A THEORETICAL ANALYSIS OF THE ECONOMIC STRUCTURE OF THE FACTORY TOWN

BY

LES WITCHARD

A THESIS SUBMITTED FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

OF THE AUSTRALIAN NATIONAL UNIVERSITY

OCTOBER 1977

In compliance with the requirements relating to Admission to Examination, and Submission of Theses, for the Degree of Doctor of Philosophy of the Australian National University, it is affirmed that, unless otherwise stated, the work that follows is my own.

ACKNOWLEDGEMENTS

I am greatly indebted to Professor J.D. Pitchford and Dr Neil Vousden for their supervision, assistance and encouragement throughout the preparation of this thesis. I also acknowledge, with thanks, the helpful comments by participants of seminars at the Department of Economics, School of General Studies, Australian National University, and by my colleagues in the Department of Economics.

I would especially like to thank Beryl Palmer for her skill and patience in typing this thesis.

ABSTRACT

This thesis is a theoretical investigation of the structure of the factory town. The study considers both the positive and normative economic aspects of an isolated town. The optimum industrial state, consisting of a factory town and its economically associated agricultural zone, is also investigated.

The structure of the residential zone of the town is expressed in terms of household preferences in respect to consumption of the factory produced good, the services of residential space and leisure time, the allocation of land to transport and transport congestion. The areas of the factory and the town and the population of the town are obtained as implicit functions of the values of the marginal products of land and town population. Expressions for returns to scale in factory production at the optimum and equilibrium points are derived.

Some comparative static analyses of the optimum town are presented using the opportunity cost of land, a transport parameter and population density as shift parameters.

Conditions for equilibrium in a company town are derived, and it is shown that a general equilibrium involving production in factories cannot be competitive.

CONTENTS

CHAPTER 1

INTRODUCTION

From the beginnings of the industrial revolution there has been a steady decline in the relative importance of the rural sector, and a major shift towards urban economic activity in almost all countries. Industrial production has moved out of the home and into the factory. Yet economic theory has remained almost untouched by this important change in economic structure. Except for the specialized field of urban economics (which, in any case, has been more concerned with the application of existing theory to urban problems than with the extension of that theory), economics has not been structured to take account of how production and consumption are modified by the urban environment.

It is not altogether surprising that economic theory has not been developed in this direction. It appears to be a specialization which would not modify the substance of most branches of economics. In macroeconomic theory, for example, we already make heroic assumptions in respect to the aggregation of commodities, yet obtain important and meaningful conclusions. It is unlikely, therefore, that consideration of the fine structure of urban areas will give added depth to this theory.

Nevertheless, it is easy to miscalculate the narrowness of the specialization. In respect to the theory of value, the Arrow-Debreu Theorem is confined in its application to:

> "elementary regions......small enough for all the points of one of them to be indistinguishable from the point of view of the analysis."[1]

In other words, it applies only to regions in which all economic activity, that is, production, marketing and consumption, are uniformly distributed. By contrast, a crucial feature of the economics of urban regions is the spatial distribution of these three economic activities. Production takes place in factories, consumption takes place in households, perforce located outside the factory area, and there is transportation of producers and consumers to and from the market places. Clearly, the Arrow-Debreu Theorem is not a valid description of economic equilibrium in an urban environment.

At the same time, this same competitive equilibrium theorem fails to accommodate the increasing returns to scale which are pervasive in factory operations. It is necessary to assume constant returns, at least in some neighborhood of equilibrium:

> "If competition is perfect, and if no frictions prevent firms from growing to their equilibrium size, then falling average costs for the individual firm cannot occur."[2]

> "Yet on an empirical level, nobody doubts that in any economic activity which involves the processing or transportation of basic materials - in other words, in industry - increasing returns dominate the picture ......"[3]

Equilibrium theories developed within these spatial and production limitations have not been immune from criticism. In particular, Kornai [1971] and Kaldor [1972] have declared all such equilibrium theory to be irrelevant as a means of describing the operation of economic forces, and they call for its demolition. Kornai's wide ranging attack includes a strong explicit criticism of the assumption of constant returns. Kaldor concentrates most of his attack on this same assumption.

---

1 Debreu [1959, p.29].

2 Joan Robinson [1932, p.544].

3 Kaldor [1972, p.1242].

However, both are silent in respect to the spatial assumption. This is surprising for the following reasons. We have a theory in which, for conceptual, not merely technical reasons, we have to assume that returns to scale in production are constant (or at least non-increasing), and we observe that, in this regard, the theory often fails to describe production in factories. At the same time, the theory postulates a spatial uniformity which conflicts with the spatial structure that is observed to be associated with factory production. One would expect that the critics of equilibrium theory would look for a nexus between these two contradictions of reality by the theory.

That they have not is probably related to a widespread belief among economists that the costs of the transport systems, which are necessary to bring the spatially separated parts of an economy together, are no more than passive economic forces, that is, "frictions", which simply retard and modify the approach towards competitive equilibrium, and do not actively drive the system away from it. This belief has prevailed notwithstanding Sraffa's very clear warning of 50 years ago that it may not be valid.[1] Sraffa was writing about any phenomenon which gave firms comparative advantages with respect to some market places, and was arguing for the abandonment of the concept of perfect competition so that the co-existence of equilibrium and increasing returns in production could be explained. The monopoly power derived from transport costs, specifically, had been explored 100 years earlier by von Thünen, whose theory of agricultural rent is derived in terms of the cost of transporting output per acre.[2]

In regard to consideration of economies of scale in production and locational effects, normative economics has been somewhat further

---

1 Sraffa [1926, p.188].

2 For a translation of von Thünen's works see Wartenberg [1966].

developed than its positive counterpart. Starrett [1974] has presented a theory of optimal location which takes into consideration transport costs. In particular, he obtains a relationship between the optimal average degree of increasing returns in production and the transport function. However, Starrett's model consists of a system of zones, in each of which production and consumption are independent of location. This describes the essential features of an agricultural economy in which specialization takes place, or of cottage industry, but it does not capture urban economic activity where production and consumption must be separated. In other words, it cannot be claimed that Starrett has solved the problem of finding the optimum allocation of resources when production takes place in factories.

A promising means of introducing urban structure into the theory of resource allocation (positive or normative) is by the use of the presently evolving theory of the factory town. Mills and MacKinnon [1973] have called this theory the "New Urban Economics". It examines the fundamental economic forces which operate in an urban area using a simple abstraction of a factory town of the following basic structure. The circular core of the town, called the central business district (CBD), is wholly occupied by a single factory. This factory produces a consumption good from inputs of land and labor. The CBD is surrounded by an annular residential zone, the land of which is allocated either to transport or to residential purposes. The factory workforce lives in the residential area and derives utility from the consumption of the produced good, residential space (or housing services) and perhaps leisure.

The theory of the factory town has proved very useful in studying the structure of the residential zone, particularly in investigating the problems of allocating land between household residential space and the transport system. However, in its present form, it is not sufficiently general to be used for a definitive investigation of returns to scale in production at either the optimum or equilibrium points. This is because the mathematical problems of using the area of the CBD and the population of the town as control parameters have not been solved. Consequently, the land input to factory production plus CBD transport and the workforce (although not the labor input to production) have hitherto always been taken as given.

The primary aim of this thesis is to use economic theory to investigate the interrelationships between the location of economic activity, transport costs and returns to scale in production, particularly in an urban environment. The model used is the factory town, and to achieve this primary aim it is generalized to make CBD size and town population free parameters.

Secondary aims are to generalize current knowledge of the structure of the residential zone and to examine the structure of the optimum isolated state of the von Thünen type.

The point-by-point approach is used throughout. That is to say, each model used is made as simple as possible by omitting consideration of all influences which are thought to be extraneous to the relationships being investigated. This approach comes under fire from time to time. Within urban economics, Richardson [1973], for example, has criticized early models of the optimum town for the large number of important phenomena they have failed to take into consideration. At the same time, the approach has had many defenders. Mirrlees [1973] and Solow [1973c], the authors of the early models, acted as their own apologists against Richardson's direct attack. A general case for using the point-by-point approach has been made by Nagel [1963]. He divides unrealistic theoretical statements into three different types. It is his third type which interests us here, and in respect to it he argues:

> "In many sciences, relations of dependence between phenomena are often stated with reference to so-called "pure cases" or "ideal types" of the phenomena being investigated. That is, such theoretical statements (or "laws") formulate relations specified to hold under highly "purified" conditions between highly "idealized" objects or processes, none of which is actually encountered in experience. For example, the law of the lever in physics is stated in terms of the behavior of absolutely rigid rods turning without friction about dimensionless points: similarly, a familiar law of pricing in economics is formulated in terms of the exchange of perfectly divisible and homogenous commodities under conditions of perfect competition. Statements of this kind contain what have previously been called "theoretical terms", which connote what are in effect the limits of various non-terminating series and which are not intended to designate anything actual. Such statements may be said to be unrealistic but in a sense different from the two previously noted. For they are not distinguished by their failure to provide exhaustive descriptions, nor are they literally false of anything; their distinguishing mark is the fact that when they are strictly construed, they are applicable to nothing actual.
>
> However, laws of nature formulated with reference to pure cases are not therefore useless. On the contrary, a law so formulated states how phenomena are related when they are unaffected by numerous factors whose influence may never be completely eliminable but whose effects generally vary in magnitude with differences in the attendant circumstances under which the phenomena actually recur. Accordingly, discrepancies between what is asserted for the pure case and what actually happens can be attributed to the influence of factors not mentioned in the law."[1]

In Nagel's language, we wish to formulate laws relating transport costs, production and urban structure, and in using the point-by-point approach our formulation is in terms of "pure cases"; none of our models will attempt to depict a real town because of factors not considered in their formulations.

The thesis is divided into eight chapters, the first of these being this introduction, and the last a summary of the main conclusions.

Chapter 2 surveys the published literature. It is convenient

---

1 Nagel [1963, p.215].

to divide this literature into three topics, although many articles span more than one of them. The first topic is the development of the von Thünen theory of land rent. The second topic is the equilibrium factory town, and the third is the optimum town. It becomes clear that the CBD and residential zones are von Thünen "rings" in an expanded form of his theory of the *Isolated State*.

The next three chapters examine the optimum town. Historically, the theory of the equilibrium town was developed before that of the optimum town. However, as is so frequently the case, normative models are easier to formulate and solve than positive models, and it is for this reason that the optimum town is explored first. Chapter 3 examines the optimum town in which the household's leisure time is held constant. This leisure constraint is conventional in the literature. Like earlier researchers, we find that, unless some kind of "morality" is built into the model, equals are not treated equally in the optimum town. To explore around this result the alternative polar case is examined, that is, the case in which equality is deemed to be paramount. An interesting feature of the chapter is the extent to which the two cases can be handled with a single calculus.

Leisure time as a variable in the optimization process is introduced in Chapter 4. Most interesting here is the second best town in which factory work hours are fixed, because of its accord with the real world. In this model leisure time is a state variable in the solution. This solution is a further polar case to be compared with the models of Chapter 3. We find that several of the strong unambiguous results obtained in Chapter 3 are not robust to the change in the constraint.

Chapter 5 is a comparative static analysis of the optimum town using the opportunity cost of land, the inverse of transport velocity and the population density as shift parameters. Throughout this chapter population density is taken to be independent of location, and the householder's leisure time is held fixed. The analysis is conducted in a general equilibrium framework, and some of its conclusions contradict earlier, partial equilibrium analyses.

In Chapter 6 the constant density, constant leisure model is extended to include the cost of transporting an agricultural good consumed by the urban households. The extension leads us to generalize from the optimum *town* to the optimum *Isolated State*. There are three "rings" in this von Thünen model: the CBD; the residential zone; and the agricultural zone.

Chapter 7 contains our only sortie into positive economics (although we earlier showed that market realizations of some optima are possible). We examine the structure of a typical town in a general equilibrium model of a large economy. We make all markets "as near to perfect as they can be", and we discover that at equilibrium our town will be wholly owned by a single firm. Furthermore, competitive equilibrium cannot occur, even when goods transport costs are neglected. Inevitably, there is an "apparent" monopsony in the factory labor market and monopolistic competition in the urban land market. Returns to scale in factory production are increasing at equilibrium, but there is no income distribution problem, because the factory-employed factors are not paid the values of their marginal products.

Altogether, the research described in Chapters 3 to 7 gives a set of theoretical statements which relate factory production, the structure of the residential area, per capita utility, the spatial structure of the agricultural hinterland and transport costs.

In the residential zone we find, as did earlier writers, that equals are treated unequally in the optimum town if the area of the household's residential site is an instrument of optimization. Furthermore, under the usual assumptions on the shape of the utility function, the household allocation of *every* consumption good increases with the distance of residential location from the centre of the town. The cause underlying these results is explained in terms of the transport costs which are saved by moving the centre point of the location of every household toward the centre of the town and the positive sign of the cross partial derivatives of the utility function.

When an equality constraint is imposed upon the model, the structure of the residential zone depends crucially upon whether household leisure time is assumed to be constant or is treated as an instrument of optimization.

The conclusions reached in respect to returns to scale in factory production in both the equilibrium and optimum towns are new. They contradict the intuitively derived results of earlier writers that transport can be internalized into factory production so as to obtain constant returns to scale in the aggregate production. In fact, returns to scale in both factory and aggregate production in both the optimum and equilibrium towns may be decreasing, constant or increasing,depending upon the structure of the model.

When the transport costs of agricultural goods from the town's hinterland are included in the model, the nature of the central business district and the residential zone,as von Thünen "rings" of production, becomes clear. In this context the residential zone is identified as the ring in which labor is produced for use in the factory production of the CBD, and von Thünen's model of the isolated state is generalized so as to give the town a more extensive role than that of a simple market.

Finally, the comparative static results obtained from a general equilibrium model of the optimum town refute those obtained from partial equilibrium analyses. We find ambiguity of sign in almost all our results. These ambiguities are explained in terms of shifts in the optimum degree of increasing returns to scale combined with the shifts in the optimum factor proportions.

CHAPTER 2

THE STRUCTURE OF THE FACTORY TOWN: THE LITERATURE

In this chapter we survey literature directly related to the theory of the factory town. Two earlier surveys of this field are available. Goldstein and Moses [1973] presented a comprehensive appraisal of theoretical urban economics, the second section of their paper surveying housing and land values, and the third section, being on intra-urban land use, together review most of the literature on the structure of the factory town, then published. Mills and MacKinnon [1973] have a much shorter analysis intended primarily to give an overview of models of the factory town.

It is convenient to divide this chapter into three parts. The first reviews the development of the von Thünen theory of land rent in its application to urban areas. The second part surveys equilibrium models of the factory town, and the third examines normative models and comparisons between the structures of equilibrium and optimum towns.

2.1 THE THEORY OF URBAN LAND RENT

Broadly speaking, the theory of the factory town is concerned with the allocation of land in urban areas. Therefore, a land rent function, dependent upon a location variable and the characteristics of the transport system, will normally be a crucial part of the solution to any urban model. It is the development of this type of rent theory that we now review.

The theory of land rent can be traced back at least to the Physiocrats of the 18th century. At the beginning of the 19th century Ricardo presented his well known treatment of land rent, based upon soil fertility, which is the foundation of modern agricultural land rent theory. Although he and Adam Smith before him recognised that location with respect to markets would influence land values, the credit for originating the formal treatment of agricultural rent arising from transport costs and location belongs to Johann von Thünen.

Writing over the period 1826-1863, von Thünen analysed the problem of the efficient allocation of *uniformly fertile* agricultural land around a single market place.[1] The land rent is obtained in the solution to his problem as the producer's surplus per unit area of production net of freight costs to the market. The main thrust of the von Thünen theory can be captured by a simple model. Assume fixed coefficient production functions, and let all producers be price takers in the goods and labor markets. In general, more than one good can be produced from the land. We consider the i^{th} of the set of possible goods. For this good let the freight rate (that is, the cost of transporting unit quantity unit distance), be τ_i, a constant. Furthermore, let area s_i be required to produce unit quantity of it. Then, the producer's surplus per unit area under crop i is a linear function of the distance from the market of slope $-\tau_i/s_i$. The landowner maximizes his profit by the allocation of his land between crops. The agricultural land rent, r(x) say, where x is the distance from the market, is the envelope of the set of producer's surplus functions. If we number crops outward from the market, this envelope will be made up of straight line segments whose slopes

---

1 See Wartenberg [1966].

$-\tau_i/s_i$, $i = 1,2,\ldots$, decrease in absolute magnitude with x when profits are maximized. The envelope, $r(x)$, generates the so-called von Thünen "rings" of production, because, associated with each segment of $r(x)$, there is an annulus of land which is allocated exclusively to the production of a single crop. The process is illustrated in Figure 1.

For each ring we can write[1]

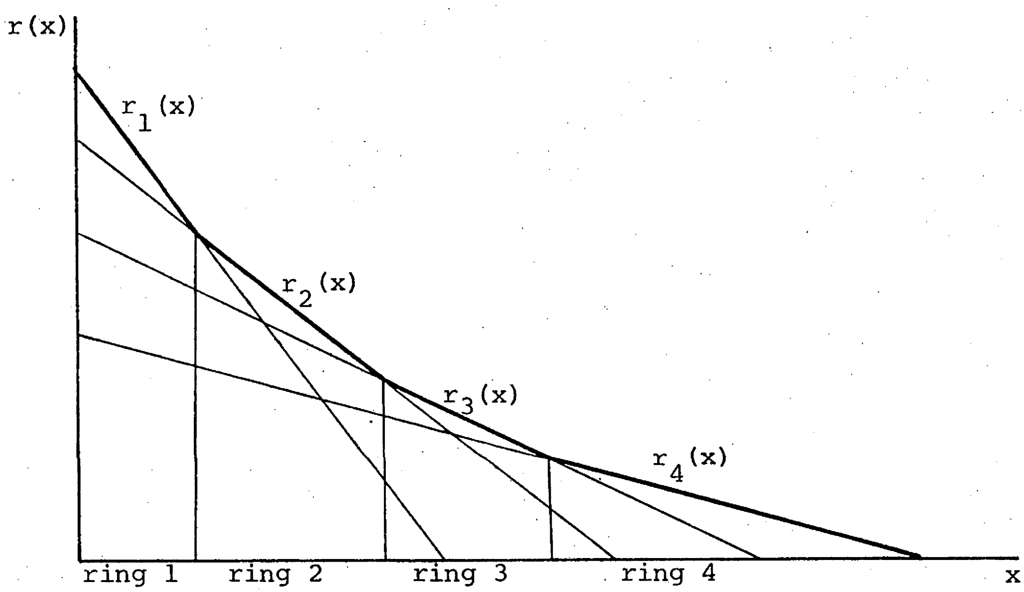

Figure 1

$$\dot{r}(x) = -\tau_i/s_i. \tag{2.1}$$

The integral of (2.1) depends upon the prices of neighboring crops, and therefore, the dimensions of each "ring", depend, among other things, upon commodity prices. All land is allocated out to the distance at which the von Thünen rent becomes zero.

Beckmann [1972] introduced neo-classical production functions

---

1 Throughout this thesis we will use the convention $\dot{r} \equiv \frac{dr}{dx}$.

into the von Thünen model. He showed that the land continues to be allocated in discrete rings. The major effect of the substitution between factors, that the neo-classical production function allows, is to make the producer's surpluses downward sloping, *strictly convex* functions (for a constant freight rate), and this, of course, implies the possibility that one good might be produced in two separated rings. The rent function continues to be the envelope of the producer's surplus functions. Some details of Beckmann's analysis were re-examined by Renaud [1972].

Strangely, almost exactly 100 years were to pass between von Thünen's last publication and the first application of his theory to residential areas in an attempt to explain the character of urban land rent. In the interim, attempts were made to develop a theory of rent for retail premises based upon the "profitability" of location. These attempts had only limited explanatory power, because profitability was treated as being determined exogenously from the model.

Alonso [1964] has given a brief review of these theories.[1] It is sufficient here to take a critical look at a representative example, the retailing rent theory presented by Chamberlin [1948]. This writer examines the retailing industry *in isolation*. He argues that the land rent for retailing premises is determined by the favorable location of the site in respect to profitability in trading. He regards this rent as being different in kind from agricultural rent, because the agriculturalist is always at a distance from the market, whereas the retailer is the market:

> "The ordinary rent reasoning does not fit at all. Rent is not paid in order to save transportation charges. It is paid in order to secure a larger volume of sales."[2]

---

1 Alonso [1964, Ch.1].

2 Chamberlin [1948, p.243].

However, in treating retailing in isolation, Chamberlin fails to take into consideration the transport costs of buyers. Clearly, *from an economic point of view, it does not matter whether it is the buyer or the seller who travels to the market.* In Chamberlin's model buyers' transport costs are implicit in his concept of exogenously determined profitability, but in a general equilibrium model containing both retailers and buyers, the land rent is determined by the competition among the buyers for housing favorably located relative to the market place, as well as by the competition between the retailers themselves for the market locations. Given a perfect land market, the rent on retailing land must equal the residential rent at the boundary of the market area. The rent function generated by this competition is not different in kind from, but is precisely a von Thünen rent, and, in a market town without industrial production, the boundary of the market is the limit of the first von Thünen ring.

Muth [1961] examined this von Thünen type rent function within the residential zone and at its boundary with the rural area. In a pure model, the residential zone is the second von Thünen ring. Muth's method was to find equilibrium conditions for housing production. A land rent function can be obtained from these conditions. However, the transport costs, which determine the favorable quality of each location, are subsumed in an exogenous housing demand function in his model, and, as a consequence, his solution lacks generality. He does, however, obtain a housing rent profile.

It was Alonso [1964] who developed the theory of the von Thünen rent, which is generated by the demand for residential space in relation to the location of some centre of economic activity. He expressed demand for land by a series of "bid-price" curves. We limit our interest to the monocentric factory town, and in this simple model, his theory can be succinctly stated in the following way. Consider a large number of identical households living in a monocentric factory town. We assume that the typical household derives utility from the consumption of a vector of produced goods, f_i, and the enjoyment of residential space of area, s. We also assume that every household must travel to the centre of the town (to the factory) each day to derive an income w. The household budget, therefore, will have the form

$$\Sigma\, p_i f_i + r(x)s + \int_0^x \tau(z)dz = w, \qquad (2.2)$$

where the p_i are prices, r(x) is the residential land rent, $\tau(x)$ is the cost of travelling unit distance at x, and x is the distance between the residential site and the centre of the town. Household utility maximization is effected by the usual choices of the quantities consumed of the produced goods and of residential space, and, in addition, by the *choice of residential location*. This choice of residential location implies the equilibrium condition

$$\dot{r}(x) = -\frac{\tau(x)}{s}. \qquad (2.3)$$

Comparison between (2.1) and (2.3) shows that the Alonso urban rent is a von Thünen rent, and that the residential zone is a von Thünen ring. In the context of the von Thünen model, this ring is used for the production of labor for sale at the central market.

The integration of (2.2) requires the complete solution of the equilibrium conditions. This leads us into an examination of models of the equilibrium factory town.

2.2 THE EQUILIBRIUM FACTORY TOWN

Throughout this section and the next we will assume that the land on which the town is located is perfectly uniform in quality. In the literature, several authors made their towns pie-slice in shape, implying that a regularly shaped section of the land, viz. the remainder of the pie, is perfectly inaccessible. This is a very simple and straightforward generalization (requiring no more than the substitution of 2π by $\theta < 2\pi$ whenever it occurs), but not a very powerful one, and for the sake of clarity in the exposition, we do not consider it when reviewing the individual articles.

Equilibrium models examine the interactions of some or all of the following economic activities:

(i) household consumption;

(ii) factory production;

(iii) housing services production;

(iv) transport production;

(v) land transactions.

There has been a tendency amongst authors to specify the functional form of some of these activities. Usually the Cobb-Douglas function with constant returns to scale is chosen. The solution is then obtained as a set of functional relationships of specified forms.

All the models have the following features in common. First the opportunity cost of land, identified as the agricultural rent, is independent of the size and population of the town, and is paid to absentee landlords. We will represent this opportunity cost by the rent, r_a. Second, the population of the town is an exogenously determined constant, which is measured as the number of identical households living in the town. We represent this number by N. Each of the households occupies its own residential site and sells one unit of labor to the town. The only reason for travel in the town, which is considered in the models, is to supply labor to the factory. Third, the form of the transport production function is always such that, given the uniformity of land quality, it guarantees circular symmetry in the structure of the town. This symmetry makes possible the designation of location by a single variable, x, the distance between the location and the centre of the town. Finally, the area of the central business district, CBD, is assumed constant. We designate the radius of the CBD by x_c. The area between x_c and the town boundary, x_t say, is the residential zone.

In the first model of the factory town of which I am aware, Mills [1967] used Cobb-Douglas functions of land, labor and capital to represent the factory good and housing productions. Returns to scale are constant in both activities. The factory has a monopoly, and all its profits pass to the *urban* landlords. The housing industry is competitive. Transport production is proportional to the single input, land.

The demand side of his model is represented by the following assumptions. Demand for the factory good is characterized by constant own-price elasticity and zero cross elasticities. The demand for housing is uniform across the residential zone and independent of price. Travellers pay the cost of their transport, and their demand is insensitive to price.

Mills obtained equilibrium conditions for the allocation of CBD land between transport and the factory and showed that the area of land allocated to transport per annulus of unit width increases at an increasing rate with x, while the area allocated to the factory increases at a decreasing rate. For the residential zone, household residential site area increases linearly with x.

Colin Clark [1951] plotted population density data for 20 cities of the world against distance from the city centre, and showed that, in practically every case, the population density outside the CBD approximated closely to a negative exponential function of distance from the centre of the city. No special case of Mills' model predicts this functional form, although his population density function is strictly decreasing.

The land rent function for the residential zone that Mills obtained is a decreasing convex function of x, which decreases more steeply than exponential. It is, however, a negative exponential function of x when land is the only input to housing.

Mills [1969] abandoned the concept of a CBD and examined the implications of the dispersion of production throughout the town. His transport production function is Cobb-Douglas and the demand for transport at a given location is proportional to the output of the produced good at that location. The rent profile he obtains in the solution to this model is of the same functional form as that described in Mills [1967].

Muth [1969] used partial equilibrium analysis to examine three aspects of the structure of the residential zone of a simple town. First, he assumed household utility to be derived from the consumption of a factory produced good and housing services, and found the first and second order conditions for household equilibrium. In this model the household's transport cost is assumed to depend upon the distance travelled and household income. The price of the factory good, and the housing services supply function are exogenous to the model. Muth showed that the price of housing is a decreasing function of distance, which is strictly convex if the equilibrium is stable.

Second, the conditions for equilibrium in housing production under conditions of perfect competition are obtained, given a housing demand function. Land and labor are the factors of production. The first order conditions derived by Muth can be separated into two classes. The first class is the well known pair of conditions which relate factor prices to the value of the marginal products. In this urban model these relationships are location dependent. The second class is an additional condition which is necessary for locational equilibrium. This condition is also a function of location. Muth showed that, at any location, land rent is an increasing function of the price of housing and a decreasing function of the wage rate. It is also a decreasing function of the distance x.

Finally, making assumptions in respect to the price elasticity of demand for housing, the equilibrium conditions derived from the first two models were used to obtain the equilibrium population density function. This function is a negative exponential (and thus accords with the empirical results of Clark [1951]) when the housing price elasticity is -1 and the housing production function is Cobb-Douglas with constant returns to scale.

Hochman and Pines [1971] used a model very similar to Mills [1967] to examine the effects of the commuter's choice of the quantity of transport purchased upon the equilibrium rent and population density functions. Their exogenously determined demand for housing function has the same constant elasticity form as that used by Mills, and the transport and housing production functions are Cobb-Douglas with constant returns to scale. The cost of a household's transport per unit distance is made up of two terms. The first is proportional to the quantity of transport purchased. This term is entirely standard. The second term is proportional to the reciprocal of the quantity purchased, and it represents the opportunity cost of travel time. Thus, speed of transport is assumed to increase with the quantity of transport consumed, so that an increase in the quantity of transport purchased reduces travel time and, hence, the opportunity cost of travel.

The authors obtained a solution to their model in terms of specific functions. Qualitatively, their rent and density functions do not differ significantly from those obtained by Mills [1967] and Muth [1969].

Hochman and Pines generalized the treatment of household transport, but their approach did not enable them to examine the important urban problem of traffic congestion. This is because transport velocity in their model depends only upon the quantity of transport purchased, and is independent of the number of travellers. Mills [1972] introduced the transport price function

$$p(x) \quad = \quad p_0 + p_1 \left(\frac{n(x)}{g_1(x)} \right)^{\alpha} \tag{2.4}$$

into a model similar in essentials to that used by Mills [1967]. In (2.4) $p(x)$ is the price charged for transporting one traveller unit distance at x, and p_0, a constant, is this price when travel is uncongested. In the second term of the right hand side $n(x)$ is the number of travellers at x, $g_1(x)$ is the land input to the transport facility at x, and p_1 and α are positive constants. This term, therefore, represents the cost of congestion. The functional form of $p(x)$ was first suggested by Vickrey [1965].

Mills obtained some numerical solutions to his model for the rent function and the price of transport, but he was unable to find an analytical solution.

Solow [1972] presented a model which is demand orientated. His household utility function is a transform of the Cobb-Douglas function, and its arguments are a vector of produced goods and residential site area. Utility maximization gives the equilibrium values for the total demand for produced goods, the land rent and the household residential site area, each expressed as a function of income net of expenditure on transport, which itself is a function of location. He shows that goods consumption per household and the rent function are both downward sloping functions of x, while the residential site area has a positive slope. This means that households substitute residential area for goods consumption as the distance of their residential location from the centre of the town is increased.

In formulating his model, Solow assumed that, at all locations, a constant proportion of land is allocated to transport, and he used the Vickrey transport price function (2.4), but set $p_0 = 0$. The solution to his model was obtained in the form of Bessel functions.

To simplify his analysis, Solow used parameter values which he considered to be unrealistic. Later, Solow [1973a] used more realistic values and obtained numerical solutions to the model. These and his earlier solutions suggested that transport congestion increases the convexity of the rent function.

In all the models reviewed so far, it has been assumed that households are identical and receive the same income. Beckmann [1969] investigated the locational distribution of income in a factory town. He used a demand-oriented model similar to Solow [1972] except that location, x, was included as an explicit argument of the utility function. He assumed household income to have a Pareto distribution with respect to population. His transport cost per unit distance is constant.

Beckmann showed that household income increases unambiguously with x. That is to say, the richer households live further from the factory than the poorer households.

Delson [1970] and Montesano [1972] showed that Beckmann's analysis was faulty. Montesano re-examined the Beckmann model. He confirmed Beckmann's conclusions for the case of non-free transport, but showed that an ambiguity, not recognised by Beckmann, appears in the polar case of free transport.[1]

---

[1] The assumption, in this model, that households derive utility from location (with $\frac{\partial u}{\partial x} < 0$) is the *raison d'etre* for a residential zone, rather than the dispersion of households, when transport is free.

The investigation of transport congestion and of traffic nuisance costs have been carried forward in two articles by Oron, Pines and Sheshinski. The first of these studies compares equilibrium and optimum towns. It is convenient, therefore, to defer discussion of these articles until we have reviewed the development of normative models in the next section of this chapter.

Comparative static analysis of the factory town is clearly an important aspect of the study of urban structure. However, as we will see from the work of Oron, Pines and Sheshinski [1973], this analysis is made difficult by the need to use general equilibrium models and the complexity of the functional dependences between the variables which describe the structure of the town. They showed that partial equilibrium analyses which do not take account of some of these dependences may lead to erroneous conclusions. We outline their discussion in the next section of this chapter.

Only one extensive comparative static analysis of the equilibrium town appears to be available. This is a partial equilibrium analysis presented by Wheaton [1974]. He examined the effects of parameter shifts on the structure of the residential area of two equilibrium towns. His basic model is demand orientated with households consuming a factory produced good and the services of residential space. In his first town population is fixed, and his shift parameters are household income, population, agricultural rent and the price of transport. He investigates the effects of shifts in these parameters on the rent and population density profiles and on household utility. In the second town household income is fixed, and the causal relationship between household utility and population is reversed.

Wheaton obtains unambiguous signs for all his derivatives. However, his partial equilibrium framework, particularly his assumption that household income is independent of the population of the town, seriously limits the generality of his conclusions.

With the exception of the two Oron, Pines and Sheshinski articles referred to above, this completes our review of the published articles on equilibrium models of the factory town. The state of knowledge of the equilibrium town can be summarized as follows. The spatial structure of the residential zone of a town of fixed population is now fairly well understood. The shapes of the rent and population density functions have been investigated, as has the allocation of land between residential use and transport under conditions of traffic congestion.

There remain, however, a number of aspects of the equilibrium factory town which require more extensive investigation. For example, in all the models reviewed CBD size and population are fixed. These models are therefore unsuitable for an investigation of equilibrium in the goods production sector. Consequently, this aspect of urban structure remains untouched. It has significance, because, when land quality is uniform, there must be economies of scale in factory production to justify the transport costs inevitably associated with the existence of the town. The relationships at equilibrium, between returns to scale, transport costs and the wage rate is of fundamental importance to our understanding of city size.

Furthermore, each of the several economic activities of the town is assumed to be controlled by an independent entrepreneur. However, in a locational model, one activity may create monopoly power in another. For example, factory production creates monopoly power for the land and transport suppliers. Therefore, equilibria in which a single entrepreneur controls several activities in an attempt to obtain monopoly profits are of special interest. A polar case of this horizontal integration is the *company* town. An examination of the economic structure of the company town is, therefore, of considerable interest.

The incompleteness of our knowledge of comparative static analysis has already been mentioned. A final point we might make is that authors have assumed the agricultural rent at the boundary of the town to be independent of town size. However, the town must be viewed as a market for agricultural products, and given non-zero freight costs, the agricultural rent must depend upon town size and population. The need to integrate the town and its agricultural hinterland into a single von Thünen model remains.

2.3 NORMATIVE MODELS

We now turn to a survey of the development of the theory of the optimum town. A convenient starting point is a model due to Solow and Vickrey [1971]. This model is different from all others in two respects. Firstly, it has no residential zone. Secondly, the CBD is long and narrow, and is treated as having only one dimension. All other models are circular in form, and tend to concentrate upon the structure of the residential zone. The Solow and Vickrey model examines the relationship between *freight* costs and the allocation of land between production and transport. The supply of labor and its transport cost is not considered.

It is assumed that each unit of land area allocated to production generates a fixed amount of goods for transport. The destinations of these goods are uniformly distributed over the land allocated to production. The authors found the conditions for optimal land allocation. Later, Hochman and Pines [1972] showed that the Solow and Vickrey analysis was incomplete. Combining the two sets of results, we find that there is an optimum length for the town, and that, within this optimum town, transport land (i.e.,the "road") is of a symmetric cigar shape. That is, the width of the road decreases at an increasing rate with distance from the centre of the town. A shadow rent function is also derived. Kanemoto

[1975] presented a similar analysis for a circular CBD.

The minimization of the congestion costs of passenger transport in a circular town has been the subject of four articles. Consider a CBD in which land is allocated either to factory production or (road) transport, and a surrounding zone in which land is allocated either to residential sites of uniform size or to transport, and let the population be fixed. Assume that the household's transport cost per unit distance is given by (2.4) with p_0 set to zero. The problem to be solved is: what is the road width profile which minimizes total transport cost in the town?

Mills and de Ferranti [1971] solved this problem for the residential area only, given a CBD of fixed area. Later, Livesey [1973] and Sheshinski [1973] independently extended this solution to encompass a CBD of optimum size.

It is found that the optimum road width is a strictly increasing, convex function of x in the CBD, and a strictly decreasing, concave function in the residential zone. The road width is zero at the centre of the CBD and at the town boundary, and at no point does it occupy all available land. It is also found that optimum congestion increases linearly from the centre of the town to the CBD boundary, after which it decreases linearly to the town boundary.

Legey, Ripper and Varaiya [1973] introduced capital into this congestion cost model to derive the profile of the intensity of land use for transport across the town.

The explanatory power of the models used in all these analyses is limited by their failure either to include goods production, or to optimize household utility. The role of the factory town is to produce goods for household consumption. Congestion minimization has a role in the maximization of household utility in a town, but this role is only a part of the process, and models which do not include production and the utility derived from the consumption of that production are necessarily incomplete.

The first examination of the structure of the town in which household utility is maximized is due to Mirrlees [1972]. In his model household utility is a function of the consumption of a produced good, residential space and distance from the town centre. The last of these arguments serves as a proxy for transport costs. There is no CBD or factory production, and population and total income are fixed. The residential site area, however, is a control variable in the optimization. In the previous models it was assumed constant.

Mirrlees derived a downward sloping shadow rent profile, but he was unable to find the slope of the population density function.[1] He showed that competitive realization of his optimum town was possible through an appropriate distribution of income. However, in general, equals are not treated equally in his town, and household utility depends upon the distance between the residential site and the centre of the town.

The discovery that equals are treated unequally in the optimum town has evoked considerable comment in the literature. It is well known that, when the consumption set includes choice of location, the feasible set is non-convex.[2] Levhari, Oron and Pines [1972] discuss the consequence of this non-convexity, and argue that a lottery for location would ensure *a priori* equality of treatment. Stern [1973] has discussed utility functions which, in the context of the Mirrlees model, would lead to equal treatment.

Riley [1972] derived an expression for the income distribution

---

1 There is an error in the derivation of his equation (23).

2 See, for example, Malinvaud [1972, p.22].

necessary for competitive realization of the optimum town when the utility function is Cobb-Douglas and the product of household utilities is maximized. In his model goods and transport production require no land inputs, and their outputs are linear functions of their labor inputs. The distribution of household income with respect to population for competitive realization was shown to be a truncated gamma function.

A feature of Riley's model is that leisure time is an argument of the household utility function. There has been a tendency in other models to assume that all households are allocated the same, exogenously determined amount of leisure per day, so that each provides a fixed amount of time to the town in the form of the sum of its travel and labor times. Given that households enjoy leisure, Riley generalizes the optimization process by allocating leisure optimally.

Riley [1974] introduced proxies for congestion and transport technology into the model. He found that, in treating equals unequally, household allocations of all consumer goods increase with distance from the centre of the town.

Dixit [1973] explored the implications of unequal treatment using the minimization of the sum of the m^{th} powers of the reciprocals of the household utilities as the criterion of optimality. In his model m is treated as a parameter, and in the limit, m tends to infinity, his criterion becomes equivalent to the maximization of household utility when equals are treated equally by constraint. In the analytical solution he obtained, Dixit's town contains a CBD of fixed area, which is wholly allocated to the factory. Transport is free in the CBD. In the residential zone the time spent in travelling unit distance is given in the form of the Vickrey equation (2.4). Household utility is a Cobb-Douglas function of the consumption of the factory produced good and of residential space.

Dixit obtained an analytical solution for the optimum town when the population is fixed. He obtained rent and density profiles across the residential zone and the optimal allocation of land to transport.

We now turn to comparisons between optimum and equilibrium towns. Oron, Pines and Sheshinski [1973] found the solution to the equilibrium town in the presence of transport congestion, and compared this solution with the price structure for competitive realization of the optimum town when equals are treated equally. In this model households derived utility from the consumption of the factory good and housing services. Housing services are produced from inputs of land and labor. Travel in the CBD is free and the size of the CBD (which is equal to the area of the factory) is fixed. The population of the town is also fixed. The household's cost of travel is the opportunity cost of its travel time plus a congestion toll imposed by the transport authority. Transport services are produced from an input of land only.

The authors formulate a similarly structured normative model in which household utility is maximized subject to an equality constraint. They do not find the solution to this normative problem, although they are able to show that, if the optimum exists, it could be realized by a competitive price system in which the traveller is charged a congestion toll equal to the external congestion cost of his journey. In the equilibrium model it is shown that efficient allocation of resources requires that the congestion toll shall have this same value. The authors call it the warranted congestion toll.

Using their equilibrium model, Oron,Pines and Sheshinski make a comparison of the conclusions derived from partial and general equilibrium analyses of the way in which population density varies with the magnitude of the congestion toll. In the partial equilibrium analysis the price of the factory good and household income are assumed fixed as the toll varies. In the general equilibrium model these quantities are endogenously determined. The authors show that, in the partial equilibrium model, increasing the congestion toll to its warranted value induces a fall in population density at the CBD boundary. They were not able to solve the equivalent general equilibrium model analytically, but their computed solutions show that, contrary to the conclusion from the partial equilibrium model, increasing the congestion toll to its warranted value results in an *increase* in the population density at the CBD boundary. Of course, the partial equilibrium model fails to take into consideration the fact that an outward migration of population results in increased transport time, and therefore, reduced factory production. In this respect it fails to provide an adequate description of the equilibrium town. We referred to these results when discussing comparative static analyses in the previous section of this chapter.

In a second article, Oron, Pines and Sheshinski [1974] extend their treatment of traffic nuisance by including environmental quality and leisure time among the arguments of the household utility function. In this model residential site area replaces housing services as a consumption good. Environmental quality is assumed to be a function of the distance from the centre of the town and of the number of travellers.

The authors show that, given no tax on traffic nuisance, household utility in an equilibrium town can be increased by re-locating all households closer to the centre of the town. Furthermore, if the utility function is Cobb-Douglas, and if the environmental quality is defined as the reciprocal of the traffic density, the equilibrium town, in which the congestion toll is less than the external cost of the traveller's journey, is always greater in area than the town in which household utility is maximized. In other words, a town in which travellers pay less than the marginal social cost of their travel will always be sub-optimal, and it will always be more dispersed than the optimum town.

This completes the survey of the normative models of the factory town. It is clear that the comments made at the end of the last section in respect to the state of knowledge of the equilibrium town are equally valid for the optimum town. No comparative static analysis appears to be available and in every model the agricultural rent at the boundary is an exogenously determined constant. The CBD size (but not the population) is a control parameter in the Livesey [1973] and Sheshinski [1973] models. However, there is no goods production in these models, and hence they throw no light on optimum factory output.

To conclude this survey we consider the role of "housing services" in both the equilibrium and optimum towns. It is clear that there has been a growing tendency to use residential site area as a proxy for housing services as the theory has developed, although this trend is not commented upon in the literature. However, the concept of housing services as a *single* commodity implies that consumers would not make separate choices of land space and capital structure in an equilibrium or optimum town.[1] Not only does this appear to be contrary to experience, but also the production of housing services complicates the structure of the model. It appears more realistic and more convenient, therefore, firstly, to use residential site area as a control variable in its own right and not as a proxy for housing services, and secondly, to introduce the services of capital as an additional consumption good. However, we will show in the next chapter that malleable capital adds very little to the power of the model of the factory town, and having established this fact, capital will be ignored throughout the remainder of this thesis.

---

[1] The earlier writers' inclusion of labor among the factors of production of housing services seems to have arisen from a confusion between house building and housing services. Clearly, the labor input to housing services, in which the residential structure is the capital input, is negligible.

CHAPTER 3

THE OPTIMUM SIZE FOR A FACTORY TOWN

In this chapter we attempt to extend the research of Mirrlees [1972] and Oron, Pines and Sheshinski [1973] in the examination of the structure of the optimum factory town. These writers used a model in which the population is given, and they maximized total utility. Average utility in their solutions is, of course, 1/N times the total utility they obtain. We will treat population as a control parameter in the optimization, and use average utility as the welfare function. Thus, our optimum town will be a direct extension of their work.

In solving our model, we have three primary aims. The first of these is to examine household consumption patterns and the distribution of household utility throughout the residential zone of the optimum factory town. We will compare the results of this examination with those obtained by Mirrlees and Oron, Pines and Sheshinski. Our second aim is to examine optimal transport congestion and the optimal allocation of land to transport in the residential zone. The results of this examination will be compared with those obtained by Mills and de Ferranti [1971], Livesey [1973] and Sheshinski [1973], who derived conditions for transport congestion minimization in a monocentric town. Our third aim is to find the optimum size for the factory town, and to find expressions for optimal returns to scale in factory production. We have already referred to the role of population in earlier models, and stated that it will be treated as a control parameter in ours. However, optimum size for a factory town involves consideration of spatial dimensions as well as population, and it is therefore necessary for us also to treat CBD area as a control parameter. There is no CBD in Mirrlees' model, and Oron, Pines and Sheshinski assumed its area to be fixed. So far as optimum size is concerned, our treatment is new. In examining optimal returns to scale in factory production, we will find it necessary to develop new concepts in marginal productivity.

3.1 THE MODEL

We assume that land is perfectly uniform in quality. Its opportunity cost is r_a per unit area, which, in this model, is assumed to be independent of the size and population of the town. To fix ideas we will describe r_a as the agricultural rent. This rent is paid to absentee landlords.

The town has circular symmetry. It consists of a central business district (CBD) of area L and radius x_c in which factory production takes place. The CBD is surrounded by an annular residential zone of outer radius x_t. The area of the town is A. Thus,

$$L = \pi x_c^2, \tag{3.1}$$

$$A = \pi x_t^2. \tag{3.2}$$

Given the circular symmetry of the town, a location can be specified by its distance from the centre of the CBD. We use the variable x to designate this distance and location.

The population of the town is divided into households which are identical in the sense that their attitudes to work and their utility functions are the same. These households reside in the residential zone, where each occupies a residential site. They are the suppliers of labor for production in the town. Each provides just one laborer, who commutes daily to work in the CBD. Given our assumptions in respect to the households, we can define the household as the unit of population. Let this population be N. It follows that there are N laborers in the town's workforce.

In respect to transport costs, we concentrate our attention in this model upon the commuting costs of the workforce. We assume that transport in the CBD is free. This assumption is to some extent justified by the observation that central business districts are small compared with their associated residential zones. Put another way, workers are observed to require much more residential space than work space.

We assume that the labor input to transport in the residential zone, T say, is given by

$$T = - \int_{x_c}^{x_t} \dot{n}(x) t(x) dx, \tag{3.3}$$

where n(x) is the number of commuters travelling beyond x and t(x) is the time taken for a journey from x to the CBD boundary.[1] If we set

$$\tau(x) = \dot{t}(x), \tag{3.4}$$

we can integrate (3.3) by parts to obtain

$$T = \int_{x_c}^{x_t} n(x) \tau(x) dx, \tag{3.5}$$

since $t(x_c) = 0$ and $n(x_t) = 0$. In (3.4) $\tau(x)$ is the time taken for one

---

[1] This time includes the time of the return journey. Throughout this thesis our transport relationships will refer to the round trip.

commuter to travel unit distance at x. We assume that

$$\tau(x) = \tau(n(x), g_1(x)) \tag{3.6}$$

where $g_1(x)$ is the "width" of the transport facility. That is, $2\pi g_1(x)dx$ is the land input to transport in the thin ring lying between x and x + dx.

The transport time function $\tau(x)$ defined in (3.6) captures congestion effects. It is clear that we can set $\tau_n > 0$, $\tau_{g_1} < 0$.[1]

Land in the residential zone is allocated either to transport or to residential purposes, or it may be left vacant. Let $2\pi g_2(x)dx$ be the land in the thin ring between x and x + dx, which is allocated to residential purposes. Then,

$$x - g_1(x) - g_2(x) \geq 0. \tag{3.7}$$

The non-negativity constraints

$$g_1(x) \geq 0, \; g_2(x) \geq 0, \tag{3.8}$$

must apply, however, we will assume that travel is not possible without transport land, and therefore, the shape of $\tau(n,g_1)$ will ensure that $g_1(x)$ is strictly positive except, possibly, at x_t, where it may be zero, because no travel occurs through that point.

We write the factory production function in the form F = F(L,W) where L is defined in (3.1) and W is the labor input to production.[2] We assume that both marginal products are strictly positive.

---

1 We adopt the convention of using subscripts to denote partial derivatives. Furthermore, we will assume that all necessary derivatives exist.

2 We will consider capital as a factor of production later when analysing the solution to our model.

To find the labor input to factory production we assume that all households are allocated the same *fixed* amount of leisure time. Let the worker resident at x_c supply unit labor to the factory. Then,

$$N - W - T \geq 0, \tag{3.9}$$

that is,

$$N - W - \int_{x_c}^{x_t} n(x)\tau(n,g_1)dx \geq 0, \tag{3.10}$$

with the equality holding when all labor is employed.

Households derive utility from the consumption of the factory good and the services of residential space. We will discuss the role of housing capital in consumption later on. Let f and s be the household's allocations of the factory good and residential space, respectively. Then we write the household's utility as

$$u(x) \;=\; u(f(x),s(x)),$$

and assume that the marginal utilities are strictly positive and diminishing. We wish to exclude the possibility of a household being allocated zero consumption or occupying zero residential space, so we will assume that the shape of $u(f,s)$ guarantees $f > 0$, $s > 0$ in the optimum town.

We can now write

$$\dot{n}(x) = 2\pi\rho(x)g_2(x), \tag{3.11}$$

$$n(x_c) = N,\ n(x_t) = 0, \tag{3.12}$$

where, for convenience of presentation, we have introduced the population density function $\rho(x) = 1/s(x)$.

Using the factory good as numeraire, the income constraint on the town is

$$F - Ar_a - \int_{x_c}^{x_t} 2\pi\rho(x)g_2(x)f(x)dx \geq 0. \quad (3.13)$$

This completes the formulation of the model.

3.2 THE WELFARE PROBLEM

The selection of a welfare function necessarily takes us beyond economics and into the realm of politics. However, if we assume that our optimum town is the archetype of so large a number that no household is excluded from living in optimum conditions, and if we further assume that maximization of utility is the ultimate goal of the typical household, maximization of average utility is an interesting and plausible criterion of optimality for a factory town.

Nevertheless, the selection of a criterion of optimality for the factory town poses an additional problem not normally found in economic theory. Mirrlees, assuming population to be fixed, maximized total household utility, and found that equals are treated unequally in his optimum town. We will maximize average utility, and confirm that equals are also treated unequally in our optimum town. The additional problem is associated with an implicit assumption underlying the Mirrlees criterion (and ours) to the effect that households are influenced only by the absolute values of their individual consumptions, and are indifferent to the utility derived by others. This seems to be an implausible representation of human behavior. Rather, households appear to take a moral stance in respect to inequality,particularly when it is they who are underpriveleged, with respect to those they judge to be their peers. This being so, a welfare function, which, when used in conjunction with a model containing the assumption $u = u(f,s)$, yields an optimum in which equals are treated unequally, cannot be entirely convincing.

Of course, the assumption that the consumer is indifferent to the utility of others is widespread in economics, as is the adoption of welfare functions based upon total or average utility. However, in almost all problems, convexity assumptions can be, and are, made that ensure that optimization results in equality among equally endowed consumers. We will show that our model contains an intrinsic non-convexity, which removes us from the general run of normative economics. It is this non-convexity which lies at the basis of our difficulty in finding a fully convincing welfare function.

In an investigation of unequal treatment of equals in his study of the optimum factory town, Dixit [1973] introduced a measure of "morality" into his criterion of optimality. Like Mirrlees, he treated population as fixed, and he optimized by minimizing the sum of the $-m^{th}$ powers of the household utilities. The exponent, m, is a parameter of his model, and its role is to provide a weight to each household's utility in the summation. Oron, Pines and Sheshinski [1973] chose to maximize average utility, subject to equals being treated equally. This is equivalent to assuming that equality is paramount, that is, to be realized without regard to price. The approach we will adopt is to consider two polar cases. The first of these being to assume that households are indifferent to the utility of others. This case is directly related to the criterion adopted by Mirrlees. The second polar case is that in which equality is assumed to be paramount. This case is directly related to the criterion adopted by Oron, Pines and Sheshinski. Dixit's polar case, where m tends to infinity, is equivalent to the equality paramount case, but his model does not capture the equality indifferent case.

To distinguish between the two cases we will refer to the equality indifferent case as FIBOT, this being an acronym of first best optimum town, and to the equality paramount case as SEBOT, because it represents a second best optimum town in the mathematical sense that an additional constraint is imposed in its formulation.

The two welfare problems are contained in the following formal statement. Let $\bar{u}$ be average household utility. Then, we wish to

$$\max_{g_1(x),g_2(x),f(x),s(x),\bar{u},W,A,L,N} \bar{u} \; ;$$

subject to:

$$\frac{1}{N}\int_{x_c}^{x_t} 2\pi\rho(x)g_2(x)u(f,s)dx - \bar{u} = 0; \tag{3.14}$$

$$u(f,s) - \bar{u} = 0; \tag{3.15}$$

the constraints (3.7), (3.8), (3.10), (3.11) and (3.13) and the boundary conditions (3.12). The FIBOT solution is obtained from this formulation by setting the Lagrange multiplier associated with the constraint (3.15) identically equal to zero.

We define the Lagrangian of the systems

$$\begin{aligned}\mathcal{L}(x) \equiv 2\pi\rho(x)g_2(x)\left\{\frac{\lambda_1}{N}u(f,s) - \lambda_2 f(x) - \phi(x)\right\} \\ - \lambda_3 n(x)\tau(n,g_1) + \mu_1(x)\{u(f,s) - \bar{u}\} \\ + \mu_2(x)\{x - g_1(x) - g_2(x)\} + \mu_3(x)g_2(x),\end{aligned} \tag{3.16}$$

and the function

$$J \equiv (1 - \lambda_1)\bar{u} + \lambda_2\{F - Ar_a\} + \lambda_3\{N - W\}, \tag{3.17}$$

where $\phi(x)$ is the co-state variable associated with the state variable $n(x)$, λ_1, λ_2 and λ_3 are the constant Lagrange multipliers associated with the integral constraints (3.14), (3.13) and (3.10), respectively, and $\mu_1(x)$, $\mu_2(x)$ and $\mu_3(x)$ are the variable Lagrange multipliers associated with the constraints (3.15), (3.7) and (3.8), respectively.[1]

The maximization is a standard problem in optimal control theory.[2] The first order necessary conditions for a maximum are equation (3.11) and

$$\dot{\phi}(x) = \lambda_3\{\tau(n,g_1) + n(x)\tau_n\}, \tag{3.18}$$

$$2\pi\rho(x)\left\{\frac{\lambda_1}{N}u(f,s) - \lambda_2 f(x) - \phi(x)\right\} - \mu_2(x) + \mu_3(x) = 0, \tag{3.19}$$

$$-\lambda_3 n(x)\tau_{g_1} - \mu_2(x) = 0, \tag{3.20}$$

$$2\pi\rho(x)g_2(x)\left\{\frac{\lambda_1}{N}u(f,s) - \lambda_2 f(x) - \psi(x)s(x)u_s - \phi(x)\right\} = 0, \tag{3.21}$$

$$2\pi\rho(x)g_2(x)\{\psi(x)u_f - \lambda_2\} = 0 \tag{3.22}$$

$$\mu_1(x)\{u(f,s) - \bar{u}\} = 0, \tag{3.23}$$

$$\mu_2(x) \geq 0, \quad \mu_2(x)\{x - g_1(x) - g_2(x)\} = 0, \tag{3.24}$$

---

1 We have neglected to include the constraint $g_1(x) \geq 0$ because the impossibility of travel without transport land means that the shape of $\tau(n,g_1)$ ensures that it cannot fail to hold.

2 See, for example, Long and Vousden [1977].

$$\mu_3(x) \geq 0, \quad \mu_3(x)g_2(x) = 0, \tag{3.25}$$

$$1 - \lambda_1 = \int_{x_c}^{x_t} \mu_1(x)\,dx, \tag{3.26}$$

$$-\lambda_2 F_2 + \lambda_3 = 0, \tag{3.27}$$

$$\lambda_2 r_a - \frac{1}{2\pi x_t}\mathcal{L}(x_t) = 0, \tag{3.28}$$

$$-\lambda_2 F_1 + \frac{1}{2\pi x_c}\mathcal{L}(x_c) = 0, \tag{3.29}$$

$$\lambda_3 + \phi(x_c) = \frac{\lambda_1}{N}\bar{u}, \tag{3.30}$$

$$\lambda_2 \geq 0, \; \lambda_2\left\{F - Ar_a - \int_{x_c}^{x_t} 2\pi\rho(x)g_2(x)f(x)\,dx\right\} = 0, \tag{3.31}$$

$$\lambda_3 \geq 0, \; \lambda_3\left\{N - W - \int_{x_c}^{x_t} n(x)\tau(n,g_1)\,dx\right\} = 0, \tag{3.32}$$

where

$$\psi(x) = \frac{\lambda_1}{N} + \frac{\mu_1(x)}{2\pi\rho(x)g_2(x)} \quad . \tag{3.33}$$

In (3.29) and (3.27) we have used F_1 and F_2 to denote $\left(\frac{\partial F}{\partial L}\right)_W$ and $\left(\frac{\partial F}{\partial W}\right)_L$, respectively. This is desirable, because we wish to reserve the symbol F_L for $\left(\frac{dF}{dL}\right)_N$ which we will see is of fundamental significance in the optimum towns.

3.3 EMPLOYMENT IN THE OPTIMUM TOWNS

We now prove that factors are fully employed, and that all income is consumed in the optimum towns.

Divide (3.22) by u_f and integrate over the residential zone. Then, using (3.26) and (3.33),

$$\lambda_2 \int_{x_c}^{x_t} \frac{2\pi g_2(x)\rho(x)}{u_f} dx = 1. \tag{3.34}$$

Therefore, $\lambda_2 > 0$. It follows from (3.27) that $\lambda_3 > 0$ and by (3.20) that $\mu_2(x) > 0$ throughout the residential zone (except, possibly, at x_t). Now, from (3.24), (3.31) and (3.32),

$$x - g_1(x) - g_2(x) = 0, \tag{3.35}$$

$$N - W - \int_{x_c}^{x_t} n(x)\tau(n,g_1)dx = 0, \tag{3.36}$$

$$F - Ar_a - \int_{x_c}^{x_t} 2\pi\rho(x)g_2(x)f(x)dx = 0, \tag{3.37}$$

which proves the proposition.

3.4 THE EMPLOYMENT OF CAPITAL

We digress briefly to examine the employment of capital in the optimum towns. Let K be the input of capital to factory production, and let k(x) be the capital value of the housing structure occupied by the household resident at x. Assume that capital is borrowed from external sources at a fixed price, i. The income constraint (3.37) now becomes

$$F - Ar_a - K_i - \int_{x_c}^{x_t} 2\pi\rho(x)g_2(x)\{f(x) + ik(x)\}dx = 0. \tag{3.38}$$

In the maximization process we obtain the two additional first order conditions:

$$2\pi\rho(x)g_2(x)\{\psi(x)u_k - \lambda_2 i\} = 0; \tag{3.39}$$

$$F_K - i = 0. \tag{3.40}$$

Using (3.22), (3.39) becomes

$$\frac{u_k}{u_f} - i = 0. \tag{3.41}$$

Equations (3.40) and (3.41) are standard conditions for profit and utility maximization. Thus, we see that the theory of the factory town adds nothing that is new to our knowledge of the optimal employment of capital. This result is easily predicted. Land has locational uniqueness, and labor must live at one location and work at another. Neither of these properties is considered in conventional theories of the allocation of scarce resources, and we would expect that the theory of the factory town would provide new results in respect to the employment of land and labor. Capital, on the other hand, has no such special property in the factory town, and as a consequence, entirely standard conditions for its optimal employment are derived.

In many of the earlier models housing services has been treated as a single good produced from inputs of land, labor and capital. This treatment complicates the mathematical structure of the models. However, apart from its mathematical inconvenience, this approach is open to criticism on two scores. First, the inclusion of labor among the factors of production is inappropriate. Housing is made up of a capital structure called a house which is located upon a residential site. The only labor input to housing services is for such minor issues as the time taken to collect rents and the time taken to oversee maintenance. Second, it does not correctly portray the consumer's problem in normative and equilibrium studies. This is because in either the optimum town or the equilibrium town consumers will choose their consumptions of the services of housing capital and residential land area separately, subject only to the budget constraints. Thus, our treatment of the services of residential site area as a separate consumption good in the optimum town is not a limitation on the generality of our model, but is a more satisfactory formulation of the consumer's problem than those which assume that household utility is a function of housing services.

The employment of capital will not be considered in the remainder of this thesis.

3.5 HOUSEHOLD CONSUMPTION

In this section it is convenient to treat the two towns separately.

In FIBOT $\mu_1(x) \equiv 0$. Therefore, by (3.26), $\lambda_1 = 1$, and, by (3.33), $\psi(x) = \frac{1}{N}$. Consumption occurs only at locations where $g_2(x) > 0$, and at these locations, (3.21) and (3.22) give

$$\frac{1}{N}\, u(x) - \lambda_2 f(x) - \frac{1}{N}\, s(x) u_s = \phi(x), \tag{3.42}$$

$$\frac{u_f}{N} = \lambda_2. \qquad (3.43)$$

Differentiating (3.43) totally with respect to x,

$$u_{ff}\dot{f}(x) + u_{fs}\dot{s}(x) = 0. \qquad (3.44)$$

Differentiating (3.42) and substituting from (3.44)

$$s(x)\frac{d}{dx}u_s = -N\,\ddot{\phi}(x), \qquad (3.45)$$

which on expanding out the derivative and substituting from (3.44) gives

$$\left\{u_{ff}\,u_{ss} - u_{fs}^2\right\}\dot{s}(x) = -\frac{u_{ff}N}{s(x)}\,\dot{\phi}(x). \qquad (3.46)$$

We have assumed u(f,s) to be strictly concave, and it is clear from (3.18) that $\dot{\phi} > 0$, therefore, $\dot{s} > 0$. It now follows from (3.44) that $\dot{f}$ has the same sign as u_{fs}. Furthermore, differentiating u(f,s) totally with respect to x and using (3.44) to eliminate $\dot{f}(x)$,

$$\dot{u}(x) = \left\{u_s - u_f\,\frac{u_{fs}}{u_{ff}}\right\}\dot{s}(x). \qquad (3.47)$$

If we restrict ourselves to the interesting case of $u_{fs} > 0$, we have shown that consumptions of both the factory good and the services of residential space increase with distance from the centre of the town, and that equals are treated unequally in FIBOT. We will discuss further this unequal treatment in the next section. In SEBOT,

$$\dot{u} = \dot{f}u_f + \dot{s}u_s = 0. \qquad (3.48)$$

Eliminating $\psi(x)$ from (3.21) by use of (3.22), differentiating totally with respect to x and using (3.48),

$$s(x)\ \frac{d}{dx}\left(\frac{u_s}{u_f}\right) = -\frac{1}{\lambda_2}\,\dot{\phi}(x). \tag{3.49}$$

Expanding out the derivative and using (3.48) again to eliminate $\dot{f}(x)$,

$$s(x)\ \frac{\lambda_2}{u_f^3}\left\{u_f^2\,u_{ss} + u_s^2\,u_{ff} - 2u_f\,u_s\,u_{fs}\right\}\dot{s}(x) = -\,\dot{\phi}(x). \tag{3.50}$$

The assumption of strict concavity of the utility function ensures that u(f,s) is quasi-concave, which, in turn, ensures that the coefficient of $\dot{s}(x)$ on the left hand side of (3.50) is non-positive. Equation (3.18) shows that $\dot{\phi}(x)$ is strictly positive. Therefore, $\dot{s}(x)$ is strictly positive, and by (3.48), $\dot{f}$ is strictly negative. Thus we conclude that, in SEBOT, consumption of the services of residential site area increases and consumption of the factory good decreases with distance of residential location from the centre of the town. This result is independent of the sign of u_{fs}.

We see from (3.43) that the household's marginal utility in factory good consumption is constant in FIBOT. This conventional result is to be expected. In the optimum town in which households are concerned only with their absolute levels of consumption the factory good will be consumed to the point where its marginal utility equals its shadow price, and, since goods transport is free, this shadow price must be independent of location. However, we see from (3.22), that u_f is location dependent in SEBOT. If we multiply (3.33) by $2\pi\rho(x)g_2(x)$ and integrate across the residential zone,

$$\int_{x_c}^{x_t} 2\pi\rho(x)g_2(x)\psi(x)\,dx = 1, \tag{3.51}$$

by (3.26). Thus, $\psi(x)$ is a weighting function on household consumption, and (3.22) implies that each household consumes the factory good to the point at which the weighted household marginal utility of the factory good is constant. Integrating (3.22) across the residential zone, the average marginal utility of the factory good, $\bar{u}_f$ say, is given by

$$\bar{u}_f = N\ \lambda_2, \tag{3.52}$$

which is the SEBOT analogue of (3.43) of FIBOT.

Finally, a comparison between (3.45) and (3.49) using (3.43) shows that (3.49) is valid for both towns.

3.6 THE UNEQUAL TREATMENT OF EQUALS

Mirrlees [1972] was first to show that equals are treated unequally in the optimum town. So far as I am aware, a fully satisfactory explanation of this result has not yet been presented. The result arises from the fact that households who occupy inner locations impose an external resource cost, in the form of transport, *upon the town as a whole*. Outer residents have to travel beyond these inner residents, and the town, including the inner residents, has to bear the cost of this transport. Inequality in the optimum town is the manifestation of the internalization of this externality, and the presence of non-convexity in the consumption set is the reason that the inequality does not conflict with conclusions derived from conventional, non-locational economic theory.

To explain the optimality of inequality it is convenient first to consider a factory town containing only two households.[1] Referring to Figure 2, assume that x_c and x_t are exogenously determined constants, and that the residential site areas s_1 and s_2 are equal. Now, if we

---

[1] I am grateful to Neil Vousden for suggesting this model.

increase s_2 and decrease s_1, both households move closer to the CBD, and therefore, the resource cost of transport to the town is decreased. This implies that factory production is increased. At the same time, $u_{fs} > 0$ implies that average utility will be increased if household 2, whose residential site area is greater than that of household 1, is allocated more of the factory good than household 1. In other words, average utility will be increased if household 2 derives a higher level of utility than household 1. There is a trade-off in the process described if the utility

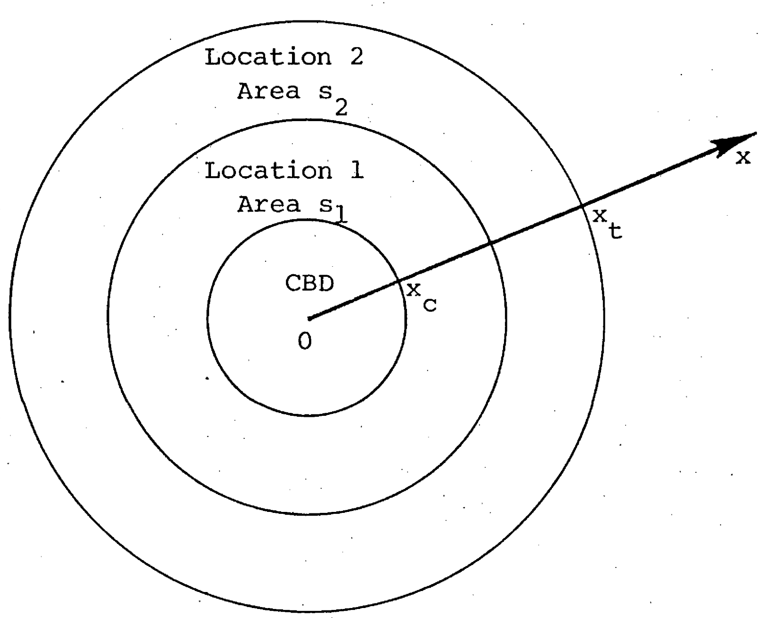

Figure 2

function is strictly concave, because decreasing marginal utility implies that the increment in the allocations to household 2 necessary to compensate for the decrease in the utility derived by household 1 become increasingly greater as the difference between the two utilities increases. Thus, average utility will be maximized when household 2 derives greater utility than household 1, but, given a strictly concave utility function, it will not be optimum to give household 2 all consumption and household

1 none at all.

It is easy to extend this explanation to the town of fixed x_c and x_t in which there are N households, because, by analogy, we can see that, starting from the configuration in which all residential site areas are equal, an inward migration of the centrepoints of all residential sites will result in a reduction in the resource cost of transport. Again the trade-off process between increased factory output and diminishing marginal utility will result in household utility being an increasing function of x in the optimum town.

In developing our explanation of the optimality of inequality, we have had to depend upon strict concavity of the utility function to justify non-zero allocations to more than one household. In this regard, it is interesting to note that our FIBOT model fails when the utility function is linear homogeneous. Equation (3.21), for example, implies that $\phi(x) \equiv 0$ in FIBOT for linear homogeneous utility functions. To anticipate the next section, $\phi(x)$ is related to transport cost. Thus, superficially at least, a linear homogeneous utility function implies zero transport costs. The reason for this anomalous result is that, without the trade-off described in the preceding paragraph, the residential area does degenerate so that N - 1 households are allocated f = 0, s = 0 at x_c, while the remaining household is allocated $f = F - Ar_a$, s = A - L also at x_c. No transport is then required. However, this analysis is not rigorous. Equation (3.21) was derived on the assumption that $\rho(x)$ was everywhere finite, whereas it is unbounded at x_c in this special case. The assumption of finite $\rho(x)$ was justified by our requiring the utility function to reflect the need of a household for both living space and the consumption of the factory good. This requirement is not satisfied by the linear homogeneous function, because u_f and u_s may be finite at (0,0).

It remains to identify the nature of the non-convexity in our model. It is well known that, when a household's consumption is confined to fewer than the total available locations, the consumption set is non-convex.[1] In our model the typical household's feasible consumption of the factory good is not affected by its residential location, but it can occupy only one location. Starting with a town with a population of 2 households, the consumption set is illustrated in Figure 3. A household

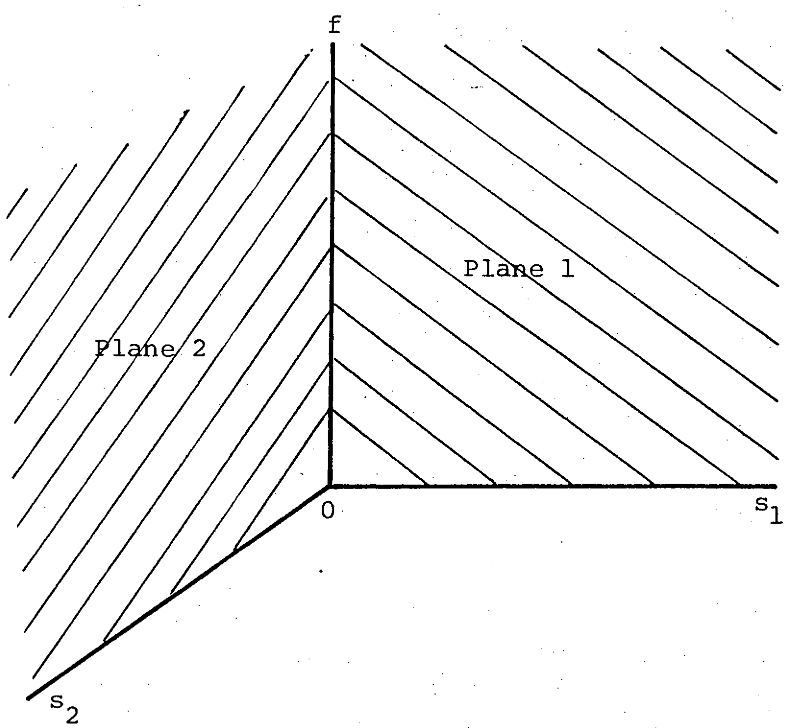

Figure 3

may live at location 1 where it can consume the factory good and residential space on plane 1, or it may live at location 2 and consume on plane 2. Consumption off these planes is infeasible, because it implies residing at both locations. Therefore, the feasible consumption set is non-convex.

The two planes are, of course, orthogonal, and share the f axis. When the population is N households, the feasible consumption set can be represented geometrically by N orthogonal, two dimensional plane surfaces

---

1 See, for example, Malinvaud [1972, p.22].

in N + 1 dimensional space, and it is, therefore, also non-convex. Note that the non-convexity does not depend upon the household being limited to only one location. It exists so long as the permissible number of residential locations per household is less than the total number of locations available.

3.7 THE MEANING OF THE CO-STATE VARIABLE AND THE SHADOW RESIDENTIAL RENT

Arrow and Kurz have given an intuitive proof that a co-state variable at any point is equal to the marginal contribution of its state variable to the maximum value of the objective functional.[1] In our model the state variable, $n(x)$, is the number of travellers at x, and the objective functional is average household utility. We would, therefore, expect our co-state variable $\phi(x)$ to be related to the socially valued cost of adding one more traveller at x in the optimum town.

To explore this relationship we first note from (3.43) and (3.52) that $N\lambda_2$ is the conversion factor between value measured in marginal social terms (averaged over all locations in the case of SEBOT) and in terms of the numeraire good. It follows from (3.27) that $N\lambda_3$ is the marginal social value of labor, and hence of travel time. Furthermore, $N\lambda_3 n(x)\tau(x)dx$ is the marginal social value of the time spent by all travellers while crossing the thin ring between x and x + dx, and $N\lambda_3\, dx \frac{\partial}{\partial n}(n\tau)$ is the increase in this value caused by adding one more traveller to the optimum transport system. However, from (3.18)

$$N\lambda_3\, dx \frac{\partial}{\partial n}(n\tau) = N\dot{\phi}(x)dx. \tag{3.53}$$

---

1 In Arrow and Kurz [1970, pp.33-37].

Therefore, integrating,

$$\phi(x) - \phi(x_c) = \lambda_3 \int_{x_c}^{x} \frac{\partial}{\partial n}(n\tau)dz \qquad (3.54)$$

is the socially valued marginal commuting cost from x to the CBD on the optimum transport system.

Later we will need to identify the economic meaning of $\frac{1}{\lambda_2}\{\phi(x_t) - \phi(x_c)\}$. We have shown that all land in the residential zone is allocated. Therefore, if one more household is added to the optimum town, *ceteris paribus*, this additional household must be located at x_t, and its contribution to the total transport cost in the town, T_N say, is given by

$$T_N = \int_{x_c}^{x_t} \frac{\partial}{\partial n}(n\tau)dx. \qquad (3.55)$$

Therefore, by (3.27) and (3.54),

$$F_2 T_N = \frac{1}{\lambda_2}\left\{\phi(x_t) - \phi(x_c)\right\} . \qquad (3.56)$$

Equation (3.56) makes the economic meaning of $\frac{1}{\lambda_2}\{\phi(x_t) - \phi(x_c)\}$ clear; it is the marginal household's cost of transport in the optimal town. However,we need to stress that the meaning of T_N,which has been derived from our solution,is such that, in performing the partial differentiation, both L and the structure of the residential zone as a function of x are held constant. The dimensions of the optimum town, however, are not held constant, since x_t must increase to accommodate the marginal household. Finally, it is obvious that

$$T_N - \frac{T}{N} > 0. \tag{3.57}$$

In both towns the shadow residential rent, $r(x)$ say, can be defined as

$$r(x) = \frac{u_s}{u_f}. \tag{3.58}$$

Therefore, from (3.49),

$$\dot{r}(x) = -\frac{1}{\lambda_2 s(x)} \dot{\phi}(x), \tag{3.59}$$

$$= -\frac{1}{s(x)} F_2 \frac{\partial}{\partial n} (n(x)\tau(x)), \tag{3.60}$$

by (3.54). Clearly, $r(x)$ is a von Thünen rent.

Equations (3.58) and (3.60) are necessary conditions for household equilibrium in the competitively realised optimum town, provided each household pays the marginal social cost of its travel. These equations plus the budget constraint (3.13), form the sufficient conditions for competitive realization if the area of the CBD and the population of the town are fixed. The proof of sufficiency is not substantially different from that given by Mirrlees [1972]. Thus, we have established that, if x_c and N are fixed, competitive realization of both towns is possible. There does not seem to be a sufficiency theorem available for the case where both end points, x_c and x_t, are free.

Comparison between SEBOT, with fixed CBD size and population, and the optimum town formulated, but not solved, by Oron, Pines and Sheshinski [1973] shows that their optimum town is similar to SEBOT. Both have a form similar to their equilibrium town in which each household pays the marginal cost of travel from its residential location.

It is clear from (3.18), (3.58) and (3.59) that the shadow rent is a strictly positive, strictly decreasing function of the distance from the centre of the town. The sign of its second derivative depends upon the second order partial derivatives of $\tau(n,g_1)$. In the next section of this chapter we will assume $\tau(n,g_1)$ to have the Vickrey form given in equation (2.4), and show that $\ddot{r}(x)$ is positive for the special case $\tau_0 = 0$. If $\tau_0 > 0$, the sign of $\ddot{r}(x)$ cannot be determined without reference to the second order sufficiency conditions for maximization. Thus, $\tau_0 = 0$ implies that the shadow rent is a positive, decreasing convex function of x. This result is in qualitative agreement with the empirical results of Clark [1951].

This is a convenient point at which to pause and review the results obtained so far. In both towns we have found the expression for the downward sloping von Thünen shadow rent. This downward sloping rent expresses the fact that the shadow price of land, relative to the shadow price of the factory good, falls with distance from the centre of the town. The falling relative price implies that there will be a substitution of land for the factory good in consumption, which is an increasing function of x. In other words, we have shown $\dot{r}(x) < 0$ and $\dot{s}(x) > 0$ in both towns. In FIBOT, $\dot{s}(x) > 0$ implies $\dot{f}(x) > 0$ if the second order cross partial derivatives are positive. In SEBOT, $\dot{u}(x) = 0$ implies $\dot{f}(x) < 0$ when $\dot{s}(x) > 0$, irrespective of the sign of u_{fs}. These results summarise the location dependent structure of consumption patterns in our two optimum towns.

3.8 OPTIMAL CONGESTION AND ROAD WIDTH

To solve for optimal congestion it is necessary to specify the transport function. We select the Vickrey form given in (2.4), but expressed in terms of travel time. That is,

$$\tau = \tau_0 + \tau_1 \theta^\alpha, \tag{3.61}$$

where τ_0, τ_1 and α are positive constants and $\theta = \dfrac{n(x)}{g_1(x)}$.

Now, in regions where $g_2(x) > 0$, (3.18), (3.27) and (3.61) imply that (3.60) can be written

$$s(x)\dot{r}(x) = - F_2 \{ \tau_0 + (\alpha + 1)\tau_1 \theta^\alpha(x) \} . \tag{3.62}$$

Also, (3.19), (3.20) and (3.25) give

$$2\pi\rho(x) \left\{ \frac{\lambda_1 u}{N} - \lambda_2 f(x) - \phi(x) \right\} = \alpha\tau_1\lambda_3\theta^{\alpha+1}(x), \tag{3.63}$$

and, from (3.21) and (3.22),

$$\frac{\lambda_1}{N} u(x) - \lambda_2(f(x) + s(x)r(x)) - \phi(x) = 0. \tag{3.64}$$

From (3.63) and (3.64),

$$r(x) = \frac{\alpha\tau_1 F_2}{2\pi} \theta^{\alpha+1}(x). \tag{3.65}$$

Differentiating (3.65) and using (3.62) to eliminate $\dot{r}$,

$$\dot{\theta}(x) = - \frac{2\pi}{\alpha} \rho(x) \left\{ 1 + \frac{\tau_0}{\tau_1 (\alpha+1) \pi\theta^\alpha(x)} \right\}. \tag{3.66}$$

The right hand side of (3.66) is strictly negative, therefore optimal congestion is a decreasing function of distance from the centre of the town.

It is interesting to note that Mills and de Ferranti [1971] and later writers found that optimal congestion is a linear function of x when τ_0 is zero. They assumed residential site area constant, and we see from (3.66) that the linear relationship they derived depends upon their assumption of the constancy of s. In our model it is easy to see that $\tau_0 = 0$ implies $\ddot{\theta}(x) > 0$. Furthermore, (3.65) then implies that $\ddot{r}(x) > 0$. This result was anticipated in the previous section of this chapter.

When $\tau_0 = 0$, (3.66) can be integrated by quadrature. We write the integral in the form

$$\theta(x) = \frac{2\pi}{\alpha}\int_{x_c}^{x_t} \rho(z)dz - \frac{2\pi}{\alpha}\int_{x_c}^{x} \rho(z)dz + \theta(x_t). \tag{3.67}$$

To evaluate $\theta(x_t)$, (3.16) and (3.28) yield

$$\lambda_2 r_a = \frac{\rho(x_t)g_2(x_t)}{x_t}\left\{\frac{\lambda_1 u(x_t)}{N} - \lambda_2 f(x_t) - \phi(x_t)\right\}, \tag{3.68}$$

from which we see that $g_2(x_t) > 0$, and hence, our analysis is valid in the neighborhood of x_t. In fact, $g_2(x_t)$ must equal x_t, because, by (3.58), r(x) is strictly positive at x_t. However, $n(x_t) = 0$, and therefore by (3.65), $r(x_t) > 0$ implies $g_1(x_t) = 0$. Thus, (3.68) simplifies to

$$\frac{\lambda_1}{N} u(x_t) - \lambda_2\{ f(x_t) + s(x_t)r_a \} - \phi(x_t) = 0. \tag{3.69}$$

Comparison between (3.64) at x_t and (3.69) shows that

$$r(x_t) = r_a. \tag{3.70}$$

That is, the residential and agricultural rents are equal at the boundary of the town. Putting $r(x_t) = r_a$ in (3.65), we obtain

$$\theta(x_t) = \left\{\frac{2\pi r_a}{\alpha\tau_1 F_2}\right\}^{\frac{1}{\alpha+1}}. \tag{3.71}$$

Therefore, congestion is non-zero at x_t for the case $\tau_0 = 0$.

To examine the slope of $g_1(x)$, in regions where $g_2(x) \geq 0$,

$$\dot{\theta}(x) = \frac{d}{dx}\left(\frac{n(x)}{g_1(x)}\right) = -\frac{2\pi\rho(x)(x - g_1(x))}{g_1(x)} - \frac{\dot{g}_1(x)}{g_1(x)}\theta(x). \qquad (3.72)$$

Therefore, eliminating $\dot{\theta}$ from (3.66),

$$\dot{g}_1 = -\frac{2\pi\rho(x)}{\alpha\theta(x)}\left\{\alpha x - (\alpha + 1)g_1(x) - \frac{\tau_0 g_1(x)}{\tau_1(\alpha+1)\pi\theta^{\alpha}(x)}\right\}. \qquad (3.73)$$

From (3.16) and (3.29),

$$2\pi x_c\lambda_2 F_1 = 2\pi\rho(x_c)g_2(x_c)\left\{\frac{\lambda_1 u(x_c)}{N} - \lambda_2 f(x_c) - \phi(x_c)\right\}$$

$$- \lambda_3 N\tau(x_c). \qquad (3.74)$$

Therefore, $g_2(x_c) > 0$, and our analysis is valid in the neighborhood of x_c. Equations (3.63) at x_c and (3.74) yield

$$2\pi x_c \frac{\lambda_2}{\lambda_3} F_1 = \alpha\tau_1 g_2(x_c)\theta(x_c)^{\alpha+1} - N\tau(x_c),$$

$$= \tau_1\theta(x_c)^{\alpha+1}\left\{\alpha x_c - (\alpha + 1)g_1(x_c) - \frac{\tau_0}{\tau_1}\frac{g_1(x_c)}{\theta(x_c)^{\alpha}}\right\}. \qquad (3.75)$$

The right hand side of (3.75) is strictly positive, and hence, by (3.73), $\dot{g}_1(x) < 0$ at x_c. Furthermore, if $\tau_0 = 0$, $\dot{g}_1(x)$ can change sign only if $\alpha x - (\alpha + 1)g_1(x)$ changes sign. However, the sign of the slope of $\alpha x - (\alpha + 1)g_1(x)$ changes only when $g_1(x) > \frac{\alpha}{\alpha+1}$. It follows that $\dot{g}_1(x) < 0$ and $g_1(x) < x$ throughout the residential zone. In the more general case where $\tau_0 > 0$ an ambiguity exists in the sign of $\dot{g}_1(x)$. Furthermore, we cannot rule out the possibility that $g_1(x) = x$ for some locations within the residential zone. In the following analysis we will restrict our attention to solutions in which $g_1(x) < x$ everywhere.

Livesey [1973] and Sheshinski [1973] were able to show that, for $\tau_0 = 0$, $g_1(x)$ is a concave function. Differentiation of (3.73) shows that their result depends upon their assumption that $\rho(x)$ is constant.

This completes our analysis of the structure of the residential zone of the optimum towns. We will now obtain implicit relationships between their geographical dimensions and populations and the value of the variables of the system at the boundaries.

3.9 BOUNDARY VALUES IN THE OPTIMUM TOWNS

We now examine the value of variables at the CBD and town boundaries of the optimum towns, and relate these values to the value of the marginal products in factory production at the optimum point. Note that, in equation (3.70), we have already shown that the residential rent is equal to the agricultural rent at the town boundary.

Consider first the marginal product of labor, F_2. Equations (3.27) and (3.30) give

$$F_2 = \frac{\lambda_1}{N\lambda_2}\bar{u} - \frac{1}{\lambda_2}\phi(x_c). \tag{3.76}$$

Dividing (3.21) by λ_2, using (3.22) to eliminate $\psi(x)$, and integrating,

$$\frac{\lambda_1}{N\lambda_2}\bar{u} = \bar{f} + \overline{r(x)s(x)} - \frac{1}{\lambda_2}\bar{\phi}, \tag{3.77}$$

where the bars denote values averaged over all households. Therefore,

$$F_2 = \bar{f} + \overline{r(x)s(x)} + \frac{1}{\lambda_2}\{\bar{\phi} - \phi(x_c)\} = \bar{w}. \tag{3.78}$$

At equation (3.54) we showed that $\frac{1}{\lambda_2}\{\phi(x) - \phi(x_c)\}$ is the marginal cost of commuting from x to the CBD. Therefore, $\frac{1}{\lambda_2}\{\bar{\phi} - \phi(x_c)\}$ is the average of these marginal commuting costs, and $\bar{w}$ is the average shadow wage when the shadow price of transport is equal to its shadow marginal cost.

In SEBOT it is not necessary to integrate (3.21), and we are able to conclude that

$$F_2 = f(x) + r(x)s(x) - \frac{1}{\lambda_2}\{\phi(x) - \phi(x_c)\} = w(x) = \bar{w}. \tag{3.79}$$

Equation (3.79) is the conventional relationship obtained in the non-locational theory of the firm. At first sight its meaning seems straightforward, but this straightforwardness is fortuitous, and arises from the fact that, so far as labor in SEBOT is concerned, the shadow marginal factor price, the shadow marginal factor price from each residential location, and the shadow average factor price are all equal. Equation (3.79) is, therefore, equivocal. The importance of recognising the ambiguity of meaning of (3.79) becomes clear when we examine (3.78), because, in respect to FIBOT, this is a relationship between the marginal product and a shadow factor price which is the *average* of the shadow *marginal* factor prices at each location. We conclude, therefore that, notwithstanding the simplicity of form of (3.78) and (3.79), we have not, in fact, obtained a direct and conventional relationship between shadow price and marginal product.

The failure of (3.78) to express a conventional relationship is not altogether surprising. It is not *labor* which is allocated to the town, but *population*. In order to increase the labor input to factory production it is necessary to add households. These, in turn, increase transport costs in the town. Thus, the partial derivative, F_2, is not fundamental in the optimal factory town, but somewhat contrived. The marginal product of population, on the other hand, does have a fundamental economic meaning.

We define the marginal product of population as the partial derivative of F with respect to N when L and the structure of the residential zone as a function of x are held constant. In this definition W and the dimension A of the town are functions of N, and it is consistent with the partial derivative with respect to N defined in (3.55). Thus,

$$F_N = \frac{\partial F}{\partial N} + \left(\frac{dW}{dN}\right)_L F_2, \tag{3.80}$$

$$= (1 - T_N)F_2. \tag{3.81}$$

Equation (3.81) simply states that $1/(1 - T_N)$ households must be added to the optimum town to increase factory labor by one unit.

In FIBOT, (3.56), (3.79) and (3.81) give

$$F_N = \bar{w} - \frac{1}{\lambda_2}\{\phi(x_t) - \phi(x_c)\}, \tag{3.82}$$

$$= \bar{f} + \overline{r(x)s(x)} - \frac{1}{\lambda_2}\{\phi(x_t) - \bar{\phi}\}. \tag{3.83}$$

Equations (3.21) with $x = x_t$, (3.77) and (3.83) yield

$$F_N = -\frac{1}{u_f}\{u(x_t) - \bar{u}\} + f(x_t) + r_a s(x_t). \tag{3.84}$$

In SEBOT,

$$F_N = f(x_t) + r_a s(x_t), \tag{3.85}$$

so that (3.84) holds in both optimum towns.

Unlike (3.78), the meaning of (3.85) is straightforward. The right hand side is equal to the shadow cost of the marginal household's consumption, that is, the marginal cost of a household to the town. Therefore, (3.85) simply states that the value of the marginal product of population is equal to its marginal cost in the optimum town. There is an additional term, $-\frac{1}{u_f}\{u(x_t) - \bar{u}\}$, in (3.84), which is non-zero in FIBOT. This term takes account of the marginal household making a more than average contribution to the total utility in FIBOT. This is an interesting result because it means that, in FIBOT, *the marginal household produces less than the value of its own consumption.*

Later it will be convenient to have the marginal conditions in a different form. Integrate (3.37), and divide by N,

$$\bar{f} = \frac{F - Ar_a}{N} = \frac{F - Lr_a}{N} - \bar{s}r_a - \frac{Rr_a}{N}, \tag{3.86}$$

where $\bar{s}$ is the average area of a residential site and R is the land area allocated to the transport system. From (3.84) and (3.86),

$$F_N = \frac{F - Lr_a}{N} - \frac{1}{u_f}\{u(x_t) - \bar{u}\} + f(x_t) - \bar{f}$$

$$+ r_a\{s(x_t) - \bar{s}\} - \frac{Rr_a}{N}. \tag{3.87}$$

To make clear the economic meaning of (3.87) and some results which follow, we define the following functions: let U be the sum of all household utility in the optimum town, and let V be the shadow cost to the town of all household consumption. Then

$$\bar{u} = \frac{U}{N}, \qquad u(x_t) = U_N, \tag{3.88}$$

$$V = \int_{x_c}^{x_t} 2\pi\rho(x) g_2(x) f(x)\,dx + r_a \int_{x_c}^{x_t} 2\pi g_2(x)\,dx, \tag{3.89}$$

$$\bar{f} + r_a \bar{s} = \frac{V}{N}, \qquad f(x_t) + r_a s(x_t) = V_N, \tag{3.90}$$

where the subscript N denotes a partial differentiation with respect to N which is consistent with our earlier usage.

Now (3.87) may be written

$$F_N = \frac{F - Lr_a}{N} - \frac{1}{u_f}\{U_N - \frac{U}{N}\} + \{V_N - \frac{V}{N}\}$$

$$+ r_a\{R_N - \frac{R}{N}\} = (1 - T_N)F_2. \tag{3.91}$$

In (3.91) we have included the term $r_a R_N$ to complete the symmetry. This is possible because

$$R_N = \frac{\partial R}{\partial N} + \frac{\partial R}{\partial A}\frac{\partial A}{\partial N} = 0, \tag{3.92}$$

because $g_1(x_t) = 0$. We will use (3.91) in the next section of this chapter.

We now find the marginal product of factory land in the optimum towns. After some manipulation, (3.16), (3.29) and (3.69) yield

$$F_1 = r_a + \frac{g_2(x_c)\rho(x_c)}{x_c}\left\{-\frac{\lambda_1}{N\lambda_2}\{u(x_t) - u(x_c)\} + f(x_t) - f(x_c)\right.$$

$$\left. + r_a\{s(x_t) - s(x_c)\} + \frac{1}{\lambda_2}\{\phi(x_t) - \phi(x_c)\}\right\}$$

$$- \frac{g_1(x_c)r_a}{x_c} - \frac{F_2 N\tau(x_c)}{2\pi x_c} . \tag{3.93}$$

The first term on the right hand side of (3.93) is the opportunity cost of factory land to the town. The second last term is the saving in rent on transport land when one unit of land is transferred from the residential area to the CBD. That is,

$$R_L r_a = - \frac{g_1(x_c)r_a}{x_c} . \tag{3.94}$$

The last term is the value of the travel time saved when, without regard to the location of households, unit area is transferred from the residential area to the CBD. To understand the remaining terms observe that $\rho(x_c)g_2(x_c)/x_c$ is the number of households resident on unit area of the residential zone at the CBD boundary. If unit area is transferred to the CBD, and the structure of the residential zone is unchanged, this number of households is displaced from x_c to x_t. It can now be deduced that the term

$$\frac{\rho(x_c)g_2(x_c)}{x_c}\{u(x_t) - u(x_c)\} = U_L, \tag{3.95}$$

where the partial derivative implies N and the structure of the residential zone constant, but not W. This term is non-zero only in FIBOT, and its coefficient in (3.93) is then equal to $1/u_f$. Furthermore,

$$\frac{\rho(x_c)g_2(x_c)}{x_c}\left\{f(x_t) - f(x_c) + r_a\{s(x_t) - s(x_c)\}\right\} = V_L, \quad (3.96)$$

$$\frac{1}{\lambda_2} \frac{\rho(x_c)g_2(x_c)}{x_c}\{\phi(x_t) - \phi(x_c)\} - \frac{F_2 N\tau(x_c)}{2\pi x_c} = F_2 T_L. \quad (3.97)$$

Equation (3.93) may now be written

$$F_1 = r_a - \frac{1}{u_f} U_L + V_L + F_2 T_L + R_L r_a. \quad (3.98)$$

The meaning of (3.98) is that, in the optimum town, when labor is held constant, land is employed in factory production to the point where the value of its marginal product is equal to the opportunity cost of land plus its marginal contributions to the shadow cost of household consumption and transport and minus the shadow value of its marginal contribution to the total utility of the town.

There are two terms in F_1 which require further discussion. These are $R_L r_a$ and $\frac{F_2 N\tau(x_c)}{2\pi x_c}$. These terms arise from our assumption that transport is free in the CBD, and they introduce an ambiguity in the signs of terms on the right hand side of (3.98). This is because, as the CBD area increases, the width of the annular residential zone, *ceteris paribus*, decreases, and with transport in the CBD free, transport costs may thus fall. The assumption of free CBD transport is unrealistic. It is made for mathematical convenience, and had we made the much more plausible assumption of continuity of transport velocity across the CBD boundary, the two terms under discussion would have been negligible. We can, therefore, attach no theoretical significance to the ambiguity of sign which flows from their existence. As a consequence, where the ambiguity arises, we will facilitate our exposition by ignoring it.

We have already defined F_N. Its pair is

$$F_L \equiv \left(\frac{dF}{dL}\right)_N = F_L + \left(\frac{dW}{dL}\right)_N F_2 + r_a R_L,$$

$$= F_1 - T_L F_2 + r_a R_L,$$

$$= r_a - \frac{1}{u_f} U_L + V_L + r_a R_L. \qquad (3.99)$$

Equation (3.99) states that, given constant population and residential structure in the optimum town, the value of the marginal product of land is equal to the opportunity cost of land plus the marginal change in the shadow cost of total household consumption plus the value of the marginal change in the land allocated to transport minus $\frac{1}{u_f}$ times the marginal change in total utility in the optimum town.

3.10 RETURNS TO SCALE IN PRODUCTION

It has frequently been observed that there must be economies of scale in factory production to justify the costs of urban transport. This point cannot be disputed, and in any case, our model is not suitable for discovering corner solutions. However, this necessity for economies of scale to justify the existence of a town has frequently been assumed to imply that returns to scale must be increasing *at the optimal point*, and Starrett [1974, pp.420-1] has given an intuitive "proof" that returns to scale in factory production net of transport costs must be constant in the optimum town. Nevertheless, upon reflection we can see that it is not obvious that the need for economies to justify the existence of the optimum town precludes the possibility of their being exhausted at the optimum point, and Starrett's proof contains unrecognized assumptions which seriously limit the generality of his conclusions.

In this section we evaluate returns to scale using two measures. In the first case we obtain the measure which is conventional in non-locational models. This is returns to scale in terms of the factors directly employed in the factory. The resources necessary to assemble labor at the factory each day are ignored in this measure, and it can, therefore, be described as returns to scale in gross factory production (gross of transport costs). The second measure is returns to scale in net factory production.

In the neighborhood of the optimum point we write

$$\Theta(L,W)F(L,W) = LF_1 + WF_2 = LF_1 + NF_2 - TF_2. \tag{3.100}$$

Equations (3.91), (3.98) and (3.100) yield

$$(\Theta - 1)F = -\frac{1}{u_f}\{ LU_L + NU_N - U \} + LV_L + NV_N - V$$

$$+ F_2 \{ LT_L + NT_N - T \} + r_a \{LR_L + NR_N - R \} . \tag{3.101}$$

We see from (3.101) that the local degree of increasing returns to scale with respect to the factors L and W, at the optimum point, is expressed in terms of the degree of homogeneity of U, V, T and R, expressed as functions of L and N. From (3.88) and (3.90),

$$-\frac{1}{u_f}\{ U_N - \frac{U}{N}\} + V_N - \frac{V}{N} + F_2 \{ T_N - \frac{T}{N}\}$$

$$= -\frac{\lambda_1}{N\lambda_2}\{ u(x_t) - \bar{u} \} + \{ f(x_t) - \bar{f} \} + r_a \{s(x_t) - \bar{s} \}$$

$$+ \frac{1}{\lambda_2}\{ \phi(x_t) - \bar{\phi} \} ,$$

$$= \overline{r(x)s(x)} - r_a\bar{s} > 0, \tag{3.102}$$

by (3.21). Furthermore,

$$\frac{x_c}{\rho(x_c)g_2(x_c)}\{-\frac{1}{u_f}LU_L + LV_L + F_2LT_L\}$$

$$= -\frac{\lambda_1}{N\lambda_2}\{u(x_t) - u(x_c)\} + f(x_t) - f(x_c)$$

$$+ r_a\{s(x_t) - s(x_c)\} + \frac{1}{\lambda_2}\{\phi(x_t) - \phi(x_c)\},$$

$$= (r(x_c) - r_a)s(x_c) > 0. \tag{3.103}$$

However,

$$r_a\{LR_L + NR_N - R\} < 0. \tag{3.104}$$

It follows that returns to scale in gross factory production may be decreasing, constant or increasing in the neighborhood of the optimum point. The ambiguity arises from the economies of scale which exist in the use of the transport network.

We now examine returns to scale in net factory production. Analogous to (3.100) we write

$$\Phi(L,N)F(L,N) = LF_L + NF_N. \tag{3.105}$$

From (3.91), (3.99) and (3.105),

$$(\Phi - 1)F(L,N) = \frac{1}{u_f}\{LU_L + NU_N - U\} + LV_L + NV_N - V$$

$$+ r_a\{LR_L + NR_N - R\}. \tag{3.106}$$

Now define

$$\mathcal{F}(L,N) \equiv F(L,N) - Rr_a. \tag{3.107}$$

Then $\mathcal{F}(L,N)$ is factory production net of all transport costs, and from (3.106) and (3.107),

$$L\mathcal{F}_L + N\mathcal{F}_N - \mathcal{F} = -\frac{1}{u_f}\{ LU_L + NU_N - U \} + LV_L + NV_N - V. \tag{3.108}$$

From (3.21),

$$u(x) - u_f\{ f(x) + r_a s(x)\} = N\phi(x) + (u_s - r_a u_f)s(x). \tag{3.109}$$

Differentiating (3.109) totally, and using (3.45), we find that in FIBOT

$$\frac{d}{dx}\left\{u(x) - u_f\{ f(x) - r_a s(x)\}\right\} = (u_s - r_a u_f)\dot{s}(x) \geq 0, \tag{3.110}$$

with equality holding at x_t only.

In SEBOT, $\dot{u}(x) = 0$, hence,

$$\frac{d}{dx}\{ f(x) + r_a s(x) \} = \dot{f} + r_a \dot{s}(x),$$

$$= -\{ r(x) - r_a\}\dot{s}(x) \leq 0, \tag{3.111}$$

again with equality holding only at x_t.

It follows from the definitions of U and V that

$$-\frac{1}{u_f} LU_L + LV_L < 0, \tag{3.112}$$

$$-\frac{1}{u_f}\{NU_N - U\} + NV_N - V < 0, \tag{3.113}$$

and

$$\Phi - 1 < 0. \tag{3.114}$$

Therefore, *returns to scale in net factory production in the neighborhood of the optimum point are unambiguously decreasing.* The reason for this result is as follows. As we move outwards in the residential zone the relative price of residential space falls. There is, therefore, an increasing substitution of residential space for the factory good in household consumption as x increases. In SEBOT, this substitution has a form such that the shadow cost to the town of a household's consumption falls with x, and it is optimal to increase population beyond the point where economies of scale are exhausted. In FIBOT the explanation is similar, but involves additional discussion concerning the household's contribution to total utility in the town. In his proof that returns to scale in net production are constant in the optimum town, Starrett [1974] failed to take into consideration the substitution of living space for the factory good in household consumption. If this substitution does not take place,Starrett's conclusion is correct. In fact, in Chapter 5, where we assume s constant, we will see that returns to scale in net factory production are indeed constant.

This concludes the discussion of the optimum towns in which household leisure is constant.

3.11 CONCLUSIONS

In some respects our analysis has confirmed the conclusions of earlier normative studies of urban areas. For example, in respect to the structure of the residential zone, we confirm the conclusions of Mirrlees [1972] and Dixit [1973] that, unless an equality constraint is imposed upon the town, equals will be treated unequally at the optimum. However, we are also able to make some new, quite general statements with respect to population density as a function of distance from the centre of the town. Furthermore, we are able to solve the normative model of the factory town in which equals are treated equally. This model has not, hitherto, been solved in a general form, although Dixit's polar case is a restricted formulation of it.

Our results in respect to household consumption in the optimum towns may be summarised as follows. In both towns $\dot{r}(x) < 0$ and $\dot{s}(x) > 0$. In FIBOT $\dot{f}(x) > 0$ and $\dot{u}(x) > 0$. In SEBOT $\dot{u}(x) = 0$ by constraint and $\dot{f}(x) < 0$.

Also in the residential zone, we confirm the conclusions of Mills and de Ferranti [1971], Livesey [1973] and Sheshinski [1973] in respect to optimum congestion and the optimum allocation of land to the transport network. Some of their strong results, however, are shown to depend upon their simplifying assumptions. For example, their conclusion that optimal congestion is a linear function of distance from the centre of the town depends upon their assumption that residential site area is independent of location. Also, their conclusion that not all land will be allocated to roads depends upon their assumption that uncongested travel takes zero time.

Our conclusions in respect to town size, population and factory production are entirely new. In this field Starrett has offered an intuitive proof that returns to scale in factory production net of transport costs are constant. We show that this proof contains an implicit assumption which significantly reduces its generality. We find, in fact, that returns to scale in gross factory production may be increasing, constant or decreasing, but returns in net factory production are unambiguously decreasing. However, if the households are constrained to live on residential sites of exogenously determined area, Starrett's proof is valid, and returns to scale in net production are constant.

Throughout this chapter we have assumed that leisure time is constant with respect to household location. In the next chapter we will examine the role of leisure in the optimum town, and discover how our present conclusions depend upon this assumption of the constant leisure time.

CHAPTER 4

THE ROLE OF LEISURE IN THE OPTIMUM TOWN

In Chapter 3 we assumed that households are allocated the same, exogenously determined amount of leisure time regardless of their residential locations. This has become a standard assumption in the literature, and it probably has its origin in the implicit assumption that travel time is indistinguishable from work time to the laborer, since they both represent leisure time foregone. Nevertheless, the relevence of the uniform leisure time constraint to the theory of the optimum factory town remains debatable for two reasons. First, leisure time is consumed by households in the same sense as they consume the factory good and the services of residential space, and, by symmetry, it should be treated as a control variable in the optimization process. Second, if leisure time is not to be treated as a control variable, the optimum towns in which it is constrained to be independent of household location are second best optima of subordinate interest when viewed as representations of reality, because factory laborers are commonly observed to work fixed hours, independent of their residential locations, rather than enjoy fixed leisure time. In other words, the second best towns in which laborers work fixed hours are probably of more relevence to reality than the FIBOT and SEBOT analyzed in Chapter 3.

One writer who has not followed the general trend, but has treated leisure time as a control variable is Riley.[1] In two articles

1 In Riley [1973] and Riley [1974].

he has developed a model of the first best optimum town assuming household utility to be a function of the factory good, residential site area and leisure time. However, the generality of his conclusions is limited by his use of functions of specific form in his model. Furthermore, CBD size and population are constants in his model, and as a consequence, he does not derive relationships between marginal products and factor shadow costs.

In this chapter we will extend the model of Chapter 3 so as to treat leisure as a consumption good. To begin with we will reformulate the Riley problem in a more general context, and show that Riley's basic conclusions remain valid. We will find that the structure of his residential zone is not very different from the structure of FIBOT, except that leisure time, rather than being constant, is a strictly increasing function of distance from the centre of the town. This means that those households which live furthest from the CBD, and therefore spend most time in commuting, are allocated the greatest amount of leisure time.

The generalized Riley town is, strictly speaking, *the* optimum factory town, and therefore to be regarded as the datum against which the second best solutions are to be measured. For this reason we will derive the marginal products of population and factory land, and evaluate returns to scale in net factory production at the optimum point to confirm that these relationships have a functional form qualitatively similar to those derived for FIBOT and SEBOT.

We will then go on to examine the second best optimum in which equals are treated equally by constraint and every household contributes the same fixed amount of labor to the factory. This is the model which seems to correspond most closely to reality. However, its interest does not end there, because it also serves as another polar case, to be compared with the polar case, SEBOT, in which leisure time is fixed and equals are treated equally.

In the solution to this fixed labor time model we will find that, so far as the structure of the residential zone is concerned, some of the unambiguous conclusions we obtained in Chapter 3 no longer hold. We will continue to find that the shadow residential rent is unambiguously strictly decreasing with respect to distance from the centre of the town. This, of course, is as it must be, since the residential rent is a von Thünen rent. However, household residential site area and consumption of the factory good may each be of either slope in some parts of the residential zone. Furthermore, the shadow price of leisure and optimal congestion may be either increasing or decreasing within the residential zone. Finally, returns to scale in net factory production will be found to be increasing at the optimum point, and not decreasing as we have hitherto found.

We concentrate our attention on the solution to the equality case, because it seems to be the more interesting, and because we wish to avoid repetitious analysis. The method of solving the model when the equality constraint is relaxed is straightforward. However, for completeness, towards the end of this chapter, we state the major results obtained from the model of the optimum factory town when work hours are fixed and equals are not necessarily treated equally.

4.1 THE GENERALIZED RILEY PROBLEM

Using the notation established in Chapter 3, we assume that a consumption good is produced in a factory of area $L = \pi x_c^2$ according to the production function $F(L,W)$, and that the time taken for a laborer to commute across unit distance at x is $\tau(n(x), g_1(x))$. These are the assumptions we made in Chapter 3. We choose the day as the unit of time. The time constraint on the town is, therefore,[1]

$$N - W - \int_{x_c}^{x_t} n(x)\tau(n,g_1)dx - \int_{x_c}^{x_t} 2\pi\rho(x)g_2(x)\ell(x)dx = 0, \qquad (4.1)$$

where $\ell(x)$ is the amount of leisure time allocated to the household residing at the CBD boundary. In Chapter 3 we assumed ℓ to be constant, and defined the unit of time so as to make $\ell = 0$.

In (4.1)

$$T \equiv \int_{x_c}^{x_t} n(x)\tau(n,g_1)dx, \qquad (4.2)$$

is the total commuting time in the town, and

$$\Lambda \equiv \int_{x_c}^{x_t} 2\pi\rho(x)g_2(x)\ell(x)dx, \qquad (4.3)$$

is the total leisure time.

The income and land constraints on the town continue to be

$$F - Ar_a - \int_{x_c}^{x_t} 2\pi\rho(x)g_2(x)f(x)dx = 0, \qquad (4.4)$$

$$x - g_1(x) - g_2(x) = 0. \qquad (4.5)$$

Furthermore, equation (3.11) must continue to hold. That is,

---

[1] Following the analysis of Chapter 3, we will take as proven the proposition that it is always optimal to employ factors fully and to distribute all output.

$$\dot{n}(x) = -\ 2\pi\rho(x)g_2(x). \tag{4.6}$$

However, now we assume that households derive utility from the consumption of the factory good, the services of residential site area and leisure time. That is to say, $u = u(f,s,\ell)$.

Therefore, choosing the maximization of average household utility as the criterion of optimality, Riley's problem becomes: we wish to

$$\max_{g_1(x)g_2(x),f(x),s(x),\ell(x),L,A,N,W} \frac{1}{N}\int_{x_c}^{x_t} 2\pi\rho(x)g_2(x)u(f,s,\ell)dx,$$

subject to the constraints (4.1) and (4.3) - (4.6).

Defining[1]

$$\mathcal{L}(x) \equiv 2\pi\rho(x)g_2(x)\left\{\frac{u(f,s,\ell)}{N} - \lambda_1 f(x) - \lambda_2\ell(x) - \phi(x)\right\}$$

$$- \lambda_2 n(x)\tau(n,g_1) + \mu(x)\{x - g_1(x) - g_2(x)\}, \tag{4.7}$$

$$J \equiv \lambda_1(F - Ar_a) + \lambda_2(N - W), \tag{4.8}$$

the first order conditions for the maximum are:

$$\frac{u_f}{N} - \lambda_1 = 0, \tag{4.9}$$

$$\frac{u_\ell}{N} - \lambda_2 = 0, \tag{4.10}$$

---

[1] To avoid repetition of analysis, we will take as proven that $g_1(x) > 0$ except at x_t when it is zero, and limit our analysis to solutions in which $g_2(x) \geq 0$. Thus, we are able to omit the nonnegativity constraints on $g_1(x)$ and $g_2(x)$.

$$\frac{u(x)}{N} - \lambda_1 f(x) - \lambda_2 \ell(x) - \phi(x) - \frac{s(x)u_s}{N} = 0, \tag{4.11}$$

$$\lambda_2 n(x)\tau_{g_1} - \mu(x) = 0, \tag{4.12}$$

$$\frac{u(x)}{N} - \lambda_1 f(x) - \lambda_2 \ell(x) - \phi(x) - \frac{s(x)\mu(x)}{2\pi} = 0, \tag{4.13}$$

$$\dot{\phi}(x) = \lambda_2 \frac{\partial}{\partial n}(n(x)\tau(n,g_1)), \tag{4.14}$$

$$\lambda_1 F_2 - \lambda_2 = 0, \tag{4.15}$$

$$\lambda_1 F_1 - \frac{1}{2\pi x_c}\mathcal{L}(x_c) = 0, \tag{4.16}$$

$$\lambda_1 r_a - \frac{1}{2\pi x_t}\mathcal{L}(x_t) = 0, \tag{4.17}$$

$$\lambda_2 + \phi(x_c) - \frac{\bar{u}}{N} = 0. \tag{4.18}$$

In equations (4.7) to (4.18), $\phi(x)$ is the co-state variable and $\mu(x)$, λ_1 and λ_2 are the Lagrange multipliers.

We see from (4.9), (4.10) and (4.15) that the shadow price of leisure,

$$p \equiv \frac{u_\ell}{u_f} = \frac{\lambda_2}{\lambda_1} = F_2 \tag{4.19}$$

is constant across the residential zone and equal to the marginal product of labor.

Differentiating (4.11) totally with respect to x, and using (4.9) and (4.10)

$$s(x)\dot{r}(x) = s(x)\,\frac{d}{dx}\left(\frac{u_s}{u_f}\right) = \frac{1}{\lambda_1}\,\dot{\phi}(x), \tag{4.20}$$

$$= p\,\frac{\partial}{\partial n}\,(n\tau) < 0, \tag{4.21}$$

by (4.14). Similarly differentiating (4.9) and (4.10), and expanding out the derivative $\frac{d}{dx}\left(\frac{u_s}{u_f}\right)$ in (4.20),

$$\begin{pmatrix} u_{ff} & u_{fs} & u_{f\ell} \\ u_{sf} & u_{ss} & u_{s\ell} \\ u_{\ell f} & u_{\ell s} & u_{\ell\ell} \end{pmatrix} \begin{pmatrix} \dot{f} \\ \dot{s} \\ \dot{\ell} \end{pmatrix} = \begin{pmatrix} 0 \\ -\dfrac{\lambda_2 N}{s(x)}\,\dfrac{\partial}{\partial n}\,(n\tau) \\ 0 \end{pmatrix}. \tag{4.22}$$

Equation (4.22) has the same form as the equivalent matrix equation derived by Riley [1974, p.236]. Thus, Riley's basic qualitative results concerning the slopes of $f(x)$, $s(x)$, $\ell(x)$ and $r(x)$ in the residential zone are valid in a much more general context. The rent function is downward sloping, and $\dot{s}(x) > 0$. Furthermore, if we continue to make the quite reasonable assumption that the second order cross partial derivatives are all positive, $\dot{f}(x)$ and $\dot{\ell}(x)$ are both strictly positive, and so, therefore, is $\dot{u}(x)$. In other words, equals are treated unequally in Riley's optimum town.

The continuation of the solution to the Riley model is straightforward. It produces no results which are qualitatively different from the results obtained in Chapter 3, and therefore, we will not pursue the solution in detail. However, given that the Riley town is the first best town, and therefore of fundamental importance, we will evaluate the marginal products F_L and F_N and the local degree of returns to scale in the neighborhood of the optimum point. These results are of particular interest, because, although they do not differ significantly from those obtained in Chapter 3, where we assumed leisure time to be uniform and exogenously determined, they are different in form from the results we are to obtain when we constrain the work time to be fixed independent of household location.

From (4.15) and (4.18) we find, as we did in Chapter 3, that

$$F_2 = \frac{\bar{u}}{N} - \phi(x_c). \tag{4.23}$$

However, now, from (4.1) - (4.3),

$$\begin{aligned} F_N &= \{1 - T_N - \Lambda_N\} F_2 \\ &= \frac{1}{\lambda_1}\{\phi(x_t) - \phi(x_c)\} - p\ell(x_t). \end{aligned} \tag{4.24}$$

Therefore, from (4.11), (4.23) and (4.24)

$$\begin{aligned} F_N &= -\frac{u(x_t)}{u_f} + f(x_t) + r_a s(x_t) \\ &= \frac{F - Lr_a}{N} - \frac{1}{u_f}\left(U_N - \frac{U}{N}\right) + \left(V_N - \frac{V}{N}\right) + r_a\left\{R_N - \frac{R}{N}\right\}, \end{aligned} \tag{4.25}$$

where U and V are defined in (3.88) and (3.89), R is the land area allocated to transport and, as we showed in (3.92), $R_N = 0$. Similarly,

$$F_L = r_a - \frac{1}{u_f} U_L + V_L + r_a R_L, \tag{4.26}$$

and it is now easy to show that returns to scale in net factory production are unambiguously decreasing in the neighborhood of the optimum point.

4.2 THE OPTIMUM TOWN WITH FIXED WORKING HOURS AND EQUALITY BY CONSTRAINT

We now examine the structure of the optimum town in which factory hours are exogenously determined, and independent of the worker's residential location. Non-working time per day is shared between leisure and travel. Therefore,

$$\dot{\ell}(x) + \tau(n,g_1) = 0. \tag{4.27}$$

In addition, we can choose the unit of labor so as to make W = N. Therefore, the production function can be written $F = F(L,N)$.

Continuing to make the maximization of average household utility the criterion of optimality, we can now state our welfare problem in the following way. We wish to:

$$\max_{g_1(x),g_2(x),f(x),h(x),\bar{u},L,A,N} \bar{u},$$

subject to: (4.2), (4.3), (4.4), (4.27) and

$$\frac{1}{N}\int_{x_c}^{x_t} 2\pi\rho(x)g_2(x)u(f,s,\ell)dx - \bar{u} = 0, \tag{4.28}$$

$$u(f,s,\ell) - \bar{u} = 0. \tag{4.29}$$

We now have a problem in optimal control theory in which there are two state variables, $n(x)$ and $\ell(x)$. The Lagrangian is defined as

$$\mathcal{L}(x) \equiv 2\pi\rho(x)g_2(x)\{\lambda_0 \frac{u(f,s,\ell)}{N} - \lambda_1 f(x) - \phi_1(x)\}$$

$$- \phi_2(x)\tau(n,g_1) + \mu_1(x)\{x - g_1(x) - g_2(x)\}$$

$$+ \mu_2(x)\{u(f,s,\ell) - \bar{u}\}, \tag{4.30}$$

and

$$J \equiv \bar{u}(1 - \lambda_0) + \lambda_1\{F(L,N) - Ar_a\}. \tag{4.31}$$

In equations (4.30) and (4.31) we have again made use of the essential non-negativity of $g_1(x)$ with $g_1(x_t) = 0$, and assumed that $g_2(x)$ is strictly positive. We have also continued to use the notation established in Chapter 3 for the Lagrange multipliers, but now there are two co-state variables, $\phi_1(x)$ and $\phi_2(x)$ associated with the state variables $n(x)$ and $\ell(x)$, respectively.

The first order necessary conditions for a maximum are:

$$\phi_2(x)\tau_{g_1} + \mu_1(x) = 0, \tag{4.32}$$

$$2\pi\rho(x)\{\lambda_0 \frac{U}{N} - \lambda_1 f(x) - \phi_1(x)\} - \mu_1(x) = 0, \tag{4.33}$$

$$\psi(x)u_f - \lambda_1 = 0, \tag{4.34}$$

$$\lambda_0 \frac{u(x)}{N} - \lambda_1 f(x) - \psi(x)s(x)u_s - \phi_1(x) = 0, \tag{4.35}$$

$$\dot{\phi}_1(x) = \phi_2(x)\tau n, \tag{4.36}$$

$$\dot{\phi}_2(x) = -2\pi\rho(x)g_2(x)\psi(x)u_\ell, \tag{4.37}$$

$$1 - \lambda_0 - \int_{x_c}^{x_t} \mu_2(x)\,dx = 0, \tag{4.38}$$

$$\lambda_1 F_N - \lambda_0 \frac{u}{N} + \phi_1(x_c) = 0, \tag{4.39}$$

$$\lambda_1 F_L = \frac{\rho(x_c) g_2(x_c)}{x_c} \left\{ \lambda_0 \frac{u}{N} - \lambda_1 f(x_c) - \phi_1(x_c) \right\}$$

$$- \frac{1}{2\pi x_c} \phi_2(x_c) \tau(x_c), \tag{4.40}$$

$$\lambda_1 r_a = \rho(x_t) \left\{ \lambda_0 \frac{u}{N} - \lambda_1 f(x_t) - \phi_1(x_t) \right\}$$

$$- \frac{1}{2\pi x_t} \phi_2(x_t) \tau(x_t), \tag{4.41}$$

where

$$\psi(x) = \frac{\lambda_0}{N} + \frac{\mu_2(x) s(x)}{2\pi g_2(x)} . \tag{4.42}$$

Multiplying (4.34) by $\dfrac{2\pi\rho(x) g_2(x)}{u_f}$, using (4.42) and integrating across the residential zone, then using (4.38), we find $\lambda_1 > 0$. It follows from (4.34) that $\psi(x) > 0$. Now, from (4.35), $\lambda_0 \frac{u}{N} - \lambda_1 f(x) - \phi_1(x) > 0$, and, hence, by (4.33), $\mu_1(x) > 0$. Furthermore, from (4.32), $\phi_2(x) \geq 0$ throughout the residential zone. Now, dividing (4.35) by (4.34), and differentiating,

$$s(x)\dot{r}(x) = s(x) \frac{d}{dx}\left(\frac{u_s}{u_f}\right) = -\frac{1}{\lambda_1}\left\{ \dot{\phi}_1(x) + \lambda_1 \dot{f}(x) + r(x)\dot{s}(x) \right\}. \tag{4.43}$$

However,

$$\frac{1}{u_f}\frac{du}{dx} = \dot{f}(x) + r(x)\dot{s}(x) + p(x)\dot{\ell}(x) = 0, \tag{4.44}$$

where $p(x) = \dfrac{u_\ell}{u_f}$. Therefore, by (4.27), (4.43) and (4.44),

$$s(x)\dot{r}(x) = -\frac{1}{\lambda_1}\left\{\dot{\phi}_1(x) + \lambda_1 p(x)\tau(x)\right\} < 0. \tag{4.45}$$

Using the definition of r(x) and (4.44),

$$u_f\dot{r}(x) = -\begin{vmatrix} 0 & u_f & u_s \\ u_f & u_{ff} & u_{fs} \\ u_s & u_{sf} & u_{ss} \end{vmatrix}\dot{s}(x) - \begin{vmatrix} 0 & u_f & u_\ell \\ u_f & u_{ff} & u_{f\ell} \\ u_s & u_{sf} & u_{s\ell} \end{vmatrix}\dot{\ell}(x) < 0. \tag{4.46}$$

The utility function is quasi-concave, therefore, the determinant coefficient of $\dot{s}(x)$ is non-negative. However, we cannot put a sign on the coefficient of $\dot{\ell}(x)$, and therefore $\dot{s}(x)$ can take either sign. It follows, again because $\dot{u}(x) = 0$, that $\dot{f}(x)$ can take either sign.

We have proved that, although the price of residential land falls unambiguously with distance from the centre of the town, residential site area does not necessarily increase. The possibility of $\dot{s}(x) < 0$ arises from the fact that household leisure decreases unambiguously, and the way in which the optimum consumption bundle varies with distance to maintain equality depends upon the relative magnitudes of the second order partial derivatives of the utility function.

If we represent the coefficient of $\dot{s}(x)$ by Δ, expand out the coefficient of $\dot{\ell}$, and use (4.27) and (4.45), equation (4.46) may be written

$$\Delta\dot{s}(x) = -\{u_f u_s u_{f\ell} + u_f u_\ell u_{fs} - u_s u_\ell u_{ff}\}\dot{\ell}(x)$$
$$+ \frac{u_f^3}{\lambda_1 s}\dot{\phi}_1(x) - u_f^2\left\{\frac{u_\ell}{s} - u_{s\ell}\right\}\dot{\ell}(x). \qquad (4.47)$$

Thus, a sufficient condition for $\dot{s}(x) > 0$ is

$$\frac{u_\ell}{s} - u_{s\ell} = \frac{u_\ell}{s} - \frac{\partial u_\ell}{\partial s} \geq 0,$$

which is equivalent to the inequality

$$\frac{\partial}{\partial s}\left(\frac{u_\ell}{s}\right) \leq 0. \qquad (4.48)$$

In other words, $\dot{s}(x)$ will certainly be positive if the shape of the household utility function is such as to ensure that the marginal utility of leisure time per unit of residential area is a monotone decreasing function of residential site area. However, there does not seem to be any particular reason why the utility function should necessarily have this shape.

Since $p(x) = \dfrac{u_\ell}{u_f}$,

$$u_f^3\dot{p}(x) = -\begin{vmatrix} 0 & u_f & u_\ell \\ u_f & u_{ff} & u_{\ell f} \\ u_\ell & u_{f\ell} & u_{\ell\ell} \end{vmatrix}\dot{\ell}(x) - \begin{vmatrix} 0 & u_f & u_\ell \\ u_f & u_{ff} & u_{\ell f} \\ u_s & u_{fs} & u_{\ell s} \end{vmatrix}\dot{s}(x).$$

(4.49)

In (4.49) $\dot{s}(x)$ and its coefficient can take either sign, and therefore, so can $\dot{p}(x)$. Comparing (4.46) and (4.49), we see that it is only in the special case where the coefficient of $\dot{\ell}(x)$ in (4.46) is zero that the signs of $\dot{p}(x)$ and $\dot{s}(x)$ are determined to be unambiguously positive.

We will now assume that (3.61) holds. That is,

$$\tau(n,g_1) = \tau_0 + \tau_1\theta^{\alpha}(n,g_1), \tag{4.50}$$

where $\theta = n(x)/g_1(x)$, and τ_0, τ_1 and α are positive constants. Congestion must be bounded within the residential zone. Assume, further, that it is bounded at x_t where $n(x)$ and $g_1(x)$ are both zero. Then, $\lim_{x \to x_t} \theta(x)$ is bounded above, and therefore,

$$\lim_{x \to x_t} \tau_{g_1}(x) = -\infty. \tag{4.51}$$

Now putting $x = x_t$ in (4.32), (4.33) and (4.35), we find $\phi_2(x_t) = 0$. Equation (4.37) may now be integrated to give

$$\phi_2(x) = \lambda_1 n(x)\,\bar{p}(x), \tag{4.52}$$

where

$$n(x)\,\bar{p}(x) = \int_x^{x_t} 2\pi\rho(z)g_2(z)p(z)\,dz. \tag{4.53}$$

By definition, $p(x) \equiv u_\ell(x)/u_f(x)$ is the subjective value of leisure, at the margin, to the household which resides at x. Therefore $n(x)\bar{p}(x)$ is the total subjective value of the leisure time lost when all commuters passing through x are delayed by unit time, and $\bar{p}(x)$ is the marginal subjective value of leisure averaged over all those households which commute through x. It follows that $\frac{1}{\lambda_1}\phi_2(x)\tau_n(x)dx$ is that part of the shadow cost of adding one more commuter between x and x + dx which is external to that marginal commuter. That is to say, it is the congestion cost imposed upon the rest of the town by the marginal commuter's trip from x to x + dx.

Using (4.36), and using an argument similar to that used in Section 7 of Chapter 3, we deduce that

$$\frac{1}{\lambda_1}\left\{\phi_1(x_t) - \phi_1(x_c)\right\} + \int_{x_c}^{x_t} p(x)\tau(n,g_1)dx = F_N T_N. \tag{4.54}$$

is the marginal transport cost of population. As in Chapter 3, this partial derivative with respect to N implies that L and the structure of the residential zone remain constant.

Equations (4.36), (4.45) and (4.52) yield

$$s(x)\dot{r}(x) = -\{\, p(x)\tau(x) + n(x)\tau_n\bar{p}(x)\} . \tag{4.55}$$

Also, from (4.32) - (4.35), (4.50) and (4.52),

$$r(x) = \frac{\alpha\tau_1}{2\pi}\,\theta^{\alpha+1}(x)\bar{p}(x). \tag{4.56}$$

From (4.52)

$$\frac{d}{dx}\,(\bar{p}(x)) = -\,\frac{\dot{n}(x)}{n(x)}\left\{\bar{p}(x) - p(x)\right\}. \tag{4.57}$$

Therefore, differentiating (4.56),

$$\dot{r}(x) = \frac{\alpha\tau_1}{2\pi}(\alpha+1)\,\theta^{\alpha}(x)\dot{\theta}(x)\bar{p}(x) - \frac{\dot{n}(x)}{n(x)}\theta^{\alpha+1}(x)\{\bar{p}(x) - p(x)\}. \tag{4.58}$$

Eliminating $\dot{r}(x)$ from (4.55) and (4.58)

$$\frac{1}{2\pi}\alpha(\alpha+1)\tau_1\bar{p}(x)\theta^{\alpha+1}(x)\dot{\theta}(x) = -\tau_1\alpha\theta^{\alpha}(x)\bar{p}(x) - p(x)\tau(x) - \alpha\tau_1\frac{g_2(x)}{g_1(x)}\theta^{\alpha}(x)\{\bar{p}(x) - p(x)\}. \tag{4.59}$$

From (4.53), $\dot{p}(x) \geq 0$ implies $\bar{p}(x) \geq p(x)$. Therefore, by (4.59), $\dot{p}(x) \geq 0$ implies $\dot{\theta}(x) < 0$, unambiguously. However, if $\dot{p}(x) < 0$, the sign of $\dot{\theta}(x)$ may be positive. In other words, we are unable to rule out the possibility, as we were in Chapter 3, that optimal congestion could increase with distance from the centre of the town. However, optimal congestion can be upwards sloping only where the shadow price of leisure is downward sloping.

Our results suggest that, in regions where the shadow price of leisure is falling, it may be optimal to increase the allocation of land to residential purposes and decrease the allocation of land to transport, even though congestion and hence travel time per unit distance may be thus increased.

4.3 RETURNS TO SCALE IN FACTORY PRODUCTION

We have already shown that returns to scale in net factory production in Riley's optimum town are unambiguously decreasing. This result agrees in qualitative terms with the results obtained for FIBOT and SEBOT in Chapter 3. We now derive expressions for the marginal products of land and population at the optimum point when all households supply the same exogenously determined amount of labor to the factory.

We have proven that $\phi_2(x_t) = 0$. It follows from a comparison of (4.35), when $x = x_t$, and (4.41) that

$$r(x_t) = r_a. \tag{4.60}$$

Equations (4.34), (4.35), (4.39) and (4.60) yield

$$F_N = f(x_t) + r_a s(x_t) + \frac{1}{\lambda_1}\{\phi_1(x_t) - \phi_1(x_c)\} . \tag{4.61}$$

In the notation of Chapter 3, the first two terms on the right hand side of (4.61) equal V_N, and, given that household utility is independent of location in our present town, these are the terms we obtained for the Riley town at equation (4.25), and for FIBOT and SEBOT in Chapter 3. However, we now have an additional term on the right hand side, which, after (4.53), we identified as being that part of the marginal household's shadow cost of transport that is external to it. The meaning of (4.61), therefore, is that the shadow value of the marginal product of labor is equal to the shadow value of the marginal household's consumption *plus* the shadow congestion cost the marginal household imposes upon the other households of the town. The difference between our earlier results and this present result arises from the fact that the uniform working hours constraint implies an identity between the marginal product of labor and the marginal product of population, which did not exist in our earlier models. In the earlier models, the marginal product of population was defined in terms of output per *day*, while the marginal product of labor was defined in terms of output per *working hour*. When travel times were increased by the addition to the town of the marginal household, labor hours were reduced, either because leisure was constrained to remain constant, or because the subjective price of leisure was constant. As a result F_2 increased and F_N decreased. Now there is a fixed relationship between the hour and the day, and working time cannot be adjusted to compensate for the disutility experienced by households when their travel times are increased by the addition of the marginal household. The marginal household must, therefore, produce the shadow value of this lost utility in addition to the shadow value of its own consumption.

The marginal product of factory land, F_L, is obtained directly from (4.40). Adding (4.41) to (4.40), after setting $\phi_2(x_t) = 0$, and using (4.52), we obtain

$$F_L = r_a + \frac{\rho(x_c)g_2(x_c)}{x_c}\left\{ f(x_t) - f(x_c) + r_a\{s(x_t) - s(x_c)\} + \frac{1}{\lambda_1}\{\phi_1(x_t) - \phi_1(x_c)\}\right\} - \frac{g_1(x_c)}{x_c} r_a - \frac{N}{2\pi x_c}\bar{p}(x_c)\tau(x_c), \tag{4.62}$$

$$= r_a + V_L + \frac{1}{\lambda_1}\frac{\rho(x_c)g_2(x_c)}{x_c}\{\phi_1(x_t) - \phi_1(x_c)\} - \frac{N}{2\pi x_c}\bar{p}(x_c)\tau(x_c) - r_a R_L. \tag{4.63}$$

Compared with equations (3.98) and (4.25) we have the additional terms

$$\frac{1}{\lambda_1}\frac{\rho(x_c)g_2(x_c)}{x_c}\{\phi_1(x_t) - \phi_1(x_c)\} - \frac{N}{2\pi x_c}\bar{p}(x_c)\tau(x_c)$$

in (4.63). These terms represent those transport shadow costs, which are gene ated by transferring unit area of land from the residential zone to the CBD, and which are external to the $\rho(x_c)g_2(x_c)/x_c$ households who are displaced from x_c to x_t by the transfer. It is only the external costs which appear in (4.63), because the loss of the displaced households' leisure time is compensated for in the change in their consumptions. This latter change is implicit in the term V_L.

We can use (4.61) and (4.63) to evaluate the local degree of increasing returns to scale at the optimum point. We let

$$\Phi \mathcal{F} = L\mathcal{F}_L + N\mathcal{F}_N, \tag{4.64}$$

where

$$\mathcal{F} = F - r_a R. \tag{4.65}$$

Then, neglecting the term

$$-\frac{N}{2\pi x_c}\bar{p}(x_c)\tau(x_c),$$

because it has no theoretical interest,

$$(\Phi - 1)\mathcal{F} = LV_L + NV_N - V$$

$$+ \left\{\frac{N}{\lambda_1} + \frac{L}{\lambda_1}\frac{\rho(x_c)g_2(x_c)}{x_c}\right\}\left\{\phi_1(x_t) - \phi_1(x_c)\right\}. \tag{4.66}$$

To evaluate the sign of $\Phi - 1$ we have, using (4.35) and (4.36),

$$V_N - \frac{V}{N} + \frac{1}{\lambda_1}\{\phi_1(x_t) - \phi_1(x_c)\} > f(x_t) + r_a s(x_t) + \frac{1}{\lambda_1}\phi_1(x_t)$$

$$- \{\bar{f} + r_a \bar{s} + \frac{1}{\lambda_1}\bar{\phi}\}$$

$$= \overline{r(x)s(x)} - r_a \bar{s},$$

$$= \int_{x_c}^{x_t} 2\pi\rho(x) g_2(x)\{r(x) - r_a\} s(x) dx,$$

$$> 0. \tag{4.67}$$

Similarly,

$$\frac{x_c}{\rho(x_c)g_2(x_c)}\left\{V_L + \frac{1}{\lambda_1}\frac{\rho(x_c)g_2(x_c)}{x_c}\{\phi_1(x_t) - \phi_1(x_c)\}\right\},$$

$$= f(x_t) + r_a(x_t) + \frac{1}{\lambda_1}\phi_1(x_t)$$

$$- \{f(x_c) + r_a s(x_c) + \frac{1}{\lambda_1}\phi_1(x_c)\},$$

$$= (r(x_c) - r_a)s(x_c) > 0. \tag{4.68}$$

It follows that

$$(\Phi - 1) > 0, \tag{4.69}$$

and returns to scale in net factory production are increasing in the neighborhood of the optimum point. This is a fundamentally different result from those obtained from our earlier models, where returns to scale were found to be unambiguously decreasing. We will discuss this difference in Section 4.5.

4.4 THE OPTIMUM TOWN WITH FIXED WORKING HOURS AND NO EQUALITY CONSTRAINT

We will now state the main results derived from the model of the optimum town in which working hours are fixed and no equality constraint is imposed in respect to the treatment of the resident households. The necessary conditions for an optimum in this case can be derived from equations (4.32) - (4.42) by setting $\mu_2(x) \equiv 0$.

From (4.27), (4.34) and (4.35) we then obtain

$$\begin{pmatrix} u_{ff} & u_{fs} & u_{f\ell} \\ u_{sf} & u_{ss} & u_{s\ell} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{f}(x) \\ \dot{s}(x) \\ \dot{\ell}(x) \end{pmatrix} = \begin{pmatrix} 0 \\ -X(x) \\ -Y(x) \end{pmatrix} \tag{4.70}$$

where X and Y are strictly positive functions of x. Solving (4.70), we find that $\dot{s}(x)$ and $\dot{f}(x)$ can take either sign. These are the conclusions we obtained when the equality constraint was imposed.

In respect to the value of the marginal products in the optimum town, we find that, analogous to (4.61),

$$F_N = -\frac{1}{u_f}\{ u(x_t) - \bar{u} \} + f(x_t) + r_a s(x_t)$$

$$+ \frac{1}{\lambda_1}\{ \phi_1(x_t) - \phi_1(x_c) \}, \tag{4.71}$$

and analogous to (4.62),

$$F_L = r_a + \frac{\rho(x_c) g_2(x_c)}{x_c} \Bigg\{ - \frac{1}{u_f} \{ u(x_t) - u(x_c) \}$$

$$+ f(x_t) - f(x_c) + r_a \{ s(x_t) - s(x_c) \}$$

$$+ \frac{1}{\lambda_1} \{ \phi_1(x_t) - \phi_1(x_c) \} \Bigg\} - \frac{g_1(x_c)}{x_c} r_a$$

$$- \frac{N}{2\pi x_c} \bar{p}(x_c) \tau(x_c). \qquad (4.72)$$

Equations (4.71) and (4.72) differ in form from (4.61) and (4.62) only in the terms in $(u(x_t) - \bar{u})$ and $\{u(x_t) - u(x_c)\}$, which, of course, are zero when the equality constraint is imposed. Finally, it follows from (4.71) and (4.72) that returns to scale in factory production net of transport costs are increasing at the optimum point. This is the result we obtained for the case where working hours are fixed and equality of treatment of households is imposed as a constraint.

4.5 CONCLUSIONS

This is an appropriate point to make comparisons between the solutions we have obtained from our four main models. Firstly, comparing the Riley town with FIBOT, we observe that leisure time is allocated to households differently. In FIBOT, the equality of leisure constraint is equivalent to assuming that an exogenously determined amount of leisure is paramount to each household, and, as a consequence, a shadow price of leisure is not discovered in the solution. In the Riley town, on the other hand, leisure time is allocated optimally, and because it is optimal to treat households unequally, the leisure allocation is a strictly increasing function of distance from the centre of the town. Furthermore, the shadow price of leisure is discovered in the solution. Since leisure has no special locational properties this price is found to be independent of location. Apart from these differences, FIBOT and the Riley town are seen to have the same qualitative form.

Comparing FIBOT and SEBOT, we find that, apart from the obvious difference that household utility is location dependent in FIBOT and location independent in SEBOT, the two solutions differ qualitatively only in so far as the household allocation of the factory good increases with respect to distance in FIBOT, and decreases in SEBOT. This difference is directly related to the inequality in FIBOT, and the equality in SEBOT.

In all three models compared so far, the functions representing household allocations of the consumption goods and transport congestion in the residential zone have been shown to have slopes whose signs can be determined unambiguously for the quite general production, utility and transport functions we have used. Furthermore, returns to scale in net factory production are found to be strictly decreasing.

In our fourth model, that is, the town in which labor works fixed hours and equals are treated equally, the strong results with respect to the structure of the residential zone are not obtained. In fact, the only unambiguous result we obtain in our description of the residential zone is that the von Thünen, residential rent is strictly decreasing. This result, of course, was an inevitable consequence of our observation that $\tau(x) > 0$. In addition, returns to scale in net factory production are shown to be strictly increasing.

The ambiguities in the structure of the residential zone can be directly related to the working hours constraint. That constraint implies that leisure time falls with distance from the centre of the town. With a downward sloping leisure time allocation, household utility may be held constant within the residential zone by upward sloping factory goods and residential site area allocations, or by making just one of these upward sloping and the other downward sloping. The shapes of the optimal allocations from these three possibilities will depend upon the shape of the household utility function. The ambiguities arise from this dependence. None of these ambiguities could occur in the earlier models, because leisure time is constant in FIBOT and SEBOT, and the optimality of inequality ensures that all allocations increase with distance from the centre of the town in Riley's optimum town.

The difference found in optimal returns to scale in net production arises from the way in which household utility is adjusted against the transport congestion costs imposed by the marginal household. In FIBOT and SEBOT leisure time is fixed, and therefore, transport congestion is reflected in working hours and the marginal product of labor. The marginal household must produce the value of its consumption (with an adjustment for differences in household utility in the case of FIBOT) while working comparatively short hours. In the Riley optimum town the process is much the same, since the shadow price of leisure is equated to the marginal product of labor. In our fourth model, however, the constant working hours constraint rules out any adjustment process through maintaining leisure time. The transport congestion costs imposed upon the town by the marginal household can only be compensated for by production of the factory good by the marginal household. However, in this case the marginal household works comparatively long hours. The marginal product of factory land is expressible in terms of the way in which land allocation to the factory forces the displacement of residents from the CBD boundary to the town boundary, thus making them "marginal" households. For this reason we find that F_L for the fourth model contains a transport congestion term which makes it different in form from that found for the first three models. We then find that these additional terms in the marginal conditions ensure that the local returns to scale at the optimum point are strictly increasing when working hours are fixed.

This concludes our analysis of the optimum factory town.

CHAPTER 5

COMPARATIVE STATIC ANALYSES OF THE OPTIMUM TOWN

We now turn to some comparative static analyses of the optimum factory town. During our survey of the literature we recorded that Wheaton [1974] has presented comparative static analyses of two equilibrium towns. In one of these the population is determined exogenously and the town is closed to the movement of households. The other is an open town, and population is determined endogenously. In his closed town Wheaton uses household income, the opportunity cost of land, population and the transport rate (that is,the cost of transporting one household unit distance) as shift parameters, and he examines the effects of these shifts on the residential rent and the population density, both of which are variables in his model, and on the household utility and the town's radius, which are parameters. In his open town he changes the roles of population and household utility. That is to say, household utility is used as a shift parameter and population is endogenous.

It is noteworthy that Wheaton is able to put unambiguous signs on all his results. However, this lack of ambiguity seems very largely to be a consequence of the assumptions which underlie his model. For example, his assumption that per capita income is exogenously determined (and therefore independent of population) is too strong. It rules out justification for the existence of the town in the usual sense, because, if CBD size were endogenously determined, transport costs could be saved without reduction in per capita income by limiting the population of the town to one household. That is, by organizing the economy for individual production by households. Therefore, Wheaton's result, that the derivative of household utility with respect to population is negative for all values of population in the equilibrium town, appears to be almost a re-statement of his income assumption, and several more of his results appear also to rest directly upon the lack of justification for the existence of the town.

In this chapter we will attempt to avoid this pitfall by setting up a simple model in which per capita income is determined endogenously, and use it for some comparative static analyses. Normative models are usually easier to formulate and to solve than equilibrium models, and, for this reason, we will restrict ourselves to an examination of the optimum town. However, subject to some restrictions, competitive realization of the optimum town has been shown to be possible, hence our results will have fairly general application.

Since income must be endogenous, it is necessary to use a model in which the land input to factory production and the population are also endogenous. This implies complexity in the structure of the model. However, considerable simplification of the models developed in Chapters 3 and 4 is possible. For the purpose of establishing principles we will use FIBOT, because we have seen that optimal consumption of leisure does not add much of significance to the model. Furthermore, we will assume that the population density in the residential zone, $\rho(x)$, is equal to the constant ρ, independent of location. We will assume there is no land input to transport, and that the time taken for a household to travel unit distance at x is τ, also independent of location. This relation holds in the CBD as well as the residential zone.[1]

Our simplifying assumptions are not entirely artificial. The assumption of uniform population density has been used by Mills and de Ferranti [1971], Livesey [1973] and Sheshinski [1973] in their studies of optimal congestion costs. The optimum town in which population density is uniform is an interesting second best town in its own right, because when we observe new towns or new suburbs of old towns, we tend to find a uniformity in residential site area, which is equivalent to a uniform population density. This uniformity arises, not so much from the market forces which determine the residential land rent, but from the controls of planning authorities. It would be going too far to argue that a residential area which was developed during a period of constant planning fashion has precisely uniform population density, but we do find that population densities can be fairly easily fitted into a very few classes; low density detached housing, medium density low rise apartment housing and high density high rise apartment housing for example. Given this observation, the uniform residential density optimum town becomes a first approximation to the optimum town under planning control - one more polar case.

The assumption that land is not an input to transport is also of interest, because, given uniform population density, it is equivalent to assuming that each household requires the same road area. We very quickly reach the point of indivisibility in the design of residential streets. The large majority of roadways in residential areas are two lanes wide because they cannot be narrower. Therefore, in assuming that there is no land input to transport, and choosing a population density

---

1 In much of our analysis we could generalize by putting $\tau = \tau(x)$. However, this device does not alter in any substantial way the results we obtain. We could also generalize by considering the case where laborers travel only to the CBD boundary, and the case where worker density in the CBD is constant. Again it can be shown that these cases do not add to our conclusions.

which takes into consideration a constant road area per household, we capture much of the effects of indivisibilities in roadways.

5.1 THE OPTIMUM TOWN

In adopting FIBOT to our purposes, equations (3.1) and (3.2) continue to hold. That is,

$$L = \pi x_c^2, \tag{5.1}$$

$$A = \pi x_t^2. \tag{5.2}$$

However, if we assume that all workers commute to the centre of the town, we now have

$$T = \int_{x_c}^{x_t} 2\pi x^2 \rho dx, \tag{5.3}$$

because the time taken for a household resident at x to commute to the centre of the town is $\int_0^x \tau dz = \tau x$. The population constraint on FIBOT continues to be

$$N - W - T = 0. \tag{5.4}$$

All households reside in the residential zone. Therefore, the residential land constraint may be written

$$\rho(A - L) - N = 0. \tag{5.5}$$

Households derive utility from the consumption of the factory good and the services of residential site area. However, since the residential density is constant, household utility is a function of the single variable $f(x)$. That is, $u = u(f(x))$. The income constraint on the town is

$$F - Ar_a - \int_{x_c}^{x_t} 2\pi\rho x f(x)\,dx = 0. \tag{5.6}$$

The welfare problem can now be stated as: we wish to

$$\max_{f(x),W,A,L,N} \frac{1}{N}\int_{x_c}^{x_t} 2\pi\rho x u(f(x))\,dx,$$

subject to the constraints (5.3) - (5.6).

It is easy to show that one of the necessary conditions for the maximization implies

$$\dot{f}(x) = 0. \tag{5.7}$$

That is, factory output net of rent payments to absentee landlords must be equally distributed between households. This implies

$$\dot{u}(x) = 0. \tag{5.8}$$

Equals are treated equally in this optimum town. We would have expected to obtain this result, since the inward movement of the centres of household residential sites, which reduced transport costs, and thus made it optimal to treat equals unequally in the general FIBOT, cannot occur when the population density is constrained to be uniform across the residential zone.

The necessary condition (5.7) makes it possible for us to restate the welfare problem in a form which will enable us to derive the second order sufficiency conditions. Average utility is maximized if we

$$\max_{W,A,L,N} \frac{F - Ar_a}{N},$$

subject to the constraints (5.3) - (5.5).

Before proceeding to the solution it will be useful to set down some notation. We continue to use F_1 and F_2 to represent the partial derivatives of F(L,W) with respect to its arguments. However, we can use (5.4) to eliminate W, in which case we can obtain the derivatives

$$\frac{\partial F}{\partial L} \equiv \frac{\partial}{\partial L} F(L,A,N), \quad \frac{\partial F}{\partial A} \equiv \frac{\partial}{\partial A} F(L,A,N), \quad \frac{\partial F}{\partial N} \equiv \frac{\partial}{\partial N} F(L,A,N). \tag{5.9}$$

The derivatives F_L and F_N defined in Chapter 3 may now be written

$$F_L \equiv \left(\frac{dF}{dL}\right)_N = \frac{\partial F}{\partial L} + \frac{\partial F}{\partial A}, \tag{5.10}$$

$$F_N \equiv \left(\frac{dF}{dN}\right)_L = \frac{\partial F}{\partial N} + \frac{1}{\rho}\frac{\partial F}{\partial A}. \tag{5.11}$$

Similarly,

$$T_L \equiv \left(\frac{dT}{dL}\right)_N = \frac{\partial T}{\partial L} + \frac{\partial T}{\partial A}, \tag{5.12}$$

$$T_N \equiv \left(\frac{dT}{dN}\right)_L = \frac{1}{\rho}\frac{\partial T}{\partial A}. \tag{5.13}$$

The welfare problem is a simple problem in constrained maximization. The first order necessary conditions are by now familiar:

$$F_1 = r_a + T_L F_2; \tag{5.14}$$

$$(1 - T_N)F_2 = \frac{F - Lr_a}{N}. \tag{5.15}$$

It is obvious from these conditions that returns to scale in gross factory production are unambiguously increasing.

The conditions (5.14) and (5.15) can be re-stated as

$$F_L = r_a, \tag{5.16}$$

$$F_N = \frac{F - Lr_a}{N}, \tag{5.17}$$

which means that returns to scale in net factory production are constant. Again, this could be expected, since household substitution between consumption goods cannot occur.

Finally, the second order sufficient conditions are obtained in the standard way. We find these reduce to

$$|H| \equiv F_{LL}F_{NN} - F_{LN}^2 > 0, \tag{5.18}$$

$$F_{LL} < 0, \quad F_{NN} < 0, \tag{5.19}$$

That is $F(L,N)$ must be strictly concave. For the remainder of this chapter we will assume that (5.18) and (5.19) hold, and that $F_{LN} \geq 0$.

5.2 COMPARATIVE STATIC ANALYSES

5.2.1 The Opportunity Cost of the Land

If the population of the town is fixed, differentiating (5.16) with respect to r_a gives

$$\frac{dL}{dr_a} = \frac{1}{F_{LL}} < 0. \tag{5.20}$$

Hence,

$$\frac{dW}{dr_a} = - T_L \frac{dL}{dr_a} > 0, \quad \frac{dA}{dr_a} < 0. \tag{5.21}$$

Similarly, when the area of the CBD is fixed, (5.17) yields

$$\frac{dN}{dr_a} = - \frac{1}{F_{NN}} \frac{L}{N} > 0, \tag{5.22}$$

$$\frac{dW}{dr_a} = (1 - T_N) \frac{dN}{dr_a} > 0 , \frac{dA}{dr_a} > 0. \tag{5.23}$$

The meanings of the results (5.20) - (5.23) are straightforward: an increase in the opportunity cost of land, *ceteris paribus*, will always result in a decrease in the land/labor ratio in factory production at the optimum. However, this does not mean that the town will occupy less land. If the area of the CBD is fixed, the land/labor ratio in production is decreased by increasing the population of the town. This implies an increase in the residential zone, and hence, in the area of the town.

When both L and N are free, (5.16) and (5.17) yield

$$\frac{dL}{dr_a} \quad \frac{1}{N|H|}\left(LF_{LN} + NF_{NN}\right), \tag{5.24}$$

$$\frac{dN}{dr_a} \quad \frac{-1}{N|H|}\left(LF_{LL} + NF_{LN}\right). \tag{5.25}$$

The right hand sides of both (5.24) and (5.25) can be of either sign. Equation (5.24), therefore, implies that an increase in r_a can result in an increase in the use of land as a factor of production in the optimum town. However,

$$|H| \frac{dN}{dr_a} = -\left(LF_{LL} + NF_{LN}\right),$$

$$> -\frac{F_{LL}}{F_{LN}}\left(LF_{LN} + NF_{NN}\right) = -\frac{F_{LL}}{F_{LN}} |H| \frac{dL}{dr_a}. \tag{5.26}$$

That is, if an increase in the opportunity cost of land induces an increase in the use of land as a factor of production, it also induces an increase in the population of the town. Furthermore, if $\frac{dL}{dr_a} > 0$,

$$\frac{L}{N} \frac{\frac{dN}{dr_a}}{\frac{dL}{dr_a}} - 1 > -\frac{1}{F_{LN}}\left(LF_{LL} + NF_{LN}\right) > 0. \tag{5.27}$$

That is to say, if an increase in r_a results in increased use of land as a factor of production, this increase is always associated with a decrease in the land/labor ratio in production. It follows that the land/population ratio of the town also decreases.

It is easily shown that both $\frac{dW}{dr_a}$ and $\frac{dA}{dr_a}$ can take either sign. We, therefore, conclude that, when both L and N are free, no general statement can be made concerning those changes in the dimensions and population of the optimum town which are induced by a change in the opportunity cost of land. Observe that we have one additional parameter over those usually found in a problem in comparative statics. It is the optimal degree of increasing returns to scale. It can easily be checked that the induced change in this parameter due to an upward shift in the opportunity cost of land can be of either sign. The implication of this result is that a shift in r_a not only induces a change in the optimal factor ratio in production, but it also induces a change in the *scale* of production. It is this latter induced change which leads to the ambiguous results we obtain when both factors of production are free.

Using the labor and land constraints (5.4) and (5.5),

$$\frac{d}{dr_a}\left(\frac{F - Ar_a}{N}\right) = \frac{\partial}{\partial r_a}\left\{\frac{F(L,A,N) - Ar_a}{N} + \lambda\{\rho(A - L) - N\}\right\},$$

$$= -\frac{A}{N} < 0 , \tag{5.28}$$

where λ is the Lagrange multiplier associated with the constraint (5.5). Thus, an increase in r_a always results in a fall in household consumption of the factory produced good (and, therefore, in household utility), irrespective of the signs of the changes in the structure of the town. This result was, of course, to be expected.

5.2.2 Travel Time

We will now examine shifts in τ, which represents the time taken for a commuter to travel unit distance. Integrating (5.3)

$$T = \frac{2}{3}\pi\rho\tau\left(x_t^3 - x_c^3\right). \tag{5.29}$$

Therefore,

$$T_L = \rho\tau(x_t - x_c), \quad T_N = \tau x_t. \tag{5.30}$$

Using (5.16) and (5.17),

$$F_{LL}\frac{dL}{d\tau} + F_{LN}\frac{dN}{d\tau} = -\frac{\partial}{\partial\tau}F_L = \frac{T}{\tau}(F_{12} - T_L F_{22}) + \frac{T_L F_2}{\tau}, \tag{5.31}$$

$$= \frac{T}{\tau(1-T_N)}F_{LN} + \frac{1}{\tau}\left(T_L + \frac{TT_{NN}}{1-T_N}\right)F_2. \tag{5.32}$$

$$F_{LN}\frac{dL}{d\tau} + F_{NN}\frac{dN}{d\tau} = -\frac{\partial}{\partial\tau}\left(F_N - \frac{F}{N}\right),$$

$$= \frac{T}{\tau(1-T_N)}F_{NN} + \frac{1}{\tau}\left(T_N - \frac{T}{N} + \frac{TT_{NN}}{1-T_N}\right)F_2. \tag{5.33}$$

Therefore,

$$\frac{dL}{d\tau} = \frac{F_2}{\tau|H|}\begin{vmatrix} T_L + \dfrac{TT_{NN}}{1-T_N} & F_{LN} \\ T_N - \dfrac{T}{N} + \dfrac{TT_{NN}}{1-T_N} & F_{NN} \end{vmatrix} < 0, \tag{5.34}$$

$$\frac{dN}{d\tau} = \frac{T}{\tau(1-T_N)} + \frac{F_2}{\tau|H|}\begin{vmatrix} F_{LL} & T_L + \dfrac{TT_{NN}}{1-T_N} \\ F_{LN} & T_N - \dfrac{T}{N} + \dfrac{TT_{NN}}{1-T_N} \end{vmatrix} \gtrless 0, \tag{5.35}$$

$$\frac{dW}{d\tau} = \left(\frac{dW}{dL}\right)_N \frac{dL}{d\tau} + \left(\frac{dW}{dN}\right)_L \frac{dN}{d\tau} + \frac{\partial W}{\partial \tau}$$

$$= -\frac{T}{\tau}\frac{dL}{d\tau} + \frac{F_2(1-T_N)}{\tau|H|}\begin{vmatrix} F_{LL} & T_L + \dfrac{TT_{NN}}{1-T_N} \\ F_{LN} & T_N - \dfrac{T}{N} + \dfrac{TT_{NN}}{1-T_N} \end{vmatrix} \gtrless 0. \qquad (5.36)$$

Before we interpret these results, we note that, if L is fixed,

$$\frac{dN}{d\tau} = \frac{T}{\tau(1-T_N)} + \frac{F_2}{\tau F_{NN}}\left\{T_N - \frac{T}{N} + \frac{TT_{NN}}{1-T_N}\right\} \gtrless 0. \qquad (5.37)$$

However,

$$\frac{dW}{d\tau} = \frac{(1-T_N)F_2}{\tau F_{NN}}\left\{T_N - \frac{T}{N} + \frac{TT_{NN}}{1-T_N}\right\} < 0. \qquad (5.38)$$

The interpretation of (5.37) and (5.38) is straightforward. Referring to (5.38) first, if L is fixed, an increase in τ increases the shadow price of labor. Therefore, the optimal labor input to factory production falls unambiguously. Now referring to (5.37), there are two parts to the shift in optimal population. *Ceteris paribus*, the fall in optimal labor implies a *fall* in optimal population, but the increase in τ also implies that commuters will spend more time travelling, and, *ceteris paribus*, this effect causes an *increase* in optimal population. The combination of these two effects makes the induced shift in optimal population ambiguous in sign.

Now returning to (5.34) - (5.36), an increase in τ induces an unambiguous fall in optimal L. We have seen that, given constant L, optimal W falls. However, the fall in L, *ceteris paribus*, reduces the shadow price of labor, and, therefore, acts to increase optimal W. As a result, the change in optimal W may be of either sign. Finally, the ambiguity of sign in $\frac{dW}{d\tau}$ implies ambiguity in the sign of $\frac{dN}{d\tau}$.

It is interesting to note that, while $\frac{dL}{d\tau} < 0$ when population is free, this ambiguity does not carry through to the fixed population case. Fixing population, (5.31) becomes

$$F_{LL}\frac{dL}{d\tau} = \frac{T}{\tau}(F_{12} - T_L F_{22}) + \frac{T_L}{\tau}F_2 ,$$

$$= \frac{1}{\tau}\frac{d}{dL}(TF_2). \tag{5.39}$$

In (5.39), TF_2 is the opportunity cost of the total travel time of the population of the town, and the meaning of the relationship is that $\frac{dL}{dr_a}$ will be positive if and only if a marginal increase in L causes a decrease in this opportunity cost. If returns to scale in gross factory production were non-increasing, the sign of the right hand side of (5.39) would be unambiguously positive. However, returns to scale are increasing, and F_{22} may, therefore, be positive. If it is, $\frac{d}{dL}(TF_2)$ may be negative.

Finally,

$$\frac{d}{d\tau}\left(\frac{F - Ar_a}{N}\right) = \frac{\partial}{\partial\tau}\left\{\frac{F - Ar_a}{N} + \lambda\{\rho(A - L) - N\}\right\},$$

$$= -\frac{T}{\tau N}F_2 < 0, \tag{5.40}$$

which implies that household consumption and utility always fall when the unit travel time increases.

5.2.3 Population Density

We can write

$$\frac{dF_L}{d\rho} = \frac{\partial F_L}{\partial L}\frac{dL}{d\rho} + \frac{\partial F_L}{\partial N}\frac{dN}{d\rho} + \frac{\partial F_L}{\partial A}\frac{dA}{d\rho} + \frac{\partial F_L}{\partial \rho},$$

$$= F_{LN}\frac{dL}{d\rho} + F_{LN}\frac{dN}{d\rho} - \frac{N}{\rho^2}\frac{\partial F_L}{\partial A} + \frac{\partial F_L}{\partial \rho}. \qquad (5.41)$$

Therefore, total differentiation of (5.16) with respect to ρ yields

$$F_{LL}\frac{dL}{d\rho} + F_{LN}\frac{dN}{d\rho} = -\frac{N}{\rho}\left(T_N - \frac{T}{N}\right)\frac{\partial}{\partial W}\left(F_1 - T_L F_2\right) - \left(NT_{NN} - \frac{T_L}{\rho}\right)F_2,$$

$$= -\frac{N}{\rho}\frac{T_N - \frac{T}{N}}{1 - T_N}F_{LN} - \left\{\frac{WT_{NN}}{1-T_N} - \frac{T_L}{\rho}\right\}F_2. \qquad (5.42)$$

Similarly,

$$F_{LN}\frac{dL}{d\rho} + F_{NN}\frac{dN}{d\rho} = -\frac{N}{\rho}\left(T_N - \frac{T}{N}\right)\frac{\partial}{\partial W}\left(F_N - \frac{F}{N}\right) + \frac{1}{\rho}\left(T_N - NT_{NN}\right)F_2,$$

$$= -\frac{N}{\rho}\frac{T_N - \frac{T}{N}}{1 - T_N}F_{NN} + \frac{1}{\rho}\left\{2T_N - \frac{T}{N} - \frac{WT_{NN}}{1-T_N}\right\}F_2. \qquad (5.43)$$

After some manipulation, (5.42) and (5.43) yield

$$\frac{dL}{d\rho} = \frac{F_2}{|H|}\begin{vmatrix} 2T_N - \frac{T}{N} - \frac{T_L}{\rho} & F_{LN} - \rho F_{NN} \\ \frac{1}{\rho}\left\{2T_N - \frac{T}{N} - \frac{WT_{NN}}{1-T_N}\right\} & -F_{NN} \end{vmatrix}, \qquad (5.44)$$

$$\frac{dN}{d\rho} = -\frac{N}{\rho}\,\frac{T_N - \frac{T}{N}}{1 - T_N} + \frac{F_2}{|H|}\begin{vmatrix} 2T_N - \frac{T}{N} - \frac{T_L}{\rho} & \rho F_{LN} - F_{LL} \\ \frac{1}{\rho}\left\{2T_N - \frac{T}{N} - \frac{WT_{NN}}{1-T_N}\right\} & F_{LN} \end{vmatrix}. \quad (5.45)$$

All the elements of the determinants are unambiguously positive with the exception of $\frac{1}{\rho}\left\{2T_N - \frac{T}{N} - \frac{WT_{NN}}{1-T_N}\right\}$, which may take either sign. Therefore, both $\frac{dL}{d\rho}$ and $\frac{dN}{d\rho}$ may take either sign. It can also be shown that $\frac{dA}{d\rho}$ and $\frac{dW}{d\rho}$ can take either sign. Furthermore, none of these ambiguities is removed when either N or L is assumed fixed.

The effects of an increase in ρ can be divided into two parts. The increased density increases the shadow residential rent and the opportunity cost of factory land. In this respect an increase in ρ is equivalent to an increase in τ. At the same time, however, the increase in density, *ceteris paribus*, results in a decrease in the area of the residential zone, which implies a fall in the travel time of households. This effect has no parallel in the case of a shift in the unit travel time. Comparisons between (5.34) and (5.44), and between (5.35) and (5.45) show a correspondence between the right hand sides of (5.34) and (5.35) and some of the terms of (5.44) and (5.45), but (5.44) and (5.45) contain additional terms which arise from the second of the above effects. The ambiguous results we have obtained are to some extent due to the fact that the two effects tend to work in opposite directions.

However,

$$\frac{d}{d\rho}\left(\frac{F - Ar_a}{N}\right) = \frac{\partial}{\partial\rho}\left\{\frac{F - Ar_a}{N} - \lambda\{\rho(A - L) - N\}\right\},$$

$$= -\frac{TF_2}{N} + \lambda N. \quad (5.46)$$

λ can be evaluated by differentiating the Lagrangian with respect to A; thus,

$$\lambda N = \frac{r_a}{\rho} + T_N F_2 . \tag{5.47}$$

Therefore,

$$\frac{d}{d\rho}\left(\frac{F - Ar_a}{N}\right) = F_2\left(T_N - \frac{T}{N}\right) + \frac{r_a}{\rho} > 0. \tag{5.48}$$

That is, in the optimum town, household consumption of the factory produced good increases as the population density increases.

5.3 CONCLUSIONS

The striking difference between the results of our comparative static analyses and those of Wheaton [1974] is that very few of our results are unambiguous in sign, whereas all of his are. The reason for this difference is that his is a partial equilibrium analysis in the sense that per capita income is treated as being exogenous, while ours was carried out in a general equilibrium framework. Having made this observation, it becomes clear that the most important lesson to be learned from our analysis is how crucial it is always to regard a town as a single indivisible unit. The importance of this essential unity on the results of our analysis is emphasized by the existence of increasing returns to scale at the optimum point.

It is worthwhile to re-state our main results briefly. So far as the opportunity cost of land is concerned, we find that, if population is fixed, $\frac{dL}{dr_a} < 0$, and, if CBD size is fixed, $\frac{dN}{dr_a} > 0$. The effect of both of these results is to make the factor ratio L/W decrease when the opportunity cost of land increases. This result could easily be predicted. When L and N are both free to vary, the sign of all induced shifts are ambiguous. This is because the local degree of increasing returns at the optimum may change, however it does produce the apparently paradoxical result that an increase in r_a may lead to an increased use of land as a factor of production. The paradox is only apparent, however, since we show that $\frac{dL}{dr_a} > 0$ implies a fall in the land/labor ratio in production.

Using the time taken for a commuter to travel unit distance, τ, as a shift parameter we obtain one unambiguous result when L and N are both free. This is that the area of the CBD decreases with an increase in τ. The sign of $\frac{dW}{d\tau}$, on the other hand, is ambiguous. The ambiguity arises from the way in which the decrease in L reduces transport times for the laborers. When N is fixed, $\frac{dL}{d\tau}$ can take either sign. It will certainly be negative if there are constant returns to scale in production, but this condition is not necessary for an optimum. When L is fixed, $\frac{dW}{d\tau} < 0$, because the shadow price of labor increases unambiguously.

Finally, using the population density in the residential area, ρ, as a shift parameter, we obtain a series of ambiguous results. Population density appears in two constraints on the optimization. In the population constraint (5.4) its effect is identical to τ, and it appears with τ in the slope of the von Thünen rent. It also appears in the constraint (5.5) where it is the coefficient of proportionality between residential zone area and population. Its effect on the comparative static results through one of the constraints is in the opposite direction to its effect through the other, and an intuitive analysis of the comparative static relationships becomes complex.

CHAPTER 6

THE OPTIMUM INDUSTRIAL STATE

In Chapter 2 we reviewed the development of the von Thünen theory of rent. Writing over the period 1826-63, von Thünen examined agricultural land use patterns around an isolated market place when land and labor are factors of production. Assuming both factors to be of uniform quality, he found that profit maximization requires that, at equilibrium, the land will be divided into contiguous annular zones. In each of these zones, one, and only one, crop will be produced. These zones are the so-called von Thünen "rings", and they and the market make up the von Thünen *Isolated State*.

The von Thünen analysis was based upon fixed coefficient production functions. Quite recently, Beckmann [1971] introduced neoclassical production functions into the von Thünen model, and found that the "rings" are robust to this change in the model's structure. However, he further found that, within each ring, the land/labor ratio increases strictly with distance from the market, and is continuous across the boundaries of the rings when factor substitution can occur.

Apart from Beckmann's contribution (re-examined in some details by Renaud [1972]), the von Thünen model, in its application to agricultural land, has remained almost static for more than a century, notwithstanding its limitations when judged on the standards of modern economic theory. Prices in this *isolated* community continue to be assumed to be exogenously determined, and the market place is located on a point, and has none of the other economic activities normally associated with a market town. There is no urban production, no population and no residential zone. In brief, the present status of the von Thunën *Isolated State* is that of a purely agricultural community, whose only contact with the rest of the world is at a market point, at which it is a price taker.

Just as the locational theory of agricultural land use oversimplifies the role of the market town in the economy, the theory of urban land use has been developed without regard to the structure and role of the town's agricultural environment. We have already noted that the residential zone and the CBD are typical von Thünen "rings", but we have not, so far, concerned ourselves with interactions between these rings and the agricultural rings in the surrounding rural area. We have represented the agricultural land rent by a constant independent of the size and population of the town. In respect to the consumption of produced goods, we have assumed that the residents of the town derive utility from the factory good alone.

This separation of the two theories is convenient in developing pure theories, but it is artificial, and it omits to examine interesting relationships of dependence between the two sectors of the economy. A most casual observation of reality reveals that towns and their agricultural environments produce different goods, and that the economic interdependence between the two sectors is profound. There is a need, therefore, to integrate the two theories, and to describe in formal terms the structure of the isolated industrial state. Quite clearly, if we assume a von Thünen-like price for goods transport, the agricultural rent (including the agricultural rent at the town boundary) will depend, *inter alia*, upon the distance from the centre of the town, and if our state is truly isolated, relative prices will be determined endogenously. In a normative context, these observations suggest to us that the optimum size for the factory town in an isolated industrial state will depend upon the shadow costs of transporting agricultural goods to the urban market, and that there must be an optimum size for the industrial state. We now proceed to examine these ideas in a formal model of the optimum state.

6.1 THE MODEL

In Chapters 3 and 4 we examined in some detail the spatial structure of the residential zone of the optimum town. There is no need to re-traverse this ground, and at this point our problem can be simplified by the use of the model of the factory town introduced in Chapter 5. This is the model in which population density in the residential zone is constant, and no land is required for transport. The analytical convenience of this model is manifest in three ways. First, the constant residential density simplifies the analysis of the residential zone. Second, the fact that equals are treated equally in this optimum town simplifies the choice of a welfare criterion. Finally, we are able to avoid the difficulties which exist in presenting a suitable definition of the residential site area of a farm worker. These difficulties arise from environmental differences between the town and rural areas, because it seems probable that urban and farm households have different tastes in regard to residential space. The urban household needs space for two reasons. First, it needs living space in the direct sense. Second, it needs space to separate itself from neighboring households, and thus obtain privacy. The farming household requires living space in the same way as the urban household, but its privacy is obtained from the size of its farm. Thus, it would appear that farm households will demand less residential space than an otherwise identical urban household. We avoid the issues raised by this difference in taste by assuming that every household in the state is allocated a residential site of area $1/\rho_h$, where the subscript "h" is used to denote residential density on land allocated to household residential purposes.

In setting up the model we will consider only one agricultural good. Our state, therefore, is made up of three zones: the CBD, which contains the factory; the residential zone, which houses the factory workforce; and the agricultural zone, which contains the farms and the residences of the farm workforce. Following our earlier notation we write

$$L = \pi x_c^2, \quad A = \pi x_t^2, \quad S = \pi x_s^2, \tag{6.1}$$

where L, A, x_c and x_t have the meanings already assigned to them in Chapter 3, and S and x_s are the area and radius of the state, respectively. The factory production function is now written in the form $F(L,W_u)$ where W_u is the (urban) labor input to factory production. The factory good will again be nominated as the numeraire good of the state.

The population of the state is assumed to be N identical households. Each of these contributes a total of unit time to labor and (possibly) travel in the state. Of the N households, N_u reside in the residential zone of the town. This residential zone is wholly allocated to residential purposes, therefore,

$$N_u - \rho_h(A - L) = 0. \tag{6.2}$$

We assume that the only input to the transportation of the factory workforce is its travel time. Let $t_u(x)$ be the time taken for a laborer resident at x to make a round trip to the centre of the town. Then we assume that the total travel time per day of the residents of the town is given by

$$T_u = \int_{x_c}^{x_t} 2\pi\rho_h x\, t_u(x)\,dx. \tag{6.3}$$

The labor input to factory production is, therefore, given by

$$W_u = N_u - T_u. \tag{6.4}$$

The agricultural good is produced on farms from inputs of land and labor. In our examinations of factory production we have had to take into account the economies of scale which must exist to justify the existence of a town. In agricultural production, however, the labor workforce per productive unit is usually observed to be small (compared with the factory workforce), and it does not have to travel to its place of work. It seems reasonable, therefore, to assume that returns to scale in agricultural production are typically very nearly constant. Accordingly, we will assume constant returns to scale in agriculture and write the production function in the form

$$\phi_a(x) = \phi_a(\rho_a(x)), \tag{6.5}$$

where ϕ_a is the output per acre from land allocated to farming and $\rho_a(x)$ is the labor input per acre of farm land. Since farm laborers do not have to travel to work, $\rho_a(x)$ is also the number of farm laborers per acre of farm land. It is well known that, when returns to scale are constant, $\phi_a'(\rho_a(x)) \equiv \dfrac{\partial\phi_a}{\partial\rho_a}$ is the marginal product of labor, and $\phi_a - \rho_a\phi_a'$ is the marginal product of land. We make the additional assumption that $\phi_a''(\rho_a)$ is strictly negative.

We assume that $t_a(x)$ units of labor are required to transport one unit of the agricultural good from x to the centre of the town. Therefore, each acre of farm land requires $t_a(x)\phi_a(\rho_a(x))$ households to transport its output to market. Where should these agricultural transport households be housed? Since their journeys to market are always round trips, they will not have to travel to work if they reside between the town centre and x. It will, however, be sub-optimal to locate them in the residential zone of the town, because, located within the town, they will increase T_u by displacing urban households outward from the CBD. Furthermore, since the von Thünen rent must be downward sloping, the opportunity cost of their residential sites must decrease as their residential locations are moved outward. Therefore, we will locate them at x. In this way we make the opportunity cost of agricultural transport workers' residential sites the least it can be without our introducing labor transport in the agricultural zone. It follows that the area of residential land associated with one acre of farm at x is $\{\rho_a(x) + t_a(x)\phi_a(x)\}/\rho_h$, and the total output of the agricultural good, Φ_a say, is given by

$$\Phi_a = \int_{x_t}^{x_s} 2\pi x \left\{ \frac{\rho_h}{\rho_h + \rho_a(x) + t_a(x)\phi_a(x)} \right\} \phi_a(x)\,dx. \tag{6.6}$$

Furthermore, the population density in the agricultural zone, $\rho_r(x)$ say, is given by

$$\rho_r(x) = \frac{\rho_h\{\rho_a(x) + t_a(x)\phi_a(x)\}}{\rho_h + \rho_a(x) + t_a(x)\phi_a(x)}, \tag{6.7}$$

and the population of the agricultural zone, N_a say, is given by,

$$N_a = \int_{x_t}^{x_s} 2\pi x \rho_r(x)\,dx. \tag{6.8}$$

In deriving equations (6.4) - (6.8) we have not allocated labor for the transport of goods from the market to the consumers. This is because, in the optimum state, urban households can combine shopping and work journeys, while the transport system set up to carry the agricultural good to market can be used, without additional cost, to carry goods from the market to the agricultural households. The population of the state is, therefore,

$$N = N_u + N_a. \tag{6.9}$$

The N identical households of the state are assumed to derive utility from the consumption of the factory and agricultural goods and from the services of residential space. However, since the area of residential sites is constant throughout the state we can write $u = u(f(x), a(x))$, where $f(x)$ and $a(x)$ are the household's allocation of the factory and agricultural good, respectively. From the model developed in Chapter 5 it is easy to show that f and a are constant in the residential zone of the town. Furthermore, there can be no discontinuity of price ratio or of household utility across the town boundary, because households are identical and goods transport is costless. Therefore, we can write the urban household's allocations as $f(x_t)$, $a(x_t)$, this representing the allocations to an agricultural household located at x_t.

Now, writing the agricultural household's allocations as $f(x)$, $a(x)$, where x lies in the closed interval $x_t \leq x \leq x_s$, the income constraints on the state can be written

$$F - Sr_H - N_u f(x_t) - \int_{x_t}^{x_s} 2\pi x \rho_r(x) f(x)\,dx = 0, \tag{6.10}$$

and

$$\int_{x_t}^{x_s} 2\pi x \frac{\rho_h}{\rho_h + \rho_a(x) + \tau_a(x)\phi_a(x)} \phi_a(x)dx - N_u a(x_t)$$

$$- \int_{x_t}^{x_s} 2\pi x \rho_r(x) a(x) dx = 0. \qquad (6.11)$$

Finally, average utility may be written

$$\bar{u} = \frac{1}{N}\{ N_u u(f(x_t), a(x_t)) + \int_{x_t}^{x_s} 2\pi x \rho_r(x) u(f,a) dx \}. \qquad (6.12)$$

This completes the specification of the model.

6.2 THE STRUCTURE OF THE AGRICULTURAL ZONE

We again choose as our criterion of optimality the maximization of average utility. The control variables in the maximization are $f(x)$, $a(x)$, $\rho_r(x)$ and $\rho_a(x)$ and the control parameters are L, A, S, N, N_u, N_a, W_u and T_u. The constraints on the maximization are (6.2) - (6.4) and (6.7) - (6.11). We define the Lagrangian

$$\mathcal{L}(x) \equiv 2\pi x \rho_r(x) \left\{ \frac{u(x)}{N} - \lambda_1 f(x) - \lambda_2 a(x) - \lambda_6 \right\}$$

$$+ 2\pi x \frac{\lambda_2 \rho_h}{\rho_h + \rho_a(x) + t_a(x)\phi_a(x)} \phi_a(x)$$

$$+ 2\pi x \mu(x) \left\{ \rho_r(x) - \frac{\rho_h(\rho_a(x) + t_a(x)\phi_a(x))}{\rho_h + \rho_a(x) + t_a(x)\phi_a(x)} \right\}, \qquad (6.13)$$

and the function

$$J \equiv \frac{1}{N} N_u u(f(x_t), a(x_t)) + \lambda_1 \{ F - Sr_H - N_u f(x_t) \}$$

$$- \lambda_2 N_u a(x_t) + \lambda_3 \{ N_u - \rho_h (A - L) \} + \lambda_4 \{ T_u - \int_{x_c}^{x_t} 2\pi x \rho_h t_u(x) dx \}$$

$$+ \lambda_5 \{ N_u - W_u - T_u \} + \lambda_6 N_a + \lambda_7 \{ N - N_a - N_u \}, \qquad (6.14)$$

where the λ's are constant Lagrange multipliers and $2\pi x\, \mu(x)$ is a location dependent Lagrange multiplier.

Choice of the variables $f(x)$ and $a(x)$ gives the necessary conditions

$$\frac{1}{N} u_f - \lambda_1 = 0, \qquad (6.15)$$

$$\frac{1}{N} u_a - \lambda_2 = 0. \qquad (6.16)$$

Differentiating (6.15) and (6.16) totally with respect to x, and using the strict concavity of $u(f,a)$, we obtain

$$\dot{f}(x) = \dot{a}(x) = 0. \qquad (6.17)$$

So that equals are treated equally in the agricultural zone. This result was to be expected, given the explanation in Chapter 3 of why equals are treated unequally. It also follows from (6.15) and (6.16) that the shadow price of the agricultural good (at the market and to the consumer), p_a say, is given by

$$p_a = \frac{\lambda_2}{\lambda_1} . \tag{6.18}$$

Choice of the parameters N and N_a give

$$\lambda_6 = \lambda_7 = \frac{u}{N} . \tag{6.19}$$

Using (6.19), choice of the variable $\rho_r(x)$ gives

$$\mu(x) = \lambda_1 f + \lambda_2 a , \tag{6.20}$$

so that μ is, in fact, a constant. Using (6.18) and (6.20), choice of $\rho_a(x)$ gives

$$\begin{aligned} f + p_a a[\{1 + t_a(x)\phi_a'(\rho_a)\}\{\rho_h + \rho_a(x) + t_a(x)\phi_a(\rho_a)\} \\ -\{\rho_a(x) + t_a(x)\phi_a(\rho_a)\}\{1 + t_a(x)\phi_a'(\rho_a)\}] \\ = p_a[\phi_a'(\rho_a)\{\rho_h + \rho_a(x) + t_a(x)\phi_a(\rho_a)\} \\ - \phi_a(\rho_a)\{1 + t_a(x)\phi_a'(\rho_a)\}], \end{aligned} \tag{6.21}$$

which simplifies to

$$\rho_h(f + p_a a)\{1 + t_a(x)\phi_a'(x)\} = p_a[\{\rho_h + \rho_a(x)\}\phi_a'(x) - \phi_a(x)]. \tag{6.22}$$

Let w(x) be the shadow wage at x in the agricultural zone. Then the shadow cost of transporting one unit of the agricultural good from x to the market is $w(x)t_a(x)$. The shadow value of the agricultural good at the farm is, therefore, $p_a - w(x)t_a(x)$, and, therefore, $p_a - w(x)t_a(x)$ must be strictly positive in the agricultural zone. The value of the marginal product of farm land at x is, therefore, $\{p_a - w(x)t_a(x)\}\{\phi(x) - \rho_a\phi_a'(x)\}$. The shadow value of a residential site is $\frac{1}{\rho_h}$ times the value of the marginal product of farm land. Therefore,

$$w(x) = f + p_a a + \frac{1}{\rho_h}\{p_a - w(x)t_a(x)\}\{\phi_a(x) - \rho_a(x)\phi_a'(x)\}. \quad (6.23)$$

Eliminating $f + p_a a$ in (6.22) and (6.23)

$$\rho_h\{1 + t_a(x)\phi_a'(x)\}w(x) = p_a[\{\rho_h + \rho_a(x)\}\phi_a'(x) - \phi_a(x)]$$

$$+ \{p_a - w(x)t_a(x)\}\{\phi_a(x) - \rho_a(x)\phi_a'(x)\}\{1 + t_a(x)\phi_a'(x)\},$$

$$= p_a\phi_a'(x)[\rho_h + \{\phi_a(x) - \rho_a(x)\phi_a'(x)\}t_a(x)]$$

$$- w(x)t_a(x)\{\phi_a(x) - \rho_a(x)\phi_a'(x)\}\{1 + t_a(x)\phi_a'(x)\}. \quad (6.24)$$

Therefore,

$$w(x)\{1 + t_a(x)\phi_a'(x)\} = p_a\phi_a', \quad (6.25)$$

which implies that the shadow wage at x is equal to the value of the marginal product of farm labor at x. That is

$$w(x) = \{p_a - w(x)t_a(x)\}\phi_a'. \quad (6.26)$$

It follows, from the properties of $\phi(\rho_a(x))$, that the shadow rent at x is equal to the value of the marginal product of farm land at x. That is,

$$r(x) = \{p_a - w(x)t_a(x)\}\{\phi_a(x) - \rho_a(x)\phi'_a(x)\}. \tag{6.27}$$

Furthermore, the shadow wage of transport labor is equal to the value of its marginal product.

Using (6.27) we can re-write (6.26) in the form

$$\phi'_a(x) = \frac{f + p_a a}{p_a - w(x)t_a(x)} + \frac{1}{\rho_h}\{\phi_a(x) - \rho_a(x)\phi'_a(x)\}. \tag{6.28}$$

Differentiating (6.28) totally with respect to x,

$$\dot{\rho}_a\phi''_a(x)\left\{1 + \frac{\rho_a(x)}{\rho_h}\right\} = \frac{f + p_a a}{\{p_a - w(x)t_a(x)\}^2}\frac{d}{dx}\{w(x)t_a(x)\}. \tag{6.29}$$

We have $\phi''_a(x) < 0$. Therefore, the slope of the density of farm labor, $\dot{\rho}_a(x)$, is opposite in sign from the slope of the shadow cost of transporting one unit of farm output to market. Beckmann found that $\dot{\rho}_a(x) < 0$ in his equilibrium model. He used fixed factor and commodity prices, so that, in his solution $\frac{d}{dx}w(x)\,t_a(x) = w\frac{d}{dx}t_a(x) > 0$. In our model further analysis is necessary before we can put a sign on $\dot{\rho}_a(x)$. Differentiating (6.23) totally,

$$\dot{w}(x)[\rho_h + t_a(x)\{\phi_a(x) - \rho_a(x)\phi'_a(x)\}]$$

$$= -w(x)\,\dot{t}_a(x)\{\phi_a(x) - \rho_a(x)\phi'_a(x)\}$$

$$-\{p_a - w(x)t_a(x)\}\rho_a(x)\dot{\rho}_a(x)\phi''_a(x). \tag{6.30}$$

Therefore,

$$\rho_h + t_a(x)\{\phi_a(x) - \rho_a(x)\phi_a'(x)\}\frac{d}{dx}\{w(x)t_a(x)\}$$

$$= \rho_h w(x)\dot{t}_a(x) - \{p_a - w(x)t_a(x)\}\rho_a(x)\dot{\rho}_a(x)\phi_a''(x). \quad (6.31)$$

Equations (6.29) and (6.31) yield

$$\dot{\rho}_a(x) < 0, \qquad \frac{d}{dx}\{w(x)t_a(x)\} > 0. \quad (6.32)$$

Thus Beckmann's condition, $\dot{\rho}_a(x) < 0$, also holds when all prices are endogenously determined. Furthermore,

$$\frac{d}{dx}\{\phi_a(x) - \rho_a(x)\phi_a'(x)\} < 0, \qquad \dot{r}(x) < 0, \quad (6.33)$$

and by (6.30)

$$\dot{w}(x) < 0. \quad (6.34)$$

In Beckmann's model $\dot{w}(x)$ was, of course, equal to zero. It is interesting to note that, although $\dot{w}(x) < 0$ in our model, $\frac{d}{dx}(wt_a)$ is, nevertheless, unambiguously negative.

The structure of the agricultural zone of the optimum industrial state may, therefore, be summarized as follows. At all locations factors are employed to the point where the values of their marginal products are equal to their shadow prices. Both shadow factor prices are then downward sloping functions of distance from the centre of the state. In addition, the land/labor ratio in farming and the shadow cost of transporting unit output of the agricultural good to the market are both upward sloping functions of x.

We now obtain an implicit expression for the radius of the optimum state. Choice of S gives the condition

$$\lambda_1 r_H - \frac{1}{2\pi x_s} \mathcal{L}(x_s) = 0. \tag{6.35}$$

Equations (6.13), (6.18) - (6.20) and (6.35) yield

$$r_H + \rho_r(x_s)(f + p_a a) - \frac{\rho_h p_a \phi_a(x_s)}{\rho_h + \rho_a(x_s) + t_a(x_s)\phi_a(x_s)} = 0. \tag{6.36}$$

Using (6.7) and (6.23), (6.36) may be written

$$\frac{\rho_h + \rho_a(x_s) + t_a(x_s)\phi_a(x_s)}{\rho_h} r_H = \{p_a - (f + p_a a)t_a(x_s)\}\phi_a(x_s)$$

$$+ \rho_a(x_s)(f + p_a a),$$

$$= \{p_a - w(x_s)t_a(x_s)\}\left[\phi_a(x_s) + \frac{t_a(x_s)\phi_a(x_s)}{\rho_h}\{\phi_a(x_s) - \rho_a(x_s)\phi_a'(x_s)\}\right]$$

$$+ \rho_a(x_s)\left[\{p_a - w(x_s)t_a(x_s)\}\frac{\phi_a(x_s) - \rho_a(x_s)\phi_a'(x_s)}{\rho_h} - w(x_s)\right]$$

(6.37)

Now, using (6.26), (6.37) may be written

$$\frac{\rho_h + \rho_a(x_s) + t_a(x_s)\phi_a(x_s)}{\rho_h} r_H = \{p_a - w(x_s)t_a(x_s)\}$$

$$\left\{\phi_a(x_s) + \frac{\phi_a(x_s) - \rho_a(x_s)\phi'_a(x_s)}{\rho_h} t_a(x_s)\phi_a(x_s)\right.$$

$$\left. + \rho_a(x_s)\frac{\phi_a(x_s) - \rho_a(x_s)\phi'_a(x_s)}{\rho_h} - \rho_a(x_s)\phi'_a(x_s)\right\},$$

$$= \frac{\rho_h + \rho_a(x_s) + t_a(x_s)\phi_a(x_s)}{\rho_h}\{p_a - w(x_s)t_a(x_s)\}$$

$$\{\phi_a(x_s) - \rho_a(x_s)\phi'_a(x_s)\},$$

$$= \frac{\rho_h + \rho_a(x_s) + t_a(x_s)\phi_a(x_s)}{\rho_h} r(x_s).$$

Therefore,

$$r(x_s) = r_H. \tag{6.37}$$

The meaning of (6.37) is that the optimum state expands until the shadow rent equals the opportunity cost of land.

We have assumed that returns to scale in agricultural production are constant. Also, is is clear, from the form we have assumed, that returns to scale in agricultural transport are constant. We have also proved that factors are employed to the point where their shadow values are equal to the values of their marginal products in the agricultural zone. It follows that, in shadow terms, income in the agricultural zone is precisely distributed between factors. Therefore,

$$(f + p_a a)N_a + (S - A)r_H - p_a\Phi_a = 0. \qquad (6.38)$$

It follows from (6.38) and the income constraints (6.10) and (6.11) that

$$\frac{F - Ar_H}{N_u} - (f + p_a a) = 0. \qquad (6.39)$$

6.3 THE SIZE OF THE TOWN

Choices of the parameters T_u, W_u, N_u and A yield

$$\lambda_1 F_2 = \lambda_4 = \lambda_5, \qquad (6.40)$$

$$\lambda_3 = \lambda_1 f + \lambda_2 a - \lambda_5, \qquad (6.41)$$

$$\lambda_3\rho_h + \lambda_4\rho_h t_u(x_t) - \lambda_1\rho_r(x_t)(f + p_a a)$$

$$+ \frac{\lambda_2\rho_h\phi_a(x_t)}{\rho_h + \rho_a(x_t) + t_a(x_t)\phi_a(x_t)} = 0. \qquad (6.42)$$

Equations (6.40) - (6.42) yield

$$F_2\{1 - t_u(x_t)\} = f + p_a a + \frac{1}{\rho_h}\left\{\frac{p_a\rho_h\phi_a(x_t)}{\rho_h + \rho_a(x_t) + t_a(x_t)\phi_a(x_t)}\right.$$

$$\left. - \rho_r(x_t)(f + p_a a)\right\}. \qquad (6.43)$$

It is clear from the analysis following (6.36) that

$$\frac{p_a \rho_h \phi_a(x_t)}{\rho_h + \rho_a(x_t) + t_a(x_t)\phi_a(x_t)} - \rho_r(x_t)(f + p_a a) = r(x_t). \qquad (6.44)$$

Therefore, (6.43) reduces to

$$F_2\{1 - t_u(x_t)\} = w(x_t), \qquad (6.45)$$

where $w(x_t)$ is the agricultural wage at x_t. However, we know from our study of the optimum town that the left hand side of (6.45) is also the value of the marginal product of population in the town and hence the factory wage. Therefore, the meaning of (6.45) is that the town expands to the point where there is continuity of wage across the town boundary. This is equivalent to saying that there is continuity of the value of marginal product at the town boundary.

Equations (6.39) and (6.45) give

$$F_2 = \frac{F - Lr_H}{N_u} + t_u(x_t)F_2 + \frac{1}{\rho_h}\{r(x_t) - r_H\}. \qquad (6.46)$$

Now,

$$t_u(x_t) = \left(\frac{dT_u}{dN_u}\right)_L, \qquad (6.47)$$

so that the first two terms on the right hand side of (6.46) are familiar from our analysis of the optimum town. The final term is new, and arises from the fact that the marginal household in the residential area of the town displaces the agricultural activity which could take place on $\frac{1}{\rho_h}$ acres at x_t from x_t to x_s.

Choice of L yields

$$\lambda_1 F_1 + \lambda_3 \rho_h + \lambda_4 \rho_h t_u(x_c) = 0. \tag{6.48}$$

Equations (6.40) - (6.44) and (6.48) give

$$F_1 = r(x_t) + \rho_h \{ t_u(x_t) - t_u(x_c) \} F_2$$

$$= r_H + \rho_h \{ t_u(x_t) - t_u(x_c) \} F_2 + (r(x_t) - r_H). \tag{6.49}$$

In (6.49)

$$\rho_H \{ t_u(x_t) - t_u(x_c) \} = \left(\frac{dT_u}{dL} \right)_{N_u} , \tag{6.50}$$

so that the first two terms are familiar from our analysis of the optimum town. However, as in (6.46), we have an additional term in $(r(x_t) - r_H)$. In this case it arises from the fact that an increase of one acre in the area of the CBD causes a displacement of urban households from x_c to x_t, and, ultimately, one acre of agricultural activity displaced from x_t to x_s.

Now, using a notation similar to that established in Chapter 3,

$$F_L \equiv \left(\frac{dF}{dL} \right)_{N_u} = F_1 - F_2 \left(\frac{dT_u}{dL} \right)_{N_u} , \tag{6.51}$$

$$F_{N_u} \equiv \left(\frac{dF}{dN_u} \right)_L = F_2 (1 - \left(\frac{dT_u}{dN_u} \right)_L). \tag{6.52}$$

Therefore, from (6.46), (6.47), (6.49) - (6.52),

$$LF_L + N_u F_{N_u} - F = \frac{N_u}{\rho}(r(x_t) - r_H) + L(r(x_t) - r_H),$$

$$= A(r(x_t) - r_H). \tag{6.53}$$

We see from (6.53) that returns to scale in factory production net of transport costs are increasing in the optimum state. The additional term, compared with the relationship we obtained in Chapter 5 after equations (5.16) and (5.17), that is, the term $A(r(x_t) - r_H)$, is the product of the area of the optimum town and the difference between the opportunity cost of land at the town boundary and in the hinterland. This term arises from the transport cost of the agricultural good not being zero.

6.4 CONCLUSIONS

In this chapter we have examined the inter-relationship between the factory town and its agricultural environment when they make up an optimum isolated industrial state. We find that, given constant returns to scale in agricultural production, factors are employed in the agricultural zone to the point where their marginal products are equal to their shadow costs. Equal households are treated equally throughout the optimum state, and the shadow wage is constant in the urban area and continuous across the town boundary. However, in the agricultural zone, the household wage is a decreasing function of the distance from the centre of the state to the residential location of the household. This is because the shadow rent on residential land (which equals the shadow rent on farming land at the same location) is a decreasing function of the same distance. This decreasing land rent is a von Thünen rent which has its origins in the transport cost of the agricultural good. The downward sloping land rent is associated with an upward sloping land/labor ratio in production. This result implies that the opportunity cost of land decreases more rapidly with distance than does the opportunity cost of labor.

The solution to the residential zone of the town is obtained directly from the solution to the simple model of Chapter 5. In respect to the CBD, the expressions obtained for the marginal products of land and labor continue to show that these factors are not paid the value of their marginal products, although town households are treated as equal to the agricultural households. In Chapter 5 we found that, although returns to scale in factory production gross of the transport costs of its work force were increasing, returns to scale in net factory production were constant. In our present model, however, we find that returns to scale in net factory production are unambiguously increasing in the optimum state. The difference in the two results arises from the fact that in the optimum state some of the shadow cost of transporting the agricultural good is "charged" against factory production through a term involving the difference between the shadow rent at the town boundary and the opportunity cost of land to the state.

Finally, we can interpret (6.38) and (6.39) as meaning that, in the agricultural zone and the town separately, the shadow cost to the state in the form of rent payments to absentee landlords and the shadow value of household consumption is exactly equal to the shadow value of output. In other words both productive zones separately "pay their own ways".

This completes our analysis of the optimum state and of normative models in general.

CHAPTER 7

EQUILIBRIUM AND RETURNS TO SCALE IN THE COMPANY TOWN

In Chapter 1 we quoted the locational assumption upon which the Arrow-Debreu Theorem is based, and pointed out that this assumption implies locational homogeneity of economic activity throughout the region of application of the theorem. We then went on to argue that, since the factory is *essentially* an example of locational inhomogeneity of economic activity, application of the Arrow-Debreu Theorem to the analysis of factory production cannot be valid. Competitive equilibrium theory may have relevence to agricultural production or to cottage industry, but so far as factory production is concerned, a different, spatially structured theory is required.

We also drew attention to the way in which one prediction of the Arrow-Debreu Theorem, i.e.,that returns to scale in production must be constant at a competitive equilibrium, has been thought to be in conflict with observation - at least so far as multi-firm industrial (i.e., factory) production is concerned. Whenever many firms participate, it has been supposed that an equilibrium must be, at least to a close approximation, competitive. Yet at the same time, and as the quotation on page 2 attributed to Kaldor indicates, it is also widely believed that industrial producers, in equilibrium, are frequently observed not to have expanded to the point where all the economies of scale available to them are exhausted. Apparently, there is a fundamental conflict between observation and the theory of the firm in equilibrium.

This apparent conflict has led to a debate on the possibility of the co-existence of increasing returns and competitive equilibrium, and hence, by implication, the validity of the Arrow-Debreu Theorem. Of course, the debate pre-dates the development of the formal statement of the Arrow-Debreu Theorem. In fact, its origins go back at least fifty years, and a very considerable literature has accumulated on the subject.[1] The general consensus of this literature seems to be that, separately both increasing returns in production and competitive equilibrium are feasible, and that, subject to the effects of some *passive* forces, which retard or modify the path to equilibrium, they are observed to occur *together* in industrial production, notwithstanding the predictions of the Arrow-Debreu Theorem that such a state of affairs is not possible. The conflict implicit in this consensus is summarized in the two quotations given on page 2 of this thesis.

Sraffa [1926], however, adopted a fundamentally different stance on this question. More than 50 years ago he argued that not all economic forces could be dismissed as being merely passive in respect to the movement towards a competitive equilibrium. Rather, some would actively advantage a producer in some markets and disadvantage him in others. Such forces are inimical to perfect competition, and, as a result, Sraffa argued, the explanation for the observed co-existence of equilibrium and increasing returns lies in abandoning the belief that the equilibrium is competitive. Interestingly, in developing his argument, Sraffa was describing product differentiation and its associated imperfect competition before the formal theory of monopolistic competition was presented.[2]

---

[1] For a recent contribution see Koopmans [1974].

[2] See Chamberlin [1946, p.5n].

Sraffa's ideas stimulated discussion,[1] but in the long run he failed to draw economic thought away from the acceptance of the feasibility of competitive equilibrium in industrial production. It was generally accepted that, while Sraffa may have explained some special cases, in the majority of instances equilibria can be regarded as being competitive. Sraffa failed to identify an active economic force (or system of forces) which was essentially associated with industrial production, and as a result, his ideas could be regarded as being somewhat nebulous.

In his discussion of the relevance of competitive equilibrium theory, Koopmans [1974] suggested that, so far as the existence of increasing returns to scale at competitive equilibrium are concerned:

> "this important point can be met at least half way, by introducing further assumptions that bear on the way time and space enter into the problem. .... A relevent spatial factor is the cost of transporting the product from producer to user."

However, if the freight rate per unit distance is strictly positive, the goods transport costs to which Koopmans refers are *active*, not passive forces. They create product differentiation based upon the producer's locational advantage in respect to markets, and thus, monopolistic competition. They are, therefore, a particular example of the general class of forces discussed by Sraffa, and where their magnitudes are significant, an equilibrium cannot be competitive. Furthermore, the von Thünen rent, which has its origin in these transport costs, is the monopoly profit associated with the imperfect competition.

Nevertheless, there is room to doubt that goods transport costs can explain, in quantitative terms, the degree of increasing returns to scale observed in factory production at equilibrium. This is

---

1 See, for example, "Increasing Returns and the Representative Firm. A Symposium", *Economic Journal*, Vol. 40, 1930, pp.79-116.

because such costs can be expected to affect the nature of the equilibrium in an industry without regard to how production is organized. However, returns to scale in agricultural production seem to be nearly constant at equilibrium, while in factory production, they seem often to be markedly increasing. It would appear, therefore, that, while the costs of goods transport, viewed as economic forces, act in the right direction to explain the co-existence of increasing returns and equilibrium in goods markets, they are of the wrong magnitude. Otherwise, increasing returns would be as likely to be observed in some agricultural production as they are in industry. There is a need to explore the matter further in order to find some active force, which is peculiar to industrial production. Our studies of the optimum town and the optimum state suggest that the opportunity cost of the workforce's travel is an important active force, which is very much more important in industrial production than it is in agricultural production, and in this chapter we will develop this idea to examine the nature of a general equilibrium when production takes place in factories, and markets are assumed to be as perfect as they can be. We will start with an heuristic analysis in order to explore the nature of the problem. We will conclude from this analysis that, given pure competition, all industrial production takes place in *company* towns at equilibrium. We will then move on to construct and solve a formal model of the company town. Throughout our analysis we will assume that all goods transport is free.

7.1 HEURISTIC ANALYSIS

We consider a closed economy in which only two goods are produced, a factory good and an agricultural good. We will assume constant returns to scale in the production of the agricultural good. We will also assume that labor and land are of uniform quality, that there are large numbers of farms and factories, and make any other assumptions necessary to ensure that the prices of the produced goods and the agricultural wage and rent are competitively determined. Among these latter assumptions will be included the assumptions that farm laborers live at their jobs and that labor is perfectly mobile between the manufacturing and rural sectors and within each of these sectors. Our assumption of free goods transport is, of course, crucial to the validity of our model.

Labor is supplied by identical households. Each household contributes one laborer to the workforce, and he supplies one unit of labor per day. Households are the only consumers in the economy. They derive utility from the consumption of the two produced goods and from the enjoyment of leisure time and residential space. Each of these is indispensible to the households. To simplify the exposition we will assume that each household is allocated a residential site of fixed area, and discuss a consequence of this assumption later. Households are assumed to be indifferent to the choice of urban or rural life styles.

Factory laborers cannot live at their jobs. Therefore, to minimise labor transport costs, each factory will be surrounded by a compact residential zone. Given homogeneous land, the factory and its associated residential zone will make up a circular factory town. The only labor transport cost we will consider is the subjective cost of commuting time, which of course, applies only to the factory workforce.

Although we have been able to construct an economy in which the market for agricultural labor is perfect, we cannot make this assumption in respect to the factory labor market. This is because the marginal urban household, that is, the household which supplies the marginal unit of labor to the factory, has the choice of earning the agricultural wage, w_a say, working on a farm, or of living at the boundary of a town and earning the factory wage. If it chooses factory employment, it will forego some leisure in commuting, and therefore, it will be indifferent to the choice if and only if the factory wage exceeds w_a by its evaluation of the leisure time it foregoes. This excess will depend upon the spatial dimensions of the town, which implies that it will depend upon the size of the workforce and the area of the factory. Every laborer in a town will demand the same wage, and hence, at equilibrium, the factory wage is an increasing function of the quantity of labor employed. In respect to this relationship between the price of labor and the quantity employed, each factory owner appears to have a monopsony in the labor market. He faces a wage rate which increases with the quantity of labor he employs, and he is able to effect cost minimization by choice of production scale. However, the monopsony is apparent, not real. There are no barriers to entry to factory production, labor is mobile, and hence, entrepreneurs will set up factories to the point where the supernormal profits, associated with monopsony power, vanish.

A necessary condition for equilibrium is that identical households must derive equal utility. Given that all urban households will receive the same wage, this condition will be realized through the residential land market. Households will bid for residential sites until utility is uniform throughout the town. At the same time, any firm which does not own the town in which its factory is located will have to compete in this same land market for factory space. The nature of the urban land market is, therefore, such that, if we view the households as producers of labor, the residential rent is a von Thünen rent, by means of which the urban landlords extract all producer's surplus from the households. Furthermore, no firm could survive by producing the factory good in a town it did not own, because, so far as production costs are concerned, it would be doubly disadvantaged with respect to company towns. First, it would have to pay a rent on its factory land, which exceeded the agricultural rent, second, it could not, itself, extract the producer's surplus from its workforce in the way that a company town can. Given the perfect agricultural land market and the perfect mobility of households, any firm could rent sufficient broad acres at the agricultural rent to set up its own town, and hence, *all towns in the equilibrium economy will be company towns.*

The structure of our general equilibrium can be summarized as follows. In spatial terms the economy consists of a number of company towns set in an agricultural environment. Each company town consists of a central, circular factory surrounded by an annular,residential zone. The market structure is such that perfect competition prevails in the goods markets and in the agricultural factor markets. There is monopolistic competition in the urban land market and an apparent monopsony in the urban labor market. Additional conditions for equilibrium are that the urban and rural household utilities must be equal, and the urban and agricultural rents must be equal at town boundaries. Note that in introducing spatial considerations into the economy we have had to abandon the concept of competitive equilibrium even though we have assumed goods transport free. Notwithstanding this movement away from perfect competition, our general equilibrium has a fairly simple structure.

Before going on with our solution we observe that, although we have shown that all towns are company towns in our general equilibrium model, there are, in reality, very few company towns. Our pure model is, therefore,not a plausible description of the real world. Why should this be so? There appear to be many reasons. Our assumption of perfect mobility of labor is unrealistic. So also is the assumption, implicit in our essentially static analysis, that new towns can be constructed instantaneously. An effect of choosing more realistic assumptions in respect to these facets of the model would be to make the decision to set up a factory in an existing town more attractive to a firm. In any case, there are many institutional barriers to the development of towns by the private sector, and a firm which wished to construct its own town would probably find that government regulations prevented it from controlling the town as a company town. In addition, we have assumed only one factory produced good. In reality, there are many such goods, and economies of agglomeration exist in their production. As a result, the efficient unit of industrial production is often a large city. Imperfections in the capital market, uncertainty and the possibility of bankruptcy make it difficult for a single firm to accumulate the capital represented by one of these cities.

Given the evident lack of realism of some of our assumptions, the question arises as to whether our model has any relevance to the examination of industrial production. The answer appears to be in the affirmative. The crucial difference between agricultural and industrial production lies in the fact that the factory can only increase its workforce by persuading laborers to commute from more remotely located households, whereas the agricultural producer can distribute his workforce uniformly throughout his farm. In idealizing the structure of urban land and labor markets we may fail to identify where monopoly profits go, but we do not fail to capture this crucial difference.

Continuing with our analysis, we can use the factory good as numeraire, and write the profit of the typical town-owning company as

$$\Pi = F - Lr_a - w_a N - C(L,N), \tag{7.1}$$

where, as before, F is factory output and L and N are the land and labor inputs to factory production, but now w_a is the exogenously determined agricultural wage and C(L,N) is the total subjective value of the commuting time of urban laborers. In formulating (7.1), we have taken into account the fact that every rural household will pay from its wage, w_a, r_a multiplied by the area of its residential site for its consumption of the services of residential space. This means that the term w_aN will include all rent payments to absentee landlords for the residential zone of the town. The form of the term $C(L,N)$ arises from the fact that, in extracting all (labor) producer's surplus from households, the company will pay only the subjective value of commuting time, and not N times the subjective value of the marginal household's commuting time.

Since there is perfect competition in the agricultural land market (i.e., the land market in which companies are buyers), and no barriers to entry in factory production, there can be no monopoly profits to the companies, and we can set

$$\Pi = 0. \tag{7.2}$$

Companies will employ factors in factory production to the point where the marginal conditions

$$F_L = r_a + C_L, \tag{7.3}$$

$$F_N = w_a + C_N, \tag{7.4}$$

hold. Equations (7.1) - (7.4) yield

$$LF_L + NF_N - F = LC_L + NC_N - C. \tag{7.5}$$

Equation (7.5) can be written in the form

$$L\mathscr{F}_L + N\mathscr{F}_N - \mathscr{F}_N = 0, \tag{7.6}$$

where $\mathcal{F} = F - C$ is factory output net of transport costs.

In our model, in which the area of a residential site is constant, the marginal household will make a greater than average contribution to the total transport cost. Furthermore, we can realistically assume that V_L is positive. It follows that the right hand side of (7.5) is strictly positive and that returns to scale in factory production are increasing at equilibrium. However, we see from (7.6) that returns to scale in net factory production are constant at equilibrium. These results, which are derived from profit and utility maximization, are just the results we obtained from our normative study when residential site area was assumed constant and no land was required for transport production. It is to be noted that our conclusion that the right hand side of (7.5) is strictly positive is not necessarily true if households are free to maximize utility by choice of residential site area. This is because the residential rent (i.e., the price ratio of consumption goods) will, in general, depend upon the population of the town. Therefore, when residential site area is a variable, a marginal increase in the population may induce a substitution of the factory good for residential space such that the right hand side of (7.5) is negative.

7.2 MATHEMATICAL ANALYSIS

We now present a formal mathematical analysis of an equilibrium company town. We will continue to assume perfect competition in all goods and agricultural factor markets, and that all goods transport is free. We will continue to use the notation established in earlier chapters. The The CBD and town radii and areas are x_c, x_t, L and A, respectively, and

$$L = \pi x_c^2, \quad A = \pi x_t^2. \tag{7.7}$$

The time taken to travel unit distance at x is $\tau(n(x), g_1(x))$, where,

$$\dot{n}(x) = -2\pi\rho g_2(x), \tag{7.8}$$

the population density on land allocated to residential purposes, ρ, is constant, and

$$x - g_1(x) - g_2(x) = 0. \tag{7.9}$$

At the factory gate each worker sells one unit of labor. The remainder of his day, D say, is divided between leisure and travel. Thus

$$D - \ell(x) - \int_{x_c}^{x} \tau(n, g_1)\, dz = 0, \tag{7.10}$$

where $\ell(x)$ is the leisure available to a laborer who lives at x. The typical household utility function will be assumed to take the form $u = u(f, \ell)$. In this formulation we neglect to include the residential site area, $s = \frac{1}{\rho}$, because it is constant. We could also have included the household's consumption of the agricultural good, a say. However, when goods transport is free, choice of the consumption, $a(x)$, for household utility maximization will clearly give the standard, non-locational conditions, so it is not necessary to pursue this aspect of the urban household's consumption.

Equilibrium in the company town requires that the following conditions hold simultaneously:

(i) equilibrium between land markets, that is, $r(x_t) = r_a$;

(ii) maximization of company profit;

(iii) maximization of the urban household's utility;

(iv) equality of the utility derived by urban and agricultural workers.

Since land has a non-zero cost, and nobody travels at x_t, profit maximization by the company ensures that

$$g_1(x_t) = 0. \tag{7.11}$$

Let w be the factory wage. Then the income constraint on the household at x is

$$w - f - sr(x) = 0. \tag{7.12}$$

Utility maximization is effected by choice of f, ℓ and x, subject to the constraints (7.10) and (7.12). The first order necessary conditions give:

$$\dot{r}(x) = - \rho p(x)\tau(x), \tag{7.13}$$

$$p(x) = p(f(x),\ell(x)) = u_\ell/u_f, \tag{7.14}$$

and equations (7.10) and (7.12). In equations (7.13) and (7.14), p(x) is the household's marginal subjective price of leisure. It is easy to see that these first order conditions ensure that $\dot{u}(x) = 0$, which implies that the company extracts all producer's surplus from the households, its producers of labor.

All agricultural workers, and the factory workers who reside at x_c, have leisure time D. Equality between the utilities of these households, therefore implies that they consume the same quantity of the factory good, f_a say. Thus,

$$w - sr(x_c) = w_a - sr(x_t). \tag{7.15}$$

Equations (7.13) and (7.15) imply

$$w = w_a + \int_{x_c}^{x_t} p(x)\tau(x)dx, \tag{7.16}$$

which means that the factory wage exceeds the agricultural wage by the subjective value of the marginal urban household's commuting time. We derived this relationship intuitively in the previous section of this chapter.

It will be convenient to have (7.10) and (7.12) in differential equation form. That is,

$$\dot{\ell}(x) = -\tau(x), \tag{7.17}$$

$$\dot{f}(x) = p(x)\tau(x). \tag{7.18}$$

The company's profit is given by

$$\Pi = F - wN - r_a A + \int_{x_c}^{x_t} 2\pi g_2(x)r(x)dx, \tag{7.19}$$

and its instruments for profit maximization are $g_1(x)$, $g_2(x)$, w, N, L and A. To determine the conditions for equilibrium in the company town, therefore, we must solve the following problem in Control Theory:

Max Π,

subject to (7.8), (7.9), (7.13), (7.14), (7.16) - (7.19). We have four state variables in this problem: $n(x)$, $r(x)$, $\ell(x)$ and $f(x)$.

The Lagrangian is defined as

$$\begin{aligned}\mathcal{L}(x) \equiv\ & 2\pi\rho g_2(x) \{ sr(x) - \phi_1(x) \} \\ & - \tau(n,g_1) \{ \rho p(f,\ell)\Psi(x) + \phi_2(x) \} \\ & + \mu(x) \{ x - g_1(x) - g_2(x) \}, \end{aligned} \tag{7.20}$$

where

$$\Psi(x) = \phi_3(x) - s\phi_4(x) + \lambda s. \tag{7.21}$$

We also define

$$J \equiv F - wN - Ar_a + \lambda(w - w_a). \tag{7.22}$$

In equations (7.20) and (7.22) we have again neglected the non-negativity constraints on $g_1(x)$ and $g_2(x)$, used λ to indicate the constant Lagrange multiplier and $\mu(x)$ to indicate the variable multiplier. The $\phi_i(x), i = 1,2,3,4$, are the co-state variables associated with the state variables $n(x)$, $\ell(x)$, $r(x)$ and $f(x)$, respectively.

The first order conditions for equilibrium are:

$$\tau_{g_1} \{ \rho p(f,\ell)\Psi(x) + \phi_2(x) \} + \mu(x) = 0; \tag{7.23}$$

$$2\pi\rho \{ sr(x) - \phi_1(x) \} - \mu(x) = 0; \tag{7.24}$$

$$\lambda - N = 0; \tag{7.25}$$

$$F_N - w + \phi_1(x_c) = 0; \tag{7.26}$$

$$F_L - \frac{1}{2\pi x_c}\mathcal{L}(x_c) = F_L - \frac{g_2(x_c)}{x_c}\{ r(x_c) - \rho\phi_1(x_c) \}$$

$$+ \frac{\tau(x_c)}{2\pi x_c}\{ \rho p(x_c)\Psi(x_c) + \phi_2(x_c) \} = 0; \tag{7.27}$$

$$r_a - \frac{1}{2\pi x_t}\mathcal{L}(x_t) = \rho\phi_1(x_t) + \frac{\tau(x_t)}{2\pi x_t}\{ \rho p(x_t)\Psi(x_t) + \phi_2(x_t) \} = 0; \tag{7.28}$$

$$\dot{\phi}_1(x) = \tau_n\{ \rho p(x)\Psi(x) + \phi_2(x) \}; \tag{7.29}$$

$$\dot{\phi}_2(x) = \rho\tau(x)\Psi(x)\frac{\partial p}{\partial \ell}; \tag{7.30}$$

$$\rho\dot{\phi}_3(x) = -2\pi\rho g_2(x); \tag{7.31}$$

$$\dot{\phi}_4(x) = \rho\tau(x)\Psi(x)\frac{\partial p}{\partial f}. \tag{7.32}$$

Using (7.14), (7.17), (7.18), (7.30) and (7.32),

$$\dot{p}(x) = \frac{\partial p}{\partial \ell}\dot{\ell} + \frac{\partial p}{\partial f}\dot{f} = \frac{s}{\Psi}\ \{ p(x)\dot{\phi}_4(x) - \dot{\phi}_2(x) \}. \tag{7.33}$$

Now,

$$\Big[\rho p(x)\Psi(x) + \phi_2(x) \Big]_x^{x_t} = \int_x^{x_t} \rho\dot{p}(z)\Psi(z) + \rho p(z)\dot{\Psi}(z) + \dot{\phi}_2(z) \}dz, \tag{7.34}$$

therefore, using (7.22), (7.33) and (7.34),

$$\rho p(x)\Psi(x) + \phi_2(x) = \{ \rho p(x_t)\Psi(x_t) + \phi_2(x_t) \} - \int_x^{x_t} \rho p(z)\dot{\phi}_3(z)dz. \tag{7.35}$$

In earlier chapters we assumed that the form of $\tau(n,g_1)$ is such that

$$\lim_{x \to x_t} \tau_{g_1} = -\infty. \tag{7.36}$$

Continuing this assumption, we see from (7.23) that

$$\rho p(x_t)\Psi(x_t) + \phi_2(x_t) = 0. \tag{7.37}$$

Therefore, (7.35) reduces to

$$\rho p(x)\Psi(x) + \phi_2(x) = -\int_x^{x_t} \rho p(z)\dot{\phi}_3(z)dz,$$

$$= \int_x^{x_t} 2\pi\rho g_2(z)p(z)dz. \qquad (7.38)$$

In Chapter 4 we identified the right hand side of (7.38) as the total subjective value of the leisure time lost when all workers travelling through x are delayed by unit time. Therefore, using (7.29) and (7.38),

$$-\phi_1(x_c) = \int_{x_c}^{x_t} \tau_n \int_x^{x_t} 2\pi\rho g_2(z)p(z)dzdx, \qquad (7.39)$$

is the value of the leisure time lost by all other commuters when, *ceteris paribus*, one more household is added to the town. The value of the marginal household's leisure time lost in commuting is

$$\int_{x_c}^{x_t} p(x)\tau(x)dx,$$

therefore,

$$-\phi_1(x_c) + \int_{x_c}^{x_t} p(x)\tau(x)dx = C_N, \qquad (7.40)$$

where C is the total subjective value of leisure time lost in commuting in the equilibrium town, and the subscript N refers, as it has done throughout this thesis, to partial differentiation with respect to N with L and the functional structure of the residential zone held constant.

In the heuristic analysis we did not consider a land input to transport. However, in this formal model, the area of transport land, R, is given by

$$R = \int_{x_c}^{x_t} 2\pi g_1(x)\,dx, \tag{7.41}$$

so that, once again,

$$R_N = 0, \tag{7.42}$$

because $g_1(x_t) = 0$. Equations (7.26), (7.40) and (7.42) yield

$$F_N = w_a + C_N + r_a R_N, \tag{7.43}$$

which is analogous to (7.4).

Equation (7.27) may be written in the form

$$F_L = r_a + \frac{\rho g_2(x_c)}{x_c}\left\{\frac{1}{\rho}\{\, r(x_c) - r(x_t)\,\} - \phi_1(x_c)\right\}$$

$$- \frac{\tau(x_c)}{2\pi x_c}\{\, \rho p(x_c)\Psi(x_c) + \phi_2(x_c)\,\} - \frac{g_1(x_c)}{x_c} r_a. \tag{7.44}$$

Integrating (7.13), (7.44) becomes

$$F_L = r_a + \frac{\rho g_2(x_c)}{x_c}\{ - \phi_1(x_c) + \int_{x_c}^{x_t} p(x)\tau(x)dx \}$$

$$- \frac{\tau(x_c)}{2\pi x_c}\{ \rho p(x_c)\Psi(x_c) - \phi_2(x) \} - \frac{g_1(x_c)}{x_c} r_a. \tag{7.45}$$

The last term on the right hand side of (7.45) is familiar from earlier chapters. We can write

$$r_a R_L = - \frac{g_1(x_c)}{x_c} r_a. \tag{7.46}$$

The second last term is also familiar. It and the second term represent C_L. Therefore, (7.45) can be written

$$F_L = r_a + C_L + r_a R_L, \tag{7.47}$$

which is analogous to (7.3).

Using (7.43), (7.47) and the profit equation

$$\Pi = F - Lr_a - Rr_a - w_a N - C(L,N), \tag{7.48}$$

we obtain the analogue of (7.5). That is,

$$LF_L + NF_N - F = LC_L + NC_N - C + \{ LR_L + NR_N - R \} r_a. \tag{7.49}$$

If we ignore the ambiguity of sign in C_L and the term $LR_L r_a$, the sign of the right hand side of (7.49) remains ambiguous, because $-Rr_a$ is of opposite sign to $\{LC_L + NC_N - C\}$. This is the conclusion we reached in Chapter 3; returns to scale in gross factory production may be increasing, decreasing or constant, because there are economies of scale in the use of the transport network. However, net factory production, $\mathcal{F}$, is given by

$$\mathcal{F} = F - C - r_a R, \tag{7.50}$$

and

$$L\mathcal{F}_L + N\mathcal{F}_N - \mathcal{F} = 0. \tag{7.51}$$

In other words,returns to scale in net factory production are constant in the equilibrium factory town. Of course, we know from Chapters 3 and 4 that the lack of ambiguity of this last result depends crucially upon our assumption that residential site area is independent of location.

To this point our solution has been concerned exclusively with production in the equilibrium company town. Looking at the residential zone, it is clear from (7.13) that $\dot{r}(x) < 0$,and from (7.17) and (7.18) that $\dot{\ell}(x) < 0$ and $\dot{f}(x) > 0$. Furthermore, having identified the meaning of $\rho p(x)\Psi + \phi_2(x)$ in (7.38), it is clear from (7.23) and (7.24) that land is allocated to transport in the equilibrium company town according to the same economic criteria as apply in the optimum town of Chapter 4. Thus, apart from the loss of generality which arises from our assumption of constant household residential site area, our equilibrium company town has the same qualitative form as the optimum town in which equals are treated equally and every household provides the same exogenously determined amount of labor to the factory.

7.3 CONCLUSIONS

In this chapter we set out to examine the nature of equilibrium in a pure model of industrial production when markets are as perfect as they can be. A particular aim was to examine the nature of returns to scale in factory production at the equilibrium point. We find that, if markets are as perfect as they can be, all towns are company towns, and the urban factor markets are necessarily imperfect. There is monopolistic competition in the urban land market, which is to say that the residential rent is a von Thünen rent. Furthermore, there is a relationship between the urban wage rate and the quantity of labor employed,which gives the urban labor market the mathematical form of monopsony. As a result we are forced to abandon the concept of competitive equilibrium. We also find that returns to scale in gross factory production will not in general be constant at equilibrium.

We did, however, show that, in our model, returns to scale in net factory production are constant at equilibrium. If that result were generally true, the relevance of competitive equilibrium theory to industrial production could be established by treating the company town as the fundamental and indivisible unit of production. However, knowledge gained from the normative analyses of Chapters 3 and 4 suggests that returns to scale will not be constant, in general, at equilibrium, and that the conclusion of this chapter depends crucially on the simplifying assumption that residential site area is constant across the residential zone. Certainly, the proof we presented that returns to scale are constant at equilibrium depended upon site area being constant. In other words, if we assume away all goods transport costs, if we redefine the problem so that the town is the unit of production, and if we assume all markets to be perfect, we are still unable to show that returns to scale in production will, in general, be constant at equilibrium.

If we go back to the Arrow-Debreu Theorem we can see that this result is predictable. The very strong assumption on the spatial homogeneity of economic activity is crucial to the proof of this theorem, and cannot be weakened without loss of generality. Therefore, when we examine the spatial structure of an economy, we cannot expect that the conclusions of the Arrow-Debreu Theorem will hold in the large scale when the economy is spatially non-homogeneous in the small scale. The imperfect urban factor markets, which have been subsumed in the production unit, the company town, continue to influence the production scale which maximizes company profit.

This concludes our analysis of the equilibrium company town and of the factory town in general.

CHAPTER 8

CONCLUSION

The aim of this thesis has been to investigate the structure of the factory town, and to discover how production in factories modifies the conclusions derived from the theory of the firm. The thesis is, therefore, a study of some aspects of the way in which the location of economic activities influences the conclusions of microeconomic theory. The basic model which has been used is that of a monocentric town divided into two zones. The circular, inner zone is wholly occupied by a factory which produces a single good. The annular, outer zone is the residential area of the factory's workforce, and its land is allocated either to residential purposes or to transport facilities.

The land on which the town is located is of uniform quality, and the labor supplied by the resident households of the town is perfectly homogeneous. These are the factors of production employed in the town's factory. We did introduce capital as a factor employed in factory and housing services production in Chapter 3, but showed that the criteria which determine its optimal employment were not different from those derived from the conventional, non-locational theory of production. For this reason it was excluded from examination in the remainder of the thesis.

Both positive and normative economic aspects of the factory town have been considered, and these two parts of the study are brought together by our showing that, subject to some limitations on the generality of the model, competitive realization of the optimum town is possible. The reduced generality in this regard arises from an apparent lack of a suitable sufficiency theorem when both the inner and outer radii of the residential zone are control parameters in the optimization. However, when the land area of the factory is constant, it was shown in Chapter 3 that the optimum town in which equals are treated equally is the equilibrium town in which all labor is paid the same wage and each commuter pays a transport toll equal to the value of the external transport cost imposed upon the rest of the community by the marginal traveller's journey from his residential location to the central business district.

We chose as our criterion of optimality the maximization of average household utility, and assumed that household utility is derived from the consumption of the factory good, the services of residential space and leisure time. It is implicit in this assumption of the form of the household utility function that each household is indifferent to the level of utility derived by its peers. We confirmed that, in general, it is optimal to treat equal households unequally. Although this result is, by now, familiar in normative studies of the factory town, it is not normally discovered in conventional, non-locational investigations into microeconomic theory. Usually, convexity assumptions in non-locational models guarantee that average household utility is maximized when equals are treated equally. However, in a locational model non-convexity arises from the locational uniqueness of land. In Chapter 3 we described the nature of the locational non-convexity in our model in terms of the non-convex household consumption set. This consumption set takes the form of N mutually orthogonal 2-dimensional plane surfaces in N + 1 dimensional space, where N is the number of household locations available in the town. Points not on these planes do not belong to the consumption set, because they require residence at more than one location, and this is not permitted in the model.

The lack of familiarity of unequal treatment of equals in conventional economic theory gives it a special interest when it is discovered in a study of the optimum factory town. We attempted to examine its implications by investigating polar cases. We studied two cases in respect to the criterion of optimality. First, we examined the case where each household is assumed to be indifferent to the utility derived by others. This is the case already alluded to. The second case, where it differs from the first, is the case where equals are treated equally by constraint.

In setting out the results of these studies it is convenient to divide the description of the structure of the optimum town into three parts. These are:

(i) the distribution of household consumption allocations as functions of residential distance from the centre of the town;

(ii) the allocation of land to transport, and transport congestion across the residential zone; and

(iii) the values of the marginal products of factors employed in the factory.

It was shown in Chapters 3 and 4 that, unless equals are treated equally by constraint, it is optimal to make household allocations of *all* consumption goods, which are treated as control variables in the optimization process, strictly increasing functions of distance from the centre of the town. This result holds irrespective of whether household leisure time is treated as a variable or a constant, and is subject only to the additional assumption (as a sufficient condition) that the second order cross partial derivatives of the utility function are all positive. The reason for this result was presented in Chapter 3. Since the shadow residential rent is a von Thünen rent, it must be a decreasing function of distance. It will, therefore, be optimal to allocate residential site area as an increasing function of distance. However, if the second order cross partials of the utility function are positive, the optimality of an upward sloping residential site area function implies upward sloping allocations of all other consumption goods. In a non-locational model in which the utility function is strictly convex, this argument is negated by the optimality of equality. However, we showed that the upward sloping residential site area function results in a saving in the resource cost of transport, which results in a trade-off against the sub-optimality of unequal treatment when transport is free.

When the equality constraint is imposed upon the optimization of average utility and household leisure time is fixed, it continues to be true that the residential site area is a strictly increasing function of distance, and equality of household utility is achieved by the household allocation of the factory good being a decreasing function of distance. However, we found in Chapter 4 that, when factory work hours, rather than leisure time, are fixed, these unambiguous conclusions are not necessarily valid. Leisure time, in this model, is necessarily a decreasing function of distance, and substitution between the three consumption goods depends upon the shape of the utility function to the extent that there may be regions within the residential zone where residential site area decreases with distance from the town centre, notwithstanding the fact that the residential rent is unambiguously downward sloping. It was also found that this same ambiguity appears in the town of unequals when work hours are fixed.

It was shown in Chapter 5 that, when residential site area is fixed, equals are treated equally in the optimum town. The reason for this result is that fixing the residential site area rules out the possibility of the trade-off between savings in transport resource cost and the sub-optimality of *ceteris paribus* unequal treatment.

Our analysis of the optimal allocation of land to transport was presented in Chapters 3 and 4. It was found that, if leisure time is a control variable or is fixed, optimal transport congestion is unambiguously downward sloping with respect to distance from the centre of the town. Furthermore, for the simple specific transport congestion function examined,the land allocated to transport is also downward sloping. It is never optimal to allocate all land to transport at the boundary of the central business district.

When factory working hours are fixed by constraint, the lack of ambiguity in the slope of the transport congestion function is lost, and it may be optimal for congestion to increase with distance in some regions of the residential zone. The ambiguity in this result is associated with ambiguity in the slope of the shadow price of leisure function. A downward sloping shadow price of leisure function is a necessary condition for the optimal transport congestion function to have a positive slope.

The marginal products of factors employed in factory production were also evaluated in Chapters 3 and 4. It was argued that the marginal product of labor is a contrived concept, and is not of fundamental importance in the optimum factory town. The basis of this argument is that, whereas the marginal product of labor is derived from the effect of an increment in the labor input to production, all other things remaining constant, in our models the labor input to production is increased through an increase in population, and this increase in population affects travel time in the town. Accordingly, we have placed most interest in the marginal products of population and factory area.

The marginal product of population was shown to be equal to the shadow value of the marginal household's consumption in the optimum town in which equals are treated equally. This simply means that households are added to this town to the point where the marginal household just produces the value of its consumption. This result is, perhaps, a predictable extension of the conventional theory of the firm. However, when the equality constraint is removed we find an additional term appears in the equation for the marginal product of population. This term is proportional to the difference between the utilities of the average and marginal households. Its effect is to ensure that, in this optimum town, the marginal household produces less than the value of its consumption.

When equals are treated equally by constraint and factory work hours are fixed, the value of the marginal product of population was shown, in Chapter 4, to be equal to the value of the consumption of the marginal household *plus* the value of the transport congestion caused by the marginal journey from the boundary of the town to the CBD. Since every householder works fixed hours in this model, regardless of his travel time, the entry of the marginal household into the town results in a reduction in the consumption of leisure time by the rest of the town. Household utility can only be maintained by a substitution of the factory good and/or residential space for leisure time. The marginal household, therefore, must not only produce the value of its own consumption, but also the value of the reduction in the town's leisure time its commuting has caused.

Compared with the predictions of the conventional theory of the firm, the value of the marginal product of land derived from the theory of the optimum factory town contains additional terms. These arise from the necessity of displacing households from residential locations at the CBD boundary to the boundary of the town in order to increase the land area allocated to factory production. Thus, in all our models we found that, at the optimum, the value of the marginal product of land equals the opportunity cost of land to the town *plus* terms which are proportional to the differences between the values of the consumptions and utilities of the households which reside at the town and CBD boundaries. When work hours are fixed, there is one further term, which is proportional to the increase in transport congestion brought about by this displacement of households.

Several measures of the degree of increasing returns to scale can be obtained from these evaluations of the marginal products of land and population. The measure most directly significant to economic theory is probably the local degree of homogeneity, at the optimum point, of the production function net of all transport costs. We found from this measure that, except for the town in which work hours are fixed, returns to scale in the optimum town are decreasing regardless of whether or not equals are treated equally by constraint. We found, in Chapter 5, that if residential site area is constant, these returns to scale are constant. Removing this constraint, we found, in Chapters 3 and 4, that returns to scale are strictly decreasing. However, we also found in Chapter 4 that, when factory work hours are fixed, returns to scale in net factory production are strictly increasing. In all our models, returns to scale are expressed in terms of marginal products which refer to outputs per day. The introduction of the fixed work hours constraint results in the marginal products having somewhat different meanings, because, in the other models, the labor input per day could vary. This completes the summary of conclusion in respect to the structure of the optimum town.

A number of comparative static results in respect to the optimum town were obtained in Chapter 5. In contrast to earlier studies in which household income was assumed to be constant, almost all of these results were ambiguous in sign. This ambiguity can be related to the existence of increasing returns to scale in gross factory output. The optimum degree of increasing returns to scale may change in either direction in response to a change in a shift parameter. Thus, the scale of production may increase or decrease, and this, of course, implies that factor inputs, town size and population may increase or decrease.

Using the opportunity cost of land as a shift parameter, we found that the induced changes in the optimum values of the areas of the factory and the town, the population and the workforce are all ambiguous in sign. In respect to the area of the factory, this means that an increase in the price of land may induce an increase in the employment of land. However, this result is not paradoxical, because we were able to show that, if the area of the factory increased, labor also increased, and in such a way that the land/labor ratio decreased.

The time taken for a commuter to travel unit distance and the population density, both assumed to be constants with respect to location, were also used as shift parameters, and the induced changes in the factory and the town areas, the population and the workforce were evaluated. All of these results were found to be ambiguous in sign with the single exception that an increase in the transport time induces an unambiguous decrease in the area of the factory in the optimum town.

It is well known that the factory town is a von Thünen model; the factory and the residential zone being two von Thünen "rings". However, these rings had not, hitherto, been integrated into a model containing the classical, agricultural rings. This was done in Chapter 6, in a normative context, in order to examine the optimum industrial state.

The constant residential density model formulated in Chapter 5 was used with one agricultural ring added. It was shown that the town and the agricultural zone each separately "paid its own way". The agricultural production function was assumed to exhibit constant returns to scale, and marginal productivity conditions of standard form in respect to shadow prices were derived. However, the shadow prices of the factors employed in this zone were shown to be location dependent. Household utility is constant across the profile of the state, but the shadow wage rate in the agricultural zone is a decreasing function of distance from the centre of the state. This is because the von Thünen, agricultural rent is a downwards sloping function of distance from the market, and therefore, the residential rents of the agricultural workers must be downward sloping. Thus, the shadow wage of agricultural labor must be downward sloping when the household utility is constant.

The equilibrium town was examined in Chapter 7. The ultimate purpose of this examination was to make a contribution to the long standing debate on the possibility of the co-existence of increasing returns to scale and competitive equilibrium. For this purpose a pure model of an industrial economy in which markets are "as perfect as they can be" was chosen. It was shown that, at equilibrium, all towns are company towns, and, while the goods and agricultural factor markets may be competitive if freight costs are zero, imperfections in the urban factor markets are inevitable. This is because homogeneous labor must be paid the same wage in the equilibrium town, notwithstanding the fact that a worker's travel time depends upon the location of his residential site. The market imperfections are manifest, in a pure model, as an apparent monopsony in the urban labor market and monopolistic competition in the urban land market. These imperfections mean that the equilibrium cannot be competitive.

It was shown that returns to scale in gross factory production - the measure of production usually considered in discussions on the co-existence of increasing returns and equilibrium - may be increasing, constant or decreasing. However, since the debate has been based upon the assumption that observed equilibria are approximately competitive, the debate is misconceived, and no anomaly between observation and theory in respect to competitive equilibrium and returns to scale has been demonstrated.

This completes our summary of the results obtained in this thesis.

It remains to suggest some directions for further work. Three different lines for further research are apparent.

First, in respect to the models in this thesis, there is a need for further examination of the relationship between the optimum and equilibrium towns. It appears that this programme would require further development of control theory, either by extending the generality of sufficiency and existence theorems or by treating the central business district and residential zone together in the formulation of the welfare function. This latter approach implies that the lower limit of the welfare function is constant at zero, but it introduces mathematical difficulties in respect to the description of the state variables. In a similar mathematical vein, there is a need to generalize the company town by making household residential site area a control variable. It could be treated either as an instrument of company profit maximization or of household utility maximization. We saw in Chapters 3 and 4 compared with Chapters 5 and 6 that, when site area is a variable, relationships in the optimal town take more general forms. However, we also saw that this generalization is associated with greater complexity in the mathematical analysis. Second, our models were all based upon the assumption of uniform land quality. There is a need to apply the techniques developed in the theory of the factory town to the Weberian point location theory. This theory is based upon differences in the spatial distribution of resources, but, being a point theory, fails to provide a satisfactory rent theory. An integration of these two approaches to spatial economics would offer significant advances to our present knowledge of economic theory.

Finally, our purely theoretical approach begs the question of how our conclusions may be applied by policy makers in the discovery of solutions to urban problems. A great deal has been said about the need to improve urban life style, but we might suspect that many of the current *ad hoc* remedies, which involve subsidies, serve to aggravate the problems they are intended to cure. For example, New York, a large, heavily transport dependent city is said by its administrators to face bankruptcy, and therefore, to require subsidy. Our results suggest that there is an optimum size for such a city, and that this size depends upon transport cost. Our limited view of transport can be extended in a qualitative way to include such costs as garbage removal and the reticulation of other services, and we must suspect that the size of New York is very much greater than optimum, and that this excessive size is the fundamental cause of its financial problems.

Of course, the non-malleability of urban capital precludes the possibility of the rapid restructuring of New York into the optimum town. However, if our theory has application, it suggests that subsidies intended directly to overcome New York's immediate problems will compound them by encouraging further *expansion*. If our theory supports the policy of subsidization at all, that support would be for subsidizing the transfer of industry from New York to new towns. In other words, for subsidizing the *contraction* of New York.

BIBLIOGRAPHY

Alonso, W., [1960], "A Theory of the Urban Land Market", *Regional Science Association Papers*, Vol.6, pp.149-157.

Alonso, W., [1964], *Location and Land Use*, Harvard University Press, Cambridge.

Arrow K.J. and M. Kurz, [1970], *Public Investment, The Rate of Return, and Optimal Fiscal Policy*, Johns Hopkins, Baltimore.

Beckmann, M.J., [1969], "On the Distribution of Urban Rent and Residential Density", *Journal of Economic Theory*, Vol.1, pp.60-67.

Beckmann, M.J., [1972], "Von Thünen's Model Revisited: A Neo-Classical Land Use Model", *Swedish Journal of Economics*, Vol.74, pp.1-7.

Chamberlin, E.H., [1948], *The Theory of Monopolistic Competition: A Re-orientation of the Theory of Value*, Harvard University Press, Cambridge.

Clark, C., [1951], "Urban Population Densities", *Journal of the Royal Statistical Society*, Series A. Vol.114, pp.490-496.

Debreu, G., [1959], *Theory of Value: An Axiomatic Analysis of Economic Equilibrium*, Yale University Press, New Haven.

Delson, J.K. [1970], "Corrections on the Boundary Conditions in Beckmann's Model on Urban Rent and Residential Density", *Journal of Economic Theory*, Vol.2, pp.314-318.

Dixit, A., [1973], "The Optimum Factory Town", *Bell Journal of Economics and Management Science*, Vol.4, pp.637-651.

Goldstein, G. and L.N. Moses, [1973], "A Survey of Urban Economics", *Journal of Economic Literature*, Vol.11, pp.471-515.

Haig, R.M., [1926], "Towards an Understanding of the Metropolis", *Quarterly Journal of Economics*, Vol.40, pp.179-208.

Haig, R.M., [1927], *Regional Survey of New York and Its Environs*, New York City Plan Commission, New York.

Hochman, O. and D. Pines,[1971], "Competitive Equilibrium of Transportation and Housing in the Residential Ring of an Urban Area", *Environment and Planning*, Vol.3, pp.51-62.

Hochman, O. and D. Pines, [1972], "Note on Land Use in a Long Narrow City", *Journal of Economic Theory*, Vol.5, pp.540-541.

Hurd, R.M., [1903], *Principle of City Land Values*, The Record and Guide, New York.

Kaldor, N., [1972], "The Irrelevence of Equilibrium Economics", *Economic Journal*, Vol.82, pp.1237-1255.

Kanemoto, Y. [1975], "Congestion and Cost Benefit Analysis in Cities", *Journal of Urban Economics*, Vol.2, pp.246-264.

Koopmans, T.C., [1974], "Is the Theory of Competitive Equilibrium With It?", *American Economic Review Papers and Proceedings*, pp.325-329.

Kornai, J., [1971], *Anti-Equilibrium: On Economic Systems Theory and the Task of Research*, North Holland, Amsterdam.

Legey, L., M. Ripper and P. Varaiya, [1973], "Effects of Congestion on the Shape of Cities", *Journal of Economic Theory*, Vol.6, pp.162-179.

Levhari, D., Y. Oron and D. Pines, [1972], "In Favor of a Lottery in Cases of Non-Convexity", Working Paper, Centre for Urban and Regional Studies, Tel-Aviv University.

Livesey, D.A., [1973], "Optimal City Size: A Minimum Congestion Approach", *Journal of Economic Theory*, Vol.6, pp.144-161.

Long, N.V., and N. Vousden [1977], "Optimal Control Theorems", in J.D. Pitchford and S. Turnovsky, eds., *Applications of Control Theory to Economic Analysis*, North Holland, Amsterdam, Chapter 1.

Malinvaud, E., [1972], *Lectures on Microeconomic Theory*, North Holland, Amsterdam.

Mills, E.S., [1967], "An Aggregative Model of Resource Allocation in a Metropolitan Area", *American Economic Review*, Vol.57, pp.197-210.

Mills, E.S., [1969], "The Value of Urban Land", in H.S. Perloff, ed., *The Quality of the Urban Environment*, Resources for the Future, Washington.

Mills, E.S., [1972], *Studies in the Structure of the Urban Economy*, Johns Hopkins, Baltimore.

Mills, E.S., and D.M. de Ferranti, [1971], "Market Choice and Optimum City Size", *American Economic Review*, Vol.61, pp.340-345.

Mills, E.S. and J. MacKinnon, [1973], "Notes on the New Urban Economics", *Bell Journal of Economics and Management Science*, Vol.4, pp.593-601.

Mirrlees, J., [1972], "The Optimum Town", *Swedish Journal of Economics*, Vol.74, pp.114-135.

Mirrlees, J., [1973], "Rejoinder to Richardson - II", *Urban Studies*, Vol. 10, pp.267-269.

Montesano, A., [1972], "A Restatement of Urban Rents and Residential Density", *Journal of Economic Theory*, Vol.4, pp.329-354.

Muth, R.F., [1961], "Economic Change and Rural-Urban Land Conversion", *Econometrica*, Vol.29, pp.1-23.

Muth, R.F., [1969], *Cities and Housing*, University of Chicago Press, Chicago.

Nagel, N., [1963], "Assumptions in Economic Theory", *American Economic Review*, Vol.53, pp.211-219.

Oron, Y., D. Pines and E. Sheshinski, [1973], "Optimum vs Equilibrium Land Use Pattern and Congestion Toll", *Bell Journal of Economics and Management Science*, Vol.4, pp.619-636.

Oron, Y., D. Pines and E. Sheshinski, [1974], "The Effects of Nuisances Associated with Urban Traffic on Suburbanization and Land Values", *Journal of Urban Economics*, Vol.1, pp.382-394.

Renaud, B.M. [1972], "On a Neo-Classical Model of Land Use", *Swedish Journal of Economics*, Vol. 74, pp.400-405.

Richardson, H.W., [1973], "A Comment on Some Uses of Mathematical Models in Urban Economics", *Urban Studies*, Vol.10, pp.259-270.

Riley, J.G., [1973], "Gammaville: An Optimum Town", *Journal of Economic Theory*, Vol.6, pp.471-482.

Riley, J.G., [1974], "Optimal Residential Density and Road Transportation", *Journal of Urban Economics*, Vol.1, pp.230-249.

Robinson, J., [1932], "Imperfect Competition and Falling Supply Price", *Economic Journal*, Vol.42, pp.544-554.

Sheshinski, E., [1973], "Congestion and Optimum City Size", *American Economic Review*, Vol.63, pp.61-66.

Solow, R., [1972], "Congestion, Density and the Use of Land in Transportation", *Swedish Journal of Economics*, Vol.74, pp.161-1973.

Solow, R., [1973a], "Congestion Cost and the Use of Land for Streets", *Bell Journal of Economics and Management Science*, Vol.4, pp.602-618.

Solow, R. [1973b], "On Equilibrium Models of Urban Location", in J.M. Parkin, ed., *Essays in Modern Economics*, Longmans, London.

Solow, R., [1973c], "Rejoinder to Richardson - I", *Urban Studies*, Vol.10, p.267.

Solow, R., and W. Vickrey, [1971], "Land Use in a Long Narrow City", *Journal of Economic Theory*, Vol.3, pp.403-447.

Sraffa, P., [1926], "The Laws of Returns Under Competitive Conditions", *Economic Journal*, Vol.36, pp.180-197.

Starrett, D., [1974], "Principles of Optimal Location in a Large Homogeneous Area", *Journal of Economic Theory*, Vol.9, pp.418-448.

Stern, N.H., [1973], "Homogeneous Utility Functions and Equality in 'The Optimum Town'", *Swedish Journal of Economics*, Vol.75, pp. 204-207.

Vickrey, W., [1965], "Pricing as a Tool in Coordination of Local Transportation", in J. Meyer, ed., *Transportation Economics*, National Bureau of Economic Research, New York.

Wartenberg, C.M., [1966], *von Thünen's Isolated State*, Pergamon, London.

Wheaton, W.C., [1974], "A Comparative Static Analysis of Urban Spatial Structure", *Journal of Economic Theory*, Vol.9, pp.223-237.

Wingo, L., [1961], *Transportation and Urban Land Use*, Resources for the Future, Washington.