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CYLINDRICAL ESTIMATES FOR HYPERSURFACES MOVING BY CONVEX CURVATURE FUNCTIONS
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We prove a complete family of cylindrical estimates for solutions of a class of fully nonlinear curvature flows, generalising the cylindrical estimate of Huisken and Sinestrari [Invent. Math. 175:1 (2009), 1–14, §5] for the mean curvature flow. More precisely, we show, for the class of flows considered, that, at points where the curvature is becoming large, an \((m+1)\)-convex (\(0 \leq m \leq n - 2\)) solution either becomes strictly \(m\)-convex or its Weingarten map becomes that of a cylinder \(\mathbb{R}^m \times S^{n-m}\). This result complements the convexity estimate we proved with McCoy [Anal. PDE 7:2 (2014), 407–433] for the same class of flows.

1. Introduction

Let \(M\) be a smooth, closed manifold of dimension \(n\), and \(X_0 : M \rightarrow \mathbb{R}^{n+1}\) a smooth hypersurface immersion. We are interested in smooth families \(X : M \times [0, T) \rightarrow \mathbb{R}^{n+1}\) of smooth immersions \(X(\cdot, t)\) solving the initial value problem

\[
\begin{align*}
\partial_t X(x, t) &= -F(W(x, t))\nu(x, t), \\
X(\cdot, 0) &= X_0,
\end{align*}
\]

(CF)

where \(\nu\) is the outer normal field of the evolving hypersurface \(X\) and \(W\) the corresponding Weingarten curvature. In order that the problem (CF) be well-posed, we require that \(F(W)\) be given by a smooth, symmetric function \(f : \Gamma \rightarrow \mathbb{R}\) of the principal curvatures \(\kappa_i\) which is monotone increasing in each argument. The symmetry of \(f\) ensures that \(F\) is a smooth, basis-invariant function of the components of the Weingarten map (or an orthonormal frame-invariant function of the components of the second fundamental form) [Glaeser 1963]. Monotonicity ensures that the flow is (weakly) parabolic. This guarantees local existence of solutions of (CF), as long as the principal curvature \(n\)-tuple of the initial data lies in \(\Gamma\); see [Langford 2014].

For technical reasons, we require some additional conditions:

**Conditions.**

(i) \(f\) is homogeneous of degree one.

(ii) \(f\) is convex.

Since the normal points out of the region enclosed by the solution, we may assume, by condition (ii), that \((1, \ldots, 1) \in \Gamma\). Thus, by condition (i), we may further assume that \(f\) is normalised such that \(f(1, \ldots, 1) = 1\).

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The additional conditions (i)–(ii) have several consequences. Most importantly, they allow us to obtain a preserved cone $\Gamma_0 \subset \Gamma$ of curvatures for the flow (Lemma 2.2). This allows us to obtain uniform estimates on any degree-zero homogeneous function of curvature along the flow (Lemma 2.3); in particular, we deduce a uniform parabolicity condition (Corollary 2.4). The convexity condition then allows us to apply the second derivative Hölder estimate of [Evans 1982; Krylov 1982] to deduce that the solution exists on a maximal time interval $[0, T), T < \infty$, such that $\max_{M \times [t]} F \to \infty$ as $t \to T$; see [Andrews et al. 2014a, Proposition 2.6]. Thus, it is of interest to study the behaviour of solutions as $F \to \infty$. Let us recall the following curvature estimate [Andrews et al. 2014b] (cf. [Huisken and Sinestrari 1999a; 1999b]).

**Theorem 1.1** (convexity estimate). Let $X : M \times [0, T) \to \mathbb{R}^{n+1}$ be a solution of (CF) such that $f$ satisfies conditions (i)–(ii). Then, for all $\varepsilon > 0$, there is a constant $C_\varepsilon < \infty$ such that

$$G(x, t) \leq \varepsilon F(x, t) + C_\varepsilon \quad \text{for all } (x, t) \in M \times [0, T),$$

where $G$ is given by a smooth, nonnegative, degree-one homogeneous function of the principal curvatures of the evolving hypersurface that vanishes at a point $(x, t)$ if and only if $W(x, t) \geq 0$.

We remark that the constant $C_\varepsilon$ depends only on $\varepsilon$, the dimension $n$, the choice of speed function $f$, the preserved curvature cone $\Gamma_0$, and bounds for the initial volume and diameter [Langford 2014].

Theorem 1.1 implies that the ratio of the smallest principal curvature to the speed is almost positive wherever the curvature is large. Combining it with the differential Harnack inequality of [Andrews 1994b] and the strong maximum principle [Hamilton 1986] yields useful information about the geometry of solutions of (CF) near singularities [Andrews et al. 2014b] (cf. [Huisken and Sinestrari 1999a; 1999b]):

**Corollary 1.2.** Any blow-up limit of a solution of (CF) is weakly convex. In particular, any type-II blow-up limit about a type-II singularity is an eternal solution of the form $X_\infty : (\mathbb{R}^k \times \Gamma^{n-k}) \times \mathbb{R} \to \mathbb{R}^{n+1}$, $k \in \{0, 1, \ldots, n-1\}$, such that $X_\infty|_{\mathbb{R}^k}$ is flat, and $X_\infty|_{\Gamma^{n-k}}$ is a strictly convex translation solution of the corresponding flow in $\mathbb{R}^{n-k+1}$.

Motivated by the surgery construction of [Huisken and Sinestrari 2009, §5] for 2-convex mean curvature flow, we will apply Theorem 1.1 to obtain the following family of cylindrical estimates for solutions of (CF):

**Theorem 1.3** (cylindrical estimate). Let $X$ be a solution of (CF) such that conditions (i)–(ii) hold. Suppose also that $X$ is uniformly $(m+1)$-convex for some $m \in \{0, 1, \ldots, n-2\}$. That is, $\kappa_1 + \cdots + \kappa_{m+1} \geq \beta F$ for some $\beta > 0$. Then, for all $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that

$$G_m(x, t) \leq \varepsilon F(x, t) + C_\varepsilon \quad \text{for all } (x, t) \in M \times [0, T),$$

where $G_m : M \times [0, T) \to \mathbb{R}$ is given by a smooth, nonnegative, degree-one homogeneous function of the principal curvatures that vanishes at a point $(x, t)$ if and only if

$$\kappa_1(x, t) + \cdots + \kappa_{m+1}(x, t) \geq \frac{1}{c_m} f(\kappa_1(x, t), \ldots, \kappa_n(x, t)),$$

where $c_m$ is the value $F$ takes on the unit radius cylinder $\mathbb{R}^m \times S^{n-m}$.
We note that the constant $C_{\varepsilon}$ will only depend on $\varepsilon$, $\beta$, $m$, the dimension $n$, the choice of speed function $f$, the preserved curvature cone $\Gamma_0$, and upper bounds for the initial volume and diameter. Theorem 1.3 implies that the ratio of the quantity

$$K_m := \kappa_1 + \cdots + \kappa_{m+1} - \frac{1}{c_m} F$$

to the speed is almost positive wherever the curvature is large. Observe that this quantity is nonnegative on a weakly convex hypersurface $\Sigma$ only if either $\Sigma$ is strictly $m$-convex or $\Sigma = \mathbb{R}^m \times S^{n-m}$. In particular, we find that, whenever $\kappa_1(x, t) + \cdots + \kappa_m(x, t)$ is small compared to the speed, the Weingarten curvature is close to that of a thin, round cylinder $\mathbb{R}^m \times S^{n-m}$. We therefore obtain a refinement of Corollary 1.2:

**Corollary 1.4.** Any blow-up limit of an $(m+1)$-convex, $0 \leq m \leq n-2$, solution of (CF) is either strictly $m$-convex, or a shrinking cylinder $\mathbb{R}^m \times S^{n-m}$. In particular, if the blow-up is of type-II, then this limit is of the form $X_\infty : (\mathbb{R}^k \times \Gamma^{n-k}) \times \mathbb{R} \to \mathbb{R}^{n+1}$ for $k \in \{0, 1, \ldots, m-1\}$, such that $X_\infty|_{\mathbb{R}^k}$ is flat and $X_\infty|_{\Gamma^{n-k}}$ is a strictly convex translation solution of the corresponding flow in $\mathbb{R}^{n-k+1}$.

The $m = 0$ case of the cylindrical estimates demonstrates that convex hypersurfaces become umbilic at points where the curvature is blowing up, generalising a result of Huisken [1984, Theorem 5.1] for the mean curvature flow (we note that the convergence result of [Huisken 1984] has been obtained by the first author for the class of flows considered here without the need for such an estimate [Andrews 1994a]). Moreover, Huisken and Sinestrari [2009] have recently obtained the $m = 1$ case of the cylindrical estimates for the mean curvature flow, making spectacular use of it through their surgery program, which yields a classification of 2-convex hypersurfaces. The convexity and cylindrical estimates stated above, in addition to generalising the Huisken–Sinestrari cylindrical estimate to all $m$ in $\{0, \ldots, n-2\}$, constitute a first step towards improving upon such results by allowing a larger class of evolution equations.

### 2. Preliminaries

We will follow the notation used in [Andrews et al. 2014b]. In particular, we recall that a smooth, symmetric function $g$ of the principal curvatures gives rise to a smooth function $G$ of the components $h_i^j$ of the Weingarten map. Equivalently, $G$ is an orthonormal frame invariant function of the components $h_{ij}$ of the second fundamental form. To simplify notation, we denote $G(x, t) \equiv G(W(x, t)) = g(\kappa(x, t))$ and use dots to denote derivatives of functions of curvature as follows:

$$\dot{g}^k(z)v_k = \frac{d}{ds} \bigg|_{s=0} g(z + sv), \quad \dot{G}^{kl}(A)B_{kl} = \frac{d}{ds} \bigg|_{s=0} G(A + sB),$$

$$\ddot{g}^{pq}(z)v_pv_q = \frac{d^2}{ds^2} \bigg|_{s=0} g(z + sv), \quad \ddot{G}^{pq,rs}(A)B_{pq}B_{rs} = \frac{d^2}{ds^2} \bigg|_{s=0} G(A + sB).$$

The derivatives of $g$ and $G$ are related in the following way:

**Lemma 2.1** [Gerhardt 1996; Andrews 1994a; 2007]. Let $g : \Gamma \to \mathbb{R}$ be a smooth, symmetric function. Define the function $G : \mathcal{G}_\Gamma := \mathbb{R}$ by $G(A) \equiv g(\lambda(A))$, where $\lambda(A)$ denotes the eigenvalues of $A$ (up to order) and $\mathcal{G}_\Gamma$ denotes the set of symmetric matrices with eigenvalues in $\Gamma$. Then, for any diagonal $A \in \mathcal{G}_\Gamma$,
\[ \dot{G}^{kl}(A) = \dot{g}^k(\lambda(A))\delta^{kl}, \]  
(2-1)

and, for any diagonal \( A \in S_\Gamma \) with distinct eigenvalues and any symmetric \( B \in \text{GL}(n) \),

\[ \dot{G}^{pq,rs}(A)B_{pq}B_{rs} = \dot{g}^{pq}(\lambda(A))B_{pp}B_{qq} + 2\sum_{p>q} \frac{\dot{g}^p(\lambda(A)) - \dot{g}^q(\lambda(A))}{\lambda_p(A) - \lambda_q(A)}(B_{pq})^2. \]  
(2-2)

We note that \( \dot{g} \geq 0 \) if and only if \((\dot{g}^p - \dot{g}^q)(z_p - z_q) \geq 0 \) for all \( p, q \) [Andrews et al. 2014b, Lemma 2.2], so Lemma 2.1 implies that \( G \) is convex if and only if \( g \) is convex.

The following useful lemma was proved in [Andrews et al. 2014b]:

**Lemma 2.2.** Let \( f : \Gamma \to \mathbb{R} \) be a flow speed for (CF) satisfying Conditions (i)–(ii). Then, for any admissible initial datum \( X_0 : M \to \mathbb{R}^{n+1} \) there exists a cone \( \Gamma_0 \subset \mathbb{R}^n \) satisfying \( \overline{\Gamma}_0 \setminus \{0\} \subset \Gamma \) such that the principal curvatures of the solution \( X : M \times [0,T] \to \mathbb{R}^{n+1} \) of the initial value problem (CF) satisfy

\[ \kappa(x,t) := (\kappa_1(x,t), \ldots, \kappa_n(x,t)) \in \Gamma_0 \text{ for all } (x,t) \in M \times [0,T). \]

We refer to such a cone \( \Gamma_0 \) as a preserved cone for the solution \( X \). As mentioned in the introduction, the existence of a preserved cone allows us to obtain bounds for homogeneous functions of the curvature:

**Lemma 2.3.** Let \( X : M \times [0,T) \to \mathbb{R}^{n+1} \) be a solution of (CF) such that \( f \) satisfies conditions (i)–(ii). Let \( g : \Gamma \to \mathbb{R} \) be a smooth, degree-zero homogeneous symmetric function. Then there exists \( c > 0 \) (depending only on \( n, f \) and \( M_0 \)) such that

\[ -c \leq g(\kappa_1(x,t), \ldots, \kappa_n(x,t)) \leq c \text{ for all } (x,t) \in M \times [0,T). \]

If \( g > 0 \), then there exists \( c > 0 \) such that

\[ \frac{1}{c} \leq g(\kappa_1(x,t), \ldots, \kappa_n(x,t)) \leq c. \]

**Proof.** Let \( \Gamma_0 \) be a preserved cone for the solution \( X \). Then \( K := \overline{\Gamma}_0 \cap S^n \) is compact. Since \( g \) is continuous, the required bounds hold on \( K \). But these extend to \( \overline{\Gamma}_0 \setminus \{0\} \) by homogeneity. The claim follows since \( \kappa(x,t) \in \overline{\Gamma}_0 \setminus \{0\} \) for all \( (x,t) \in M \times [0,T) \).

By condition (i), the derivative \( \dot{f} \) of \( f \) is homogeneous of degree zero. Since \( f^k > 0 \) for each \( k \), we obtain uniform parabolicity of the flow:

**Corollary 2.4.** There exists a constant \( c > 0 \) (depending only on \( n, f \) and \( M_0 \)) such that, for any \( v \in T^*M \), it holds that

\[ \frac{1}{c}|v|^2 \leq \dot{F}^{ij}v_iv_j \leq c|v|^2, \]

where \( |\cdot| \) is the (time-dependent) norm on \( M \) corresponding to the (time-dependent) metric induced by the flow.

We now recall the following evolution equation (see for example [Andrews et al. 2013]).
Lemma 2.5. Let $X: M \times [0, T) \to \mathbb{R}^{n+1}$ be a solution of (CF) such that $f$ satisfies conditions (i)–(ii). Let $G: M \times [0, T) \to \mathbb{R}$ be given by a smooth, symmetric, degree-one homogeneous function $g$ of the principal curvatures. Then $G$ satisfies the evolution equation

$$(\partial_t - \mathcal{L})G = (\dot{G}^{kl} F^{pq,rs} - \dot{F}^{kl} G^{pq,rs}) \nabla h_{pq} \nabla h_{rs} + G |\nabla |^2_F,$$  (2-3)

where $\mathcal{L} := \dot{F}^{kl} \nabla_k \nabla_l$ is the linearisation of $F$, and $|\nabla |^2_F := \dot{F}^{kl} h_k^r h_r^l$.

In particular, the speed function $F$ satisfies $(\partial_t - \mathcal{L}) F = F |\nabla |^2_F$.

As we shall see, in order to obtain Theorem 1.3, it is crucial to obtain a good upper bound on the term

$Q(\nabla W, \nabla W) := (\dot{G}^{kl} F^{pq,rs} - \dot{F}^{kl} G^{pq,rs}) \nabla_k h_{pq} \nabla_l h_{rs}$

for the pinching functions $G_m$ which we construct in the following section. The following decomposition of $Q$ is crucial in obtaining this bound.

Lemma 2.6. For any totally symmetric $T \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n$, we have

$$\left(\dot{G}^{kl} F^{pq,rs} - \dot{F}^{kl} G^{pq,rs}\right)_{T_{kpq} T_{lrs}} = \left(\dot{g}^k \dot{f}^{pq} - \dot{f}^k \dot{g}^{pq}\right)_{T_{kpq} T_{lrs}} + 2 \sum_{p > q} \frac{(\dot{f}^p \dot{g}^q - \dot{g}^p \dot{f}^q)}{z_p - z_q} (T_{pq})^2 + 2 \sum_{k > p > q} \frac{(\dot{g}_{kpq} \times \dot{f}_{kpq})}{z_p - z_q} z_{kpq} (T_{kpq})^2$$  (2-4)

at any diagonal matrix $B$ with distinct eigenvalues $z_i$, where “$\times”$ and “$\cdot$” are the three-dimensional cross and dot product respectively, and we have defined the vectors

$$\ddot{f}_{kpq} := (\dot{f}^k, \dot{f}^p, \dot{f}^q),$$

$$\ddot{g}_{kpq} := (\dot{g}^k, \dot{g}^p, \dot{g}^q),$$

$$\ddot{z}_{kpq} := \left(\frac{z_p - z_q}{(z_k - z_p)(z_k - z_q)}, \frac{z_k - z_q}{(z_k - z_p)(z_k - z_q)}, \frac{z_k - z_p}{(z_k - z_p)(z_k - z_q)}\right).$$

Proof: Since $B$ is diagonal, Lemma 2.1 yields (suppressing the dependence on $B$)

$$(\dot{G}^{kl} F^{pq,rs} - \dot{F}^{kl} G^{pq,rs})_{T_{kpq} T_{lrs}} = \sum_{k, p, q} (\dot{g}^k \dot{f}^{pq} - \dot{f}^k \dot{g}^{pq})_{T_{kpq} T_{lrs}} + 2 \sum_k \sum_{p > q} \left(\dot{g}^k \dot{f}^{p-q} - \dot{f}^k \dot{g}^{p-q}\right) (T_{kpq})^2.$$  (2-4)

We now decompose the second term into the terms satisfying $k = p, k = q, k > p, p > k > q,$ and $q > k$ respectively:

$$\sum_k \sum_{p > q} \left(\dot{g}^k \dot{f}^{p-q} - \dot{f}^k \dot{g}^{p-q}\right) (T_{kpq})^2$$

$$= \sum_{p > q} \left(\dot{g}^p \dot{f}^{p-q} - \dot{f}^p \dot{g}^{p-q}\right) (T_{pq})^2 + \sum_{p > q} \left(\dot{g}^q \dot{f}^{p-q} - \dot{f}^q \dot{g}^{p-q}\right) (T_{pq})^2$$

$$+ \left(\sum_{k > p > q} + \sum_{p > k > q} + \sum_{p > q > k}\right) \left(\dot{g}^k \dot{f}^{p-q} - \dot{f}^k \dot{g}^{p-q}\right) (T_{kpq})^2.$$
\[
\begin{align*}
&= \sum_{p>q} \frac{\dot{f}^p \dot{g}^q - \dot{g}^p \dot{f}^q}{z_p - z_q} ((T_{pq})^2 + (T_{qp})^2) \\
&\quad + \sum_{k>p>q} \left( \frac{\dot{g}^k \dot{f}^p - \dot{f}^k \dot{g}^p}{z_p - z_q} - \frac{\dot{f}^k \dot{g}^q - \dot{g}^k \dot{f}^q}{z_k - z_q} - \frac{\dot{f}^p \dot{g}^q - \dot{g}^p \dot{f}^q}{z_k - z_p} + \frac{\dot{g}^q \ddot{f}^k - \ddot{f}^q \dot{g}^k}{z_k - z_p} \right) (T_{kpq})^2 \\
&= \sum_{p>q} \frac{\dot{f}^p \dot{g}^q - \dot{g}^p \dot{f}^q}{z_p - z_q} ((T_{pq})^2 + (T_{qp})^2) + \sum_{k>p>q} \left( \frac{\dot{g}^k \dot{f}^p - \dot{f}^k \dot{g}^p}{z_p - z_q} \left( \frac{1}{z_k - z_q} - \frac{1}{z_k - z_q} \right) \right) - \left( \frac{\dot{f}^k \dot{g}^q - \dot{g}^k \dot{f}^q}{z_k - z_q} \right) \left( \frac{1}{z_k - z_p} - \frac{1}{z_k - z_q} \right) \right) (T_{kpq})^2 \\
&= \sum_{p>q} \frac{\dot{f}^p \dot{g}^q - \dot{g}^p \dot{f}^q}{z_p - z_q} ((T_{pq})^2 + (T_{qp})^2) + \sum_{k>p>q} (\ddot{g}_{kpq} \times \ddot{f}_{kpq} \cdot z_{kpq} (T_{kpq})^2). \quad \Box
\end{align*}
\]

We complete this section by proving that \((m+1)\)-convexity is preserved by the flow \((\text{CF})\), so that this assumption need only be made on initial data:

**Proposition 2.7.** Let \(X\) be a solution of \((\text{CF})\) such that conditions (i)–(ii) are satisfied. Suppose that there is some \(m \in \{1, \ldots, n-1\}\) and some \(\beta > 0\) such that

\[
\kappa_{\sigma(1)}(x, 0) + \cdots + \kappa_{\sigma(m)}(x, 0) \geq \beta F(x, 0)
\]

for all \(x \in M\) and all permutations \(\sigma \in P_n\). Then this estimate persists at all later times.

**Proof.** Denote by \(SM\) the unit tangent bundle over \(M \times [0, T)\) and consider the function \(Z\) defined on \(\bigoplus^m SM\) by

\[
Z(x, t, \xi_1, \ldots, \xi_m) = \sum_{\alpha=1}^m h(\xi_{\alpha}, \xi_\alpha) - \beta F(x, t).
\]

Since we have

\[
\inf_{\xi_1, \ldots, \xi_m \in S(x, t)M} Z(x, t, \xi_1, \ldots, \xi_m) = \kappa_{\sigma(1)}(x, t) + \cdots + \kappa_{\sigma(m)}(x, t) - \beta F(x, t)
\]

for some \(\sigma \in P_n\), it suffices to show that \(Z\) remains nonnegative. First fix any \(t_1 \in [0, T)\) and consider the function \(Z_\varepsilon(x, t, \xi_1, \ldots, \xi_m) := Z(x, t, \xi_1, \ldots, \xi_m) + \varepsilon e^{(1+C)t}\), where \(C := \sup M \times [0, t_1] \|W\|^2_F\). Note that \(C\) is finite since \(M\) is compact and \(\dot{F}\) is bounded. Observe that \(Z_\varepsilon\) is positive when \(t = 0\). We will show that \(Z_\varepsilon\) remains positive on \(M \times [0, t_1]\) for all \(\varepsilon > 0\). So suppose to the contrary that \(Z_\varepsilon\) vanishes at some point \((x_0, t_0, \xi_{01}, \ldots, \xi_{0m})\). We may assume that \(t_0\) is the first such time. Now extend the vector \(\xi^0 := (\xi^0_1, \ldots, \xi^0_m)\) to a field \(\xi := (\xi_1, \ldots, \xi_n)\) near \((x_0, t_0)\) by parallel translation in space and solving

\[
\frac{\partial \xi^i}{\partial t} = F \xi^j \, h_j^i.
\]

Since the metric evolves according to

\[
\frac{\partial g_{ij}}{\partial t} = -2 F h_{ij}
\]
the resulting fields have unit length. Now recall (see for example [Andrews 1994a]) the following evolution equation for the second fundamental form:

\[ \partial_t h_{ij} = \mathcal{L} h_{ij} + \dot{F}^{pq,rs} \nabla_i h_{pq} \nabla_j h_{rs} + |\mathcal{W}|^2_F h_{ij} - 2 F h_{ij}^2, \]

where \( \mathcal{L} := \dot{F}^{kl} \nabla_k \nabla_l \) and \( |\mathcal{W}|^2_F := \dot{F}^{kl} h_{kl}^2 \). It follows that

\[ (\partial_t - \mathcal{L})(Z_\varepsilon(x, t, \xi)) = \varepsilon (1 + C) e^{(1+C) t} + \sum_{a=1}^{m} \dot{F}^{pq,rs} \nabla_{\xi_a} h_{pq} \nabla_{\xi_a} h_{rs} + |\mathcal{W}(x, t)|^2_F Z(x, t, \xi) \geq \varepsilon (1 + C) e^{(1+C)t_0} - C \varepsilon e^{(1+C)t_0} = \varepsilon e^{(1+C)t_0} > 0. \]

Since the point \((x_0, t_0, \xi_{t_0})\) is a minimum of \( Z_\varepsilon \), we obtain

\[ 0 \geq (\partial_t - \mathcal{L})|_{(x_0, t_0)} (Z_\varepsilon(x, t, \xi)) \geq \varepsilon (1 + C) e^{(1+C)t_0} - C \varepsilon e^{(1+C)t_0} = \varepsilon e^{(1+C)t_0} > 0. \]

This is a contradiction, implying that \( Z_\varepsilon \) cannot vanish at any time in the interval \([0, t_1]\). Since \( \varepsilon > 0 \) was arbitrary, we find \( Z \geq 0 \) at all times in the interval \([0, t_1]\). Since \( t_1 \in [0, T) \) was arbitrary, we obtain \( Z \geq 0. \)

\[ \square \]

### 3. Constructing the pinching function

In this section we construct the pinching functions \( G_m \) satisfying the conditions in **Theorem 1.3**. Let us first introduce the *pinching cones*

\[ \Gamma_m := \{ z \in \Gamma : z_{\sigma(1)} + \cdots + z_{\sigma(m+1)} > c_{m}^{-1} f(z) \text{ for all } \sigma \in H_m \}, \]

where \( H_m \) is the quotient of \( P_n \), the group of permutations of the set \( \{1, \ldots, n\} \), by the equivalence relation

\[ \sigma \sim \omega \quad \text{if} \quad \sigma([1, \ldots, m+1]) = \omega([1, \ldots, m+1]). \]

Using the methods of [Huisken 1984], and their adaptations to 2-convex flows in [Huisken and Sinestrari 2009] and fully nonlinear flows in [Andrews et al. 2014b], we will see that, in order to prove **Theorem 1.3**, it suffices to construct a smooth function \( g_m : \Gamma \to \mathbb{R} \) satisfying the following properties.

**Properties.**

(i) \( g_m(z) \geq 0 \) for all \( z \in \Gamma \) with equality if and only if \( z \in \overline{\Gamma}_m \cap \Gamma \).

(ii) \( g_m \) is smooth and homogeneous of degree one.

(iii) For every \( \varepsilon > 0 \) there exists \( c_\varepsilon > 0 \) such that

\[ (C_{m}^{kl} \dot{F}^{pq,rs} - \dot{F}^{kl} \dot{G}_{m}^{pq,rs}) \big|_{B} T_{kpq} T_{lrs} \leq -c_\varepsilon |T|^2 \frac{F}{F} \]

for all \( B \in \mathcal{B}_{\Gamma_0} \) satisfying \( G_m(B) \geq \varepsilon F(B) \) and all totally symmetric \( T \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n \), where \( G_m \) is the matrix function corresponding to \( g_m \) as described in **Section 2**, and \( \Gamma_0 \) is a preserved cone for the flow.
(iv) For every $\delta > 0$, $\epsilon > 0$, and $C > 0$, there exist $\gamma_1 > 0$, $\gamma_2 > 0$ and $\gamma_3 > 0$ such that
\[
|G_m \dot{F}^{kl} - F \dot{G}_m^{kl}|_B B_{kl}^2 \leq -\gamma_1 F^2 (G_m - \delta \gamma_2 F) |_B + \gamma_3 C F^2 |_B
\]
for all $(m + 1)$-positive $B \in \mathcal{T}_{\Gamma_0}$ satisfying $G_m(B) \geq \epsilon F(B)$ and
\[
\lambda_{\min}(B) \geq -\delta F(B) - C.
\]

Our construction of the pinching function $g_m$ will be similar for each choice of $m$. So let us fix $m \in \{0, 1, \ldots, n - 2\}$ and assume that the flow is $(m+1)$-convex. We first consider the preliminary function $g : \Gamma \to \mathbb{R}$ defined by
\[
g(z) := f(z) \sum_{\sigma \in H_m} \varphi \left( \frac{\sum_{i=1}^{m+1} z_{\sigma(i)} - c_{m-1} f(z)}{f(z)} \right),
\]
(3-1)
where $\varphi : \mathbb{R} \to \mathbb{R}$ is a smooth
1 function which is strictly convex and positive, except on $\mathbb{R}^+ \cup \{0\}$ where it vanishes identically. Such a function is readily constructed; for example, we could take
\[
\varphi(r) = \begin{cases} 
  r^4 e^{-1/r^2} & \text{if } r < 0, \\
  0 & \text{if } r \geq 0.
\end{cases}
\]
We note that such a function necessarily satisfies $\varphi(r) - r \varphi'(r) \leq 0$ and $\varphi'(r) \leq 0$ with equality if and only if $r \geq 0$.

Now define the scalar $G : M \times [0, T) \to \mathbb{R}$ by
\[
G(x, t) := g(\kappa_1(x, t), \ldots, \kappa_n(x, t)).
\]
Then $G$ is a smooth, degree-one homogeneous function of the components of the Weingarten map which is invariant under a change of basis. Moreover, $G$ is nonnegative and vanishes at, and only at, points for which the sum of the smallest $(m+1)$-principal curvatures is not less than $c_{m-1} F$. Thus properties (i) and (ii) are satisfied by $g$.

We now show that property (iii) is satisfied weakly by $g$:

**Lemma 3.1.** Let $G$ be the matrix function corresponding to the function $g$ defined by (3-1). Then, for any symmetric matrix $B$ and totally symmetric 3-tensor $T$,
\[
(\dot{G}^{kl} \dot{F}^{pq,rs} - \dot{F}^{kl} \dot{G}^{pq,rs}) |_B T_{kpq} T_{lrs} \leq 0.
\]

**Proof.** We will show that each of the terms in the decomposition (2-4) in Lemma 2.6 is nonpositive. Note that, by the invariance properties of $G$ and $F$, it suffices to prove the claim for diagonal $B$. In fact, we can also assume that $B$ has distinct eigenvalues, since the result at an arbitrary diagonal matrix $B$ may then be

---

1 In fact, $\varphi$ need only be twice continuously differentiable.
obtained by taking a limit $B^{(k)} \to B$ such that each matrix $B^{(k)}$ has distinct eigenvalues. We first compute

$$
\ddot{g}^k = \dot{f}^k \sum_{\sigma \in H_m} \varphi(r_{\sigma}) + \sum_{\sigma \in H_m} \varphi'(r_{\sigma}) \sum_{i=1}^{m+1} \left( \delta^k_{\sigma(i)} - \frac{z_{\sigma(i)}}{f} \dot{f}^k \right)
$$

$$
= \dot{f}^k \sum_{\sigma \in H_m} \left( \varphi(r_{\sigma}) - \varphi'(r_{\sigma}) \sum_{i=1}^{m+1} \frac{z_{\sigma(i)}}{f} \right) + \sum_{\sigma \in H_m} \sum_{i=1}^{m+1} \varphi'(r_{\sigma}) \delta^k_{\sigma(i)}
$$

and

$$
\dddot{g}^{pq} = \left( \sum_{\sigma \in H_m} \varphi(r_{\sigma}) - \sum_{\sigma \in H_m} \varphi'(r_{\sigma}) \sum_{i=1}^{m+1} \frac{z_{\sigma(i)}}{f} \right) \dddot{g}^{pq}
$$

$$
+ \sum_{\sigma \in H_m} \frac{\varphi''(r_{\sigma})}{f} \sum_{i=1}^{m+1} \left( \delta^p_{\sigma(i)} - \frac{z_{\sigma(i)}}{f} \dot{f}^p \right) \sum_{i=1}^{m+1} \left( \delta^q_{\sigma(i)} - \frac{z_{\sigma(i)}}{f} \dot{f}^q \right),
$$

where we have set

$$
r_{\sigma}(z) := \sum_{i=1}^{m+1} \frac{z_{\sigma(i)} - c^{-1}_{m} f(z)}{f(z)}.
$$

It follows that

$$
\dddot{g}^{pq} \dot{f}^q - \dddot{g}^{pq} \dot{f}^k = \sum_{\sigma \in H_m} \sum_{i=1}^{m+1} \varphi'(r_{\sigma}) \delta^k_{\sigma(i)} \dddot{g}^{pq} \dot{f}^q - \dot{f}^k \sum_{\sigma \in H_m} \frac{\varphi''(r_{\sigma})}{f} \sum_{i=1}^{m+1} \left( \delta^p_{\sigma(i)} - \frac{z_{\sigma(i)}}{f} \dot{f}^p \right) \sum_{i=1}^{m+1} \left( \delta^q_{\sigma(i)} - \frac{z_{\sigma(i)}}{f} \dot{f}^q \right).
$$

If we fix the index $k$ and set $\xi = T_{kpp}$, then, by convexity of $\varphi$ and positivity of $\dot{f}^k$, we have

$$
- \dot{f}^k \sum_{\sigma \in H_m} \frac{\varphi''(r_{\sigma})}{f} \sum_{i=1}^{m+1} \left( \delta^p_{\sigma(i)} - \frac{z_{\sigma(i)}}{f} \dot{f}^p \right) \sum_{i=1}^{m+1} \left( \delta^q_{\sigma(i)} - \frac{z_{\sigma(i)}}{f} \dot{f}^q \right) \xi \dot{\xi} \leq 0.
$$

On the other hand, since $\varphi$ is monotone nonincreasing, and $f$ is convex, we have

$$
\varphi'(r_{\sigma}) \sum_{i=1}^{m+1} \delta^k_{\sigma(i)} \dddot{g}^{pq} \xi \dot{\xi} \leq 0
$$

for each $\sigma$. Since both inequalities hold for all $k$, we deduce that

$$
\sum_{k,p,q} \left( \dddot{g}^{pq} \dot{f}^q - \dddot{g}^{pq} \dot{f}^k \right) T_{kpp} T_{qq} \leq 0.
$$

We next consider

$$
\dot{f}^p \dddot{g}^q - \dot{f}^p \dddot{g}^q = \sum_{\sigma \in H_m} \varphi'(r_{\sigma}) (\delta^q_{\sigma(i)} \dot{f}^p - \delta^p_{\sigma(i)} \dot{f}^q) = \sum_{\sigma \in O_q} \varphi'(r_{\sigma}) \dot{f}^p - \sum_{\sigma \in O_p} \varphi'(r_{\sigma}) \dot{f}^q.
$$
where we have introduced the sets
\[ O_a := \{ \sigma \in H_m : a \in \sigma([1, \ldots, m+1]) \}. \]

If \( z_p > z_q \), we obtain
\[
\hat{f}^p \hat{g}^q - \hat{g}^p \hat{f}^q \leq \hat{f}^p \left( \sum_{\sigma \in O_q} \varphi'(r_\sigma) - \sum_{\sigma \in O_p} \varphi'(r_\sigma) \right).
\]

We now show that the term in brackets is nonpositive whenever \( z_p > z_q \).

**Lemma 3.2.** If \( z_p \geq z_q \), then
\[
\sum_{\sigma \in O_p} \varphi'(r_\sigma) - \sum_{\sigma \in O_q} \varphi'(r_\sigma) \geq 0.
\]
Moreover, equality holds only if either \( z_p = z_q \) or \( r_\sigma(z) \geq 0 \) for all \( \sigma \in O_{q,p} := O_q \setminus O_p \).

**Proof of Lemma 3.2.** First note that
\[
\sum_{\sigma \in O_p} \varphi'(r_\sigma) - \sum_{\sigma \in O_q} \varphi'(r_\sigma) = \sum_{\sigma \in O_{p,q}} \varphi'(r_\sigma) - \sum_{\sigma \in O_{q,p}} \varphi'(r_\sigma),
\]
where \( O_{a,b} := O_a \setminus O_b \). Next observe that, if \( \sigma \in O_{p,q} \), then
\[
z_{\sigma(1)} + \cdots + z_{\sigma(m+1)} = z_p + z_\hat{\sigma}((i_1)) + \cdots + z_\hat{\sigma}((i_m))
\]
for some \( \hat{\sigma} \in H_{m-2}(p, q) := P_{n-2}(p, q) / \sim \), where \( P_{n-2}(p, q) \) denotes the set of permutations of \( \{1, \ldots, n\} \setminus \{p, q\} ; i_1, \ldots, i_m \) are \( m \) distinct elements of \( \{1, \ldots, n\} \setminus \{p, q\} \); and \( \sim \) is defined by
\[
\hat{\sigma} \sim \hat{\omega} \quad \text{if} \quad \hat{\sigma}((i_1, \ldots, i_m)) = \hat{\omega}((i_1, \ldots, i_m)).
\]

Observe also that the converse holds (that is, (3-2) defines a bijection), so that
\[
\sum_{\sigma \in O_{q,p}} \varphi'(r_\sigma) - \sum_{\sigma \in O_{p,q}} \varphi'(r_\sigma) = \sum_{\hat{\sigma} \in H_{m-2}(p, q)} \left( \varphi' \left( \frac{z_p + \sum_{k=1}^m z_\hat{\sigma}(i_k) - c_m^{-1} f}{f} \right) - \varphi' \left( \frac{z_q + \sum_{k=1}^m z_\hat{\sigma}(i_k) - c_m^{-1} f}{f} \right) \right).
\]
Since \( z_p \geq z_q \), the claim follows from (strict) convexity of \( \varphi \) (where it is positive).

Thus,
\[
\sum_{p > q} \frac{\hat{f}^p \hat{g}^q - \hat{g}^p \hat{f}^q}{z_p - z_q} \left( (T_{pq})^2 + (T_{qpp})^2 \right) \leq 0.
\]

We now compute
\[
\hat{g}_{kpq} = \left( \frac{g}{f} - \sum_{\sigma \in H_m} \varphi'(r_\sigma) \sum_{i=1}^{m+1} \frac{z_{\sigma(i)}}{f} \right) \hat{f}_{kpq} + \sum_{\sigma \in H_m} \varphi'(r_\sigma) \sum_{i=1}^{m+1} \left( \delta^k_{\sigma(i)} , \delta^p_{\sigma(i)} , \delta^q_{\sigma(i)} \right),
\]
Applying Lemma 3.2 yields

\[ (\vec{g}_{kpq} \times \vec{f}_{kpq}) \cdot \vec{z}_{kpq} = \sum_{\sigma \in H_m} \sum_{i=1}^{m+1} \phi'(r_\sigma) \left[ (\delta^k_{\sigma(i)} f^q - \delta^q_{\sigma(i)} f^p)(z_p - z_q) \right] \cdot \vec{z}_{kpq} \]

so that

\[ (\vec{g}_{kpq} \times \vec{f}_{kpq}) \cdot \vec{z}_{kpq} = \sum_{\sigma \in H_m} \sum_{i=1}^{m+1} \phi'(r_\sigma) \left[ \frac{(\delta^k_{\sigma(i)} f^q - \delta^q_{\sigma(i)} f^p)(z_p - z_q)}{(z_k - z_p)(z_k - z_q)} + \frac{(\delta^q_{\sigma(i)} f^k - \delta^k_{\sigma(i)} f^q)(z_k - z_q)}{(z_k - z_p)(z_k - z_q)} \right]. \]

Removing the positive factor \( \alpha_{kpq} := [(z_k - z_p)(z_k - z_q)(z_p - z_q)]^{-1} \) and setting

\[ P_a := \sum_{\sigma \in O_a} \phi'(r_\sigma), \]

we obtain

\[ (\vec{g}_{kpq} \times \vec{f}_{kpq}) \cdot \vec{z}_{kpq} = \alpha_{kpq} \left[ (P_p f^q - P_q f^p)(z_p - z_q)^2 + (P_q f^k - P_k f^q)(z_k - z_q)^2 + (P_k f^p - P_p f^k)(z_k - z_p)^2 \right]. \]

Applying Lemma 3.2 yields

\[ (\vec{g}_{kpq} \times \vec{f}_{kpq}) \cdot \vec{z}_{kpq} \leq \alpha_{kpq} \left[ (P_p f^k - P_q f^q)(z_k - z_q)^2 - (z_k - z_p)^2 - (z_p - z_q)^2 \right]. \]

Since the term in square brackets is nonnegative, applying Lemma 3.2 once more yields

\[ (\vec{g}_{kpq} \times \vec{f}_{kpq}) \cdot \vec{z}_{kpq} \leq 0. \]

This completes the proof of the lemma.

\[ \square \]

**Corollary 3.3.** There exists \( C < \infty \) (depending only on \( n, f \) and \( M_0 \)) such that \( G / F \leq C \) along the flow.

**Proof.** In view of Lemma 3.1 and the evolution equation (2-3), this is a simple application of the maximum principle.

\[ \square \]

In order to obtain the uniform estimate required by property (iii), we modify \( G \) in order to obtain a function with a strict convexity property. A well-known trick (cf. [Andrews 1994b, Lemma 7.10; Huisken and Sinestrari 1999a, Theorem 2.14; Andrews et al. 2014b, Lemma 3.3]) then allows us to extract the required uniform estimate. First, we relabel the preliminary pinching function \( g \to g_1 \) (\( G \to G_1 \)), and consider the new pinching function \( g \) defined by

\[ g := K(g_1, g_2) := \frac{g_1^2}{g_2} \] \hspace{1cm} (3-3)

where \( g_2(z) = M \sum_{i=1}^n z_i - |z| \) for some large constant \( M \gg 1 \), for which \( g_2 \) is positive along the flow. That there is such a constant follows from applying the maximum principle to the evolution equation (2-3) for the function \( G_2(x, t) := g_2(\kappa(x, t)) \) as in [Andrews et al. 2014b, Lemma 3.1]. Note that \( \dot{K}^1 > 0 \), \( \dot{K}^2 < 0 \) and \( \ddot{K} > 0 \) wherever \( g_1 > 0 \).
Observe that properties (i) and (ii) are not harmed in the transition from \( g_1 \) to \( g \). We now show that the estimates listed in properties (iii) and (iv) are satisfied by the curvature function defined in (3-3).

**Proposition 3.4.** Let \( g \) be the pinching function defined by (3-3) and \( G \) its corresponding matrix function. Then, for every \( \varepsilon > 0 \), there exists \( c_\varepsilon > 0 \) (depending only on \( \varepsilon \), \( n \), \( f \) and \( \Gamma_0 \)) such that

\[
(\hat{G}^{kl} \hat{F}_{pq,rs} - \hat{F}^{kl} \hat{G}^{pq,rs})|_B T_{kpq} T_{irs} \leq -c_\varepsilon \frac{|T|^2}{F}
\]

for all \( B \in \mathcal{H}_{\Gamma_0} \) satisfying \( G(B) \geq \varepsilon F(B) \) and all totally symmetric \( T \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n \).

**Proof.** First note that (suppressing dependence on \( B \))

\[
(\hat{G}^{kl} \hat{F}_{pq,rs} - \hat{F}^{kl} \hat{G}^{pq,rs}) T_{kpq} T_{irs} = \hat{K}^a (\hat{G}^{kl} \hat{F}_{pq,rs} - \hat{F}^{kl} \hat{G}^{pq,rs}) T_{kpq} T_{irs} - \hat{F}^{kl} \hat{K}^\alpha \hat{G}^{pq,rs} T_{kpq} T_{irs} \\
\leq \hat{K}^2 (\hat{G}_{2}^{kl} \hat{F}_{pq,rs} - \hat{F}^{kl} \hat{G}^{pq,rs}) T_{kpq} T_{irs} \\
\leq -\hat{K}^2 \hat{F}^{kl} \hat{G}_{2}^{pq,rs} T_{kpq} T_{irs},
\]

where we used Lemma 3.1, convexity of \( K \), and the inequalities \( \hat{K}^1 \geq 0 \) and \( \hat{F} \geq 0 \) in the first inequality, and the inequalities \( \hat{G}_2 \geq 0 \) and \( \hat{K}^2 \leq 0 \), and convexity of \( F \) in the second. Since \( \hat{K}^2 < 0 \) whenever \( G_1 > 0 \) and \( G_2 \) is strictly concave in nonradial directions, the claim follows exactly as in [Andrews et al. 2014b, Lemma 3.3].

The uniform estimate of **Proposition 3.4** yields a good bound for the term \( Q(\nabla W, \nabla W) \) in the evolution equations for the pinching functions. This is a crucial component in obtaining the \( L^p \)-estimates of the following section. This is the starting point for the Stampacchia–de Giorgi iteration argument. The second crucial estimate is the Poincaré-type inequality, Lemma 4.2 (see also [Huisken and Sinestrari 2009, §§4–5; in particular, Lemma 5.5]), which we can obtain with the help of property (iv). This estimate (corresponding to [Huisken and Sinestrari 2009, Lemma 5.2]) provides an estimate on the zero order term that occurs in contracting the Simons-type identity for \( \hat{F}^{pq} \nabla_p \nabla_q h_{ij} \) with \( \hat{G}^{ij} \) (see [Andrews et al. 2014b, Proposition 4.4]).

**Proposition 3.5.** Let \( g \) be the pinching function defined by (3-3) and \( G \) its corresponding matrix function. Then for every \( \delta > 0 \), \( \varepsilon > 0 \), and \( C > 0 \) there exist \( \gamma_1 > 0 \), \( \gamma_2 > 0 \) and \( \gamma_3 > 0 \) (depending only on \( \delta \), \( \varepsilon > 0 \), \( C \), \( n \), \( m \), \( f \) and \( \Gamma_0 \)) such that

\[
Z(B) := (F \hat{G}^{kl} - G \hat{F}^{kl})|_B B_{kl}^2 \geq \gamma_1 F^2(G - \delta F) - \gamma_3 F^2|_B
\]

for all symmetric, \((m+1)\)-positive matrices \( B \) satisfying \( \lambda(B) \in \Gamma_0 \), \( \lambda_{\text{min}}(B) \geq -\delta F(B) - C \), and \( G_m(B) \geq \varepsilon F(B) \).

**Proof.** From the definition of \( G \) we have

\[
Z = \hat{K}^1 Z_1 + \hat{K}^2 Z_2,
\]

where

\[
Z_i(B) := (F \hat{G}_i^{kl} - G_i \hat{F}^{kl})|_B B_{kl}^2.
\]
Thus, since $\dot{K}^2 = 2g_1/g_2$ is uniformly bounded below when $g \geq \varepsilon f$, it suffices to prove the estimate for $Z_1$.

So let $B$ be a symmetric, $(m+1)$-positive matrix with eigenvalues $z_1 \leq \cdots \leq z_n$. Then

$$Z_1(B) = f \sum_{p>q} (\dot{g}_1^p \dot{z}_p - g_1 \dot{f}_p \dot{z}_p) = \sum_{p>q} (\dot{g}_1^p \dot{f}_p - \dot{g}_1 \dot{f}_p) z_p z_q (z_p - z_q) = \sum_{p>q} (P_p \dot{f}_p - q \dot{f}_p) z_p z_q (z_p - z_q) = \left( \sum_{p>q} + \sum_{p=l \geq q} + \sum_{l \geq p > q} \right) (P_p \dot{f}_p - q \dot{f}_p) z_p z_q (z_p - z_q),$$

where we recall the notation $P_n := \sum_{\sigma \in O_q} \psi'(r_{\sigma})$ and we have defined $l \leq m$ as the number of nonpositive eigenvalues $z_i$. Recalling that $P_p \dot{f}_p - q \dot{f}_p \geq 0$ whenever $z_p \geq z_q$, we discard the final sum and part of the first to obtain

$$Z_1(B) \geq \sum_{p=m+2}^{m+1} \sum_{q=l+1}^{l+1} (P_p \dot{f}_p - q \dot{f}_p) z_p z_q (z_p - z_q) + \sum_{p=l+1}^{p=m+1} \sum_{q=1}^{l+1} (P_p \dot{f}_p - q \dot{f}_p) z_p z_q (z_p - z_q) \geq \sum_{i=1}^{l} \sum_{p=m+2}^{m+1} \sum_{q=l+1}^{l+1} (P_p \dot{f}_p - q \dot{f}_p) z_p z_q (z_p - z_q) - f \sum_{i=1}^{l} z_i + \sum_{p=l+1}^{p=m+1} \sum_{q=1}^{l} (P_p \dot{f}_p - q \dot{f}_p) z_p z_q (z_p - z_q).$$

So consider the term

$$S_1(z) := \sum_{p=m+2}^{m+1} \sum_{q=l+1}^{l+1} (P_p(z) \dot{f}_p(z) - q(z) \dot{f}_p(z)) z_p z_q (z_p - z_q) - f(z)^2 \sum_{i=1}^{l} z_i.$$

Observe that $S_1 \geq 0$. We claim that $S_1(z) > 0$ for all $z$ in the cone

$$\Gamma_{\varepsilon,l} := \{ z \in \Gamma_0 : g(z) \geq \varepsilon f(z), \ z_1 \leq \cdots \leq z_l \leq 0 < z_{l+1} \leq \cdots \leq z_n \}.$$

Suppose, to the contrary, that $S_1(z) = 0$ for some $z \in \Gamma_{\varepsilon,l}$. Then $z_1 = \cdots = z_l = 0$ and, for all $p > m+1 \geq q > l$, $(P_p(z) \dot{f}_p(z) - q(z) \dot{f}_p(z)) z_p z_q (z_p - z_q) = 0$. But, by Lemma 3.2, the latter implies that, for all $p > m+1 \geq q > l$, either $z_p = z_q$, or $r_{\sigma}(\lambda) \geq 0$ for all $\sigma \in O_q, p$. Note that the latter case cannot occur: since $p > m+1 \geq q$, there is a permutation $\sigma \in O_q, p$ such that $0 \leq r_{\sigma}(z) = (z_1 + \cdots + z_{m+1} - c_m^{-1} f(z))/f(z)$, which implies $g_1(z) = 0$, contradicting $z \in \Gamma_{\varepsilon,l}$. On the other hand, if $z_p = z_q$ for all $p > m+1 \geq q > l$, then we again obtain the contradiction $g_1(z) = 0$. Thus, $S_1 > 0$ on $\Gamma_{\varepsilon,l}$. Since $S_1$ is homogeneous of degree three, it follows that

$$S_1 \geq c_1 f^2 g$$

on $\Gamma_{\varepsilon,l}$, where $c_1 := \min_l \min_{\Gamma_{\varepsilon,l}} S_1(f^2 g) > 0$.

Now consider

$$S_2 := f^2 \sum_{i=1}^{l} \lambda_i + \sum_{p=l+1}^{p=m+1} \sum_{q=1}^{l} (P_p \dot{f}_p - q \dot{f}_p) z_p z_q (z_p - z_q).$$
Note that, by homogeneity, \( c_2 := \sup \{ P_p(z) \hat{f}^q(z) - P_q(z) \hat{f}^p(z) : z \in \Gamma_0, \ 1 \leq p, q \leq n \} < \infty \). Thus, \( S_2 \) is easily controlled using the “convexity estimate” \( \lambda_1 \geq -\delta f - C \):

\[
S_2 \geq -I f^2 (\delta f + C) + (n - l) c_2 z_n \sum_{q=1}^l z_q (z_n - z_q) \geq -n f^2 (\delta f + C) + 2nc_2 c_3^f \sum_{q=1}^l z_q
\]

\[
\geq -n f^2 (\delta f + C) - 2nc_2 c_3^f (\delta F + C) \geq -n (1 + 2c_2 c_3^f) f^2 (\delta f + C),
\]

where \( c_3 := \max \{|z_i|/f(z) : z \in \Gamma_0, 1 \leq i \leq n\} \).

The claim follows.

We note that the above estimate is only useful in the presence of the convexity estimate Theorem 1.1, since then, for any \( \delta > 0 \), there is a constant \( C_\delta > 0 \) for which \( \Gamma_{\delta, C_\delta} := \{ z \in \Gamma_0 : z_i > -\delta f(z) - C_\delta \text{ for all } i \} \) is preserved by the flow.

**4. Proof of Theorem 1.3**

In order to prove Theorem 1.3, it suffices to obtain, for any \( \epsilon > 0 \), an upper bound on the function

\[
G_{\epsilon, \sigma} := \left( \frac{G}{F} - \epsilon \right) F^\sigma
\]

for some \( \sigma > 0 \). We will use the estimates of Propositions 3.5 and 3.4 to obtain bounds on the spacetime \( L^p \)-norms of the positive part of \( G_{\epsilon, \sigma} \), so long as \( p \) is sufficiently large and \( \sigma \) sufficiently small, just as in [Huisken and Sinestrari 1999b; 1999a; 2009] (see also [Andrews et al. 2014b] where these techniques are applied in the fully nonlinear setting). A Stampacchia–de Giorgi iteration procedure similar to that used in [Huisken 1984] (see also [Huisken and Sinestrari 1999b; Andrews et al. 2014b]) then allows us to extract a supremum bound on \( G_{\epsilon, \sigma} \).

We begin with an evolution equation for \( G_{\epsilon, \sigma} \):

**Lemma 4.1 [Andrews et al. 2014b].** The function \( G_{\epsilon, \sigma} \) satisfies the evolution equation

\[
(\partial_t - \mathcal{L})G_{\epsilon, \sigma} = F^{\sigma - 1} (\hat{G}^k G_{pq, rs} + \hat{G}^k G_{pq, rs}) \nabla_k h_{pq} \nabla_l h_{rs}
\]

\[
+ \frac{2(1 - \sigma)}{F} (\nabla G_{\epsilon, \sigma}, \nabla F)_F - \frac{\sigma(1 - \sigma)}{F^2} |\nabla F|^2_F + \sigma G_{\epsilon, \sigma} |W|^2_F,
\]

where \( (u, v)_F := \hat{\nabla}^k u_k u_l \).

Now set \( E := \max \{ G_{\epsilon, \sigma}, 0 \} \). We need to obtain spacetime \( L^p \)-estimates for \( E \). Let us first observe that integration by parts and application of Young’s inequality, in conjunction with Lemma 2.3 and Proposition 3.4, yields the estimate (cf. [Andrews et al. 2014b])

\[
\frac{d}{dt} \int E^p \ d\mu \leq - (A_1 p(p - 1) - A_2 p^{\frac{3}{2}}) \int E^{p-2} |\nabla G_{\epsilon, \sigma}|^2 \ d\mu
\]

\[
- (B_1 p - B_2 p^{\frac{1}{2}}) \int E^{p - \frac{3}{2}} |\nabla W|^2_F \ d\mu + C_1 \sigma p \int E^p |W|^2 \ d\mu
\]

(4-2)

for some positive constants \( A_1, A_2, B_1, B_2, C_1 \) (which depend only on \( \epsilon, n, m, f \) and \( M_0 \)).
To estimate the final term, we make use of Proposition 3.5 in a similar manner to [Huisken and Sinestrari 2009, §5]. We first observe:

**Lemma 4.2.** There are positive constants \(A_3, A_4, A_5, B_3, B_4, C_2\), independent of \(p\) and \(\sigma\), such that

\[
\int E^p \frac{Z(W)}{F} \, d\mu \leq \left( A_3 p^{\frac{3}{2}} + A_4 p^{\frac{1}{2}} + A_5 \right) \int E^{p-2} |\nabla G_{\varepsilon, \sigma}|^2 \, d\mu + \left( B_3 p^{\frac{1}{2}} + B_4 \right) \int E^p \frac{|\nabla W|^2}{F^2} \, d\mu.
\]

**Proof.** As in [Andrews et al. 2014b, §4], contraction of the commutation formula for \(\nabla^2 W\) with \(\dot{F}\) and \(\dot{G}\) yields the identity

\[
\mathcal{L} G_{\varepsilon, \sigma} = -F^{\sigma-1} Q(\nabla W, \nabla W) + F^{\sigma-1} Z(W) + F^{\sigma-2}(\dot{F} G_{kl} - G \dot{F}_{kl}) \nabla_k \nabla_l F + \frac{\sigma}{F} G_{\varepsilon, \sigma} \mathcal{L} F - 2(1-\frac{\sigma}{F}) (\nabla F, \nabla G_{\varepsilon, \sigma})_F + \frac{\sigma}{F^2} |\nabla F|^2.
\]

The claim is now proved using integration by parts and Young’s inequality, with the help of Lemma 2.3 and Proposition 3.4 (see [Andrews et al. 2014b, Lemma 4.2]).

**Corollary 4.3.** For all \(\varepsilon > 0\) there exist constants \(\ell > 0\) and \(L < \infty\) (depending only on \(\varepsilon, n, m, f\) and \(M_0\)) such that for all \(p > L\) and \(0 < \sigma < \ell p^{-\frac{1}{2}}\) there is a constant \(K = K_{\varepsilon, \sigma, p}\) (depending only on \(\varepsilon, n, m, f, M_0, \sigma\) and \(p\)) for which the following estimate holds:

\[
\int (G_{\varepsilon, \sigma})_+^p \, d\mu \leq \int (G_{\varepsilon, \sigma} (\cdot, 0))_+^p \, d\mu_0 + t K \mu_0(M),
\]

where \(\mu_0\) is the measure induced on \(M\) by the initial immersion.

**Proof.** Recall Proposition 3.5. Setting \(\delta = \varepsilon/(2\gamma_2)\) and applying the convexity estimate, we obtain

\[
\frac{Z(W)}{F} \geq \frac{\varepsilon}{2} \gamma_1 F^2 - \gamma_3 C_{\varepsilon/(2\gamma_2)} F
\]

whenever \(G - \varepsilon F > 0\). We now use Young’s inequality to obtain (cf. [Huisken and Sinestrari 2009, §5])

\[
F = F^{-\sigma p} F^{1+\sigma p} \leq F^{-\sigma p} \left( b^q \frac{q}{q} F^{q(1+\sigma p)} + \frac{b^{-q'}}{q'} \right)
\]

for any \(b > 0\) and \(q > 0\), where \(q'\) is the Hölder conjugate of \(q\): \(\frac{1}{q} + \frac{1}{q'} = 1\). Choosing \(q = \frac{2+\sigma p}{1+\sigma p}\), so that \(q' = 2 + \sigma p\), we obtain

\[
F \leq b^{(2+\sigma p)/(1+\sigma p)} \frac{1}{2+\sigma p} F^2 + b^{-(2+\sigma p)} \frac{2+\sigma p}{2+\sigma p} F^{-\sigma p} \leq b^{(2+\sigma p)/(1+\sigma p)} F^2 + b^{-(2+\sigma p)} F^{-\sigma p}.
\]

Now choose \(b := \left( \frac{\varepsilon \gamma_1}{4 \gamma_3 C_{\varepsilon/(2\gamma_2)}} \right)^{\frac{1+\sigma p}{2+\sigma p}}\), so that

\[
\gamma_3 C_{\varepsilon/(2\gamma_2)} F \leq \frac{\varepsilon \gamma_1}{4} F^2 + K F^{-\sigma p},
\]

where

\[
K := \gamma_3 C_{\varepsilon/(2\gamma_2)} \left( \frac{\varepsilon \gamma_1}{4 \gamma_3 C_{\varepsilon/(2\gamma_2)}} \right)^{-(1+\sigma p)}.
\]
Returning to Equation (4-3), we find
\[ \frac{\varepsilon y_1}{4} F^2 \leq K F^{-\sigma p} + \frac{Z(W)}{F}. \]
Estimating \( G_{\varepsilon,\sigma} \leq c_1 F^\sigma \) and \( |W|^2 \leq c_2 F^2 \), we obtain
\[ E^p |W|^2 \leq \tilde{K} + c_3 E^p \frac{Z(W)}{F} \]
for some constants \( \tilde{K} > 0 \) (depending on \( F, M_0, \varepsilon, \sigma \) and \( p \)) and \( c_3 > 0 \) (depending on \( F, M_0, \) and \( \varepsilon \)).

Combining Lemma 4.2 and inequality (4-2) now yields
\[ \frac{d}{dt} \int E^p d\mu \leq K_{\varepsilon,\sigma,p} \mu_0(M) - (\alpha_0 p^2 - \alpha_1 \sigma p^{5/2} - \alpha_2 p^{3/2} - \alpha_3 p) \int E^{p-2} |G_{\varepsilon,\sigma}|^2 d\mu \]
\[ \quad - (\beta_0 p - \beta_1 \sigma p^{3/2} - \beta_2 \sigma p - \beta_3 p^{1/2}) \int E^p \frac{\nabla^2 W}{F^2} d\mu \]
for some positive constants \( \alpha_i \) and \( \beta_i \), which depend on \( \varepsilon \) but not on \( \sigma \) or \( p \), and \( K_{\varepsilon,\sigma,p} \), which depends on \( \varepsilon, \sigma \) and \( p \).

It is clear that \( L > 0 \) and \( \ell > 0 \) may be chosen such that
\[ (\alpha_0 p^2 - \alpha_1 \sigma p^{5/2} - \alpha_2 p^{3/2} - \alpha_3 p) \geq 0 \quad \text{and} \quad (\beta_0 p - \beta_1 \sigma p^{3/2} - \beta_2 \sigma p - \beta_3 p^{1/2}) \geq 0 \]
for all \( p > L \) and \( 0 < \sigma < \ell \frac{p}{2} \). The claim then follows by integrating with respect to the time variable. \( \square \)

The proof of Theorem 1.3 is completed by proceeding with Huisken’s Stampacchia–de Giorgi iteration argument. We omit these details as the arguments required already appear in [Andrews et al. 2014b, §5] with no significant changes necessary.

References


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