rate of growth of dividends prompted Gordon\(^9\) to seek a new index. This time he restricted his sample to a group of firms in the food and machinery industries and used data for the years 1954, 1955, 1956 and 1957. He also used new measures of firm size and earnings uncertainty. The coefficient for the growth index was still very small but it was significant for all eight samples at one percent level of significance. Dividends themselves, however, increased in significance. The size coefficients were significant in six of the eight samples and the uncertainty coefficients were significant in five samples (both at five percent level of significance). No measure of leverage was included. The coefficients of determination were higher than in the previous studies.

Gordon conducts his most exhaustive empirical study of the determinants of share prices in his book *The Investment, Financing and Valuation of the Corporation*.\(^{10}\) The sample is made up of 48 food companies and 48 machinery companies and the data relate to the five years 1954 to 1958. The two models developed here are an extension of this model in his 1962 article. The first is called the 'simple leverage model' and the measure of leverage is the ratio of net debt per share to the corporation's net worth per share. The results of this

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\(^{10}\) M.J. Gordon: *The Investment, Financing and Valuation of the Corporation*.
model were as follows: dividend coefficients performed about the same as in the previous model; the performance of growth coefficients did not materially improve; earnings uncertainty coefficients were significant in two of the five food samples and three of the five machinery samples; only one in five of the food samples and one in five of the machinery samples didn't have a significant size coefficient. In the case of the leverage coefficient, three of the five food samples had a significant coefficient and none of the machinery samples had a significant coefficient.

The other model uses Modigliani and Miller's leverage proposition in its construction and the results differ significantly between the two samples. In the food sample dividends, growth of dividends and leverage were highly significant and fluctuated within a small range of values. The index of firm size was significant for four years but it was not large and uncertainty variables performed less well. By contrast, in the machinery sample only the dividends coefficient was highly significant for the five years. The growth coefficient performs unevenly and is not significant in one year. Earnings uncertainty and size coefficients are significant in only three years, and the leverage coefficient has the wrong sign for two of the years.

Gordon's results are important for our purposes because he uses dividends as returns, and because he has concerned himself with the determinants of the rate of capitalization. His results favour the use of dividends rather than earnings and
throw serious doubts on leverage as a determinate of the rate of capitalization. His results also indicate that the size of the corporation is not an unambiguous determinant of the rate of capitalization and this serves to justify the assumption of independence of the two, which will be made later.

Most empirical research on the determinants of the rate of capitalization have assumed that earnings were the returns to shareholders, and consequently the measure of the rate of capitalization was the earnings-price ratio. It is very likely, of course, that the earnings-price ratio and the dividends-price ratio are related and so the results of these studies are of some interest to us. They do not warrant a detailed discussion and the results have been tabulated, at some injustice to the authors, in Table A.

The first set of results are for models tested by S.H. Archer and L.G. Faerher. The data relate to 238 U.S. firms and the three years 1960 to 1962. Model A in the table gives the results for a model which included all of the variables shown as 'yes' or 'no'. Model B omitted 'size of the issue' as an explanatory variable and Model C omitted 'size of corporation'. The level of significance for the t-test at which the coefficients were found to be significant are stated after 'yes'. The authors also included a measure of 'investor sentiment' but the results relating to this variable are of no interest and they have been omitted.
### Table A

**Statistical Significance of Explanatory Variables**

<table>
<thead>
<tr>
<th>Authors</th>
<th>Regressand</th>
<th>Size of Corporation</th>
<th>Leverage</th>
<th>Growth in Earnings</th>
<th>Variation in Earnings</th>
<th>Age of Corporation</th>
<th>Dividend Payout Ratio</th>
<th>Size of the Share Issue</th>
<th>Growth in Equity Value</th>
<th>Stability of Equity Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. S.H. Archer and L.G. Faerber</td>
<td>A Earnings - Price Ratio</td>
<td>Yes, 1%</td>
<td>No</td>
<td>Yes, 5%</td>
<td>No</td>
<td>No</td>
<td>-</td>
<td>Yes, 1%</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>B Earnings - Price Ratio</td>
<td>Yes, 5%</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>Yes, 5%</td>
<td>-</td>
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<tr>
<td></td>
<td>C Earnings - Price Ratio</td>
<td>-</td>
<td>No</td>
<td>No</td>
<td>Yes, 1%</td>
<td>-</td>
<td>Yes, 1%</td>
<td>-</td>
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<tr>
<td>2. A. Bargess</td>
<td>Earnings - Price Ratio</td>
<td>-</td>
<td>No</td>
<td>-</td>
<td>-</td>
<td>-</td>
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<tr>
<td>3. H. Benishay</td>
<td>A Earnings - Price Ratio</td>
<td>Yes</td>
<td>Yes in one sample</td>
<td>No</td>
<td>-</td>
<td>Yes, in two samples</td>
<td>-</td>
<td>Yes, in No samples</td>
<td>-</td>
<td>-</td>
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<tr>
<td></td>
<td>B Earnings - Price Ratio</td>
<td>Yes</td>
<td>Yes in one sample</td>
<td>-</td>
<td>-</td>
<td>Yes, in three samples</td>
<td>-</td>
<td>Yes, in No samples</td>
<td>-</td>
<td>-</td>
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<tr>
<td></td>
<td>C Earnings - Price Ratio</td>
<td>Yes</td>
<td>Yes in two samples</td>
<td>No</td>
<td>-</td>
<td>Yes, in one sample</td>
<td>-</td>
<td>-</td>
<td>Yes, in two samples</td>
<td>-</td>
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<tr>
<td>4. R.F. Wippern</td>
<td>A Earnings - Price Ratio</td>
<td>Yes, 5% in three samples, no in one</td>
<td>Yes, 5%</td>
<td>Yes, 5%</td>
<td>-</td>
<td>-</td>
<td>Yes, 5% in three samples, 10% in one</td>
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<td></td>
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<td>Yes, 0.5%</td>
<td>Yes, 0.5%</td>
<td>-</td>
<td>-</td>
<td>Yes, 0.5%</td>
<td>-</td>
<td>-</td>
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**Sources:**
- A. Bargess: "The Effect of Capital Structure on the Cost of Capital" (Prentice-Hall, Inc., 1963);
A. Barrie has conducted the most exhaustive test of the relationship between the rate of capitalization and leverage. The sample was made up of 61 Class I railway companies, 63 department store companies, and 34 cement companies. Two measures of leverage were used, one including preferred stock in the numerator of the debt-equity ratio and the other including it in the denominator. The book values of debt and equity were used. In all of the tests the leverage coefficient was found to be not significantly different from zero.

H. Benishay used a sample of 56 companies and obtained data for the four years 1954 to 1957. He fitted three types of models for each year, making twelve samples in all. Model A included all of the variables not left blank in the table. 'Yes' means that the variable was significant in each of the four samples. A variable is said to be significant if the modulus of its coefficient is more than half its standard error. Model B is the same as Model A, excepting 'growth of earnings' has been deleted as an explanatory variable. Model C is the same as Model A, excepting 'growth in equity value' has been deleted.

R.F. Wippern chose a sample of 50 companies from a wide range of industries and used data relating to the four years 1956, 1958, 1961 and 1963. He fitted the model to the data for each year separately and the results are shown alongside 'A' in the table. 'Yes' under growth in earnings means yes for all four samples. He also pooled the information for the four years and the results relating to these are alongside 'B'. 
Although the measures of the particular concepts involved differ between authors some generalization from the table is possible. Clearly leverage performed poorly although it is more significant in Wippern's tests than the others. Size is the only variable which performs well over all the tests which include it and this shakes faith in Gordon's proposition that the rate of capitalization is little affected by size. It must be remembered, however, that the measures of the rate of capitalization are different and these results aren't conclusive.
Chapter III

STRUCTURE OF THE MODELS

1. Dynamics

Theories of financing and investment of the firm by such writers as D. Durand, F. Modigliani and M.H. Miller, M.J. Gordon and R. Marris are concerned with deriving optimum stationary conditions. With varying degrees of completeness they answer this question: assume that product and factor market conditions and the goals of the firm are given, what is the optimum capital stock and financial structure when all adjustments have been completed? This question is of undoubted interest, but in a world in which adjustment is not instantaneous the path of adjustment from a non-optimum position to an optimum one is important to a description of firm behaviour. A firm is seldom at a stationary optimum and at any time, for given goals and market conditions, it is the costs of capital and the speed with which each type can be obtained and retired that determine investment policies and capital structure. The models to follow will attempt to recognize the varying dynamic characteristics of each type of finance.

---

For dynamic problems the maximization conditions of ordinary calculus are inadequate and models must state variables as explicit functions of time. A model may be set up as a system of difference equations or as a system of differential equations. Which procedure is more realistic is not clear. If difference equations are used then it must be assumed that events occur at distinct points of time and while this is in part true, the intervals between these 'action times' are not as regular as the model builder is forced to assume. Unless the cumbersome practice is followed of selecting very small time intervals, a point of time can always be found at which an event is likely to occur but which has no corresponding time unit in the model, and for which rounding to the nearest t would offend reality. The opposite problem causes concern when differential equations are used. Such a model overstates the time continuity of events. Motion of the system is never stopped often enough to correspond to reality. Clearly it is always possible to construct a difference equation model which is more realistic than a differential equation model for a particular problem, and vice versa.

Differential equations, however, are in general much more powerful tools of analysis than are difference equations and it is essential to use them in the analysis of the problems with which we are concerned.
The models will involve the following mathematical problem:

\[
\begin{align*}
T & \quad \text{maximize} \int_0^T f(x(t), u(t), t) \, dt + S[x(T)] \quad \ldots (1) \\
\text{subject to} \quad x(t) &= f(x(t), u(t), t), \quad \frac{dx}{dt} = f(x(t), u(t), t) \quad \ldots (2) \\
x(0) &= x_0, \quad x(t) \in X, \quad u(t) \in U \quad \ldots (3)
\end{align*}
\]

where \(x(t)\) is a continuous \(n\)-dimensional vector, \(u(t)\) is a piecewise continuous \(r\)-dimensional vector, \(f\) is a vector of \(n\) functions, and \(S[x(T)]\) is the value of the state vector at the terminal time. The usual technique for solving such problems is the calculus of variations and this technique is capable of providing solutions for most of the problems to follow. It is, however, cumbersome and instead Pontryagin's Maximum Principle in some of its forms will be used.

2. **Assumptions**

One of the most controversial assumptions that will be made in the following chapters is that of reversibility of capital. It will be assumed that the firm can sell its capital stock at the purchase price less depreciation. This assumption is objectional because depreciation is charged at the technical rate - that rate which is required to maintain the equipment's operating capacity.

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See also the appendix to this chapter.
Whether reversibility or irreversibility is the more appropriate assumption depends on the type of capital under consideration. Land and buildings sometimes yield a capital gain or no capital loss on sale because they were materially better located (through external economies) than when purchased. Specialized equipment and equipment subject to rapid obsolescence, however may yield significant capital losses on sale. The reversibility of capital will vary with general market conditions as well.

Apart from the assumption having some empirical validity it is also a simplifying assumption. K.J. Arrow and associates have investigated optimal investment policy under the assumption of irreversibility. Diagram I illustrates their problem. Suppose the firm wants to maximize the discounted profits from its operations, then there will be some optimal

![Diagram I](image-url)

relationship between market conditions and capital stock.
Suppose also that market conditions fluctuate so that the
optimal time pattern of capital is shown as $K^*(t)$. Because
of the irreversibility of capital any pattern of investment
will result in periods of deficient capacity and/or excess
capacity. Which is the optimal pattern? There is no simple
rule to follow and Arrow et al. could only offer an algorithm
which gives the optimal pattern as a result of iterating
possible solutions. Their solution is too complex to be used
in the models to be developed here, which are 'realistic' in
another way. The solutions obtained here, however, like theirs
are myopic. They will be stated in terms of the current values
of the relevant variables, and not in terms of expected values.

Much of the notation and many detailed assumptions
are common throughout the following chapters and these will be
stated here.

The notation is

$P(t) =$ market value per dollar of the 'paid-up' portion of
an ordinary share at time $t$;

$S(t) =$ total of paid-up ordinary capital at time $t$, and for
this the term 'shares' will be used;

$D(t) =$ dividends paid per $1 of paid-up ordinary share at
time $t$ (total dividends paid divided by total paid-up
capital);

$N(t) =$ book value of accumulated funds at time $t$;

$B(t) =$ book value of the firm's debt at time $t$;

$K(t) =$ book value of capital stock at time $t$;
\[ F[K(t)] = \text{net profit plus depreciation charged and interest at time } t; \text{ this will be called 'operating profit' even though it includes administrative costs; } \]

\[ r = \text{average rate of interest payable on debt; } \]

\[ \lambda = \text{depreciation rate, assumed to be a constant; } \]

\[ \Pi(t) = \text{net profit at time } t; \]

\[ \delta = \text{market rate of capitalization of future dividends; } \]

\[ x(t) = \text{level of output at time } t. \]

Other notation will be defined as it is introduced.

The total market value of a firm's shares at any time \( t = 0 \) is equal to the sum of discounted dividends paid on those shares plus the discounted capital distribution made on those shares when the firm liquidates. If no capital loss is assumed then

\[
P(0) \cdot S(0) = \int_0^T D(t) \cdot S(0) e^{-\delta t} \, dt + \left[ S(0) + \frac{N(T)}{S(T)} S(0) \right] e^{-\delta T} \quad \ldots (4)
\]

and so the price of a share at \( t = 0 \) is

\[
P(0) = \int_0^T D(t) e^{-\delta t} \, dt + \left[ 1 + \frac{N(T)}{S(T)} \right] e^{-\delta t} \quad \ldots (5)
\]

Profits can either be paid as dividends, in part or in full, or they can be ploughed back. Dividends, however, can be paid in excess of profits by running down accumulated funds. Thus if \( U(t) \) is the change in accumulated funds
at \( t \), we can write

\[ N(t) = U_1(t), \quad \frac{d}{dt} \]

and

\[ D(t) = \left[ \Pi(t) - U_1(t) \right] \frac{1}{S(t)} \]

\[ \Pi(t) - U_1(t) \geq 0 \]

Firms are legally prevented from 'buying back' ordinary shares and so we may write

\[ S(t) = U_2(t), \quad \frac{d}{dt} \]

\[ U_2(t) \geq 0 \]

Shares are assumed to be issued at par and it is assumed that the firm can obtain share finance at par if it wants it.

As defined in (8) and (9) above the control sets for \( U_1(t) \) and \( U_2(t) \) are unbounded in one direction with the implication that \( N(t) \) can be a step function in a downwards direction and \( S(t) \) can be a step function in an upwards direction. However, a jump in \( N(t) \) or \( S(t) \) can only occur at \( t = 0 \). The theorems of the maximum principle have been developed for two types of models. Pontryagin et.al. assume that the control sets are compact and this implies

\[ ^5 \text{L.S. Pontryagin et.al.: The Mathematical Theory of Optimal Processes.} \]
that the state variables are continuous functions of time. Hestenes, however, does not require the control sets to be compact but defines the state variables as being continuous over time. This implies that the control sets have limits other than at $t = 0$ and $t = T$, and jumps in $N(t)$ or $S(t)$ at $t = T$ are of no significance in the models to be developed. Hestenes' formulation will be used here.

The rate of capitalization, $\delta$, will be assumed to be a constant and the size of the firm's debt will be constrained to

$$0 \leq B(t) \leq k[N(t) + S(t)] \quad \ldots(10)$$

The following accounting identities will be used

$$K(t) = N(t) + B(t) + S(t) \quad \ldots(11)$$

$$\Pi(t) = F(t) - \lambda K(t) - rB(t) \quad \ldots(12)$$

and

$$F(K, x) = R(K, x) - C(K, x) \quad \ldots(13)$$

where $R(K, x)$ is gross revenue and $C(K, x)$ is total non-financial cost. The following assumptions will be made about the operative profit function

$$F_K > 0, \quad F_{KK} < 0 \quad \ldots(14)$$

when output is optimal.

Finally, $K(t)$, $N(t)$, $S(t)$, and $x(t)$ must all be non-negative.

---

3. **Optimal Terminal Date**

The share price maximizing models allow for shareholders’ discretion in determining the date at which the firm ceases to exist in its present form. Although the phenomenon of shareholders accepting takeover bids or, to lesser extent, voluntarily liquidating the firm are common, there is negligible discussion of the optimality of these decisions in the theoretical literature. A few comments on the subject will be made here.

If the firm is to maximize its shares the following proposition is true:

**Proposition 1**

The optimal value of $T$ is that which satisfies

$$
\max_{D} \int_{0}^{T} D e^{-\delta t} dt + \eta(T) e^{-\delta T} \quad \ldots \quad (15)
$$

where $D = D(t)$ and $\eta(T)$ is the capital payment per share paid to shareholders for their equity in the firm on liquidation.

It can be shown that the value of $T$ which satisfies Proposition 1 also maximizes the price of shares at any $t = t'$, $t' \in [0, T]$. Notice that maximization of $P(0)$ requires

$$
\max_{D} \left\{ \int_{0}^{t'} D e^{-\delta t} dt + \int_{t'}^{T} D e^{-\delta t} dt + \eta(T) e^{-\delta T} \right\} \quad \ldots \quad (16)
$$
and the maximization of \( P(t') \) requires

\[
\max_D \int_{t'}^T D e^{-\delta t} dt + \eta(T)e^{-\delta T} \quad \ldots (17)
\]

Now, at \( t = t' \) the firm is in a particular state. The first term in (16) is a constant so that the policy which maximizes \( P(0) \) will be decided in relation to the second and third terms of (16). Apart from the term \( e^{\delta t'} \), which is not affected by the policy chosen, these are the same terms which appear in (17). Thus the policy over \( t \in [t',T] \) and the choice of \( T \) which maximizes \( P(0) \) also maximizes \( P(t') \). This result is dependent on the additivity of the objective function \( P(0) \). This is a demonstration of the principle of optimality.\(^7\) It holds that any part of an optimal path, is itself an optimal path providing the objective function remains unchanged.

The term \( \eta(T) \) has two possible interpretations. If a raiding firm attempts to take over the firm the \( \eta(T) \) represents the price the raider is willing to pay the shareholders for their shares. Takeover bids are usually made without notice so as to avoid the price of the shares increasing in anticipation of capital gains and \( T \) is usually \( T = 0 \). Alternatively, the firm may decide to wind up operations and sell its capital stock, and in this case \( \eta(T) \) is the per share distribution of net assets.

Let the liquidated value of the capital stock be

\[
\Theta(T) K(T) \quad \ldots (18)
\]

If the firm is accepting a takeover bid then this represents

the price the raider pays for the firm's capital stock. This value must be greater than the value of capital to the firm and/or the cost of capital to the raider must be less, if the takeover is to be optimal to the raider. If the firm, however, goes into voluntary liquidation then this represents the total returns from the sale of the assets. Clearly if \( \theta(T) > 1 \) then the firm experiences a capital gain on the book value of its assets and if \( \theta(T) < 1 \) then it experiences a capital loss. If \( \theta(T) = 1 \) then there is no capital gain or loss.

The total capital returnable to shareholders on liquidation is

\[
\theta(T)K(T) - B(T)
\]

... (19)

and the per-share value is

\[
\theta(T) \left[ 1 + \frac{N(T)}{S(T)} \right] + \left[ \theta(T) - 1 \right] \frac{B(T)}{S(T)}
\]

... (20)

so that

\[\eta(T) = \theta(T) \left[ 1 + \frac{N(T)}{S(T)} \right] + \left[ \theta(T) - 1 \right] \frac{B(T)}{S(T)} \]

... (21)

Thus Proposition 2:

The value of \( \eta(T) \) is given by

\[\eta(T) = \theta(T) \left[ 1 + \frac{N(T)}{S(T)} \right] + \left[ \theta(T) - 1 \right] \frac{B(T)}{S(T)} \]

where \( \theta(T) \) the value of the firm's assets to the purchaser divided by the book value of the firm's assets.

Further progress can be made if it is assumed that \( \theta(T) = 1 \) so that

\[\eta(T) = \left[ 1 + \frac{N(T)}{S(T)} \right] \]

... (22)
Now, if product and factor market conditions are constant and the firm exists long enough the optimum values of \( D(t) \), \( N(t) \) and \( S(t) \) will (normally) become constants. Suppose the constants are \( D^*, N^* \) and \( S^* \). It is possible that a model with these market assumptions might be such that there are no long run equilibrium values but such a result does not accord with observed behaviour and it would indicate some deficiency in the model. Thus attention will be restricted to the case where the optimal values do become constants.

The question is now asked: if the time at which all variables attain their long run equilibrium values is \( t = t_1 \) on the assumption that \( T = \infty \), should the firm liquidate at \( t = t_1 \) or remain in existence forever?

Clearly, the firm should liquidate at \( t = t_1 \) if

\[
\int_{t_1}^{\infty} D^* e^{-\delta t} dt < \left(1 + \frac{N^*}{S^*}\right) e^{\delta t_1} 
\]

...(23)

or if

\[
\frac{D^*}{\delta} < \left(1 + \frac{N^*}{S^*}\right) 
\]

...(24)

The firm should set \( T = \infty \) if the inequality in (24) is reversed or if it holds with equality (providing inertia is assumed).

Thus Proposition 3 has been shown.

**Proposition 3:**

Suppose that for \( T \) fixed at \( T = \infty \) it is optimal for a firm to reach a stationary position given by \( (D^*, N^*, S^*) \) at \( t = t_1 \) and to remain there for all \( t \in [t_1, \infty) \), then if it is
optimal to continue operations until $t = t_1$, the optimal terminal date, $T^*$, is

(a) $T^* = t_1$ if $D^* < \delta(1 + \frac{N^*}{S^*})$

(b) $T^* = \infty$ if $D^* \geq \delta(1 + \frac{N^*}{S^*})$

The rate of capitalization, $\delta$, is the highest alternative rate of return which can be obtained from an alternative investment with similar characteristics in terms of security, prestige, etc. Thus condition (a) above says that if the dividend per share is less than the alternative return which can be obtained by investing shareholders' equity elsewhere then the firm should liquidate immediately. Condition (b) says that if this is not so then the firm should last forever.

Since, for any $t \in [t_1, \infty]$ the price of shares, $P^*$, is

$$P^* = e^{-\delta t} \int_{t_1}^{\infty} D^* e^{-\delta t} dt$$

$$P^* = \frac{D^*}{\delta}$$

Condition (b) can be written

$$P^* \geq (1 + \frac{N^*}{S^*})$$

Thus it is optimal for $T$ to be $T = \infty$ if the price of shares is greater than the per-share capital distribution which would be made if the firm was liquidated.

This completes the discussion of the optimal terminal date for the firm. The optimal terminal date in cases other than the one examined can be decided only by a comparison of integrals and terminal distributions and by applying Proposition 1.
Appendix A

CALCULUS OF VARIATIONS AND THE MAXIMUM PRINCIPLE

The purpose of this appendix is to draw attention to certain similarities and differences between the calculus of variations and the maximum principle. Much more systematic and technical accounts are available in the works by L.S. Pontryagin et al., and M.R. Hestenes.¹

The calculus of variations is concerned with extreme values of functionals. A functional, or a function of a function, is a correspondence which assigns a particular value to each function in a particular class. It is often written

\[ I = \int_{t_0}^{t_1} f(t, x, \dot{x}) \, dt, \quad \dot{x} = \frac{dx}{dt} \quad \ldots (1) \]

The value of \( I \) depends on the 'form' of the function \( x(t) \). The time horizon, \((t_1 - t_0)\) and the endpoints \((x(t_0), x(t_1))\) may be free or fixed and side conditions or constraints may be imposed to limit the set of admissible functions, \( x(t) \).

One classical example in the calculus of variations is John Bernoulli's brachistochrone problem. In 1696 he attracted general attention to the problem of minimizing a functional by posing the question: suppose there is an object which is to slide, by the force of gravity alone but in the absence of friction, from one position to a lower one in the minimum time. What is the optimum path?

If the object's position can be defined in two dimensions, position \( x \) and time \( t \), then we can represent the problem in Diagram I. In the diagram the object must slide from \( x(0) \) to \( x(1) \) so as to minimize \( t_1 - t_0 \) and \( A, B, C \) are some of possible paths. Since it can be shown that \( t_1 - t_0 \) is proportional to

\[
I = \int_{t_0}^{t_1} \frac{1}{x} \sqrt{1 + (\dot{x})^2} \, dt \tag{2}
\]

the problem can be solved by finding the function \( x(t) \) which minimizes this integral.

Another example is the isoperimetric problem. This is the problem of finding the closed curve of a given length which encloses the largest possible area. The answer is a circle.

In one of the simplest problems Euler's differential equation gives a necessary condition for a functional to have a (weak) extremum. Let \( f(t, x, \dot{x}) \) be a function with continuous first derivatives with respect to \( t, x, \) and \( \dot{x} \) and
subject to:

\[ \dot{x} = f(x,u) \]
\[ u \in U, \quad x \in X \]  \hspace{1cm} \ldots (7)

The functions \( f(x,u) \) are assumed to be continuous in \( x \) and \( u \) and continuously differential in \( x \). The necessary condition for a weak extremum involves the maximization of a Hamiltonian, \( H \), with respect to \( u \) for \( u \in U \) and \( x \in X \), when

\[ H \equiv cf^0(x,u) + \lambda f(x,u) \]  \hspace{1cm} \ldots (8)

and \( \lambda \equiv (\lambda_1, \lambda_2, \ldots, \lambda_n) \) are continuous with piecewise continuous derivatives and \( c > 0 \) if the integral is to be maximized and \( c < 0 \) if it is to be minimized. The motion of the auxiliary variables is given by

\[ \frac{d\lambda_i}{dt} = - \frac{\partial H}{\partial x_i}, \quad i = 1,2,\ldots,n \]  \hspace{1cm} \ldots (9)

Another condition which must be met is that either the initial and terminal values of the state variables must attain specific values or transversality conditions hold. The value of \( H \) is often known. The problem may also include terminal functions involving \( x \) in the objective function.

Hestenes' extension of Pontryagin's formulation of the maximum principle has enabled many problems to be solved more simply. The principal differences are that the control set, \( U \), needs no longer be compact, that equality and inequality constraints are treated explicitly in the Hamiltonian, and constraints in the form of integrals can be handled. He used an isoperimetric problem with constraints

in his formulation because it brings out all of the constraints his formulation can handle. Thus the problem is to maximize (or minimize)

$$I = g_0(b) + \int_{t_0}^{t_1} f^0(x, u, b) \, dt \quad \ldots (10)$$

subject to:

$$\dot{x} = f(x, u, b) \quad \ldots (11)$$

$$Q_\alpha (x, u, b) \leq 0, \quad 1 \leq \alpha \leq m' \quad \ldots (12)$$

$$Q_\alpha (x, u, b) = 0, m' < \alpha \leq m \quad \ldots (13)$$

$$I_{\gamma} \leq 0, \quad I \leq \gamma \leq p' \quad \ldots (14)$$

$$I_{\gamma} = 0, \quad p' < \gamma \leq p \quad \ldots (15)$$

when

$$I_{\gamma} = g_\gamma(b) + \int_{t_0}^{t_1} f_\gamma(x, u, b) \, dt \quad \ldots (16)$$

Time and/or endpoints may be free or fixed and the constraints $Q_\alpha$ may not include $u$. The additional symbol, $b$, refers to control parameters. The only restriction on the constraint functions $Q_\alpha$ is that the matrix

$$\frac{\partial Q_\alpha}{\partial u^k} \quad \ldots (17)$$

has rank $k$ when $\alpha_1, \alpha_2, \ldots, \alpha_k$ are the indices at which
\[ Q_\alpha (\bar{x}, \bar{u}, \bar{b}) = 0 \] 

...(18)

Arrow\textsuperscript{4} appears to have generalized this condition so that all that has to be met is the Kuhu-Tucker Constraint Qualification.\textsuperscript{5} Hestenes' rank condition is a sufficient condition for compliance with the Constraint Qualification.

The necessary condition for a weak extremum now involves an expanded Hamiltonian, or a Lagrangian, composed of the Pontryagin Hamiltonian and other terms which include the constraints. Now the Lagrangian, L, must be maximised with respect to \( u \), when

\[ (a) \quad L = H + \mu_\alpha Q_\alpha + \phi_y \] 

...(19)

\( (b) \mu_\alpha \) are piecewise continuous and continuous at each point of continuity of \( u^* \), and for each \( \alpha = m' \),
\[ \mu_\alpha(t) = 0 \] holds and \( \mu_\alpha(t)^T Q_\alpha = 0. \]

\[ (c) \phi_y \] are constants and \( \phi_y > 0, 1 \leq \gamma \leq p', \)
with \( \phi_y = 0 \) when \( I_y < 0 \). Since the maximization of \( L \) with respect to \( u \) is interior, \( u^* \) meets the condition

\[ \frac{\partial L}{\partial u^*} = 0 \] 

...(20)

The motion of \( \lambda \) is now given by

\[ \frac{d\lambda_i}{dt} = -\frac{\partial L}{\partial x_i}, \quad i = 1, 2, \ldots, n \] 

...(21)

---


Other conditions are concerned with the transversality condition or the specification of endpoints, and the value of the Lagrangian and these are similar to those of the Pontryagin formulation.

These results can be obtained from the calculus of variations but the control theory formulation is more convenient. Throughout the discussion in the following chapters only Hestenes' formulation will be used because it allows the greatest flexibility in setting up the models.

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Appendix B

TERMINAL CONDITIONS ON SHADOW PRICES

Two types of share price maximization models, in terms of time horizons, will be developed. The first will involve the maximization of an integral over the finite horizon, \( t \in [0, T] \), and the second involves maximization over the infinite horizon, \( t \in [0, \infty] \). When solving the second type of problem a number of economists\(^1\) have used the transversality conditions on the shadow prices when there was no justification for doing so. These conditions were developed for finite horizons and they may not be necessary conditions for an optimal solution when the horizon is infinite. Engineers either fix the endpoint and use an infinite horizon or leave the endpoint free and fix the horizon as large but finite.\(^2\)

This error had gone unnoticed until recently when the mathematician H. Halkin\(^3\) drew attention to it with an example. Since this is such an important point and still largely unrecognized by economists we will repeat the example here.


\(^3\) Footnote 1, pg. 31 of Chapter II in K.J. Arrow and M. Kurz: *Public Investment, the Rate of Return, and Optimal Fiscal Policy*: August 1968, (Institute for Mathematical Studies in the Social Sciences, Stanford University).
The problem is

\[
\text{maximize: } \int_0^\infty x \, dt \quad (1)
\]

subject to:

\[
x = (1 - x)v \quad (2)
\]

\[-x \leq v \leq 1 \quad (3)
\]

\[x(0) = 0 \quad (4)
\]

Now

\[
\int_0^\infty x \, dt = \lim_{t \to \infty} x(t) \quad (5)
\]

and by direct integration

\[
x(t) = 1 - e^{-v(t)} \quad (6)
\]

where

\[
v(t) = \int_0^t v(u) \, du \quad (7)
\]

Thus \(x(t) < 1\) for \(t \in [0, \infty)\) and the maximum value of the objective function, as \(t \to \infty\), is 1. Any \(v(t)\) which results in \(v(t)\) going to infinity as \(t \to \infty\) is optimal. One such control is \(v(t) = v_0, -1 < v_0 < 1\).

Now the Hamiltonian for this system is

\[
H = (1 + p) (1 - x)v \quad (8)
\]

where \(p\) is the auxiliary variable or shadow price. Optimization of the control implies

\[
\frac{\partial H}{\partial v} = 0 = (1 + p) (1 - x) \quad (9)
\]
and so \( p(t) = -1 \) for \( t \in [0, \infty) \). Thus as \( t \to \infty \), \( x(t) \to 1 \) and \( p(t) \to -1 \).

This violates the transversality condition which requires the terminal value of \( p(t) \) to be \( p(t) = 0 \) and since \( p(t) \) must be continuous, no jump is possible.

Thus the solution to a problem with an infinite horizon and free endpoint must be obtained without the use of a transversality condition and this increases the difficulty considerably. In particular problems the 'overtaking criterion'\(^4\) may assist in obtaining a solution.

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Chapter IV

A SHARE-PRICE MAXIMISATION MODEL WITH A GIVEN NUMBER OF SHARES

1. Introduction

This chapter is concerned with the special case where the firm does not increase the number of shares in the hands of the public. It warrants special attention for the following reasons.

Firstly, it will be assumed in Chapter V that shares are issued at par. Obviously there may not exist a policy which will raise the price of shares to par and so this assumption is inappropriate. Also, even if the firm can sell shares at par, it may not be optimal to do so and this chapter provides a systematic description of these cases.

Secondly, this chapter will assist in understanding the issues involved in deciding whether to issue shares. The act of issuing shares has two effects on dividends per share. It tends to reduce dividends per share by increasing the number of shares between which total dividends must be divided but it may increase dividends per share by allowing the firm to achieve the optimal capital stock sooner. How these factors compare is more easily understood once the choice between debt and ploughback finance is analysed.

Thirdly, the model may be interpreted as applying to firms under the particular circumstances where the existing owners do not wish to dilute their control by allowing new owners and where the existing owners are unable to or do not
wish to contribute more ownership capital themselves. This may be the case where the firm is a limited liability company but where the control is in the hands of a few shareholders, or to a partnership, or it may conceivably be considered to apply to some government owned enterprises. A change in the interpretation of P(0) may be necessary in the last case.

The model remains unchanged from that outlined in Chapter III excepting that

\[ S(t) = U_2(t) = 0 \] \hspace{1cm} \cdots (1)

and so

\[ S(t) = S(0) > 0. \] \hspace{1cm} \cdots (2)

The assumptions \( \delta > r \) and \( L \in [0,k] \) will remain.

2. The Problem

Mathematically, the problem is to find the optimal control vector \((U_1(t), B(t), x(t))\) and the trajectory of the state variable \(N(t)\), given \( t \in [0,T] \) for \( T \) and \( N(T) \) free which maximize

\[
\int_{0}^{T} \left[ \Pi(t) - U_1(t) \frac{e^{-\delta t}}{S(t)} \right] dt + \left[ 1 + \frac{N(T)}{S(T)} \right] e^{-\delta T} \hspace{1cm} \cdots (3)
\]

subject to:

\[ N(t) = U_1(t) \] \hspace{1cm} \cdots (4)

\[ \Pi(t) - U_1(t) \leq 0 \] \hspace{1cm} \cdots (5)

\[ 0 \leq B(t) \leq k \left[ N(t) + S(t) \right] \] \hspace{1cm} \cdots (6)
\( N(t), x(t) \geq 0 \) \hspace{1cm} \ldots(7)

\( N(0) = N_0, S(t) = S(0) > 0 \) \hspace{1cm} \ldots(8)

\( F_K > 0, F_{KK} < 0 \) \hspace{1cm} \ldots(9)

This is a non-autonomous free-time, free-endpoint problem and the necessary conditions for a weak extremum are given in Appendix A to this chapter.

The Hamiltonian, \( H \), for the problem is

\[ H = e^{-\delta t} \left\{ \frac{1}{S}[P(U_1)] + q_1 U_1 \right\} \] \hspace{1cm} \ldots(10)

where \( p_1(t) = q_1(t)e^{-\delta t} \) is the auxiliary variable relating to the state variable \( N(t) \). If \( P^*(0) \) is the maximum value of \( P(0) \) subject to the constraints given then the auxiliary variable is so defined \(^3\) that

\[ p_1(0) = \frac{3P^*}{3N} \bigg|_{N=N(0)} \] \hspace{1cm} \ldots(11)

and for any \( t_1 \in [0,T] \) when \( N(t_1) \) is optimal

\[ p_1(t_1) = \frac{3}{3N} \max \left\{ \int_{t_1}^{T} D(t)e^{-\delta t} dt + \left[ 1 + \frac{N(T)}{S(T)} \right] e^{-\delta T} \right\} \] \hspace{1cm} \ldots(12)

\(^1\) The \( t \) argument will normally be deleted from the rest of the text.

\(^2\) The subscript '1' is used to distinguish this auxiliary variable from that relating to \( S(t) \) which will be introduced in the next chapter. To simplify the comparison of results an attempt is made to use consistent notation throughout the text.

Thus

\[ q_1(t_1) = e^{\frac{\delta t}{N}} \max_{t_1} \left\{ \int_{t_1}^{T} D(t)e^{-\delta t}dt + \left[1 + \frac{N(T)}{S(T)}\right]e^{\delta T} \right\} \]

\[ = \frac{\partial}{\partial N} \max_{t_1} \left\{ \int_{t_1}^{T} D(t)e^{-\delta (t-t_1)}dt + \left[1 + \frac{N(T)}{S(T)}\right]e^{-\delta (T-t_1)} \right\} \]

\[ = \frac{\partial}{\partial N} \max P(t_1) \quad \ldots(13) \]

Thus the shadow price \( p_1(t_1) \) is the marginal contribution \( N(t) \) makes to the price of an ordinary share at \( t=t_1 \) when discounted back to \( t=0 \); and \( q_1(t_1) \) is the marginal contribution \( N(t) \) makes to the price of shares at \( t = t_1 \).

By the principle of optimality, a trajectory which maximises the objective function over \( t \in [0, T] \) also maximises the objective function over \( t \in [t_1, T] \) for \( t_1 \in [0, T] \). The trajectory which maximises \( e^{-\delta t_1}P(t_1) \) is part of the trajectory which maximises \( P(0) \). Thus

\[ p_1(t_1), N(t_1) \quad \ldots(14) \]

is the discounted time rate of change in the price of shares at \( t = t_1 \) due to varying \( N \) (when the discounted price at \( t = t_1 \) has been maximised for given initial conditions at \( t = t_1 \)).

The Hamiltonian, then, which can be written

\[ H(t_1) = D(t_1)e^{-\delta t_1} + p_1(t_1)N(t_1) \quad \ldots(15) \]

is discounted dividends plus the increase in the discounted value of the price due to increasing \( N \).

\(^4\) See Chapter III.
This can also be written

\[ H(t_1) = e^{-\delta t_1} \{ D(t_1) + q_1(t_1)N(t_1) \} \]  

... (16)

that is, the discounted value of the sum of dividends and the increase in the price due to increasing \( N \). The Hamiltonian is a function of the control variables and so the condition that the Hamiltonian must be maximised with respect to the controls means that the firm must behave in such a way as to maximise the present value of its actions at each time. The firm's actions are, of course, constrained and these constraints are included explicitly by adding Lagrangian terms to the Hamiltonian. The composite function will be called the Lagrangian, \( L \). Thus where the Lagrangian terms are defined as

\[ Z_1(t) = \phi_1(t)e^{-\delta t}, \]  

... (17)

and

\[ \phi_1 \geq 0, \phi_1(\Pi - U_1) = 0 \]  

... (18a)

\[ \phi_3 \geq 0, \phi_3 B = 0 \]  

... (18b)

\[ \phi_4 \geq 0, \phi_4[k(N+S) - B] = 0 \]  

... (18c)

\[ \phi_5 \geq 0, \phi_5 x = 0 \]  

... (18d)

\[ \phi_6 \geq 0, \phi_6 N = 0, \phi_6 U_1 = 0 \]  

... (18e)

\( ^5 \) The term \( \phi_2 \) has been deleted because this will be used for the Lagrangian term relating to the control variable \( U_2 \), in the next chapter.
the Lagrangian is
\[
L = e^{-\delta t} \left\{ \frac{1}{S} [\Pi - U_1] + q_1 U_1 + \phi_1 [\Pi - U_1] \right. \\
+ \left. \phi_3 B + \phi_4 \{k(N+S) - B\} + \phi_5 x + \phi_6 U_1 \right\} \quad \text{...(19)}
\]

The necessary conditions for an optimum policy are given by the optimal conditions relating to the controls
\[
\frac{\partial L}{\partial U_1} = 0 = -\frac{1}{S} + q_1 - \phi_1 + \phi_6 \quad \text{(20).a}
\]
\[
\frac{\partial L}{\partial B} = 0 = (\frac{1}{S} + \phi_1)(F_K - \lambda - r) + \phi_3 - \phi_4 \quad \text{..b}
\]
\[
\frac{\partial L}{\partial x} = 0 = (\frac{1}{S} + \phi_1) F_x + \phi_5 \quad \text{..c}
\]

the motion of the state variable
\[
\dot{N} = U_1 \quad \text{(21).a}
\]
\[
N(0) = N_0, \quad N(T) \text{ free} \quad \text{..b}
\]

and the motion of the auxiliary variable
\[
\dot{q}_1 = q_1 \delta - (\frac{1}{S} + \phi_1)(F_K - \lambda) - \phi_4 k \quad \text{...(22)}
\]

with \(q_1(0)\) free and \(q_1(T)\) given by the transversality condition
\[
q_1(T) = \frac{1}{S} \quad \text{...(23)}
\]

where \(T\) is finite.
The optimum rules relating to output and debt policy are independent of the auxiliary variable and they can be discussed now. The rule relating to output 20(c) can be written

\[ F_x = \frac{-\phi_5}{S + \phi_1} \]  \(\ldots(24)\)

or

\[ \frac{\partial R(K, x)}{\partial x} - \frac{\partial C(K, x)}{\partial x} = \frac{-\phi_5}{S + \phi_1} \]  \(\ldots(25)\)

when \(x > 0\) the left hand side is zero. Notice that the value of \(x\) which meets (25) with equality depends on \(K\). Proposition 4 has been shown.

**Proposition 4**

The optimum output for the firm is that at which marginal revenue equals marginal cost.

Research and development expenditure, \(z\), as a determinant of demand and costs in the way described by R. Marris\(^6\) may be included with no difficulty. The revenue function becomes \(R(x, K, z)\) and the cost function becomes \(C(x, K, z)\) with \(R_z > 0, C_z > 0\). If \(z\) is treated as a control variable the optimum condition is

\[ (\frac{1}{S + \phi_1} ) (R_z - C_z) + \phi_7 = 0 \]  \(\ldots(26)\)

\[ \phi_7 > 0, \phi_7z = 0 \]  \(\ldots(27)\)

---

and so the marginal revenue from z should be equal to its marginal cost.

From condition 20(b)

\[ F_X - \lambda - r = \frac{\phi_4 - \phi_3}{1 + \phi_1} \quad \ldots (28) \]

Thus if the left hand side is greater than zero,

\[ \phi_4 > \phi_3 \geq 0 \quad \ldots (29) \]

Since \( \phi_4 \) and \( \phi_3 \) cannot both be non-zero, \( \phi_4 > 0 \) and \( \phi_3 = 0 \). Thus, if \( B^* \) is the optimum value of \( B \), then \( B^* = k(N+S) \). If the left hand side is less than zero

\[ \phi_3 > \phi_4 \geq 0 \quad \ldots (30) \]

and so \( \phi_3 > 0, \phi_4 = 0 \) and \( B^* = 0 \). If the left hand side equals zero then \( \phi_4 = \phi_3 = 0 \) and \( 0 \leq B^* \leq K(N+S) \). Thus Proposition 5.

**Proposition 5**

If \( B^* \) is the optimum value of \( B \) then

(a) \( B^* = 0 \) if \( F_K - \lambda - r < 0 \)

(b) \( B^* = k(N+S) \) if \( F_K - \lambda - r > 0 \)

(c) \( 0 \leq B^* \leq k(N+S) \) if \( F_K - \lambda - r = 0 \)

It will now be shown that \( \Pi > 0 \) when

\[ F_K - \lambda - r = 0 \quad \ldots (31) \]

In Diagram I \((F_K - \lambda)\) is drawn according to the assumption of the model. Since \( \Pi = F(K) - rB - K \), \((F_K - \lambda - r)\) represents the addition to profits from increasing \( K \) when \( K=B \). Thus the
shaded area indicates total profit when $K = B$. Since $K > B$, total net profit will in fact be greater than this area. In any case $\Pi > 0$ when

$$F_K - \lambda - r = 0 \quad \ldots (32)$$

If $K$ increases so that

$$F_K - \lambda - r < 0 \quad \ldots (33)$$

then $B^* = 0$ and eventually

$$\Pi = F(K) - \lambda K = 0 \quad \ldots (34)$$

will be reached. This value of $K$ is shown in Diagram II as $K = K^*$. Also

$$F_K - \lambda = 0 \quad \ldots (35)$$

when $K = K_2$ and

$$F_K - \lambda - r = 0 \quad \ldots (36)$$

when $K = K_1$. 
Optimal output policy enters only incidentally in the problem and it can be ignored from now on. The remaining policies available to the firm are six and they are shown in Table A.

### Table A

<table>
<thead>
<tr>
<th>Policy</th>
<th>$u_1$</th>
<th>$B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$\pi$</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>$\pi$</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>$\pi$</td>
<td>$k(N+S)$</td>
</tr>
<tr>
<td>D</td>
<td>$\pi$</td>
<td>$k(N+S)$</td>
</tr>
<tr>
<td>E</td>
<td>$\pi$</td>
<td>$0 &lt; B &lt; k(N+S)$</td>
</tr>
<tr>
<td>F</td>
<td>$\pi$</td>
<td>$0 &lt; B &lt; k(N+S)$</td>
</tr>
</tbody>
</table>

The conditions prevailing under each of these policies will be considered in turn.
A. $U_1 = \pi, B = 0$

Clearly

$q_1 = \frac{1}{S}$  \hspace{1cm} \ldots(37)

$\phi_4 = 0$  \hspace{1cm} \ldots(38)

$F_K - \lambda - r \leq 0$  \hspace{1cm} \ldots(39)

$q_1 = q_1 \delta - \left(\frac{1}{S} + \phi_1\right)(F_K - \lambda)$  \hspace{1cm} \ldots(40)

and for $N > 0$,

$q_1 = q_1 (\delta + \lambda - F_K) > 0$  \hspace{1cm} \ldots(41)

B. $U_1 = \bar{\pi}, B = 0$

Now,

$q_1 = \frac{1}{S}$  \hspace{1cm} \ldots(42)

$\phi_4 = 0$  \hspace{1cm} \ldots(43)

$F_K - \lambda - r < 0$  \hspace{1cm} \ldots(44)

$q_1 = q_1 \delta - \left(\frac{1}{S} + \phi_1\right)(F_K - \lambda)$  \hspace{1cm} \ldots(45)

and when $N > 0$

$q_1 = q_1 (\delta + \lambda - F_K) > 0$  \hspace{1cm} \ldots(46)

This policy can last for only an instant because $q_1 > \frac{1}{S}$ will occur. This may, however, be a significant policy since $U_1$ may be $U_1 = -\infty$, that is, $N$ may be reduced in a block at one instant. If $N$ is not able to jump then this policy has no significance because it holds for only an instant and the state variable is not able to change in an instant. Since Policy B must involve a jump, the assumption of the continuity of state variables in time implies that it can only hold at $t = 0$. 
C. \( U_1 = \Pi, B = k(N+S) \)

In this case,

\[
q_1 = \frac{1}{s} \quad \ldots (47)
\]

\[
\phi_3 = 0 \quad \ldots (48)
\]

\[
F_K - \lambda - r > 0 \quad \ldots (49)
\]

\[
q_1 = q_1^\delta - \left(\frac{1}{s} + \phi_1\right) [ (F_K - \lambda) + k(F_K - \lambda - r) ] \quad \ldots (50)
\]

and when \( N > 0 \)

\[
q_1 = q_1 [ (\delta + \lambda - F_K) - k(F_K - \lambda - r) ] \quad \ldots (51)
\]

Thus

\[
\begin{cases}
> 0 \text{ when } F_K - \lambda < \frac{\delta + rk}{1 + k} \\
= 0 \text{ when } F_K - \lambda = \frac{\delta + rk}{1 + k} \\
< 0 \text{ when } F_K - \lambda > \frac{\delta + rk}{1 + k}
\end{cases}
\]

D. \( U_1 < \Pi, B = k(N+S) \)

Now

\[
q_1 = \frac{1}{s} \quad \ldots (53)
\]

\[
\phi_3 = 0 \quad \ldots (54)
\]

\[
F_K - \lambda - r > 0 \quad \ldots (55)
\]

\[
q_1 = q_1^\delta - \left(\frac{1}{s} + \phi_1\right) [ (F_K - \lambda) + k(F_K - \lambda - r) ] \quad \ldots (56)
\]

and when \( N > 0 \)

\[
q_1 = q_1 [ (\delta + \lambda - F_K) - k(F_K - \lambda - r) ] \quad \ldots (57)
\]
thus

\[
q_1 = \begin{cases} 
> 0 & \text{when } F_K - \lambda < \frac{\delta + r_k}{1 + k} \\
= 0 & \text{when } F_K - \lambda = \frac{\delta + r_k}{1 + k} \\
< 0 & \text{when } F_K - \lambda > \frac{\delta + r_k}{1 + k} 
\end{cases}
\]  

...(58)

This policy can be sustained for more than an instant if

\[
U_1 = 0 
\]  

...(59)

and

\[
F_K - \lambda = \frac{\delta + r_k}{1 + k} 
\]  

...(60)

that is, if the system is at a stationary point. Otherwise it must be an initial policy (holding only at \( t = 0 \)) and must involve a jump to be a significant policy.

E. \( U_1 = \Pi, B \in (0, k(N+S)) \)

Now

\[
q_1 > \frac{1}{S} 
\]  

...(61)

\[
\phi_4 = \phi_3 = 0 
\]  

...(62)

\[
F_K - \lambda - r = 0 
\]  

...(63)

\[
q_1 = q_1^\delta - \left(\frac{1}{S^\phi_1}(F_K - \lambda)\right) 
\]  

...(64)

and when \( N > 0 \)

\[
q_1 = q_1^\delta + \lambda - F_K > 0 
\]  

...(65)

This policy can be held for more than an instant because although \( N > 0 \), Proposition 5 requires that \( B \) falls thus maintaining (63) until \( B = 0 \).
This policy implies

\[ q_1 = \frac{1}{S} \]  
\[ \phi_4 = \phi_3 = 0 \]  
\[ F_K - \lambda - r = 0 \]  

and when \( N > 0 \)

\[ q_1 = q_1(\delta + \lambda - F_K) > 0 \]  

This policy can only last for an instant. Either \( U_1 = -\infty \)
and \( N \) jumps from the region of \( U_1 \) switches to \( U_1 = \Pi \) because \( q_1 > 0 \) and \( q_1 > \frac{1}{S} \) will follow. In the latter case Policy F
is not significant.

This completes the statement of the conditions under
which the various policies hold. The next step in the analysis
is to derive conditions under which any one policy may switch
into any other. These will be called the switching surfaces
and are given in Table B. Their explanations are given in
Appendix B to this chapter. In order that unnecessary
repetition is avoided the following conventions will be used:

1. To reduce the number of policies the expression \( U_1 < \Pi \)
will be used to represent the case when \( N \) jumps, so that
\( U_1 = -\infty \), as well as the cases where \( U_1 = 0 \) and where \( N \)
varies continuously. If a particular value of \( U_1 \) is required
for a switch to take place then it is stated in the table.

2. It may be possible to switch from one policy to another
by a jump in \( N \), provided this second policy also involves
another jump into a third policy at the same instant. Clearly
this is really a switch from the first to the third policy and it will be recorded as such. If a policy involves a jump then the conditions stated indicate the conditions prevailing before the jump.

3. Policies B and F can last for only an instant because \( q_1 = \frac{1}{5} \), \( q_1 > 0 \) holds. These policies are significant if \( U_1 = -\infty \) but are not if \( U_1 = -\infty \) because \( N \) does not change and the firm must switch immediately into Policies A and E respectively. These switches will be ignored because the firm must in fact start with Policy A or E. Similarly, a non-stationary Policy D can only last an instant and must switch into Policy C or E unless \( U_1 = -\infty \), and so these switches will be ignored unless \( U_1 = -\infty \) is involved.

4. A zero in a cell indicates that there is no condition under which a switch can occur.
### Table B
**Switching Surfaces**

<table>
<thead>
<tr>
<th>Switches From</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>$U_1 = \frac{K}{1+k}$ (1) \quad $F_{K - \lambda - r} \leq 0$ (4)</td>
<td>-</td>
<td>$U_1 = \frac{K}{1+k}$ (1) \quad $r &lt; F_{K - \lambda - r} \leq \frac{\delta + r k}{1+k}$ (4)</td>
<td>$U_1 = \frac{K}{1+k}$ (1) \quad $U_1 = 0$ (3) \quad $F_{K - \lambda - r} = 0$ (4)</td>
<td>$U_1 = \frac{K}{1+k}$ (1) \quad $F_{K - \lambda - r} = 0$ (4)</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>D</td>
<td>0</td>
<td>0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>E</td>
<td>$F_{K - \lambda - r} \leq 0$ (4)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>F</td>
<td>0</td>
<td>0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0</td>
</tr>
</tbody>
</table>

**Footnotes:**
1. The policy required for a switch to take place.
2. The condition which must hold immediately before the switch.
3. The policy required immediately after the switch.
4. The condition which must hold immediately after the switch.
4. The Phase Plane

The phase plane of the state variable \( N \) and its auxiliary variable \( q_1 \) is shown in Diagram III.

The value of \((F_K - \lambda)\) falls continuously with increases in \( N \) with the exception of a range of \( N \) during which \( N \) is being substituted for \( B \). If

\[
F_K - \lambda - r > 0
\]

then

\[
K = (1+k)(N+S)
\]

and so

\[
\frac{\partial (F_K - \lambda)}{\partial N} = \frac{d(F_K - \lambda)}{dK} \cdot \frac{\partial K}{\partial N} = F_{KK}(1+k) < 0 \quad \text{...(71)}
\]

When

\[
F_K - \lambda - r = 0
\]

then

\[
\frac{\partial (F_K - \lambda)}{\partial N} = \frac{d(F_K - \lambda)}{dK} \cdot \frac{\partial K}{\partial N} = 0 \quad \text{...(72)}
\]

because \( N \) is replacing \( B \). Finally, when

\[
F_K - \lambda - r < 0
\]

then \( B = 0 \) and

\[
\frac{\partial (F_K - \lambda)}{\partial N} = \frac{d(F_K - \lambda)}{dK} \cdot \frac{\partial K}{\partial N} = F_{KK} < 0 \quad \text{...(73)}
\]

Thus if the value of \( N \) at which

\[
F_K - \lambda = \frac{\delta + rk}{1 + k}
\]

holds is

\[
N^* = \frac{K^* - (1+k)S}{1+k}
\]

\text{...(74)}
and \( N^* > 0 \) then to the right of \( N^* \) will be the range of 
\( N \in [N_1, N_2] \) at which (74) holds. Further to the right there 
will be \( N = \hat{N} \) which holds when profits are zero. The \((N, q_1)\) 
plane can now be divided into regions corresponding to the 
policies stated earlier. The trajectories are given by 
\( q_1 \) and \( U_1 \).

It is possible to eliminate most of these 
trajectories as being non-optimal. If \( T \) is finite then the 
transversality condition

\[
q_1(T) = \frac{1}{S} \quad \ldots (80)
\]
must be met. Trajectories to the left of \( N^* \) which do not 
reach \( q_1 = \frac{1}{S} \) are thus non-optimal, and the trajectories to 
the right of \( N^* \) and in the region \( q_1 > \frac{1}{S} \) are also non-optimal. 
Thus if \( T \) is finite and \( N(0) < N^* \), it may be optimal to follow 
Policy C to increase \( N \) and then liquidate at \( N \leq N^* \). If 
\( N(0) > N^* \) then it will not be optimal for \( T \) to be finite 
unless \( T = 0 \). The firm must jump to \( N \leq N^* \). If it goes to 
\( N < N^* \) then it must liquidate immediately. If it goes to 
\( N^* \) and does not liquidate immediately then \( T = \infty \) by 
Proposition 3 of Chapter III.

If \( T \) is infinite then again it is not optimal to 
follow trajectories in the region \( N > N^* \), \( q_1 > \frac{1}{S} \). Policies 
C, E and A all involve zero dividends and so any trajectory 
in this region will involve no dividends being paid. Under 
the reasonable assumption that the capital distribution per 
share does not tend to infinity, its discounted value will 
fall to zero as \( T \to \infty \). Thus any such policy involves the 
price of shares falling to zero and it would be preferable
for the firm to liquidate. Thus if \( N(0) > N^* \) and \( T = \infty \),
the firm should jump to \( N^* \). If \( N(0) < N^* \) and \( T = \infty \) then the
firm must go to \( N^* \) and stay there. If this position cannot
be reached in a finite time then the firm should liquidate at
t = 0 because no dividends will be paid (under Policy C) and
the discounted value of capital distribution will be zero.
Thus Proposition 6 has been shown.

**Proposition 6**

If \( T = \infty \) then it is optimal for a firm to follow
policies which involve the firm achieving the capital stock
given by

\[
F_K - \lambda = \frac{\delta + rk}{1 + k}
\]

as quickly as possible.
None of the analysis is changed if $N = 0$ when $N^* > 0$ because $q_1 < 0$ also holds at $N = 0$. If $N^* < 0$ doesn't exist then the firm must liquidate at $t = 0$ because all trajectories result in the price of shares falling to zero at $t > 0$.

Since

$$\frac{\partial^2 L}{\partial N^2} = e^{-\delta t} \left( \frac{1}{S} + \phi_1 \right) F_{KK} < 0$$  \hspace{1cm} \ldots (81)$$

$L$ is concave in the state variable $N$ and by Theorem 2 of Appendix A the necessary conditions for optimality are also sufficient.

Finally all of the potentially optimal policies can be represented in a $N,b$ plane. In Diagram IV the line I
shows the solution to
\[ F_K - \lambda = \frac{\delta + rk}{1 + k} \]  
...(82)

in terms of \( N \) and \( B \) and line II shows the solution to
\[ F_K - \lambda - r = 0 \]  
...(83)

The solution of \( B = k(N+S) \) in terms of \( N \) is shown as the line \( \frac{(B-kS)}{k} \). The horizontal lines show jumps in \( B \) at \( t = 0 \) so that Proposition 5 holds for the initial value of \( N \). These jumps will always take the firm to the line wxyz. If the firm begins with \( N(0) > N^* \) then this jump will result in the firm following Policy B or F or D, all of which must involve another jump. Thus if the firm begins in a position like M or O the optimal policy is simply to jump to \( B = k(N^*+S), N = N^* \). In the diagram this is represented as a jump to Policy F and then a jump to Policy D. If, however, \( N(0) < N^* \) then the firm should jump to \( B = k(N(0)+S), N = N(0) \) and follow Policy C. If \( T \) is finite then it may be optimal to liquidate somewhere along Policy C. If \( T \) is infinite then it must go to the intersection of I and \( \frac{(B-kS)}{k} \) lines.

The following is a Solution Algorithm for the problem.

**Solution Algorithm**

1. Calculate \( K^* \). This is the value of \( K \) corresponding to the condition
   \[ F_K - \lambda = \frac{\delta + rk}{1 + k} \]

2. If \( K(0) > K^* \) when \( B = B^* \) then \( N(0) > N^* \) and if \( N^* > 0 \) does not exist then liquidate at \( t = 0 \). The return on shareholders' funds from the firm is less than the opportunity
3. If $K(0) > K^*$ when $B = B^*$ then $N(0) > N^*$ and if $N^* = 0$ does exist then reduce $N$ to $N^*$ at $t = 0$. Calculate

$$P_1(0) = \left[ N(0) - N^* \right] \frac{1}{S} + \int_0^\infty \frac{\Pi^*}{S} e^{-\delta t} dt$$

$$P_1(0) = \left[ N(0) - N^* \right] \frac{1}{S} + \frac{1}{\delta} \frac{\Pi^*}{S}$$

and

$$P_2(0) = \left[ 1 + \frac{N(0)}{S} \right]$$

when $\Pi^*$ is the level of profits corresponding to $K = K^*$.

If $P_1(0) \geq P_2(0)$ then jumps to $N^*$ and remain forever ($T = \infty$) and if $P_1(0) < P_2(0)$ liquidate at $t = 0$.

4. If $K(0) < K^*$ when $B = B^*$ then $N(0) < N^*$.

(a) Consider, firstly, policies which involve a finite $T$. From any $N(0)$ the time for which Policy C is followed will determine $N(T)$. Thus consider all possible $N(T_1)$ and calculate

$$P_3(0) = \left[ 1 + \frac{N(T_1)}{S} \right] e^{-\delta T_1}$$

No general statement can be made about which $T_1$ will give the maximum $P_3(0)$ without further knowledge of the operating profit function.

(b) Assume $T = \infty$. The share price is now

$$P_4(0) = \int_{T_1}^\infty \frac{\Pi^*}{S} e^{-\delta t} dt$$
where \( t = t_1 \) is the time at which \( N \) reaches \( N^* \) when following Policy C. The optimal policy is that which gives a maximum of \( P_i(0) \) and \( P_4(0) \).

5. Comments

Little work has yet been done on models which use dividends rather than earnings as returns but the models which have been developed treat the retention rate as a parameter. An example is J.J. Gordon's model.\(^5\) He writes

\[
P(0) = \int_0^\infty D(t) e^{-\delta t} dt \quad \cdots (84)
\]

and

\[
D(t) = (1 - U_1) \Pi(0).
\]

He assumes that \( U_1 \) is a constant. If \( r \) is the internal rate of return on investment (assumed to be a constant) then

\[
\Pi(t) = \Pi(0)(1 + rb)^t \quad \cdots (85)
\]

\[
= \Pi(0) e^{rbt} \quad \cdots (86)
\]

and

\[
P(0) = \int_0^\infty (1 - U_1) \Pi(0) e^{-t(\delta + rb)} dt \quad \cdots (87)
\]

\[
= \frac{(1 - U_1) \Pi(0)}{\delta - rb} \quad \cdots (88)
\]

---

Now $P(0)$ is maximized in relation to $U_1$. This relegates ploughback policy to a very inferior role and it has been found in this chapter once the more realistic assumption is made that $U_1$ can vary over time, then it is optimal to vary it very significantly over time.

It is also typical of models of this type to assume that $T$ is fixed at $T = \infty$. The model in this chapter has been able to solve for the optimum $T$ as part of a general solution to the problem.

Finally, it was found that the long-run optimum value of the capital stock was such that the marginal return on capital is equal to the average cost of capital. This 'average' was the sum of costs of borrowing and the opportunity cost on shareholders' capital when weighted by the proportions in which borrowed finance and shareholders' finance are used. This conclusion agrees with the results of the Traditional Theory\(^6\) even though it assumes that the returns to shareholders are profits per share. The same results are obtained because in the model developed in this chapter dividends per share are equal to profits per share in the long run.

---

Appendix A

NECESSARY AND SUFFICIENT CONDITIONS FOR OPTIMALITY

The problem to be solved in Chapter IV is a free-time free-endpoint non-autonomous problem in optimal control theory and M.R. Hestenes' formulation of Pontryagin's maximum principle is used.

The solution to the problem involves finding the piecewise continuous r-dimensional control vector \( U(t) = (U_1(t), U_2(t), \ldots, U_r(t)) \) and the continuous n-dimensional state vector \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \) which maximise

\[
I = \int_{t_0}^{t_1} f(x(t), U(t), t) \, dt + S[x(t_1)]
\]

subject to \( x_i(t) = f_i(x(t), U(t)), \quad i = 1, 2, \ldots, n \)

\( \psi(x(t), U(t)) \geq 0, \quad \alpha = 1, 2, \ldots, m \)

\( x_\beta(t) \geq 0, \quad \beta = 1, 2, \ldots, \ell \)

\( x_i(0) = x_{i0}, \quad i = 1, 2, \ldots, n \)

given \( t_0 \) fixed and \( t_1 \) free.

Further, assume that

\[
\frac{\partial \psi_\alpha(x(t), U(t))}{\partial U(t)}, \quad \alpha = 1, 2, \ldots, m
\]

has rank \( m' \) if the constraints \( \alpha = 1, 2, \ldots m' \) hold with equality at some

\( t \in [t_0, t_1] \).

---

The necessary conditions for a weak extremum are the following when $t_1$ is finite:

**Theorem 1**

If $x^*(t)$ and $U^*(t)$ maximize (1) subject to (2) to (5) then there exist multipliers

\[ p_0 > 0 \]
\[ p_i(t), \ i = 1, 2, \ldots, n \]
\[ u_\alpha(t), \ \alpha = 1, 2, \ldots, m \]
\[ \phi_\beta(t), \ \beta = 1, 2, \ldots, \ell \]

not vanishing simultaneously on $t \in [t_0, t_1]$ and the function

\[ L = p_0 f^0 + \langle p_i, f_i \rangle + \langle u_\alpha, \psi_\alpha \rangle + \langle \phi_\beta, f_\beta \rangle \]

such that

(a) \ $u_\alpha(t) \geq 0, \ u_\alpha \psi_\alpha = 0, \ \alpha = 1, 2, \ldots, m$

\[ \phi_\beta(t) \geq 0, \ \phi_\beta x_\beta = 0, \phi_\beta f_\beta = 0, \ \beta = 1, 2, \ldots, \ell \]

(b) \ $p_i(t)$ are continuous on $t \in [t_0, t_1]$ and have the piecewise continuous derivatives

\[ p_i(t) = -\frac{\partial L}{\partial x_i}, \ i = 1, 2, \ldots, n \]

(c) \ $U^*(t)$ is given by

\[ \frac{\partial L}{\partial U} = 0 \]
Theorem 2

(d) \[ L^* = L^*(t_1) - \int \left[ \frac{\partial L^*}{\partial t} + \frac{\partial S}{\partial t^2} \right] dt, L^*(t_1) = \frac{\partial K}{\partial t} |_{t_1} \]

(e) \[ p_i(t) = \frac{\partial S(t_1)}{\partial x_i(t_1)}, \quad i = 1, 2, \ldots, n \]

Condition (e) does not apply if \( t_1 \) is infinite.

A theorem\(^{2}\) on sufficiency is the following:

**Theorem 2**

If

\[ L^*[x(t), p(t), U(t), \phi(t)] = \max U(t) \]

is a concave function of \( x(t) \) for given \( p(t), \mu(t), \phi(t) \) then the necessary conditions are also sufficient.

---

Appendix B

POLICY SWITCHES

Because Policies B, D, (involving a jump in N) and F involve jumps they can only be initial policies. Thus switches into them are not optimal and these switches will not be referred to in the following discussion.

Switches of Policy A

Under Policy A

\[ F_K - \lambda - r \leq 0 \]  \hspace{1cm} \ldots(1)

holds and \( K > 0 \) in the neighbourhood of (1) holding with equality. Thus it is not optimal to switch into Policy C, D or E which require

\[ F_K - \lambda - r \geq 0 \]  \hspace{1cm} \ldots(2)

Switches of Policy B

(i) Policy B can last only an instant and it must involve a jump in N to be significant. It may be optimal to switch into Policy A if

\[ F_K - \lambda - r \leq 0 \]  \hspace{1cm} \ldots(3)

after the jump; or into Policy E if

\[ F_K - \lambda - r = 0 \]  \hspace{1cm} \ldots(4)

after the jump.

(ii) It may be optimal to switch into Policy C if

\[ r < F_K - \lambda < \frac{\delta + rk}{1 + k} \]  \hspace{1cm} \ldots(5)

holds after the jump. If the left hand inequality doesn't hold then the switch into Policy C, if it is feasible, can
only last an instant because \( K > 0 \) and the firm will switch into Policy E. If the right hand equality-inequality doesn't hold then Policy C can last only an instant because \( q_1 = \frac{1}{S} \) and \( q_1 < 0 \).

(iii) A switch into Policy D may be maintained if

\[
F_K - \lambda = \frac{\delta + rk}{1 + k}
\]  

...(6)

and

\[
U_1 = 0
\]  

...(7)

after the jump.

Swatches of Policy C

(i) Under Policy C

\[
F_K - \lambda - r \geq 0
\]  

...(8)

and \( B = k(N+S) \). Before the firm can switch into a policy involving \( B = 0 \) it must follow a policy involving

\[
0 < B < k(N+S).
\]  

...(9)

Thus a switch into Policy A is not optimal.

(ii) It may be optimal to switch into Policy D involving \( U_1 = 0 \) if

\[
F_K - \lambda = \frac{\delta + rk}{1 + k}
\]  

...(10)

(iii) A switch into Policy E is possible if

\[
F_K - \lambda - r = 0
\]  

...(11)
Switches of Policy D

(i) Under Policy D

\[ F_K - \lambda - r \geq 0 \] 

\[ \text{...(12)} \]

If Policy D involves \( U_1 = -\infty \) then

\[ F_K - \lambda - r > 0 \] 

\[ \text{...(13)} \]

and a switch into a policy requiring

\[ F_K - \lambda - r < 0 \] 

\[ \text{...(14)} \]

is not optimal. Also, if Policy D involves \( U_1 = 0 \) then

\[ F_K - \lambda = \frac{\delta + r\kappa}{1 + k} \] 

\[ \text{...(15)} \]

and the firm can reach

\[ F_K - \lambda - r \leq 0 \] 

\[ \text{...(16)} \]

only by following a policy involving \( q_1 > \frac{1}{S} \) because \( q_1 > 0 \)

when

\[ F_K - \lambda < \frac{\delta + r\kappa}{1 + k} \] 

\[ \text{...(17)} \]

Thus a switch from Policy D, whether \( U_1 = 0 \), into Policy A or E is not optimal.

(ii) A switch from Policy D involving \( U_1 = 0 \) to Policy C may be optimal at

\[ F_K - \lambda = \frac{\delta + r\kappa}{1 + k} \] 

\[ \text{...(18)} \]

If Policy D involves \( U_1 = -\infty \) then a switch into Policy C may be optimal (and significant) if

\[ r < F_K - \lambda < \frac{\delta + r\kappa}{1 + k} \] 

\[ \text{...(19)} \]

after the jump.
(iii) Policy D involving $U_1 = -\infty$ may switch into Policy D involving $U_1 = 0$ if
\[ r < F_K - \lambda < \frac{\delta + rk}{1 + k} \] ... (20)
before the jump and
\[ F_K - \lambda = \frac{\delta + rk}{1 + k} \] ... (21)
after the jump.

Switches of Policy E
(i) A switch into Policy A may occur when $B = 0$ and
\[ F_K - \lambda - r < 0 \] ... (22)
holds after the switch.

(ii) Under Policy E $q_1 \geq \frac{1}{S}$, $q_1 > 0$ so that if Policy E is significant a switch into a policy involving $U_1 \neq \Pi$ is not optimal (because $q_1 > \frac{1}{S}$). This switch into Policy D is not optimal.

(iii) Policy E involves $0 < B < k(N+S)$ and $B < 0$. Thus a switch into a policy involving $B = k(N+S)$ is not optimal. Thus a switch into Policy C is not optimal.

Switches of Policy F
(i) Policy F must involve a jump reduction in $N$ and since
\[ F_K - \lambda - r = 0 \] ... (23)
and $0 < B < k(N+S)$, a switch into Policy A which requires
\[ F_K - \lambda - r \leq 0 \] ... (24)
and $B = 0$, is not optimal.
(ii) A switch into Policy C may be optimal if the firm jumps to
\[ r < F_K - \lambda < \frac{\delta + rk}{1 + k} \] \hspace{1cm} \ldots (25)

If the left hand inequality does not hold then \( B \) will change to \( 0 < B < k(N+S) \) in an instant, and if the right hand equality-inequality doesn't hold \( q_1 < \frac{1}{S} \) will follow and a switch into another policy is required.

(iii) A switch into Policy D involving \( U_1 = 0 \) may be optimal if the firm jumps to
\[ F_K - \lambda = \frac{\delta + rk}{1 + k} \] \hspace{1cm} \ldots (26)

(iv) A switch into Policy E may be optimal if the jump is such that
\[ F_K - \lambda - r = 0 \] \hspace{1cm} \ldots (27)

still holds.
Chapter V

A SHARE-PRICE MAXIMIZATION
MODEL OF THE FIRM

1. Introduction

In the last chapter it was assumed that the firm did not engage in a share-issue policy. This assumption will be relaxed in this chapter and the most general of the share-price maximization models will be developed. It will be assumed here that the firm issues shares at par, when issuing at par means that dividends are paid on the full extent of the shareholders' contribution. Although this is obviously a simplification it is not as bad as it first appears to be. Firstly, it may in fact be the most realistic assumption when the firm is a limited liability company but where the shareholders are few, or where the firm is a partnership, or where the 'firm' is a government corporation. Secondly, it is no worse than one of the alternative assumptions that all shares are issued at the current market price. This assumption ignores the costs of floating a new issue and the difficulty of attracting new shareholders, at the current price, which sometimes results in the price received by the company for its shares being substantially below the current market price.\(^1\) Also it is inappropriate when capital is obtained from making calls on shares issued in the past because the price received by the company relates to conditions of a previous

time. These matters will be discussed further at the end of this chapter.

The specification of the problem now differs by the inclusion of the differential equation

\[ \dot{S}(t) = U_2(t) \]  

...(1)

and the constraint

\[ U_2(t) \geq 0. \]  

...(2)

As was explained in Chapter III, \( S(t) \) is restricted to functions which are continuous on the time domain.

2. The Problem

Now, the problem is to find the optimal control vector \((U_1(t), U_2(t), B(t), x(t))\) and the trajectory of the state vector \((N(t), S(t))\) given \( t \in [0, T] \) for \( T, N(T), S(T) \) all free which maximise

\[
\int_0^T \left[ \Pi(t) - U_1(t) \right] \frac{e^{-\delta t}}{S(t)} \, dt + \left[ 1 + \frac{N(T)}{S(T)} \right] e^{-\delta T} \]  

...(3)

subject to:

\[ . \quad \dot{N}(t) = U_1(t) \]  

...(4)

\[ . \quad \dot{S}(t) = U_2(t) \]  

...(5)

and

\[ . \quad \Pi(t) - U_1(t) \geq 0 \]  

...(6)

\[ 0 \leq B(t) \leq k \left[ N(t) + S(t) \right] \]  

...(7)

\[ N(t), x(t), U_2(t) \geq 0 \]  

...(8)
Again this is a non-autonomous free-time free-endpoint problem and the necessary conditions for a weak extremum are provided in Appendix A to Chapter IV.

The Hamiltonian, $H$, is now

$$H = e^{-\delta t} \{ \frac{1}{S} \left[ p_{1} U_{1} + q_{1} U_{1} + q_{2} U_{2} \right] \} \ldots (11)$$

where $p_{1}(t) = q_{1}(t)e^{-\delta t}$ is the auxiliary variable relating to the state variable $N(t)$ and $p_{2}(t) = q_{2}(t)e^{-\delta t}$ is the auxiliary variable relating to the state variable $S(t)$. The interpretation of the auxiliary variables is the same as that in Chapter IV. The term $p_{1}(t_{1})$ can be interpreted as the marginal contribution $N(t)$ makes to the price of an ordinary share at $t = t_{1}$ when discounted back to $t = 0$; and $q_{1}(t_{1})$ is the marginal contribution $N(t)$ makes to the price of shares at $t = t_{1}$. Similar interpretations are applicable in relation to $p_{2}(t_{1})$ and $q_{2}(t_{2})$ only in this case $S(t_{1})$ is varied.

Thus

$$p_{1}(t_{1}).N(t_{1}) \ldots (12)$$

is the discounted time rate of change in the price of shares at $t = t_{1}$ due to varying $N(t_{1})$ and

$$p_{2}(t_{1}).S(t_{1}) \ldots (13)$$

is the discounted time rate of change in the price of shares
at $t = t_1$ due to varying $S(t_1)$. The price of shares has been maximized, for the 'initial' conditions at $t = t_1$. The Hamiltonian, then, is the discounted value of the sum of dividends and increases in the price of shares due to varying $N$ and $S$. If the outer brackets are removed from expression (11) and its terms are multiplied through by $e^{-\delta t}$ then the Hamiltonian becomes the discounted dividends plus the increases in the discounted value of the price due to varying $N$ and $S$.

An additional Lagrangian multiplier must be introduced to reflect the constraint $U_2 = 0$. Thus where the Lagrangian terms are $Z_i(t) = \phi_i(t)e^{-\delta t}$ and

$$\phi_1 = 0, \phi_1(\Pi - U_1) = 0 \hspace{2cm} \text{(14)a}$$

$$\phi_2 = 0, \phi_2 U_2 = 0 \hspace{2cm} \text{(b)}$$

$$\phi_3 = 0, \phi_3 B = 0 \hspace{2cm} \text{(c)}$$

$$\phi_4 = 0, \phi_4 [k(N+S) - B] = 0 \hspace{2cm} \text{(d)}$$

$$\phi_5 = 0, \phi_5 x = 0 \hspace{2cm} \text{(e)}$$

$$\phi_6 = 0, \phi_6 N = 0, \phi_6 U_1 = 0 \hspace{2cm} \text{(f)}$$

the Lagrangian is

$$L = e^{-\delta t} \left[ \frac{1}{S(\Pi - U_1)} + q_1 U_1 + q_2 U_2 + \phi_1(\Pi - U_1) + \phi_2 U_2 + \phi_3 B + \phi_4 [k(N+S) - B] + \phi_5 x + \phi_6 U_1 \right] \hspace{2cm} \text{(15)}$$
The necessary conditions for optimality are given by the optimal conditions relating to the controls

\[
\frac{\partial L}{\partial U_1} = 0 = -\frac{1}{S} + q_1 - \phi_1 + \phi_6 \quad \ldots (16) a
\]

\[
\frac{\partial L}{\partial U_2} = 0 = q_2 + \phi_2 \quad b
\]

\[
\frac{\partial L}{\partial B} = 0 = \left(\frac{1}{S} + \phi_1\right)(F_K - \lambda - r) + \phi_3 - \phi_4 \quad c
\]

\[
\frac{\partial L}{\partial x} = 0 = \left(\frac{1}{S} + \phi_1\right)F_{x} + \phi_5 \quad d
\]

the motion of the state variables

\[
\dot{N} = U_1 \quad \ldots (17) a
\]

\[N(0) = N_0, \quad N(T) \text{ free} \quad b\]

and

\[
\dot{S} = U_2 \quad \ldots (18) a
\]

\[S(0) = S_0 > 0, \quad S(T) \text{ free} \quad b\]

and the motion of the auxiliary variables

\[
\dot{q}_1 = q_1 - \left(\frac{1}{S} + \phi_1\right)(F_K - \lambda) - \phi_4 k \quad \ldots (19)
\]

and

\[
\dot{q}_2 = q_2 \delta - \left(\frac{1}{S} + \phi_1\right)(F_K - \lambda) + (\Pi - U_1)\frac{1}{S_2} - \phi_4 k \quad \ldots (20)
\]

with \(q_1(0), q_2(0) \) free and \(q_1(T), q_2(T) \) given by the
transversality conditions

\[ q_1(T) = \frac{1}{S(T)} \] \hspace{1cm} \text{...(21)}

\[ q_2(T) = -\frac{N(T)}{[S(T)]^2} \] \hspace{1cm} \text{...(22)}

when \( T \) is finite.

The optimum rules relating to output and debt are unaffected by the introduction of the second state variable or the additional control variable. Since the rule in relation to output is

\[ F_x = \frac{-\phi_5}{1 + \phi_1} \] \hspace{1cm} \text{...(23)}

output must be such that marginal revenue equals marginal cost. Thus Proposition 4 of Chapter IV holds in this model as well. Similarly, since

\[ F_K - \lambda - r = \frac{\phi_4 - \phi_3}{1 + \phi_1} \] \hspace{1cm} \text{...(24)}

Proposition 5 of Chapter IV also holds. Thus

\[ B^* = 0 \text{ if } F_K - \lambda - r < 0 \] \hspace{1cm} \text{...(25a)}

\[ B^* = k(N+S) \text{ if } F_K - \lambda - r > 0 \] \hspace{1cm} \text{b}

\[ 0 < B^* < k(N+S) \text{ if } F_K - \lambda - r = 0 \] \hspace{1cm} \text{c}

In Chapter IV it was shown that \( \Pi > 0 \) when 25(c) holds. This proposition still holds in this model and the explanation is unaffected by allowing \( S(t) \) to vary.
3. **Policy Alternatives**

Again optimal output policy enters only incidentally in the problem and it can be ignored. There are twelve policy alternatives available and these are shown in Table A.

<table>
<thead>
<tr>
<th>Policy</th>
<th>$U_1$</th>
<th>$U_2$</th>
<th>$N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$\Pi$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>$&lt;\Pi$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>$\Pi$</td>
<td>0</td>
<td>$k(N+S)$</td>
</tr>
<tr>
<td>D</td>
<td>$&lt;\Pi$</td>
<td>0</td>
<td>$k(N+S)$</td>
</tr>
<tr>
<td>E</td>
<td>$\Pi$</td>
<td>0</td>
<td>$0&lt;B&lt;k(N+S)$</td>
</tr>
<tr>
<td>F</td>
<td>$&lt;\Pi$</td>
<td>0</td>
<td>$0&lt;B&lt;k(N+S)$</td>
</tr>
<tr>
<td>G</td>
<td>$\Pi$</td>
<td>$&gt;0$</td>
<td>0</td>
</tr>
<tr>
<td>H</td>
<td>$&lt;\Pi$</td>
<td>$&gt;0$</td>
<td>0</td>
</tr>
<tr>
<td>I</td>
<td>$\Pi$</td>
<td>$&gt;0$</td>
<td>$B=k(N+S)$</td>
</tr>
<tr>
<td>J</td>
<td>$&lt;\Pi$</td>
<td>$&gt;0$</td>
<td>$B=k(N+S)$</td>
</tr>
<tr>
<td>K</td>
<td>$\Pi$</td>
<td>$&gt;0$</td>
<td>$0&lt;B&lt;k(N+S)$</td>
</tr>
<tr>
<td>L</td>
<td>$&lt;\Pi$</td>
<td>$&gt;0$</td>
<td>$0&lt;B&lt;k(N+S)$</td>
</tr>
</tbody>
</table>
The conditions prevailing under each of these policies will be considered in turn.

A. \( U_1 = \Pi, U_2 = 0, B = 0 \)

Now

\[ q_1 \geq \frac{1}{S} \]  
\[ \phi_4 = 0 \]  
\[ F_K - \lambda - r \leq 0 \]  
\[ q_1 = q_1^\delta - \left( \frac{1}{S} + \phi_1 \right) (F_K - \lambda) \]

and for \( N > 0 \)

\[ q_1 = q_1^\delta - (\frac{1}{S} + \phi_1)(F_K - \lambda) \]

Further

\[ q_2 = q_2^\delta - \left( \frac{1}{S} + \phi_1 \right)(F_K - \lambda) \]

and for \( N > 0 \)

\[ q_2 = q_2^\delta - q_1(F_K - \lambda) \]

\[ q_2 = \begin{cases} >0 \text{ when } F_K - \lambda < \frac{\delta q_2}{q_1} \\ =0 \text{ when } F_K - \lambda = \frac{\delta q_2}{q_1} \\ <0 \text{ when } F_K - \lambda > \frac{\delta q_2}{q_1} \end{cases} \]

Since \( q_1 > 0 \) and \( U_1 = \Pi \) under this policy, the transversality condition is not met while following this policy and no dividends are paid. Thus if this policy is optimal then it must switch into a policy involving \( q_1 = \frac{1}{S} \).
or \( U_1 < \Pi \), or into one which allows other switches so that either of these conditions hold.

**B.** \( U_1 < \Pi, U_2 = 0, B = 0 \)

Now
\[
q_1 = \frac{1}{s} \quad \ldots (35)
\]
\[
\phi_4 = 0 \quad \ldots (36)
\]
\[
F_K - \lambda - r < 0 \quad \ldots (37)
\]
\[
q_1 = q_1 \delta - \left( \frac{1}{s} + \phi_4 \right) (F_K - \lambda) \quad \ldots (38)
\]

and for \( N > 0 \)
\[
q_1 = q_1 (\delta + \lambda - F_K) > 0 \quad \ldots (39)
\]

Further
\[
q_2 = q_2 \delta - \left( \frac{1}{s} + \phi_4 \right) (F_K - \lambda) + (\Pi - U_1) \frac{1}{s^2} \quad \ldots (40)
\]

and if \( N > 0 \)
\[
q_2 = \begin{cases} 
> 0 \text{ when } (\Pi - U_1) \frac{1}{s^2} > q_1 (F_K - \lambda) - q_2 \delta \\
= 0 \text{ when } (\Pi - U_1) \frac{1}{s^2} = q_1 (F_K - \lambda) - q_2 \delta \\
< 0 \text{ when } (\Pi - U_1) \frac{1}{s^2} < q_1 (F_K - \lambda) - q_2 \delta 
\end{cases} \quad \ldots (41)
\]

This policy can last for only an instant because it requires \( q_1 = \frac{1}{s} \) but this cannot be maintained because \( q_1 > 0 \) and \( U_2 < 0 \) cannot hold. Thus if \( U_1 = -\infty \) then this policy is significant. If this is not so then this policy must switch into some other policy at the instant Policy B is followed.
and since $N$ and $S$ cannot change under Policy $B$, a non-jump policy is not significant. Hence this case will be ignored.

If $U_1 = -\infty$ then the $q_2$ equation is undefined. Since it must involve a jump to be significant it can only be an initial policy (holding at $t = 0$) by the continuity of state variables on the time domain.

C. $U_1 = \Pi$, $U_2 = 0$, $B = k(N+S)$

Now

\[
q_1 = \frac{1}{S} \quad \ldots (42)
\]

\[
\phi_3 = 0 \quad \ldots (43)
\]

\[
F_K - \lambda - r > 0 \quad \ldots (44)
\]

\[
q_1 = q_1 \delta - (\frac{1}{S} + \phi_1) [(F_K - \lambda) + k(F_K - \lambda - r)] \ldots (45)
\]

and for $N > 0$

Further

\[
q_1 = q_1 \left( \delta + \lambda - F_K \right) - k(F_K - \lambda - r) \quad \ldots (46)
\]

\[
q_1 = \begin{cases} 
> 0 \text{ when } F_K - \lambda < \frac{\delta + rk}{1 + k} \\
= 0 \text{ when } F_K - \lambda = \frac{\delta + rk}{1 + k} \\
< 0 \text{ when } F_K - \lambda > \frac{\delta + rk}{1 + k}
\end{cases} \ldots (47)
\]

Further

\[
q_2 = q_2 \delta - (\frac{1}{S} + \phi_1) [(F_K - \lambda) + k(F_K - \lambda - r)] \ldots (48)
\]

and for $N > 0$

\[
q_2 = q_2 \delta + q_1 (\lambda - F_K) - k(F_K - \lambda - r) < 0 \quad \ldots (49)
\]
D. \( U_1 < \Pi, U_2 = 0, B = k(N+S) \)

Now

\[ q_1 = \frac{1}{S} \] \hspace{1cm} \ldots (50)

\[ \phi_3 = 0 \] \hspace{1cm} \ldots (51)

\[ F_K - \lambda - r > 0 \] \hspace{1cm} \ldots (52)

\[ q_1 = q_1 \delta - \left( \frac{1}{S} + \phi_1 \right) \left[ (F_K - \lambda) + k(F_K - \lambda - r) \right] \] \hspace{1cm} \ldots (53)

and when \( N > 0 \)

\[ q_1 = q_1 \left[ (\delta + \lambda - F_K) - k(F_K - \lambda - r) \right] \] \hspace{1cm} \ldots (54)

\[ q_1 = \begin{cases} > 0 \text{ when } F_K - \lambda < \frac{\delta + rk}{1 + k} \\ = 0 \text{ when } F_K - \lambda = \frac{\delta + rk}{1 + k} \\ < 0 \text{ when } F_K - \lambda > \frac{\delta + rk}{1 + k} \end{cases} \] \hspace{1cm} \ldots (55)

Further,

\[ q_2 = q_2 \delta - \left( \frac{1}{S} + \phi_1 \right) \left[ (F_K - \lambda) + k(F_K - \lambda - r) \right] + (\Pi - U_1) \frac{1}{S_2} \] \hspace{1cm} \ldots (56)

and when \( N > 0 \)

\[ q_2 = q_2 \delta + q_1 \left[ (\lambda - F_K) - k(F_K - \lambda - r) \right] + (\Pi - U_1) \frac{1}{S_2} \] \hspace{1cm} \ldots (57)

\[ q_2 = \begin{cases} > 0 \text{ when } (\Pi - U_1) \frac{1}{S_2} > q_1 \left[ (F_K - \lambda) + k(F_K - \lambda - r) \right] - q_2 \delta \\ = 0 \text{ when } (\Pi - U_1) \frac{1}{S_2} = q_1 \left[ (F_K - \lambda) + k(F_K - \lambda - r) \right] - q_2 \delta \\ < 0 \text{ when } (\Pi - U_1) \frac{1}{S_2} < q_1 \left[ (F_K - \lambda) + k(F_K - \lambda - r) \right] - q_2 \delta \end{cases} \] \hspace{1cm} \ldots (58)
This policy can last for more than an instant only if

\[ U_1 = 0 \]  
\[ \text{and} \]  
\[ F_K - \lambda = \frac{\delta + rk}{1 + k} \]

That is, if this policy keeps the firm in a stationary position in the \( N, q_1 \) plane. This policy may be such that \( U_1 = -\infty \) so that \( N \) jumps to a smaller value. If neither of these policies is followed then Policy D must immediately switch into some other policy and so such a policy is of no significance. If \( U_1 = -\infty \) then the \( q_2 \) equation is undefined and it can only be an initial policy.

E. \( U_1 = \pi, \ U_2 = 0, \ 0 < B < k(N+S) \)

Now

\[ q_1 > \frac{1}{s} \]  
\[ \phi_4 = \phi_3 = 0 \]  
\[ F_K - \lambda - r = 0 \]  
\[ q_1 = q_1 \delta - (\frac{1}{s} + \phi_1)(F_K - \lambda) \]

and when \( N > 0 \)

\[ q_1 = q_1(\delta + \lambda - F_K) > 0 \]

Further

\[ q_2 = q_2 \delta - (\frac{1}{s} + \phi_1)(F_K - \lambda) \]

and when \( N > 0 \)

\[ q_2 = q_2 \delta - q_1(F_K - \lambda) < 0 \]
This policy may last for more than an instant because \( N > 0 \) and \( B \) may be reduced to sustain (63).

\[ F_1 U_1 < \Pi, U_2 = 0, 0 < B < k(N+S) \]

Now

\[ q_1 = \frac{1}{S} \quad \ldots(68) \]

\[ \phi_4 = \phi_3 = 0 \quad \ldots(69) \]

\[ F_K - \lambda - r = 0 \quad \ldots(70) \]

\[ q_1 = q_1 \delta - \left( \frac{1}{S} + \phi_1 \right)(F_K - \lambda) \quad \ldots(71) \]

and when \( N > 0 \)

\[ q_1 = q_1 (\delta + \lambda - F_K) > 0 \quad \ldots(72) \]

Further

\[ q_2 = q_2 \delta - \left( \frac{1}{S} + \phi_1 \right)(F_K - \lambda) + (\Pi - U_1) \frac{1}{S^2} \quad \ldots(73) \]

and when \( N > 0 \)

\[ q_2 = q_2 \delta - q_1 (F_K - \lambda) + (\Pi - U_1) \frac{1}{S^2} \quad \ldots(74) \]

\[ q_2 = \begin{cases} > 0 \text{ when } (\Pi - U_1) \frac{1}{S^2} > q_1 (F_K - \lambda) - q_2 \delta \\ = 0 \text{ when } (\Pi - U_1) \frac{1}{S^2} = q_1 (F_K - \lambda) - q_2 \delta \\ < 0 \text{ when } (\Pi - U_1) \frac{1}{S^2} < q_1 (F_K - \lambda) - q_2 \delta \end{cases} \quad \ldots(75) \]

This policy can last only an instant because it requires \( q_1 = \frac{1}{S} \) and \( q_1 > 0 \). This is a significant policy only if \( U_1 = -\infty \). In this case the \( q_2 \) equation is undefined and it can only be an initial policy.
G. $U_1 = \pi$, $U_2 > 0$, $B = 0$

Now

$$q_1 = \frac{1}{S} \quad \ldots (76)$$

$$q_2 = 0 \quad \ldots (77)$$

$$\phi_4 = 0 \quad \ldots (78)$$

$$F_K - \lambda - r < 0 \quad \ldots (79)$$

$$q_1 = q_1 \delta - \left(\frac{1}{S} + \phi_1\right)(F_K - \lambda) \quad \ldots (80)$$

and when $N > 0$

$$q_1 = q_1(\delta + \lambda - F_K) > 0 \quad \ldots (81)$$

Further

$$q_2 = q_2 \delta - \left(\frac{1}{S} + \phi_1\right)(F_K - \lambda) \quad \ldots (82)$$

and when $N > 0$

$$q_2 = -q_1(F_K - \lambda) \quad \ldots (83)$$

$$q_2 = \begin{cases} >0 & \text{when } F_K - \lambda < 0 \\ =0 & \text{when } F_K - \lambda = 0 \\ 0 & \text{when } F_K - \lambda > 0 \end{cases} \quad \ldots (84)$$

This policy can hold for only an instant. If it did hold longer then $q_2 = q_2 = 0$ must hold for more than an instant. However $q_2 = q_2 = 0$ holds only when

$$F_K - \lambda = 0 \quad \ldots (85)$$
Since $U_1 = \Pi > 0$ and $U_2 > 0$, $K > 0$, (85) will hold for only an instant. Since the policy only holds for an instant it is significant only if $U_2 = +\infty$, that is, unless $S$ jumps. Thus it can only be an initial policy.

This policy cannot hold when

$$F_K - \lambda < 0 \quad \ldots(86)$$

because $K > 0$ and (86) will continue and $q_2 > 0$ will follow.

H. $U_1 < \Pi$, $U_2 > 0$, $B = 0$

Now

$$q_1 = \frac{1}{S} \quad \ldots(87)$$

$$q_2 = 0 \quad \ldots(88)$$

$$\phi_4 = 0 \quad \ldots(89)$$

$$F_K - \lambda - r < 0 \quad \ldots(90)$$

$$q_1 = q_1(\delta - (\frac{1}{S} + \phi_1)(F_K - \lambda)) \quad \ldots(91)$$

and when $N > 0$

$$q_1 = q_1(\delta + \lambda - F_K) > 0 \quad \ldots(92)$$

Further

$$q_2 = - (\frac{1}{S} + \phi_1)(F_K - \lambda) + (\Pi - U_1) \frac{1}{S^2} \quad \ldots(93)$$
and when $N > 0$

$$q_2 = - q_1 (F_K - \lambda) + (\Pi - U_1) \frac{1}{S^2}$$  \hspace{1cm} \ldots(94)$$

$$q_2 = \begin{cases} > 0 \text{ when } (\Pi - U_1) \frac{1}{S^2} > \frac{1}{S} (F_K - \lambda) \\ = 0 \text{ when } (\Pi - U_1) \frac{1}{S^2} = \frac{1}{S} (F_K - \lambda) \\ < 0 \text{ when } (\Pi - U_1) \frac{1}{S^2} < \frac{1}{S} (F_K - \lambda) \end{cases}$$  \hspace{1cm} \ldots(95)$$

This policy can hold for only an instant because it requires $q_1 = \frac{1}{S}$ and since $q_1 > 0$ and $U_2 < 0$ is not admissible $q_1 > \frac{1}{S}$ follows. It may, however, be a significant policy because it may involve $U_1 = - \infty$ or $U_2 = + \infty$ or both. Thus it can only be an initial policy. If this policy implies a jump in only one state variable then it will be ignored because it is equivalent to Policy B or Policy G.

I. $U_1 = \Pi$, $U_2 > 0$, $B = k(N+S)$

Now

$$q_1 \geq \frac{1}{S}$$  \hspace{1cm} \ldots(96)$$

$$q_2 = 0$$  \hspace{1cm} \ldots(97)$$

$$\phi_3 = 0$$  \hspace{1cm} \ldots(98)$$

$$F_K - \lambda - r \geq 0$$  \hspace{1cm} \ldots(99)$$

$$q_1 = q_1 \delta - \left( \frac{1}{S} + \phi_1 \right) \left[ (F_K - \lambda) + k(F_K - \lambda - r) \right]$$  \hspace{1cm} \ldots(100)$$

and when $N > 0$

$$q_1 = q_1 \left[ (\delta + \lambda - F_K) - k(F_K - \lambda - r) \right]$$  \hspace{1cm} \ldots(101)$$
Further

\[ q_2 = - \left( \frac{1}{S} + \phi_1 \right) \left[ (F_K - \lambda) + k(F_K - \lambda - r) \right] \]  \hspace{1cm} (103)

and when \( N > 0 \)

\[ q_2 = - q_1 \left[ (F_K - \lambda) + k(F_K - \lambda - r) \right] < 0 \]  \hspace{1cm} (104)

This policy can only last for an instant and must therefore involve \( U_2 = + \infty \) to be significant. Thus this is so because \( q_2 < 0 \). Thus it can only be an initial policy.

\[ J. \quad U_1 < \Pi, \quad U_2 > 0, \quad B = k(N+S) \]

Now

\[ q_1 = \frac{1}{S} \]  \hspace{1cm} (105)

\[ q_2 = 0 \]  \hspace{1cm} (106)

\[ \phi_3 = 0 \]  \hspace{1cm} (107)

\[ F_K - \lambda - r = 0 \]  \hspace{1cm} (108)

\[ q_1 = q_1 \delta - \left( \frac{1}{S} + \phi_1 \right) \left[ (F_K - \lambda) + k(F_K - \lambda - r) \right] \]  \hspace{1cm} (109)

and when \( N > 0 \)

\[ q_1 = q_1 \left[ (\delta + \lambda - F_K) - k(F_K - \lambda - r) \right] \]  \hspace{1cm} (110)
Further,

$$q_2 = -\left(\frac{1}{S} + \phi_1\right)[(F_K - \lambda) + k(F_K - \lambda - r)] + (\Pi - U_1) \frac{1}{S^2} \quad \cdots (112)$$

and when \( N > 0 \)

$$q_2 = -q_1[(F_K - \lambda) + k(F_K - \lambda - r)] + (\Pi - U_1) \frac{1}{S^2} \quad \cdots (113)$$

$$q_2 = \begin{cases} >0 & \text{when } (\Pi-U_1) \frac{1}{S} > (F_K - \lambda) + k(F_K - \lambda - r) \\ =0 & \text{when } (\Pi-U_1) \frac{1}{S} = (F_K - \lambda) + k(F_K - \lambda - r) \\ <0 & \text{when } (\Pi-U_1) \frac{1}{S} < (F_K - \lambda) + k(F_K - \lambda - r) \end{cases} \quad \cdots (114)$$

This policy may be followed for more than an instant. If it is then

$$q_1 = \left(\frac{1}{S}\right) \quad \cdots (115)$$

or

$$\frac{1}{S}[(\delta + \lambda - F_K) - k(F_K - \lambda - r)] = \frac{U_2}{S^2} \quad \cdots (116)$$

and so

$$U_2 = S[(F_K - \lambda - \delta) + k(F_K - \lambda - r)] \quad \cdots (117)$$
Since
\[ q_2 = 0 \]  \quad \ldots (118)

must also hold
\[ U_1 = \pi - S \left[ (F_K - \lambda) + k(F_K - \lambda - r) \right] \]  \quad \ldots (119)
or
\[ D(t) = (\pi - U_1) \frac{1}{S} = (F_K - \lambda) + k(F_K - \lambda - r) \]  \quad \ldots (120)

Thus when following Policy J for more than an instant, \( U_1 \) and \( U_2 \) must be the values given in (119) and (117) above.

If Policy J lasts only an instant then it is significant only if \( U_1 = -\infty \) and \( U_2 = +\infty \). Otherwise it is really following Policy D or Policy I when only \( U_1 = -\infty \) or \( U_2 = +\infty \). If \( U_1 \) and \( U_2 \) are both finite then this policy (because it only holds for an instant) results in no change in \( N \) or \( S \). If Policy J involves jumps then the \( q_2 \) equation is undefined and it can only be an initial policy.

K. \( U_1 = \pi, \quad U_2 > 0, \quad 0 < B < k(N+S) \)

Now
\[ q_1 > \frac{1}{S} \]  \quad \ldots (121)
\[ q_2 = 0 \]  \quad \ldots (122)
\[ \phi_4 = \phi_3 = 0 \]  \quad \ldots (123)
\[ F_K - \lambda - r = 0 \]  \quad \ldots (124)
\[ q_1 = q_1 \delta - \left( \frac{1}{S} + \phi_1 \right)(F_K - \lambda) \]  \quad \ldots (125)
and where $N > 0$

$$q_1 = q_1 (\delta + \lambda - F_K) > 0 \quad \ldots (126)$$

Further

$$q_2 = - (\frac{1}{s} + \phi_1)(F_K - \lambda) \quad \ldots (127)$$

and when $N > 0$

$$q_2 = - q_1 (F_K - \lambda) < 0 \quad \ldots (128)$$

Thus this policy can last for only an instant because $q_2$ will fall to $q_2 < 0$. For this policy to be significant $U_2$ must be $U_2 = + \infty$ and so it can only be an initial policy.

$L. \quad U_1 < \Pi, \quad U_2 > 0, \quad 0 < B < k(N+S)$

Now

$$q_1 = \frac{1}{s} \quad \ldots (129)$$

$$q_2 = 0 \quad \ldots (130)$$

$$\phi_4 = \phi_3 = 0 \quad \ldots (131)$$

$$F_K - \lambda - r = 0 \quad \ldots (132)$$

$$q_1 = q_1 \delta - (\frac{1}{s} + \phi_1)(F_K - \lambda) \quad \ldots (133)$$

and for $N > 0$

$$q_1 = (\delta + \lambda - F_K) > 0 \quad \ldots (134)$$

Further

$$q_2 = - (\frac{1}{s} + \phi_1)(F_K - \lambda) + (\Pi - U_1) \frac{1}{s^2} \quad \ldots (135)$$
and when $N > 0$

$$q_2 = -q_1(F_K - \lambda) + (\Pi - U_1)\frac{1}{S^2} \quad \ldots (136)$$

$$q_2 = \begin{cases} 
> 0 & \text{when } (\Pi - U_1)\frac{1}{S} > (F_K - \lambda) \\
= 0 & \text{when } (\Pi - U_1)\frac{1}{S} = (F_K - \lambda) \\
< 0 & \text{when } (\Pi - U_1)\frac{1}{S} < (F_K - \lambda)
\end{cases} \quad \ldots (137)$$

This policy can last for only an instant because $q_1$ will rise to $q_1 > \frac{1}{S}$. It may, however, be significant if $U_1 = -\infty$ and $U_2 = +\infty$. If $U_1$ is finite then this policy is the same as Policy K and it will be treated as such. If both $U_1$ and $U_2$ are finite then this policy can be ignored because there are no changes in the state variables. Thus this policy is significant only if $U_1 = -\infty$, $U_2 = +\infty$.

Here, the $q_2$ equation is undefined. It can only be an initial policy.

Diagram I shows a $S,N$ plane and the regions in which each of the above policies may operate. In the diagram line $K_1$ shows combinations of $S$ and $N$ which meet the conditions

$$F_K - \lambda - r = 0 \quad \ldots (138)$$

$$B = 0 \quad \ldots (139)$$

Line $K_2$ shows combinations of $S$ and $N$ which meet the conditions

$$F_K - \lambda - r = 0 \quad \ldots (140)$$

$$B = k(N + S) \quad \ldots (141)$$
and $K_3$ shows combinations of $S$ and $N$ which meet the conditions:

$$F_k - \lambda = \frac{\delta + rk}{1 + k} \quad \ldots (142)$$

$$B = k(N + S) \quad \ldots (143)$$
The value of \( S \) at \( t = 0 \) is shown as a vertical line and all initial policies must begin on that line.

Because of the requirement that state variables are continuous on the time domain Policies B, D, F, G, H, I, J, and K can only be initial policies or terminal policies. In this model these policies have no significance as terminal policies because all variables fall to zero at \( t = T \).

4. Policy Switches

The switching surfaces are shown in Table B and brief explanations of their derivations are given in Appendix A to this Chapter. In order that unnecessary repetition is avoided the following conventions have been used:

1. Policies which hold for only an instant and do not involve a jump in any of the variables will be ignored because these policies involve no change in the state of the firm.

2. If Policies H, J and L hold for only an instant then they will be regarded as being significant only if they involve jumps in both \( N \) and \( S \). Otherwise they will be treated as some other policy in which either \( U_1 = \infty \) or \( U_2 = 0 \).

3. There are combinations of policies such as that shown in Diagram II. Policy 1 involves \( U_1 = -\infty \), \( U_2 \) finite; Policy 2 involves \( U_1 \) finite, \( U_2 = +\infty \); Policy 3 involves \( U_1 = -\infty \), \( U_2 = +\infty \); Policy 4 involves \( U_1 \) finite, \( U_2 = +\infty \). Provided all of these policies last for only an instant this combination means that the firm jumps from position A to
position B in an instant. Clearly this combination is the same as Policy 5 which involves $U_1 = -\infty$, $U_2 = +\infty$ and it will be treated as Policy 5. Thus all of these switches will be ignored.

4. When a policy involves a jump, the conditions stated indicate the conditions prevailing before the jump.

5. Infinite Horizon

Analysis of the possible optimal policy chains will be begun by considering the case where $T$ is infinite. Table A shows a number of policy chains but many of these can be eliminated on the following grounds:

(1) A policy which leads to a situation where dividends are not paid after the adoption of the policy is not optimal because the price of shares from that time
<table>
<thead>
<tr>
<th>Switches into</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
<th>I</th>
<th>J</th>
<th>K</th>
<th>L</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>$U_1=±1$</td>
<td>$F_k=λ−r&lt;0^b$</td>
<td>$U_1=±1$</td>
<td>$F_k=λ−r&lt;0^b$</td>
<td>$U_1=±1$</td>
<td>$F_k=λ−r&lt;0^b$</td>
<td>$U_1=±1$</td>
<td>$F_k=λ−r&lt;0^b$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$U_1=±1$</td>
<td>$F_k=λ−r&lt;0^b$</td>
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<td>0</td>
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<tr>
<td>D</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$U_1=±1$</td>
<td>$F_k=λ−r&lt;0^b$</td>
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<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>E</td>
<td>$F_k=λ−r&lt;0^b$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>F</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$U_1=±1$</td>
<td>$F_k=λ−r&lt;0^b$</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>G</td>
<td>$U_2=±1$</td>
<td>$F_k=λ−r&lt;0^b$</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>H</td>
<td>$U_1=±1$</td>
<td>$U_2=±1$</td>
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<tr>
<td>I</td>
<td>$U_1=±1$</td>
<td>$F_k=λ−r&lt;0^b$</td>
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<tr>
<td>J</td>
<td>$U_1=±1$</td>
<td>$U_2=±1$</td>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>K</td>
<td>$U_2=±1$</td>
<td>$F_k=λ−r&lt;0^b$</td>
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</tr>
<tr>
<td>L</td>
<td>$U_1=±1$</td>
<td>$U_2=±1$</td>
<td>0</td>
<td>$U_1=±1$</td>
<td>$F_k=λ−r&lt;0^b$</td>
<td>0</td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

1. The policy required for a switch to take place.
2. The condition which must hold immediately before the switch.
3. The policy required immediately after the switch.
4. The condition which must hold immediately after the switch.
will fall to zero. Thus a switch into Policy A or into Policy E or into Policy C when

\[ F_K - \lambda < \frac{\delta + rk}{1 + k} \]

is not optimal.

(2) In infinite chain of policies with recurring policies involving \( U_2 > 0 \) will not be optimal because eventually \( S \), and hence \( K \), will be so large that \( \Pi < 0 \) and this will remain because of the irreversibility of \( S \). The price of the shares will eventually fall to zero because \( N \) will fall to zero and \( \Pi < 0 \) will hold. Now, an examination of Policies A to F (which are the same as those in Chapter IV) will show that an infinite sequence of policies from this set cannot be constructed. Thus if all policies apart from stationary ones last a finite time, an infinite sequence of policies must also have an infinite number of policies involving \( U_2 > 0 \). However, an infinite number of policies involving \( U_2 > 0 \) is not optimal. Thus it has been shown that an infinite sequence of policies is not optimal.

(3) With the assumption that any non-stationary policy must switch in a finite time, and the proposition that an infinite sequence of policies is not optimal, the firm must eventually reach the stationary point \( N = S = 0 \). This is Policy \( D_nj \) when \( U_1 = U_2 = 0 \) and

\[ F_K - \lambda = \frac{\delta + rk}{1 + k} \]

...(144)
By Proposition 3 of Chapter III, since \( T = \infty \)

\[
\frac{\pi}{S} = \delta 
\]

must hold at this stationary point.

However, since the terminal Policy \( D_{nj} \) (the subscript "nj" indicates that the policy does not involve a jump in the state variables) lasts an infinite time, \( q_2 \leq 0 \) must hold during Policy \( D_{nj} \) to prevent \( q_2 \) rising to \( q_2 = 0 \).

Thus

\[
q_2 = q_2 \delta - \frac{1}{S} [(F_k - \lambda) + k(F_k - \lambda - r)] + \frac{\pi}{S^2} \leq 0 
\]

implies

\[
\frac{\pi}{S} \leq \delta - \frac{q_2 \delta}{S} 
\]

when \( K = K^* \). Conditions (145) and (147) can only hold simultaneously if

\[
\frac{\pi}{S} = \delta 
\]

and

\[
q_2 = 0 
\]

when terminal Policy \( D_{nj} \) is followed. This means two things. Firstly, a switch into terminal Policy \( D_{nj} \) cannot be from Policy C because \( q_2 < 0 \) under Policy C. A switch to terminal Policy \( D_{nj} \) must either be from an initial jump policy, Policy B, D, or F, or a policy involving \( U_2 > 0 \). If, after following a jump policy

\[
\frac{\pi}{S} > \delta 
\]

when Policy \( D_{nj} \) is followed then this is not an optimal policy because, as shall be seen, switches out of
Policy $D_{nj}$ are not optimal and
\[ \frac{\Pi}{S} = \delta \tag{151} \]
does not hold. The optimum strategy instead is to follow a policy sequence which results in $S$ being increased until (151) holds when Policy $D_{nj}$ is followed. Secondly, a policy involving a jump in $N$ is followed so that $K=K^*$ and
\[ \frac{\Pi}{S} < \delta \tag{152} \]
then the firm should have a finite terminal date.

Now there exists a very significant difference between the optimal policies of Chapter IV and those in this chapter. In Chapter IV it was always optimal to reach $K=K^*$ in the shortest possible time and to stay there providing
\[ \frac{\Pi}{S} > \delta \tag{153} \]
In this chapter it is optimal to jump to $K=K^*$ only if when it is reached
\[ \frac{\Pi}{S} = \delta \tag{154} \]
If, however,
\[ \frac{\Pi}{S} > \delta \tag{155} \]
then the firm should engage in policies which increase $S$ until (154) holds when following Policy $D_{nj}$.
The optimal policy will result in the price of the share falling to par when the firm reaches this stationary position. If the firm reaches the stationary position at \( t = t' \) then the price of shares at that time is

\[
P(t') = e^\delta t' \int_t^{t'} \frac{\Pi(t')}{S(t')} e^{-\delta t} \ldots (156)
\]

and this result is clearly different from that obtained in Chapter IV. However, the price of the share may be higher at \( t=0 \) because issuing shares may allow dividends closer to \( t=0 \) to be higher and may allow the optimal capital stock to be achieved sooner.

(4) It is not optimal to switch into Policy \( D_{nj} \) unless this is a terminal policy. If the firm switches from Policy \( D_{nj} \) to Policy C then at the switch

\[
F_k - \lambda = \frac{\delta + rk}{1 + k} \ldots (158)
\]

and it has already been shown that this is not optimal ((1) above).

(5) It is now possible to eliminate more policies. It is never optimal to follow the following Policies:

(a) Policy A
(b) Policy C (switching into Policy \( D_{nj} \) or E is not optimal
(c) Policy E (it can only switch into Policy A)
(d) Policy G (it can only switch into Policy A)
(e) Policy H (it can only switch into Policy A, C or E)
(f) Policy I (it can only switch into Policy A, C or E)
(g) Policy K (it can only switch into Policy A or E)
(h) Policy L (it can only switch into Policy A, C or E)
(i) Policy J (it can only switch into Policy A, C and E).

(6) The only policies which need to be considered are Policies B, D, F and J. Policies B, D, F and J must be initial policies and only Policies Dnj and Jnj can operate at $t > 0$.

(7) Notice that if Policy Jnj is followed it can only switch into Policy Dnj and must do so when

$$F_K - \lambda = \frac{\delta + rk}{1 + k} \quad \cdots (159)$$

We will now consider policies in each of the regions of the $S$, $N$ space. The designation of these regions assumes that $B = B^*$ and they refer to the position of the firm at $t=0$.

(i) $F_K - \lambda - r < 0$

By (6) above, only Policy B may be followed in this region. It may switch into Policy Dnj. If it does and

$$\frac{\Pi}{S} = \delta \quad \cdots (160)$$

then this is the optimal strategy. If

$$\frac{\Pi}{S} < \delta \quad \cdots (161)$$

then $T = \infty$ is not optimal. If, however,

$$\frac{\Pi}{S} > \delta \quad \cdots (162)$$
then the firm should follow the alternative policy. That is, it shows switch from Policy B to Policy N at
\[ F_k - \lambda > \frac{\delta + rk}{1 + k} \quad \ldots (163) \]
and follow this policy of issuing shares and paying dividends until
\[ F_k - \lambda = \frac{\delta + rk}{1 + k} \quad \ldots (164) \]
and then follow Policy N. The size of the jump under Policy B must be such that when Policy N is followed
\[ \frac{S}{S} = \delta \quad \ldots (165) \]
See Diagram III (a). These two alternatives are also shown in Diagram VI, where A is the initial position at \( t = 0 \), \( N^* \) is the endpoint in first policy and \( N_2^* \) is the endpoint in the second.

The objective function with the first policy is
\[ P_1(0) = (N(0) - N_1^*) \frac{1}{S} + 1 \quad \ldots (166) \]
and with the second
\[ P_2(0) = (N(0) - N_1^*) \frac{1}{S} + \left\{ \left[ (F_k - \lambda) + k(F_k - \lambda - r) \right] e^{-\delta t} + e^{-\delta t} \right\} \quad (167) \]
where \( t = t_1 \) when \( N = N_2^* \).
(ii) $F^K = \lambda - r = 0$

Only Policy F is relevant in this region and again Policy F may switch into Policy D_{nj} or Policy J_{nj}.

The former case is optimal if

$$\frac{\pi}{S} = \delta$$

...(168)

after the switch and $T = \infty$ is not optimal if

$$\frac{\pi}{S} < \delta$$

...(169)
If, however,

\[
\frac{\Pi}{S} > \delta
\]  

...(170)

after the switch then the firm should switch from Policy F to Policy J_{nj} and then to Policy D_{nj} when

\[
\frac{\Pi}{S} = \delta
\]  

...(171)

must hold. See Diagram IV. These policies are also shown in Diagram VI only this time the initial point is B. The price of the shares are given by (166) and (167) above. The value of \(N(0)\) is now less.

(iii) \(r < \frac{K - \lambda}{1 + k} = \frac{\delta + rk}{1 + k}\)

In this region only Policies \(D_{nj}\) and \(D_j\) can be optimal. If

\[
\frac{F}{K} - \lambda = \frac{\delta + rk}{1 + k}
\]  

...(172)

then it will be optimal to stay there (follow Policy \(D_{nj}\)) if

\[
\frac{\Pi}{S} = \delta
\]  

...(173)

If, however,

\[
\frac{\Pi}{S} > \delta
\]  

...(174)

then the firm should follow Policy \(D_j\), so that

\[
\frac{F}{K} - \lambda > \frac{\delta + rk}{1 + k}
\]  

...(175)
and then Policy $J_{nj}$ until

$$F_k - \lambda = \frac{\delta + rk}{1 + k} \quad \ldots (176)$$

and

$$\frac{\Pi}{S} = \delta \quad \ldots (177)$$

hold. Then switch to Policy $D_{nj}$. See Diagram V and

Diagram IV where the initial point is $N = N_1^*$, and $S = S(0)$.

If

$$F_k - \lambda < \frac{\delta + rk}{1 + k} \quad \ldots (178)$$

then Policy $D_j$ must be followed. It is optimal to

switch immediately into Policy $D_{nj}$ if

$$\frac{\Pi}{S} = \delta \quad \ldots (179)$$

holds when this is done. If

$$\frac{\Pi}{S} > \delta \quad \ldots (180)$$

then the firm should switch from Policy $D_j$ to Policy $J_{nj}$

and then to Policy $D_{nj}$ so that (179) holds. See

Diagram V and Diagram VI where the initial situation

is $N > N_1^*$, $S = S(0)$. Again the price of the shares

are given by (165) and (167) alone, with the appropriate

values for $N(0)$.

(iv) $F_k - \lambda > \frac{\delta + rk}{1 + k}$

Now the initial policies may be Policy $J_{nj}$ or $D_j$.

If the firm begins with Policy $J_{nj}$ and follows it until
\[ F_K - \lambda = \frac{\delta + r k}{1 + k} \]  \hspace{1cm} \ldots (181)

and switch into Policy \( D_{nj} \) then it will be optimal to do so if

\[ \frac{\pi}{S} = \delta \]  \hspace{1cm} \ldots (182)

after the switch it will not be an optimal policy if

\[ \frac{\pi}{S} < \delta \]  \hspace{1cm} \ldots (183)

after the switch if the inequality in (183) is reversed it will be optimal to follow policy sequence Policy \( D_j \), then Policy \( J_{nj} \) then Policy \( D_{nj} \) with

\[ \frac{\pi}{S} = \delta \]  \hspace{1cm} \ldots (184)

holding after the switch. Again these alternatives are represented in Diagram VI. The price of the share when Policies \( J_{nj} \) and \( D_{nj} \) is

\[ P_3(0) = \left \{ \begin{array}{ll}
\int_0^{t_1} \{ (F_K - \lambda) + k(F_K - \lambda - r) \} e^{-\delta t} dt + e^{-\delta t_1} \\
0
\end{array} \right \} \]  \hspace{1cm} \ldots (185)

and when Policies \( D_j, J_{nj} \) and \( D_{nj} \) are followed the price of shares is

\[ P_4(0) = \left \{ N(0) - N_1 \right \} \frac{1}{S} + \left \{ \int_0^{t_1} \{ (F_K - \lambda) + k(F_K - \lambda - r) \} e^{-\delta t} dt + e^{-\delta t_1} \right \} \]  \hspace{1cm} \ldots (186)
6. **Finite Horizon**

A number of points can now be made about optimal sequences of policies when $T$ is finite.

(1) A policy which involves a jump is not significant if it applies at $t = T$ since all variables can be considered to jump to zero at $T$. In Diagram VII it is the value $x_2(T)$ and not $x_1(T)$ which is relevant in this model.

![Diagram VII](image)

(2) The terminal policies must meet the transversality conditions

$$q_1(T) = \frac{1}{S(T)} \quad \ldots (187)$$

$$q_2(T) = -\frac{N(T)}{[S(T)]^2} \quad \ldots (188)$$

Some policies meet these conditions only at an instant and hence are of no significance. Under Policy A, E or C when

$$F_k - \lambda < \frac{\delta + rk}{1 + k} \quad \ldots (189)$$

$q_1 \geq \frac{1}{S}$, $q_1 > 0$ so that although the transversality condition (187) may be met, these policies can only hold at that instant.
(3) The three remaining policies which do not involve jumps are Policy C when
\[ F_K - \lambda \geq \frac{\delta + r k}{1 + k} \]  \hspace{1cm} (190)
Policy \text{D}_n_j and Policy \text{D}_n_j and either of these first two may be terminal policies.

If Policy C is followed while
\[ F_K - \lambda > \frac{\delta + r k}{1 + k} \]  \hspace{1cm} (191)
then \( q_1 \) may fall so that \( q_1(T) = \frac{1}{S(T)} \) at \( t = T \) and since \( q_2 < 0 \), \( q_2 \) may fall to be
\[ q_2(T) = -\frac{N(T)}{[S(T)]^2} \]  \hspace{1cm} (192)
Policy \text{D}_n_j can be undertaken only if
\[ F_K - \lambda > \frac{\delta + r k}{1 + k} \]  \hspace{1cm} (193)
with
\[ U_1 = \Pi - S[(F_K - \lambda) + k(F_K - \lambda - r)] \]  \hspace{1cm} (194)
\[ U_2 = S[(F_K - \lambda - \delta) + k(F_K - \lambda - r)] \]  \hspace{1cm} (195)
If Policy \text{J}_n_j is a terminal policy then
\[ q_2(T) = -\frac{N(T)}{[S(T)]^2} \]  \hspace{1cm} (196)
so that \( N(T) = 0 \). Thus this requires that \( U_1 < 0 \) holds while Policy \text{J}_n_j is followed. Thus
\[ \frac{\Pi}{S} < (F_K - \lambda) + k(F_K - \lambda - r) \]  \hspace{1cm} (197)
from (194) above.
Policy $D_{nj}$ cannot be a terminal policy because if it is optimal to switch into Policy $D_{nj}$ and retain it for some finite period until $t = T$, the optimal terminal time will be $T = \infty$. See Proposition 3 of Chapter III.

(4) Thus the terminal policy must be Policy C or $J_{nj}$. The terminal capital stock, therefore, must be such that

$$F - A K > c S + rk$$

This is the same result as was obtained in Chapter IV.

(5) A number of policies can be eliminated because they result in $q_1 = \frac{1}{s}$ and a switch into a policy which allows $q_1 = \frac{1}{s}$ is not optimal.

(a) Following Policy I when

$$F_K - \lambda \geq \frac{\delta + rk}{1 + k}$$

is not optimal because the firm must switch into Policy A, C or E and each of these results in $q_1 > \frac{1}{s}$ and remaining.

(b) It is never optimal to switch into Policy $D_{nj}$ because it is not a terminal policy and Policy $D_{nj}$ can only switch into Policy I or C when

$$F_K - \lambda \leq \frac{\delta + rk}{1 + k}$$

(c) A list of policies which are never optimal are

(a) A
(b) $D_{nj}$
(c) E
(d) G
(e) K
(6) Thus the optimal policies may be Policy B, C, D, F, H, I, J, L. Of those Policy B, D, F, H, I, J, L can only be initial policies.

(7) If Policy C is followed then it must be a terminating policy because it can only switch into Policy D or E and neither switches are optimal.

We will now consider policies in each of the regions of the S, N space. The designation of these regions assumes that $B = B^*$ at $t = 0$.

(i) $F_K - \lambda - r < 0$

By (6) above Policies B and H may be optimal initial policies. Policy chains which may be optimal are shown in Diagram VIII. When a chain ends with $T, N > 0$ this means that the last policy was a terminal policy and $N > 0$ held at $t = T$. The first type of policy is shown in Diagram VIII(a) as line I and the second as line II. The terminal values of $N$ and $S$ are $N = N(T)$ and $S = S(T)$ in the latter case. The price of shares in the former case is given by

$$P_5(0) = \left[\frac{N(0) - N_1}{S}\right]^{\frac{1}{2}} + \int_{1}^{T} \left[ (F_K - \lambda) + k(F_K - \lambda - r)\right] e^{-\delta t} dt + e^{-\delta T}$$

and in the latter case by

$$P_6(0) = \left[\frac{N(0) - N_2}{S}\right]^{\frac{1}{2}} + \left[1 + \frac{N(T)}{S(T)}\right] e^{-\delta T}$$

Which of the policies is optimal is determined by a comparison of $P_5(0)$ and $P_6(0)$. 
**DIAGRAM VIII**

B

J

N = 0

T

H

C

N = 0

**DIAGRAM IX**

F

J

N = 0

T

C

L

N > 0

T

N > 0
Policies F or L may be optimal initial policies. The Policy chains are shown in Diagram IX. Diagram VIII (a) shows the alternative policies in the $N$, $S$ plane, when 'A' lies between lines $K_1$ and $K_2$ on the $S(0)$ line. The price of the shares for each type of policy is given by (201) and (202) above.

$P_{K} = \lambda - \tau = 0$

(iii) $\tau < P_{K} - \lambda < \lambda + \tau$

Policies $D_j$ and $J_j$ may be initial policies. See Diagram X. Again Diagram VIII (a) represents these alternatives, where 'A' lies between $K_2$ and $K_3$ on the $S(0)$ line. The price of shares is given by (201) and (202) and the optimal policy is that which maximises this price.
\[ F_K - \lambda > \frac{\partial + r k}{1 + k} \]

Policies \( C, D_j, J_{nj}, J_j \) may be initial policies. See Diagram XI. Policies beginning with Policy \( D_j, J_{nj} \) or \( J_j \) are already represented in Diagram VII (a) and the price of shares in this case is given by (201) or (202). The policy involving Policy \( C \) can be represented as arrows along the \( S(0) \) line pointing upwards but stopping before line \( K_3 \). The price of shares under this policy is simply

\[ P_7(0) = \left[ 1 + \frac{N(T)}{S(T)} \right] e^{-\delta T} \quad (203) \]

Now it is a simple matter to determine the optimal policy for the firm. Firstly, the position in the \( N, S \) plane in which the firm is located at \( t = 0 \), \( B = B^* \) must be identified. Secondly, the price of shares for each of the policies, involving \( T \) finite or infinite, must be calculated. Thirdly, that policy which yields a maximum \( P(0) \) is the optimal one.

Comments

Few writers have distinguished between accumulated funds and share contributions when discussing the firm's financing policy. In 1962 M.J. Gordon\(^1\) summarised the position then and there has been little change since then:

The question of outside equity financing is difficult, and its examination is not made easier by the fact that the existing theory on the subject is still at an elementary level. The theory can be stated quite briefly. If a corporation's stock is selling at an earnings yield \( y \), and the corporation's return on investment is \( r > y \), the issue of new common stock will raise the price of the stock, and vice versa, if \( r < y \).

Gordon elaborates on this principle, including premiums on share issues, but does not radically improve it. In both earnings and dividend models the real question in relation to share issues is when should share issues be used rather than accumulated funds to finance investment? To answer this question a model must allow ploughback and shares to vary simultaneously over time. The assumption that the retention rate is a constant obscures the issues.\(^2\)

The discussion above has shed some light on the conditions under which it may be optimal to issue shares. Firstly, it was found that it may be optimal to issue shares in a block (have jump in \(S\)) if it is possible to do so. In some cases it may be optimal to have jumps in both accumulated funds and shares at \(t = 0\) and then to vary accumulated funds and/or shares continuously over time.

Secondly, if it is optimal to follow a policy of issuing shares and paying dividends at the same time (Policy \(J_{nj}\)) then

\[
D(t) = (F_K - \lambda) + k(F_K - \lambda - r)
\]  \((204)\)

and from (117) above

\[
D(t) - \delta = \frac{dS}{dt} \cdot \frac{1}{S}
\]  \((205)\)

That is, the shareholder's 'Profit' (surplus of the dividend rate over the opportunity cost) should be equal to the rate at which new shares are issued. Further, from (204) and (205)

the rate at which new shares should be issued is

\[
\frac{dS}{dt} = \frac{1}{S} (F_K - \lambda - \delta) + k(F_K - \lambda - r)
\]  

(206)

The first term on the right hand side is the addition to 'profit' (after allowing for the opportunity cost, \(\delta\)) from adding to capital stock by issuing another share. The second term is the addition to profit which comes indirectly because the increase in shares allows an increase in debt. The sum of these terms is the total addition to 'profits' (after allowing for the opportunity cost of \(S\)) from issuing shares.

As \(K\) approaches \(K^*\) so that

\[
F_K - \lambda = \frac{\delta + rk}{1 + k}
\]  

(207)

holds the first term of (206) becomes negative and begins to offset the second. Condition (206) is similar to the condition given in the quote by M.J. Gordon. Condition (206) says that shares may be issued if the issue of shares gives a return greater than their cost. Here, however, 'earnings yield' and the rates of interest are costs because it includes the indirect benefit of loosening the leverage constraint. Clearly, no indirect benefits are included in the condition in the quote.

If it is optimal to switch from Policy \(J_n\) to Policy \(D_{nj}\) then at the switch

\[
D(t) = \delta
\]  

(208)
\[
\frac{\pi}{S} = \delta \quad \ldots (209)
\]

In the process (206) converges to zero. This, of course, is required to happen only if \( T = \infty \). The condition that a policy which yields

\[
\frac{\pi}{S} < \delta \quad \ldots (210)
\]

when \( K = K^* \) is non-optimal if \( T = \infty \) is not unexpected. It simply says that if a policy results in the long-run profit per share falling below the earnings yield then the policy is not optimal. This condition does not apply when \( T \) is finite because the firm may not exist long enough for the long-run profit or dividend rate to become relevant. The long-run consequences of a policy can be ignored.

Another result which is important is that, whether \( T \) is finite or infinite it may be optimal initially to move further away from the value of \( K(t) \) at \( t = T \). This is because by doing so the firm can increase the share price by increasing dividends in years closer to \( t = 0 \) at the expense of those further away. This result is unlikely to occur in a model which had profits in the objective function and not dividends, because in the latter case profits are sacrificed to boost dividends from selling capital stock.

An obvious and major shortcoming of the model is the assumption that shares must be sold at par. In effect the assumption is that the amount received by the firm from an issue of share certificates is equal to the addition to
share capital between which total dividends must be divided. The shares need not be issued as fully paid-up, but the eventual total paid-up value must be equal to the par value of the shares.

This assumption can be relaxed by changing the differential equation concerning accumulated funds. If the firm sells shares at a price greater than par then this surplus is an addition to accumulated funds. It is not an addition to shares as defined in the model because no dividends are paid on the surplus. Similarly, when a firm sells shares at a price less than par the firm can be regarded as selling shares at par but refunding the new shareholders for part of their contribution from accumulated funds. Thus if $P(t)$ is the price of a one dollar share (or the price of one dollar's worth of contribution) and is a factor which takes into account the costs of floating a new issue then the state equations for the system can be written

\[ N = U_1 + \left[ a P(t) - 1 \right] U_2 \]  \hspace{1cm} (211)
\[ S = U_2 \]  \hspace{1cm} (212)

The difficulty with this formulation is that $P(t)$ is itself an integral. The case where a state equation includes an integral is not treated in the standard literature on optimal control and it is likely that theorems have not been developed for this case. Any rewriting of the $N$ equation

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(for example, working in terms of $N$) still includes $P(t)$.

It is very likely, however, that the nature of the switching policies in this more realistic model are similar to those discovered in this chapter but that the details are different. It also seems likely that if the firm is making a surplus on issuing shares then it is more likely to be optimal to issue shares than would otherwise be the case.

M.H. Miller and F. Modigliani have been able to show that under particular assumptions the dividend rate and share issue policies are irrelevant to the price of shares and that the price depends only upon 'real' factors such as investment policy and profitability. This result does not hold in the case where there are costs associated with the issue of shares. A simple example will be used to demonstrate this. Suppose that the firm is at a stationary position and that the associated profit level is $\Pi$. Now if $T = \infty$

$$P(0) = \int_{0}^{\infty} D e^{-\delta t} \, dt \quad \text{(213)}$$

or

$$P(0) = \frac{1}{\delta} - \frac{\Pi}{S} \quad \text{(214)}$$

Suppose now that the firm engages in the following financial transaction. It issues shares equal to $\Delta S$ and, after paying the costs of floating the issue, it receives $\alpha \cdot P(0) \cdot \Delta S$ where $\alpha < 1$. Now, it uses the receipts from the issue to pay higher dividends at $t = 0$. This policy leaves the 'real' consideration unchanged since the capital stock and the level

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of profit are unchanged. The price of the share is now

\[ P'(0) = \frac{\alpha P'(0)}{S} + \frac{\Delta S}{\Delta S} + \int_0^\infty \left( \frac{\Pi}{S + \Delta S} \right) e^{-\delta t} \, dt \quad \ldots (215) \]

or

\[ P'(0) = \frac{1}{\delta} \left[ \frac{\Pi}{S + \Delta S (1-\alpha)} \right] + \frac{1}{\delta} \cdot \frac{\Pi}{S} \quad \ldots (216) \]

and so this purely financial policy has the effect of reducing the price of shares. The basic result is not affected by the assumption that the additional dividends are not paid on the new shares. It is clear from (216) above that if the assumption is made that there are no administrative or underwriting costs (which are sometimes substantial) then \( \alpha = 1 \) and the Miller and Modigliani result holds. In addition, of course, it must be assumed that share issues do not affect the rate of discount. If these restrictive assumptions are not imposed then it is clear that the firm cannot be indifferent between retained profits and share finance and the dynamics of the situation, in the terms discussed above, must be taken into account.

Finally, the process of obtaining optimal policies for a firm for given initial conditions is more difficult in the model analysed in this chapter than that in Chapter IV. This reflects the fact that the assumptions made, when shares are able to increase, result in the necessary conditions being no longer sufficient for optimality. The necessary conditions are sufficient if the Hessian of the Lagrangian is negative definite. This condition imposes particular conditions on the operating profit function and these are not required by economic theory. Consequently, it was decided to analyse this more general case.
POLICY SWITCHES

Because of the large number of switches to be considered, explanation will be kept to a minimum. As already explained in the text, switches of any policy into Policy B, D, F, G, H, I, J, K, or L is not optimal because of the requirement that the state variables are continuous on the time domain. A switch may occur at \( t = T \) but this is of no significance. Consequently, such switches will not be referred to in the following discussion.

Switches of Policy A

Policies C, D, E or J require that

\[ F_K - \lambda - r \geq 0 \]  \hspace{1cm} \ldots (1)

However, under Policy A, in the neighbourhood of

\[ F_K - \lambda - r = 0 \]  \hspace{1cm} \ldots (2)

\( U_1 = \Pi > 0 \) and \( K > 0 \). Thus if equality in (1) holds it only holds for an instant. Thus a switch into these policies is not optimal.

Switches of Policy B

(i) The switch from Policy B to Policy A may be optimal if

\[ F_K - \lambda - r < 0 \]  \hspace{1cm} \ldots (3)

after the switch.

(ii) The switch into Policy C may be optimal if \( N \) jumps so that

\[ r < F_K - \lambda \leq \frac{\delta + r K}{1 + k} \]  \hspace{1cm} \ldots (4)
(iii) The switch into Policy D may be optimal providing
\[ F_K - \lambda = \frac{\delta + rk}{1 + k} \] ...
holds after the switch.

(iv) The switch into Policy E may be optimal if the firm jumps to
\[ F_K - \lambda - r = 0 \] ...

(v) A switch into Policy J may be significant because this policy may last for more than an instant. Thus the switch is significant if
\[ F_K - \lambda > \frac{\delta + rk}{1 + k} \] ...

Switches of Policy C

(i) Under Policy C
\[ F_K - \lambda - r \geq 0 \] ...
and \( N > 0 \). Since \( U_2 = 0 \) and \( N \) cannot jump upwards the firm must go through a policy involving \( 0 < B < k(N+S) \) before following a policy involving \( B = 0 \). Thus a switch into Policy A is not possible.

(ii) A switch into Policy D is possible provided
\[ F_K - \lambda = \frac{\delta + rk}{1 + k} \] ...

(iii) A switch into Policy E can occur at
\[ F_K - \lambda - r = 0 \] ...

(iv) Under Policy C, \( q_2 < 0 \) and so \( q_2 < 0 \) provided the policy is significant (lasts more than an instant). Thus a switch into Policy J is not possible.
Switches of Policy D

(i) Since \( N \) cannot be increased in a jump under Policy D the firm must go through a policy involving \( 0 < B < k(N+S) \) to reach

\[
F_K - \lambda - r < 0 \quad \text{... (11)}
\]
or

\[
F_K - \lambda - r = 0, \quad B = 0 \quad \text{... (12)}
\]

Thus a switch into Policy A is not possible.

(ii) A switch into Policy C is possible and can be sustained for more than an instant if

\[
F < F_K - \lambda < \frac{\delta + rk}{1 + k} \quad \text{... (13)}
\]

If \( U_1 = 0 \) before the switch then

\[
F_K - \lambda = \frac{\delta + rk}{1 + k} \quad \text{... (14)}
\]

before the switch.

(iii) Provided Policy D lasts for more than an instant \( K \) will be such that the firm must go through a policy involving \( U_1 = \pi \) while \( B = k(N+S) \). Thus a switch into Policy E is not optimal.

(iv) If Policy D involved a jump then Policy J may be sustained without a jump if

\[
F_K - \lambda > \frac{\delta + rk}{1 + k} \quad \text{... (15)}
\]

If Policy D involved \( U_1 = 0 \) then a switch into Policy J requires

\[
F_K - \lambda > \frac{\delta + rk}{1 + k} \quad \text{... (16)}
\]
after the switch but

\[ F_K - \lambda = \frac{\delta + rk}{1 + k} \]

under Policy \( D_n \). Thus a switch is not optimal.

(v) Policy \( D_j \) can switch into Policy \( D_n \) at

\[ F_K - \lambda = \frac{\delta + rk}{1 + k} \]

Switches of Policy \( E \)

(i) A switch into Policy \( A \) can occur when

\[ F_K - \lambda - r < 0 \]

just after the switch.

(ii) In Policy \( E \), \( q_1 > \frac{1}{S} \), \( q_1 > 0 \) so that providing this policy is significant (lasts more than an instant) \( q_1 > \frac{1}{S} \) and the firm cannot switch into a policy involving \( U_1 < \Pi \).

Thus a switch into Policy \( D \) or \( J \) is not possible.

(iii) When

\[ F_K - \lambda - r = 0 \]

\( \Pi > 0 \) so that \( N > 0 \) and \( B < 0 \) under Policy \( E \) and a return to \( B = k(N+S) \) is not possible. Thus the firm cannot switch into Policy \( C \).

Switches of Policy \( F \)

(i) Policy \( F \) requires a reduction in \( N \) by a jump to be significant. Since \( K \) must be reduced the firm cannot switch into Policy \( A \).
(ii) The firm can switch into Policy C and this is significant if

\[ r < F_K - \lambda < \frac{\delta + rk}{1 + k} \]  \(\ldots(21)\)

(iii) The firm can jump to Policy D at

\[ F_K - \lambda = \frac{\delta + rk}{1 + k} \]  \(\ldots(22)\)

and remain there.

(iv) A switch into Policy E is possible if

\[ F_K - \lambda - r = 0 \]  \(\ldots(23)\)

(v) A switch into Policy J is possible and this policy may be sustained for more than an instant provided the firm jumps to

\[ F_K - \lambda > \frac{\delta + rk}{1 + k} \]  \(\ldots(24)\)

and

\[ U_1 = \pi - S[(F_K - \lambda) + k(F_K - \lambda - r)] \]  \(\ldots(25)\)

\[ U_2 = S[(F_K - \lambda - \delta) - k(F_K - \lambda - r)] \]  \(\ldots(26)\)

after the switch.

**Switches of Policy G**

(i) If the switch into Policy A occurs then \(q_2 < 0\) must hold after the switch and so

\[ F_K - \lambda > 0 \]  \(\ldots(27)\)

must hold. However, if the firm switches into Policy A when

\[ F_K - \lambda = 0 \]  \(\ldots(28)\)
$q_2 > 0$ will follow and $q_2 > 0$ will follow. Thus

$$F_K - \lambda > 0$$

must hold after the switch. In order that $B = 0$

$$0 < F_K - \lambda < r$$

must hold after the switch.

(ii) Policy C necessarily involves an increase in $K$ and since $B = 0$ and

$$F_K - \lambda - r < 0$$

the policy results in

$$F_K - \lambda - r > 0$$

Thus the firm cannot switch into any policy involving $0 < B < k(N+S)$ or $B = k(N+S)$. Thus a switch into Policy C, D, E or J is not optimal.

Switches of Policy H

(i) Policy H involves $U_1 = -\infty$, $U_2 = +\infty$ and the jumps may be such that

$$F_K - \lambda - r \leq 0$$

and so a switch into Policy A is possible. However, Policy A cannot be sustained unless $q_2 \leq 0$ for more than an instant. Thus

$$F_K - \lambda > 0$$

The switch can take place if

$$0 < F_K - \lambda < r$$
(ii) In Policy H, \( q_1 = \frac{1}{S} \). However, because \( U_2 = +\infty \), \( q_1 > \frac{1}{S} \) follows and the firm cannot switch into a policy involving \( U_1 \neq \Pi \). Thus the firm cannot switch into Policy D or J.

(iii) The firm may switch into Policy C even if \( q_1 < 0 \) because Policy H results in \( q_1 > \frac{1}{S} \). Further, because \( q_i = \frac{1}{S} \) at the switch, the switch is possible and significant if

\[
F_K - \lambda - r > 0 \tag{36}
\]

(iv) A switch may occur into Policy E if the firm jumps to the value of \( K \) at which

\[
F_K - \lambda - r = 0 \tag{37}
\]

Switches of Policy I

(i) The firm can switch into Policy A if just after the switch

\[
F_K - \lambda - r > 0 \tag{38}
\]

and if Policy A is to last more than an instant \( q_2 < 0 \) must hold after the switch and so

\[
F_K - \lambda > 0 \tag{39}
\]

Thus a significant switch can occur if

\[
0 < F_K - \lambda < r \tag{40}
\]

(ii) Policy I requires \( q_1 \geq \frac{1}{S} \) and since \( U_2 > 0 \), this policy results in \( q_1 > \frac{1}{S} \). Thus the firm cannot switch into a policy involving \( U_1 \neq \Pi \). This means that the firm cannot
switch into Policy D or J.

(iii) A switch into Policy C is possible and significant provided

\[ F_K - \lambda - r > 0 \] \hspace{1cm} \text{...(41)}

after the jump.

(iv) A switch into Policy E is possible and significant if

\[ F_K - \lambda - r = 0 \] \hspace{1cm} \text{...(42)}

after the switch.

Switches of Policy J

Policy J stands for policies in which \( U_1 = -\infty \)
and \( U_2 = +\infty \), as well as policies involving \( U_1 \) and \( U_2 \) finite. Switches out of each of these policies will be considered separately.

(i) Suppose Policy J involves a jump in N and S.

(a) Since Policy J involves a jump in S,

\[ q_1 > \frac{1}{S} \]

follows and so a switch into a policy involving \( U_1 \neq \Pi \) is not optimal. Thus the firm cannot switch into Policy B or J_{nj}.

(b) The firm can switch into Policy A if

\[ 0 < F_K - \lambda < r \] \hspace{1cm} \text{...(43)}

after the switch.

(c) The firm can also switch into Policy C if

\[ F_K - \lambda - r > 0 \] \hspace{1cm} \text{...(44)}

after the switch.
(d) A switch into Policy E is possible if
\[ F_K - \lambda - r = 0 \] (45)
after the switch.

(ii) Assume that Policy J involves \( U_1 \neq - \infty \) and \( U_2 \neq 0 \).

(a) Policy J requires \( q_1 = \frac{1}{S} \) and since \( U_2 > 0, q_1 < 0 \)
and
\[ F_K - \lambda \geq \frac{\delta + rk}{1 + k} \] (46)
must hold. Since
\[ F_K - \lambda - r > 0 \] (47)
the firm cannot switch into Policy A or E.

(b) If Policy J switches into Policy C while
\[ F_K - \lambda > \frac{\delta + rk}{1 + k} \] (48)
then \( q_1 < 0 \) and \( q_1 < \frac{1}{S} \) will follow. A switch may occur, however, at
\[ F_K - \lambda = \frac{\delta + rk}{1 + k} \] (49)

(c) A switch into Policy D is possible when
\[ F_K - \lambda = \frac{\delta + rk}{1 + k} \] (50)
and \( U_1 = 0 \). Since \( q_2 < 0 \) must hold after the switch,
\[ \frac{\Pi}{S} \leq \delta \] (51)
when following Policy D.
Switches of Policy K

(i) A switch into Policy A is possible and significant if

\[ 0 < F_K - \lambda < r \]  

... (52)

... (52)

after the switch.

(ii) In Policy K \( q_1 \geq \frac{1}{S} \) and since \( U_2 > 0 \) \( q_1 > \frac{1}{S} \) will follow. Thus the firm cannot switch into a policy involving \( U_1 \neq \Pi \). The firm cannot switch into Policy D or J.

(iii) Policy K involves increasing \( K \) and so the firm cannot switch into a policy involving \( B = k(N+S) \). Thus a switch into Policy C is not optimal.

(iv) The firm can switch into Policy E providing

\[ F_K - \lambda - r = 0 \]  

... (53)

Switches of Policy L

(i) A switch into Policy A is possible and significant if

\[ 0 < F_K - \lambda < r \]  

... (54)

after the switch.

(ii) A switch into Policy C is possible and significant if

\[ F_K - \lambda - r > 0 \]  

... (55)

after the switch.

(iii) Policy L results in \( q_1 > \frac{1}{S} \) so that the firm cannot switch into policies involving \( U_1 \neq \Pi \). Thus a switch into Policy D or J is not optimal.

(iv) It may be optimal to switch into Policy E if

\[ F_K - \lambda - r = 0 \]  

... (56)

after the switch.
Chapter VI

EXTENSIONS OF THE
SHARE-PRICE MAXIMIZATION MODEL OF THE FIRM.

A. Introduction

This chapter is an aside in the main development of the ideas presented in this thesis. Its aim is to discuss extensions of the share-price maximization model presented in Chapter IV. Firstly it will be assumed that the operating profit function varies over time and secondly that the cost of capital assumptions are other than those used in that chapter. In both cases it will be assumed that $U_2(t) = 0$, $t \in [0, T]$. In this case the model has only one state variable and so only one auxiliary variable and the problem can be adequately analysed in terms of the phase plane in $(N, q_1)$. The discussions of these extensions are brief and a detailed analysis is not attempted.
B. DYNAMIC MARKET CONDITIONS

1. Introduction

The analysis of the share price maximizing firm in Chapter IV assumed that the product and factor market conditions were constant. The firm's task was to optimally adjust to the constant environment. In this section it is assumed that product and/or factor market conditions change continuously over time. The capital market conditions will remain unchanged so that \( \delta, r \) and \( k \) remain constants. The assumptions that \( \delta > r \), \( L \in [0,k] \) are retained.

Since it is assumed that only those things vary which enter into the \( R \) and \( C \) functions, variations in market conditions, both product and factor, can be represented in variations in \( F(K) \). The operating profit function, then, is written \( F(K,t) \) and it is assumed to be a continuous function of time. It will be assumed firstly that \( F(K,t) \) moves in an unrestricted way over time and secondly that

\[
F(K,t) = F(K)e^{\rho t}
\]  

where \( \rho > 0 \) and then \( \rho < 0 \). When \( \rho > 0 \) the firm is experiencing continuous improvement in market conditions, and when \( \rho < 0 \) it is experiencing continuous decline. This is analogous to a neo-classical production function of growth theory with neutral technical progress or regress.

The problem is the same as that in equations (3) to (9) of Chapter IV.
The constraints remain

\[ \phi_1 \geq 0, \quad \phi_1 [\Pi - U_1] = 0 \quad \ldots (2) a \]

\[ \phi_3 \geq 0, \quad \phi_3 B = 0 \quad b \]

\[ \phi_4 \geq 0, \quad \phi_4 [k(N+S) - B] = 0 \quad c \]

\[ \phi_5 \geq 0, \quad \phi_5 x = 0 \quad d \]

\[ \phi_6 \geq 0, \quad \phi_6 N = 0, \quad \phi_6 U_1 = 0 \quad e \]

And the Lagrangian is

\[ L = e^{-\delta t} \left( \frac{1}{S} [\Pi - U_1] + q_1 U_1 + \phi_1 [\Pi - U_1] + \phi_3 B + \phi_4 [k(N+S) - B] + \phi_5 x + \phi_6 U_1 \right) \quad \ldots (3) \]

Where the auxiliary variable is \( p_1 = q_1 e^{-\delta t} \). Theorem 1 of Appendix A to Chapter IV which states the necessary conditions of optimality is still relevant.1

The conditions relating to optimal controls are

\[ \frac{\partial L}{\partial U_1} = 0 = \frac{1}{S} + q_1 - \phi_1 + \phi_6 \quad (4) a \]

\[ \frac{\partial L}{\partial B} = 0 = \left( \frac{1}{S} + \phi_1 \right) (F_K(t) - \lambda - r) + \phi_3 - \phi_4 \quad b \]

\[ \frac{\partial L}{\partial x} = 0 = \left( \frac{1}{S} + \phi_1 \right) F_x(t) + \phi_5 \quad c \]

And the auxiliary equation is

\[ q_1 = q_1 \delta - (\frac{1}{S} + \phi_1) (F_K(t) - \lambda) - \phi_4 k \quad (5) \]

---

1 As in Chapter IV the \( t \) argument will be deleted unless it is important to stress the time dependence of the variable.
the transversality condition is

\[ q_1(T) = \frac{1}{S(T)} \]  \hspace{1cm} \text{(6)}

when T is finite.

The principles of optimal output and debt policy are the same as those stated in Chapter IV. Proposition 4 remains unchanged excepting that \( x^* \) is a function of \( t \) and Proposition 5 remains unchanged excepting that \( B^* \) is a function of \( t \).

It is in ploughback policies that the major changes occur.

2. Unrestricted Variations in \( F(K,t) \) Over Time

We will begin by considering the most general case where \( F(K,t) \), although a continuous function of time, varies in an unrestricted way. No attempt will be made to solve the problem in detail but some light will be thrown on the optimal solutions.

The phase plane at some time \( t = t_1, t_1 \in [0,T] \) for a policy involving \( U_2(t) = 0, t \in [t_1,T] \) is the same as Diagram III of Chapter IV. If \( F(K,t) \) does vary over time then the phase plane changes correspondingly. The value of \( N^* \) becomes a function of time because

\[ N^*(t) = K^*(t) - B^*(t) - S(0) \]  \hspace{1cm} \text{(7)}

or

\[ N^*(t) = \frac{K^*(t) - S(0)(1+k)}{1 + k} \]  \hspace{1cm} \text{(8)}
where \( K^*(t) \) is given by

\[
F_K(t) - \lambda = \frac{\delta + rK}{1 + k}
\]

The \( q_1 = 0 \), while remaining vertical, shifts to the left if market conditions are deteriorating \((N^*(t) \) falls) and to the right if conditions are improving \((N^*(t) \) increases). Corresponding shifts occur in \( N(t) \).

Now, suppose at \( t = 0, N(0) > N^*(0) \). There exist two possible optimal policies. Since \( N(0) > N^*(0) \), \( q_1(0) > 0 \) and either \( U_1 = \Pi, q_1 > \frac{1}{s} \) can be followed or \( U_1 = -\infty \), \( q_1 = \frac{1}{s} \).

(i) Consider the first case where \( U_1(0) = \Pi(0) \).
Regardless of whether \( \Pi > 0 \) this may be an optimal policy. The conditions under which this policy may be optimal are specific. Since \( q_1(0) > 0 \)

1. \( N^*(t) > U_1(t) = \Pi(t) \) \( \ldots \) (10)

over some range of \( t \) in order that \( N(t) < N^*(t) \). If this is not the case then either the transversality condition is never met, if \( T \) is finite, or the policy involves \( D(t) = 0, t \in [0, \infty] \) if \( T \) is finite. In both cases the policy would not be optimal. Once \( N(t) < N^*(t) \), \( q_1(t) < 0 \) and \( q(t) \) may fall until \( q_1(t) = \frac{1}{s} \), then, either the firm winds up at that \( t \):

2.(a) \( q_1(t) = \frac{1}{s}, N(t) \leq N^*(t), t = T < \infty \) \( \ldots \) (11)

or

2.(b) \( N^*(t) < U_1(t) = \Pi(t), T \) infinite \( \ldots \) (12)

over some range of \( t \) in the future so that eventually \( q_1(t) = \frac{1}{s} \) and \( N(t) = N^*(t) \). Once this is achieved, \( U_1(t) = N^*(t) \).
If at some future time \( N^*(t) > \bar{N} \) this condition cannot hold and \( q_1(t) < \frac{1}{S} \) will follow. Consequently this particular policy is non-optimal and a new one, along the same lines, must be devised. This type of policy, which involves chasing \( N^*(t) \) will be called a 'pursuit policy'.

(ii) If a pursuit policy is not optimal then the optimal policy, if it exists, is the same as that described in Chapter IV. Set \( U_1 = -\infty \) at \( t = 0 \) until \( N(0) = N^*(0) \) and \( q_1(t) = \frac{1}{S} \). Once \( N = N^* \) this must be maintained by setting \( U_1(t) = N^*(t) \). If at some \( t \in [0,T] \), \( N^*(t) > \bar{N}(t) \) then \( q_1(t) < 0 \) and \( q_1(t) < \frac{1}{S} \). This policy is non-optimal and a pursuit policy, or one involving \( u_2(t) > 0 \) may be optimal.

Suppose, instead, that \( N(0) < N^*(0) \). If we assume that the optimal policy involves \( U_2(t) = 0 \), then \( q(0) > \frac{1}{S} \) must hold.

(iii) Firstly, a pursuit policy may be optimal. This involves \( q_1 > \frac{1}{S} \) and \( U_1 = \bar{N} \) until \( N(t) > N^*(t) \). When this holds \( q_1 > \frac{1}{S} \) and so in order that a pursuit policy be optimal

\[
N^*(t) > U_1(t) = \bar{N}(t) 
\]

must eventually hold long enough for \( N(t) < N^*(t) \) and at this stage \( q_1 < 0 \). Now either of the following conditions may follow:

2. (a) \( q_1(t) = \frac{1}{S}, \ N(t) < N^*(t), \ t = T < \infty \) \( \ldots \ldots \ldots \ldots (14) \)

or

2. (b) \( N^*(t) < U_1(t) = \bar{N}(t), \ T \) infinite \( \ldots \ldots \ldots \ldots (15) \)

over some range of \( t \) in order that \( N = N^*(t) \).
Notice that conditions (1) and 2(a), 2(b) apply whether $N(0) > N^*(0)$ if a pursuit policy is optimal.

(iv) If a pursuit policy is not optimal, and $U_2(t) = 0$, $t \in [0, T]$, then the optimal policy, if it exists, is that described in Chapter IV for the case $N(0) < N^*$, $U_2(t) = 0$, $t \in [0, T]$. That is,

$$U_1 = \Pi$$

until $N \leq N^*$ with equality holding for $T$ infinite.

Once $N(t) = N^*(t)$, if $T$ is infinite then the firm should follow the policy

$$U_1 = N^*$$

No attempt has been made to analyse the case where $U_2(t) > 0$. It is not the sole purpose of this thesis to analyse the share-price maximization case and so this has been dealt with briefly.

3. **Exponential Improvement in Market Conditions**

Now it is assumed that

$$F(K, t) = e^{\rho t} F(K), \quad \rho > 0$$

This is the case where market conditions are believed to be improving exponentially over the indefinite future. The substitution for $F(K, t)$ must now be made in the system 4(a) - 4(c) to (6). Proposition 4 remains unchanged from its original form in Chapter IV because 4(c) becomes

$$F_x(K) = \frac{-\frac{1}{S} e^{-\rho t}}{\frac{1}{S} + \phi_1}$$

$$\cdots (19)$$
and Proposition 5 is given by
\[ e^{\phi t} F_K(K) - \lambda - r = \frac{\phi 4 - \phi 3}{1 + \phi 1} \] \hspace{1cm} \cdots (20)

The auxiliary equation is now
\[ \phi 1 = q_1 \delta - (\frac{1}{S} + \phi 1) \left[ e^{\phi t} F_K(K) - \lambda \right] - \phi 4 k \] \hspace{1cm} \cdots (21)

It follows from the analysis of Chapter IV that because \( r \) and \( \delta \) are constants and \( r < \delta \), \( q_1 = 0 \) for \( N > 0 \) when
\[ e^{\phi t} F_K(K) - \lambda = \frac{\delta + rk}{1 + k} \] \hspace{1cm} \cdots (22)

Now, however, because the operating profit shifts over time the value of \( K \) associated with (21), \( K = \dot{K}^* \) also is a function of time. By differentiating (21) with respect to time (23) is obtained
\[ \dot{K}^*(t) = \frac{-\phi F_K(K)}{F_{KK}(K)} \] \hspace{1cm} \cdots (23)

and \( \dot{K}^*(t) > 0 \) for \( \phi > 0 \). By substituting (22) into (23), we find
\[ \dot{K}^*(t) = \frac{-\phi e^{-\phi t} \left[ \frac{\delta + rk}{1 + k} + \lambda \right]}{F_{KK}(K)} \] \hspace{1cm} \cdots (24)

Since
\[ N^*(t) = K^*(t) - B^*(t) - S(t) \] \hspace{1cm} \cdots (25)

and
\[ B^*(t) = k[N^*(t) + S(t)] \text{ when } K = K^*(t) \]

when \( S(t) \) is a constant \( (U_2(t) = 0) \)
The motion of \( N^*(t) \) is independent of policy, provided

\[
U_2(t) = 0
\]

Suppose that \( N(0) < N^*(0) \) then \( K(0) < K^*(0) \) and

Either the non-pursuit or the pursuit may be optimal. Consider the former. The optimal policy involves \( q_1(0) > \frac{1}{S} \)

and \( U_1(t) = \Pi(t) \) until \( N(t) < N^*(t) \). If \( T \) is finite then

\( N(T) < N^*(T) \) may be optimal. Consider the case, however, where it is optimal not go to \( N(t) = N^*(t) \). This is the case where \( T > t_1 \) where \( t_1 \) is the time at which \( N(t) = N^*(t) \) when 

\( U_1(t) = \Pi(t) \). Once \( N(t) = N^*(t) \) this must be maintained and so

\[
U_1(t) = N^*(t)
\]

where \( N^*(t) \) is given by (26 and 27) above. If this policy

is optimal then \( N^*(t) < \Pi(t) \) with the inequality holding over some range of \( t \in [t_1, T] \). The price of shares when such a policy is optimal is

\[
P_1(0) = \left\{ \begin{array}{ll}
\int_{t_1}^{T} \left[ \frac{\rho F_K(K)}{(1+k)F_{KK}(K)} e^{-\delta t} dt + \left[ 1 + \frac{N(T)}{S(T)} \right] e^{-\delta T} \right] \\
\int_{t_1}^{T} \left[ \frac{\rho F_K(K)}{(1+k)F_{KK}(K)} e^{-\delta t} dt + \left[ 1 + \frac{N(T)}{S(T)} \right] e^{-\delta T} \right]
\end{array} \right.
\]

If a pursuit policy were optimal then the following conditions must exist:

1. \( U_1(t) = \Pi(t) > N^*(t), \quad N(t) < N^*(t) \) \hspace{1cm} \ldots (30)
2. \( U_1(t) = \Pi(t) < N^*(t), \quad N(t) > N^*(t) \) \hspace{1cm} \ldots (31)
Firstly, \( N(t) \) must move to the right faster (but not over the whole range of time for which \( N(t) < N^*(t) \)) than \( N^*(t) \) so that \( N(t) \) will become greater than \( N^*(t) \). Then, for some range of time after that \( N^*(t) \) must move to the right more quickly than \( N(t) \) so that \( N(t) \) will become less than \( N^*(t) \). Since at \( N(t) = N^*(t) \), \( q_1 > \frac{1}{S} \), \( N(t) \) must actually fall below \( N^*(t) \) so that \( q_1 < 0 \) and \( q_1 \) can fall to \( q_1 = \frac{1}{S} \). Finally \( N(t) \) must again increase until \( N(t) = N^*(t) \).

If these necessary conditions, (30) to (32) hold then there is no question of whether to use a non-pursuit or a pursuit policy because the former is not optimal. If a non-pursuit policy was followed, \( N(t) \) would equal \( N^*(t) \) at the same time as this occurs in a pursuit policy. Then

\[
N(t) = N^*(t)
\]  

would be retained. When \( N(t) = N^*(t) \) and \( U_1(t) = \Pi(t) < N^*(t) \) holds in a pursuit policy.

\[
\Pi(t) < N^*(t)
\]  

also holds for a non-pursuit policy and so \( N(t) \) falls short of \( N^*(t) \) and \( q_1 < \frac{1}{S} \) follows. This is not optimal and a pursuit policy may be optimal.

A mixture of a non-pursuit and a pursuit policy may be optimal. For example, it may be optimal for a firm to follow a non-pursuit policy over some interval of time, then to follow a pursuit policy, and then to follow a non-pursuit...
policy. If only a pursuit policy is followed then the price of ordinary shares is calculated according to (29) above. If a more complicated pattern is followed then the computation is basically the same.

If \( N(0) > N^*(0) \) then again a non-pursuit policy or a pursuit policy or a mixture may be optimal. If a non-pursuit policy is optimal for \( u_2(t) = 0, \ t \in [0,T] \), then set \( U_1 = - \infty \) and \( N(0) \) jumps to \( N(0) = N^*(0) \) and \( N(t) = N^*(t) \) is maintained by holding

\[
U_1(t) = N^*(t)
\]

where \( N^*(t) \) is given by (26) or (27) above.

If a pursuit policy was followed from \( t = 0 \) then \( q_1 \) would rise to \( q_1 > \frac{1}{S} \) and \( U_1(t) = \Pi(t) \). The following necessary conditions are required:

1. \[
U_1(t) = \Pi(t) < N^*(t), \ N > N^*(t)
\]

2. \[
U_1(t) = \Pi(t) > N^*(t), \ N < N^*(t)
\]

and the same interpretation should be applied to these statements as was applied to (31) and (32). Again, consider a non-pursuit policy under these circumstances. Now \( N(0) \) jumps to \( N^*(0) \) and remains at \( N(0) = N^*(0) \). However, if (36) holds a non-pursuit policy is non-optimal because \( N(t) \) falls below \( N^*(t) \) and \( q_1 \) will fall below \( q_1 = \frac{1}{S} \). Thus these policies are not alternatives if \( u_2(t) = 0, \ t \in [0,T] \).
4. Exponential Decline in Market Conditions

Now the operating function becomes

$$F(K, t) = e^{\rho t} F(K), \quad \rho < 0$$

...(38)

and it is assumed that market conditions are in a continual state of decline. The values of $K^*(t)$ and $N^*(t)$ are still given by (22) and (25), and $\dot{K}^*(t)$ and $\dot{N}^*(t)$ for $U_2(t) = 0$ are still given by (23), (24), (26), (27). The signs of $K^*(t)$ and $N^*(t)$ are both negative.

None of the pursuit policies are available to the firm in this case. If $N(0) < N^*(0)$ and a policy is followed which results in $N(t) > N^*(t)$ then there is no way for $N(t)$ to return to $N^*(t)$ and so a pursuit policy is not optimal. Similarly, if $N(0) > N^*(0)$ and if $N(0)$ jumps to $N(0) < N^*(0)$ this is not an optimal policy because $q_1 = \frac{1}{S}$ must hold and $q_1 < 0$ at $N(t) < N^*(t)$ and $q_1$ will fall below $q_1 = \frac{1}{S}$ unless $U_2(t) > 0$.

The non-pursuit policies are available whether $N(0) > N^*(0)$. If $N(0) < N^*(0)$ then the firm may follow the policy

$$U_1(t) = \Pi(t).$$

...(39)

Then maintain $N(t) = N^*(t)$ with $U_1(t)$ being given by (26) and (27). Once $N(t) = 0$, $q_1 > 0$ and $N(t)$ rises to $N(t) = \hat{N}(t)$. Now $\hat{N}(t)$ shifts to the left and eventually $\Pi(t) < 0$ for all $N \geq 0$. Thus $N(t) = 0$ again and the firm continues to make losses. Since these losses cannot be financed indefinitely the firm must liquidate. Thus $T$ is finite when $\rho < 0$. 
If \( N(0) > N^*(0) \) and a non-pursuit policy is followed \( N(0) \) jumps to \( N^*(0) \) and remains there by holding \( U_1(t) = N^*(t) \) as before.

The purpose of this analysis is to demonstrate the effects of alternative cost of capital assumptions on the results obtained in Chapter 19. This would serve to underscore the dependence of the results on the assumptions and that models may have contended measures.

The particular models to be developed are:

Model I: 

\[
(1) \quad \text{Model I:} \sum_{i=0}^{t} \left( \frac{1}{2} \right) \quad \text{Model I:} \sum_{i=0}^{t} \left( \frac{1}{2} \right)
\]

Model II: 

\[
(1) \quad \text{Model II:} \sum_{i=0}^{t} \left( \frac{1}{2} \right) \quad \text{Model II:} \sum_{i=0}^{t} \left( \frac{1}{2} \right)
\]

Model III: 

\[
(1) \quad \text{Model III:} \sum_{i=0}^{t} \left( \frac{1}{2} \right) \quad \text{Model III:} \sum_{i=0}^{t} \left( \frac{1}{2} \right)
\]

This model is the one actually used in Chapter 19 with some modifications. The modifications are given in Appendix 2.
C. ALTERNATIVE COST OF CAPITAL ASSUMPTIONS

The assumptions used in Chapters IV and V regarding the cost of capital were very specific. It was argued in Chapter II that the most reasonable assumptions are that \( \delta \) and \( r \) are constants, that \( B \in [0, k(N+S)] \) and that \( \delta > r \) and throughout the remaining chapters these assumptions will remain. The purpose of this section is to investigate the effects of alternative cost of capital assumptions on the results obtained in Chapter IV. This should assist in understanding the dependency of the results on the assumptions and these models may have empirical relevance.

The particular models to be developed are:

Model 1. \( B \in [0, k(N+S)] \); \( r, \delta \) constants

(i) \( r = \delta \)

(ii) \( r > \delta \)

Model 2. \( B \in (0, \bar{B}] \); \( r, \delta \) constants

(i) \( r < \delta \)

(ii) \( r = \delta \)

(iii) \( r > \delta \)

Model 3. \( r' (B) > 0 \); \( \delta \) constant.

Model 1. \( B \in (0, k(N+S)] \); \( r, \delta \) constants

This model is the one already discussed in Chapter IV with minor differences. The motion of the system is given by:

\[
q_1 - \frac{1}{5} - \phi_1 + \phi_6 = 0
\]
\[
\left(\frac{1}{S} + \phi_1\right)\left(F_K - \lambda - r\right) + \phi_3 - \phi_4 = 0 \tag{40(b)}
\]

\[
\left(\frac{1}{S} + \phi_1\right)F_x + \phi_5 = 0 \tag{c}
\]

\[q_1(0) \text{ free}, \quad N(0) = N_0,
\]

and

\[q_1 = q_1^\delta - \left(\frac{1}{S} + \phi_1\right)\left(F_K - \lambda\right) - \phi_4 k \tag{41}
\]

with

\[q_1(T) = \frac{1}{S} \tag{42}
\]

for \(T\) finite.

Propositions 4 and 5, as well as the preceding Propositions remain valid.

(4) \(r = \delta\)

It will now be assumed that benefits to shareholders are such that the rate of capitalization and the rate of interest are the same. This might be the case where the firm is very secure so that over the relevant leverage, \(L \in [0, k]\), the uncertainties regarding income and capital are approximately the same for owners and creditors. As leverage tends to \(L = k\), however, the uncertainties must differ but this difference may be offset if shareholders have strong and direct control over the firm. This effect is conceivably strong enough to cause \(r \geq \delta\).

Discussion will take place in relation to Diagram I of this chapter. By 40(a)

\[q_1 = \frac{1}{S} \tag{43}
\]
if $N > 0$, and $U_1 = \Pi$ if $q_1 > \frac{1}{S}$. Thus $N > 0$ for $N < N$ and $q_1 > \frac{1}{S}$.

Again, for $N > 0$,

$$q_1 = q_1 (\delta + \lambda - F_K) - \phi_4 k$$

...(44)

If

$$F_K - \lambda - \delta > 0$$

...(45)

then $\phi_4 > 0$ from (40)(b) and $r = \delta$.

Thus $q_1 < 0$. If

$$F_K - \lambda - \delta < 0$$

...(46)

then $\phi_3 > 0$ and $\phi_4 = 0$ from (40)(b) and $r = \delta$,

thus $q_1 > 0$. Thus $q_1$ is stationary when

$$F_K - \lambda - \delta = 0$$

...(47)

and $q_1$ disappears from (44) when $q_1 = 0$.

There will be unique value of $K$ meeting (47) and this will be called $K^*$. Now since $r$ and $\delta$ are constants with regard to leverage and

$$F_K - \lambda - r = 0$$

...(48)

when (47) holds, the optimal value of $B$ is $B^* \in [0, k(N+S)]$.

Thus the value of $N$ corresponding to (47) being met is

$$N^* \in [N^*, \bar{N}^*]$$

when

$$\bar{N}^* = K^* - S$$

...(49)
The boundary conditions satisfy, however, only when the optimal \( \hat{N} \), and we must then take a direct approach

\[ \dot{\hat{N}} = \begin{cases} \frac{k(N+S)}{S} & \text{for } N > \hat{N} \\ \frac{q_1}{S} & \text{for } N < \hat{N} \end{cases} \]
The trajectories meeting the necessary conditions are derived as in Chapter IV and are shown in Diagram I. For an optimal programme to exist, $N^* > 0$.

If $T = \infty$ then under realistic adjustment assumptions, the firm will always be able to reach $N^* = [N^*, \bar{N}]$ in finite time. Thus it will be optimal for the firm to reach that range (on the basis of arguments put forward in Chapter IV). The necessary conditions do not, however, help select the optimum $N^*$, and we must consider $P(O)$ directly. Suppose $N(0) > N^*$ then it is optimal to jump to $N^* = [N^*, \bar{N}]$. The value of $P(O)$ is

$$P(O) = (N^* - N(0)) + \int_{0}^{t_1} (\Pi - U/L) \frac{S}{\delta t} dt + \int_{t_1}^{\infty} \Pi S e^{-\delta t} dt \quad \ldots (52)$$

where the first term is the dividend distribution related to the jump at $t = 0$, the second term is dividends distributed while moving in the range $N^* = [N^*, \bar{N}]$ and the third term is the dividends distributed when $N$ is constant. Clearly $\frac{\Pi}{S}$ falls as the firm moves from $N = \bar{N}$ to $N = N^*$ because $K$ is constant and $r_B$ increases. Thus the firm should jump to $N = \bar{N}$. 

\begin{align*}
K^* &= N^* + S + k(N^* + S) \quad \ldots (50) \\
\frac{\bar{N}}{N^*} &= \frac{K^* - (1 + k)S}{1 + k} \quad \ldots (51)
\end{align*}
If $N < N^*$ then

\[
P(0) = \int_{t_1}^{\infty} e^{-\delta t} \frac{(\mu - U_1) e^{-\delta t}}{S} dt \quad \ldots (53)
\]

where $t_1$ is $t_1 \epsilon[t', \infty)$ when $t_1$ is the time at which $N = N^*$.

Whether it is optimal to reduce $B$ and increase $N$ until $N = \frac{N^*}{N}$ depends on a comparison of particular rates of change of $N$.

We have no intention of attempting to solve this problem in detail. The main result for our purposes is obvious. In this model, when $r = \delta$ the average cost of capital is $r = \delta$. See Diagram II.

(ii) $r > \delta$.

If $r > \delta$ then the motion of the system is still given by (40(a) - (c) to (42) above.

When $N > 0$

\[
q_1 = q_1(\delta + \lambda - F_K) - \phi_4 k \quad \ldots (54)
\]
or

\[
q_1 = q_1(\delta + \lambda - F_K) - k[q_1(F_K - \lambda - r) + \phi_3] \quad \ldots (55)
\]

and when $q_1 = 0$

\[
F_K - \lambda - \delta = -\frac{\phi_4 k}{q_1} \quad \ldots (56)
\]

by (54) and by (55)

\[
q_1(F_K - \lambda - \delta) + k(F_K - \lambda - r) = -k\phi_3 \quad \ldots (57)
\]

Now, if $\phi_3 > 0$ then $B^* = 0$ and $\phi_4 = 0$ from (56)

\[
F_K - \lambda - \delta = 0 \quad \ldots (58)
\]
Diagram III

Diagram IV
From (57)

$$F_{\lambda} - \lambda - r < 0$$

(59)

and since $r > \delta$ this is feasible.

If, however, $\phi_4 > 0$ then $B^* = k(N+S)$ and $\phi_3 = 0$

and from (56)

$$F_{\lambda} - \lambda - \delta < 0$$

(60)

But from (57)

$$F_{\lambda} - \lambda - r > 0$$

(61)

and this is inconsistent with $r > \delta$. It is simple to confirm

that $\phi_3 = \phi_4 = 0$ cannot hold when $q_1 = 0$. Thus $q_1$ is stationary

when

$$F_{\lambda} - \lambda - \delta = 0$$

(62)

and the cost of capital is $\delta$, the minimum of $\delta$ and $r$. It

is not an average of $\delta$ and $r$ because $B^* = 0$.

The trajectories meeting the necessary conditions

for $U_2(t) = 0$, $t \geq 0$, are shown in Diagram III. This is the

same as Diagram III in Chapter IV, only the value of $N^*$ is

different and $B^* = 0$ when $N = N^*$.

Diagram IV shows policies meeting the necessary

conditions. Line A represents combinations of $B$ and $(N+S)$

which meet

$$F_{\lambda} - \lambda - \delta = 0$$

(63)

and line B represents combinations which meet

$$F_{\lambda} - \lambda - r = 0$$

(64)
Model 2. \( B \in [0, B] \); \( r, \delta \) constants.

This is the most simple model so far considered but it need not be the most unrealistic. Both the rate of capitalization and the rate of interest are assumed to be constants. The amount of credit available to the firm is not determined by leverage but by a complex of factors, none of which can be singled out in the model. These factors may include the firm's history as a debtor, or profit prospects.

The constraints in this model are

\[ \phi_1 \geq 0, \quad \phi_1 \{ \Pi - U_1 \} = 0 \]  
\[ \phi_3 \geq 0, \quad \phi_3 B = 0 \]  
\[ \phi_4 \geq 0, \quad \phi_4 (\overline{B} - B) = 0 \]  
\[ \phi_5 \geq 0, \quad \phi_5 x = 0 \]  
\[ \phi_6 \geq 0, \quad \phi_6 \overline{N} = 0, \quad \phi_6 U_1 = 0 \]

and the Lagrangian is

\[ L = e^{-\delta t} \left\{ \frac{1}{S} \{ \Pi - U_1 \} + q_1 U_1 + \phi_1 \{ \Pi - U_1 \} + \phi_3 B + \phi_4 (\overline{B} - B) + \phi_5 x + \phi_6 U_1 \right\} \]  

The optimal controls are given by

\[ \frac{3L}{3U_1} = 0 = \frac{1}{S} + q_1 - \phi_1 + \phi_6 \]  
\[ \frac{3L}{3B} = 0 = (\frac{1}{S} + \phi_1) (F_K - \lambda - r) + \phi_3 - \phi_4 \]  
\[ \frac{3L}{3x} = 0 = (\frac{1}{S} + \phi_1) F_x + \phi_5 \]
and the state equations are

\[ q_1 = q_1 \delta - \left( \frac{1}{S} + \phi_1 \right) (F_K - \lambda) \]  

\[ \ldots (68) \]

The transversality conditions are

\[ q_1(T) = \frac{1}{S} \]  

\[ \ldots (69) \]

The only difference the new boundary of \( B \) has explicitly made on the system is that \( q_1 \) is not dependent on whether \( B \) is on its upper boundary.

When \( N > 0 \)

\[ q_1 = q_1(\delta + \lambda - F_K) \]  

\[ \ldots (70) \]

so that when \( q_1 = 0 \)

\[ F_K - \lambda - \delta = 0 \]  

\[ \ldots (71) \]

Let the value of \( K \) holding when (71) holds be \( K^* \) and the corresponding value of \( N \), for given \( S \), be \( N^* \). For \( N^* > 0 \) the trajectories in this phase plane remain the same as those in Diagram III of Chapter IV.

What difference does the relationship between \( \delta \) and \( r \) make to the results?

If \( \delta > r \) then \( B^* = \bar{B} \) and if

\[ N^* = K^* - B^* - S > 0 \]  

\[ \ldots (72) \]

then the phase plane is as described above and policies meeting the necessary conditions are shown in Diagram V(a).

If \( N^* \geq 0 \) does not exist then the firm must liquidate immediately because if \( N \) jumps to \( N = 0 \) and remains there,
$q_1 > 0$ and the transversality condition is never met. This is the case where $B^* < \bar{B}$, that is, the limit of available credit is not exhausted when $N = 0$. Clearly it would be optimal for this firm to liquidate and then lend at interest rate $r$.

If $r = \delta$ then the phase space is the same as Diagram I in this chapter. Policies meeting the necessary conditions are shown in Diagram V(b). Again, if $N^* \geq 0$ does not exist then it is optimal for the firm to liquidate at $t = 0$.

If $r > \delta$ then the phase diagram is the same as Diagram III in Chapter IV. The optimal value of $B$ when $K = K^*$ is $B^* = 0$. If $N^* \geq 0$ does not exist then it is optimal to liquidate at $t = 0$. Diagram V(c) shows policies meeting the necessary conditions for this case.

**Model 3.** $r'(B) > 0; \delta$ constant

This case may be thought of as the 'large firm' case. It is assumed that the rate of interest increases with the total of the firm's debt. It is common practice to write the total interest bill as $w(B)$

$$w(B) = r.B$$  \hspace{1cm} \ldots(73)

where $r$ is the average rate of interest, and

$$w'(B) = r + B.r'$$  \hspace{1cm} \ldots(74)

Expression (74) however, implies that all interest rates are renegotiated as the level of debt changes and this is clearly unrealistic. We can, however, write
DIAGRAM V (a)

\[ \delta > r \]

\[ 0 \leq (N + S) \leq N + S \]

DIAGRAM V (b)

\[ \delta = r \]
without specifying the derivatives in detail. In this model, the average rate of interest or the availability of credit is dependent on leverage but may be dependent on the firm's history and general prospects. The firm need not, of course, be large in the debt market but it must be a monopsonist or an oligopsonist in the particular capital markets in which it operates.
The constraints in this model are

\[ \phi_1 \geq 0, \quad \phi_1[\Pi - U_1] = 0 \]  
\[ \phi_3 \geq 0, \quad \phi_3 B = 0 \]  
\[ \phi_5 \geq 0, \quad \phi_5 x = 0 \]  
\[ \phi_6 \geq 0, \quad \phi_6 N = 0, \quad \phi_6 U_1 = 0 \]

and the Lagrangian is

\[ L = e^{\delta t} \{ \frac{1}{S} [\Pi - U_1] + q_1 U_1 + \phi_1[\Pi - U_1] + \phi_3 B + \phi_5 x + \phi_6 U_1 \} \ldots (73) \]

The constraint defining the upper limit to \( B \) is not included because the limit is defined by the rate of interest, through (75) and (76).

The optimal controls are given by

\[ \frac{\partial L}{\partial U_1} = 0 = -\frac{1}{S} + q_1 - \phi_1 + \phi_6 \]  
\[ \frac{\partial L}{\partial B} = 0 = (\frac{1}{S} + \phi_1)(F - \lambda - w'(B)) + \phi_3 \]  
\[ \frac{\partial L}{\partial x} = 0 = (\frac{1}{S} + \phi_1)F_x + \phi_5 \]

and the state equation is

\[ q_1 = q_1 \delta - (\frac{1}{S} + \phi_1)(F - \lambda) \ldots (80) \]
The transversality condition is,

\[ q_1(T) = \frac{1}{S} \]  \hspace{1cm} \text{(81)}

The differences between this system and that of Model 1 are: that \( w'(B) \), which varies with \( B \) replaces \( r \), that \( \phi_k \) is deleted from the state equations and \( \phi_k \) from the control equation relating to \( B \).

From 79(c) Proposition 4 in Chapter IV relating to optimal output policy remains unchanged. However, Proposition 5 is changed slightly because

\[ F_K - \lambda - w'(B) = \frac{-\phi_3}{1 + \phi_1} \]  \hspace{1cm} \text{(82)}

Thus if \( B^* = 0 \)

\[ F_K - \lambda - w'(B) \leq 0 \]  \hspace{1cm} \text{(83)}

and if \( B^* > 0 \)

\[ F_K - \lambda - w'(B) = 0 \]  \hspace{1cm} \text{(84)}

From control equation 79(a) the dependence of \( N \) on \( q_1 \) remains as in Model 1.

When \( N > 0 \)

\[ q_1 = q_1 (\delta + \lambda - F_K) \]  \hspace{1cm} \text{(85)}

so that when \( q_1 = 0 \)

\[ F_K - \lambda - \delta = 0 \]  \hspace{1cm} \text{(86)}

From (85) \( q_1 < 0 \).
Diagram VI(a)

Diagram VI(b)
for $N < N^*$ and $q_1 > 0$ for $N > N^*$. The phase plane for $U_2(t) = 0 \, t \geq 0$, $N^* > 0$ is the same as Diagram III in Chapter IV. The optimal paths to $N = N^*$ or $N \leq N^*$ when $T$ is finite, as described in Chapter IV remain unchanged. If $N^* = 0$ does not exist then it is optimal for the firm to liquidate at $t = 0$, otherwise the transversality condition will not be met.

Clearly, if an optimal program exists

$$F_K - \lambda - \delta = 0 \quad \ldots(88)$$

$$F_K - \lambda - w'(B) \leq 0 \quad \ldots(89)$$

when the firm is at its stationary optimum, with $B^* = 0$ if (88) holds with inequality. In Diagram VI (a) and VI (b) line A shows combination of $B$ and $N+S$ for which (87) holds with equality and line B shows combinations of $B$ and $N+S$ which meet (88) with equality. Diagram VI (a) shows policies which meet the necessary conditions when $N^* > 0$, $B = 0$ when the firm is adjusted and $T = \infty$, and Diagram VI (b) shows the case when $N^* > 0$, $B^* > 0$.

This completes the discussion of the affects of varying the cost of capital assumptions on the model developed in Chapter IV. Table I records the long-run average cost of capital and optimal debt for each model, including the basic model.
### TABLE I

**COST OF CAPITAL AND OPTIMAL DEBT**

<table>
<thead>
<tr>
<th>MODEL</th>
<th>$\delta &gt; r$</th>
<th>$\delta = r$</th>
<th>$\delta(L) \geq r$</th>
<th>$\delta &lt; r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Be $0, k(N+S)$, $r, \delta$ constants.</td>
<td>$\frac{\delta + kr}{1 + k}$</td>
<td>$\delta = r$</td>
<td>$B = 0, k(N+S)$</td>
<td>$\delta$</td>
</tr>
<tr>
<td>Average cost of capital.</td>
<td>$B^*$</td>
<td></td>
<td></td>
<td>$0$</td>
</tr>
<tr>
<td>2. Be $0, B$, $r, \delta$ constants.</td>
<td>$\overline{\delta}$</td>
<td>$\delta = r$</td>
<td>$B \in 0, k(N+S)$</td>
<td>$\delta$</td>
</tr>
<tr>
<td>Average cost of capital.</td>
<td>$B^*$</td>
<td></td>
<td></td>
<td>$0$</td>
</tr>
<tr>
<td>3. $r'(B) &gt; 0$, $\delta$ constant</td>
<td>$N &lt; 0$ and so no optimal policy exists.</td>
<td>$\delta = r(B)$</td>
<td></td>
<td>$\delta$</td>
</tr>
<tr>
<td>Average cost of capital.</td>
<td>$B^*$</td>
<td></td>
<td></td>
<td>$0$</td>
</tr>
<tr>
<td>Liquidate at $t = 0$.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Chapter VII

A GROWTH MAXIMIZATION MODEL WITH
A GIVEN NUMBER OF SHARES.

1. Introduction

In this chapter and the following one it will be assumed that the firm is under managerial control in the sense discussed in Chapter I. That is, it will be assumed that managers are in a position to follow policies which conflict with the owners' best interests. It has already been seen that this assumption can justify numerous objective functions. One of the most favoured was the maximization of the discounted rate of growth over an infinite horizon but this was rejected because it made managers' benefits independent of size. We will use instead

$$\maximize \int_0^\infty K(t) e^{-\rho t} \, dt, \quad \rho > 0$$

(1)

as the objective because this incorporates both size and impatience. It will be assumed in this chapter that $U_2(t) = 0$, $t \in [0, \infty)$ and the justification for this is similar to that given in the introduction to Chapter IV. Firstly, it assists understanding the case where shares may be issued at par by allowing an examination of the choice between ploughback and debt finance on their own. Secondly, it may be the most relevant model when the firm is a partnership or a joint stock company - or even a government enterprise, where the owners will not contribute more ownership capital. The case where $U_2(t) > 0$ holds will be examined in the next chapter.
The rate of discount, \( \rho \), reflects management's time preference regarding the distribution of benefits and the relationship between the capital stock and benefits. The size of \( \rho \) can only be a subject of speculation. Suppose the following assumptions hold:

1. All benefits received by managers are in terms of money income.
2. The money income of each manager is a constant proportion of the capital stock. Thus if \( y_i \) is the money income of the \( i \)th manager and his income is \( \varepsilon_i \) proportion of the capital stock, then

\[
y_i(t) = \varepsilon_i K(t)
\]  

(2)

3. The capital stock at \( t = 0 \), \( K(0) \), can be converted into capital stock at \( t = 1 \), \( K(1) \) at a decreasing rate. Thus the transformation frontier of \( K(0) \) into \( K(1) \) is concave to the origin.
4. The lending and borrowing rates of interest are equal to each other and are equal for all managers. Let the interest rate be \( r_m \).

Consider Diagram I. The line TT is the income transformation frontier for the \( i \)th manager and the slope of the line A,B is \( -(1+r_m) \). The indifference curves, \( I \), are the \( i \)th manager's indifference curves between income at \( t = 0 \) and income at \( t = 1 \). Clearly the optimal position is where his net income at \( t = 0 \) is \( y_{i1}(0) \), income received from the firm is \( y_{i2}(0) \) and where he borrows \( [y_{i1}(0) - y_{i2}(0)] \). Similarly at \( t = 1 \) he should receive \( y_{i2}(1) \) from the firm and repay \( [y_{i2}(1) - y_{i1}(1)] \). Now the rate of discount he applied to
his income from the firm was \( r_m \). Since this rate is the same for all managers and the same for borrowing and lending, all managers, whether they borrow or lend or balance their budgets at \( t = 0 \) will apply the same rate of discount to their incomes received from the firm.

For any individual

\[
y_1(t) = \varepsilon K(t)
\]

and so an income pattern of \( y_1(0), y_1(1) \) in Diagram II implies a capital pattern of \( K_1(0), K_1(1) \). Because the income transformation curves have the same shape for all managers and \( r_m \) is the same they all desire \( K_1(0), K_1(1) \). Thus

\[
\rho = r_m .
\]

All this holds providing the uncertainty associated with borrowing and lending is the same as the uncertainty relating to the receipt of future income. In fact most of the perquisites associated with the growth of the firm are much more uncertain than a borrowing or lending contract entered into by the manager. They include promotion, expense accounts and 'gifts' which are usually highly contingent by their nature. The interest rate used in the above analysis must, then, relate to lending and borrowing involving a high degree of risk. If this is \( r_u \) then on the assumption that people are risk avoiders

\[
r_u > r_m .
\]

A reinterpretation of Diagram I is now necessary. If \( TT \) refers to the expected value of benefits then the line \( A,B \) should have a larger (negative) slope. Similarly, in Diagram II,
\( K_1(0) \) should be increased and \( K_1(1) \) decreased. The rate of discount should be

\[
\rho = r_u .
\]

... (6)

The fact that uncertainty of receipts of future benefits differs between individuals does not affect the conclusion since (5) is very likely to hold for all managers.

Finally, it is very likely that

\[
\rho > \delta
\]

unless the firm's future for shareholders is very uncertain. Since it is assumed that managers operate under a share-price constraint condition (7) seems justifiable.

The assumptions stated in Chapter III concerning the supply of capital and its costs remain unchanged. Perhaps the most desirable constraint on managers actions, apart from those already stated, is a minimum share price. The constraint would have to be of the form

\[
\int_0^\infty \left[ \Pi(t) - U(t) \right] \frac{e^{-\delta t}}{S(t)} dt \geq z \]

... (8)

since the theorems\(^1\) for problems involving integral constraints have been developed only for the case where the integration is over \( t \in [0, T] \) when the planning horizon is over \( t \in [0, T] \).

Such a problem, however, makes only the price at \( t = 0 \) the constraint and does not restrict the price of shares at \( t \in (0, T) \). For example, a policy may be optimal involving a large initial dividend payment and thereafter no dividend payments, but this policy will result in the price of shares falling to zero at \( t > 0 \). A constraint such as a share price constraint

must be binding on managers over \( t \in [0, T] \). Thus in this and the next chapter managers will be constrained by a minimum dividend payment

\[ \frac{\Pi(t) - U_1(t)}{S(t)} > z \]

where \( z \) is a constant, rather than a minimum share price.

2. The Problem

Mathematically, the problem in this chapter is to find the optimal control vector \((U_1(t), B(t), x(t))\) and the trajectory of the state variable \(N(t)\), given \( t \in [0, \infty] \) and \( N(\infty) \) free, which maximize

\[ J = \int_0^\infty K(t) e^{-pt} \, dt \]

subject to:

\[ \dot{N}(t) = U_1(t) \]

and

\[ \frac{\Pi(t) - U_1(t)}{S} > z \]

\[ 0 \leq B(t) \leq k [N(t) + S] \]

\[ N(t), x(t) > 0 \]

\[ N(0) = N_0, S = S_0. \]

This is a non-autonomous fixed-time, free endpoint problem and the necessary conditions for a weak extremum are given in Appendix A to this chapter.
The interpretation of the shadow price for the state variable \( N(t) \), \( p_1(t) \), is now given by

\[
\sum \max \left[ K(t) e^{-\rho t} dt \right] \frac{\partial}{\partial N} \left|_{N=N(t_1)} \right. 
\]

That is, \( p_1(t_1) \) is the present value of the marginal contribution \( N(t) \) makes to the managers' objective function at \( t = t_1 \). Similarly, if \( p_1(t) \) is defined as

\[
p_1(t) = q_1(t) e^{-\rho t_1} 
\]

then

\[
\sum \max \left[ K(t) e^{\rho t} dt \right] \frac{\partial}{\partial N} \left|_{N=N(t_1)} \right. 
\]

That is, \( q_1(t_1) \) is the marginal contribution \( N(t) \) makes to the managers' objective function at \( t = t_1 \). Now, if the Lagrangian terms are defined as

\[
Z_1(t) = \phi_1(t) e^{-\rho t} 
\]

and

\[
\phi_1 \geq 0, \quad \phi_1 \left( \frac{\Pi-U}{S} - z \right) = 0 
\]

\[
\phi_3 \geq 0, \quad \phi_3 B = 0 
\]

\[
\phi_4 \geq 0, \quad \phi_4 [k(N+S) - B] = 0 
\]

\[2\] The 't' argument will be deleted from the rest of this chapter unless time dependence is being stressed. Notice that \( q_2 \) and \( \phi_2 \) are not used - these will be used to refer to the state variable \( S \) and the control variable \( U_2 \) respectively in the next chapter.
\[ \dot{\phi}_5 = 0, \; \dot{\phi}_5 x = 0 \] ...21(d)

\[ \dot{\phi}_6 > 0, \; \dot{\phi}_6 N = 0, \; \dot{\phi}_6 U_1 = 0 \] (e)

The Lagrangian is

\[ L = e^{-\phi t} \left( N + S + B + q_1 U_1 + \phi_1 \left( \frac{\Pi - U_1}{S} - z \right) \right) \]

\[ + \phi_3 B + \phi_4 \left[ k(N+S) - B \right] + \phi_5 x + \phi_6 U_1 \] ... (22)

The constraint on \( U_1 \) given by (12) is never binding because (13) requires

\[ \Pi - U_1 > zS > 0 \] ... (23)

and so this constraint has been deleted.

The necessary conditions for an optimum policy are given by the constraints on the control and state variables, the optimal condition relating to the controls

\[ \frac{\partial L}{\partial U_1} = 0 = q_1 - \frac{\phi_1}{S} + \phi_6 \] ... 24(a)

\[ \frac{\partial L}{\partial B} = 0 = 1 + \frac{\phi_1}{S} (F_x - \lambda - \gamma) + \phi_3 - \phi_4 \] (b)

\[ \frac{\partial L}{\partial x} = 0 = \frac{\phi_1}{S} F_x + \phi_5 \] (c)

the motion of the state variable

\[ N = U_1 \] ... 25(a)

\[ N(0) = N_0 \] (b)
and the motion of the auxiliary variable

\[ q_1 = q_1^0 - \frac{\phi_1}{S}(F_k - \lambda) - \phi_4 k - 1 \]  

\[
\text{with } q_1(0) \text{ and } q_1(\infty) \text{ free.}
\]

The condition relating to the optimal output policy remains the same as that given in Proposition 4 of Chapter IV, providing $\phi_1 > 0$. If $\phi_1 > 0$ then the condition is defined.

From 24(c)

\[ F_x = \frac{-\phi_5 S}{\phi_1} \]  

\[
\text{Thus if } x^* > 0 \text{ then}
\]

\[ F_x = 0 \]  

\[
\text{and if } x^* = 0 \text{ then}
\]

\[ F_x \leq 0 \]  

The condition concerning optimal debt policy is different from that given in Proposition 5 of Chapter IV.

From 24(b)

\[ \frac{\phi_1}{S}(F_k - \lambda - r) + 1 = \phi_4 - \phi_3 \]  

and if $\phi_1 > 0$ and $B^* = k(N+S)$ then $\phi_4 > 0$, $\phi_3 = 0$ and

\[ \frac{\phi_1}{S}(F_k - \lambda - r) + 1 > 0 . \]  

\[
\text{If } B^* \in (0, k(N+S)) \text{ then } \phi_4 = \phi_3 = 0 \text{ and}
\]

\[ \frac{\phi_1}{S}(F_k - \lambda - r) + 1 = 0 \]  

\[
\text{... (27)}
\]

\[
\text{... (28)}
\]

\[
\text{... (29)}
\]

\[
\text{... (30)}
\]

\[
\text{... (31)}
\]

\[
\text{... (32)}
\]
If $B^* = 0$ then $\phi_3 = 0$, $\phi_4 = 0$ and

$$\frac{\phi_1}{S} (F_K - \lambda - r) + 1 \leq 0. \quad \cdots (33)$$

If $\phi_1 = 0$ then

$$l = \phi_4 - \phi_3 \quad \cdots (34)$$

and so $B^* = k(N+S)$.

It will now be shown that if $N > 0$ then

$$\frac{\Pi - U_1}{S} = z \quad \cdots (35)$$

or that $N$ jumps ($U_1 = -\infty$) to $N = 0$. Suppose that this proposition is not true.

When $N > 0$ then $\phi_1 = 0$ and

$$q_1 = 0 \quad \cdots (36)$$

Now, since $B^* = k(N+S)$

$$q_1 = -k - 1 < 0 \quad \cdots (37)$$

and so the condition $q_1 > 0$ when $N > 0$ is violated.

If $N$ jumps to $N = 0$, however, then

$$q_1 = -\frac{\phi_1}{S} (F_K - \lambda) - \phi_4 k - 1 > 0 \quad \cdots (38)$$

may hold after the switch. Hence the following proposition must hold.
Proposition 7

(i) If $N > 0$ then

$$\frac{\Pi - U_1}{S} = z$$

or

$$U_1 = -\infty$$

so that $N = 0$. That is, either dividends must be a minimum or the firm must reduce $N$ to zero.

(ii) If $N = 0$ then

$$\frac{\Pi - U_1}{S} \geq z$$

and $B^* = k(N+S)$ if the inequality holds.

3. Policy Alternatives

Optimal output policy enters only incidentally in the problem and it can be ignored from now on. The remaining policies are six and they are shown in Table A.

<table>
<thead>
<tr>
<th>Policy</th>
<th>$U_1$</th>
<th>$B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$\Pi - zS$</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>$&lt;\Pi - zS$</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>$\Pi - zS$</td>
<td>$k(N+S)$</td>
</tr>
<tr>
<td>D</td>
<td>$&lt;\Pi - zS$</td>
<td>$k(N+S)$</td>
</tr>
<tr>
<td>E</td>
<td>$\Pi - zS$</td>
<td>$0 &lt; B &lt; k(N+S)$</td>
</tr>
<tr>
<td>F</td>
<td>$&lt;\Pi - zS$</td>
<td>$0 &lt; B &lt; k(N+S)$</td>
</tr>
</tbody>
</table>
The conditions under which each of these policies may hold are considered in turn.

A. $U_1 = \Pi - zS$, $B = 0$

Clearly
\[ q_1 = \frac{\phi_1}{S} - \phi_6 \]  
\[ \phi_4 = 0 \]  
\[ \frac{\phi_1}{S} = (F_K - \lambda - r) + 1 \leq 0 \]  
\[ q_1 = q_1^0 - \frac{\phi_1}{S} (F_K - \lambda) - 1. \]

If $N > 0$ then
\[ q_1 = q_1^0 (\rho + \lambda - F_K) - 1 > 0 \]

Policy A may be optimal when $N = 0$ if
\[ \Pi - zS > 0 \]

If
\[ \Pi - zS = 0 \]
then $N = 0$, $N = 0$ remains and $B = 0$ remains. If
\[ \Pi - zS > 0 \]
then $N > 0$ and $N > 0$ will occur. If, however,
\[ \Pi - zS < 0 \]
occurs when $N = 0$ then the firm must switch into a policy involving $U_1 = 0$. However the firm can only switch into a policy involving
\[ U_1 < \Pi - zS \]
or $B > 0$ and in both cases $N < 0$ remains.

If Policy A is a terminal stationary policy then

$$U_1 = \Pi - zS = 0 \quad \ldots (49)$$

and $B = 0$. Further,

$$q_1(F_K - \lambda - r) + 1 \leq 0. \quad \ldots (50)$$

Now

$$\frac{d}{dt} [q_1(F_K - \lambda - r) + 1] = q_1(F_K - \lambda - r) < 0 \quad \ldots (51)$$

and so Policy A may be a stationary policy when $N > 0$.

B. $U_1 < \Pi - zS$, $B = 0$

Since

$$U_1 < \Pi - zS \quad \ldots (52)$$

$\phi_1 = 0$ and from

$$\frac{\phi_1}{S}(F_K - \lambda - r) + 1 = \phi_4 - \phi_3 > 0 \quad \ldots (53)$$

$\phi_4 > 0$ and so $B^* = k(N + S)$. Thus Policy B cannot exist.

C. $U_1 = \Pi - zS$, $B = k(N + S)$

Now

$$q_1 = \frac{\phi_1}{S} - \phi_6 \quad \ldots (54)$$

$$\phi_3 = 0 \quad \ldots (55)$$

$$\frac{\phi_1}{S}(F_K - \lambda - r) + 1 \geq 0 \quad \ldots (56)$$
If $N > 0$ then

$$q_1 = q_1 \left[ (\rho + \lambda - F_K) - (F_K - \lambda - r) \right] - k - 1 \quad (57)$$

and

$$q_1 = \begin{cases} >0 & \text{when } q_1 > \frac{k + 1}{(\rho + \lambda - F_K) - k(F_K - \lambda - r)} \\ =0 & \text{when } q_1 = \frac{k + 1}{(\rho + \lambda - F_K) - k(F_K - \lambda - r)} \\ <0 & \text{when } q_1 < \frac{k + 1}{(\rho + \lambda - F_K) - k(F_K - \lambda - r)} \end{cases} \quad (58)$$

If $N = 0$ then Policy C may be optimal if

$$\Pi - zS > 0 \quad (59)$$

If

$$\Pi - zS > 0 \quad (60)$$

then $N > 0$ and $N > 0$ will occur. If

$$\Pi - zS = 0 \quad (61)$$

then $N = 0$, $N = 0$. This policy may be maintained if

$$\frac{\phi_1}{s} (F_K - \lambda - r) + 1 \ge 0 \quad (62)$$

is maintained. Since $K = 0$

$$F_K - \lambda - r = \text{const.} \quad (63)$$

and whether $B = k(N+S)$ can be maintained depends on the motion of $\phi_1$. Since

$$q_1 = \frac{\phi_1}{s} - \phi_6 \quad (64)$$

the knowledge of $q_1$ and the continuity of $q_1$ does not allow
a statement on the motion of $\phi_1$, because $\phi_6$ is undetermined.

If

$$\Pi - zS < 0 \quad \text{...(65)}$$

then $N < 0$, $N = 0$ and the firm must switch into a policy resulting in $N \geq 0$.

Policy C may be a stationary policy when $N > 0$.

Policy C requires

$$q_1(F_K - \lambda - r) + 1 \geq 0 \quad \text{...(66)}$$

and so

$$\frac{d}{dt} [q_1(F_K - \lambda - r) + 1] = q_1(F_K - \lambda - r) \geq 0 \quad \text{...(67)}$$

must hold. Thus if

$$F_K - \lambda - r > 0 \quad \text{...(68)}$$

$q_1 \geq 0$ must hold. Now

$$F_K - \lambda - e^{\gamma} < 0 \quad \text{...(69)}$$

and from (58) this is consistent with $q_1 \geq 0$. Further if

$$F_K - \lambda - r < 0 \quad \text{...(70)}$$

then $q_1 \leq 0$ must hold. If $q_1 < 0$ holds then eventually $q_1 < 0$ will occur and so $q_1 = 0$ must hold. If

$$F_K - \lambda - r = 0$$

then $q_1 \geq 0$ is consistent. Thus Policy C may be a terminal policy when $N > 0$. 
D. \( U_1 < \Pi - zS, B = k(N+S) \)

Now

\[ q_1 = -\phi_6 \]  
...(71)

and from Proposition 7 above, \( N = 0 \), or \( N > 0 \) and \( U_1 = -\infty \).

\[ \phi_3 = 0 \]  
...(72)

\[ q_1 = q_1^P - 1 < 0 \]  
...(73)

Thus this policy can only be a terminal policy if it does not involve a jump because \( q_1 < 0 \) and remains, and if a policy leading to \( N > 0 \) is followed,

\[ q_1 \geq 0 \]  
...(74)

is required. Policy F will be shown to be non-optimal.

E. \( U_1 = \Pi - zS, B \in (0,k(N+S)) \)

Now

\[ q_1 = \frac{\phi_1}{S} - \phi_6 \]  
...(75)

\[ \phi_4 = \phi_3 = 0 \]  
...(76)

\[ \frac{\phi_1}{S} (F_K - \lambda - r) + 1 = 0 \]  
...(77)

If \( N > 0 \)

\[ q_1 = q_1^P (\rho + \lambda - F_K) - 1 > 0. \]  
...(78)

The firm can be in a stationary position when following Policy E if

\[ U_1 = \Pi - zS = 0 \]  
...(79)
Since \( \Pi \) falls as \( B \) increases, \( B = 0 \) is also required. Now

\[
\frac{\phi_1}{S} (F_K - \lambda - r) + 1 = 0 \quad \ldots (80)
\]

must remain. Because \( K = 0 \)

\[
F_K - \lambda - r = \text{const.} \quad \ldots (81)
\]

and \( \phi_1 \) must be a constant. Now if Policy E leads to \( N = 0 \) then

\[
q_1 = \frac{\phi_1}{S} > 0 \quad \ldots (82)
\]

when \( N = 0 \) and

\[
q_1 = q_1^0 - \frac{\phi_1}{S} (F_K - \lambda) - 1 > 0. \quad \ldots (83)
\]

Since

\[
q_1 = \frac{\phi_1}{S} - \phi_6 \quad \ldots (84)
\]

\[
\frac{d\phi_1}{dt} \neq 0 \text{ at some time after } N = 0.
\]

Thus (80) cannot continue to hold. Thus the firm cannot continue to follow Policy E when \( N = 0 \).

Policy E cannot be a terminal policy when \( N > 0 \).

If Policy E is a stationary policy then

\[
U_1 = \Pi - zS = 0 \quad \ldots (85)
\]

and \( B = 0 \). Now

\[
q_1 (F_K - \lambda - r) + 1 = 0 \quad \ldots (86)
\]

must continue to hold and so

\[
\frac{d}{dt} [q_1 (F_K - \lambda - r) + 1] = \dot{q}_1 (F_K - \lambda - r) < 0 \quad \ldots (87)
\]

and so Policy E cannot be a terminal policy.
Since

\[ U_1 < \Pi - zS \]  

\[ \phi_1 = 0 \] and from

\[ \frac{\phi_1}{s} (F_K - \lambda - r) + 1 = \phi_4 - \phi_3 \]  

\[ \phi_4 > 0 \] and so \( B = k(N+S) \). Since

\[ 0 < B < k(N+S) \]  

in Policy F, this policy is never optimal.

Now the conditions under which any one policy may switch into any other are given in Table B. The explanation of the switches is given in Appendix B to this chapter.

4. Optimal Policies

The problem being examined specifies \( N(0) = N_0 \) and so the value of \( q_1(0) \) is free and must be determined as part of the solution. If the planning horizon if finite then the necessary conditions for optimality include a transversality condition specifying \( q_1(T) \). This will sometimes enable the specification of \( q_1(0) \). In this problem the value of \( q_1(0) \) is crucial because \( q_1(t) \) and \( q_1(t) \) enter explicitly as part of the switching surface. The way to handle this is to choose \( q_1(0) \) experimentally, follow the appropriate policies and examine

\[ \int_0^\infty K e^{-p t} dt \]  

\[ \cdots \]
<table>
<thead>
<tr>
<th>Switches From</th>
<th>A</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>$\frac{-\varphi_1(F_{K-1-r})}{q_1 F_{KK}} &lt; 0^2$ $q_1(F_{K-1-r}) + 1 = 0^4$ $N &gt; 0$</td>
</tr>
<tr>
<td>C</td>
<td>$\Pi - zS &lt; 0^2$ $\frac{\varphi_1}{S}(F_{K-1-r}) + 1 \leq 0$ $N = 0$</td>
<td>-</td>
<td>0</td>
<td>$\frac{\varphi_1}{S}(F_{K-1-r}) + 1 = 0^4$ $N = 0$ $\Pi - zS &gt; -\frac{\varphi_1(F_{K-1-r})}{q_1 F_{KK}(1+k)}$ $q_1(F_{K-1-r}) + 1 = 0^4$ $N &gt; 0$</td>
</tr>
<tr>
<td>D</td>
<td>$U_1 = -M^1$ $\Pi - zS \geq 0^4$ $N = 0^4$</td>
<td>$U_1 = -M^1$ $\Pi - zS \geq 0^4$ $N = 0^4$</td>
<td>$U_1 = -M^1$ $\Pi - zS \geq 0^4$ $N = 0^4$</td>
<td>$U_1 = -M^1$ $\Pi - zS \geq 0^4$ $N = 0^4$</td>
</tr>
<tr>
<td>E</td>
<td>$\frac{\varphi_1}{S}(F_{K-1-r}) + 1 \leq 0$ $\Pi - zS \geq 0^4$ $N = 0$</td>
<td>$\Pi - zS &gt; -\frac{\varphi_1(F_{K-1-r})}{q_1 F_{KK}(1+k)}$ $q_1(F_{K-1-r}) + 1 \geq 0^4$ $N &gt; 0$</td>
<td>0</td>
<td>-</td>
</tr>
</tbody>
</table>

1. The policy which must hold before the switch.
2. The condition which must hold before the switch.
3. The policy which must hold after the switch.
4. The condition which must hold after the switch.
The optimal value of \( q_1(0) \), and policies, are those which maximize (91).

In this section various combinations of policies which meet the necessary conditions for optimality will be shown in the \( N, B \) plane. As a first step, the shape of the loci of \( N, B \) which meet

\[
\Pi - zS = 0 \quad \ldots (92)
\]

and

\[
q_1 (F_K - \lambda - r) + 1 = 0 \quad \ldots (93)
\]

will be established. Observe Diagram III.

If

\[
\Pi - zS = 0 \quad \ldots (94)
\]

while \( N \) and \( B \) are varying then

\[
d(\Pi-zS) = \frac{\partial (\Pi-zS)}{\partial N} dN + \frac{\partial (\Pi-zS)}{\partial B} dB = 0 \quad \ldots (95)
\]

and so

\[
\frac{dB}{dN} = \frac{-(F_K - \lambda)}{F_K - \lambda - r} \quad \ldots (96)
\]

Thus the loci has a shape dependent on the value of the numerator and denominator.

If

\[
q_1 (F_K - \lambda - r) + 1 = 0 \quad \ldots (97)
\]

while \( N \) and \( B \) are varying then

\[
\frac{d}{dt} [q_1 (F_K - \lambda - r) + 1] = q_1 (F_K - \lambda - r) + q_1 F_K K = 0 \quad \ldots (98)
\]
Thus

\[ \dot{K} = -q_1 \left( \frac{F_K}{q_1 F_{KK}} - \lambda - r \right) < 0 \]

under Policy E. Thus

\[ \ddot{K} = \dot{N} + B < 0 \]

and so

\[ \frac{d\delta}{dN} < -1 \]

when (97) holds. The position of these two lines in relation to each other cannot be known without knowledge of \( z \) and \( q_1(0) \).

Now, it was found in the previous section that only Policies A, C or D may be terminal stationary policies. It will now be shown (as a simple application of the overtaking principle) that Policy A is always a superior terminal policy to the others. When the system is stationary the only constraint which involves the absolute size of \( K(t) \) is the constraint
\[ \pi - zS > 0 \] ...

If \( B^*(t) \) is chosen profit becomes a function of capital and such a relation is shown as \( \pi(K) \) in Diagram IV. Now if Policy D is chosen as the terminal policy \( B = k(N+S) \) and

\[ \pi - zS < 0 \] ...

so that the terminal capital stock is less than \( K_1 \). If Policy C is chosen as the terminal policy then

\[ \pi - zS = 0 \] ...

so that the terminal capital stock is \( K_1 \). However, if the firm switches out of either of these policies into others which lead to Policy A, then \( B \) is reduced to zero so that the \( \pi(K) \) function rises and the long run capital stock becomes greater than \( K_1 \). Thus it is always optimal to choose Policy A as the terminal policy.

Now

\[ \pi - zS > 0 \] ...

and \( N > 0 \) at \( t = 0 \) then Diagram V shows a sequence of policies which might be optimal. Of course it may be optimal to use Policy E or Policy A as an initial policy. It is not optimal to follow Policy \( D_j \) as an initial policy because it involves a reduction in \( K(0) \) without any offsetting increase in \( K(t) \) at \( t > 0 \). See Diagram VI.

Other combinations of policies which may be optimal can be constructed from the table but explicit solutions cannot, of course, be obtained without knowledge of the operating profit function and other data. However, regardless
of the combinations of policies which may be optimal, it is always optimal to end with Policy A. That is, the longrun stationary position of the firm is where debt is zero and the dividend constraint holds.
NECESSARY CONDITIONS FOR OPTIMALITY

The problem to be solved in Chapter VII is a non-autonomous fixed-time free-endpoint problem in optimal control theory and M.R. Hestenes' formulation of Pontryagin's maximum principle will be used.

The solution to the problem involves finding the piecewise continuous r-dimensional control vector $U(t) = (U_1(t), U_2(t), \ldots, U_r(t))$ and the continuous n-dimensional state vector $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ which maximize

$$ I = \int_{t_0}^{\infty} f^0(x(t), U(t), t) \, dt \quad \ldots (1) $$

subject to $\dot{x}_i(t) = f_i(x(t), U(t)), \ i = 1, 2, \ldots, n \quad \ldots (2)$

$$ \psi_{\alpha}(x(t), U(t)) \geq 0, \ \alpha = 1, 2, \ldots, m \quad \ldots (3) $$

$$ x_\beta(t) \geq 0, \quad \beta = 1, 2, \ldots, 1 \quad \ldots (4) $$

$$ x_i(0) = x_{i0}, \quad i = 1, 2, \ldots, n \quad \ldots (5) $$

given $t_0$ fixed. Further assume that

$$ \frac{\partial \psi_{\alpha}(x(t), U(t))}{\partial U(t)}, \ \alpha = 1, 2, \ldots, m \quad \ldots (6) $$

---

has rank \( m' \) if the constraints \( \alpha = 1, 2, \ldots, m' \) hold with equality at some \( t \in [t_0, \infty] \).

The necessary conditions for a weak extremum are contained in the following theorem:

**Theorem**

If \( x^*(t) \) and \( U^*(t) \) maximize (1) subject to (2) to (5) then there exist multipliers

\[
p_0 > 0
\]

\[
p_i(t), \quad i = 1, 2, \ldots, n
\]

\[
\mu_\alpha(t), \quad \alpha = 1, 2, \ldots, m
\]

\[
\phi_\beta(t), \quad \beta = 1, 2, \ldots, l
\]

not vanishing simultaneously on \( t \in [t_0, \infty] \) and the function

\[
L = p_0 f^0 + p_i f_i + < \mu_\alpha, \psi_\alpha > + < \phi_\beta, \beta >
\]

such that

(a) \( \mu_\alpha(t) \geq 0, \quad \mu_\alpha \psi_\alpha = 0, \quad \alpha = 1, 2, \ldots, m \)

(b) \( \phi_\beta(t) \geq 0, \quad \phi_\beta x_\beta = 0, \quad \phi_\beta f_\beta = 0, \quad \beta = 1, 2, \ldots, l \)

(b) \( p_i(t) \) are continuous on \( t \in [t_0, \infty] \) and have the piecewise continuous derivatives

\[
\dot{p}_i(t) = - \frac{\partial L}{\partial x_i}, \quad i = 1, 2, \ldots, n
\]

(c) \( U^*(t) \) is given by

\[
\frac{\partial L}{\partial U} = 0.
\]
Appendix B

POLICY SWITCHES

It has already been shown in the body of the chapter that Policies B and F are never optimal and so switches into or out of these policies are of no relevance. Also, Policy D involving a jump in N can only be an initial policy so that switches into this policy cannot occur.

Switches of Policy A

(i) If the firm follows Policy A then before it can switch into a policy involving \( B = k(N+S) \) it must follow a policy involving

\[
0 < B < k(N+S) \quad \cdots (1)
\]

Suppose that \( N > 0 \) and

\[
q_1(F_K - \lambda - r) + 1 = 0 \quad \cdots (2)
\]

while \( B = 0 \). If \( B \) jumps to \( B = k(N+S) \) then

\[
q_1(F_K - \lambda - r) + 1 < 0 \quad \cdots (3)
\]

must hold because of the time continuity of \( q_1 \). Thus it is not optimal for the firm to switch into Policy C or D.

(ii) A switch into Policy E when \( N > 0 \) requires

\[
q_1(F_K - \lambda - r) + 1 = 0 \quad \cdots (4)
\]

Under Policy A

\[
q_1(F_K - \lambda - r) + 1 \neq 0 \quad \cdots (5)
\]
Thus a switch requires

\[
\frac{1}{K} \leq \frac{-q_1(F - \lambda - r)}{q_1 F_{KK}} < 0 \quad \ldots (7)
\]

or

\[
\frac{-q_1(F - \lambda - r)}{\Pi - zS} \geq \frac{1}{q_1 F_{KK}} < 0 \quad \ldots (8)
\]

(iii) If Policy A is followed and \( N = 0 \) is reached then when \( N = 0 \)

\[
\Pi - zS \leq 0. \quad \ldots (9)
\]

If the firm switches into any other policy then

\[
U_1 < \Pi - zS \quad \ldots (10)
\]

or \( B \neq 0 \) will hold and \( N < 0 \) will occur. Since this is not possible when \( N = 0 \) a switch into Policy C, D, or E is not optimal.

**Switches of Policy C**

(i) Under Policy C and \( N > 0 \)

\[
q_1(F - \lambda - r) + 1 \geq 0 \quad \ldots (11)
\]

and \( B = k(N+S) \). By the reasoning provided above the continuity of \( q_1 \) and \( K \) requires the firm to follow a policy involving

\[
0 < B < k(N+S) \quad \ldots (12)
\]
before it can follow a policy involving $B = 0$. Thus it is not optimal to switch into Policy A.

(ii) If $N > 0$ then a switch into Policy E requires that

$$q_1 (F_K - \lambda - r) + 1 = 0 \quad \ldots (13)$$

Thus under Policy C

$$\frac{d}{dt} [q_1 (F_K - \lambda - r) + 1] = q_1 (F_K - \lambda - r) + q_1 F_{KK} K \leq 0 \quad \ldots (14)$$

must hold. Since

$$\dot{K} = U_1 (1+k) \quad \ldots (15)$$

$$\dot{K} = (\Pi - zS)(1+k) \quad \ldots (16)$$

this implies

$$\Pi - zS \geq \frac{-q_1 (F_K - \lambda - r)}{q_1 F_{KK} (1+k)} < 0 \quad \ldots (17)$$

before the switch.

(iii) If $N = 0$ then

$$\Pi - zS < 0 \quad \ldots (18)$$

must hold while $N > 0$ and

$$\Pi - zS \leq 0 \quad \ldots (19)$$

when $N = 0$. If the firm switches into Policy D then $\Pi$ will fall and

$$U_1 < \Pi - zS < 0 \quad \ldots (20)$$
cannot hold while $N = 0$. Thus it is not optimal to switch into Policy D.

**Switches of Policy D**

(i) Under Policy D

$$q_1 = -\phi_6$$

...(21)

and $q_1 < 0$ and so the firm cannot switch into a policy involving $N > 0$.

Also

$$l = \phi_4 - \phi_3$$

...(22)

and so $\phi_4 > 0$, $\phi_3 = 0$ and $B = k(N+S)$. A switch into Policy A, C, or E will result in $N > 0$ and so such switches are not optimal.

(ii) If Policy D involves a jump in $N$ then it can switch into any of the policies providing $N = 0$ and

$$\Pi - z_S \geq 0$$

...(23)

after the switch. Strict inequality must hold if the switch is into Policy D.

**Switches of Policy E**

(i) If $N > 0$ then a switch into Policy A requires a change from

$$q_1(F_K - \lambda - r) + 1 = 0$$

...(24)
to

\[ q_1(F_K - \lambda - r) + 1 \leq 0. \]  

...(25)

Thus

\[ q_1(F_K - \lambda - r) + q_1 F_{KK} K \leq 0 \]  

...(26)

must hold under Policy E

\[ K = \frac{-q_1(F_K - \lambda - r)}{q_1 F_{KK}} \]  

...(27)

Now

\[ K = U_1 + B \]  

...(28)

and in the time neighbourhood of the switch \( B < 0 \) because \( B \)
changes from

\[ 0 < B < k(N+S) \]  

...(29)

to \( B = 0 \). Thus

\[ K < U_1 \]  

...(30)

and

\[ U_1 = \pi - zS > -\frac{q_1(F_K - \lambda - r)}{q_1 F_{KK}} < 0 \]  

...(31)

must hold before the switch.

(ii) If \( B > 0 \) then a switch into Policy C requires

\[ q_1(F_K - \lambda - r) + 1 \geq 0 \]  

...(32)

and so

\[ q_1(F_K - \lambda - r) + q_1 F_{KK} K \geq 0 \]  

...(33)
Now

\[ K < -q_1(F_K - \lambda - r) \]
\[ q_1 F_{KK} \]

(34)

and so

\[ K = U_1 + B \]

(35)

and since B must switch from

\[ 0 < B < k(N+S) \]

(36)

to \( B = k(N+S), B < 0 \) before the switch. However at the

switch B, when defined from above, in time, is

\[ \dot{B} = kU_1 \]

(37)

and a switch into Policy C requires

\[ \Pi - zS < -q_1(F_K - \lambda - r) \]
\[ q_1 F_{KK} (1+k) \]

(39)

immediately after the switch.

(iii) If \( N = 0 \) under Policy D then

\[ U_1 = \Pi - zS < 0 \]

(40)

must have held immediately before \( N = 0 \). Now if

\[ \Pi - zS < 0 \]

(41)

then the firm cannot switch into Policy C because \( \Pi \) falls as

B is increased and
\[ U_1 = \Pi - zS < 0 \] 

when \( N = 0 \) cannot be maintained. Thus a switch into Policy C when \( N = 0 \) is not optimal.

(vi) When \( N = 0 \)

\[ q_1 = \frac{\phi_1}{S} - \phi_6 \] 

with \( \phi_6 \) undetermined. Thus \( q_1 \) can jump upwards and

\[ \frac{\phi_1}{S} (F_K - \lambda - r) + 1 < 0 \] 

can occur. Thus the firm may switch into Policy A providing \( B \) is reduced so that

\[ \Pi - zS \geq 0 \] 

(v) It is not optimal for the firm to switch into Policy D because increasing \( B \) will reduce \( \Pi \) and cause

\[ \Pi - zS < 0. \]
Chapter VIII

A GROWTH MAXIMIZATION MODEL
OF THE FIRM

1. Introduction

In the last chapter it was assumed that the firm did not engage in share issue policies. Now a growth maximization model will be developed in which share issues can be undertaken. The constraints and cost of capital assumptions of the previous chapter will be retained and the specification of the problem differs only by the inclusion of the differential equation

\[ S(t) = U_2(t) \quad \ldots \quad (1) \]

and the constraint

\[ U_2(t) \geq 0 \quad \ldots \quad (2) \]

The set of functions \( S(t) \) is, of course, restricted to those which are continuous on the time domain.

2. The Problem

The problem is to find the optimal control vector \((U_1(t), U_2(t), B(t), x(t))\) and the trajectory of the state vector \((N(t), S(t))\) given \( t \in [0, \infty] \) and \( N(\infty), S(\infty) \) free, which maximize
subject to:

\[
\begin{align*}
\dot{N}(t) &= U_1(t) \\
\dot{S}(t) &= U_2(t)
\end{align*}
\]

and

\[
\frac{\Pi(t) - U_1(t)}{S(t)} \geq z
\]

**(...6)**

\[
0 \leq B(t) \leq k[N(t) + S(t)]
\]

**(...7)**

\[
N(t), U_2(t), x(t) \geq 0
\]

**(...8)**

\[
N(0) = N_0, S(0) = S_0.
\]

**(9)**

This is a non-autonomous fixed-time, free-endpoint problem and the necessary conditions for a weak extremum are given in Appendix A to Chapter VII. The interpretation of the shadow price of \(N(t)\) is the same as that given in the previous chapter, and the shadow price of \(S(t)\) can be interpreted in the same way, only \(S\) and not \(N\) is being varied. The shadow price of \(S(t)\) is defined to be

\[
p_2(t) = q_2(t) e^{-\rho t}.
\]

**(10)**

Now, the Lagrangian terms are defined as

\[
z_1(t) = \phi_1(t) e^{-\rho t}.
\]

**(11)**
and

\[ \phi_1 \geq 0, \quad \phi_1 \left( \frac{\Pi - U_1}{S} - z \right) = 0 \]  \quad \ldots 12(a)

\[ \phi_2 \geq 0, \quad \phi_2 U_2 = 0 \]  \quad (b)

\[ \phi_3 \geq 0, \quad \phi_3 B = 0 \]  \quad (c)

\[ \phi_4 \geq 0, \quad \phi_4 [k(N+S) - B] = 0 \]  \quad (d)

\[ \phi_5 \geq 0, \quad \phi_5 x = 0 \]  \quad (e)

\[ \phi_6 \geq 0, \quad \phi_6 N = 0, \quad \phi_6 U_1 = 0 \]  \quad (f)

and the Lagrangian is

\[
L = e^{-\phi t} \{ N + S + B + q_1 U_1 + q_2 U_2 + \phi_1 \left( \frac{\Pi - U_1}{S} - z \right)
+ \phi_2 U_2 + \phi_3 B + \phi_4 [k(N+S) - B] + \phi_5 x + \phi_6 U_1 \}. \quad \ldots 13
\]

The necessary conditions for an optimum policy are given by the constraints on the controls and state variables, the optimal conditions relating to the controls

\[
\frac{\partial L}{\partial U_1} = 0 = q_1 - \phi_1 \left( \frac{1}{S} \right) + \phi_6 \]  \quad \ldots 14(a)

\[
\frac{\partial L}{\partial U_2} = 0 = q_2 + \phi_2 \]  \quad (b)

\[
\frac{\partial L}{\partial B} = 0 = 1 + \frac{\phi_1}{S} (F_K - \lambda - r) + \phi_3 - \phi_4 \]  \quad (c)

\[
\frac{\partial L}{\partial x} = 0 = \frac{\phi_1}{S} F + \phi_5 \]  \quad (d)
the motion of the state variables

\[ N = U_1 \quad \ldots 15(a) \]

\[ N(0) = N_0 \quad (b) \]

and

\[ S = U_2 \quad \ldots 16(a) \]

\[ S(0) = S_0 \quad (b) \]

and the motion of the auxiliary variables.

\[ \dot{q}_1 = q_1^\rho - \frac{\phi_1}{S} (F_K - \lambda) - \phi_4 k - 1 \quad \ldots (17) \]

\[ \dot{q}_2 = q_2^\rho - \frac{\phi_1}{S} [(F_K - \lambda) - \frac{\Pi - U_1}{S}] - \phi_4 k - 1 \quad \ldots (18) \]

with \( q_1(0) \) and \( q_2(0) \) free.

Since

\[ F_x = \frac{-\phi_5 S}{\phi_1} \quad \ldots (19) \]

from (14)(d), the optimum output rule is the same as that in Chapter VII. That is, if \( \phi_1 > 0 \), then

\[ F_x = 0 \quad \ldots (20) \]

if \( x^* > 0 \), and

\[ F_x < 0 \quad \ldots (21) \]

if \( x^* = 0 \).

The condition concerning optimal debt is also the same as in that chapter. If \( \phi_1 > 0 \) and \( B^* = k(N+S) \) then \( \phi_4 > 0, \phi_3 = 0 \) and
If $B^* \in (0, k(N+S))$ then $\phi_4 = \phi_3 = 0$

and

\[
\frac{1}{S}(F_K - \lambda - r) + 1 > 0
\]  

...(22)

If $B^* = 0$ then $\phi_3 > 0$, $\phi_4 = 0$ and

\[
\frac{1}{S}(F_K - \lambda - r) + 1 = 0
\]  

...(23)

Now, if $\phi_1 = 0$ then

\[1 = \phi_4 - \phi_3
\]  

...(25)

and so $B^* = k(N+S)$.

In the last chapter it was shown the Proposition 7 holds. That is, if $N > 0$ then either

\[
\frac{\Pi - U_1}{S} = z
\]  

...(26)

or $N$ jumps to $N = 0$. Also, if $U_1 = -\infty$ then $B^* = k(N+S)$.

This Proposition remains true in this chapter and the proof is the same.

3. Policy Alternatives

As with all of the previous models output enters only incidentally in the problem and the policy alternatives involving output will be ignored. The remaining policies are twelve and they are shown in Table A.
The conditions under which each of these policies may hold are considered in turn. By (25) above, Policies B, F, H or L are never optimal.

A. \( U_1 = \pi - zS, U_2 = 0, B = 0 \)

Clearly

\[
q_1 = \phi \frac{1}{S} - \phi_6
\]

\( \ldots (28) \)

\[
q_2 = -\phi_2 < 0
\]

\( \ldots (29) \)

\[
\phi_4 = 0
\]

\( \ldots (30) \)
Now if \( N > 0 \) then \( \phi_6 = 0 \) and \( \phi_1 > 0 \) by the proof relating to Proposition 7. Further

\[
q_1 = q_1^\rho - \frac{\phi_1}{S} (F_K - \lambda) - 1 \quad \ldots (32)
\]

\[
q_2 = q_2^\rho - \frac{\phi_1}{S} (F_K - \lambda - z) - 1 \quad \ldots (33)
\]

Policy A can be a stationary terminal policy when \( N > 0 \). If it is then \( N = B = S = 0 \). This requires

\[
\Pi - zS = 0 \quad \ldots (36)
\]

and

\[
q_1(F_K - \lambda - r) + 1 \leq 0 \quad \ldots (37)
\]

to remain. Thus

\[
\frac{d}{dt} [q_1(F_K - \lambda - r) + 1] = \dot{q}_1(F_K - \lambda - r) \leq 0 \quad \ldots (38)
\]

and this holds. The condition \( q_2 \leq 0 \) must remain, and so

\[
q_2 = q_2^\rho + q_1^\rho (z + \lambda - F_K) - 1 \leq 0 \quad \ldots (39)
\]

or

\[
F_K - \lambda - z \geq \frac{q_2^\rho - 1}{q_1^\rho} < 0 \quad \ldots (40)
\]
is sufficient to ensure this. Condition (40) is consistent with the circumstances of this policy.

Policy A can be a terminal stationary policy if

\[ N = 0 \text{ providing } \]

\[ U_1 = \Pi - zS = 0 \quad \ldots(41) \]

and

\[ \frac{\phi_1}{S}(F_K - \lambda - r) + 1 \leq 0 \quad \ldots(42) \]

\[ q_2 = q_2 \phi - \frac{1}{S}(F_K - \lambda - z) - 1 \leq 0 \quad \ldots(43) \]

hold simultaneously. Now

\[ \frac{d}{dt}\left[ \frac{\phi_1}{S}(F_K - \lambda - r) + 1 \right] = \left( \frac{\phi_1}{S} \right)(F_K - \lambda - r) \leq 0 \quad \ldots(44) \]

must hold. Since

\[ F_K - \lambda - r < 0 \quad \ldots(45) \]

\[ \phi_1 = 0 \text{ must hold. From } \]

\[ q_1 = \frac{\phi_1}{S} - \phi_6 \quad \ldots(46) \]

(42) may hold indefinitely. If

\[ q_2 \phi = \frac{\phi}{S}(F_K - \lambda - z) + 1 < 0 \quad \ldots(47) \]

holds then \( q_2 \leq 0 \) will continue to hold. Thus Policy A may be a terminal stationary policy when \( N = 0 \).
C. \( U_1 = \Pi - zS, \ U_2 = 0, \ B = k(N+S) \)

Now

\[
q_1 = \frac{\phi_1}{S} - \phi_6 \quad \ldots (48)
\]

\[
q_2 = -\phi_2 \leq 0 \quad \ldots (49)
\]

\[
\phi_3 = 0 \quad \ldots (50)
\]

\[
\frac{\phi_1}{S}(F_K - \lambda - r) + 1 \geq 0 \quad \ldots (51)
\]

\[
The_1 = q_1^0 \quad \ldots (52)
\]

\[
The_2 = q_2^0 \quad \ldots (53)
\]

If \( N > 0 \) then \( \phi_1 > 0 \) and

\[
q_1 = -q_1[(F_K - \lambda - \rho) + k(F_K - \lambda - r)] - k - 1 \quad \ldots (54)
\]

\[
q_2 = q_2^0 - q_1[(F_K - \lambda - z) + k(F_K - \lambda - r)] - k - 1 \quad \ldots (55)
\]

Policy C may be a terminal stationary policy when

\( N > 0 \). As with the previous case, this requires

\[
\Pi - zS = 0 \quad \ldots (56)
\]

and

\[
q_1(F_K - \lambda - r) + 1 \geq 0 \quad \ldots (57)
\]

must remain. Thus

\[
q_1(F_K - \lambda - r) \geq 0 \quad \ldots (58)
\]
and
\[ \dot{q}_2 = q_2^0 - q_1[(F_K - \lambda - z) + k(F_K - \lambda - r)] - k - 1 \leq 0 \] \quad \ldots (59)
must hold. Since \( \rho > z \) (59) does not restrict \( \dot{q}_1 \). If
\[ F_K - \lambda - r > 0 \] \quad \ldots (60)
then \( q_1 \geq 0 \) must hold if (57) is to remain. If
\[ F_K - \lambda - r < 0 \] \quad \ldots (61)
then \( q_1 \leq 0 \) must hold. If
\[ F_K - \lambda - r = 0 \] \quad \ldots (62)
then \( q_1 \) is unrestricted. Thus conditions (56), (57), and (59) may all hold. Thus Policy C may be a terminal stationary policy if \( N > 0 \).

Policy C may also be a stationary terminal policy when \( N = 0 \).

Now this requires
\[ \Pi - zS = 0 \] \quad \ldots (63)
\[ \frac{1}{S}(F_K - \lambda - r) + 1 \geq 0 \] \quad \ldots (64)
and
\[ \dot{q}_2 = q_2^0 - \frac{1}{S}[(F_K - \lambda - z) + k(F_K - \lambda - r)] - k - 1 \leq 0 \] \quad \ldots (65)
when \( q_2 = 0 \), holding simultaneously.
Now
\[ q_1 = \frac{\phi_1}{S} - \phi_6 \] \hfill \ldots (66)

and \( \phi_6 \geq 0 \). For any given \( q_1 \), \( \phi_1 \) increases as \( \phi_6 \) increases.

If \( (\phi_1) \) exists, condition (64) requires
\[ \frac{\phi_1}{S}(F_K - \lambda - r) \geq 0 \] \hfill \ldots (67)

Since \( S \) is a constant under this policy, if
\[ F_K - \lambda - r > 0 \] \hfill \ldots (68)

then \( \phi_1 \geq 0 \). This may hold regardless of \( q_1 \). If, however,
\[ F_K - \lambda - r < 0 \] \hfill \ldots (69)

then \( \phi_1 \leq 0 \) and this requires \( q_1 \leq 0 \). If
\[ F_K - \lambda - r = 0 \] \hfill \ldots (70)

the \( q_1 \) is unrestricted. These conditions are consistent with (63), (64) and (65). Thus Policy C may be a terminal stationary policy when \( N = 0 \).

D. \( U_1 < R - zS \), \( U_2 = 0 \), \( B = k(N+S) \)

Now
\[ q_1 = -\phi_6 \leq 0 \] \hfill \ldots (71)
\[ q_2 = -\phi_2 \leq 0 \] \hfill \ldots (72)
\[ \phi_4 = 1 \] \hfill \ldots (73)
As already mentioned, if this policy holds when $N > 0$ then $N$ must jump to $N = 0$. If it holds when $N = 0$ then it may be a terminal stationary policy.

E. $U_1 = \Pi - zS$, $U_2 = 0$, $B \in (0, k(N+S))$. 

Now

\[ q_1 = \frac{\phi_1}{S} - \phi_6 \]  
\[ q_2 = - \phi_2 \leq 0 \]
\[ \phi_4 = \phi_3 = 0 \]
\[ \frac{\phi_1}{S}(F_K - \lambda - r) + 1 = 0 \]
\[ \hat{q}_1 = q_1 \rho - \frac{\phi_1}{S}(F_K - \lambda) - 1 \]
\[ \hat{q}_2 = q_2 \rho - \frac{\phi_1}{S}(F_K - \lambda - z) - 1 \]

If $N > 0$ then $\phi_1 > 0$ and

\[ \hat{q}_1 = q_1 (\rho + \lambda - F_K) - 1 > 0 \]
\[ \hat{q}_2 = q_2 \rho + q_1 (z + \lambda - F_K) - 1 \]

Policy E cannot be a terminal stationary policy when $N > 0$. 

One of the conditions required for this is

\[ q_1 (F_K - \lambda - r) + 1 = 0 \]  \hspace{1cm} \ldots (84)

or

\[ \frac{d}{dt} [q_1 (F_K - \lambda - r) + 1] = \dot{q}_1 (F_K - \lambda - r) = 0. \] \hspace{1cm} \ldots (85)

However, since \( q_1 > 0 \) and

\[ F_K - \lambda - r < 0 \] \hspace{1cm} \ldots (86)

condition (85) does not hold. Thus Policy E cannot be a terminal stationary policy if \( N > 0 \).

Similarly, Policy E cannot be a terminal stationary policy when \( N = 0 \). One of the conditions required for Policy E is

\[ \frac{\dot{\phi}_1}{S} (F_K - \lambda - r) + 1 = 0 \] \hspace{1cm} \ldots (87)

Now

\[ F_K - \lambda - r = \text{constant} \] \hspace{1cm} \ldots (88)

because \( K \) is given. Thus \( \phi_1 \) must be a constant if Policy E is to remain.

Now

\[ \left( \frac{\dot{\phi}_1}{S} \right) = \dot{q}_1 + (\dot{\phi}_6) \] \hspace{1cm} \ldots (89)

and

\[ \dot{q}_1 = q_1 \rho - \frac{\phi_1}{S} (F_K - \lambda) - 1 > 0 \] \hspace{1cm} \ldots (90)
at the instant $N = 0$. Thus $\phi_1 > 0$ and condition (87) is violated. Thus Policy E cannot be a terminal stationary policy when $N = 0$.

G. $U_1 = \Pi - zS$, $U_2 > 0$, $B = 0$

Now

$$q_1 = \frac{\phi_1}{s} - \phi_6 \quad \ldots(91)$$

$$q_2 = 0 \quad \ldots(92)$$

$$\phi_4 = 0 \quad \ldots(93)$$

$$\frac{\phi_1}{s} (F_K - \lambda - z) + 1 \leq 0 \quad \ldots(94)$$

$$q_1 = q_1 \rho - \frac{\phi_1}{s} (F_K - \lambda) - 1 \quad \ldots(95)$$

$$q_2 = -\frac{\phi_1}{s} (F_K - \lambda - z) - 1 > 0. \quad \ldots(96)$$

If $N > 0$ then $\phi_1 > 0$ and

$$q_1 = q_1 (\rho + \lambda - F_K) - 1 > 0 \quad \ldots(97)$$

$$q_2 = q_1 (z + \lambda - F_K) - 1 > 0 \quad \ldots(98)$$

This policy can only last for an instant, otherwise the condition $q_2 \leq 0$ will be violated. Thus if this policy is significant it involves a jump in $S$ and can only occur at $t = 0$. 
I. \( U_1 = \pi - zS, \ U_2 > 0, \ B = k(N+S) \).

Now

\[
q_1 = \frac{\phi_1}{S} - \phi_6 \qquad \ldots (99)
\]

\[
q_2 = 0 \qquad \ldots (100)
\]

\[
\phi_3 = 0 \qquad \ldots (101)
\]

\[
\frac{\phi_1}{S}(F_K - \lambda - r) + 1 \geq 0 \qquad \ldots (102)
\]

\[
q_1 = q_1^0 - \frac{\phi_1}{S}[(F_K - \lambda) + k(F_K - \lambda - r)] - k - 1 \qquad \ldots (103)
\]

\[
q_2 = - \frac{\phi_1}{S}[(F_K - \lambda - z) + k(F_K - \lambda - r)] - k - 1 \qquad \ldots (104)
\]

If \( N > 0 \) then \( \phi_1 > 0 \) and

\[
q_1 = - q_1[(F_K - \lambda - z) + k(F_K - \lambda - r)] - k - 1 \qquad \ldots (105)
\]

\[
q_2 = - q_1[(F_K - \lambda - z) + k(F_K - \lambda - r)] - k - 1 \qquad \ldots (106)
\]

This policy need not be a jump policy if \( q_2 = 0 \) for more than an instant. Thus

\[
q_2 = - q_1[(F_K - \lambda - z) + k(F_K - \lambda - r)] - k - 1 = 0 \qquad \ldots (107)
\]

and \( q_1 > 0 \) because \( \phi > z \). Now \( q_2 = 0 \) implies

\[
(F_K - \lambda)(1 + k) - z - rk = - \frac{k + 1}{q_1} < 0 \qquad \ldots (108)
\]

Since \( q_1 > 0 \), \( F_K \) must be rising and so \( K < 0 \) must hold. Thus a non-jump form of Policy I can only hold if \( K < 0 \) and since
$U_2 > 0,$

$\Pi - zS < 0 \quad \ldots (109)$

must hold. This type of policy cannot be a terminal stationary policy because (109) must hold and $K < 0.$

If $N = 0$ then (109) holds and this is non-optimal.

Alternatively, Policy I may be a jump policy and in this case it can only be an initial policy.

J. $U_1 < \Pi - zS, U_2 > 0, B = k(N+S).$

Now

$q_1 = -\phi_6 \leq 0 \quad \ldots (110)$

$q_2 = 0 \quad \ldots (111)$

$\phi_4 = 1 \quad \ldots (112)$

$q_1 = q_1 \rho - 1 < 0 \quad \ldots (113)$

$q_2 = q_2 \rho - 1 < 0 \quad \ldots (114)$

Thus if this is a significant policy it must involve a jump in $N$ to $N = 0$ at $t = 0.$

K. $U_1 = \Pi - zS, U_2 > 0, B \in (0, k(N+S))$

Finally,

$q_1 = \frac{\phi_1}{B} - \phi_6 \quad \ldots (115)$
<table>
<thead>
<tr>
<th>Switches</th>
<th>A</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>G</th>
<th>I</th>
<th>J</th>
<th>K</th>
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<tbody>
<tr>
<td>From</td>
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</tr>
<tr>
<td>C</td>
<td>$\Pi - z \leq 0^2$</td>
<td>$\frac{q_1}{B}(F_{K(-1)-r}) + 1 \leq 0$</td>
<td>$N = 0$</td>
<td>$-z \leq 0^2$</td>
<td>$\frac{q_1}{B}(F_{K(-1)-r}) + 1 \leq 0$</td>
<td>$N = 0$</td>
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</tr>
<tr>
<td>D</td>
<td>$U_1 = -1$</td>
<td>$E = z \leq 0^2$</td>
<td>$\frac{q_1}{B}(F_{K(-1)-r}) + 1 \leq 0$</td>
<td>$-z \leq 0^2$</td>
<td>$\frac{q_1}{B}(F_{K(-1)-r}) + 1 \leq 0$</td>
<td>$N = 0$</td>
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<tr>
<td>E</td>
<td>$N = 0$</td>
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</tbody>
</table>

1. The policy required for a switch to take place.
2. The condition which must hold immediately before the switch.
3. The policy required immediately after the switch.
4. The condition which must hold immediately after the switch.
\[ q_2 = 0 \] \hspace{1cm} \text{...(116)}

\[ \phi_4 = \phi_3 = 0 \] \hspace{1cm} \text{...(117)}

\[ \frac{\phi_1}{S} (F_K - \lambda - r) + 1 = 0 \] \hspace{1cm} \text{...(118)}

\[ q_1 = q_1^0 - \frac{\phi_1}{S} (F_K - \lambda) - 1 \] \hspace{1cm} \text{...(119)}

\[ q_2 = -\frac{\phi_1}{S} (F_K - \lambda - z) - 1 > 0 \] \hspace{1cm} \text{...(120)}

If \( N > 0 \) then

\[ q_1 = q_1 (\rho + \lambda - F_K) - 1 > 0 \] \hspace{1cm} \text{...(121)}

\[ q_2 = q_1 (z + \lambda - F_K) - 1 > 0. \] \hspace{1cm} \text{...(122)}

Thus this policy can last for only an instant and must involve a jump in \( S \) at \( t = 0 \) if it is to be a significant policy.

Table B shows the switching surface for the problem.

4. The Optimum Policy

The essential conflict in this model is shown in Diagram I. Suppose that it is decided to hold \( S(t) = S(0) \).

Once \( B^*(t) \) is chosen, the firm's profit becomes a function of its capital stock and this relationship is shown as \( \Pi(K) \).

The dividend constraint is binding in the long run when \( K = K_1 \).

If, however, a policy is followed which involves issuing shares
the constraint is raised and the ultimate size of the capital stock is smaller, assuming that the $\Pi(K)$ function is unchanged. Issuing shares, however, may allow this capital stock to be achieved sooner. This conflict is resolved, in principle, by the rate of discount $\rho$. The conflict in the choice of optimal debt policy still remains: increasing $B(t)$ increases $K(t)$, but it may reduce the rate of change of $K(t)$ by reducing profit.

In the previous section it was found that only Policies A, C, and D can be terminal stationary policies. Policy D involves $U_2 = 0, B = k(N+S), N = 0$ and

$$
\Pi - zS > 0
$$

...(123)

This is not an optimal terminal policy because $K$ can be increased by increasing $N$ or $S$ or $B$ until the dividend constraint becomes binding. Thus the terminal stationary policies may be Policy A or C. There is an important difference between these terminal policies since debt policy before reaching the terminal state. Suppose that the constraint is met while following Policy C. If, however, the firm follows Policy E and then Policy A, so that $B$ is replaced by $N$, the profit function, $\Pi(K)$, shifts upwards thus increasing the ultimate capital stock. Thus the optimal terminal stationary policy is Policy A.
Now, only policy chains which lead to Policy A may be optimal. Thus if $N(0) > 0$ and

$$\Pi - z S > 0 \quad \ldots (124)$$

at $t = 0$ then the following chain of policies may be optimal.

**DIAGRAM I**

**DIAGRAM II**
Initially following Policy D is not optimal because it reduces $K(0)$ without affecting the objective function in any other way. Of course, the firm may start at Policy C, E, or A itself. If $N = 0$ then the policy alternatives become more complicated but the analysis is unchanged. The long run stationary position is the same as that of the previous chapter, that is, debt is zero and the dividend constraint holds.

The optimal chain of policies, of course, maximises

$$\int_0^\infty K(t) e^{-ct} \, dt.$$ 

...(125)
Appendix A

POLICY SWITCHES

It was shown in the body of this chapter that Policies B, F, H and L are never optimal and so switches into and out of these policies are not relevant. Also Policies G, J and K must be jump policies to be significant and so they must be initial policies. Switches into these policies are not relevant and so they will not be discussed.

Switches of Policy A

(i) If \( N > 0 \) then a firm following Policy A must follow a policy involving

\[
0 < B < k(N+S)
\]  

before it can switch into a policy involving \( B = k(N+S) \).

This is required by the continuity of \( q_1(t) \) and \( K(t) \) under Policy A. If Policy A is followed so that

\[
q_1(F_K - \lambda - r) + 1 = 0
\]  

and \( B = 0 \) and the firm switches into a policy with \( B = K(N+S) \) then

\[
q_1(F_K - \lambda - r) + 1 < 0
\]  

follows and this implies \( \phi_3 > 0 \). Thus it is not optimal to switch into Policy C, D or I.
(ii) A switch into Policy E when \( N > 0 \) requires 
\[
q_1 (F_K - \lambda - r) + 1 = 0 \quad \cdots (4)
\]
and so 
\[
\frac{d}{dt} [q_1 (F_K - \lambda - r) + 1] = \frac{d}{dt} q_1 (F_K - \lambda - r) + q_1 F_K \frac{d}{dt} F_K \geq 0 \quad \cdots (5)
\]
must hold before the switch. Thus 
\[
\Pi - zS \leq - \frac{q_1 (F_K - \lambda - r)}{q_1 F_K} < 0 \quad \cdots (6)
\]
(iii) If \( N = 0 \) is reached while following Policy A, then 
\[
U_1 = \Pi - zS \leq 0 \quad \cdots (7)
\]
while \( N = 0 \). Now, a switch into any policy will result in \( N < 0 \) while \( N = 0 \) and this is non-optimal. This occurs because 
\[
U_1 < \Pi - zS \quad \cdots (8)
\]
occur, or \( \Pi \) is reduced by increasing \( B \) or \( zS \) is increased by setting \( U_2 > 0 \). Thus a switch into any policy is not optimal, when \( N = 0 \).

Switches of Policy C

(i) If \( N > 0 \) under Policy C then it is not optimal to switch into a policy involving \( B = 0 \). The proof is the same as that given above and is based on the continuity of
(i) If \( N > 0 \) then a switch into Policy E requires that

\[
q_1(F_K - \lambda - r) + 1 = 0 \quad \ldots \quad (9)
\]

so that

\[
\frac{d}{dt}[q_1(F_K - \lambda - r) + 1] = q_1(F_K - \lambda - r) + q_1F_{KK} K < 0 \quad \ldots \quad (10)
\]

or

\[
q_1(F_K - \lambda - r) \frac{q_1}{q_1F_{KK}} \quad \ldots \quad (11)
\]

must hold before the switch. Since

\[
K = \rho_1(1 + k) \quad \ldots \quad (12)
\]

\[
\Pi - zS \geq \frac{-q_1(F_K - \lambda - r)}{q_1F_{KK}(1 + k)} < 0 \quad \ldots \quad (13)
\]

must hold before the switch.

(ii) A switch into Policy I can occur only if this

is a non-jump policy. If a switch takes place then \( q_2 \geq 0 \)

must hold under Policy C and \( q_2 = 0 \) must hold for more than

an instant under Policy I. Thus under Policy I

\[
q_2 = q_2 \rho - q_1[(F_K - \lambda - z) + k(F_K - \lambda - r) - k] - k \geq 0 \quad \ldots \quad (14)
\]
or

\[(F_K - \lambda - z) + k(F_K - \lambda - r) \leq - \frac{k + 1 - q_2}{q_1} < 0 \quad \cdots (15)\]

must hold before the switch. After the switch \( q_2 = 0 \) must hold so

\[(F_K - \lambda - z) + k(F_K - \lambda - r) = - \frac{k + 1}{q_1} < 0 \quad \cdots (16)\]

and \( K < 0 \) must hold since \( q_1 > 0 \). Thus

\[\Pi - zS < 0 \quad \cdots (17)\]

must hold when the switch takes place.

(iv) If \( N = 0 \) while following Policy C then

\[\Pi - zS \leq 0. \quad \cdots (18)\]

Now, a switch into Policy D requires

\[U_1 < \Pi - zS \quad \cdots (19)\]

and so \( N < 0 \) while \( N = 0 \). This is not an optimal policy.

(v) When \( N = 0 \) and Policy C is followed,

\[\Pi - zS \leq 0 \quad \cdots (20)\]

must hold. Now, under Policy C

\[F_K - \lambda - r \geq - \frac{S}{\phi_1} < 0 \quad \cdots (21)\]

and under Policy A

\[F_K - \lambda - r \leq - \frac{S}{\phi_1} < 0 \quad \cdots (22)\]
Since under Policy C, when \( N = 0 \), \( K \) is stationary a switch into Policy A requires \( \phi_1 \) to increase. Since

\[
q_1 = \frac{\phi_1}{S} - \phi_6
\]

and \( \phi_6 \geq 0 \), this requires \( q_1 > 0 \) under Policy C. Thus a switch may take place.

(vi) Can the firm switch into Policy E when \( N = 0 \)? The answer is yes, for the reason given in (v) above. Now

\[
\frac{\phi_1}{S}(F_K - \lambda - r) + 1 = 0 \quad \cdots (24)
\]

must occur and this requires a fall in \( \phi_1 \). This can occur if \( q_1 < 0 \).

(vii) Finally, it is not optimal for the firm to switch into Policy I when \( N = 0 \). Since

\[
\Pi - zS \leq 0 \quad \cdots (25)
\]

when \( N = 0 \), an increase in \( S \) under Policy I will result in \( N < 0 \) while \( N = 0 \) and this is not optimal.

Switches of Policy D

(i) Assume Policy D is a non-jump policy. Now it can only hold when \( N = 0 \) and

\[
U_1 < \Pi - zS \quad \cdots (26)
\]

and

\[
\Pi - zS > 0 \quad \cdots (27)
\]
Now \( q_1 < 0 \) so that

\[
q_1 = -\phi_6 < 0 \quad \ldots (28)
\]

The firm cannot switch into any policy involving \( N > 0 \).

Thus a switch into Policy A, C or E is not optimal. Further, since Policy I requires \( N < 0 \), this switch is not optimal.

(ii) Assume that Policy D is a jump policy. Now this policy requires that \( N \) jump to \( N = 0 \). The firm, thereafter may switch into Policy A, C or E providing

\[
\Pi - zS \geq 0 \quad \ldots (29)
\]

when \( B = B^* \) after the switch. The firm may also switch into Policy D providing

\[
\Pi - zS > 0 \quad \ldots (30)
\]

when \( B = B^* \) after the switch. The firm cannot switch into Policy I because \( N < 0 \) must hold and this is not optimal when \( N = 0 \).

Switches of Policy E

(i) If \( N > 0 \) then it may be optimal for the firm to switch into Policy A. Under Policy E

\[
q_1(F_K - \lambda - r) + 1 = 0 \quad \ldots (31)
\]

and so

\[
\frac{d}{dt}[q_1(F_K - \lambda - r) + 1] = q_1(F_K - \lambda - r) + q_{1KK} \leq 0 \quad \ldots (32)
\]
must hold before the switch. Since

\[ \dot{K} = U_1 + B \]  

...(34)

and in the neighbourhood of the switch B falls from \( B > 0 \)
to \( B = 0 \).

Thus \( B < 0 \) and

\[ U_1 > K \]  

...(35)

and

\[ \Pi - zS > -\frac{q_1(F_K - \lambda - r)}{q_1 F_{KK}} < 0 \]  

...(36)

must hold before the switch.

(ii) If \( M > 0 \) then a switch into Policy C requires

\[ q_1(F_K - \lambda - r) + q_1 F_{KK} \dot{K} \geq 0 \]  

...(37)

or

\[ \dot{K} = \frac{-q_1(F_K - \lambda - r)}{q_1 F_{KK}} < 0 \]  

...(38)

Now

\[ \dot{K} = U_1 + B \]  

...(39)

At the switch

\[ \dot{K} = U_1(1 + k) \]  

...(40)
so that

\[
\Pi - zS \leq -\frac{q_1(F_K - \lambda - r)}{q_1 F_{KK}(1 + k)} < 0 \quad \ldots (41)
\]

must hold at the switch.

(iii) A switch into Policy I may be optimal under the following conditions. Firstly \( q_2 > 0 \) if \( q_2 < 0 \) before the switch and so

\[
F_K - \lambda - z < \frac{q_2 \rho - 1}{q_1} < 0 \quad \ldots (42)
\]

and this is possible under Policy E. Also

\[
q_1(F_K - \lambda - r) + q_1 F_{KK} K \geq 0 \quad \ldots (43)
\]

or

\[
K = \frac{-q_1(F_K - \lambda - r)}{q_1 F_{KK}} \quad \ldots (44)
\]

must hold in the neighbourhood of the switch. Since

\[
K = U_1(1 + k) \quad \ldots (45)
\]

as the switch takes place

\[
\Pi - zS \leq -\frac{q_1(F_K - \lambda - r)}{q_1 F_{KK}(1 + k)} \quad \ldots (46)
\]

after the switch.
(iv) If \( N = 0 \) while following Policy E then

\[
N - zS \leq 0
\]

...(47)

when \( N = 0 \). Now a switch into Policy C, D or I is not optimal because it will cause \( N < 0 \) and this is not optimal.

(v) A switch into Policy A requires

\[
\frac{1}{S}(F_K - \lambda - r) + 1 \leq 0
\]

...(48)

to occur. Thus \( \phi_1 \) must increase. Now

\[
\frac{1}{S} = q_1 + \phi_6
\]

...(49)

where the time derivatives are defined but

\[
q_1 = -\frac{1}{S}(F_K - \lambda - r) - 1 - \phi_6 \rho
\]

...(50)

so that the sign of \( q_1 \) is ambiguous. Thus a switch may be optimal.

Switches of Policy G

(i) It is not optimal to switch from Policy G, I, J or K when \( N \geq 0 \) into Policy A because under this policy

\[
q_2 = -\frac{1}{S}(F_K - \lambda - z) - 1 > 0
\]

...(51)

and the condition \( q_2 \leq 0 \) will be violated.
(ii) Similarly, it is not optimal to switch fromPolicy G, I, J, or K when \( N \geq 0 \) into Policy A because under this policy

\[
q_2 = -\frac{1}{S}(F_K - \lambda - z) - 1 > 0 \quad \ldots(52)
\]

and the condition \( q_2 \leq 0 \) will be violated.

(iii) The switch into Policy C may be optimal if \( N > 0 \) provided

\[
q_2 = q_1[(F_K - \lambda - z) + k(F_K - \lambda - r)] - k - 1 \leq 0 \quad \ldots(53)
\]

occurs after the switch. That is

\[
F_K - \lambda - z + k(F_K - \lambda - r) \geq \frac{k + 1}{q_1} \quad \ldots(54)
\]

(iv) A switch into Policy D may be optimal if \( N = 0 \) and

\[
\Pi - zS > 0 \quad \ldots(55)
\]

after the switch.

(v) A switch into Policy C or I is not optimal. It requires

\[
q_1(F_K - \lambda - r) + q_1 F_{KK} K \geq 0 \quad \ldots(56)
\]

or

\[
K \leq \frac{-\hat{q}_1(F_{KK} - \lambda - r)}{q_1 F_{KK}} < 0 \quad \ldots(57)
\]

and \( K > 0 \) under Policy G.
(vi) The firm may switch into Policy C if \( N = 0 \). The switch requires that

\[
\frac{\phi_1}{S}(F_K - \lambda - r) + 1 \geq 0 \quad \ldots (58)
\]

occur, and so that \( \phi_1 \) must decrease. However

\[
q_1 = \frac{\phi_1}{S} - \phi_6 \quad \ldots (59)
\]

and since \( N = 0 \), \( \phi_6 > 0 \) may hold under Policy G. Thus \( \phi_6 \) and hence \( \phi_1 \) may decrease. After the switch \( q_2 \leq 0 \) or

\[
F_K - \lambda - z + k(F_K - \lambda - r) \geq -\frac{(k + 1)S}{\phi_1} \quad \ldots (60)
\]

must occur. Thus providing \( \phi_1 \) can be found to meet (58) and (60), the switch may be optimal.

(vii) A switch into Policy I is not optimal when \( N = 0 \) because Policy I requires

\[
U_1 = \Pi - zS < 0 \quad \ldots (61)
\]

and this is non-optimal while \( N = 0 \).

**Switches of Policy I**

(i) If Policy I involves a jump then a switch into Policy C may be optimal provided \( q_2 < 0 \) or

\[
F_K - \lambda - z + k(F_K - \lambda - r) \geq -\frac{k + 1}{q_1} \quad \ldots (62)
\]

holds after the switch. This is also true after a switch from
a Policy I which does not involve a jump.

(ii) If Policy I involves a jump then a switch into Policy D may be optimal provided

\[ \Pi - zS > 0 \] ...

after the switch. If Policy I does not involve a jump then

\[ \Pi - zS < 0 \]

must hold and so a switch into Policy D is not optimal when \( N = 0 \). This will result in \( N < 0 \) when \( N = 0 \).

(iii) The firm may switch from Policy I involving a jump to Policy I without a jump providing

\[ \Pi - zS < 0 \] ...

and

\[ F_K - \lambda - z + k(F_K - \lambda - r) = -\frac{k + 1}{\phi_1} \] ...

is maintained. Of course no such switch can be made if \( N = 0 \).

(iv) A switch into Policy C may be optimal if Policy I involves a jump when \( N = 0 \) provided \( \tilde{q}_2 \leq 0 \) or

\[ F_K - \lambda - z + k(F_K - \lambda - r) \geq -\frac{k + 1}{\phi_1} \] ...

holds after the switch, and, of course,

\[ \frac{\phi_1}{S}(F_K - \lambda - r) + 1 \geq 0 \] ...

and
\[ \Pi - zS \geq 0 \] \hspace{1cm} \ldots(69)

**Switches of Policy J**

(i) Policy J must result in \( N \) jumping to \( N = 0 \). Once there
\[ q_1 = \frac{\phi_1}{S} - \phi_6 = 0 \] \hspace{1cm} \ldots(70)
so that \( \phi_1 > 0 \) may occur by choosing \( \phi_6 > 0 \). After the
switch \( q_2 \leq 0 \) or
\[ F_K - \lambda - z + k(F_K - \lambda - r) \geq -\frac{k + 1}{\phi_1} \] \hspace{1cm} \ldots(71)
must hold if a switch into Policy C is optimal. This may occur.

(ii) A switch into Policy D may be optimal provided
\[ \Pi - zS > 0 \] \hspace{1cm} \ldots(72)
after the switch.

(iii) A switch into Policy I is not optimal because
\[ \Pi - zS < 0 \] \hspace{1cm} \ldots(73)
must hold after the switch and \( N = 0 \) cannot hold for more
than an instant when \( N = 0 \).

**Switches of Policy K**

(i) If \( N \geq 0 \) then Policy K requires
\[ \frac{\phi_1}{S}(F_K - \lambda - r) + 1 = 0 \] \hspace{1cm} \ldots(74)
and \( B = k(N+S) \). Since it involves a jump in \( K \)

\[
\frac{\phi_1}{S}(F_K - \lambda - r) + 1 < 0
\]

must occur. Thus the firm cannot switch into Policy C or I.

If \( N > 0 \) then \( q_1 \) cannot jump but this cannot occur. If \( N = 0 \) then \( \phi_1 \) must be decreased to result in

\[
\frac{\phi_1}{S}(F_K - \lambda - r) + 1 \geq 0.
\]

However

\[
q_1 = \frac{\phi_1}{S} - \phi_6
\]

so that \( \phi_1 \) can be increased but not decreased.

(iii) A switch into Policy D may occur providing

\[
\Pi - zS > 0
\]

after the switch, and \( N = 0 \).
Chapter IX

SOME ALTERNATIVE FORMULATIONS

The aim of this thesis has been to develop a number of models of the firm's investment and financing behaviour which differ in the assumptions regarding the motivation of the firm and its cost of capital.

The first model assumed that the goal of the firm was the maximization of the price of its ordinary shares and that only debt and ploughback finance were available to the firm. Myopic decision rules were derived. The rule relating to output required that marginal revenue equal marginal cost; and the optimal level of debt was that which made the marginal operating profit equal to the rate of interest. Accumulated funds had to be either increased at the maximum possible rate (equal to profits) or decreased at the maximum rate until the total capital stock was such that the marginal operating profit is equal to the average cost of capital.

The next model to be developed was the same as the first excepting that the firm was now able to issue shares. The optimum rules relating to output, debt and long-run capital stock were the same as in the first model. There was a problem of choice between accumulated funds
and share finance because the former slows the achievement of
the long-run capital stock but the latter reduces dividends
per share by increasing the number of shares between which
total dividends must be divided.

Variations of the first model were then examined.
Firstly product and (non-capital) factor market conditions
were allowed to vary and it was found that 'pursuit' policies
may be optimal. Secondly it was assumed that the cost of
capital conditions were different from those assumed initially
and it was found that the optimum rules in relation to output,
debt and ploughback finance were basically the same and that
the optimal long-run capital stock was, as before, that
determined by the average cost of capital. The particular
capital stock determined by that rule, however, differed
in each case. The complexity of the model in which shares
could be varied was such that it was not practical to
consider the same variations for this model.

The first model to assume the goal of the firm
was the maximization of growth or size, also assumed that
shares were not issued. The optimum rule relating to
output was unaffected by the change in the motivational
assumption and this was not an unexpected result. The optimum
rule relating to debt was so changed that optimal debt continued
to be at its maximum level even when the marginal return from
debt was less than the rate of interest. This was because
debt, itself, enters the firm's utility function as part of
the capital stock. The essential conflict in this model
was between largeness now and largeness later. Maximizing
the size of the firm's debt had the effect of maximizing
the capital stock now, but also possibly reducing profits
so that the rate of growth of capital was reduced. This
conflict was ultimately resolved by the rate of discount.
The longrun position of the firm was that where the dividend
constraint held with equality.

The last model also assumed that the goal of the
firm was the maximization of growth or size but it was
assumed that the number of shares could be increased. Again
the optimal rule for output was unaffected and the optimal
debt rule was the same as that in the previous model. The
conflict also existed between the use of accumulated funds
and shares. Accumulated funds could be increased only at
the rate given by profits less the constraint level of
dividends, and although shares could be increased in a block,
they had the effect of reducing the longrun level of capital
by increasing the constraint level of dividends and hence
the minimum level of profit.

It is freely admitted that only a very small range
of models has been considered and that the conclusions
reached are specific to these models. However the method
of analysis and the general sorts of conclusions gained
have implications for similar models.

One of the most important simplifications of the
above models is the assumption of certainty. However,
alternative formulations are unsatisfactory. One of the
most common ways of introducing uncertainty is to assume
that individuals analyse problems in terms of the expected
value of the probability distribution of outcomes, possibly with some discount for the variance of the distribution. However, without some information on the way in which individuals respond to uncertainty this approach merely provides a framework and leaves open the question of the affects of uncertainty. Another approach is that adopted by G.S. Shackle who pictures decisions as being made with regard to the focus-gain and focus-loss. This method and the game theory approach cannot readily be incorporated with the mathematics used to analyse the problems considered above. Until economists can commit themselves on how people do behave under uncertainty there is little use in providing the theoretical framework.

No attempt has been made to simulate situations although the technique makes this possible. Again lack of specific information regarding production functions and other data means that such an exercise would be no more than a demonstration.

The difficulties of obtaining solutions to problems in optimal control severely limit the types of problems which can be considered. It has already been indicated how the technique limits the development of a share-price maximization model more akin to how the market is likely to behave. This problem also applies to growth maximization model. A formulation more in line with existing theory is

$$\max \int_0^\infty \frac{K(t)}{K(t)} e^{-\rho t} dt, \quad \rho > 0$$

...(1)
subject to already existing constraints and

$$\dot{B}(t) = U_3(t)$$

Since $K(t)$ enters, the additional state variable, $B(t)$ must be included. In all but the simplest problems, three state variables make the solutions very difficult to obtain and experience with the simpler models in earlier chapters suggests that even less would be learned about the phenomenon from this formulation.

It is very difficult to tread the path of relevance and simplicity.
This bibliography contains those papers, articles and books referred to in the text.


ARROW, K.J. & KURZ, M., Public Investment, the Rate of Return and Optimal Fiscal Policy, August 1968 (Institute for Mathematical Studies in the Social Sciences, Stanford University).


