CONTROLLABILITY AND OPTIMALITY
IN
ECONOMIC STABILISATION THEORY

By

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The results presented in this Thesis are
my own work except where otherwise stated.

Preston
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CHAPTER I

INTRODUCTION

This thesis is a contribution to the theory of economic policy under certainty, viewed in abstract rather than specific terms. Concern is not for particular applications, such as the debate over monetarism and fiscalism[1], but for theoretical principles. More precisely, the thesis is built around the two fundamental issues of existence and design. By existence is meant the primary ability to stabilise a given economic system; by design, the techniques employed to construct a stabilising policy once existence is assured. This thesis contends, firstly, that the question of existence has been ignored in the theory of dynamic stabilisation; and secondly, that several aspects of dynamic design theory yield profitably to further analysis.

Analysis of existence and design is undertaken by pairing each issue with a specific concept. Thus to existence is applied the concept of controllability; and to design, the concept of optimality. To explain the relevance of these concepts and their effect on the structure of the thesis, this introductory chapter divides into three sections. Section 1.1 defines the existence and design problems to be investigated and briefly relates them to received economic theory. Section 1.2 formally states the policy framework within which these problems will be investigated. To provide a starting point for subsequent analysis, a concise and selective review of the theory of linear optimal design is provided for this framework. Section 1.3 then presents a synopsis of later chapters.

1.1 PROBLEM STATEMENT

As a unifying device, Howard's concept of the problem space may be applied to the theory of economic policy to delineate the problems to be analysed. According to Howard, three dichotomies are basic to any

[1] Cf. the critique by Fand.

decision problem, those defining whether the problem is:

(i) static or dynamic,
(ii) uni-dimensional or multi-dimensional, and
(iii) deterministic or probabilistic.

The combinations produced by these dichotomies are clarified by Figure 1.1, reproduced from Howard [p.212]. The relevance of (i) and (iii)

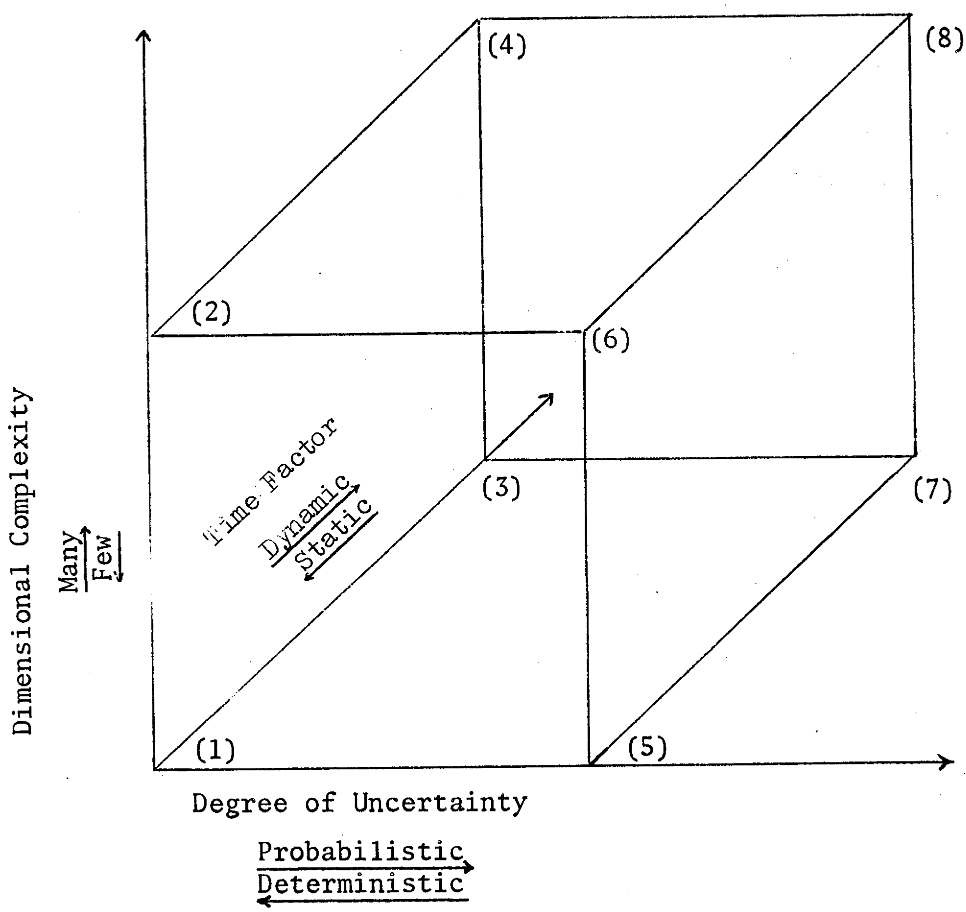

Figure 1.1
Problem Space

to the theory of economic policy is immediate; (ii) categorises the dimensional complexity of the policy problem, in the present context referring to the number of instruments and targets appearing in the reduced form of the policy model.

Each of the eight corners of the problem space of Figure 1.1 corresponds to a particular set of problems in the theory of economic policy. Corner (1) refers to static deterministic models involving a single target and single instrument; the prime example being the aggregative demand model used for Keynesian gap analysis. Corner (2) generalises this simple policy problem to a multi-dimensional context. The seminal work here is by Tinbergen [1963, 1966] who first provided necessary and sufficient conditions for existence of a solution to the general static policy problem. These two corners comprise the static theory of policy under certainty; the results of which are firmly entrenched in economics, as evidenced by the representative analyses of Bent Hansen, Nevile, and Peacock & Shaw.

Remaining in a deterministic framework, Phillips [1954, 1957] demonstrated convincingly the necessity of explicitly accounting for the dynamics of economic systems in formulating policy. This analysis, conducted with a dynamic single-target single-instrument model and therefore referring to corner (3), initiated the study of classical stabilisation policy, also contributed to by Allen [1960, 1968] and Tustin. In a first step towards corner (4), Bergstrom [chap. 6] subsequently developed classical stabilisation policies for multi-dimensional models of cyclical growth. Fox, Sengupta & Thorbecke, in a separate direction, investigated the concept of optimal stabilisation as a generalisation of classical stabilisation, both in a uni-dimensional framework - corner (3) - and in a multi-dimensional framework - corner (4).

These deterministic corners (1) to (4) constitute the domain of investigation of this thesis: the dichotomy (iii) being suppressed, with problems of uncertainty ignored. It is readily agreed that a theory of stochastic policy is the ultimate objective of research work such as this thesis. Important contributions to the excluded corners (5) to (8), representing progress towards this goal, are the certainty equivalence principle in the time domain, as used by Fox, Sengupta & Thorbecke, Holt, and Theil, and the Wiener-Hopf technique in the frequency domain, as used by Phillips [1958] and Whittle [chap. 10]. But before attempting to achieve this generality, it appears desirable to remove a fundamental gap in the theory of policy concerning dynamic existence and to consolidate the theory of classical and optimal design. The issues to be analysed are better comprehended without the complications of uncertainty, and the major conclusions lose little by this simplification.

The four deterministic corners fall naturally into the static pair (1), (2) and the dynamic pair (3), (4). Regarding this division, the idea of existence is prominent in the static theory; and the idea of design, prominent in the dynamic theory. The static bias towards existence is explained both by the simplicity of the static design problem and by the logical precedence of existence over design. Thus, provided a certain instrument coefficient matrix is invertible (existence), appropriate policy is obtained simply by inversion (design). Once existence is ascertained, design is therefore trivial. In contrast, there is no concept of dynamic existence but there are two well-defined design concepts: classical and optimal. The reasons for this are partly contextual and partly historical. Contextually, design of both classical and optimal stabilisation policy has generally referred to a uni-dimensional single-target single-instrument model; and the problems of existence and design have therefore been resolved simultaneously by exhibiting a specific policy. Historically, Phillips' application of classical control techniques to dynamic stabilisation policy preceded any analysis of existence in control theory; and this therefore provided an inbuilt bias towards design.

Although an implicit treatment of existence suffices in low-dimensional models, it is unsatisfactory in the general multi-dimensional model. Dimensionality precludes general theoretical analysis of policy and necessitates numerical analysis. It is then preferable to determine existence explicitly rather than on an ad hoc basis for each particular numerical application. More importantly, it is desirable to be able to define the general characteristics of a policy model that promote or prevent effective stabilisation.

Existence of a policy solution to the multi-dimensional problem of dynamic stabilisation is therefore a major issue analysed in this thesis. The remaining issues pertain to design theory. As noted, dynamic design theory divides into classical and optimal design; and it is natural therefore to contrast and compare these two approaches. Before this evaluation is undertaken, examination reveals a paradox underpinning the theory of optimal stabilisation. This paradox - that optimality and stability are conflicting policy attributes - has been stated by Fox, Sengupta & Thorbecke. The paradox is shown to be false. To define the consequences of optimality, and to relate optimal policy to classical policy, optimal policies are then derived for a simple uni-dimensional model. The dimensional simplicity of this analysis is balanced by an analysis of the numerical problems created by dimension in a general multi-dimensional model of optimal stabilisation.

To summarise, the major aims of this thesis are fourfold:

(1) to resolve the problem of dynamic existence,

(2) to defend and explore the concept of policy optimisation,

(3) to integrate classical and optimal design techniques, and

(4) to confront the problem of multi-dimensional design.

These aims are now given a little more flesh and substance in section 1.2, which specifies the policy framework within which the existence and design of stabilisation policy will be investigated.

1.2 POLICY FRAMEWORK

Four elements taken together comprise the problem of dynamic stabilisation studied in this thesis:

(i) a system to be stabilised,

(ii) a set of available instruments,

(iii) a stabilisation objective, and

(iv) a measure of system performance.

In particular, the stabilisation problem will be specified formally as[2]:

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[2] Throughout the thesis, x^T denotes x transpose; and $X \geqslant 0$ ($X > 0$) signifies that the nxn matrix X is positive semidefinite (definite).

$$\min_{u(t)} W = \tfrac{1}{2}z^T(T)Fz(T) + \tfrac{1}{2}\int_0^T [z^T(t)Vz(t)+u^T(t)Ru(t)]dt \tag{1}$$

<u>subject to</u>

$$\dot{z}(t) = Az(t) + Bu(t), \quad z(0)=z_o\neq 0, \quad z(T) \text{ free}, \tag{2}$$

$$\left.\begin{array}{l} T \text{ fixed}; \quad u(t),\ t \in [0,T], \text{ unconstrained}, \\ \\ F,\ V \geqslant 0, \quad R > 0, \quad F,\ V,\ R \text{ symmetric}, \end{array}\right. \tag{3}$$

with dimensions given by

$$z\text{: nx1}; \quad A,\ V,\ F\text{: nxn}; \quad u\text{: kx1}; \quad R\text{: kxk}; \quad B\text{: nxk}. \tag{4}$$

The linear constant differential system (2) depicts the dynamic behaviour of the system to be stabilised, corresponding to (i) above. In this representation, the dynamic targets $z(t)$ are assumed to exhibit first-order dynamics determined linearly by the levels of the targets $z(t)$ and instruments $u(t)$. The instruments, the element (ii) above, are assumed piecewise continuous and freely adjustable. A policy problem is created by assigning a stabilisation objective to this system. For the system (2), the stabilisation objective (iii) is stipulated as $z(t) = u(t) = 0$, $t \geqslant T > 0$, where T is the stabilisation horizon. When the dynamic motion of (2) ceases and this objective is satisfied, a desired static equilibrium is presumed to obtain. The question of existence occurs naturally at this point. Can the system (i) be stabilised with the available instruments (ii) to achieve the desired objective (iii)? Given a precise statement of this existence question, the concept of controllability will later be shown to lead to conditions necessary and sufficient for policy existence.

Assuming that a policy solution does exist, the major practical question of design must be tackled. To facilitate the logical design of policy, a measure (iv) of system performance under control is specified and a policy giving best performance chosen. Thus for the system (2), a policy $u(t)$ is to be chosen to minimise the integral (1), over the stabilisation horizon T, of a quadratic form in the target and instrument vectors. Since the integrand of (1) is of the form

$J(z-\bar{z},u-\bar{u})$, where J is a quadratic scalar function and $\bar{z} = \bar{u} = 0$, the stabilisation objective (iii) is therefore embedded in the performance measure (iv) so that deviations from the desired static equilibrium are penalised quadratically. The weighting matrix V expresses the relative weights attaching to target deviations; and the weighting matrix R, the relative weights attaching to instrument deviations. Additionally, since target behaviour is to be determined by instrument manipulation, the relation of V to R specifies the target-instrument tradeoff, the relative weighting of target equilibration against instrument equilibration. The remaining term in (1), $z^T(T)Fz(T)$, penalises deviations of the terminal target from the desired equilibrium; and is included, when T is finite, to give greater flexibility in controlling endpoint behaviour.

Traditionally the specification (1) to (3) is termed a fixed-time, free endpoint optimal regulator[3]. Although there are several other possible formulations of the design problem of optimal stabilisation, this particular formulation serves most conveniently for analysis of the design issues to be considered. Chapter VII does, however, consider some of the alternatives. Formal solution of this dynamic optimisation problem may be achieved using methods of varying degrees of generality, ranging from the classical calculus of variations to the maximum principle of Pontryagin et al. The brief review that follows, based mainly on Athans & Falb [chap.9], is of the theory of linear optimal control and is directed solely towards the construction or synthesis of the optimal policy solution to this regulator problem. To supplement this review, Kalman [1963a] provides a perspective of the historical development of optimal control theory; while Falb presents a comprehensive analysis of existence, necessity and sufficiency.

According to Kalman [1963a, p.310],

> "The theory of optimal control, under the assumption that the equations of motion are known exactly and the state can be measured instantaneously, may be regarded as a generalisation of the problem of Lagrange in the calculus of variations: minimization of an integral subject to side conditions, which may be ordinary or differential equations."

For this reason, the modern solution of the regulator problem (1) to

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3 Athans & Falb provide a detailed rationale [pp.750-6] for this regulator formulation.

(3) may be usefully approached from the classical calculus of variations. Consider, after Sage [pp.56-9], a nonlinear dynamic system operating over the fixed interval $[t_o, t_f]$:

$$\dot{z} = f(z,u,t), \quad z(t_o)=z_o, \tag{5}$$

and determine the control u(t) to minimise

$$W = \phi(z(t), t)\Big|_{t_o}^{t_f} + \int_{t_o}^{t_f} \phi(z(t), u(t), t)\, dt. \tag{6}$$

This formulation subsumes the regulator problem as a particular case of the vector functions f, ϕ. Adjoining the constraint (5), the equivalent minimisation problem is:

$$\underset{u(t)}{\text{MIN}}\ W = \phi(z,t)\Big|_{t_o}^{t_f} + \int_{t_o}^{t_f} (\phi(z,u,t) + p^T(t)\{f(z,u,t)-\dot{z}\})dt, \tag{7}$$

where p(t) is a costate vector of the same dimension as the state vector z(t). Here, the definition of the Hamiltonian function

$$H(z,u,p,t) \equiv \phi(z,u,t) + p^T.f(z,u,t) \tag{8}$$

is essential to the modern solution. Applying (8) to (7) supplies

$$\underset{u(t)}{\text{MIN}}\ W = \phi(z,t)\Big|_{t_o}^{t_f} + \int_{t_o}^{t_f} (H(z,u,p,t)-p^T\dot{z})dt, \tag{9}$$

which, after integration, is

$$\underset{u(t)}{\text{MIN}}\ W = (\phi(z,t)-p^Tz)\Big|_{t_o}^{t_f} + \int_{t_o}^{t_f} (H(z,u,p,t) + \dot{p}^Tz)dt. \tag{10}$$

Now the first variation of W is

$$\delta W = [\delta z^T(\frac{\partial\phi}{\partial z} - p)]\Big|_{t_o}^{t_f} + \int_{t_o}^{t_f} [\delta z^T(\frac{\partial H}{\partial z} + \dot{p}) + \delta u^T(\frac{\partial H}{\partial u})]\ dt. \tag{11}$$

First order conditions for a minimum of the functional W require δW to vanish identically for arbitrary variations δz, δu, so that

$$\frac{\partial H}{\partial u} = 0, \tag{12}$$

$$\dot{p} = -\frac{\partial H}{\partial z}, \qquad (\dot{z} = \frac{\partial H}{\partial p}, \text{ from (5), (8)}), \tag{13}$$

$$\delta z^T[\frac{\partial \phi}{\partial z} - p] = 0, \quad t = \{\begin{matrix} t_o \\ t_f \end{matrix}. \tag{14}$$

Without regard to existence and uniqueness, these three necessary conditions heuristically define the modern solution structure. The stationarity condition (12) indicates that minimisation of the functional (6) requires a stationary solution of the Hamiltonian function (8). This is a problem to which the traditional static optimisation apparatus may be applied. Simultaneously, the original dynamic state constraint and the additional dynamic costate constraint, as given in (13), are to be satisfied $\forall\ t \in [t_o, t_f]$. These differential equations are equivalent to the classical Euler-Lagrange necessary conditions for a minimum of W. As for any differential system, boundary conditions must be available if a unique solution is to exist: these are provided by the third necessary condition (14), and are referred to as the transversality conditions for the problem. Now by (2) the terminal endpoint z(T) is free, so that the variation $\delta z(T)$ is arbitrary; hence satisfaction of the transversality condition requires

$$\frac{\partial \phi}{\partial z(t_f)} = p(t_f). \tag{15}$$

The original minimisation problem therefore reduces to the problem of obtaining solutions to the differential system

$$\dot{z} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial z}, \tag{16}$$

with split boundary conditions

$$z(t_o) = z_o, \quad p(t_f) = \partial\phi/\partial z(t_f), \tag{17}$$

where the control u(t) and costate p(t) are linked through the stationarity condition $\partial H/\partial u = 0$.

It is the solution of this boundary value problem that particularly distinguishes the modern theory from the classical theory. Specialising to the optimal regulator model (1) to (3), the Hamiltonian becomes

$$H \equiv \tfrac{1}{2}z^TVz + \tfrac{1}{2}u^TRu + p^T(Az+Bu), \tag{18}$$

and the differential system (16) is therefore

$$\begin{cases} \dfrac{\partial H}{\partial p} = \dot{z} = Az + Bu \\[2ex] -\dfrac{\partial H}{\partial z} = \dot{p} = -Vz - A^Tp. \end{cases} \tag{19}$$

Using (15), the transversality condition is

$$\frac{\partial}{\partial z(T)}\,[\tfrac{1}{2}z^T(T)Fz(T)] = p(T) \iff Fz(T) = p(T), \tag{20}$$

and the remaining necessary condition, $\partial H/\partial u = 0$, implies

$$Ru + B^Tp = 0 \iff u(t) = -R^{-1}B^Tp(t). \tag{21}$$

Substitution of this relation between the control and costate into the differential system (19) produces the two-point boundary value problem

$$\left[\begin{array}{c} \dot{z}(t) \\ \hline \dot{p}(t) \end{array}\right] = \left[\begin{array}{c|c} A & -BR^{-1}B^T \\ \hline -V & -A^T \end{array}\right]\left[\begin{array}{c} z(t) \\ \hline p(t) \end{array}\right], \quad \left[\begin{array}{c} z(0)=z_o \\ \hline p(T)=Fz(T) \end{array}\right], \tag{22}$$

comparable to (16) and (17).

Prior to solving (22), two implicit assumptions need attention: that $\partial H/\partial u = 0$ produces a minimum of the Hamiltonian, and that this minimum corresponds to a minimum of the performance functional W. The second-order condition for a minimum of the Hamiltonian (18) with respect to u(t) is that $\partial^2H/\partial u^2 = R$ be positive definite, and this is satisfied by an assumption in (3). In classical terms, the second-order conditions for minimising W require the second variation of W to be a nonnegative definite form. Since this second variation is expressible in terms of the Hamiltonian, the second order conditions become, following Athans & Falb [pp.269-70],

$$\left[\begin{array}{c|c} \dfrac{\partial^2 H}{\partial z^2} & \dfrac{\partial}{\partial u}\dfrac{\partial H}{\partial z} \\ \hline (\dfrac{\partial}{\partial u}\dfrac{\partial H}{\partial z})^T & \dfrac{\partial^2 H}{\partial u^2} \end{array}\right] \geqslant 0, \quad F \geqslant 0. \tag{23}$$

For the regulator problem, these conditions are satisfied by the assumptions on F, V, and R in (3).

Returning to (22), observe that the differential system is linear, with boundary conditions split equally between the origin and termination of control. Suppose the transition matrix for this system, of order 2nx2n, is $\Phi(t,0)$ such that

$$\left[\begin{array}{c} z(t) \\ \hline p(t) \end{array}\right] = \left[\begin{array}{c|c} \Phi_{11}(t,0) & \Phi_{12}(t,0) \\ \hline \Phi_{21}(t,0) & \Phi_{22}(t,0) \end{array}\right] \left[\begin{array}{c} z(0) \\ \hline p(0) \end{array}\right]. \tag{24}$$

Now the initial costate p(0) is unknown, so consider the positive translation

$$\left[\begin{array}{c} z(T) \\ \hline p(T)=Fz(T) \end{array}\right] = \left[\begin{array}{c|c} \Phi_{11}(T,t) & \Phi_{12}(T,t) \\ \hline \Phi_{21}(T,t) & \Phi_{22}(T,t) \end{array}\right] \left[\begin{array}{c} z(t) \\ \hline p(t) \end{array}\right], \tag{25}$$

solution of which for p(t) provides the relation

$$\begin{cases} p(t) = K(t)z(t), \\ K(t) = -[F\Phi_{12}(T-t)-\Phi_{22}(T-t)]^{-1}[F\Phi_{11}(T-t)-\Phi_{21}(T-t)], \end{cases} \tag{26}$$

where the indicated inverse is known to exist.

At the solution level, the costate is therefore related to the state by means of the time-varying function K(t). At the lower level of the differential system (22), this relation is manifested as

$$\begin{bmatrix} \dot{z} \\ \hdashline \dot{K}z+K\dot{z} \end{bmatrix} = \left[\begin{array}{c:c} A & -BR^{-1}B^T \\ \hdashline -V & -A^T \end{array}\right] \begin{bmatrix} z \\ \hdashline Kz \end{bmatrix}, \tag{27}$$

reducing to

$$[\dot{K} + KA + A^TK - KBR^{-1}B^TK + V]z(t) = 0 \;\forall\, t \in [0,T]. \tag{28}$$

Now from (27), z(t) is the solution of the differential equation

$$\dot{z}(t) = [A - BR^{-1}B^TK(t)]z(t), \quad z(0)=z_o. \tag{29}$$

For arbitrary initial conditions z(0), and for z(t) satisfying (29), K(t) must therefore satisfy the nonlinear (Riccati) differential equation

$$\dot{K}(t) = -K(t)A - A^TK(t) + K(t)BR^{-1}B^TK(t) - V, \quad K(T)=F, \tag{30}$$

with the boundary condition deriving from (20) and (26). Athans & Falb then show that a solution K(t) of (30) exists, is symmetric positive definite for $t \in [0,T)$, and symmetric positive semidefinite for t=T; and that a solution to the optimal regulator problem therefore exists as a consequence, [pp.764-6].

Following this brief review, oriented solely towards exhibiting the optimal solutions in useable form, the solution theory applicable to the optimal stabilisation models considered in this thesis is collected in two theorems. Theorem 1.1, based on equations (21), (26), (29), and (30), specifies the solution structure of the optimal regulator when the stabilisation horizon is finite; theorem 1.2 indicates the appropriate modifications for an infinite stabilisation horizon.

Theorem 1.1

For the finite horizon regulator model as defined,

(i) an optimal control exists, is unique, and is given by

$$u(t) = -R^{-1}B^TK(t)z(t), \tag{31}$$

where

(ii) the nxn symmetric positive-definite matrix $K(t)$ is the unique solution of the matrix Riccati equation

$$\dot{K} = -KA - A^TK + KBR^{-1}B^TK - V, \quad K(T)=F, \tag{32}$$

and

(iii) the optimal state vector is the unique solution of the linear time-varying differential system

$$\dot{z}(t) = [A - BR^{-1}B^TK(t)]z(t), \quad z(0)=z_o. \tag{33}$$

...

Theorem 1.2

If, in theorem 1.1, $T = \infty$ and the dynamic system (2) is controllable, then the results of that theorem are valid provided the substitution $K(t) = \bar{K}$ is made, where $\bar{K} = \lim_{T\to\infty} K(t)$ is the constant nxn symmetric, positive definite matrix solution of the algebraic equation

$$-\bar{K}A - A^T\bar{K} + \bar{K}BR^{-1}B^T\bar{K} - V = 0. \tag{34}$$

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1.3 SYNOPSIS

The six central chapters divide into three groups: chapter II studies the effect of dimension on the design of optimal stabilisation policy; chapters III and IV analyse the problem of the existence of stabilisation policy; and chapters V, VI, and VII explore the theory of optimal stabilisation policy, abstracting from dimensional complications.

Implementation of the optimal solutions just reviewed is only possible on a numerical basis for the general multi-dimensional regulator model. But straightforward numerical application of theorems 1.1 and 1.2 proves unworkable. Chapter II investigates the reasons for this, the primary objective being the development of computational routines for the finite horizon and infinite horizon regulator models. This chapter utilises current computational research in the control literature, and contributes to this work through provision of a transformation to handle complex arithmetic due to oscillatory modes in the optimal solutions. The two principal results are the generation of workable computational procedures; and the acquisition of a deeper theoretical understanding of the regulator solution structure. After this general analysis of the multi-dimensional optimal design problem, the third group of chapters is justifiably able to specialise to the uni-dimensional design problem, making full use of the theoretical content of chapter II.

Chapters III and IV formulate and solve the problem of dynamic existence analogous to the static problem solved by Tinbergen. Solution of this problem relies on the concept of dynamic controllability developed in modern control theory, especially in the writings of Kalman. This concept is shown to provide a dynamic rank criterion necessary and sufficient for existence. Except for a special case of the target-instrument dimensions, for which static and dynamic existence criteria coincide, the dynamic criterion is not susceptible to immediate economic interpretation. Both chapter III and chapter IV therefore attempt to develop economic motivation for the criterion. Chapter III provides alternative conditions that are sufficient for satisfaction of the rank criterion, while chapter IV provides conditions that are necessary as well as sufficient. Dynamic existence is shown to involve two factors: the determination of the minimal number of instruments required for stabilisation, and specification of necessary and sufficient conditions to be satisfied by such a minimal set. As a result, these two chapters place dynamic policy-making on the same footing as static policy-making, allowing existence to be demonstrated prior to design.

Turning to the optimal design of stabilisation policy, chapter V considers the impossibility theorem enunciated by Fox, Sengupta & Thorbecke: that optimal stabilisation policies are destabilising and suboptimal policies, stabilising. Were this theorem valid, further progress in designing optimal stabilisation policy would be prevented. But the theorem is clearly invalid, contradicting the general theory of the optimal regulator. Although this impossibility theorem has had no impact on the theory of economic policy, careful study of its derivation permits not only a precise refutation but also an increased understanding of the optimal design technique. Ironically, this refutation implies the proposition that optimal policies are stabilising and suboptimal policies, destabilising; thus standing the impossibility theorem on its head.

Continuing the analysis of design, optimal stabilisation with a lagged instrument is the specific topic of chapter VI. This problem was first analysed by Phillips [1954] using classical design theory. A closer look at the rationale for control lag suggests some ambiguity in its traditional formulation; and this is discussed initially. The lagged stabilisation problem proposed by Phillips is then considered in an optimal context. Two types of optimal policy are derived: policy that passively adjusts to the dynamic effects of control lag, and policy that actively modifies these effects. This distinction emphasises the flexibility of optimisation as a design technique and clarifies the functions that optimal feedback assumes. Apart from the intrinsic interest of the results, comparison of the optimal policies with the well-known classical policies for similar models reveals a qualitative correspondence that is almost, but not quite, complete.

Chapter VII covers a miscellany of design issues, welded together as an analysis of the degrees of freedom characterising specification of the optimal stabilisation problem - an analysis more enumerative than exhaustive. Optimality, as an attribute of policy, must always be qualified as optimality with respect to a given criterion: yet many criteria appear reasonable for a given stabilisation problem. This subjective aspect of optimality cannot be entirely removed but inroads can be made by developing the consequences of alternative specifications. Three particular issues analysed in this chapter are the possible role of the control weighting matrix as a surrogate for explicit control constraints; the role of terminal target weighting as a surrogate for fixed target endpoints; and the effects of alternative disturbance classes on regulator design. The outcome of this last analysis is that the gap between classical and optimal design techniques is closed.

Finally, chapter VIII states the major conclusions and likely extensions of this present study.

CHAPTER II

LINEAR OPTIMAL STABILISATION: A COMPUTATIONAL ALGORITHM

The central objective of this chapter is the development of an efficient computational procedure for the general regulator model defined in chapter I. Optimisation techniques inevitably highlight computational problems because of one specific characteristic: each dynamic state variable is assigned a costate variable, doubling the dynamic order of the solution space. Although, given linearity, there are compensations for this doubling, analysis of the computability of solutions is therefore imperative. Concern for computation has not featured significantly in dynamic stabilisation theory, mainly because of use of the single-target, single-instrument model for dynamic analysis.

Despite the completeness of theoretical solutions to the general regulator problem computational theory is still being developed in the current control literature. To provide a coherent computational procedure, it has been necessary to integrate some of this work; and, at one point, to extend it. Section 2.1 describes a naive computational procedure, placing the computational problem for the regulator in perspective. The naive procedure is subject to severe limitations and the reasons for this are outlined. A preliminary refinement of this procedure is undertaken in section 2.2, providing additional understanding of the solution structure of the regulator model. Section 2.3 presents the basic building block of the final algorithm. This building block, the negative exponential procedure, consolidates the work of O'Donnell, and Vaughan. Defined over the complex field, this algorithm is unnecessarily extravagant in its core storage requirements; section 2.4 therefore develops a more economical version. Finally, section 2.5 summarises the computing procedure and indicates problems requiring further research.

2.1 A NAIVE COMPUTATIONAL PROCEDURE

The transition from general theoretical solution to general numerical solution is never straightforward. To illustrate the transitional problems associated with optimal regulators the following Bergstrom regulator model, derived in Appendix IIa (pp. 201-4 below), is used:

$$\underset{g}{\text{MIN}}\ W = \tfrac{1}{2}\int_0^T [z^T V z + 2\kappa^T z g + \pi g^2]\, dt + \tfrac{1}{2} z^T(T) F z(T)$$

<u>subject to</u> (1)

$$\dot{z} = Az + bg, \quad z(0)=z_0 \neq 0, \quad z(T) \text{ free},$$

$$T \text{ fixed}; \quad g(t),\ t \varepsilon [0,T],\ \text{unconstrained},$$

where $z(t)$ is a 4x1 state vector, and $g(t)$ is the control variable. In theoretical discussions below, the specification (1) will frequently be assumed to be of general state dimension n.

Formal solution of this optimisation problem proceeds in the manner described in chapter I. Define the Hamiltonian function

$$H \equiv \tfrac{1}{2} z^T V z + \kappa^T z g + \tfrac{1}{2}\pi g^2 + p^T A z + p^T b g, \tag{2}$$

where $p(t)$ is the costate vector. Then the minimising control is

$$\frac{\partial H}{\partial g} = 0 \quad \Longleftrightarrow \quad g = -\pi^{-1}(\kappa^T z + b^T p). \tag{3}$$

Evaluation of the canonical equations $\dot{z} = \partial H/\partial p$, $\dot{p} = -\partial H/\partial z$ with use of (3) provides the canonical system[1]

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[1] Because the Hamiltonian function H of (2) is not referred to again; and because, by definition 2.1 below, the coefficient matrix (4) belongs to the class of Hamiltonian matrices, it is convenient to use H to also denote this matrix.

$\dot{x} = Hx,$

$$H = \left[\begin{array}{c|c} A - \frac{b\kappa^T}{\pi} & -\frac{bb^T}{\pi} \\ \hline \frac{\kappa\kappa^T}{\pi} - V & -(A - \frac{b\kappa^T}{\pi})^T \end{array}\right], \quad x = \left[\begin{array}{c} z \\ \hline p \end{array}\right], \tag{4}$$

$z(0) = z_o, \quad p(T) = Fz(T).$

If this canonical system, of order 8x8 for the Bergstrom regulator, has the transition matrix $\Phi(t,0)$ such that, taking advantage of the constancy of H,

$$\left[\begin{array}{c} z(t) \\ \hline p(t) \end{array}\right] = \left[\begin{array}{c|c} \Phi_{11}(t) & \Phi_{12}(t) \\ \hline \Phi_{21}(t) & \Phi_{22}(t) \end{array}\right] \left[\begin{array}{c} z(0) \\ \hline p(0) \end{array}\right], \tag{5}$$

then, from (1.26), the state and costate are related by

$$p(t) = K(t)z(t), \tag{6}$$

where

$$K(t) = -[F\Phi_{12}(T-t) - \Phi_{22}(T-t)]^{-1}[F\Phi_{11}(T-t) - \Phi_{21}(T-t)], \tag{7}$$

and

$$z(t) = [\Phi_{11}(t) + \Phi_{12}(t)K(0)]\ z(0). \tag{8}$$

And from (3), the optimal control is

$$g(t) = -\pi^{-1}[\kappa^T + b^T K(t)]\ z(t). \tag{9}$$

Inspection of equations (7), (8), and (9) shows that the 8x8 transition matrix $\Phi(t)$ is a prerequisite for any solution of the Bergstrom regulator. Here the computability of solutions emerges as a separate and major problem. For instance, there is no general formula for extracting eigenvalues for systems of order 4 or greater; even apart from the difficulties of manipulating all except low order systems. Thus, in general, theoretical solutions are obtainable for one-state regulator models for $T \leqslant \infty$ (see chapter V below) and for two-state regulator models for $T = \infty$ (see chapter VI below). Otherwise, numerical solutions are necessary: as is the case for this Bergstrom regulator.

The naive computational procedure now to be presented computes the transition matrix

$$\Phi(t) = e^{Ht}, \tag{10}$$

as the solution of the matrix differential system

$$\dot{\Phi}(t) = H\Phi(t), \quad \Phi(0)=I, \tag{11}$$

where the 2nx2n constant matrix H is the coefficient matrix of (4) with n=4, or its general equivalent. Zadeh & Desoer [pp.300-10] describe a computationally oriented Laplace transform method for generating $\Phi(t)$ that is used as the basis of the naive procedure. Their method is therefore summarised <u>verbatim</u>, apart from the simplifying assumption that the eigenvalues of the canonical system (4) are distinct. There is no theoretical presumption for multiple eigenvalues; should they occur, they can be replaced by simple eigenvalues, according to an approximation theorem due to Bellman [p.199].

The Laplace transform of the linear time-invariant differential system (11) is

$$\hat{\Phi}(s) = [sI - H]^{-1} \ \forall \ s \notin E(H), \tag{12}$$

where E(H) is the set of eigenvalues of H. The Zadeh-Desoer procedure[2] recursively generates this matrix inverse - lemma 2.1, theorem 2.1 -

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2 Relevant proofs offered by Zadeh & Desoer are collected in Appendix IIb, pp.205-8 below.

proceeding then to the inverse Laplace transform - theorem 2.2.

Lemma 2.1 (Zadeh & Desoer)

$$[sI - H]^{-1} = \frac{B(s)}{d(s)}, \tag{13}$$

where d(s) is the characteristic equation

$$d(s) = |sI-H| = s^{2n}+d_1 s^{2n-1}+\ldots+d_{2n-1}s+d_{2n} \tag{14}$$

and B(s) is the adjoint matrix

$$B(s) = B_1 s^{2n-1}+B_2 s^{2n-2}+\ldots+B_{2n-1}s+B_{2n}, \tag{15}$$

for B_j a constant 2nx2n matrix $\forall$ $j = 1, \ldots, n$.

...

Theorem 2.1 (Zadeh & Desoer)

The scalar coefficients of the characteristic polynomial d(s) and the matrix coefficients of the matrix polynomial B(s) are generated recursively by

$$\begin{aligned}
B_1 &= I, \quad d_1 = -\mathrm{tr}(H), \\
B_2 &= B_1 H+d_1 I, \quad d_2 = -\tfrac{1}{2}\mathrm{tr}(B_2 H), \\
&\vdots \\
B_{k+1} &= B_k H+d_k I, \quad d_{k+1} = -\frac{1}{k+1}\mathrm{tr}(B_{k+1}H), \\
&\vdots \\
d_{2n} &= -\frac{1}{2n}\mathrm{tr}(B_{2n}H), \\
B_{2n+1} &= B_{2n}H+d_{2n}I = 0.
\end{aligned} \tag{16}$$

...

Theorem 2.2 (Zadeh & Desoer)

Let H be a constant 2nx2n matrix over the complex field, possessing distinct eigenvalues. Then:

$$[sI - H]^{-1} = \frac{B(s)}{d(s)} = \sum_{k=1}^{2n} \frac{R_k}{(s-\lambda_k)}, \tag{17}$$

$$\sum_{k=1}^{2n} R_k = I, \tag{18}$$

$$e^{Ht} = \sum_{k=1}^{2n} R_k e^{\lambda_k t}, \tag{19}$$

where $d(s) = \prod_{k=1}^{2n} (s-\lambda_k)$, and R_k are the constant 2nx2n residue matrices at the eigenvalues $\lambda_k \varepsilon E(H)$, $k = 1,\ldots,2n$.

...

Theorem 2.2 resolves the Laplace transform of (13) into the matrix partial fraction expansion (17), the transition matrix (10) following as the inverse Laplace transform (19). After partitioning $\Phi(t)$, the Riccati solution (17) is available, and subsequently the state (8) and control solution (9). This naive approach will be termed the transition matrix procedure (TMP).

TMP possesses severe limitations when applied, for example, to the Bergstrom regulator. Accurate computation appears conditioned by two factors: the length of the stabilisation horizon T, and the maximum eigenvalue modulus of E(H). For the product of these two factors above a certain limit, TMP returns nonsense results. Thus for likely parameter values, the upper limit on T ranges between two to six unit periods of supply lag. Given a supply lag of three months, this at best confines the stabilisation horizon to one and a half years: a severe limitation, even apart from the thwarted desire to investigate asymptotic behaviour ($T=\infty$).

Upon investigation, a theoretical rationale for this irregular behaviour is found in Kalman's first law of computation [1966, p.25]:

> "Optimal control computations deal with blocks of numbers. The computations are meaningful only when the ratio of the largest to the smallest number in each data block is kept within preassigned limits."

Kalman's argument is the following. Given a finite precision machine arithmetic of (d) significant decimal places, full information may be recovered from a comparison between any two of the $4n^2$ elements of $\Phi(t)$ only if the condition

$$R(\Phi, t) = \frac{\max_{ij} \left| \Phi_{ij} \right|}{\min_{ij} \left| \Phi_{ij} \right|} < 10^d \tag{20}$$

is satisfied. Now $\Phi(t)$ is similar to the diagonal matrix Γ:

$$W^{-1}\Phi(t)W = \Gamma(t), \qquad \Gamma = \mathrm{diag}(\delta_1, \ldots, \delta_{2n}), \tag{21}$$

where W is a right eigenvector matrix, and the δ_i, $i = 1,\ldots,2n$, are the distinct eigenvalues of $\Phi(t)$. Hence condition (20) may be approximated, given (21), by the eigenvalue range of $\Phi(t)$:

$$R(\Phi,t) \doteqdot \frac{\max_{i} \left| \delta_i \right|}{\min_{i} \left| \delta_i \right|}. \tag{22}$$

But $\Phi(t) = \exp(Ht)$ implies $\Gamma(t) = \exp(\Lambda_1 t)$ such that

$$W^{-1}HW = \Lambda_1, \qquad \Lambda_1 = \mathrm{diag}(\lambda_1, \ldots, \lambda_{2n}), \tag{23}$$

with the λ_i as eigenvalues of H, and $\delta_i = \exp \lambda_i t$. Therefore, the approximation (22) may be rewritten in terms of the eigenvalues of H as

$$R(\Phi,t) \doteqdot \exp \{\max_{ij} \left| \lambda_i - \lambda_j \right| t\} = \exp \{r(H).t\}, \tag{24}$$

where

$$r(H) = \max_{ij} \left| \lambda_i - \lambda_j \right|. \tag{25}$$

Thus the computability of the transition matrix depends firstly, on the maximum eigenvalue spread r(H) of the canonical matrix H; and secondly, on the stabilisation interval, $t \in [0, T]$, over which the transition matrix is required. In other words, from (20), (24), the Kalman constraint

$$r(H).T < d \log 10 \tag{26}$$

must be met if TMP is to generate valid numerical results.

The process of precision loss consequent upon violation of (26) may be illustrated with the Bergstrom regulator. For parameter values (see Appendix IIc, p.242 below) producing the eigenvalues

$$\pm\ 5.2763, \quad \pm\ .53535, \quad \pm\ .26298, \quad \pm\ .15783, \tag{27}$$

and for an horizon T=10 unit periods, the state equation (8) was computed, using TMP, for 101 discrete values of t, $t \in [0, 10]$, separated by the intervals .01xT. Values for the four state variables - consumption (c), investment (i), income (y), stocks (s) - expressed as deviations, are abstracted in Table 2.1.

Beginning from the specified initial conditions at time t=0, all four state variables behave regularly for $t \leq 6$, but subsequently exhibit increasingly irregular behaviour. The problem occurs in the calculation of the state transition matrix

$$M(t) = \Phi_{11}(t) + \Phi_{12}(t)K(0). \tag{28}$$

Numerically, K(0) and $\Phi_{12}(t=6)$ are given by

TABLE 2.1

BERGSTROM SIMULATIONS WITH NAIVE PROCEDURE

t	c	i	y	s
0.	.75	.25	1.	.5
⋮	⋮	⋮	⋮	⋮
5.0	.093	-.442	.064	.309
.1	.090	-.433	.064	.304
.2	.088	-.424	.066	.300
.3	.085	-.416	.066	.296
.4	.083	-.409	.067	.292
.5	.081	-.402	.068	.289
⋮	⋮	⋮	⋮	⋮
6.0	.073	-.346	.101	.278
.1	.075	-.400	.052	.264
.2	.059	-.005	.234	.251
.3	.080	-.110	.424	.245
⋮	⋮	⋮	⋮	⋮
7.0	.427	3.406	2.297	-.508
.2	.596	32.500	1.250	-1.469
.4	1.875	54.250	39.250	-3.125
.6	2.531	47.000	113.500	-12.250
.8	-8.250	476.000	464.000	-32.500
8.0	29.250	1120.000	792.000	-183.500

$$\Phi_{12}(6) = \begin{bmatrix} .18\ 10^{12} & -.42\ 10^{13} & -.29\ 10^{13} & .13\ 10^{13} \\ .48\ 10^{13} & .10\ 10^{15} & -.81\ 10^{14} & .36\ 10^{14} \\ .36\ 10^{13} & -.86\ 10^{14} & -.62\ 10^{14} & .27\ 10^{14} \\ -.52\ 10^{12} & .12\ 10^{14} & .88\ 10^{13} & -.39\ 10^{13} \end{bmatrix}$$

$$K(0) = \begin{bmatrix} .3746 & .5751 & -.6529 & -.1723 \\ .5751 & 1.098 & -1.419 & -.2991 \\ -.6529 & -1.419 & 2.229 & .3702 \\ -.1723 & -.2991 & .3702 & .085 \end{bmatrix}. \tag{29}$$

As Kalman's computational law argues, precision loss relates not to the absolute size of the elements of $\Phi_{12}(6)$ - maximum size for floating-point representation in the IBM 360/50 is approximately 7×10^{75} - but to the relative sizes of the elements involved in the additive and multiplicative operations in (28). To compare two floating-point numbers of differing magnitude, whether for addition or multiplication, the exponent of the smaller is increased to equality with the larger, while the mantissa is adjusted to compensate. Thus comparison of the (1,1) elements of Φ_{12} and K(0) in (29) requires

$$.18 \times 10^{12}, \qquad .\underbrace{0\ \ldots\ 0}_{12\ \text{places}}3746 \times 10^{12}. \tag{30}$$

With finite precision arithmetic, only the (d) most significant decimal places of the mantissa resulting from the comparison in (30) are retained. If $d \leq 12$, then information on the smaller number is completely lost. This is the reason for the relative nature of the criterion (20) - a minimal condition for retention of information. For the relative magnitudes shown in (29), precision loss commences in the operation $\Phi_{12} \times K(0)$, producing the irregularity observed in Table 2.1.

First attempts to implement optimal regulator solutions numerically fail because information is coded in numbers whose relative magnitudes are too dispersed for accurate machine computation. In terms of Kalman's constraint (26), since d is fixed, disparate numerical comparisons can only be avoided by adjusting the range function r(H) or the time interval T. This is the basic task of section 2.3. TMP is jettisoned as a result but lemma 2.1 and theorem 2.1 are retained for calculation of the characteristic equation d(s). Before the modified procedure is developed, section 2.2 considers the relevance of these two results to the solution structure of the regulator.

2.2 ASPECTS OF REGULATOR STRUCTURE

Lemma 2.1 and theorem 2.1 of the Zadeh-Desoer algorithm provide the characteristic equation for any 2nx2n constant matrix. The structure of the optimal regulator permits, however, certain simplifications. Thus (16) implies that the coefficients d_j, B_j, are constructed from operations on successively higher powers of the canonical matrix H:

$$\begin{cases} B_1 = I \\ d_1 = -\mathrm{tr}(H) \\ B_2 = H + d_1 I \\ d_2 = -\frac{1}{2}\mathrm{tr}(H^2+d_1H) \\ \cdots \\ B_k = H^{k-1}+d_1H^{k-2}+\ldots+d_{k-2}H+d_{k-1}\cdot I \\ d_k = -\frac{1}{k}\mathrm{tr}(H^k+d_1H^{k-1}+\ldots+d_{k-2}H+d_{k-1}H). \end{cases} \tag{31}$$

Now from (4), and in general,

$$\mathrm{tr}(H) = 0. \tag{32}$$

Examination of (31) suggests that an analysis of $\mathrm{tr}(H^k)$, $k = 2,\ldots,2n$, might prove fruitful in further simplifying (31). Lemma 2.3 justifies this intuition, using the following definition and lemma provided both by Kalman & Englar [p.196] and by O'Donnell [p.584].

Definition 2.1 (Kalman & Englar, O'Donnell)

An even-dimensional 2nx2n matrix H is said to be Hamiltonian if

$$H = JH^TJ, \tag{33}$$

where J is the 2nx2n matrix

$$J = \left[\begin{array}{c|c} 0 & -I_n \\ \hline I_n & 0 \end{array}\right], \tag{34}$$

such that

$$J^2 = -I, \qquad J^T = -J = J^{-1}. \tag{35}$$

...

Lemma 2.2 (Kalman & Englar, O'Donnell)

The canonical matrix H of the regulator model is Hamiltonian.

...

Thus the matrix H of (4) satisfies (33), and H is therefore Hamiltonian.

Lemma 2.3

For the 2nx2n Hamiltonian matrix H,

$$tr(H^k) = 0 \quad \forall \quad k \text{ odd}, \quad k = 1,3, \ldots,2n-1. \tag{36}$$

proof

(i) $H = JH^TJ$

$$H^2 = JH^TJJH^TJ = - JH^{2T}J \qquad \ldots(35)$$

$$H^3 = -JH^{2T}JJH^TJ = JH^{3T}J \qquad \text{"}$$

$$H^4 = JH^{3T}JJH^TJ = - JH^{4T}J \qquad \text{"}$$

(ii) $H^{k-1} = (-1)^{k-2}JH^{(k-1)T}J$

=> $H^k = (-1)^{k-1}JH^{kT}J$

(iii) $\mathrm{tr}(H^k) = (-1)^{k-1}\mathrm{tr}(JH^{kT}J)$

$$= (-1)^{k-1}\mathrm{tr}(H^{kT}JJ)$$

$$= (-1)^k\mathrm{tr}(H^{kT})$$

$$= - \mathrm{tr}\ (H^{kT}) \ \forall\ k \text{ odd.}$$

Hence the lemma.

...

The numerical significance of lemma 2.3 is expressed in corollary 2.1; its theoretical significance, in corollary 2.2. Thus:

Corollary 2.1

The coefficients d_j, $j = 1,\ldots,2n$, of the characteristic polynomial $d(s)$ of the Hamiltonian matrix H vanish $\forall$ j odd, $j = 1,\ldots,2n-1$.

proof

(i) $d_1 = - \mathrm{tr}(H) = 0$ ...(32)

$$d_3 = -\frac{1}{3}\mathrm{tr}(H^3+d_1H^2+d_2H) = 0 \qquad \ldots(32)\text{, lemma 2.3, (i).}$$

(ii) From (31), assume

$$d_k = - \frac{1}{k}\mathrm{tr}(\sum_{j=1}^{k} d_{k-j}H^j) = 0, \quad d_o=1,\ k \text{ odd.} \qquad (37)$$

Define $D_k = \sum_{j=1}^{k} d_{k-j}H^j$, such that

$$D_k = [D_{k-1}+d_{k-1}].H, \qquad \ldots(16),\ (31).$$

Therefore

$$d_{k+2} = -\frac{1}{k+2}\,\mathrm{tr}(D_{k+2}) \qquad \ldots(37)$$

$$= -\frac{1}{k+2}\,\mathrm{tr}([D_{k+1}+d_{k+1}].H)$$

$$= -\frac{1}{k+2}\,\mathrm{tr}([D_k+d_k].H^2+d_{k+1}H)$$

$$= -\frac{1}{k+2}\,\mathrm{tr}(D_k.H^2), \qquad \ldots(37),\ (32). \qquad (38)$$

(iii) $\mathrm{tr}(D_k.H^2) = \mathrm{tr}\,(\sum_{j=1}^{k} d_{k-j}H^{j+2})$, by definition of D_k.

Hence $\forall$ k odd,

either (a) j odd, k-j even, j+2 odd,

or (b) j even, k-j odd, j+2 even,

$j = 1,\ldots,k$. Thus $d_{k-j}\mathrm{tr}(H^{j+2}) = 0\ \forall\ j = 1,\ldots,k$,

by lemma 2.3 if (a); and by (37) if (b). From (38),

$$d_{k+2} = 0,$$

and the corollary follows by induction.

...

Corollary 2.2 (Hamiltonian Saddle Point)

The eigenvalues of the Hamiltonian matrix of optimal regulator models are distributed symmetrically about the complex axis in the complex plane.

proof

From (14), the Hamiltonian characteristic equation is

$$d(s) = \sum_{j=o}^{2n} d_j s^{2n-j}, \quad d_o = 1,$$

$$= \sum_{j=o}^{n} d_{2j} s^{2(n-j)}, \tag{39}$$

by corollary 2.1. Because (39) is even-powered in s, the root s_j implies the root $-s_j$. For real s_j, the corollary is immediate. For complex s_j, the conjugate property and evenness imply the pairs $(s_j, -s_j^*)$ and $(s_j^*, -s_j)$, also satisfying the corollary.

...

In summary, this section presents specific properties of the optimal regulator model that not only simplify computation with the Zadeh-Desoer characteristic equation algorithm but also clarify the theoretical solution structure. The knowledge that the canonical matrix H is Hamiltonian and that its characteristic equation therefore exists as a purely even-powered polynomial considerably simplifies (31). To the original statement (16) of the characteristic equation algorithm

$$d_k = -\frac{1}{k}\mathrm{tr}\ (HB_k), \qquad B_k = \sum_{j=1}^{k} d_{k-j}H^{j-1}, \quad d_o = 1,$$

$$k = 1,\ldots,2n, \tag{40}$$

is added

$$d_k = 0\ \forall\ k \text{ odd}. \tag{41}$$

Corollary 2.1 also implies that the computational disadvantage with respect to doubling of order suffered by optimal stabilisation models, when compared with classical stabilisation models, is partially alleviated. Thus (39) allows the system eigenvalues to be computed from

$$d(v) = \sum_{j=o}^{n} d_{2j} v^{n-j}, \qquad s = \pm\ v^{\frac{1}{2}}. \tag{42}$$

Although optimal stabilisation policies double the order n of the uncontrolled system (whereas classical policies increment order, according to the type of policy used), the characteristic equation (42) retains an effective order n. That this is half the optimal order 2n reflects the symmetry property of corollary 2.2 - a compensation provided by linearity.

With this theoretical appreciation of the Hamiltonian eigenvalue structure, section 2.3 now returns to the development of a computational procedure satisfying Kalman's constraint.

2.3 THE NEGATIVE EXPONENTIAL PROCEDURE

At first sight, the precision loss described in section 2.1 may be avoided by rescaling the magnitudes of the matrices Φ_{11} and Φ_{12} in the state transition matrix (8), (28). Thus, since the Hamiltonian system is linear and constant, consider the following iterative form of (8):

$$z(t) = [\Phi_{11}(t-\tau) + \Phi_{12}(t-\tau)K(\tau)]z(\tau),$$

or

$$z(\tau+\varepsilon) = [\Phi_{11}(\varepsilon) + \Phi_{12}(\varepsilon)K(\tau)]z(\tau). \tag{43}$$

The step size,

$$\varepsilon = t - \tau = T/N, \tag{44}$$

can be chosen so that the Kalman constraint is met by Φ_{11} and Φ_{12}, where N is the number of discrete computing points used to approximate the continuous solutions over the specified horizon T. Yet the precision problem cannot be suppressed. The Riccati solution $K(\tau)$ is now variable; and if computed by (7) the same form of precision loss occurs for T-t large, $t\varepsilon[0, T]$. Use of (43) therefore requires a satisfactory computational procedure for the Riccati solution. Recent work by O'Donnell and Vaughan reformulates the Riccati solution (7) so that it satisfies Kalman's constraint. The task of this section is to summarise these results; and of the following section, to present a modification more economical in its core storage requirements.

Refinement of the naive Riccati procedure (7) proceeds in two steps. The first step, due to O'Donnell, characterises the eigenstructure of the Hamiltonian system; the second step, due to Vaughan, then restructures the naive procedure.

On the assumption of distinct eigenvalues for the Hamiltonian system, (23) defines a similarity transform of H in which Λ_1 is the diagonal matrix of eigenvalues. The Hamiltonian transition matrix $\Phi(t)$ is therefore given by

$$\Phi(t) = e^{Ht} = We^{\Lambda_1 t}W^{-1}. \tag{45}$$

Now from corollary 2.2, Λ_1 possesses the saddle point structure

$$\Lambda_1 = \left[\begin{array}{c|c} -\Lambda & 0 \\ \hline 0 & \Lambda \end{array}\right], \qquad \Lambda = \text{diag } (\lambda_1, \ldots, \lambda_n), \tag{46}$$

for λ_i, i = 1,...,n, distinct but not necessarily real. Because of this saddle point structure, the eigenvector matrix W of (45) is of a particular class called simplectic:

<u>Definition 2.2</u> (Kalman & Englar, O'Donnell)

An even-dimensional matrix W is simplectic if

$$W^TJW = J, \qquad \text{or} \qquad -JW^TJ = W^{-1}. \tag{47}$$

...

O'Donnell [p.586] uses this simplectic property to construct the left eigenvector matrix W^{-1} directly from W, avoiding the need for inversion. His results are summarised in the following theorem (for proof of which see Appendix IIb, pp.208-10 below).

<u>Theorem 2.3</u> (O'Donnell)

If a Hamiltonian matrix of order 2nx2n has eigenvalues

$\pm\lambda_j$, $j = 1,\ldots,n$, assumed nonzero and distinct over the complex field, then there exists a simplectic eigenvector matrix W such that

$$W^{-1}HW = \Lambda_1 = \left[\begin{array}{c|c} -\Lambda & 0 \\ \hline 0 & \Lambda \end{array}\right], \qquad \Lambda = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n), \tag{48}$$

where

$$W = \left[\begin{array}{c|c} W_{11} & W_{12} \\ \hline W_{21} & W_{22} \end{array}\right], \quad W^{-1} = \left[\begin{array}{c|c} W_{22}^T & -W_{12}^T \\ \hline -W_{21}^T & W_{11}^T \end{array}\right]. \tag{49}$$

...

Using the Hamiltonian eigenstructure defined by theorem 2.3, Vaughan demonstrates that the Riccati solution matrix K(t) can be obtained without loss of information for any horizon $T \leq \infty$. His negative exponential procedure devised for this purpose is condensed into theorem 2.4[3].

Theorem 2.4 (Vaughan)

For the optimal regulator with $T \leq \infty$, the matrix Riccati equation associated with the Hamiltonian system is computable in the form

$$K(t) = [W_{21}+W_{22}Q(T-t)]\,[W_{11}+W_{12}Q(T-t)]^{-1}, \tag{50}$$

$$Q(T-t) = e^{-\Lambda(T-t)}Re^{-\Lambda(T-t)}, \tag{51}$$

$$R = -\,[W_{22}-K(T)W_{12}]^{-1}\,[W_{21}-K(T)W_{11}]. \tag{52}$$

...

To perceive the advantage that the negative exponential procedure (NEP) possesses over TMP, observe that the Hamiltonian transition matrix

---

3 Section 2.4 below presents a proof of a modified version of theorem 2.4; Vaughan's proof of this theorem is therefore omitted from Appendix IIb.

$\Phi(t)$ of (45) may be partitioned, after using theorem 2.3, as

$$\Phi(t) = \left[\begin{array}{c|c} W_{11}e^{-\Lambda t}W_{22}^T - W_{12}e^{\Lambda t}W_{21}^T & W_{12}e^{\Lambda t}W_{11}^T - W_{11}e^{-\Lambda t}W_{12}^T \\ \hline W_{21}e^{-\Lambda t}W_{22}^T - W_{22}e^{\Lambda t}W_{21}^T & W_{22}e^{\Lambda t}W_{11}^T - W_{21}e^{-\Lambda t}W_{12}^T \end{array}\right]. \quad (53)$$

To simplify, suppose the Riccati endpoint is $K(T) = F = 0$, so that (7) becomes, using (53),

$$\begin{aligned} K(t) &= -\Phi_{22}^{-1}(T-t)\Phi_{21}(T-t), \\ &= -[W_{22}e^{\Lambda(T-t)}W_{11}^T - W_{21}e^{-\Lambda(T-t)}W_{12}^T]^{-1} \\ &\qquad X[W_{21}e^{-\Lambda(T-t)}W_{22}^T - W_{22}e^{\Lambda(T-t)}W_{21}^T], \end{aligned} \quad (54)$$

which is the TMP version of the Riccati solution. Exercising this endpoint assumption in theorem 2.4 yields the NEP version

$$\begin{aligned} K(t) &= [W_{21} - W_{22}e^{-\Lambda(T-t)}W_{22}^{-1}W_{21}e^{-\Lambda(T-t)}] \\ &\qquad X[W_{11} - W_{12}e^{-\Lambda(T-t)}W_{22}^{-1}W_{21}e^{-\Lambda(T-t)}]^{-1}. \end{aligned} \quad (55)$$

Performance of (54) and (55) may be compared for large T-t, after first establishing the asymptotic limit $\bar{K} = \lim_{T\to\infty} K(t)$. For large T-t, the exponential functions, $\exp(-\Lambda(T-t))$, decay rapidly: hence, for the TMP version (54),

$$\begin{aligned} \bar{K} &= \lim_{T\to\infty} [W_{22}e^{\Lambda(T-t)}W_{11}^T]^{-1}[W_{22}e^{\Lambda(T-t)}W_{21}^T] \\ &= \lim_{T\to\infty} [W_{11}^{-T}e^{-\Lambda(T-t)}W_{22}^{-1}W_{22}e^{\Lambda(T-t)}W_{21}^T] \\ &= W_{11}^{-T}W_{21}^T = W_{21}W_{11}^{-1}, \end{aligned} \quad (56)$$

since $K(t)$ is symmetric. The same result follows from (55).

Although both approaches are asymptotically equivalent, their computational requirements differ markedly for finite T. The TMP version (54) grows exponentially with T while the NEP version (55) decays exponentially with T. Further, the asymptotic solution matrices W_{11}, W_{21} of (56) occur multiplicatively with respect to the exponential functions in TMP, but additively in NEP. Thus in (56), each of the products

$$e^{\Lambda(T-t)}W_{11}^T, \qquad e^{\Lambda(T-t)}W_{21}^T \tag{57}$$

suffers precision loss for T-t sufficiently large; the resulting loss of asymptotic information producing numerical inaccuracy when the TMP version (54) is used. On the other hand, the NEP version (55), with its decaying exponentials, has all information coded in comparable magnitudes, retaining full information about the asymptotic solution as $T \to \infty$.

For computation of the Riccati solution required in (43), NEP provides a numerically stable solution procedure satisfying Kalman's constraint. But as formulated in theorem 2.4, NEP is not explicitly concerned with computational problems engendered by complex eigenvalues and eigenvectors. Section 2.4 therefore proposes a transformation for handling complex arithmetic when it occurs.

2.4 REGULATOR COMPUTATION WITH COMPLEX ARITHMETIC

Allowance for complex eigenvalues and eigenvectors is obligatory; the probability of their occurrence increasing with the state dimension of the system. Yet the negative exponential procedure pays no regard to the additional core storage requirements necessitated by complex arithmetic relative to real arithmetic. Taking the polar cases of real eigenvalues only and complex eigenvalues only, the complex case doubles storage requirements. For large order systems, this is likely to be prohibitive. It is worthwhile, therefore, to attempt the reformulation of NEP to economise on core storage in the face of complex arithmetic.

For a Hamiltonian matrix H possessing real and complex eigenvalues, define

$$n = 2c + r, \tag{58}$$

where r is the number of real positive eigenvalues, and c is the number of positive real part conjugate eigenvalue pairs. The diagonal matrix Λ_1 of (48) is then partitionable as

$$\Lambda_1 = \begin{bmatrix} -\Gamma_{2c} & & & 0 \\ & -\Sigma_r & & \\ & & \Gamma_{2c} & \\ 0 & & & \Sigma_r \end{bmatrix}. \tag{59}$$

The even-dimensioned diagonal matrix Γ contains complex eigenvalues with positive real parts, stored in conjugate pairs; and Σ contains real positive eigenvalues. From Ogata [pp.143-5], the transformation

$$\Gamma_2^* = M_2^{-1}\Gamma_2 M_2, \tag{60}$$

where

$$\Gamma_2 = \begin{bmatrix} \sigma+j\omega & 0 \\ 0 & \sigma-j\omega \end{bmatrix}, \qquad M_2 = \tfrac{1}{2}\begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix}, \tag{61}$$

provides the real matrix

$$\Gamma_2^* = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}. \tag{62}$$

Information on the real part of conjugate eigenvalue pairs is retained in the diagonal elements but information on the imaginary part is now stored in the offdiagonal elements. Applying this pairwise transformation to Λ_1 in (59),

$$\Lambda_1^* = V^{-1}\Lambda_1 V, \tag{63}$$

where

$$V_{2n} = \left[\begin{array}{c|c} \Delta_n & 0 \\ \hline 0 & \Delta_n \end{array}\right], \quad \Delta_n = \left[\begin{array}{c|c} M_{2c} & 0 \\ \hline 0 & I_r \end{array}\right], \quad M_{2c} = \begin{bmatrix} M_2 & & 0 \\ & \ddots & \\ 0 & & M_2 \end{bmatrix}. \tag{64}$$

Hence

$$\Lambda_1^* = \begin{bmatrix} -\chi_{2c} & & & 0 \\ & -\Sigma_r & & \\ & & \chi_{2c} & \\ 0 & & & \Sigma_r \end{bmatrix}, \tag{65}$$

in which, from (60),

$$\chi_{2c} = (M^{-1}\Gamma M)_{2c} = \begin{bmatrix} \begin{pmatrix} \sigma_1 & \omega_1 \\ -\omega_1 & \sigma_1 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} \sigma_c & \omega_c \\ -\omega_c & \sigma_c \end{pmatrix} \end{bmatrix}. \tag{66}$$

Since both χ, Σ are real matrices, Λ_1^* in (63) is also real.

Using the similarity transformation (48) in (63) supplies

$$\Lambda_1^* = V^{-1}(W^{-1}HW)V = P^{-1}HP; \quad P = WV. \tag{67}$$

The operation of the block diagonal, complex-valued matrix V in post-multiplying the complex-valued eigenvector matrix W generates a transformed eigenvector matrix P with real elements, in which eigenvector information is stored in a single column (real eigenvalues) or in two adjacent columns (complex eigenvalues). Thus equation (67) specifies the spectral form of H in real-valued quantities; and is obtained by a simple transformation of the complex-valued form (48).

With this transformation, theorem 2.3 on the spectral form of H, and theorem 2.4 on the negative exponential Riccati procedure may be restated as follows:

Theorem 2.5

Given the Hamiltonian matrix H, with eigenvalues assumed nonzero and distinct, then there exists a similarity transformation of the spectral form $W^{-1}HW = \Lambda_1$ such that

$$P^{-1}HP = \Lambda_1^* = \left[\begin{array}{c|c} -\Lambda^* & 0 \\ \hline 0 & \Lambda^* \end{array}\right]; \quad \Lambda_n^* = \begin{bmatrix} \ddots & & & & 0 \\ & \begin{pmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{pmatrix} & & & \\ & & \ddots & & \\ & & & \lambda_{n-r+1} & \\ & & & & \ddots \\ 0 & & & & \lambda_n \end{bmatrix} \tag{68}$$

$$i = 1, \ldots, c \; ; \; n = 2c+r,$$

where

$$P = \left[\begin{array}{c|c} W_{11}\Delta & W_{12}\Delta \\ \hline W_{21}\Delta & W_{22}\Delta \end{array}\right], \quad \Delta_n = \begin{bmatrix} \ddots & & 0 \\ & \frac{1}{2}\begin{pmatrix} 1 & -j \\ 1 & j \end{pmatrix} & \\ & & \ddots \\ 0 & & I_r \end{bmatrix}, \tag{69}$$

and

$$P^{-1} = \left[\begin{array}{c|c} \Delta^* P_{22}^T & -\Delta^* P_{12}^T \\ \hline -\Delta^* P_{21}^T & \Delta^* P_{11}^T \end{array}\right]; \quad \Delta_n^* = \begin{bmatrix} \ddots & & 0 \\ & 2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \\ & & \ddots \\ 0 & & I_r \end{bmatrix}. \tag{70}$$

proof

Equations (68) and (69) follow immediately from equations (61), (64), and (67). To establish (70) - the analogue of O'Donnell's result for W^{-1}, given in (49) - observe that by the definition (47) of a simplectic matrix

$$W^{-1} = -JW^TJ. \tag{71}$$

Using the transformation $P = WV$ of (67) in (71) implies

$$P^{-1} = -(V^{-1}JV^{-T})P^TJ. \tag{72}$$

But from the definition of V in (64),

$$JV^{-T} = \left[\begin{array}{c|c} 0 & -I \\ \hline I & 0 \end{array}\right] \left[\begin{array}{c|c} \Delta^{-T} & 0 \\ \hline 0 & \Delta^{-T} \end{array}\right] = \left[\begin{array}{c|c} 0 & -\Delta^{-T} \\ \hline \Delta^{-T} & 0 \end{array}\right] = V^{-T}J. \tag{73}$$

Hence (72) is written

$$P^{-1} = V^*(-JP^TJ), \tag{74}$$

in which

$$V^*_{2n} = (V^TV)^{-1} = \left[\begin{array}{c|c} \Delta^*_n & 0 \\ \hline 0 & \Delta^*_n \end{array}\right]; \quad \Delta^* = (\Delta^T\Delta)^{-1} = \left[\begin{array}{c|c} (M^TM)^{-1}_{2c} & 0 \\ \hline 0 & I_r \end{array}\right]. \tag{75}$$

The expression for P^{-1} in (70) follows from (74), using the definitions of J, V^* in (34), (75). And from (61),

$$(M_2^TM_2)^{-1} = \left[\tfrac{1}{4}\begin{pmatrix} 1 & 1 \\ -j & j \end{pmatrix}\begin{pmatrix} 1 & -j \\ 1 & j \end{pmatrix}\right]^{-1} = 2\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{76}$$

establishing, from (75), the second result in (70).

...

Theorem 2.6

The matrix Riccati solution K(t) is computable in the real-valued form

$$K(t) = [P_{21}+P_{22}Q^*(T-t)]\ [P_{11}+P_{12}Q^*(T-t)]^{-1}, \tag{77}$$

where

$$Q^*(T-t) = e^{-\Lambda^*(T-t)}R^*e^{-\Lambda^*(T-t)}, \tag{78}$$

$$R^* = -\ [P_{22}-K(T)P_{12}]^{-1}\ [P_{21}-K(T)P_{11}]. \tag{79}$$

proof

The following proof imitates Vaughan. By (4), (67),

$$x(t) = PE(t)P^{-1}x(0), \tag{80}$$

with, from (63), (65),

$$E(t) = e^{\Lambda_1^* t} = \left[\begin{array}{c|c} e^{-\Lambda^* t} & 0 \\ \hline 0 & e^{\Lambda^* t} \end{array}\right],\quad \Lambda_n^* = \left[\begin{array}{c|c} X_{2c} & 0 \\ \hline 0 & \Sigma_r \end{array}\right]. \tag{81}$$

Given the block diagonality of Λ_1^*, application of the Laplace transform method yields (see Appendix IIb, p.210 below),

$$e^{\pm\Sigma t} = \begin{bmatrix} e^{\pm\lambda_{n-r+1}t} & & 0 \\ & \ddots & \\ 0 & & e^{\pm\lambda_n t} \end{bmatrix},\quad e^{\pm X t} = \begin{bmatrix} e^{\pm\sigma_1 t}\begin{pmatrix} \cos\omega_1 t & \pm\sin\omega_1 t \\ \mp\sin\omega_1 t & \cos\omega_1 t \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & e^{\pm\sigma_c t}\begin{pmatrix} \cos\omega_c t & \pm\sin\omega_c t \\ \mp\sin\omega_c t & \cos\omega_c t \end{pmatrix} \end{bmatrix}. \tag{82}$$

Now define the backward transformation

$$x(t) = Pr(\tau) \quad \Leftrightarrow \quad \left[\begin{array}{c} z(t) \\ \hline p(t) \end{array}\right] = \left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right] \left[\begin{array}{c} u(\tau) \\ \hline v(\tau) \end{array}\right], \tag{83}$$

$\tau = T-t; \qquad r = [\, u^T \quad v^T]^T.$

Thus

$$x(T) = PE(T-t)P^{-1}x(t), \tag{84}$$

obtained from (80), may be written

$$r(\tau) = E^{-1}(\tau)r(0) \quad \Leftrightarrow \quad \left[\begin{array}{c} u(\tau) \\ \hline v(\tau) \end{array}\right] = \left[\begin{array}{c|c} e^{\Lambda^* \tau} & 0 \\ \hline 0 & e^{-\Lambda^* \tau} \end{array}\right] \left[\begin{array}{c} u(0) \\ \hline v(0) \end{array}\right]. \tag{85}$$

Combining (83) and (85),

$$x(t) = PE^{-1}(\tau)r(0), \tag{86}$$

implying for $t=T$, $\tau=0$,

$$x(T) = Pr(0) \quad \Leftrightarrow \quad \left[\begin{array}{c} z(T) \\ \hline p(T) \end{array}\right] = \left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right] \left[\begin{array}{c} u(0) \\ \hline v(0) \end{array}\right]. \tag{87}$$

Use of the costate-state relation $p(T) = K(T)z(T)$ in (87) supplies

$$v(0) = R^* u(0), \tag{88}$$

for R^* defined in (79). Hence from (85) and (88),

$$v(\tau) = Q^*(\tau)u(\tau), \tag{89}$$

for $Q^*(T-t)$ defined in (78). Finally, use of $p(t) = K(t)z(t)$ and

(89) in (88) provides the Riccati solution (77).

...

This section therefore effects a pairwise similarity transformation of conjugate eigenvalues and eigenvectors whenever these occur in O'Donnell's specification, theorem 2.3, of the spectral form of the Hamiltonian system; and in Vaughan's negative exponential procedure, theorem 2.4, for computing the Riccati solution. As a result, both the Riccati solution and the Hamiltonian transition matrix may be computed utilising the same storage for complex as for real arithmetic. Further, in calculating the Hamiltonian transition matrix, the transformed left eigenvector matrix P^{-1} is available as (70), avoiding the need for direct inversion of P.

2.5 STATEMENT OF COMPUTATIONAL PROCEDURE

Previous sections have indicated various problems confronting attempts to compute solutions to the optimal regulator. Are there procedures available which allow accurate computation of finite trajectories subject to the Kalman constraint? How are storage requirements to be minimised in large order systems for which complex arithmetic is likely? What complications, if any, are avoided by an infinite horizon assumption? Certain answers to these questions are now collected in an integrated account of the computational algorithm so far developed in this chapter.

Two sets of routines derive from the previous theoretical developments: a finite horizon program and an infinite horizon program. Fortran IV coding for each is attached and briefly discussed in Appendix IIc (pp.212-60 below). Basically, both programs link five sequential steps:

(i) system specification,
(ii) eigen analysis,
(iii) Riccati solution,
(iv) state solution, and
(v) control solution.

Step (i) specifies two types of information: horizon information and structural information. The first refers to the value of T to which simulations are to be computed, and to the number of steps N in (44). The second refers to a numerical specification of the matrix and vector coefficients of a particular regulator model: for the Bergstrom regulator (1), a numerical set $\{A, b, z(0), V, \kappa, \pi\}$. Step (ii) then computes the eigenvalues and eigenvectors of the Hamiltonian differential system derived from the optimisation analysis. The eigenvalues are found from the Hamiltonian characteristic equation, constructed from equations (39) to (42). The right eigenvector matrix P of (69) is subsequently computed using a slightly modified version of IGVEC5, developed by Blackburn & Vaughan[4]. In the attached coding, the left eigenvector matrix P^{-1} is obtained by direct inversion; in large order systems, the expression (70) would be used.

For the finite horizon routine, step (iii) obtains the Riccati solution using equations (77), (78), and (79) of theorem 2.6. The asymptotic case is dealt with below. The optimal state solution - step (iv) - is found with the aid of (43):

$$z(\tau+\varepsilon) = [\Phi_{11}(\varepsilon) + \Phi_{12}(\varepsilon)K(\tau)]z(\tau), \tag{43}$$

in which the Hamiltonian transition matrix partitions Φ_{11}, Φ_{12} are derived from the appropriate partitions of

$$\Phi(t) = e^{Ht} = PE(t)P^{-1}, \tag{90}$$

obtained from (80). Using E(t) defined in (81), and P^{-1} defined in (70), the Hamiltonian transition matrix is

$$\left[\begin{array}{c|c} \Phi_{11}(t) & \Phi_{12}(t) \\ \hline \Phi_{21}(t) & \Phi_{22}(t) \end{array}\right] = \left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right] \left[\begin{array}{c|c} e^{-\Lambda^* t} & 0 \\ \hline 0 & e^{\Lambda^* t} \end{array}\right] \left[\begin{array}{c|c} \Delta^* P_{22}^T & -\Delta^* P_{12}^T \\ \hline -\Delta^* P_{21}^T & \Delta^* P_{11}^T \end{array}\right]. \tag{91}$$

4 The McDonnell-Douglas Co generously provided a copy of the Blackburn-Vaughan report.

Hence the two partitions are

$$\begin{aligned}\Phi_{11}(\varepsilon) &= P_{11}e^{-\Lambda^* \varepsilon_\Delta^*}P_{22}^T - P_{12}e^{\Lambda^* \varepsilon_\Delta^*}P_{21}^T,\\ \Phi_{12}(\varepsilon) &= -P_{11}e^{-\Lambda^* \varepsilon_\Delta^*}P_{12}^T + P_{12}e^{\Lambda^* \varepsilon_\Delta^*}P_{11}^T.\end{aligned} \tag{92}$$

Since $\Phi_{11}(\varepsilon)$, $\Phi_{12}(\varepsilon)$, and $K(\tau)$ are now scaled in commensurate terms, (43) provides a computationally stable method for computing the finite-horizon state solution. And with $z(\tau+\varepsilon)$ and $K(\tau)$ available, the finite-horizon control solution - step (v) - follows immediately from (9), or its general equivalent.

Nothing specific has yet been said concerning the asymptotic program. With an infinite horizon, O'Donnell [pp.582-3] shows that considerable simplification ensues. In demonstration, the Hamiltonian system $\dot{x} = Hx$ may be partitioned as

$$\left[\begin{array}{c}\dot{z}\\ \hline \dot{p}\end{array}\right] = \left[\begin{array}{c|c}H_{11} & H_{12}\\ \hline H_{21} & H_{22}\end{array}\right]\left[\begin{array}{c}z\\ \hline p\end{array}\right], \tag{93}$$

and the relation $p(t) = K(t)z(t)$ of (6) applied to the first vector differential equation to provide the state dynamics

$$\dot{z}(t) = [H_{11} + H_{12}K(t)]z(t). \tag{94}$$

For $T = \infty$, $\bar{K} = \lim_{T\to\infty} K(t)$ is, from (77), and by analogy to (56),

$$\bar{K} = \lim_{T\to\infty} K(t) = P_{21}P_{11}^{-1}. \tag{95}$$

Thus (94) is

$$\dot{z}(t) = [H_{11}P_{11} + H_{12}P_{21}]P_{11}^{-1}z(t), \qquad t\varepsilon[0,\infty]. \tag{96}$$

Now the modified spectral form of H, (68), implies

$$HP = P\Lambda_1^* \Leftrightarrow \left[\begin{array}{c|c} H_{11} & H_{12} \\ \hline H_{21} & H_{22} \end{array}\right]\left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right] = \left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right]\left[\begin{array}{c|c} -\Lambda^* & 0 \\ \hline 0 & \Lambda^* \end{array}\right], \tag{97}$$

reducing to

$$\left[\begin{array}{c|c} H_{11}P_{11}+H_{12}P_{21} & H_{11}P_{12}+H_{12}P_{22} \\ \hline H_{21}P_{11}+H_{22}P_{21} & H_{21}P_{12}+H_{22}P_{22} \end{array}\right] = \left[\begin{array}{c|c} -P_{11}\Lambda^* & P_{12}\Lambda^* \\ \hline -P_{21}\Lambda^* & P_{22}\Lambda^* \end{array}\right]. \tag{98}$$

Comparing (96) and (98), the state equation (96) may be expressed in terms of the (1,1) partition of (98) as

$$\dot{z}(t) = -P_{11}\Lambda^* P_{11}^{-1} z(t), \qquad z(0)=z_o, \tag{99}$$

with stable solution

$$z(t) = P_{11}e^{-\Lambda^* t}P_{11}^{-1}z(0). \tag{100}$$

Thus the asymptotic solution is embedded in the finite horizon solution. The asymptotic eigenvalues are the stable, finite horizon eigenvalues $-\Lambda^*$; the asymptotic eigenvectors are the upper partition of the finite horizon eigenvectors, $[P_{11}^T \mid P_{12}^T]^T$, corresponding to the stable eigenvalues. Computationally, the asymptotic Riccati solution (95) and the asymptotic state solution (100) are considerably simpler than their finite horizon counterparts.

Appendix IIc contains (p.246 below) an illustrative simulation of the Bergstrom regulator for the same parameter values generating Table 2.1 above. Comparison of these two sets of results reveals that the irregularity due to precision loss has been removed. Further, satisfactory results have been obtained with horizons of 40 unit periods; and larger horizons may be used by appropriate selection of the step size (44).

This chapter therefore presents a computational algorithm for the optimal regulator that overcomes the precision loss problem and circumvents core storage problems associated with complex arithmetic. Additional research is still required to test the algorithm for efficiency and speed in high order systems. Although the original intention was to pursue these computational problems in depth, prior theoretical problems exist in the theory of stabilisation policy; and the rest of the thesis is devoted to these problems. But the analysis of this chapter remains relevant to this later work: there is an important and valuable feedback between computation and theory that will emerge in the sequel.

CHAPTER III

A DYNAMIC GENERALISATION OF TINBERGEN'S THEOREM

A review of the corpus of economic theory broadly referred to as the theory of economic policy, and represented by the work of Tinbergen, Bent Hansen, and Fox, Sengupta & Thorbecke (FST), permits several assertions. Firstly, the static theory of quantitative economic policy is well defined. Secondly, there is full awareness, following Phillips [1954, 1957], of the importance of the dynamics of policy-making. Thirdly, a significant body of knowledge exists relating to the design of dynamic policy - Phillips, Allen, and FST. But fourthly, there are no dynamic results comparable to Tinbergen's static analysis of the relation between instruments and targets. This last assertion begets, in part exploration and part answer, this present chapter.

Accompanying these assertions is a distinction between existence and design. Existence refers to the fundamental question of stabilisation: is it possible to design at least one policy to achieve the stated policy objectives? Design refers to the practical question of method: given existence, how are policies actually designed?

Statically, the problems of existence and design were first considered by Tinbergen [1963]; and have subsequently been elaborated as part of the conventional wisdom. Yet this has not been balanced by an analogous treatment dynamically. Phillips [1954] squarely confronts the dynamic design problem but treats the problem of existence implicitly. Applying classical control techniques to low dimension models, he demonstrates that stable dynamic policies can be constructed with these design methods. Existence is tied to design in that a suitably designed classical policy must satisfy a stability criterion such as Routh-Hurwitz. To overcome this ad hoc treatment of each model, FST sought to extend the theory of design by utilising dynamic optimisation techniques. Again, no specific attention is paid to the existence question. Further, their contribution to the design problem is an impossibility theorem which argues that if policy-makers use a quadratic performance ordering to obtain an optimal policy, then that policy is necessarily unstable. If true, this impossibility theorem denies the existence of dynamic policies that are simultaneously stable and optimal. Chapter V establishes, however, that the theorem is invalid, and that it is certainly possible to design optimal policies with satisfactory properties.

Thus the state of the art dynamically is that there exist two design methods - classical and optimal - neither of which specifically considers the fundamental question of whether a given economic system can be stabilised. This lack of explicit concern is not critical when these design methods are applied to the traditional single-target, single-instrument models of dynamic stabilisation. Existence for these models is a transparent question. But progression to the general multi-target, multi-instrument model necessitates explicit analysis of the possibility of design.

Faced with this lack of a criterion for dynamic existence comparable to the static criterion, Culbertson, for example, conjectures [pp.392-5] that the utility of Tinbergen's static analysis is questionable since this analysis is apparently irrelevant to the more realistic problem of dynamic stabilisation. The major objective of this chapter is to investigate this policy problem of dynamic existence; and to provide, as a corollary, a dynamic generalisation of Tinbergen's theorem.

Section 3.1 begins by positing a vector stabilisation model and defining conditions under which it partitions into two problems, one static, the other dynamic. To achieve this, a brief review of the static theory of existence is necessary. Section 3.2 generalises the analysis of section 3.1 through an explicit consideration of the possible combinations of the dimensions of the static and dynamic target and instrument vectors. As a result, two classes of vector stabilisation model are defined: square policy systems and rectangular policy systems. Statement of the dynamic existence problem in a format similar to the static existence problem leads section 3.3 to a recognition of the economic significance of the concept of dynamic controllability. Annexed from modern control theory, this concept provides a dynamic rank criterion to complement

Tinbergen's static criterion: indeed, for square policy systems, this dynamic criterion collapses to the static criterion. Section 3.4 specialises the dynamic criterion to rectangular policy systems; in these systems, the results obtained have no static counterparts. Phase analyses of simple policy systems are subsequently presented in section 3.5, clarifying the ideas of controllability and noncontrollability. Finally, section 3.6 summarises and concludes.

3.1 STATIC CONTROLLABILITY AND EQUILIBRIUM PARTITIONING

Suppose that a given economic system is modelled by the vector differential equation

$$\dot{X}(t) = \Gamma A^{*}X(t) + \Gamma B^{*}U(t) + \Gamma D^{*}, \quad X(0)=X_{o}, \tag{1}$$

with dimensions

$$X,\ D^{*}:\ N\text{x}1,\quad \Gamma,\ A^{*}:\ N\text{x}N,\quad U:\ K\text{x}1,\quad B^{*}:\ N\text{x}K. \tag{2}$$

The vectors X, U, and D^{*} are respectively the target, instrument and autonomous expenditure vectors. The matrices A^{*}, B^{*} describe the static economic structure: A^{*} will be defined as the static structural matrix and B^{*} as the static control matrix. The nonsingular matrix Γ is a diagonal matrix of adjustment speeds reflecting the dynamic process superimposed on the underlying static structure.

As a description of the stabilisation problem, equation (1) contains several implicit assumptions, two of which may be mentioned immediately. Firstly, the economic structure is assumed fixed, an assumption captured by the time-invariant matrices A^{*}, B^{*}, Γ. Qualitative policy, as defined by Tinbergen [1963, pp.2-3] and FST [p.20], is ignored; the stabilisation problem being to specify quantitative policy for a system with known and fixed structure. A form of qualitative change could be incorporated by specifying A^{*}, B^{*}, Γ as time-varying matrices $A^{*}(t)$, $B^{*}(t)$, $\Gamma(t)$. Such a change complicates but does not invalidate the results of this chapter. Yet a particular time-varying specification of these matrices necessitates provision of an economic theory of structural evolution and is therefore avoided in this preliminary analysis.

Secondly, the description (1) abstracts from economic growth, both explicitly and implicitly. Explicitly, autonomous expenditures, a traditional if now an insufficient collage of growth factors, are represented by a constant vector D^*. In terms of the solution of (1), this constancy generates a fixed rather than a moving economic equilibrium. Implicitly, the absence of growth is reflected in the type of variable and of functional relation appearing in the specification $\mathring{X}(t) = \Gamma A^* X(t)$ of the uncontrolled system. Thus the appearance of positive eigenvalues in ΓA^* will be interpreted as a manifestation of instability rather than of growth.

This chapter therefore investigates the shortrun policy problem of stabilisation in a nongrowing, nonevolving economy. Specifically, given the representation (1) and its assumptions, can conditions be provided to establish whether this system may be stabilised? To relate this problem both to the traditional static theory of economic policy and to the observed lack of dynamic theory, it is convenient to assume what it is ultimately desired to show. Thus suppose momentarily that a dynamic control vector U(t) can be designed so that the motion of (1) eventually ceases. The system's static equilibrium behaviour is then

$$A^* X + B^* U + D^* = 0, \qquad (\Gamma \text{ nonsingular}). \tag{3}$$

Additionally, suppose that there is a desired static target vector $\bar{X}$ attainable with the static control vector $\bar{U}$, again assuming such a control to exist. Then this desired solution $(\bar{X}, \bar{U})$ must satisfy (3),

$$A^* \bar{X} + B^* \bar{U} + D^* = 0. \tag{4}$$

In moving from (1) to (4), two existence assumptions have been made: that both static and dynamic policies can be designed. In the theory of economic policy, the first assumption has been well explored, while the second has been totally ignored. A brief summary of static existence theory will therefore suffice to establish its relevance to the pending development of a corresponding dynamic theory.

Two basic approaches to the static problem exist, depending on whether, following Tinbergen [1966, Chap. 3], targets are fixed or flexible. If the policy-maker independently specifies a desired target vector $\bar{X}$ and asks for the appropriate $\bar{U}$ such that $(\bar{X}, \bar{U})$ satisfy (4), then a fixed-target model of static stabilisation has been formulated. The conditions for which such a solution pair exists constitute the cornerstone of the static theory of economic policy. Writing the fixed-target model as

$$B^*\bar{U} = R, \qquad R \equiv -(A^*\bar{X} + D^*), \tag{5}$$

suppose that the NxK static control matrix B^* has rank M, implying that

$$M \leq \min (N, K). \tag{6}$$

Linear equation theory then provides the following two theorems (as stated for example by Lancaster [pp.248-9]):

<u>Theorem 3.1</u> (strong existence)

A solution to the system $B^*U = R$ <u>exists $\forall$ R</u> if and only if (iff) B^* has full row rank: i.e.

$$M = N \leq K. \tag{7}$$

...

<u>Theorem 3.2</u> (strong uniqueness)

A unique solution to the system $B^*U = R$ <u>exists $\forall$ R</u> iff B^* is full rank square: i.e.

$$M = N = K. \tag{8}$$

...

These theorems are termed 'strong' because they are satisfied not for one possible target vector but for all possible target vectors. Here R is loosely referred to as the target vector but is strictly, by (5), a linear function of the target vector $\bar{X}$.

With respect to the numerical relation of instruments to targets, it will be assumed that $K \leqslant N$. No attention is given to an economy with a surfeit of instruments: concern is with a balanced endowment ($K=N$) or a sparse endowment ($K<N$). Removal of $K > N$ in no way alters the conclusions of the chapter.

These two theorems are the basis for Tinbergen's classic proposition that the number of instruments K be at least equal to the number of targets N, [1963, chap. 4]. Given the exclusion of $K > N$, and in order to parallel the dynamic criterion to be presented in section 3.3, Tinbergen's results on the existence of a solution to the static fixed-target model may be reformulated as:

<u>Theorem 3.3</u> (static controllability)

The static economic system $A^*X + B^*U + D^* = 0$ is statically controllable iff the control coefficient matrix is full-rank square: i.e.

$$\rho(B^*) = N, \quad B^*: NxN. \tag{9}$$

...

From theorem 3.3, economies sparsely endowed with instruments ($K<N$) are statically noncontrollable; a solution does not exist for the fixed-target problem for every $\bar{X}$. And it is here that the flexible-target model is relevant to the static existence problem. Tinbergen has argued that fixed static targets derive from an implicit preference function. Recovery of this function affords one method for removing an impasse due to static noncontrollability. Thus following Bent Hansen [pp.23-7], although the model $B^*\bar{U} = R$ with $K < N$ is noncontrollable, a solution generally exists to the constrained static optimisation problem

$$\underset{U}{\text{MAX}}\; W(R) \quad \underline{\text{subject to}} \quad B^*U = R, \tag{10}$$

where W is a preference ordering of the elements of the target vector R. First-order conditions for (10) are

$$\frac{\partial W}{\partial U} = 0, \quad (Kx1), \tag{11}$$

so that this flexible-target model automatically adjusts to the availability of instruments. Its sole function is to resolve the policy dilemma of controlling a statically noncontrollable model. A second best or constrained solution $(\bar{X}, \bar{U})$ is obtained by positing tradeoffs among the targets.

Irrespective of whether $(\bar{X}, \bar{U})$ is obtained from a fixed-target model (5) or a flexible-target model (10), this solution pair must satisfy (3), as in (4). Comparison of the actual dynamic system (1) and this desired static solution (4) allows, after premultiplying (4) by Γ, the dynamic disequilibrium description

$$\dot{x}(t) = \Gamma A^{*}x(t) + \Gamma B^{*}u(t), \quad x(0)=x_{o}\neq 0, \tag{12}$$

in terms of the deviation vectors

$$x(t) \equiv X(t) - \bar{X}, \quad u(t) \equiv U(t) - \bar{U}. \tag{13}$$

This procedure of subtracting the desired equilibrium behaviour from the actual dynamic behaviour partitions the total stabilisation problem into a static problem (4) and a dynamic problem (12); and is identified as the equilibrium partition.

To avoid misunderstanding, it is stressed that the only requirement placed on the desired equilibrium solution (from which deviations are measured) is that it be a solution of (4). This allows for the use of either global optima or constrained optima as reference targets levels for the equilibrium partition. If the economic system is statically controllable, then provided a dynamic policy can be designed, targets will settle at those levels representing a global optimum of the implicit preference function. Otherwise, targets can only equilibrate to levels representing a constrained optimum of that function. Static stabilisation is concerned with the specification of appropriate levels of targets and instruments; dynamic stabilisation, with adjustment paths. Solution of the dynamic problem will force the system to equilibrate to a given equilibrium position; the task of the static solution is to define the preferred equilibrium.

Tinbergen's theorem, summarised in theorem 3.3, is a pair of conditions: numerical equation of instruments and targets is insufficient, the control matrix B^* must also possess full row rank. Thus, even though N=K, the static system (3) or (5) may be noncontrollable because $\rho(B^*) < N$. For subsequent analysis, these static possibilities are normalised as follows.

Proposition 3.1

For the static policy system $A^*X+B^*U+D^*=0$, either (i) $\rho(B^*)=N$ or (ii) the system is equivalent to

$$A^*X + B_1^*U_1 + D^* = 0, \tag{14}$$

where

$$B_1^*: \ N\times K, \quad U_1: \ K\times 1, \quad \rho(B_1^*)=K < N. \tag{15}$$

proof

If (i) does not hold, then $\rho(B^*) < N$. Suppose $\rho(B^*) = N-1$, then one of the N columns of B^* is a linear combination of the other N-1 columns. Hence assume without loss of generality that the first column of B^* is a linear combination of the next i-1 columns. Then

$$B^*U = [\alpha_2 b_2^* + \ldots + \alpha_i b_i^* \ b_2^* \ \ldots \ b_N^*] \begin{bmatrix} U_1 \\ \vdots \\ U_N \end{bmatrix}, \tag{16}$$

where b_j^* is the j^{th} column of B^*. Hence

$$B^*U = [b_2^* \ \ldots b_i^* \ b_{i+1}^* \ \ldots b_N^*] \begin{bmatrix} U_2+\alpha_2U_1 \\ \vdots \\ U_i+\alpha_iU_1 \\ U_{i+1} \\ \vdots \\ U_N \end{bmatrix} = B_1^*U_1. \tag{17}$$

This process of reduction continues until $\rho(B_1^*)=K < N$.

...

Proposition 3.1 simply removes dependent instruments from the statement of the dynamic stabilisation problem. The derivation of the equilibrium partition (12) remains valid except that the condition $\rho(B^*)=K$, $K \leq N$, is always implied in the sequel. The subscript, as in (15), is also omitted: the dimension and rank of B^* will be clear in context.

Derivation of the equilibrium partition requires that the chosen equilibrium be attainable; that with respect to (12) a dynamic policy $u(t)$ can be designed to drive $x(t)$ to zero. Thus if $K=N$, dynamic existence for a statically controllable system is to be investigated in the context of (12); if $K < N$, dynamic existence for a statically noncontrollable system. These are the basic questions to be considered in this chapter. But before establishing necessary and sufficient conditions for which dynamic design is possible, section 3.2 removes an unnecessary assumption concealed in (12).

3.2 VARIABLE TARGET AND INSTRUMENT DIMENSIONS

Implicit in the analysis of the previous section, leading from the static description (1) to the dynamic description (12), is the assumption that the static and dynamic reduced forms are of the same dimensions. Thus the target vectors $x(t)$, $X(t)$ are both of dimension $N\text{x}1$, and the instrument vectors $u(t)$, $U(t)$ are both of dimension $K\text{x}1$, $K \leq N$. This dimensional equivalence need not be true for either instruments or targets.

Fewer instruments may be used dynamically than are statically available. Instruments are only optionally dynamic; if they do vary dynamically, it is because of a deliberate policy decision. To avoid prejudging the dynamic existence question, it is therefore desirable to permit the possibility that some, but not all, instruments $u_j(t)$, $j \varepsilon 1,\ldots,K$, are zero. That is, some instruments may be left at their desired static levels while the remaining instruments are varied dynamically. Now the effect of $u_j(t) = 0$ is to filter out the control coefficient vector b_j^*, the j^{th} column of B^*. Hence if any $u_j(t)$ are identically zero $\forall$ t, the dimensions (K, k) of (U, u) must be such that

$$K \geqslant k. \tag{18}$$

Since the existence of a dynamic control implies a static control but not conversely, the case k > K is excluded. The case K > k is defined as reduced stabilisation; and rigorous conditions validating this possibility are presented in chapter IV following. At this stage, (18) asserts that the instrument vectors U, u need not possess identical dimensions.

Equivalence of the target vector dimensions depends on the assumption in (1) that the dynamics of the given economic system are of the first order. But it is likely that realistic lag structures will produce higher order dynamics, destroying this equivalence. Since the ideas involved here are relevant to later chapters as well as to the historical development of dynamic stabilisation theory, they are now considered in some detail.

The vehicle traditionally used for analysis of dynamic stabilisation policy - for example, by Phillips [1954, 1957], Allen [1960, 1968] - may be termed the scalar policy model. Statically, this model relates a single target to a single instrument; dynamically, it extends this relation, by various lag and response assumptions, to an n^{th} order differential equation in the target, with the instrument as forcing function. The basic idea of variable target dimensions is that this differential equation can be transformed to a system of n first order differential equations, in which the nx1 vector of dynamic variables is defined as the dynamic target vector. Since the static reduced form is a scalar equation, the static and dynamic reduced forms therefore differ in their target dimensions.

For illustration, suppose that a macroeconomic system is given by

$$y(t) = Fe(t)$$

$$e(t) = g(t) + i(t) + c(t) \tag{19}$$

$$i(t) = L_i y(t)$$

$$c(t) = L_c y(t),$$

and schematically by Figure 3.1.

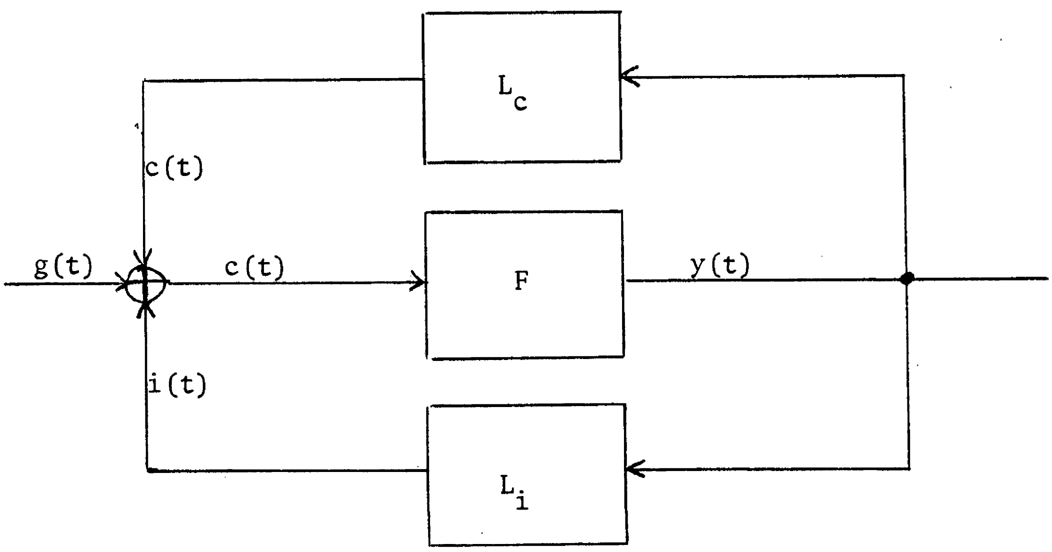

Figure 3.1

A Scalar Policy System

The variables in (19) are the target y(t), the instrument g(t), and the feedback variables c(t), i(t), functionally related to the target. The variable e(t) is the sum of the instrument and these feedbacks. F, L_c, and L_i are assumed to be linear operators. It is also assumed that (19) represents a system to which the equilibrium partition has been applied so that exogenous variables are removed.

Particular specifications of the linear operators determine the dynamic dimension acquired by (19)[1]. From that equation, target and instrument are related by

$$y(t) = [\frac{F}{1-FL_i-FL_c}]g(t). \tag{20}$$

[1] The scalar policy models of chapters V and VI below are specific examples of the system (19).

Hence for the specification

$$F = 1, \quad L_i = \alpha D, \quad L_c = \beta, \quad D \equiv d/dt, \tag{21}$$

equation (20) is the first-order differential equation

$$\dot{y}(t) = (1/\alpha)(1-\beta)y(t) - (1/\alpha)g(t). \tag{22}$$

Or if the forward operator F is the n^{th} order exponential lag operator

$$F = \frac{\lambda^n}{(D+\lambda)^n}, \tag{23}$$

with L_c, L_i defined in (21), then (20) becomes

$$[F^{-1}-L_i-L_c]y(t) = g(t), \tag{24}$$

equivalent to the n^{th} order differential equation

$$D^n y(t) + a_{n-1}D^{n-1}y(t) + \ldots + a_o y(t) = b_o g(t). \tag{25}$$

The coefficients a_i, $i=o,\ldots,n-1$, and b_o are determined from (24). A similar representation to (25) results if the control input g(t) can only be applied with an exponential lag.

Specification of the feedback operators L_c, L_i as lag operators has an additional effect. Thus suppose

$$F = 1, \quad L_i = \frac{\rho^p}{(D+\rho)^p}, \quad L_c = \frac{\sigma^q}{(D+\sigma)^q}, \quad p+q = n. \tag{26}$$

Then the system (20) is

$$[(L_iL_c)^{-1}-FL_c^{-1}-FL_i^{-1}]y(t) = (L_iL_c)^{-1}Fg(t), \tag{27}$$

which simplifies, using (26), to

$$D^n y(t) + a_{n-1}D^{n-1}y(t) + \ldots + a_o y(t)$$
$$= b_n D^n g(t) + \ldots + b_o g(t). \tag{28}$$

Lag specifications in the feedback loops of Figure 3.1 therefore induce input dynamics as well as output dynamics.

Depending on dynamic operator specifications, equations (22), (25), and (28) are thus alternative versions of the general scalar policy model. All, however, possess the static solution

$$y(\infty) = 0, \quad g(\infty) = 0, \tag{29}$$

provided the controlled system is stable. To subsume these versions in a single equation, replace the input dynamics of (28) by

$$\hat{g}(t) = b_n D^n g(t) + \ldots + b_o g(t), \tag{30}$$

as suggested by Kalman, Ho & Narendra [p.202]. Here g(t) is a solution of the differential equation (30) with forcing function $\hat{g}(t)$ and given initial conditions. Given this device, there is no loss of generality in considering the differential equation

$$D^n y(t) + a_{n-1} D^{n-1} y(t) + \ldots + a_o y(t) = b_o g(t), \tag{31}$$

for appropriate b_o, g(t), as the dynamic reduced form of the scalar policy model.

What relevance does (31) have to variable target dimensions? To answer this question, consider the canonical transformation

$$D^{i-1} y(t) = z_i(t), \qquad i = 1,\ldots,n, \tag{32}$$

applied to (31) to give

$$\begin{aligned} \dot{z}_i(t) &= z_{i+1}(t), \qquad i = 1,\ldots,n-1, \\ \dot{z}_n(t) &= D^n y(t) = -a_{n-1} z_n(t) - \ldots - a_o z_1(t) + b_o g(t). \end{aligned} \tag{33}$$

In matrix form, this system of n first-order equations is

$$\dot{z} = Az + bg,$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \\ -a_o & & \cdots & & & -a_{n-1} \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} b_o, \quad z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}. \qquad (34)$$

Equations (31) and (34) are therefore equivalent representations of the scalar policy model. The first is the classical input-output representation; the second is the modern state space representation[2].

Now equations (22), (25), and (28) correspond to particular values of n, the dimension of the state vector z in (34). Although from the classical viewpoint there is one target y(t) possessing n^{th} order dynamics, it is convenient to regard each component of the state vector as a dynamic target variable, for reasons made clear in the following section. If n is defined as the number of dynamic targets, and if N is the number of static targets, then for the scalar policy model

$$n \gtrsim N (= 1). \qquad (35)$$

Hence static and dynamic reduced forms need not possess identical target dimensions.

The general stabilisation problem will realistically involve the attainment of multiple targets using multiple instruments - the vector stabilisation problem of (1). Similar arguments with respect to variable target dimensions apply immediately to this model. Therefore, using the earlier assumption $N \gtrsim K$, and equations (18) and (35), this analysis of variable target and instrument dimensions implies the integer ordering

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2 For further analysis of these two approaches see, for example, Athans & Falb [pp.173-90] and Ogata [chap. 4].

$$n \geqslant N \geqslant K \geqslant k \tag{36}$$

on the numbers of dynamic targets (n), static targets (N), static instruments (K), and dynamic instruments (k). Given the nesting of (36), if the dynamic dimensions coincide (n=k), then

$$n = N = K = k. \tag{37}$$

To summarise, the variable target and instrument dimensions (36) arise because dynamic targets are not necessarily static targets, and because static instruments are not necessarily dynamic instruments. Consequently, the dynamic stabilisation model (12) may be respecified as

$$\dot{x}(t) = Ax(t) + Bu(t), \quad \rho(B)=k, \tag{38}$$

with dimensions

$$x: \; nx1, \quad u: \; kx1, \quad A: \; nxn, \quad B: \; nxk. \tag{39}$$

Both A and B subsume an nxn adjustment speed matrix Γ.

Two general cases of (38) are important for subsequent analysis. If k=n, then by (37), the original disequilibrium specification (12) re-emerges, after invoking the rank assumption of proposition 3.1 to give

$$\rho(B^{*}) = \rho(B) = K = k = n = N. \tag{40}$$

Dynamic systems (38) satisfying (40) will be defined as square policy systems. If k < n, then systems (38) satisfying

$$\rho(B^{*}) = k < n, \tag{41}$$

will be defined as rectangular policy systems: dynamically there are fewer instruments than targets. Rectangular systems occur not only because of static noncontrollability but also because of target and/or instrument variation; all three cases being characterised by k < n.

The following section presents the general existence criterion for the dynamic system (38), for both square and rectangular systems; and specialises it to square policy systems. Section 3.4 then considers its application in rectangular policy systems.

3.3 DYNAMIC CONTROLLABILITY IN SQUARE POLICY SYSTEMS

Section 3.1 partitioned the stabilisation problem into two problems and interpreted the central result of the static theory of economic policy as providing necessary and sufficient conditions for static controllability. This prompts inquiry as to the existence and form of corresponding results for the dynamic stabilisation problem. Is there a concept of dynamic controllability; and if so, what form does it take?

When evaluating Tinbergen's contribution to the theory of economic policy, Culbertson argues [p.394] that the Tinbergen framework

> "...has to do with equilibrium values. The framework does not appear to have any clear application to dynamic analysis, in which each variable has not a one-dimensional fixed value but a time pattern of behaviour...".

Thus Culbertson conjectures that there are no dynamic results comparable to those of static controllability; or that Tinbergen's static results may lack relevance dynamically. This theoretical asymmetry is undesirable given the importance, as shown by Phillips [1954], of the dynamics of stabilisation. Clarification of the dynamic problem is therefore a first step towards removal of this asymmetry.

The object of analysis is the dynamic system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0)=x_o \neq 0, \quad \rho(B)=k < n. \tag{42}$$

Now if Tinbergen's analysis is to be vindicated dynamically, a logical procedure is to pose the dynamic problem in a fixed-target format similar to the static problem. Thus given the dynamic system (42), does there exist a control policy $u(t)$ which drives the system from an initial position $x(0)\neq 0$ to a terminal position $x(T)=0$ over some fixed control period T? If this terminal position can be achieved and maintained, the specified static equilibrium results. The primary difference between the dynamic and static fixed-target models is that explicit recognition of the dynamics of adjustment necessitates explicit recognition of the period of adjustment.

Equation (42) has the general solution

$$x(t) = \Phi(t,0)x(0) + \int_0^t \Phi(t,\tau)Bu(\tau)d\tau, \tag{43}$$

where $\Phi(t,0)$ is the transition matrix given by, for a linear constant system,

$$\Phi(t,0) = e^{At}. \tag{44}$$

At time $t=T$, the desired solution for (43) is $x(T)=0$, implying with use of (44) that

$$- \int_0^T e^{-A\tau}Bu(\tau)\ d\tau = x(0). \tag{45}$$

Equation (45) is a fixed-target model of dynamic stabilisation analogous to the static model (5). The dynamic target vector $x(0)$ is that discrepancy between the initial position and the static equilibrium to be removed by dynamic policy action; and the integral on the left of (45) defines the dynamic policy structure in which the available instruments $u(\tau)$ are embedded.

What are the requirements for the existence of a control policy $u(\tau)$ satisfying (45)? Is it possible, with the given instruments, to move the economic system dynamically from the given initial state to the desired terminal state? This fundamental question of dynamic existence has been ignored by both the classical and the optimal schools. It is here, and not for the first time[3], that the discipline of control theory can make a basic contribution to the theory of economic policy. During the last decade, Kalman [1959, 1961, 1963a, 1963b] and Kalman, Ho & Narendra have developed the concept of dynamic controllability to a stage where it is immediately applicable to this

3 Cf. Phillips' application [1954] of classical control techniques to the dynamic design problem.

policy problem.

Linear constant systems of the form (42) are dynamically controllable if it is possible to find a control u(t) which, in specified finite time T, will transfer the system between any two arbitrary states, x(0) and x(T). For $x(0)\neq 0$ and $x(T)=0$, this definition accords with the statement of dynamic stabilisation as a fixed-target problem. Now necessary and sufficient conditions for dynamic controllability are given by (see Appendix III, pp.261-3 below):

<u>Theorem 3.4</u> (dynamic controllability)

The continuous-time linear constant system $\dot{x}(t) = Ax(t)+Bu(t)$ is dynamically controllable iff the composite nxnk matrix

$$Q = [B \quad AB \quad \dots \quad A^{n-1}B] \tag{46}$$

possesses full row rank, i.e.

$$\rho(Q) = n. \tag{47}$$

...

As demonstrated in Appendix III, theorem 3.4 is obtained by converting the integral equations of (45) into a set of algebraic equations

$$Q\beta = x(0), \qquad (nxnk.nkx1=nx1), \tag{48}$$

to which theorem 3.1 is immediately applicable. The matrix Q and the vector β are the dynamic analogues of B^* and $\bar{U}$ in (5). Whereas B^* is the static control coefficient matrix, Q is a composite function of the dynamic control matrix B and the dynamic structural matrix A; and whereas $\bar{U}$ is simply the vector of static instruments, β is a vector whose elements are functions both of the instruments and of certain linearly independent time functions. Just as theorem 3.3 provides necessary and sufficient conditions for static controllability for all possible static targets, so too theorem 3.4 provides necessary and sufficient conditions for dynamic controllability for all possible dynamic targets, or for all possible initial conditions.

It is therefore clear from (5), (48) that Tinbergen's static approach, at the very least, carries over to the problem of dynamic stabilisation. Further, if the equilibrium partition derives from a statically controllable system, so that (40) is satisfied, a striking result emerges. For then the controllability matrix Q comprises n blocks $A^{j-1}B$, $j=1,\ldots,n$, $A^o = I$, each of dimension nxn:

$$Q_S = [B \quad AB \quad \ldots \quad A^{n-1}B], \qquad (nxn^2). \tag{49}$$

The dynamic rank criterion of theorem 3.4 requires that $\rho(Q_S)=n$; or, from Ferrar [p.94], that at least one nxn minor of (49) is nonzero. From inspection of (49), an immediate sufficient condition for dynamic controllability is therefore

$$\rho(B) = \rho(\Gamma B^*) = \rho(B^*). \tag{50}$$

Hence statically controllable systems are invariably dynamically controllable. This result preserves unchanged Tinbergen's static theorem in a dynamic context; and is an appropriate denial of Culbertson's conjecture that Tinbergen's static analysis has 'no clear application to dynamic analysis'.

Before examining theorem 3.4 for the case $k < n$, an equivalent but intuitively more acceptable criterion is available[4]. Assuming that the structural matrix A possesses n distinct eigenvalues, there exists a nonsingular eigenvector matrix P such that the transformation

$$x(t) = Pz(t) \tag{51}$$

reduces (42) to

$$\dot{z}(t) = \Lambda z(t) + \hat{B}u(t), \tag{52}$$

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4 See Kalman, Ho & Narendra [pp.201-4]; and for exposition, Elgerd [pp.67-117].

where Λ, $\hat{B}$ are given by

$$\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) = P^{-1}AP, \qquad \hat{B} = P^{-1}B, \quad \lambda \varepsilon E(A), \tag{53}$$

and E(A) is the set of n distinct eigenvalues of A. Since Λ is the diagonal matrix of eigenvalues, the motion $\dot{z}_i(t)$ of the variable $z_i(t)$ is uncoupled from the variables $z_j(t)$, $j \neq i$. To control each variable $z_i(t)$, it is necessary and sufficient that each equation of (52) contains at least one control variable $u_j(t)$ with a nonzero coefficient. For suppose the i^{th} equation is

$$\dot{z}_i(t) = \lambda_i z_i(t), \qquad z_i(0) = \bar{z}_i, \tag{54}$$

with solution

$$z_i(t) = e^{\lambda_i t} \bar{z}_i. \tag{55}$$

Then the i^{th} variable is uncontrollable: for $\lambda_i > 0$, $z_i(t)$ diverges uncontrollably; and for $\lambda_i < 0$, converges uncontrollably. In either case, it cannot be forced to a desired terminal value $\bar{z}_i(T)$. Dynamic controllability therefore requires that the matrix $\hat{B} = P^{-1}B$ contains nonvanishing rows. This will be termed the coupling criterion.

Equations (54) and (55) illustrate that dynamic controllability is defined without reference to system stability. If a system is dynamically controllable, then a control can always be devised, irrespective of the stability of the uncontrolled system. If, however, a system is dynamically noncontrollable, stability is then important. From the coupling criterion, dynamic noncontrollability means that certain (transformed) states are not accessible to the given instruments. If these states are unstable, target performance as a function of these states worsens progressively with the length of the stabilisation horizon, irrespective of any control action. If these states are stable, target performance as a function of these states improves progressively with the length of the stabilisation horizon, owing to their decay.

Dynamic noncontrollability in an economic system would indicate structural areas insulated both directly and indirectly from policy. In principle, this is overcome by developing, if possible, additional control forces appropriate to the control 'gaps' in the system. Policy intransigence, therefore, in the large-scale complex system of a modern economy may be a consequence of economic structure rather than or as well as of data availability and lack of expertise, resources, and accurate system modelling.

To summarise results to this point, the theory of economic policy provides the Tinbergen theorem of static controllability but no comparable dynamic theorem. Indeed, Culbertson conjectures that such a dynamic theorem might not exist. By applying the concept of controllability to the problem of dynamic stabilisation, it is shown that a comparable theorem does exist. As a result, either of two equivalent criteria - the rank and coupling criteria - may be used to ascertain existence of a solution to the dynamic problem. This question of existence logically precedes that of design. And when the rank criterion is applied to a disequilibrium system derived from a statically controllable model, Tinbergen's static theorem generalises to provide a dynamic existence criterion for square policy systems.

Section 3.4 now considers the interpretation of theorem 3.4 in rectangular policy systems.

3.4 DYNAMIC CONTROLLABILITY IN RECTANGULAR POLICY SYSTEMS

Section 3.1 argued that, independently of whether static equilibrium represents a global or constrained optimum of the static preference function, the total stabilisation problem may be partitioned into a static problem and a dynamic problem, with theorem 3.4 of the previous section providing necessary and sufficient conditions for existence of a policy solution to the dynamic problem. The type of static solution does, however, affect the conclusions drawn from application of this theorem; as do variations in target and instrument dimensions. Thus section 3.3 specialised the rank criterion of theorem 3.4 to square policy systems; in this section, theorem 3.4 is specialised to rectangular policy systems.

From (38) and (41), rectangular policy systems are defined by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad B: nxk, \quad \rho(B)=k < n, \tag{56}$$

for which the controllability matrix (46) is

$$Q_R = [B \quad AB \quad \dots \quad A^{n-1}B], \quad Q_R: nxnk, \quad k < n. \tag{57}$$

A simple condition like (50) on the existence of an nxn minor in (57) is no longer evident. Dynamic controllability is now dependent on both the control matrix B and the structural matrix A. But neither policy-makers not policy-designers will be enlightened by the role of rank (Q_R) as arbiter of the dynamic controllability of their particular economic system. This section therefore attempts to expose the operation of this criterion in terms of properties of the structural and control matrices. To facilitate this task, the distinctness assumption on the eigenvalues of the structural matrix A will be retained[5]. The derivation and interpretation of a sufficient condition for dynamic controllability in rectangular policy systems will be given; analysis of necessity is deferred to chapter IV.

Interpretation of the rank criterion may be broached by taking the simplest case first. Thus for k=1, consider the scalar policy system

$$\dot{x}(t) = Ax(t) + bu(t), \quad b: nx1, \tag{58}$$

with square controllability matrix

$$Q_s = [b \quad Ab \quad \dots \quad A^{n-1}b], \quad (nxn). \tag{59}$$

Is the system (58) controllable with a single instrument; and if so, under what conditions?

From the discussion of the coupling criterion in section 3.3,

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5 This assumption is previously used (i) as a means of simplifying the proof of theorem 3.4; and (ii) as a sufficient condition for the existence of a diagonalising transformation of the structural matrix when defining the coupling criterion.

$\rho(Q_s) \neq n$ iff

$$\rho(\hat{Q}_s) \neq n, \qquad \hat{Q}_s = [\hat{b} \quad \Lambda\hat{b} \quad \ldots \Lambda^{n-1}\hat{b}], \tag{60}$$

where, by similarity transformation of (58),

$$\hat{b} = P^{-1}b, \qquad \Lambda = \text{diag}(\lambda_1,\ldots,\lambda_n) = P^{-1}AP. \tag{61}$$

Now the matrices Λ^j, by the distinctness assumption, are independent $\forall$ j, so that the only admissible cause of singularity of $\hat{Q}_s$ is a vanishing element (or elements) in the vector $\hat{b}$. But from (61), this requires that the corresponding row of P^{-1} is orthogonal to b; or, since P^{-1} is a normalised left eigenvector matrix for the structural matrix, that b is a linear combination of no more than n-1 of the eigenvectors of A. For example, if

$$b = \alpha_1 p_1 + \ldots + \alpha_{n-1} p_{n-1}, \qquad \alpha_i \text{ constants}, \tag{62}$$

then

$$\hat{b} = P^{-1}b = [\alpha_1 \; \alpha_2 \quad \ldots \quad \alpha_{n-1} \; 0]^T, \tag{63}$$

and the n^{th} element of $\hat{b}$ vanishes. A necessary and sufficient condition for dynamic controllability in the scalar policy system (58) may therefore be stated as

$$b \neq \sum_{i=1}^{v} \alpha_i p_i, \qquad v \leqslant n-1. \tag{64}$$

For v = 1, (64) may be rewritten as

$$Ab \neq \lambda b, \qquad \lambda \varepsilon E(A), \tag{65}$$

a requirement that the control coefficient vector b is not an eigenvector of the structural matrix A.

Returning to the general rectangular problem (56), and writing the controllability matrix (57) in expanded form as

$$Q_R = [b_1 \ldots b_k \quad Ab_1 \ldots Ab_k \ldots A^{n-1}b_1 \ldots A^{n-1}b_k], \quad k<n, \tag{66}$$

displays the existence of k scalar systems of the form (58) with associated controllability matrices (59). Hence, by (64), an immediate sufficient condition for dynamic controllability in rectangular policy systems is

$$b_j \neq \sum_{i=1}^{v_j} \alpha_{ij} p_i, \qquad v_j \leq n-1, \qquad j \in 1,\ldots,k, \tag{67}$$

satisfied for some j.

It can be shown immediately that violation of (65) $\forall$ j = 1,...,k is sufficient for dynamic noncontrollability. That is, suppose each control vector b_j is, apart from a possible scalar multiple, an eigenvector p_j of A, so that $v_j = 1 \; \forall \; j = 1,\ldots,k$ in (67). Using the transformed controllability matrix

$$\hat{Q}_R = [\hat{B} \quad \Lambda\hat{B} \quad \ldots \quad \Lambda^{n-1}\hat{B}], \qquad \hat{B} = P^{-1}B, \tag{68}$$

assume without loss of generality that the k columns of B and the first k rows of P^{-1} are biorthogonal. Then, ignoring possible scalar factors,

$$\hat{B} = P^{-1}B = \begin{bmatrix} I_{kxk} \\ \hline 0_{n-kxk} \end{bmatrix}, \quad \Lambda^j\hat{B} = \begin{bmatrix} \Lambda^j_{kxk} \\ \hline 0_{n-kxk} \end{bmatrix}, \tag{69}$$

so that

$$Q_R = \left[\begin{array}{c|c|c|c} I_{kxk} & \Lambda_{kxk} & \cdots & \Lambda^{n-1}_{kxk} \\ \hline 0_{n-kxk} & 0_{n-kxk} & \cdots & 0_{n-kxk} \end{array}\right]. \tag{70}$$

Under the assumption that each control vector b_j is an eigenvector of A, the largest nonzero minor in Q_R is thus of dimension k < n, implying noncontrollability for any number of instruments. In the general case in which (67) is violated $\forall$ j = 1,...,k so that

$$b_j = \sum_{i=1}^{v_j} \alpha_{ij} p_j, \qquad j = 1,\ldots,k, \qquad 1 < v_j \leq n-1, \tag{71}$$

the rectangular policy system (56) is noncontrollable with respect to any single instrument but the question of controllability when several instruments are jointly used remains open. This question will be examined in chapter IV.

Since the eigenvector condition (64) or (67) has no counterpart in the static theory of economic policy, the remaining part of this section seeks an interpretation[6] of this condition. Equation (45) specifying the dynamic instrument-target structure is a convenient starting point. Taking k=1, and using (53) to provide

$$e^{-At} = Pe^{-\Lambda t}P^{-1}, \tag{72}$$

equation (45) may be written

$$-\int_0^T e^{-\Lambda t}P^{-1}bu(t)\ dt = P^{-1}x(0). \tag{73}$$

Now solutions of the linear constant differential system $\dot{x} = Ax$ will consist of linear combinations of the n exponential modes exp $\{\lambda_j t\}$, $j = 1,\dots,n$, $\lambda \epsilon E(A)$. Therefore, solutions of $\dot{x} = Ax+Bu$, and hence of (73), will also consist of linear combinations of these modes, provided the feedback principle is used in designing the controller u(t) and provided no constraints are placed on the instruments. Thus, postulating the smooth controller

$$u(t) = -\sum_{j=1}^{n} \gamma_j e^{-\lambda_j t}, \tag{74}$$

where the coefficients γ_j have yet to be determined, and taking p_j (p_j^{-1}) as the j^{th} column (row) of P (P^{-1}), equation (73) is

---

6 The following analysis is suggested by the mode analysis presented by Zadeh & Desoer [pp.311-26].

$$\int_0^T \left\{ \begin{bmatrix} e^{-\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{-\lambda_n t} \end{bmatrix} \begin{bmatrix} (p_1^{-1},b) \\ \vdots \\ (p_n^{-1},b) \end{bmatrix} \begin{bmatrix} e^{-\lambda_1 t} & \cdots & e^{-\lambda_n t} \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{bmatrix} \right\} dt$$

$$= \begin{bmatrix} (p_1^{-1},x(0)) \\ \vdots \\ (p_n^{-1},x(0)) \end{bmatrix}. \tag{75}$$

After manipulation, (75) becomes

$$\begin{bmatrix} (p_1^{-1},b) & & 0 \\ & \ddots & \\ 0 & & (p_n^{-1},b) \end{bmatrix} \left\{ \int_0^T \begin{bmatrix} e^{-\lambda_1 t} e^{-\lambda_1 t} & \cdots & e^{-\lambda_1 t} e^{-\lambda_n t} \\ \vdots & & \vdots \\ e^{-\lambda_n t} e^{-\lambda_1 t} & \cdots & e^{-\lambda_n t} e^{-\lambda_n t} \end{bmatrix} dt \right\} \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{bmatrix}$$

$$= \begin{bmatrix} (p_1^{-1},x(0)) \\ \vdots \\ (p_n^{-1},x(0)) \end{bmatrix}, \tag{76}$$

or

$$B'G\,\gamma = x'(0), \qquad (n x n . n x n . n x 1 = n x 1), \tag{77}$$

with the appropriate identifications.

The problem of dynamic controllability is whether or not there exists a solution for γ, the vector of open-loop coefficients. Applying theorem (3.1) to (77), such a solution exists iff

$$\rho(B'G) = \rho(B') = \rho(G) = n. \tag{78}$$

From Zadeh & Desoer [pp.497-8], a necessary and sufficient condition for $\rho(G)=n$ is that $\exp\{-\lambda_1 t\}$, ..., $\exp\{-\lambda_n t\}$ are linearly independent functions of t; and this is guaranteed by the distinctness assumption on the eigenvalues. A necessary and sufficient condition for $\rho(B')=n$ is

$$(p_j^{-1},b) \neq 0 \;\forall\, j = 1,\ldots,n. \tag{79}$$

With respect to (79), any nx1 vector x has a representation $x = \sum_{j=1}^{n} \alpha_j p_j$ in the eigenvector basis, with the α_j as components along p_j in that basis. Hence $x = P\alpha$ implies $\alpha = P^{-1}x$, so that $\alpha_j = (p_j^{-1},x)$, with x therefore given by

$$x = \sum_{j=1}^{n} (p_j^{-1},x)p_j. \tag{80}$$

For $x \equiv b$ in (80), (79) requires that no component of b in the eigenvector basis vanishes. But if

$$b = \sum_{i=1}^{v} \alpha_i p_i, \qquad v \leq n-1, \tag{81}$$

as given in (64), then at least one such component must vanish. Thus the control coefficient vector b must not be confined to a proper subspace, of dimension n-1 or less, of the state space if dynamic controllability is to hold. For then, in the words of Kalman, Ho & Narendra [p.203], "the effect of control eventually reaches every state".

Although the eigenvector condition (67) is, in general, only sufficient for dynamic controllability, it is, by (64), both necessary and sufficient when only one instrument is available. In the following section, the notion of dynamic controllability and the significance of this eigenvector condition are therefore considered geometrically in simple scalar policy models.

3.5 PHASE ANALYSIS OF CONTROLLABILITY

Further insight into the dynamic controllability criterion may be obtained by considering a dynamic system with two target variables and a single control variable, the control variable appearing in only one of the state equations: i.e.

$$\mathring{x} = Ax + bu, \qquad x(0) \neq 0,$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ b_2 > 0 \end{bmatrix}. \tag{82}$$

This model could, for example, represent the venerable problem of internal and external balance, treated not in terms of static equilibrium but in terms of disequilibrium dynamics. The dynamic policy problem is whether or not the static equilibrium implicit in (82) is attainable with the single instrument available dynamically.

By theorem 3.4, the system (82) can be forced to the equilibrium point $x \equiv 0$ in finite time iff

$$\rho(Q) = 2, \qquad Q = [b \quad Ab] = \begin{bmatrix} 0 & a_{12}b_2 \\ b_2 & a_{22}b_2 \end{bmatrix}. \tag{83}$$

Hence Q is singular if $a_{12} = 0$ and/or $b_2 = 0$. The system (82) is therefore dynamically noncontrollable if the first target x_1 is disconnected from the second target x_2 and thus from the instrument, or if the instrument itself is unavailable.

Now suppose (83) is satisfied, and that the structural matrix A has the unstable sign pattern

$$A: \begin{pmatrix} + & - \\ - & - \end{pmatrix}. \tag{84}$$

Then the policy system (82), (84) may be represented geometrically by the phase diagram of Figure 3.2:

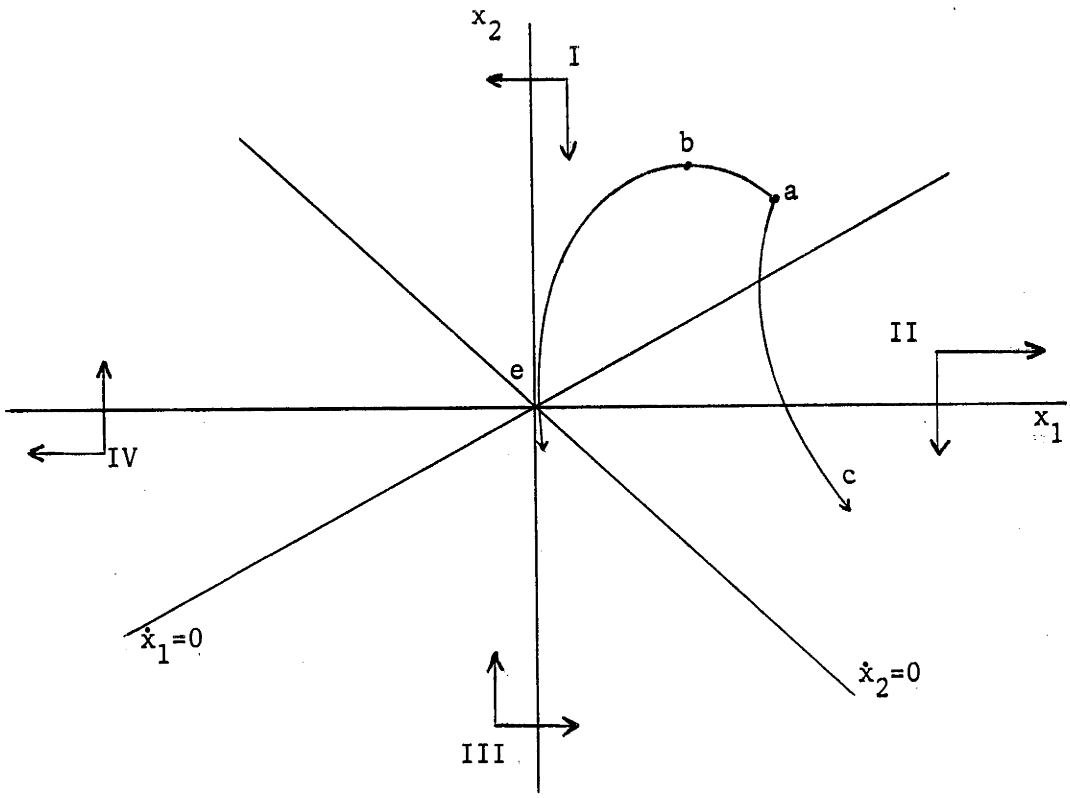

Figure 3.2

An Unstable Controllable System

The locus $\dot{x}_2 = 0$ is a parametric function of the instrument u(t), and is shown for u(t)=0. Thus the designated origin (e) is the desired static equilibrium point for the system. For initial conditions $x(0)\neq 0$ given by (a), suppose that the system, left to its own devices dynamically, would follow the trajectory (ac), reflecting the assumed instability (84). But since the system satisfies (83), and is controllable, it must therefore be possible to force the system from point (a) to point (e) in finite time. Figure 3.3 presents one construction of a trajectory accomplishing this transfer - the trajectory (abe) shown in Figure 3.2.

At the control origin t=0, the instrument is set at a constant level such that region I in Figure 3.3a is contracted eastwards, relocating (a) in region IV. If control persisted at this level, the system would diverge along the trajectory (ab); but at some time t=t' corresponding to (b), control is reset as shown in Figure 3.3b. Region I is expanded sufficiently not only to recapture the system trajectory but also to steer it through the equilibrium point (e) at

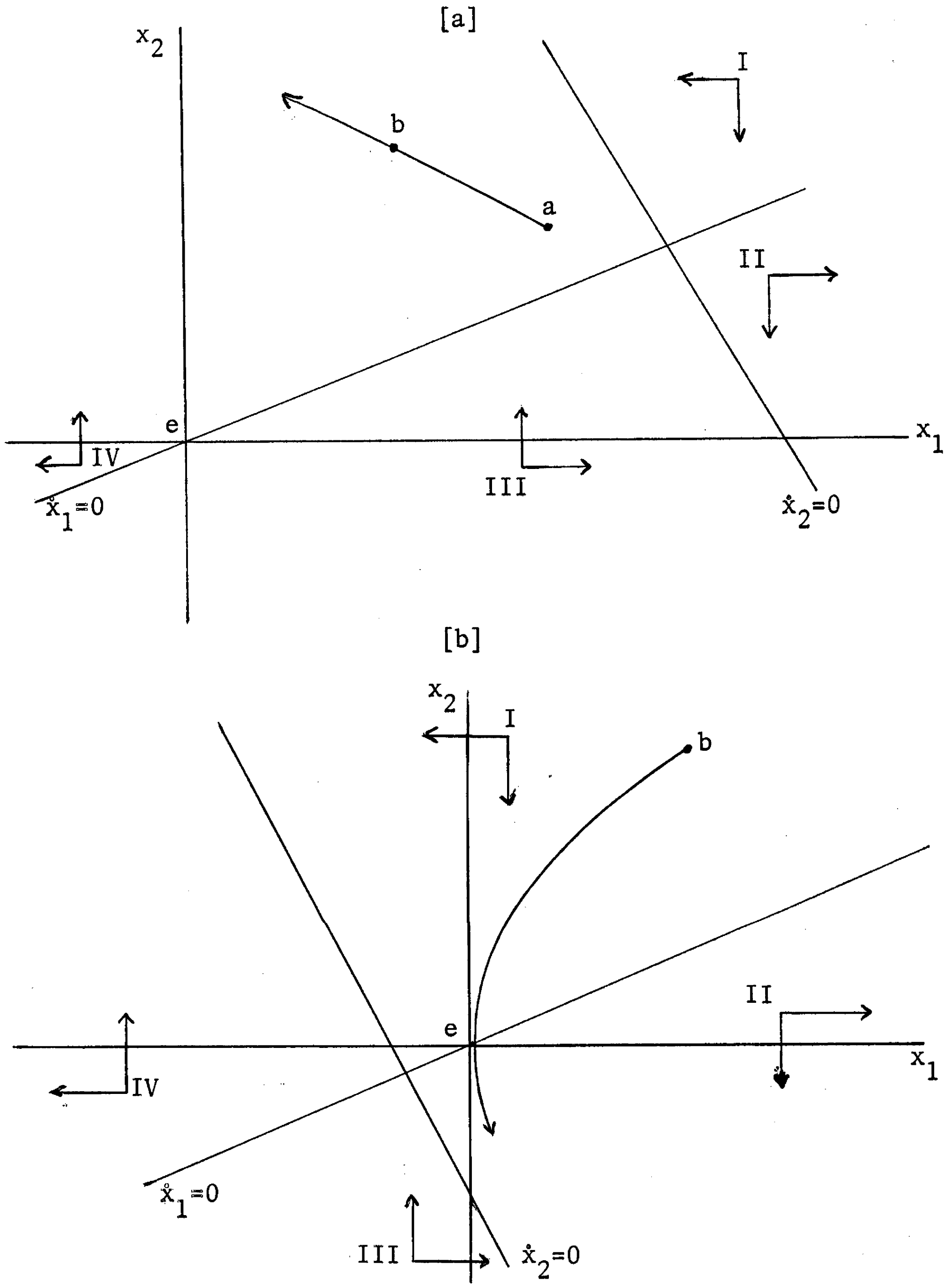

Figure 3.3

Construction of a Controlled Path

time t=T. This analysis is intended in a qualitative sense only. For $b_2 > 0$, the implied controller is the piecewise constant function of Figure 3.4:

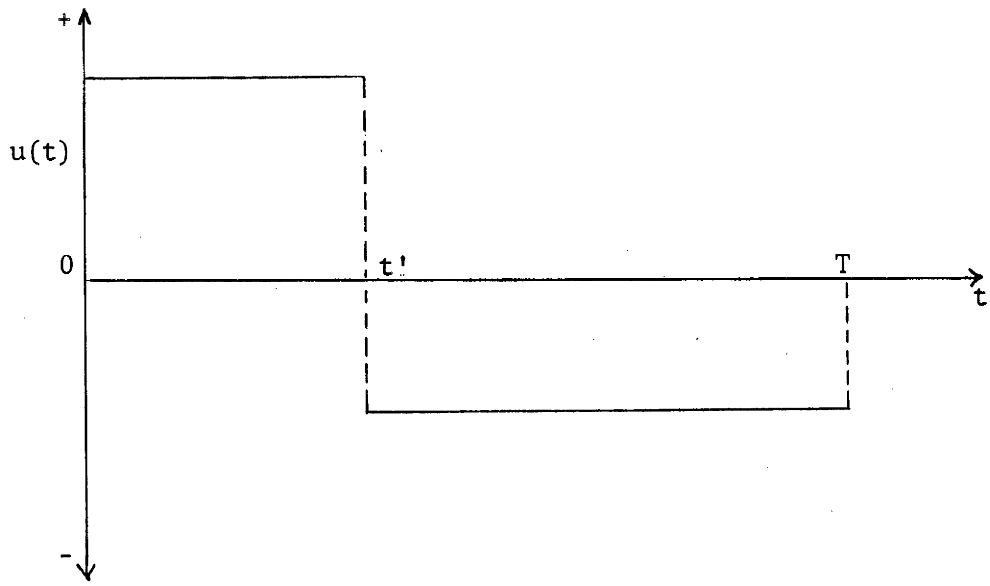

Figure 3.4
Piecewise Constant Controller

The illustrated control magnitudes and the timing of the policy switch at t' are merely assumed to determine the trajectory (abe) of Figure 3.2, but some such controller can be shown to exist. For expositional simplicity, a piecewise constant controller is chosen instead of a smoothly continuous controller, without prejudging the design question.

Controllability, as a system property, permits the policy-maker to modify the system dynamics so as to lure the resultant trajectory to the equilibrium point - irrespective of whether the system is naturally stable or naturally unstable. Yet if the controlled trajectory arrives at the equilibrium point (e) in finite time, it does so by directly hitting rather than asymptotically approaching (e). Once control lapses at the horizon $t=T$, this creates the problem of holding the system at or near equilibrium for $t > T$. With natural stability, this problem is unimportant; but with natural instability, further policy action is imperative to avoid subsequent divergence.

Following this analysis of controllability, consider now the significance of noncontrollability. Provided the single instrument is available ($b_2 \neq 0$), the system (82) is noncontrollable iff $a_{12}=0$, so that the dynamic variable x_1 is severed both directly ($b_1=0$) and indirectly ($a_{12}=0$) from control. As a result, the structural matrix A is lower triangular with eigenvalues

$$\lambda_1 = a_{11}, \qquad \lambda_2 = a_{22}, \tag{85}$$

such that

$$Ab = \lambda_2 b, \quad \forall\, b_2 \neq 0. \tag{86}$$

Therefore, in this rectangular policy system, noncontrollability implies, by (86), violation of the eigenvector condition (65). The control coefficient vector b is an eigenvector of the structural matrix A corresponding to the eigenvalue $\lambda_2 = a_{22}$.

Imposing, for example, the stable sign pattern

$$A: \begin{bmatrix} - & 0 \\ - & - \end{bmatrix}, \tag{87}$$

the significance of this violation may be analysed in terms of Figure 3.5. No matter how the instrument is varied, shifting the $\dot{x}_2 = 0$

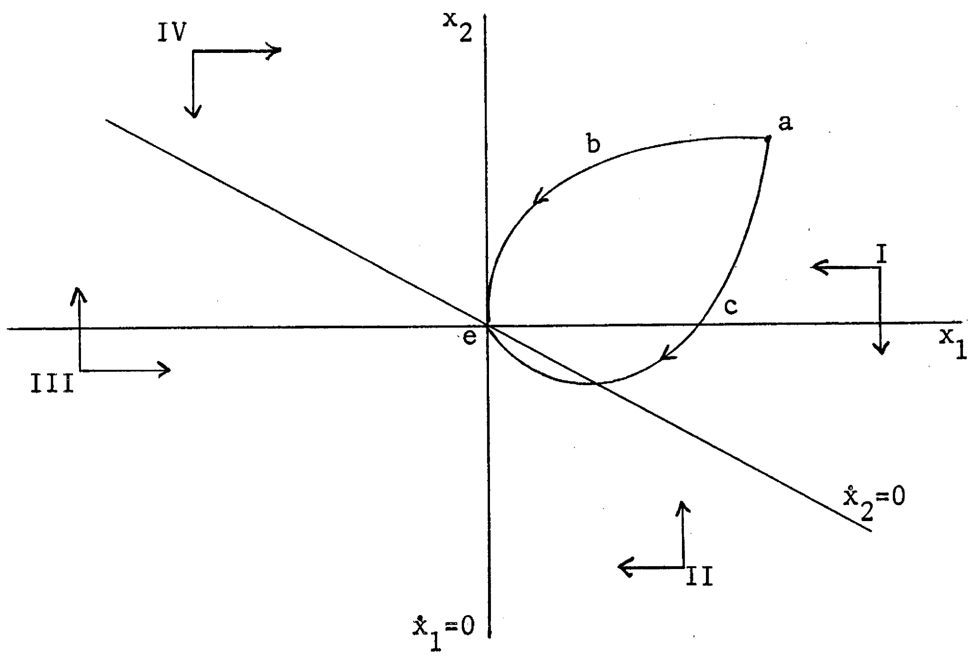

Figure 3.5
A Stable Noncontrollable System

locus up or down cannot force the system trajectory - whether (abe) or (ace) - to actually cross the vertical axis, the locus $\dot{x}_1 = 0$.

Depending on initial conditions, system response is confined to the right-hand or left-hand vertical planes of the state space. Hence no trajectory can be made to hit the equilibrium point (e) in finite time. Although speeds of adjustment may allow the system to be within a small prescribed distance of equilibrium in finite time, equilibrium is only attainable in infinite time. But because dissatisfaction with adjustment speeds is a primary rationale for active stabilisation, absence of controllability in naturally stable systems leaves no room for complacency.

These simple geometrical examples illustrate some of the basic aspects of the controllability concept. Controllability is not a function of natural stability (which is determined purely by the structural matrix), but is a function of the target-instrument structure (determined by the control matrix as well as the structural matrix). Natural stability properties are, however, relevant at the cessation of control. Noncontrollability represents the inability of the instrument to force the pair of targets to behave arbitrarily, the effect of the instrument being restricted by violation of the eigenvector condition.

3.6 CONCLUSIONS AND QUALIFICATIONS

To summarise, this chapter begins by defining the notion of an equilibrium partition in the total stabilisation problem. This partition, derived from global or constrained optima of a static preference function, separates the twin problems of static and dynamic stabilisation. It is argued that an asymmetry exists between the theoretical analyses of each problem: dynamically there is no criterion analogous to the static Tinbergen criterion. And such criteria are essential to validate the equilibrium partition. Given Culbertson's conjecture, which is sceptical about the dynamic applicability of Tinbergen's analysis, this asymmetry is even more provocative. The control concept of dynamic controllability is therefore shown to remove this asymmetry, validating the spirit and the letter of Tinbergen's approach and refuting Culbertson's conjecture.

In square policy systems, Tinbergen's static controllability theorem retains its full force; if a square system is statically controllable then it is also dynamically controllable, and conversely.

This result is independent of the assumption of distinct eigenvalues for the structural matrix, and is therefore completely general. That both static and dynamic controllability of square systems depend on the rank of the control matrix is an apt refutation of the Culbertson conjecture.

Results for rectangular policy systems are neither as general nor as immediate in interpretation. Because of the distinctness assumption on the eigenvalues of the structural matrix, the eigenvector condition (67) is not necessarily valid if multiple eigenvalues occur. In practice, if not in theory, Bellman's approximation theorem [p.199] disposes of this problem. Geometrical interpretation of the eigenvector condition has been provided for a scalar policy system in which the instrument is an eigenvector of the structural matrix. That dynamic controllability is the failure of instruments to affect all dynamic targets is obvious in this simple example.

Upon dissection, the rank criterion therefore yields sufficient conditions for dynamic controllability of the two classes of stabilisation model identified. If the stabilisation model is square, the rank criterion is equivalent to Tinbergen's static criterion; if the stabilisation model is rectangular, the rank criterion is equivalent to the eigenvector condition. Progression from the statics to the dynamics of stabilisation not only introduces a new dynamic condition - the eigenvector condition - but also retains the old static condition - the Tinbergen theorem.

Historically, lack of concern for dynamic existence is explained by use of the scalar policy model for investigations of dynamic stabilisation. That a single target with n^{th} order dynamics is controllable with a single instrument is so intuitively obvious as to suppress explicit analysis of existence. Generalising the stabilisation problem to a multi-target, multi-instrument framework creates such a need. With respect to the scalar policy model, Kalman, Ho & Narendra [p.202] demonstrate that these models are invariably dynamically controllable. Thus, from (34), the controllability matrix Q is

$$Q = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & & -a_o \\ 0 & & & -a_1+a_o^2 \\ 1 & -a_o & -a_1+a_o^2 & \vdots \end{bmatrix}, \qquad (88)$$

which is nonsingular $\forall\ a_j$. The scalar policy model is therefore well-behaved, precluding the possibility of noncontrollability and verifying the intuitive treatment of existence.

If an economic system is dynamically controllable, then, because the rank criterion is independent of the stabilisation horizon, it is possible, as indicated by Kalman [1959, p.108], to transfer the system between arbitrarily specified states as quickly as desired. But the shorter the stabilisation horizon, the greater is the necessary expenditure of stabilisation resources. This necessitates, in practice, the recognition of a lower value to the stabilisation horizon below which economically excessive resource flows are required from the public sector. Such constraints on the instruments clearly reduce the degrees of freedom associated with dynamic controllability. Alternatively, given a fixed stabilisation horizon and known control constraints, an equivalent problem is to define the region of controllability - for which see Athans & Falb [pp.197-200] and Lee & Markus [pp.68-80]. This second type of formulation is relevant, for instance, to a study of the efficacy of interest rate policy.

Chapter IV extends this analysis of controllability in two directions. Firstly, the assumption of distinct eigenvalues is relaxed; and secondly, the concept of reduced stabilisation is investigated.

CHAPTER IV

CONTROLLABILITY CRITERIA FOR STABILISATION POLICY

This chapter continues the development of chapter III. Concern is for the underlying structure of the stabilisation problem rather than for methods of solution. Existence, not design, is the theme. Because the concept of dynamic controllability is new to the theory of policy, this chapter is directed not only to an economic interpretation of the concept but also to an accessible exposition of the supporting theory.

Chapter III analysed the rank criterion for controllability on the assumption of distinct eigenvalues for the structural matrix. Relaxation of this distinctness assumption facilitates theoretical analysis of two topics left open by that chapter. The first topic refers to interpretation of the rank criterion in rectangular policy systems. Necessary and sufficient conditions are stated in section 3.4 for satisfaction of this criterion in scalar policy systems; while a sufficient condition is stated for rectangular systems. Necessity is still to be investigated when more than one instrument is used simultaneously. The second topic refers to the asserted existence, in section 3.2, of reduced models of stabilisation. Conditions under which reduction is available are still to be stipulated.

Both topics are investigated in this chapter after relaxing the distinctness assumption. Allowance for multiple eigenvalues permits a broader statement of results, and compels a deeper understanding of the controllability concept. Dissection of the rank criterion as it applies to rectangular policy systems is undertaken in two steps, separated by an analysis of reduced stabilisation. Section 4.1 investigates the effect of multiple eigenvalues on the rank criterion, and then justifies an equivalent coupling criterion for scalar policy systems. Section 4.2 pauses to apply this coupling criterion to the derivation of necessary and sufficient conditions for reduced stabilisation. Section 4.3 returns to the analysis of dynamic controllability in rectangular policy systems. A coupling criterion is presented that unifies interpretation of the rank criterion in both square and rectangular policy systems. Finally, section 4.4 summarises the conclusions of the chapter.

4.1 THE STRUCTURAL MATRIX UNRESTRICTED

Once multiple eigenvalues are admitted, the rank criterion for dynamic controllability is a function of the degree of the minimal polynomial rather than of the characteristic polynomial of the structural matrix. To define the significance of the minimal polynomial, the Jordan canonical form of A is presented and related to the minimal polynomial. Dynamic controllability is then shown to be affected by the Jordan chains of A - a concept illustrated with a simple example. Following these preliminaries, controllability is considered for scalar policy models in which there are no restrictions on the structural matrix. A rationale for a general result due to Kalman, Ho & Narendra is given, paving the way for the analysis of reduced stabilisation in section 4.2 and of general controllability in section 4.3.

4.1.1 Similarity transforms of A

Relaxing the assumption of distinct eigenvalues in the statement and proof of theorem 3.4, the general controllability criterion becomes

$$\rho(Q) = \rho[B \quad AB \quad \dots \quad A^{p-1}B] = n, \qquad (nxpk), \tag{1}$$

where p is the degree of the minimal polynomial of A. Following Ogata [pp.385-6], p is introduced into the rank criterion (1) by noting that equation (2) of Appendix III (p. 261 below) may be written

$$e^{-A\tau} = \sum_{i=0}^{p-1} \alpha_i(\tau)A^i, \tag{2}$$

and by then tracing this change through the Appendix. To investigate controllability when multiple eigenvalues occur, it is therefore necessary to assess the role of the minimal polynomial.

The distinctness assumption used in chapter III is a sufficient condition for the existence of n linearly independent eigenvectors for A; existence of these eigenvectors, in turn, is necessary and sufficient for existence of a similarity transformation of A to the diagonal eigenvalue matrix Λ:

$$P^{-1}AP = \Lambda, \qquad \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n), \qquad \lambda \varepsilon E(A), \tag{3}$$

where P is a nonsingular eigenvector matrix. Although multiple eigenvalues generally prevent A from being similar to a diagonal matrix Λ, A is always similar to a matrix Λ in Jordan canonical form. Following Noble, the possibilities are represented in Figure 4.1.

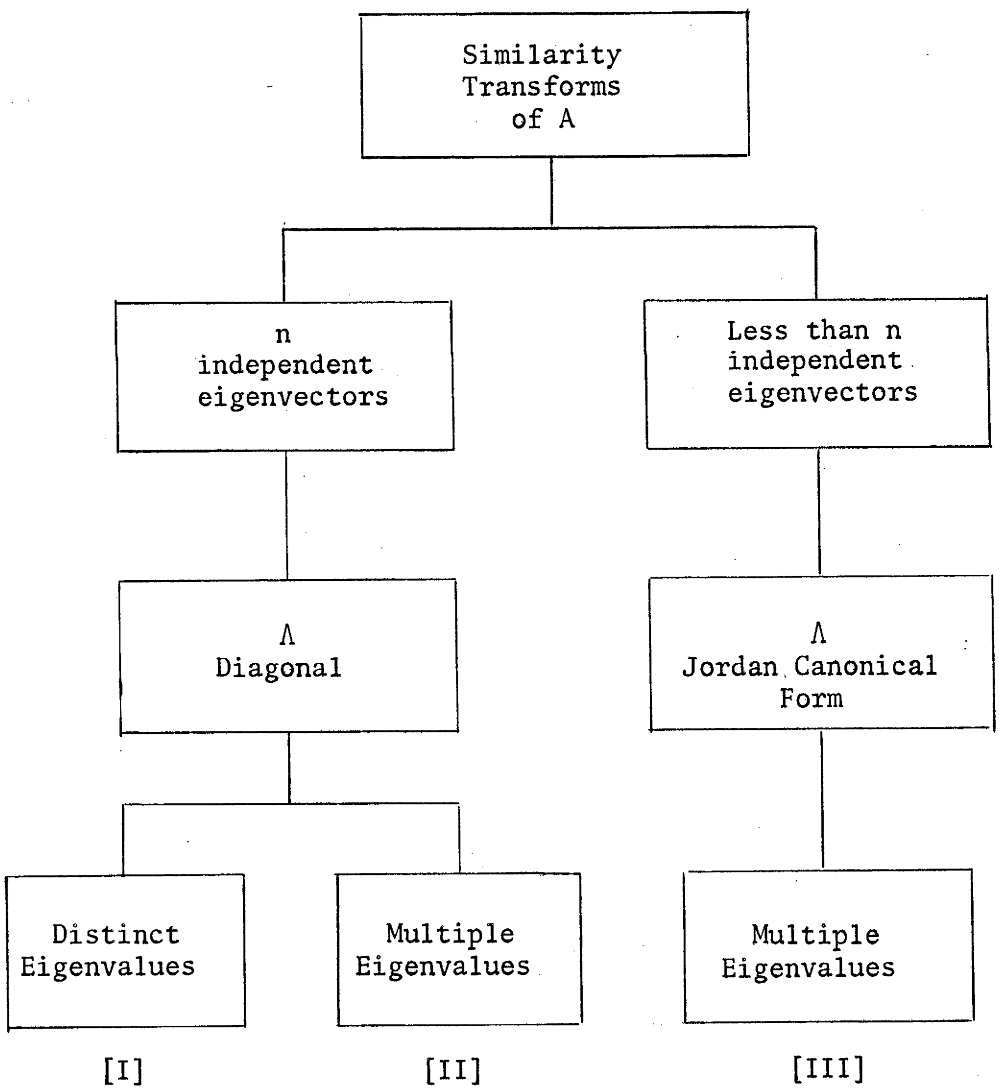

Figure 4.1

Similarity Transforms of Structural Matrix

Thus the matrix Λ is either in diagonal form or in Jordan canonical form [pp.345-65]. The distinctness assumption therefore induces the transformation (I). If no such restriction is placed on the structural matrix, then the transformations (II) and (III), corresponding to multiple eigenvalues, must be considered.

4.1.2 Jordan canonical form of A

Are systems represented by these two transformations controllable? This question may be answered after first considering the Jordan canonical form associated with (III). The following theorem, which consolidates the analysis given by Noble [chap.11], characterises this canonical form:

Theorem 4.1 (Jordan Canonical Form)

Given a general square matrix A of order n, there exists a nonsingular generalised eigenvector matrix P such that

$$P^{-1}AP = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_r \end{bmatrix} = \Lambda, \tag{4}$$

where

(i) the matrices J_i, $i=1,\ldots,r$ are Jordan blocks defined by

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda_i \end{bmatrix}, \quad \dim(J_i) \geqslant 1; \tag{5}$$

(ii) the eigenvalue λ_i may occur in different Jordan blocks;

(iii) the number r_i of such blocks corresponding to λ_i is equal to the number of independent eigenvectors associated with λ_i;

(iv) the order p_i of the largest Jordan block with a specific λ_i on its diagonal defines λ_i as an eigenvalue of index p_i;

(v) the eigenvalue λ_i occurs on the diagonal of Λ in (4) a number of times equal to its multiplicity n_i; and

(vi) simple eigenvalues λ_i occur in one and only one Jordan block, of order 1, with $n_i = p_i = 1$.

...

The relationship between the Jordan canonical form (4) and the minimal polynomial of the structural matrix A is given by the following theorem[1]:

<u>Theorem 4.2</u>

Given the nxn structural matrix A with s distinct eigenvalues, $\lambda_1, \ldots, \lambda_s$, $s \leqslant n$. Then the characteristic polynomial of A is

$$\chi(z) = (z-\lambda_1)^{n_1} \ldots (z-\lambda_s)^{n_s}, \tag{6}$$

where n_i is the multiplicity of λ_i; and the minimal polynomial

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1 Cf., e.g., Noble [pp.370-2].

of A is

$$\psi(z) = (z-\lambda_1)^{p_1} \ \ldots \ (z-\lambda_s)^{p_s}, \tag{7}$$

where p_i is the index of λ_i. The degree of $\chi(z)$ is

$$n = \sum_{i=1}^{s} n_i, \tag{8}$$

and of $\psi(z)$ is

$$p = \sum_{i=1}^{s} p_i, \tag{9}$$

where

$$n_i \geqslant p_i \quad i = 1,\ldots,s, \quad s \leqslant n. \tag{10}$$

...

Thus the degree p of the minimal polynomial is the sum of the eigenvalue indices p_i; or, by (iv) of theorem 4.1, the sum of the orders p_i of the largest Jordan block J_i associated with each eigenvalue λ_i.

4.1.3 Jordan chains and controllability

Dynamic controllability is linked to the minimal polynomial of A through the concept of a Jordan chain[2]. A multiple eigenvalue λ_i, of multiplicity n_i, and index p_i, occurs in a Jordan chain iff it appears in one and only one Jordan block on the diagonal of Λ in (4); the unit superdiagonal of this Jordan block (5) constituting the chain. By (iv) and (v) of theorem 4.1, λ_i occurs in such a Jordan chain iff

$$n_i = p_i. \tag{11}$$

To illustrate the relevance of this concept, the scalar policy system $\dot{x} = Ax+bu$ is, using k=1 in (1), controllable iff

$$\rho(Q_s) = [b \quad Ab \ \ldots, \ A^{p-1}b] = n, \qquad (nxp). \tag{12}$$

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2 Cf., e.g., Ogata [pp.249-50].

By theorem 3.1, a necessary condition for (12) is n=p, or equality of the degrees of the characteristic and minimal polynomials. From (8), (9), n=p iff

$$\sum_{i=1}^{s} n_i = \sum_{i=1}^{s} p_i \quad \Leftrightarrow \quad n_i = p_i \;\forall\, i = 1,\ldots,s, \tag{13}$$

by (10). If scalar policy systems with multiple eigenvalues are to be dynamically controllable, (11) and (13) imply the necessary condition that every distinct eigenvalue λ_i, i=1,...,s, s $\leq$ n, is linked in a Jordan chain or occurs in one and only one Jordan block.

Interpretation of the Jordan chain condition (13) will therefore aid understanding of the effect of relaxing the distinctness assumption. Consider a second-order scalar system,

$$\begin{bmatrix} \dot z_1(t) \\ \dot z_2(t) \end{bmatrix} = A \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(t) \tag{14}$$

assumed to possess a 2-fold eigenvalue and to be already in Jordan canonical form. Then A can assume one of two forms:

$$A_1 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \qquad A_2 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}. \tag{15}$$

If $A \equiv A_1$, the Jordan chain associated with λ is unbroken; if $A \equiv A_2$, the chain is broken. Solutions of (14) for the second case are given by

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = e^{\lambda t} \begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \bar u(t), \tag{16}$$

where

$$\bar u(t) = \int_0^t e^{(t-\tau)} u(\tau)\, d\tau. \tag{17}$$

Imposing the desired terminal solution $z(T)=0$ in (16) therefore requires that

$$\begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} = -e^{-\lambda T} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \bar{u}(T). \tag{18}$$

Thus with a broken Jordan chain, the scalar policy system (14) is dynamically controllable iff the initial condition vector $z(0)$ and the control coefficient vector b are collinear - a restrictive condition only met accidentally. If, however, the Jordan chain is unbroken so that $A \equiv A_1$, then $\rho(Q) = \rho[b \mid A_1 b] = 2$, and the system is dynamically controllable.

4.1.4 General scalar controllability

Using the foregoing analysis, a coupling criterion equivalent to the rank criterion (1) for k=1 is presented. For the scalar policy system $\dot{x} = Ax + bu$, consider, following (3.45), the explicit solution

$$x(0) = -\int_0^T e^{-At} bu(t)dt, \tag{19}$$

for the endpoint $x(T)=0$. For an unrestricted structural matrix A, the transition matrix is, from (4),

$$e^{At} = Pe^{\Lambda t}P^{-1}, \tag{20}$$

where Λ is the Jordan canonical form of A. To obtain a simple evaluation of (20), suppose that all eigenvalues of A are linked in Jordan chains. Then the matrix exponential (20), using Ogata [pp.152-5,308-9], may be evaluated as

$$e^{At} = P\{S(t)\Psi\}P^{-1}, \tag{21}$$

where

$$\left\{\begin{aligned} \Psi &= \begin{bmatrix} e^{\lambda_1 t} I_{n_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_s t} I_{n_s} \end{bmatrix}, \qquad S(t) = \begin{bmatrix} S_{n_1}(t) & & 0 \\ & \ddots & \\ 0 & & S_{n_s}(t) \end{bmatrix} \\ S_{n_i}(t) &= \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n_i-1}}{(n_i-1)!} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2!} \\ & & & \ddots & t \\ 0 & & & & 1 \end{bmatrix}, \quad i = 1,\ldots,s, \quad s \leqslant n. \end{aligned}\right. \tag{22}$$

Hence, from (21) and (22),

$$e^{-At} = [e^{At}]^{-1} = P\Psi^{-1}S(-t)P^{-1}, \tag{23}$$

so that (19) is

$$P^{-1}x(0) = -\int_0^T f(t)u(t)dt, \tag{24}$$

where

$$f(t) \equiv \Psi^{-1}S(-t)P^{-1}b, \qquad (n\mathrm{x}1). \tag{25}$$

Postulating the smooth controller

$$u(t) = f^T(t).\gamma, \qquad (1\mathrm{x}n.n\mathrm{x}1), \tag{26}$$

(24) is written

$$P^{-1}x(0) = G(T).\gamma, \tag{27}$$

where G(T) is the nxn matrix

$$G(T) = -\int_0^T f(t)f^T(t)dt. \tag{28}$$

Now there exists a unique solution for γ in (27), and the system is controllable, iff G(T) in (28) is nonsingular. By a result given by Zadeh & Desoer [p.497], a matrix of the form (28) is nonsingular iff the n elements of f(t) in (25) are linearly independent functions of time over the interval [0, T]. For $\hat{b} = P^{-1}b$, and using (22), f(t) may be written

$$f(t) = \begin{bmatrix} e^{-\lambda_1 t} S_{n_1}^{-1} & & 0 \\ & \ddots & \\ 0 & & e^{-\lambda_s t} S_{n_s}^{-1} \end{bmatrix} \begin{bmatrix} \hat{b}_{1.1} \\ \vdots \\ \hat{b}_{1.n_1} \\ \vdots \\ \hat{b}_{s.1} \\ \vdots \\ \hat{b}_{s.n_s} \end{bmatrix} = \begin{bmatrix} e^{-\lambda_1 t} S_{n_1}^{-1} \begin{bmatrix} \hat{b}_{1.1} \\ \vdots \\ \hat{b}_{1.n_1} \end{bmatrix} \\ \vdots \\ e^{-\lambda_s t} S_{n_s}^{-1} \begin{bmatrix} \hat{b}_{s.1} \\ \vdots \\ \hat{b}_{s.n_s} \end{bmatrix} \end{bmatrix}, \tag{29}$$

where $\hat{b}$ is partitioned compatibly with $\Psi^{-1}S^{-1}(t)$. Therefore the elements of f(t) are linearly independent iff the elements of $\exp\{-\lambda_j t\} S_{n_j}^{-1}\hat{b}_j$ are linearly independent $\forall$ j = 1,...,s. Using (22) again,

$$f_{n_j}(t) = e^{-\lambda_j t} S_{n_j}^{-1}\hat{b}_j = e^{-\lambda_j t} \begin{bmatrix} 1 & -t & \frac{t^2}{2!} & \cdots & \frac{(-t)^{n_j-1}}{(n_j-1)!} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2!} \\ & & & \ddots & -t \\ 0 & & & & 1 \end{bmatrix} \begin{bmatrix} \hat{b}_{j.1} \\ \cdot \\ \cdot \\ \cdot \\ \hat{b}_{j.n_j} \end{bmatrix}. \tag{30}$$

Evaluating (30) readily shows that the elements of $f_{n_j}(t)$ are linearly independent functions of t iff

$$\hat{b}_{j.n_j} \neq 0. \tag{31}$$

Hence if (31) is satisfied $\forall\ j = 1,\ldots,s$, then G(T) in (28) is nonsingular and the scalar system (19) is dynamically controllable.

Condition (31) and the assumption, prior to (21), that all eigenvalues are linked in Jordan chains illustrate the basis of the following theorem due to Kalman, Ho & Narendra [p.204]:

Theorem 4.3 (Kalman, Ho & Narendra)

The scalar policy system $\dot{x} = Ax + bu$ is dynamically controllable iff

$$\begin{aligned} &\text{(a)}\quad n = p \iff n_j = p_j, \\ & \qquad\qquad\qquad\qquad\qquad \forall\ j = 1,\ldots,s,\quad s \leqslant n. \\ &\text{(b)}\quad \hat{b}_{j.n_j} \neq 0. \end{aligned} \tag{32}$$

...

Condition (32a) refers to the eigenvalues of the structural matrix; condition (32b), to the eigenvectors. The eigenvalue condition is the Jordan chain condition (13). Since $n_i = p_i = 1\ \forall\ i = 1,\ldots,n$ for distinct eigenvalues, the distinctness assumption of chapter III automatically ensures satisfaction of (32a). The eigenvector condition requires that each element of $\hat{b} = P^{-1}b$ corresponding to the last row of each of the (s) Jordan blocks of A must be nonzero. This is the eigenvector condition of section 3.4, except that it applies to only $s < n$ elements when multiple eigenvalues occur.

Thus, with respect to Figure 4.1, only the transformations (I) and (III) permit dynamic controllability in scalar policy systems. The class (II) invariably violate the Jordan chain condition, with at least one eigenvalue occurring in more than one Jordan block. For scalar systems belonging to class (I), only those satisfying the eigenvector condition (32b) for s=n are controllable; and for those belonging to class (III), only those satisfying both the conditions (32) are controllable.

Although the effect of relaxing the distinctness assumption is so far considered just for scalar policy systems, theorem 4.3 is adequate for investigating the reduced stabilisation concept. Section

4.2 therefore undertakes this investigation, deferring consideration of nonscalar policy systems to section 4.3.

4.2 CRITERIA FOR REDUCED STABILISATION

Section 3.2 presented an ordering

$$n \geqslant N \geqslant K \geqslant k \tag{3.36}$$

on the numbers of static and dynamic targets and instruments, asserting that the inequality $K > k$ arises because of deliberate reduction in the usage of instruments dynamically. This section examines such a reduction in the instrument vector: in particular, conditions for which reduction is feasible are obtained, and the economic significance of the concept is elaborated.

4.2.1 Regular and reduced stabilisation

The ranking $K \geqslant k$ on the dimensions of the static and dynamic instrument vectors defines two categories of dynamic stabilisation: regular ($k=K$) and reduced ($k < K$). With regular stabilisation, the dynamic policy mix contains every instrument available statically; with reduced stabilisation, fewer instruments are used dynamically than are statically available. Reduced stabilisation is a concept novel to the theory of economic policy. Previous work[3] on dynamic stabilisation policy has focussed entirely on the problems of regular stabilisation. Again, this is principally a consequence of the scalar policy model: with a single instrument, the possibility of reduction is clearly absent. The notion of reducing the number of dynamic instruments is itself contrary to the thinking engendered by the static Tinbergen rule, which emphasises the need for more rather than fewer instruments. If the concept is of value, it is therefore necessary (i) to present conditions under which it is possible to design a reduced policy mix and (ii) to demonstrate the utility of the concept as a policy option. These two requirements are now considered.

Reduction, as defined, is a qualitative, not a quantitative

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[3] For example, Phillips [1954, 1957], Allen [1960, 1968], FST, and Bergstrom.

concept. There are k instruments available and it is desired to select only r of these; the restriction $r < k$ implying freedom within the range $1 \leqslant r < k$. To unify analysis of these possibilities, a device used by Kalman, Ho & Narendra [p.204] and Lee & Markus [p.86] is employed. Consider the regular stabilisation model

$$\dot{x} = Ax + Bu, \qquad u(t) \subset R^k, \tag{33}$$

and assume that it is dynamically controllable. Define the kx1 column vector (c) such that

$$\dot{x} = Ax + (Bc)\mu(t), \quad \mu(t) \subset R^1, \quad u(t) = c\mu(t). \tag{34}$$

Then the system (33) is scalar controllable in terms of (34) iff the two conditions of theorem 4.3 are satisfied. Hence the Jordan chain condition n=p must be satisfied; and certain elements of the column vector $P^{-1}Bc$ must be nonzero.

A system (33) satisfying theorem 4.3 in terms of (34) will be said to be reducibly controllable. Depending on the constant vector, c, two types of reduced controllability can be identified. If c is one of the column vectors ε_j of the kxk unit matrix I_k, then $\mu(t) = u_j(t)$ and $Bc = b_j$, $j \in 1,\ldots,k$. In this case, the process of reduction is to ask whether one of the original k instruments is sufficient for dynamic stabilisation. Such systems will be defined to be reducibly scalar controllable (RSC).

If, however, $c \neq \alpha\varepsilon_j$, $\forall\ j = 1,\ldots,k$, α constant, there is necessarily more than one nonzero element in c. Given that $u(t) = c\mu(t)$, then $u_j(t) = c_j\mu(t)$ for at least two values of j, $j \in 1,\ldots,k$. In this case, more than one of the original instruments are to be used for dynamic stabilisation, each instrument being determined as a constant function of the scalar controller $\mu(t)$. Such systems will be defined to be reducibly multiple controllable (RMC).

Reduced systems are therefore to be distinguished by the number of original instruments ultimately employed in implementing the scalar controller $\mu(t)$. For RSC systems, only one instrument is required; for RMC systems, more than one and possibly all of the original instruments are necessary. But for either class of system, the design problem itself is simplified: it is only necessary to design a scalar controller $\mu(t)$, the dimension of the control space being reduced from k to unity.

These two types of reduced system do not exhaust all the possibilities for designing reduced systems. Thus if a regular system is reducibly scalar controllable with respect to each of its k instruments, then there exist r' additional reduced systems,

$$r' = \sum_{j=2}^{k-1} \binom{k}{j}, \tag{35}$$

obtained from combinations of these individual instruments. But RSC and RMC systems are of basic importance in characterising the options of maximum reduction.

4.2.2 Examples of reduced controllability

Concepts of scalar and multiple controllability may be illustrated in the context of a square policy model containing two targets and two instruments. Consider the scalar policy model

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} \mu, \tag{36}$$

in which

$$\hat{\beta} = P^{-1}Bc, \qquad B: 2x2, \quad c: 2x1. \tag{37}$$

By theorem 4.3, this scalar system is controllable iff

$$\hat{\beta}_2 \neq 0, \tag{38}$$

since the Jordan chain condition n=p=2 is satisfied by the structural matrix of (36).

The role of this eigenvector condition differentiates the two classes of reduced system identified above. Provided c is not a unit vector, its elements can always be chosen so that (38) is satisfied. But if c is a unit vector, then $\hat{\beta}_2 \neq 0$ must be a structural property of the given system. Thus RMC systems exist if the Jordan chain condition is satisfied; RSC systems exist only if the eigenvector condition is satisfied as well. The more stringent option of using one rather than both of the available instruments necessitates satisfaction of more stringent requirements.

Assuming the stable sign pattern

$$A: \begin{bmatrix} - & + \\ 0 & - \end{bmatrix} \tag{39}$$

for the structural matrix of (36), the phase diagram associated with this system is given by Figure 4.2. The loci are shown for $\mu = 0$, so that (e) represents the desired equilibrium point. Given the assumed dynamics, and initial conditions corresponding to point (a), asymptotic behaviour will be described by a trajectory similar to (ae).

The significant aspect of Figure 4.2 is that the horizontal locus $\dot{z}_2 = 0$ separates the top half-plane from the bottom half-plane in the sense that no trajectory can cross that locus. Therefore, if the instrument $\mu(t)$ appears only in the first equation of (36) so that $\hat{\beta}_1 \neq 0$, $\hat{\beta}_2 = 0$, no parallel shifting of the locus $\dot{z}_1 = 0$, achieved through a policy $\mu(t)$, can induce the system to hit the equilibrium point (e) in finite time[4]. But if the instrument $\mu(t)$ appears in the second equation of (36) so that $\hat{\beta}_1 = 0$, $\hat{\beta}_2 \neq 0$, then vertical shifting of the locus $\dot{z}_2 = 0$ in response to control can induce the system to hit the equilibrium point (e). The composite path (abe) of Figure 4.3 supports this assertion. If Figure 4.3 describes an RSC system, the ability to construct the path (abe) is a structural characteristic of the system; if it describes an RMC system, this ability is then a discretionary option in the sense that the eigenvector condition $\hat{\beta}_2 \neq 0$ can always be satisfied by appropriate selection of the vector c.

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4 Cf. the analysis accompanying Figure 3.5 in section 3.5 above.

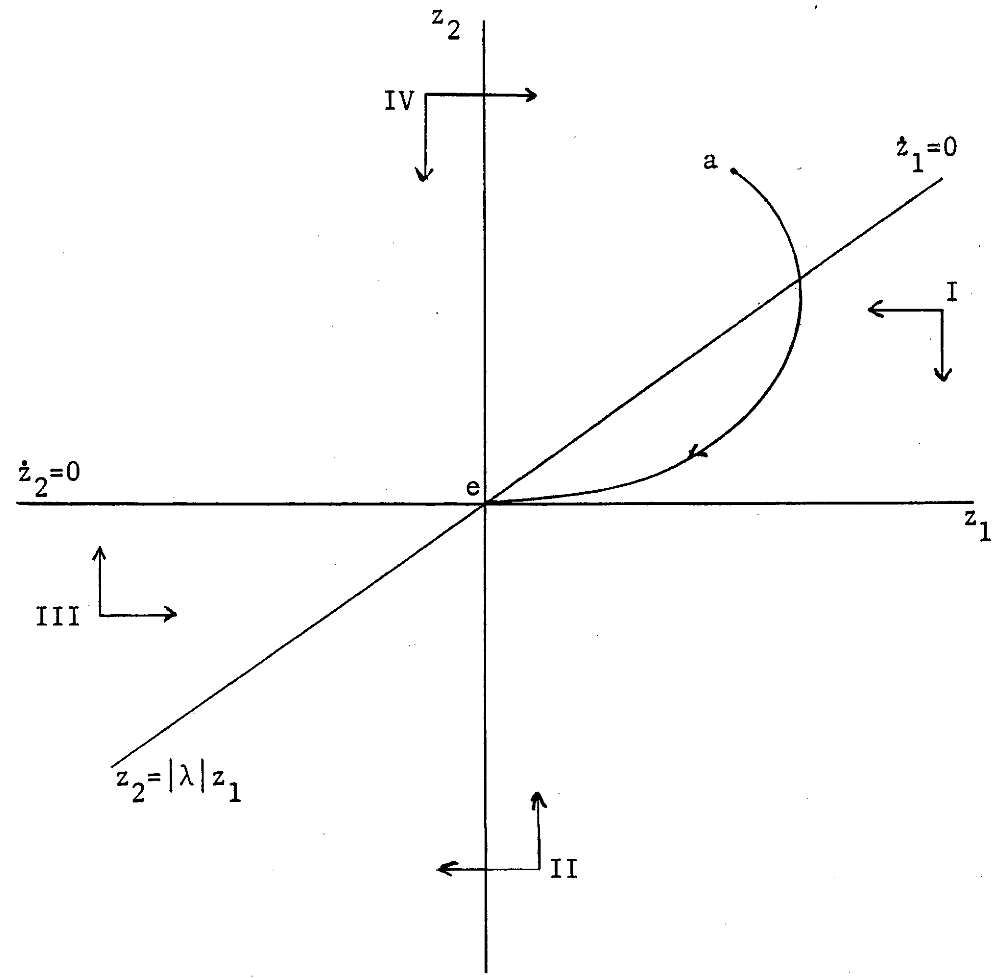

Figure 4.2

Controllability with a Double Eigenvalue

Now suppose that the Jordan chain condition is violated for the system (36), providing

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} \mu. \tag{40}$$

Assuming the stable sign pattern

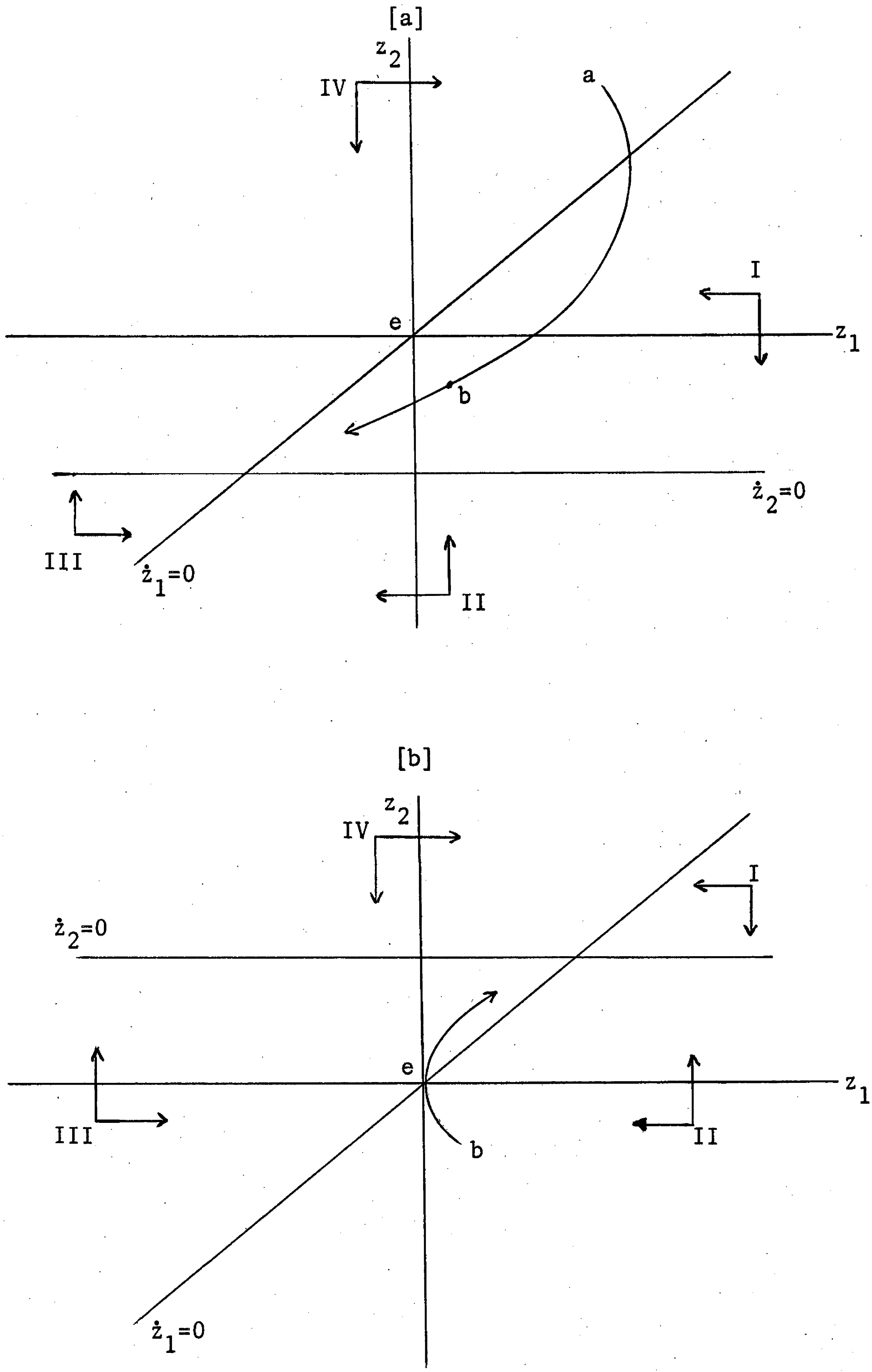

Figure 4.3

Construction of a Controlled Path

$$A: \begin{bmatrix} - & 0 \\ 0 & - \end{bmatrix} \tag{41}$$

the phase behaviour of (40) is given by Figure 4.4.

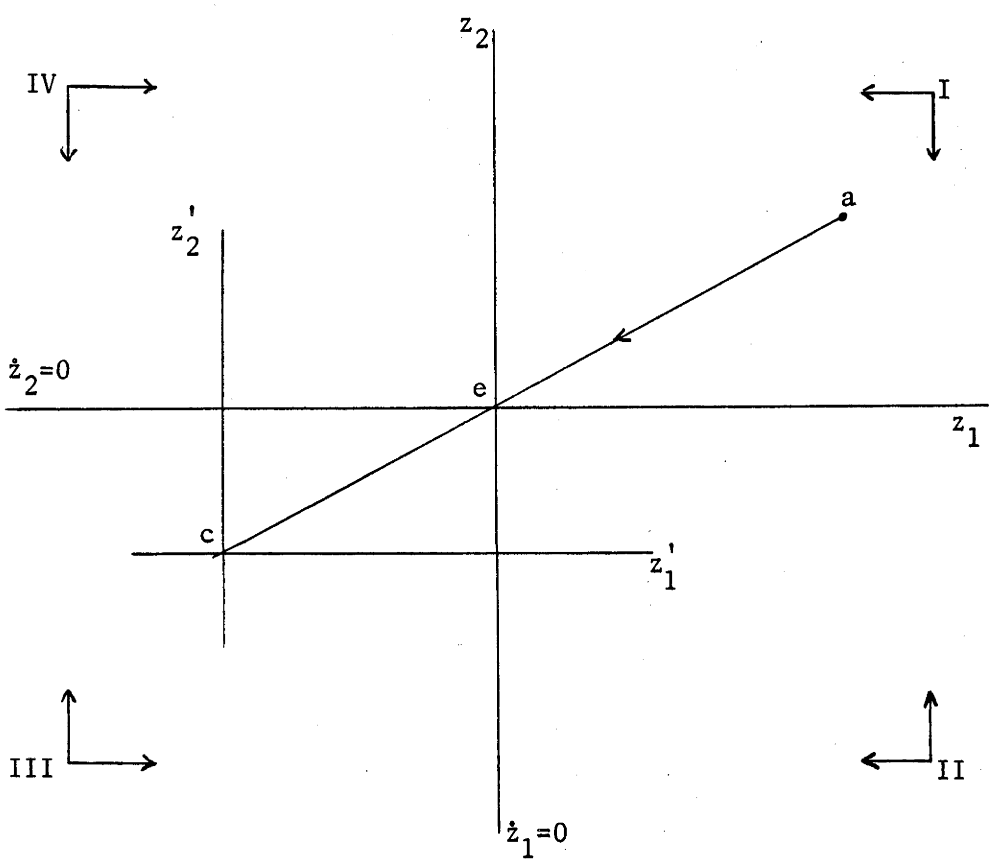

Figure 4.4
Noncontrollability with a Double Eigenvalue

For $\mu(t) = 0$, the asymptotic solution of (40), given (41), is the linear trajectory (ae), with gradient determined by the initial conditions (a). If either $\hat{\beta}_1 = 0$ or $\hat{\beta}_2 = 0$, so that the instrument is confined to just one of the state equations, the separation argument noted with respect to Figure 4.2 is also applicable to Figure 4.4. Because either possibility violates the eigenvector condition, the system is therefore noncontrollable. But suppose

$\hat{\beta}_1 \neq 0$, $\hat{\beta}_2 \neq 0$, satisfying this condition. Now although the Jordan chain condition remains violated, the type of reduction used in (40) affects the controllability or otherwise of the system. Thus if a controller $\mu(t) \neq 0$ can produce the new loci z_1', z_2', the system possesses a linear asymptotic trajectory (aec) allowing the desired equilibrium to be achieved in finite time. As equations (14) through (18) demonstrate, the trajectory (aec) requires collinearity of the initial condition vector $z(0)=a$ and the control coefficient vector $\hat{\beta}$. While collinearity is an arbitrary property of reduced scalar systems (for which the elements of $\hat{\beta}$ are fixed), it is a discretionary property of reduced multiple systems. Thus

$$\hat{\beta} = \alpha z(0),\ \alpha \text{ constant} \iff P^{-1}Bc = \alpha z(0), \tag{42}$$

and a unique choice of c always exists $\forall\, z(0)$ provided $\rho(B) = n = 2$.

In square policy systems, reduced multiple controllability is possible, even if the Jordan chain condition is broken, provided the multiple controller is constructed using $k=n$ instruments, with $\rho(B) = n$. In rectangular policy systems, the Jordan chain condition remains binding because a solution for c exists iff

$$\rho(B \,\vdots\, z(0)) = \rho(B) = k, \tag{43}$$

and this cannot be universally guaranteed. In square policy systems, reduced multiple controllability invariably exists while reduced scalar controllability need not. In rectangular systems, the Jordan chain condition is necessary and sufficient for reduced multiple controllability; but must be accompanied by the eigenvector condition to ensure reduced scalar controllability.

4.2.3 Economic significance of reduced stabilisation

Given that an economic model is regularly controllable, the primary importance of reduced stabilisation is captured by the following appeal: can the problem of designing and implementing a dynamic controller be simplified? Although the target and instrument dimensions n and k have no qualitative effect on policy design, the quantitative problems of computation, illustrated in chapter II, increase more than quadratically with these dimensions. Regular controllability defines whether an economic system can be dynamically stabilised at all; reduced controllability, whether the task can be simplifed.

Two types of reduction are defined: scalar reduction and multiple reduction. Scalar reduced controllers involve the use of only one of the original instruments. Multiple reduced controllers still involve the design of only one dynamic controller, but this is implemented by setting several, or all, of the original instruments proportional to it. In either case, the regular problem of designing k independent dynamic policies is reduced to the problem of designing one independent policy.

Assuming a reduced system exists and has been selected, the following balancing principle applies to the design of stabilisation policy: statically, it is necessary to specify K instruments each with feedbacks to N targets; whereas dynamically, it is necessary to specify only one instrument with feedbacks to n targets. This reduction in the dimension of the control space compensates for the design characteristic that, statically, only proportional feedback is employed; whereas dynamically, integral, proportional, derivative, and higher-order feedbacks are required. Loosely translated, the balancing principle is therefore: use a larger number of instruments in the simple static context, and a smaller number of instruments in the complex dynamic context.

As a cost of design simplification, there exists a concurrent choice problem. If there are several reduced systems, is there a unique choice of system? Thus suppose that there is a regular system with two instruments, generating two scalar reduced systems and one multiple reduced system. Which system will be preferred for dynamic policy-making? If the benefit of reduction is simplification of the policy design problem, it is probable that there is a resulting cost in terms of either deteriorated performance or increased resource expenditure, as measured against the regular system. This choice problem may be formalised by appropriate definition of all benefits and costs but there do not appear to be simple a priori economic rules for guiding such a choice.

Simultaneous existence of scalar and multiple reduction creates yet another choice problem: criteria for preferring one type of reduction to the other are not apparent. For example, it may be easier to meet implicit control magnitude constraints using multiple reduction rather than the more stringent scalar reduction, but this is only conjectural. If multiple reduction is used, it is then necessary to provide criteria for selection of the elements of the discretionary vector c.

Significant questions remain to be explored before the role of reduced stabilisation is fully understood. These, however, are not pursued further here; and section 4.3 now resumes the analysis of the rank criterion in nonscalar policy systems.

4.3 UNIFIED CONTROLLABILITY CRITERIA

Although the rank criterion (1) provides a fully general statement of necessary and sufficient conditions for dynamic controllability, it is, in that form, devoid of intuitive appeal. Both chapter III and this present chapter have attempted, therefore, to discern the basis for this criterion. After defining two general classes of policy system - square and rectangular - chapter III demonstrates that Tinbergen's theorem generalises dynamically to square policy systems; and then uses the distinctness assumption to isolate the role of the eigenvector condition in rectangular policy systems. Section 4.1 relaxes this distinctness assumption and provides a rationale for theorem 4.3. This theorem, applicable to scalar policy models, characterises the rank criterion in terms of the Jordan chain and eigenvector conditions.

Analysis of the rank criterion is still required for the general case when a regular controller using more than one instrument is to be designed. Thus if the nxpk controllability matrix Q of (1) is to possess full row rank n, a necessary condition is clearly

$$k \geqslant n/p. \tag{44}$$

Suppose that the Jordan chain condition is violated, with $p < n$. Then (44) implies

$$1 < k \leqslant n \qquad \text{for} \qquad n > p \geqslant 1. \tag{45}$$

With $p < n$, rectangular policy systems are not reducibly controllable: does there exist a set of $k > 1$ instruments for which these systems are regularly controllable? If so, under what conditions?

4.3.1 <u>A general coupling criterion</u>

Investigation of this problem leads to an alternative characterisation of the rank criterion valid for all pairings of the dimensions k and n. For the rectangular policy system $\dot{x} = Ax+Bu$, B: nxk, equation (19) becomes

$$x(0) = -\int_0^T e^{-At}Bu(t)dt. \tag{46}$$

Since the Jordan chain condition is not necessarily satisfied, the expression (23) for the transition matrix e^{-At} is not necessarily valid either. To allow for multiple Jordan blocks associated with a given eigenvalue, define the integer $(n_i.j)$ such that n_i, the multiplicity of λ_i, is given by

$$n_i = \sum_{j=1}^{r_i} (n_i.j), \qquad i = 1,\ldots,s, \qquad s \leqslant n, \tag{47}$$

where $(n_i.j)$ defines the dimension of the j^{th} Jordan block, $j = 1, \ldots, r_i$, associated with i^{th} eigenvalue λ_i, $i = 1,\ldots,s$; and where the integer r_i is the number of such Jordan blocks corresponding to any λ_i. Hence the matrix definitions (22) become, for $j = 1,\ldots,r_i$, $i = 1,\ldots,s$, $s \leqslant n$,

$$\left\{\begin{aligned}
\Psi &= \begin{bmatrix} e^{\lambda_1 t} I_{n_1 \cdot 1} & & & & & & 0 \\ & \ddots & & & & & \\ & & e^{\lambda_1 t} I_{n_1 \cdot r_1} & & & & \\ & & & \ddots & & & \\ & & & & e^{\lambda_s t} I_{n_s \cdot 1} & & \\ & & & & & \ddots & \\ 0 & & & & & & e^{\lambda_s t} I_{n_s \cdot r_s} \end{bmatrix}, \\
S(t) &= \begin{bmatrix} S_{n_1 \cdot 1}(t) & & & & & & 0 \\ & \ddots & & & & & \\ & & S_{n_1 \cdot r_1}(t) & & & & \\ & & & \ddots & & & \\ & & & & S_{n_s \cdot 1}(t) & & \\ & & & & & \ddots & \\ 0 & & & & & & S_{n_s \cdot r_s}(t) \end{bmatrix}, \\
S_{n_i \cdot j}(t) &= \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{(n_i \cdot j-1)}}{(n_i \cdot j-1)!} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2!} \\ & & & \ddots & t \\ 0 & & & & 1 \end{bmatrix}.
\end{aligned}\right. \tag{48}$$

Thus

$$e^{-At} = P\Psi^{-1}S(-t)P^{-1}, \tag{49}$$

and (46) may be written

$$P^{-1}x(0) = -\int_0^T \{\Psi^{-1}S(-t)\}\,\hat{B}u(t)dt, \qquad \hat{B} = P^{-1}B. \tag{50}$$

Partition $\hat{B}$ such that[5]

$$\hat{B} = \begin{bmatrix} \hat{B}_{n_1} \\ \hdashline \vdots \\ \hdashline \hat{B}_{n_s} \end{bmatrix}_{nxk}, \quad \hat{B}_{n_j} = \begin{bmatrix} \hat{B}_{n_j.1} \\ \hdashline \vdots \\ \hdashline \hat{B}_{n_j.r_j} \end{bmatrix}_{n_jxk}, \tag{51}$$

and construct the matrices

$$\bar{B}_j = \begin{bmatrix} \hat{B}(n_j.1) \\ \hat{B}(n_j.2) \\ \vdots \\ \hat{B}(n_j.r_j) \end{bmatrix}_{r_jxk}, \qquad j = 1,\dots,s, \quad s \leqslant n, \tag{52}$$

where $\hat{B}(n_j.i)$, $i = 1,\dots,r_j$, is the last row of each block of $\hat{B}_{n_j}$ in (51). Then Appendix IV (pp.264-7 below) establishes the following result:

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5 The subscript x in $\hat{B}_x$ denotes row dimension.

Theorem 4.4

The linear constant system $\dot{x} = Ax+Bu$, $u \subset R^k$, $1 \leq k \leq n$, $\rho(B) = k$, is dynamically controllable iff

$$\rho(Q) = \rho[B \quad AB \quad \ldots \quad A^{p-1}B] = n, \quad (nxpk), \tag{53}$$

or iff

$$\rho(\bar{B}_1) = r_1, \quad \ldots, \quad \rho(\bar{B}_s) = r_s, \quad s \leq n. \tag{54}$$

...

4.3.2 Some particular cases of Theorem 4.4

Equations (53) and (54) are equivalent characterisations of dynamic controllability, valid for all linear constant systems. In applying the rank criteria of (54), a distinction can be made between systems in which the Jordan chain condition is satisfied and systems in which it is violated. Consider, therefore, the following particular cases.

(i) $n = p$

If the Jordan chain condition is satisfied, then, by (13), $n_i = p_i \; \forall \; i = 1,\ldots,s$, $s \leq n$; and two subcases are possible:

(ia) $n_i = p_i = 1 \; \forall \; i = 1,\ldots,s=n$

This case corresponds to n distinct eigenvalues and represents the transformations (I) of Figure 4.1. By theorem 4.1, (ia) implies $r_1 = \ldots = r_n = 1$, and since $n_1 = \ldots = n_{s=n}=1$, the criteria of (54) require each row of $\hat{B} = P^{-1}B$ to possess rank of unity - to possess at least one nonzero element. Thus

$$\rho(\bar{B}_1) = 1, \quad \ldots, \quad \rho(\bar{B}_n) = 1, \tag{55}$$

which is the coupling criterion of section 3.3.

(ib) $n_i = p_i \geqslant 1 \ \forall \ i = 1, \ldots, s < n$

This case corresponds to multiple eigenvalues possessing unbroken Jordan chains, and belongs to the transformations (III) of Figure 4.1. Again, $r_1 = \ldots = r_s = 1$, but (54) requires that only a particular $s < n$ rows possess unity rank:

$$\rho(\bar{B}_1) = 1, \qquad \ldots, \qquad \rho(\bar{B}_s) = 1. \tag{56}$$

Scalar controllability for systems satisfying (i) therefore requires, given (54), (55), and (56), that one column of $\hat{B} = P^{-1}B$ has $s \leqslant n$ nonzero elements, the positions of these nonzero elements being significant. This is the basis of theorem 4.3.

(ii) $n > p$

Two particular subcases will suffice to illustrate operation of the criteria (54) when the Jordan chain condition is violated.

(iia) $p = 1$

This case corresponds to an n-fold eigenvalue λ whose Jordan chain vanishes completely. Hence $r_1 = n$, and $s = 1$. From (51) and (52), this implies $\bar{B} = \hat{B} = P^{-1}B$, so that (54) requires

$$\rho(\bar{B}_1) = \rho(\hat{B}) = \rho(B) = n. \tag{57}$$

Thus n instruments are necessary for stabilisation, as is clear from explicit solution of $\dot{x} = \lambda x + \hat{B}u$ for $x(T)=0$.

(iib) $n_i \geqslant 1, \ p_i = 1 \ \forall \ i = 1, \ldots, s < n$

This case is the general representation of the transformations (II) of Figure 4.1. There exist $s < n$ eigenvalues λ_i of multiplicity $n_i \geqslant 1$ and index $p_i = 1$, so that $r_1 = n_1, \ldots, r_s = n_s$. Hence (54) requires

$$\rho(\bar{B}_1) = n_1, \qquad \ldots, \qquad \rho(\bar{B}_s) = n_s. \tag{58}$$

This particular result is stated by Kalman [1963b, p.171] who also refers to a then unpublished result apparently analogous to theorem 4.4. Neither the result nor further reference to it have been sighted in the control literature. Additionally, both theorem 4.4 and (58) contradict the corollary on dynamic controllability given by Kalman, Ho & Narendra [p.204] for $k > 1$: their condition (b') should be strengthened by a rank requirement on the distribution of nonzero elements, as given here in theorem 4.4.

4.3.3 Joint controllability

To clarify the question of dynamic existence, chapter III separated the problems of static and dynamic stabilisation. Now theorems 3.3 and 4.4 provide necessary and sufficient conditions for static and dynamic existence; and it is therefore natural to ask for necessary and sufficient conditions under which linear stabilisation models are simultaneously statically and dynamically controllable - or jointly controllable. Theorem 4.5 specifies these conditions.

Theorem 4.5

Given the linear constant dynamic system $\dot{x} = Ax + Bu$, A: nxn, B: nxk, $\rho(B) = k$; and associated static system $0 = A^* X^* + B^* U^* + D^*$, A^*: NxN, B^*: NxK, where $n \geqslant N \geqslant K \geqslant k$. Then these systems are jointly controllable iff

either

$$\rho(B^*) = \rho(B) = n = N = K = k, \tag{59}$$

or

$$\rho(B^*) = N = K, \tag{60}$$

and

$$\rho(\bar{B}_1) = r_1, \quad \ldots, \quad \rho(\bar{B}_s) = r_s, \; s \leqslant n. \tag{61}$$

...

Minimum instrument requirements for dynamic stabilisation are thus set by

$$r_\alpha = \max r_i, \quad i = 1,\ldots,s, \quad s \leqslant n. \tag{62}$$

Just as K=N instruments are required for static controllability, so $k = r_\alpha$ instruments are required for dynamic controllability; and just as the control matrix for these static instruments must satisfy Tinbergen's rank condition, so, too, certain blocks of the (transformed) control matrix for these dynamic instruments must satisfy the rank criteria (54). Information on the eigenvalues of the structural matrix - on the sequence of Jordan chains of A - is codified in r_i, $i = 1,\ldots,s$; and information on the eigenvectors of the structural matrix is codified in $\rho(\bar{B}_i)$, $i = 1,\ldots,s$. Since these rank conditions refer to the row rank of matrices constructed directly from the rows of $\hat{B} = P^{-1}B$, and since $r_\alpha < n$ in general, the sufficiency of $\rho(B) = n$ for dynamic controllability, noted in chapter III, is therefore explained.

4.4 CONCLUSIONS

Theorem 4.4 provides an equivalent statement to theorem 3.4 and finalises the analysis of dynamic controllability in linear stabilisation models. The necessary and sufficient conditions of these theorems apply to any model of regular stabilisation, whether rectangular or square. The rank criterion of theorem 3.3 refers to the system $\dot{x} = Ax + Bu$; to interpret this criterion, the system is transformed to the spectral form

$$\dot{z} = \Lambda z + \hat{B}u, \qquad x = P^{-1}z, \qquad \hat{B} = P^{-1}B, \tag{63}$$

from which the rank conditions of theorem 4.4 are derived.

These criteria for dynamic controllability define two attributes of a dynamic stabilisation model necessary and sufficient for existence of a dynamic policy: the minimum number of instruments required and the conditions a minimal set must satisfy. Thus the maximum number r_α of Jordan blocks, or independent eigenvectors, associated with any eigenvalue in Λ, the Jordan canonical form of A, specifies the minimum number of instruments necessary for dynamic stabilisation. Only if the number of instruments available - the column dimension k of the control matrix B or $\hat{B}$ - is greater than this minimum can a regular controller u(t) be designed. This counting condition, $k \geq r_{\alpha}$, is determined purely by the eigenvalue structure of the structural matrix. The structural and control matrices interact, however, to specify a further condition on these instruments. Assuming sufficient instruments exist, and that $\hat{B}$ in (63) is partitioned conformably with the Jordan blocks of Λ, a matrix $\bar{B}$ is constructed from the last row of each such block of $\hat{B}$. This matrix $\bar{B}$ is also row-partitioned into blocks, each block being associated with one and only one distinct eigenvalue. The second condition for dynamic controllability is that each such block possess full row rank.

Thus necessary and sufficient conditions enabling design of regular stabilisation policy are fully developed. The possibility of reducing the number of dynamic instruments to compensate for the complexity of the dynamic design problem is also explored. Criteria for reduction are implicit in theorem 4.4. The extent of reduction is constrained by the minimal instrument requirement. When the Jordan chain condition is satisfied, maximum reduction is possible; since $r_{\alpha} = 1$, the system is then controllable with only one 'instrument', where instrument is qualified to allow for multiple as well as scalar reduction. An exception to this rule occurs in square policy systems, for which multiple reduction is available independently of the Jordan chain condition. If the Jordan chain condition is violated, then, provided $k > r_{\alpha}$, the system is controllable with more than one instrument but less than all instruments. These statements assume that the row rank criteria derived from the appropriate matrix construction $\bar{B}$ are satisfied.

Reduced stabilisation appears to be a significant aspect of a generalised theory of economic policy; its importance flowing from the matching of dimensional simplicity against design complexity. If policy-makers are to learn how to implement optimal dynamic policy, such learning is facilitated by a literal reduction in the dimensions of the stabilisation problem. The reduced controllability conditions specify those well-behaved systems for which simplification is possible. But two major problems are raised by reduction: (i) the empirical incidence of well-behaved systems; and (ii) the criteria to be applied in selecting a reduced system. Neither is considered further in this thesis.

The joint controllability conditions of theorem 4.5 emphasise that two causes of persistent policy problems exist; the first being static noncontrollability, the second, dynamic noncontrollability. The problems of static noncontrollability have been thoroughly analysed in the theory of economic policy. This present analysis therefore focusses attention on the simultaneous problem of dynamic existence. It is shown that in square policy systems both the static and the dynamic existence problems are resolved in one fell swoop. In rectangular policy systems, the rank criteria (54) refer essentially to the eigenvector condition, first described in chapter III. As an agent of dynamic noncontrollability, the significance of this condition can only be accurately evaluated empirically. But theoretically, there seems to be no economic rationale for violation of this condition to be an intrinsic property of economic structure.

Dynamic controllability is clearly relevant to the practical design of stabilisation policy using econometric models. Even after the problems of modelling are overcome, it is still necessary to confirm controllability before policy design is attempted. Noncontrollability would pose the question as to whether it is induced by the estimation process or by the underlying economic structure; some test would then need to be devised to discriminate between these two possibilities. The partitioning of large-scale econometric models also requires consideration of the controllability properties of the submodels so obtained. Controllability must be investigated explicitly in all policy models, whether theoretical or practical.

Chapters III and IV consider dynamic controllability only with respect to linear constant continuous-time models, but the concept is not restricted to these models. Controllability criteria are also available for linear time-varying models and for linear discrete-time models, as given, for example, in Kalman [1961] and Ogata [pp.370-436]. For nonlinear systems, considered by Lee & Markus [pp.364-93], the global criteria of linear systems are replaced by local criteria.

Chapters III and IV therefore combine to remove the hiatus concerning dynamic existence observed in the theory of economic policy.

Consequently, chapters V, VI, and VII move from the controllability concept in existence theory to the optimality concept in design theory. To commence, chapter V investigates an alleged conflict between optimality and stability.

CHAPTER V

A PARADOX IN THE THEORY OF OPTIMAL STABILISATION[1]

Phillips [1954] applied the classical techniques of control engineering to the problem of synthesising dynamic stabilisation policies. Given the dual objective of control over both the dynamic adjustment path and the comparative static equilibrium level of a target variable, Phillips proposed _inter alia_ a combination of proportional, derivative, and integral feedbacks for effective stabilisation. A drawback of this classical approach, however, is its inability to provide a general method for specifying the simultaneous values of these feedbacks. Fox, Sengupta & Thorbecke (FST) subsequently used the calculus of variations to generalise Phillips' approach. By assigning quadratic preferences to the policy-maker and formulating the control problem as one of dynamic optimisation, a unique set of feedback parameters could, in principle, be derived through a direct search for an optimum solution.

Now the application of dynamic optimisation techniques is a significant step in the evolution of stabilisation theory. Yet the policies proposed by FST exhibit some puzzling features. Their proposition that optimal policies necessitate unstable target solutions particularly excites comment. The necessarily stable policies prescribed by Phillips are to be replaced by 'optimal' policies which render the model unstable and thwart the very purpose behind their design - the achievement of economic stability. This paradox is surely of considerable importance for policy-making but has attracted little attention as an impossibility theorem for optimal design.

If true, a conflict between optimality and stability is a telling addition to Baumol's 'theorems for skeptics', a set of negative policy

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[1] This chapter is to be published, under the same title, by _The Review of Economic Studies_.

prescriptions defining what policy-makers cannot do; and including the second best theorem, and the Phillips proposition relating the dynamic sensitivity of economic systems to the timing and magnitude of policy. The lot of policy-makers is already onerous and all potential additions to Baumol's list warrant keen investigation before acceptance. This chapter contends optimistically that the FST paradox cannot withstand close scrutiny - that there is no impossibility theorem for the design of optimal stabilisation policy. Certainly, correct analysis still implies an addition to Baumol's list, but it is less stringent than that asserted by FST.

The argument supporting this contention proceeds in four stages. In section 5.1, the optimal Phillips model used by FST is described and correct solutions are provided using necessary conditions from the theory of linear optimal control. This leads to a consideration of the alleged stability-optimality conflict in section 5.2. There it is shown that the paradox is untenable but with little perception of the logic whereby the conflict is avoided. Accordingly, section 5.3 examines the phase portrait of the optimal system, clarifying the basis of the paradox. If true, the FST paradox applies to all regulator formulations, whatever their dimensions. Section 5.4 therefore refutes the paradox at the general level. Section 5.5 concludes, and briefly relates the underlying problem to the areas of growth theory, production theory, and filter design.

5.1 AN OPTIMAL PHILLIPS MODEL

FST employ the Phillips mutliplier-accelerator model, as exposited in Allen [1960, 1968], to represent the dynamics of an economic system for which an optimal stabilisation policy is to be designed. This model is the pair of equations

$$y'(t) = (1-s)y(t) + v\dot{y}(t) + g(t), \quad s\varepsilon(0,1),\ v > 0, \tag{1}$$

$$\dot{y}(t) = \sigma(y'(t) - y(t)), \qquad y(0)=y_o\neq 0. \tag{2}$$

Equation (1) is the Harrodian demand relation specifying current demand y' as the sum of private sector demand $((1-s)y+v\dot{y})$ and public sector demand g; with private sector demand consisting of consumption demand

(1-s)y and induced investment demand $v\dot{y}$. Equation (2) represents the dynamic adjustment mechanism of supply y to demand y', taking the form of a first-order exponential distributed lag, as defined by Allen [1960,pp.23-5]. Current excess demand (y'-y) is assumed met by a reduction in inventories; producers react by increasing the rate of production $\dot{y}$ by a proportion σ of this reduction. Provided $\sigma < \infty$, producers do not match all inventory movements immediately, but accomplish instead a 63 percent response during the period of the mean supply lag, $y_1 \equiv 1/\sigma$.

The variables y, y', and g are assumed measured as deviations from static levels defining a desired equilibrium position. The target, output y, and instrument, government demand g, are thus related by

$$\dot{y}(t) = -swy(t) + wg(t), \; y(0)=y_o, \quad w \equiv \frac{\sigma}{1-v\sigma} \neq 0. \tag{3}$$

The stabilisation problem posed by this Phillips model results from the free or natural behaviour of the uncontrolled system

$$\dot{y}(t) = -swy(t), \quad y(0)=y_o, \tag{4}$$

with explicit solution

$$y(t) = y_o e^{-swt}. \tag{5}$$

Since the marginal propensity to save is strictly positive, asymptotic stability depends on the sign of the definitional parameter

$$w \equiv \frac{1}{y_1 - v}, \quad y_1 \equiv 1/\sigma. \tag{6}$$

If w is positive (negative), the uncontrolled system is naturally stable (naturally unstable). In economic terms, the system (3) is naturally stable if the mean supply lag y_1 is greater than the accelerator coefficient v.

In the naturally unstable case (w < 0), the necessity for dynamic stabilisation is obvious - the system diverges uniformly. For the naturally stable case (w > 0), the stability factor (y_1-v > 0) is inversely related to the rate of convergence sw. Thus the policy-maker may be assumed to desire a faster rate of convergence when faced with a sufficiently large value of this factor.

Following the analysis of dynamic controllability in chapters III and IV, the basic ability to stabilise the system must also be demonstrated. The solution of (3) is

$$y(t) = e^{-swt}y_o + w\int_0^t e^{-sw(t-\tau)}g(\tau)\,d\tau. \tag{7}$$

Suppose that $y = y(T)$ is the desired value of output at time $t=T$. Is it possible, with the instrument $g(t)$, to transfer the system from its initial position y_o to the terminal position $y(T)$ in time T? From (7), it is necessary and sufficient that there exists a control $g(\tau), \tau\varepsilon[0, T]$, such that

$$w\int_0^T e^{-sw(T-\tau)}g(\tau)\,d\tau = y(T) - e^{-swT}y_o. \tag{8}$$

Consider the open-loop controller,

$$g(t) = \beta e^{-sw\tau}. \tag{9}$$

Then (8) is satisfied if β is chosen as

$$\beta = \frac{y(T)e^{swT} - y(0)}{wT} = \frac{\hat{y}(0) - y(0)}{wT}, \tag{10}$$

and this is always possible. The initial condition $\hat{y}(0)$ is that for which, given the desired endpoint, no control would be necessary. Hence β is a direct function of the difference between the 'desired' and actual starting points, and an inverse function of the fixed finite time available to achieve the desired endpoint. Since $\lim_{T\to o}\beta = \infty$, the controller (9) illustrates the principle that controllability involves a tradeoff between time and control energy. Stabilisation policy must adjust explicitly to this tradeoff to avoid violation of control constraints, a design aspect considered in chapter VII.

FST [pp. 217-9] tackle the problem of selecting a specific control policy for (3) by assuming a quadratic preference functional of the form

$$\underset{g(t)}{\text{MIN}}\ W = \tfrac{1}{2}\int_0^T [y^2(t) + \phi g^2(t)]dt,\ \phi > 0. \tag{11}$$

This criterion - the optimal regulator - requires that disequilibrium income y be kept near its desired value of zero without excessive deviation of control expenditure from its desired value, also zero. The strictly positive parameter ϕ measures the relative importance of the two performance cost elements: the greater ϕ, the greater the cost of using the control g to force y to equilibrium. The parameter T defines the length of the stabilisation period. It is assumed fixed and may be either finite or infinite: in particular, FST assume T is finite [pp.218,371]. Their optimal stabilisation problem is therefore written[2]

$$\begin{cases} \underset{g(t)}{\text{MIN}}\ W = \tfrac{1}{2}\int_0^T [y^2(t) + \phi g^2(t)]dt,\ \phi > 0, \\ \underline{\text{subject to}} \\ \dot{y}(t) = -swy(t) + wg(t),\ s\varepsilon(0,1),\ w\neq 0, \\ y(0) = y_o;\quad y(T)\ \text{free};\quad T\ \text{fixed}, \\ g(t),\ t\varepsilon[0,T],\ \text{unconstrained}. \end{cases} \tag{12}$$

FST are actually ambiguous about the target endpoint y(T): is it fixed or free? To proceed, a free endpoint is taken but it is later argued that their paradox is also invalid for the fixed endpoint assumption.

This dynamic optimisation problem is a simple example of a

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2 This specification differs from that of FST in minor details [p.219, (8.60.2)]: the performance criterion is not time-averaged; the discount function is omitted; and autonomous demand expenditures are removed by the disequilibrium specification. For simplicity, their performance weights are taken in the ratio form $\phi = \alpha_2/\alpha_1$.

fixed-time, free endpoint optimal regulator, and solutions are readily found with the aid of linear optimal control theory[3]. Thus defining the Hamiltonian

$$H(y,g,p) \equiv \tfrac{1}{2}y^2 + \tfrac{1}{2}\phi g^2 + p(-swy+wg), \tag{13}$$

where $p(t)$ is the costate variable, the canonical system ($\mathring{y} = \partial H/\partial p$; $\mathring{p} = -\partial H/\partial y$), after using the minimising control

$$\frac{\partial H}{\partial g} = 0 \iff \qquad g = -\phi^{-1}wp, \tag{14}$$

is written

$$\begin{bmatrix} \mathring{y} \\ \mathring{p} \end{bmatrix} = \begin{bmatrix} -sw & -\phi^{-1}w^2 \\ -1 & sw \end{bmatrix} \begin{bmatrix} y \\ p \end{bmatrix}, \begin{bmatrix} y(0)=y_o \\ p(T)=0 \end{bmatrix}. \tag{15}$$

Here the costate boundary condition $p(T)=0$ derives from the transversality condition. Optimal solutions for the instrument g and target y are then given by

$$g = -\phi^{-1}wky, \tag{16}$$

$$\mathring{y} = -(sw + \phi^{-1}w^2k)y, \qquad y(0)=y_o, \tag{17}$$

where from the canonical system (15) the costate is given by

$$p = ky, \tag{18}$$

with k the positive solution of the Riccati differential equation

$$\mathring{k} = \phi^{-1}w^2k^2 + 2swk - 1, \quad k(T)=0. \tag{19}$$

From Appendix V (pp.268-9 below), solutions for k and y are:

$$k(t) = \frac{\phi w^{-1}(\theta-s)\{1 - e^{-2w\theta(T-t)}\}}{1 + \frac{\theta-s}{\theta+s}e^{-2w\theta(T-t)}}, \qquad \theta = (s^2+\phi^{-1})^{\frac{1}{2}}, \quad t\in[0,T], \tag{20}$$

---

[3] FST use the classical calculus of variations but greater clarity of solution structure may be obtained with the modern theory.

$$y(t) = y(0)\ \{\frac{(\theta+s)e^{-w\theta t} + (\theta-s)e^{-w\theta(2T-t)}}{(\theta+s) + (\theta-s)e^{-2w\theta T}}\}, \qquad t\varepsilon[0,T]. \tag{21}$$

Given these instrument, feedback, and target solutions (g, k, y), the conflict between optimality and stability alleged by FST is now investigated.

5.2 THE ALLEGED STABILITY-OPTIMALITY CONFLICT

The stabilisation paradox of FST receives a concise statement in their theorem [p.378] for a regulator model of order 2n, where n is the order of a dynamic equality constraint akin to (3). Thus:

> "Theorem 2 (Optimality Theorem)
>
> If there exists a control vector u=u(t) which satisfies equation (.) of the multisector Phillips model and also optimises the performance integral equation (.) in a nondegenerate way, i.e. the Euler-Lagrange equations (.) are nondegenerate, then the optimal control system has the property that if u_o is a characteristic root of the system, $-u_o$ is also a characteristic root of the system. <u>In particular, if the root u_o has a nonzero real part and the initial condition x(t) at t=0 is quite arbitrarily fixed, the optimal control vector u=u(t) is necessarily unstable.</u>"

Without commenting as yet on the validity of this theorem for the general order 2n, its validity for the Phillips regulator, in which n=1, will be investigated.

The characteristic equation of (15) implies the validity of the statement of the theorem: the eigenvalues $\pm w\theta$ are real, of given modulus but opposite sign. However, the underlined stability proposition derived from this statement is invalid. Referring to the optimal solutions (16), (20), (21),

$$\begin{cases} y = y(t;\ \ T,\ w,\ s,\ \phi,\ y_o) \\ \\ g = g(t;\ \ T,\ w,\ s,\ \phi,\ y_o) \end{cases} \qquad t\varepsilon[0,T]. \tag{22}$$

For a finite horizon T, equations (22) are also finite, and the FST paradox cannot occur. But in the proof of their 'Optimality Theorem', FST consider the limit $\{t\to\infty\}$ of analogues of these solutions. Now since t is defined on the stabilisation interval [0, T], their limiting procedure is only justifiable if preceded by the limiting process $\{T\to\infty\}$, which provides[4]

$$\begin{cases} \lim\limits_{T\to\infty} y(t) = y^*(t) = y(0)e^{-w\theta t}, \\ \\ \lim\limits_{T\to\infty} g(t) = g^*(t) = -(\theta-s)y^*(t), \end{cases} \qquad t\varepsilon[0,\infty]. \tag{23}$$

And these infinite horizon solutions are clearly asymptotically stable

$$\lim_{t\to\infty} y^*(t) = 0, \qquad \lim_{t\to\infty} g^*(t) = 0. \tag{24}$$

There appears to be no basis, therefore, to the stability proposition contained in the FST 'Optimality Theorem'. Yet after applying the correct limiting process $\{{T\to\infty \atop t\to T}\}$, the optimal solutions provided by FST [pp.224, 379] are indeed unstable, apparently justifying their paradox. This contradiction of (24) occurs because FST fail to recognise the need for a definite endpoint assumption. This need follows immediately from the classical calculus of variations. Adjoining the dynamic equality constraint (3) to the criterion functional (11) via the costate p(t), and using the Hamiltonian defined in (13), obtain, after Sage [pp.56-7], the equivalent problem

$$\underset{g}{\text{MIN}}\ W = \int_0^T [H(y,g,p) - p.\dot{y}]dt. \tag{25}$$

Or, after integration by parts,

$$\underset{g}{\text{MIN}}\ W = -p.y\Big|_0^T + \int_0^T [H(y,g,p) + \dot{p}.y]\ dt. \tag{26}$$

Now the first variation of the functional W may be written

$$\delta W = -p.\delta y\Big|_0^T + \int_0^T [(\frac{\partial H}{\partial y} + \dot{p})\delta y + \frac{\partial H}{\partial g}\delta g]\ dt. \tag{27}$$

---

4 This asymptotic procedure is justified by Kalman's existence theorem for linear optimal control theory [1959].

And since a necessary condition for a minimum for W is that δW vanishes identically, a set of necessary conditions follows:

$$\frac{\partial H}{\partial g} = 0 \tag{28}$$

$$\frac{\partial H}{\partial y} = -\dot{p}, \qquad \left(\frac{\partial H}{\partial p} = \dot{y}\right) \tag{29}$$

$$p(t)\delta y(t) = 0, \quad t = \{^{0}_{T}. \tag{30}$$

Conditions (28) and (29) are together equivalent to the Euler-Lagrange necessary conditions used by FST. Condition (30) provides the associated transversality condition. For a fixed endpoint $y(T)$, $\delta y(T)=0$ satisfies (30); and for a free endpoint $\delta y(T)$ is arbitrary, implying $p(T)=0$ to satisfy (30). Now FST apply the Euler-Lagrange conditions (28), (29) to the optimisation problem (12) to obtain

$$\dot{y} = -swy + wg, \quad y(0)=y_o \tag{31}$$

$$\dot{g} = \phi^{-1}wy + swg, \tag{32}$$

as necessary conditions for optimality. Equation (31) is simply the constraint provided by the system dynamics (3), while (32) is the optimal specification of the controller dynamics to match these system dynamics. FST reduce this pair of equations to the second order differential equation

$$\ddot{y} - w^2\theta^2 y = 0, \qquad \theta = (s^2+\phi^{-1})^{\frac{1}{2}}, \tag{33}$$

with general solution

$$y(t) = C_1e^{w\theta t} + C_2e^{-w\theta t}. \tag{34}$$

It is here that the analysis of FST fails. One of the boundary conditions $y(T)=0$ or $p(T)=0$, satisfying the transversality condition (30), is mandatory for solution of the optimal stabilisation problem with a finite horizon T. But FST, having omitted this transversality condition, must arbitrarily specify a second boundary condition to accompany the initial target condition $y(0)\neq 0$ in solving (34). This allows the horizon parameter T to disappear from the optimal target solution (21), or prevents it from appearing in (34), so that the limits $\{t\to\infty\}$, $\{{T\to\infty \atop t\to T}\}$ are, for FST, indistinguishable. Although (21) contains the unstable mode, exp $(w\theta t)$, its effect is nullified by the explicit appearance of the horizon parameter $(2T-t > 0\ \forall\ t\epsilon[0,T])$. To remove T is to render the optimal target unstable, falsely generating the paradox.

Correct specification of finite horizon regulators determines not only the optimal policy dynamics to accompany the given target dynamics, but also the terminal condition - either on the target or the instrument - to accompany the initial condition on the target. Whether the regulator is fixed endpoint or free endpoint, the horizon parameter T will be explicitly incorporated in the optimal finite horizon solutions, disallowing the paradox as an asymptotic proposition.

Given correct solutions to the Phillips regulator, it is therefore argued that (i) the finite horizon regulator is necessarily stable despite the appearance of a saddle point $\pm w\theta$; and that (ii) the infinite horizon regulator is necessarily stable following the disappearance of this saddle point. Although the FST paradox has been exposed, the cause of confusion - in particular, the process by which the potentially unstable dynamics are filtered out of the asymptotic regulator - is better understood in terms of the phase portrait of the optimal system.

5.3 SADDLE POINT OF THE PHILLIPS REGULATOR

To visualise the elements of the FST paradox geometrically, the phase portrait of the optimal system is constructed, integrating finite and infinite horizons, free and fixed endpoints, and natural stability and instability in a single diagram. The Hamiltonian system (15) may be written

$$\dot{x} = Hx = W\Lambda W^{-1}x, \qquad x(0)=x_o$$

$$H = \begin{bmatrix} -sw & -\phi^{-1}w^2 \\ -1 & sw \end{bmatrix}, \; x = \begin{bmatrix} y \\ p \end{bmatrix}, \; \Lambda = \begin{bmatrix} -w\theta & 0 \\ 0 & w\theta \end{bmatrix}. \qquad (35)$$

Here Λ is the diagonal matrix of eigenvalues, and W is a right eigenvector matrix found from solutions of

$$[H \pm w\theta I]W_{\mp} = 0, \quad W = (W_- \; W_+). \tag{36}$$

One such solution matrix is

$$W = \begin{bmatrix} 1 & 1 \\ \frac{1}{w(\theta+s)} & \frac{-1}{w(\theta-s)} \end{bmatrix}, \tag{37}$$

for which the first column W_- is associated with the stable eigenvalue $-w\theta$. Now following Hurewicz [pp.70-86],

$$\dot{r} = \Lambda r, \; r(0)=r_o, \qquad r = W^{-1}x, \tag{38}$$

is a similarity transformation of (35) in which the axes r_1, r_2 are the asymptotes of the saddle point $\pm w\theta$. Equations for these asymptotes in the x-plane are therefore given by

$$W^{-1}x = 0 \quad \Longleftrightarrow \quad \begin{bmatrix} \frac{-1}{w(\theta-s)} & -1 \\ \frac{-1}{w(\theta+s)} & 1 \end{bmatrix} \begin{bmatrix} y \\ p \end{bmatrix} = 0, \tag{39}$$

or by

$$y_+ = -w(\theta-s)p; \qquad y_- = w(\theta+s)p. \tag{40}$$

And from (35), the equations

$$\begin{cases} \dot{y} = 0 \quad \Longleftrightarrow \quad y = -(w/s\phi)p \\ \dot{p} = 0 \quad \Longleftrightarrow \quad y = swp, \end{cases} \tag{41}$$

specify trajectory turning points with respect to y and p. Combining these results, the saddle point of the Phillips regulator is drawn in

Figure 5.1:

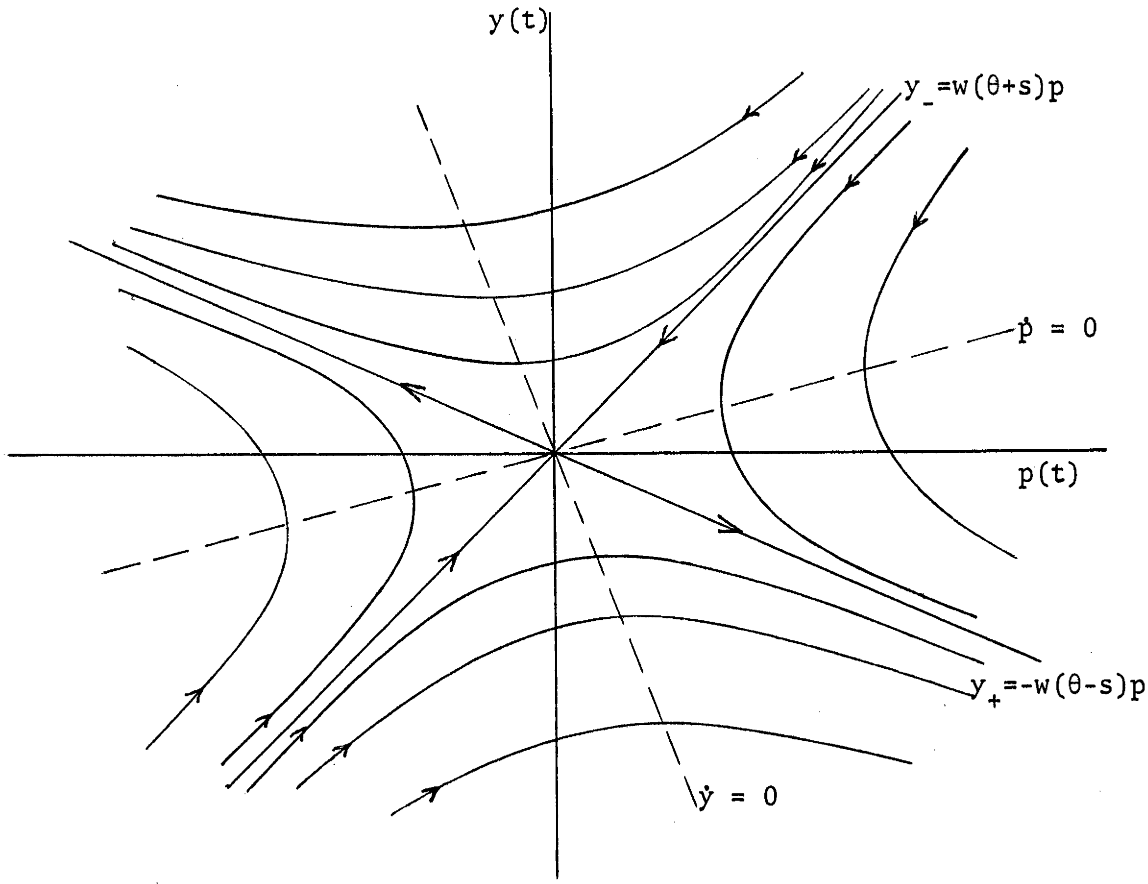

Figure 5.1
Saddle Point of Phillips Regulator (w > 0)

Figure 5.1 assumes that the Phillips model is naturally stable ($w > 0$). For completeness, natural instability is now allowed for, after which the paradox will be considered for both cases. Given that the saddle point exists $\forall\ w \neq 0$, natural instability ($w < 0$) reverses the sign of all gradients in Figure 5.1. As a first step, the revised saddle point is therefore the image of Figure 5.1 in the vertical axis y. But natural instability also switches the eigenvalue signs; as a second step, the direction of movement along trajectories must be reversed. With these changes, both cases ($w \gtrless 0$) may be unified in a single figure after specifying a particular initial target error - say $y_0 > 0$, an inflationary demand gap. Then since $p = ky$, and k is positive irrespective of w, the first quadrant of Figure 5.1, $w > 0$, may be juxtaposed with the first quadrant of a revised saddle point, $w < 0$, to give Figure 5.2, for $w \gtrless 0$, $T \leq \infty$, $y_o > 0$:

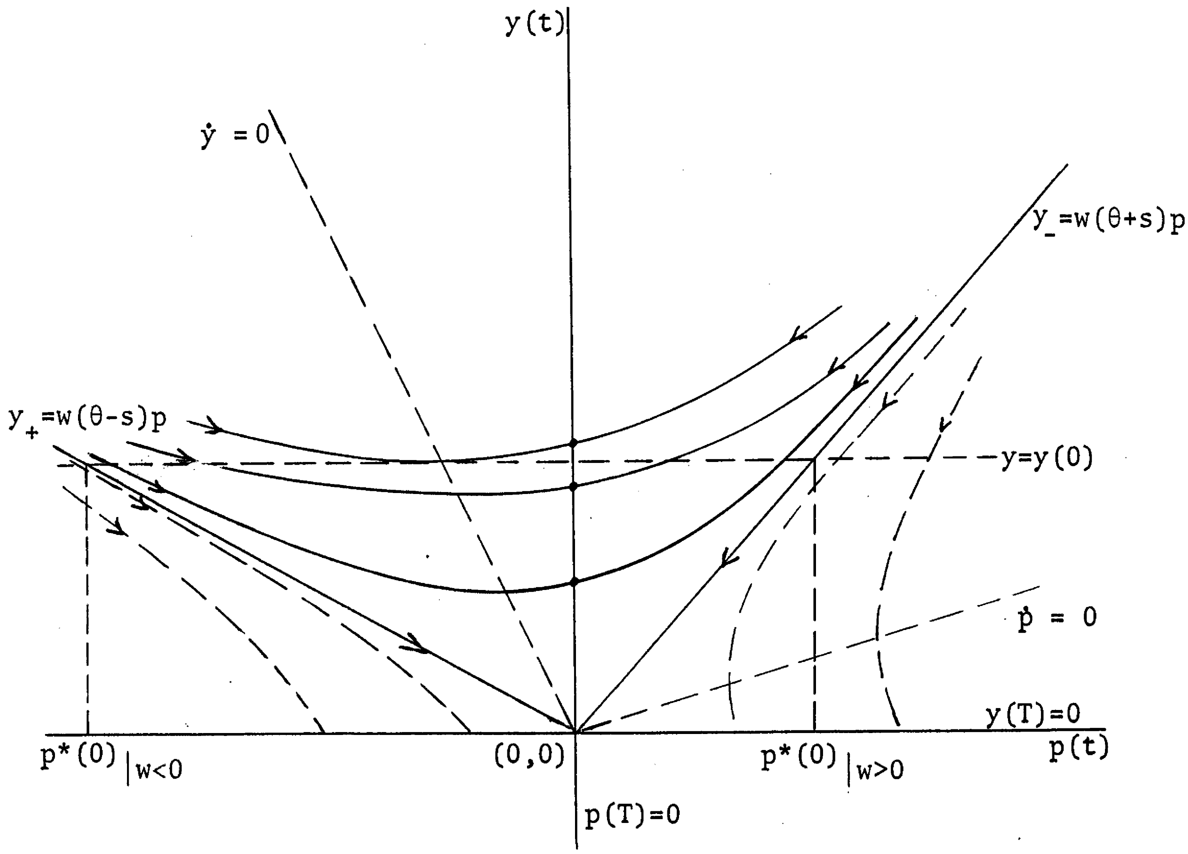

Figure 5.2
Phillips Regulator Trajectories ($w \gtrless 0$)

Identifying the infinite horizon solution for the naturally stable case first, observe that from (18), (40),

$$y^* = p^*/k^* = w(\theta+s)p^* = y_- \qquad t\varepsilon[0,\infty], \tag{42}$$

where $k^* = \lim_{T\to\infty} k(t)$ is evaluated from (20). Hence the stable asymptote y_- defines the optimal trajectory for the infinite horizon regulator with $w > 0$; and it is readily shown that $y_+ = w(\theta-s)p$, $w = |w|$, is the analogue for $w < 0$. The optimal asymptotic motions of the Phillips regulator therefore begin from the phases $\{p^*(0), y(0)\}$, $w \gtrless 0$, and proceed along the stable asymptotes to the equilibrium point (0,0). The FST paradox is clearly invalid in the infinite horizon case.

If the optimal finite horizon trajectories are to satisfy the saddle point dynamics of Figure 5.2 subject to the boundary conditions

$y(0)=y_o$, $p(T)=0$, they are necessarily those suboptimal asymptotic trajectories commencing from the initial manifold $y=y_o$, truncating at the terminal manifold $p(T)=0$, and therefore lying everywhere above the optimal asymptotic trajectories (y_-, y_+). The finite horizon assumption generates a terminal condition, the transversality condition, to express that assumption, and constrains further movement along trajectories that would be unstable under an infinite horizon assumption. And this succinctly illustrates the basis of the FST paradox. By overlooking the transversality condition and suppressing the horizon parameter, FST incorrectly define asymptotic stability in terms of an infinite traverse along an unstable curvilinear trajectory rather than in terms of a continuous shift of this trajectory towards the stable linear asymptotes.

Nor can the FST paradox be construed as referring to the behaviour of the correct optimal solutions shown in Figure 5.2. In naturally stable systems, the target and instrument converge monotonically for any stabilisation horizon, contrary to the paradox. Admittedly this is not true for naturally unstable systems: the finite horizon solutions all cross the locus $\dot{y} = 0$, after which target performance deteriorates as $t \to T$. The reason for this behaviour is that the control force applied to the system diminishes to zero as the instrument seeks to meet the terminal condition $p(T)=g(T)=0$; and the natural behaviour of the system therefore gains ascendancy over the controlled behaviour. With natural stability the target continues to converge; but with natural instability it starts to diverge, as signified by the locus $\dot{y} = 0$. This defect of the free endpoint regulator may be overcome by using terminal target weighting[5]. The addition of a quadratic term $fy^2(T)$, $f \geqslant 0$, to the performance functional (11) modifies the transversality condition to $p(T) = fy(T)$. Selection of the terminal weight f, now the gradient of the terminal manifold, at least equal to the gradient of $\dot{y} = 0$ then inhibits this endpoint deterioration. The FST paradox is therefore in no way applicable to the finite horizon regulator with a free endpoint.

The effect of the alternative endpoint specification, a fixed endpoint, may be briefly mentioned. The fixed endpoint $y(T)=0$ replaces

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[5] Cf. Athans & Falb [p.574], and chapter VII below.

the terminal condition $p(T)=0$ of the free endpoint regulator. This does not affect the dynamic saddle point structure, so that the two regulators are differentiated by boundary conditions only. Now from (35), the solution of the Hamiltonian system is

$$\begin{bmatrix} y(t) \\ p(t) \end{bmatrix} = We^{\Lambda t}W^{-1} \begin{bmatrix} y(0) \\ p(0) \end{bmatrix}, \tag{43}$$

which, using (37), becomes

$$\begin{bmatrix} y(t) \\ p(t) \end{bmatrix} = \frac{w}{2\phi\theta} \begin{bmatrix} \frac{e^{w\theta t}}{w(\theta+s)} + \frac{e^{-w\theta t}}{w(\theta-s)} & e^{-w\theta t} - e^{w\theta t} \\ \frac{e^{-w\theta t} - e^{w\theta t}}{w^2(\theta^2-s^2)} & \frac{e^{-w\theta t}}{w(\theta+s)} + \frac{e^{w\theta t}}{w(\theta-s)} \end{bmatrix} \begin{bmatrix} y(0) \\ p(0) \end{bmatrix}. \tag{44}$$

These solutions $(w > 0)$ apply to either endpoint regulator but are subject to different boundary conditions. Thus use of the fixed endpoint $y(T)=0$ in the first equation of (44) provides, for $t=T$,

$$\frac{p(0)}{y(0)} = \frac{\frac{1}{w(\theta+s)} + \frac{e^{-2w\theta T}}{w(\theta-s)}}{1 - e^{-2w\theta T}} = \left\{\frac{1 + \frac{\theta+s}{\theta-s}e^{-2w\theta T}}{1 - e^{-2w\theta T}}\right\} k^*, \tag{45}$$

where $k^* = 1/w(\theta+s)$, from (42). Hence

$$\lim_{T\to\infty} \frac{p(0)}{y(0)} = \frac{p^*(0)}{y(0)} = k^*, \tag{46}$$

and

$$p(0) \geqslant p^*(0) \;\forall\; t \in [0,\infty]. \tag{47}$$

Regarding Figure 5.2, the trajectories lying underneath the optimal asymptotic trajectory y_-, commencing from the initial manifold $y=y(0)$ and truncating at the terminal manifold $y(T)=0$, are the finite horizon solutions of the naturally stable fixed endpoint regulator. By a similar argument for $w < 0$, the trajectories lying underneath y_+ are the naturally unstable fixed endpoint solutions. The previous analysis is unaffected by this switch in boundary conditions; and the FST paradox is therefore finally subdued - at least in the Phillips regulator.

5.4 THE GENERALISED SADDLE POINT

FST also assert their paradox to hold for a regulator model of general dimension [pp.370-9]. At this level, the theoretical analysis of chapter II provides a concise refutation. Thus for the Hamiltonian system (for a finite horizon T):

$$\dot{x} = Hx, \qquad x = [\ z^T \ \ p^T\ ]^T, \qquad H: 2n\text{x}2n, \tag{48}$$

the associated spectral form is, by equation (2.48),

$$\dot{x} = W\Lambda_1 W^{-1}x, \qquad \Lambda_1 = \left[\begin{array}{c|c} -\Lambda & 0 \\ \hline 0 & \Lambda \end{array}\right], \ \Lambda = \text{diag}(\lambda_1 \ldots, \lambda_n), \ \lambda \varepsilon E(H). \tag{49}$$

If $r = W^{-1}x$, then

$$\dot{r} = \Lambda_1 r, \tag{50}$$

is the generalised saddle point for the regulator model. Solving (50) and partitioning:

$$\left[\begin{array}{c} r_1(t) \\ \hline r_2(t) \end{array}\right] = \left[\begin{array}{c|c} e^{-\Lambda t} & 0 \\ \hline 0 & e^{\Lambda t} \end{array}\right] \left[\begin{array}{c} r_1(0) \\ \hline r_2(0) \end{array}\right]. \tag{51}$$

To refute the FST paradox, it is necessary to demonstrate that the optimal transformed solution is asymptotically stable in the sense $\{{T\to\infty \atop t\to T}\}$. This is possible if, and only if, the unstable exponential matrix in (51) disappears from the infinite horizon solution. Since the influence of the horizon parameter T is exerted through the initial costate vector p(0), this suggests the hypothesis that, for T = ∞,

$$r^*(0) = W^{-1}x^*(0) \quad \Rightarrow \quad r_2^*(0) = 0, \tag{52}$$

ensuring removal of the unstable eigenvalues. Now by equation (2.49), (52) is equivalent to

$$\left[\begin{array}{c} r_1^*(0) \\ \hline r_2^*(0) \end{array}\right] = \cdot \left[\begin{array}{c|c} W_{22}^T & -W_{12}^T \\ \hline -W_{21}^T & W_{11}^T \end{array}\right] \left[\begin{array}{c} z(0) \\ \hline p^*(0) \end{array}\right]. \tag{53}$$

Therefore,

$$\begin{aligned} r_2^*(0) &= - W_{21}^T z(0) + W_{11}^T p^*(0) \\ &= 0 \quad \text{iff} \quad p^*(0) = (W_{21}W_{11}^{-1})^T z(0). \end{aligned} \tag{54}$$

But, from (2.56),

$$(W_{21}W_{11}^{-1})^T = (W_{21}W_{22}^{-1}) = K^* = \lim_{T\to\infty} K(t) \tag{55}$$

so that

$$r_2^*(0) = 0 \quad \text{iff} \quad p^*(0) = K^*z(0). \tag{56}$$

The refutation is immediate[6]. If, for an infinite horizon, the optimal policy is instituted, then $p^*(0) = K^*z(0)$, so that $r_2^*(0) = 0$. The optimal asymptotic solution of the general regulator model is therefore stable, in agreement with (2.100) and contrary to the paradox.

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[6] In a recent article, Smith considers the stability of a first-order infinite horizon model of the Phillips type. When confronted with the saddle point property, he argues heuristically [p.8] that the system can only converge if the initial condition on the unstable component is set to zero. For T = ∞, this is correct, but conceals the logic of the regulator structure as developed here in the text.

5.5 CONCLUSIONS

To recapitulate, FST apply the classical calculus of variations to the dynamic optimisation of a generalised Phillips model, proposing [p.379] that:

> "We have thus far obtained a very broad economic result, e.g., if we consider real roots (or roots having nonzero real parts) to be reasonable on a priori grounds, the optimum policy vector ... is not stable because it is optimum ... and the nonoptimal [classical] policy vector .. is stable because it is not optimum! It appears that stability and optimality are two competitive characteristics of a desirable policy of stabilisation in a generalized Phillips model ....".

Such a result would constitute an important contribution to Baumol's list of 'theorems for skeptics' were it not rejected by the preceding analysis. FST correctly observe the saddle point property of the Phillips regulator but then infer incorrect stability properties from 'optimal solutions' obtained with inappropriate boundary conditions.

It is correct to assert that the optimal asymptotic path is unstable in the sense that if $p^*(0)$ is the optimal initial costate, the selection of any other $p(0)$ will cause the system to diverge from the optimal path. According as $p(0) \gtrless p^*(0)$, the policy-maker will ultimately realise (and sooner rather than later, given the dominance of the positive eigenvalue) that he is pursuing a suboptimal, unstable policy. If, for example, $p(0) > p^*(0)$, realisation should occur when the costate starts increasing after the suboptimal trajectory crosses $\dot{p} = 0$; with a second warning occurring as the target deviation becomes negative after crossing $y = 0$. Depending on the reasons for this initial discrepancy between the actual and optimal policies, the policy-maker should be able to converge iteratively to the optimal path by inaugurating a series of suboptimal policies. It is this sensitivity to the initial policy specification, rather than the FST paradox, that rightly belongs to Baumol's list of negative policy prescriptions. In asymptotic regulator systems, the target is unstable with respect to all suboptimal controls but stable with respect to the optimal control: a proposition Kurz [pp.158-166] has shown to also have relevance to neoclassical growth theory.

As the analysis here and of Kurz makes clear, the central problem is related to the mathematics of dynamic optimisation, and is not peculiar to optimal stabilisation policy. Its emergence in yet another guise is therefore not surprising. Samuelson [1968] investigates the stability properties of a discrete-time analogue of the regulator. To quote from his introduction:

> "....The present paper shows that the characteristic roots associated with the stationary-equilibrium points of a discrete-time dynamic programming model must always come in reciprocal pairs. Since damped stability requires that no characteristic root exceed unity in absolute value, the present theorem rules out all possibility of damped stability in such models."

If by 'discrete-time dynamic programming model' Samuelson means, for instance, the type of decision model developed by Holt _et al_, then a study of their work [pp.92-101] reveals that the authors explicitly confront the reciprocal root property but reject the immediate implication of instability. In similar fashion to the FST paradox, Samuelson's firm proscription of stability should be interpreted as referring to perturbations of the optimal solution rather than to the optimal solution itself.

A much earlier article by Simon on the design of optimal linear filters correctly states [p.440] the problem bedevilling FST. After noting the saddle point property, Simon concludes

> "...it appears that straight-forward application of the calculus of variations to the filter design problem leads to the prescription of an unstable filter, and hence is not practicable ...[This] is related to the fact that the Euler equations give only a necessary and not a sufficient condition for a minimum. Hence it does not follow that a path ... that satisfies [the Euler equations] will thereby minimize [the performance functional]. The specification of appropriate initial and terminal conditions [is necessary] to guarantee a bona fide minimum...".

This chapter demonstrates that it is ~~precisely~~ the straightforward application of the calculus of variations without appropriate initial and terminal conditions that promotes the spurious conflict between optimality and stability alleged by FST.

Chapters VI and VII utilise the Phillips regulator, correctly specified, as a context for investigating some basic issues in the theory of optimal stabilisation. Thus chapter VI, immediately following, considers the effect of control lag on the design of optimal stabilisation policy for the Phillips model; and chapter VII analyses certain degrees of freedom available in the specification of the optimal Phillips regulator.

CHAPTER VI

OPTIMAL STABILISATION WITH A LAGGED INSTRUMENT

A central feature of Phillips' seminal work [1954, 1957] is the explicit treatment of control lags; and it is the principal objective of this chapter to consider the effects of control lag on the design of linear optimal stabilisation policy.

Before proceeding with this task, the concept of control lag, as employed by Phillips and exposited by Allen [1960, p.269], [1968, p.352], requires clarification. The definition used by Phillips appears to allow several interpretations; section 6.1 therefore attempts to specify the concept precisely in terms of the inside, intermediate, and outside lag concepts. Section 6.2 then examines the technical implications of policy lag for the optimal Phillips regulator of chapter V. To facilitate acquisition of general rather than numerical solutions, use is made of an infinite horizon assumption, which also accords with the implicit horizon assumption used by Phillips. The effect of the control lag on the magnitude and the timing of the optimal controller is subsequently studied in section 6.3.

To this point, the effects of policy lag are passively allowed for in designing an optimal controller - no attempt is made to actively modify these effects. But the introduction of policy lag causes concern for the relation between optimal policy and the incidence of oscillation. Section 6.4 therefore isolates conditions for the emergence and the prevention of policy-induced oscillations. Finally, the arguments and conclusions of the chapter are summarised in section 6.5.

6.1 CONTROL LAGS AND POLICY DESIGN

Phillips [1954, p.294] defines the concept of policy lag as follows:[1]

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1 Underlining added.

> "...The amount by which aggregate demand would be changed as a direct result of the stabilisation policy ... if the policy were to operate without time lag will be called the potential policy demand, and the amount by which aggregate demand is in fact changed at any time as a direct result of the policy will be called the actual policy demand ...
>
> The actual policy demand will usually be different from the potential policy demand, owing to the time required for observing changes in the (target) error, adjusting the correcting action accordingly and for the changes in the correcting action to produce their full effects ...".

Writing in 1948 on the desirability of automatic as opposed to discretionary stabilisation policy, Friedman [pp.344-77] divided the total lag between an unpredicted disturbance and offsetting policy into three separate lags: a recognition lag, a decision lag, and an effect lag. It is clear from the underlined rationale for the policy lag, that Phillips regards it as the sum of these three lags.

Following the analyses of the lags operative in monetary and fiscal policy by Kareken & Solow and Ando & Brown, the total policy lag in any stabilising action, whether monetary or fiscal, is now customarily defined as the sum of the inside, intermediate, and outside lags. The inside lag is the interval of time between the need for and implementation of policy; and is the sum of the recognition (observation) lag and the decision (administrative) lag. The recognition lag is due to the inevitable delay in statistically monitoring economic performance and hence in realising that a stabilisation problem does exist. The decision lag measures the time required to select appropriate policy action, to obtain administrative approval for it, and to construct the bureaucratic machinery for its implementation. The intermediate lag stems from the relation between proximate and ultimate instruments - a distinction defined in Rowan [p.8]. Policy action is taken by adjusting proximate instruments, those instruments directly controlled by the policy-maker, in order to affect ultimate instruments, those instruments not directly controlled by the policy-maker. In turn, these ultimate instruments affect the designated targets. A typical monetary policy example of this chain is the link between banking sector reserves, the money supply, and aggregate demand. A similar chain in fiscal policy is not as obvious, although it is possible to regard the central policy decision as the proximate instrument, the actual appearance of policy demand as the ultimate instrument, with the difference arising from an implementation or bureaucratic lag. Since ultimate instruments intermediate between proximate instruments and specified targets, the lag between movements in proximate and ultimate instruments is defined as the intermediate lag. The outside lag is the lag between movement of targets in response to movement of ultimate instruments.

What is the relevance of these three lags to the policy lag concept employed by Phillips? Consider the schematic representation in Figure 6.1. An unpredicted disturbance occurs at point A in time,

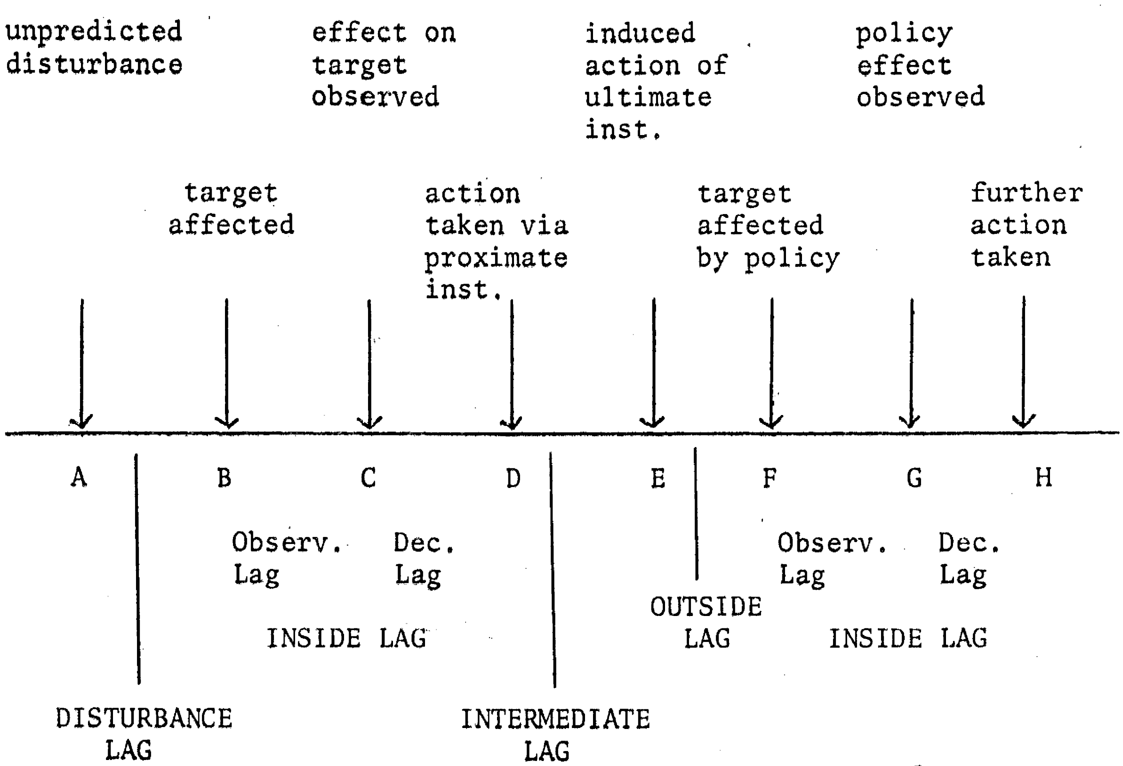

Figure 6.1

Dynamic Lag Sequence in Stabilisation Policy

with the effect on the target variable first appearing at point B. Because of the observation lag BC, this effect is not observed by the policy-maker until point C. During the period CD of the decision lag, an appropriate policy is specified and approved; action then being instituted via the proximate instrument at point D. Such action subsequently induces movement in the ultimate instrument at point E, which affects the target at point F. But, again because of the observation lag, the policy effect on the target is not observed until point G, at which stage new action is decided for point H. It is this dynamic lag sequence that Phillips wishes to model prior to designing a stabilisation policy.

Applying rational decision techniques in this framework, whether classical or optimal, requires recognition of several factors. First, the need for stabilisation policy is assumed to arise from a single unpredicted disturbance (although the policy designed must allow for further such disturbances). Second, during the period CD of the first decision lag, a policy is designed with a specific horizon in mind. Point D then becomes the control origin (t=0). The stabilisation objective is to offset, over the defined horizon, the effects of the single disturbance on the target variable. Third, since the stabilisation policy is designed in a single step CD in response to a single disturbance, the second decision lag GH is irrelevant to the stabilisation problem. Thus the crux of the stabilisation problem is to model the lag sequence represented in Figure 6.2:

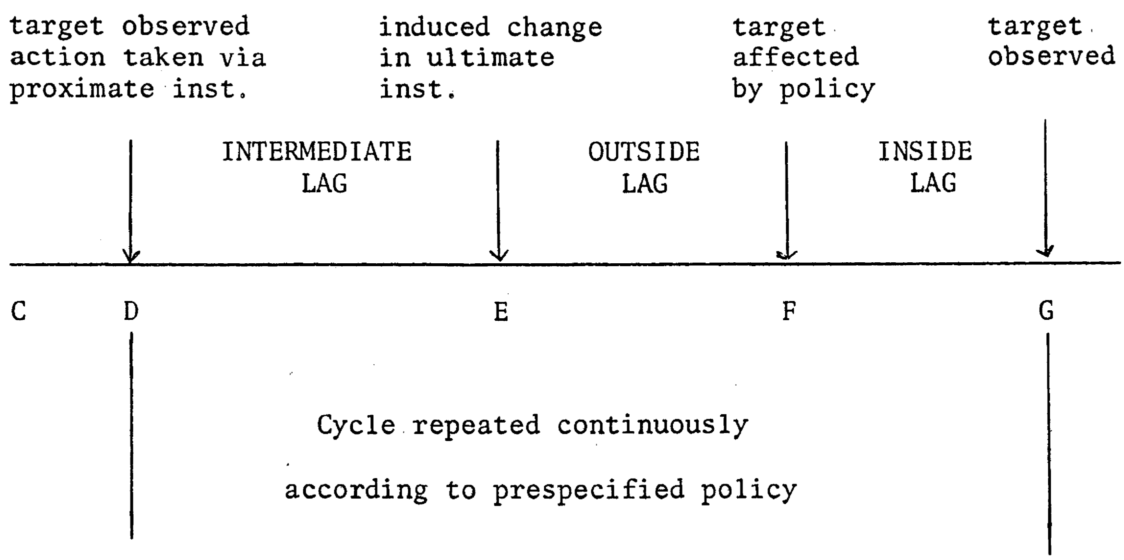

Figure 6.2

Lags and Policy Design

Here the inside lag FG is purely an observation lag. Since the stabilisation problem will be posed in the continuous domain, it is assumed following Phillips [1957,p.267] "that the regulating authorities are able to make continuous adjustments in the strength of the correcting action they take". Hence the inside and intermediate lags, provided they exist, operate at every point of time over the stabilisation period $[0,\infty]$.

To model the lags of Figure 6.2 in the single-target, single-instrument Phillips model, consider equations (1) to (4) below:

$$\dot{y} = -swy + wg^* \quad \Leftrightarrow \quad y = \frac{w}{D+sw}\, g^*, \tag{1}$$

$$g^* = \frac{\alpha}{D+\alpha}\, g, \qquad \alpha > 0, \tag{2}$$

$$g = K(D)y^*, \tag{3}$$

$$y^* = \frac{\eta}{D+\eta}\, y, \qquad \eta > 0. \tag{4}$$

Equation (1) is the first-order Phillips model, now relating the target $y(t)$ to the ultimate instrument $g^*(t)$: the operator $w/(D+sw)$ is therefore the outside lag operator. Equation (2) stipulates that the ultimate instrument $g^*(t)$ responds in lagged fashion to the proximate instrument $g(t)$, so that $\alpha/(D+\alpha)$ models the intermediate lag dynamics. Because of the inside observation lag and the likely occurrence of further unknown disturbances, the policy-maker cannot use the current target $y(t)$ as feedback but must use the currently observed target $y^*(t)$, as stated in (3). The operator $K(D)$, designed by the policy-maker in the period (CD) of the decision lag, is a polynomial in the differential operator $D \equiv d/dt$, reflecting the knowledge that linear stabilisation policy, in the absence of step disturbances, will be constructed from proportional, derivative, and higher-order feedbacks[2]. Equation (4) specifies the lag between the currently observed target $y^*(t)$ and the current target value $y(t)$: the operator $\eta/(D+\eta)$ thus being the inside lag operator. The block diagram associated with these equations is given in Figure 6.3.

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[2] If step rather than impulse disturbances occur, integral feedback is also necessary - see Phillips [1954] and chapter VII below.

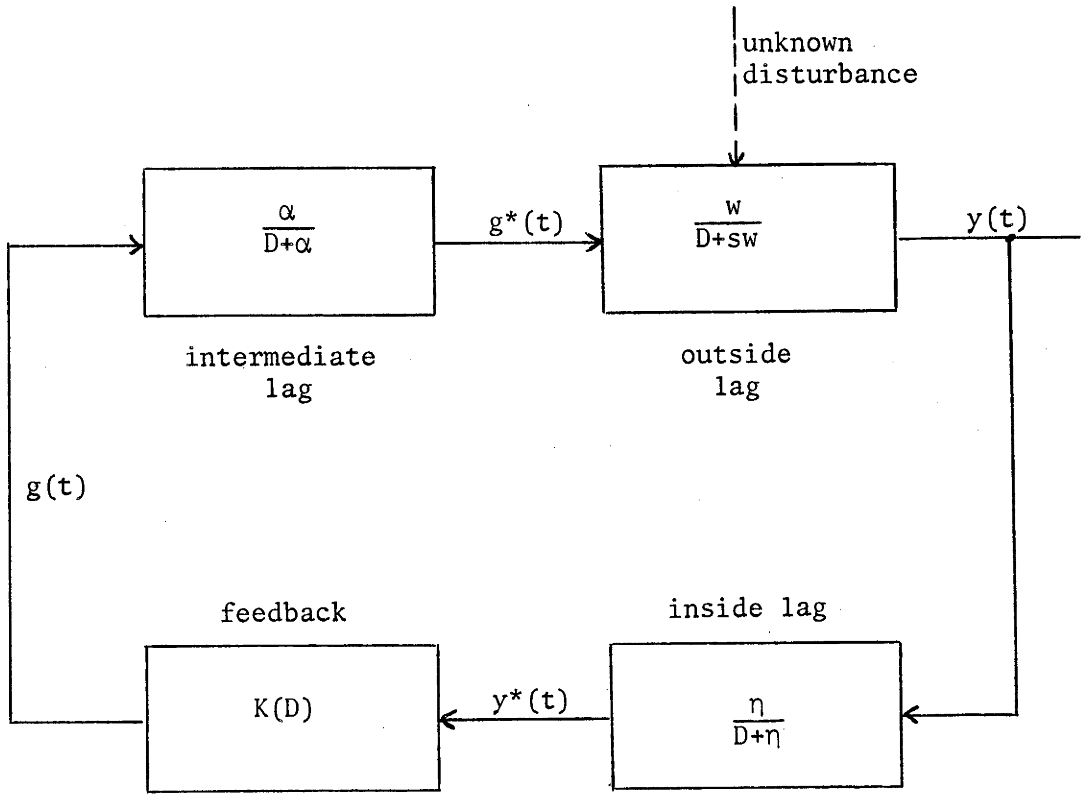

Figure 6.3

Lags in a Single-Target Single-Instrument Model

Returning to Phillips' definition of policy lag, potential policy demand is said to differ from actual policy demand because of (i) observation lag, (ii) decision lag, and (iii) effect lag; or because of the inside lag and the outside lag. Reconciling this interpretation with Figure 6.3 produces several difficulties. Since the intermediate lag is justifiably ignored initially, being part of the system dynamics rather than the policy dynamics, $g(t)$ and $g^*(t)$ are identical, and measure actual policy demand. What then is potential policy demand? If potential policy demand is a dynamic concept[3], both it and actual policy demand share the outside lag in common (focussing on the causal rather than the feedback link from instrument to target); whereas according to Phillips this lag is one of the two factors differentiating

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[3] Phillips intends it as such [1957, p.267]: "For a given correction lag the problem reduces to that of finding the most suitable way of relating the potential policy demand to the error in production".

these two policy demands. If this is a mere oversight, so that the two concepts differ only because of the inside lag, potential policy demand still remains to be identified in terms of Figure 6.3.

Now equations (1) and (2) without intermediate lag reduce to

$$\dot{y} = -swy + wg \quad \Longleftrightarrow \quad y = \frac{w}{D+sw}\, g, \tag{5}$$

while equations (3) and (4) may be redefined to

$$g = K(D)\frac{\eta}{D+\eta}\, y = \frac{\eta}{D+\eta}\, \hat{g}, \tag{6}$$

and

$$\hat{g} = K(D)y. \tag{7}$$

These equations are diagrammed as Figure 6.4. The variable $\hat{g}$,

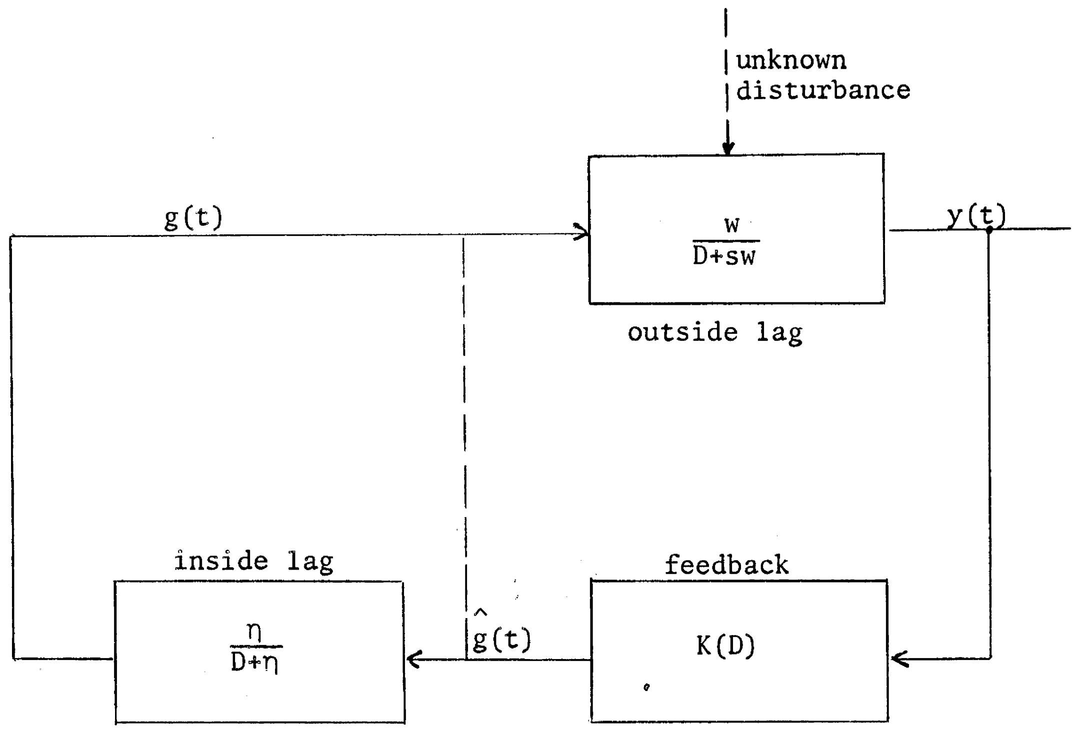

Figure 6.4

Potential and Actual Policy Demands

defined as potential policy demand, is chosen as a function of current target feedback y, with actual policy demand g exercised as a function of past $\hat{g}$ because of the inside lag. Potential policy demand $\hat{g}$ is a purely definitional and unobservable variable, serving to emphasise the conceptual effects of introducing control lag. Thus the segment to the right of the dotted line in Figure 6.4 is clearly a schematic representation of the design problem without control lag (the Phillips model of chapter V); the segment to the left representing the introduction of such a lag. If there were no control lag (that is, inside lag), the variable $\hat{g}$ would generate the observed policy; and this appears to be the sense of Phillips' definition of potential policy demand, quoted above. Ambiguity occurs because, without control lag, both actual and potential policy demands are identical; but with control lag, potential policy demand is designed in full recognition of this lag and cannot be interpreted therefore as that policy appropriate for a no-lag context.

Given these qualifications, the system (5), (6), and (7) depicted in Figure 6.4 is taken as the representation of the Phillips multiplier-accelerator model with control lag. Now considering the equivalent representation of Figure 6.3, after removing the intermediate lag, it is tempting to formulate the design problem as follows. Assuming the inside and outside lag structures to be modelled accurately, why not invert the inside lag operator, converting currently observed target values y^* into actual target values y, thus overcoming the observation lag? This would provide the design problem of Figure 6.5; which is the first-order Phillips model, and therefore a simpler design problem. The underlying assumption, however, is that it is necessary to design a controller that is optimal with respect to the sequence of ongoing and unpredictable disturbances. To invert the inside lag after one such disturbance, in order to construct y(t) as the feedback signal, is to assume that no further disturbances occur over the stabilisation period. This contradicts the purpose of designing what is essentially an automatic stabilising mechanism: although nothing can be done to offset the unknown impulses as they originate, policy is to operate continuously against their propagation.

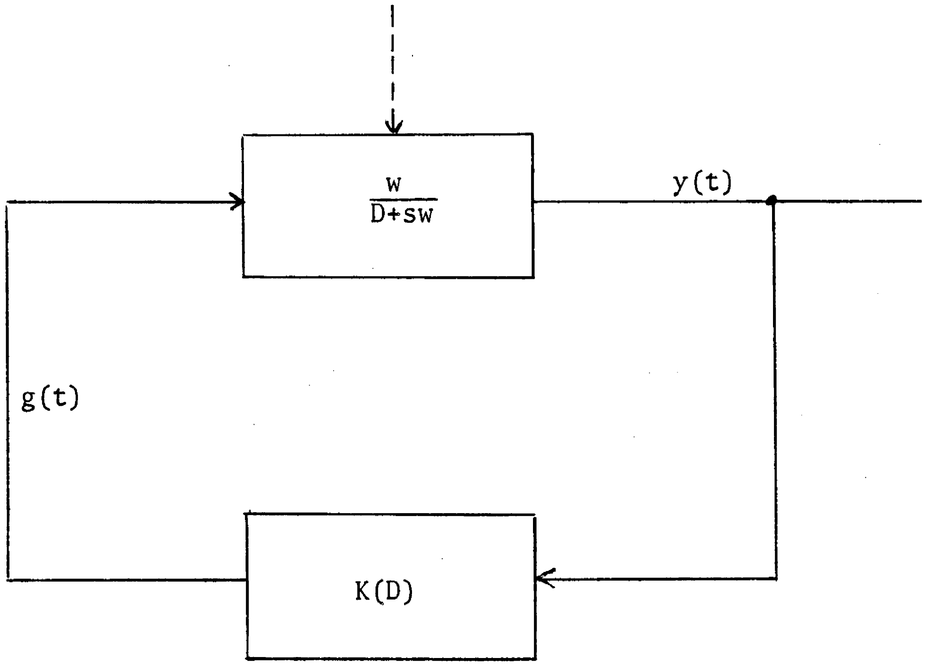

Figure 6.5
Design Problem with Inside Lag Inversion

One final comment must be made concerning the specification of the policy lag as a distributed lag function of the first-order exponential type. The justification for a distributed lag in the adjustment of supply to demand, given by Phillips [1954, p.291], reasonably relies on the aggregation of large numbers of individual responses to produce a smooth aggregate response distribution. This justification does not appear relevant to the inside lag in policy-making. A possible rationale is that the first-order exponential lag is simply a convenient approximation to a pure time delay and avoids the use of mixed difference-differential equations. A reasonable approximation to pure delay can be obtained with an n^{th} order exponential lag for n sufficiently large - Tustin [p.47] - and taking n=1 is a compromise for dimensional simplicity. Since Phillips does consider more realistic lag specifications [1957, pp.269-72], this argument will be adopted in the following. Although this is a dubious compromise between simplicity and accuracy, the previous interpretation of Figure 6.4 does not allow the alternative hypothesis that a first-order control lag is a behavioural description of policy reaction to a discrepancy between actual and potential policy demands, as Phillips argues [1954, p.294].

6.2 PHILLIPS REGULATOR WITH CONTROL LAG

Taking equations (5) to (7) as the specification of the Phillips multiplier-accelerator model with a lagged instrument, the following stabilisation problem may be considered. Supposing the policy-maker to have deterministic knowledge of the economic system apart from the impulse disturbances, what effect does the policy lag have (i) on the design of stabilisation policy, on the structure of K(D); and (ii) on the performance of the stabilised target y(t)? These comparative questions are to be answered with reference to policy design and target performance in the Phillips regulator of the previous chapter.

The first question arising from the introduction of policy lag is whether the property of dynamic controllability is altered. Now the lagged policy model may be written[4]

$$\begin{bmatrix} \dot{y} \\ \dot{g} \end{bmatrix} = \begin{bmatrix} \pm w & 0 \\ 0 & \eta \end{bmatrix} \begin{bmatrix} -s & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y \\ g \end{bmatrix} + \begin{bmatrix} \pm w & 0 \\ 0 & \eta \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{g}, \tag{8}$$

or

$$\dot{x}(t) = \Gamma A x(t) + \Gamma b \hat{g}(t). \tag{9}$$

In this state formulation, A is the static structural matrix, b is the instrument coefficient vector, and Γ is the diagonal matrix of adjustment speeds. This system is dynamically controllable iff, from (3.47),

$$\rho(Q) = \rho[\ \Gamma b \quad (\Gamma A)(\Gamma b)\ ] = 2. \tag{10}$$

Since $|Q| = \pm w\eta^2$, the lagged policy model is invariably controllable, provided only that the matrix of adjustment speeds is nonsingular; and this is satisfied by definition. Policy lag does not therefore affect the property of dynamic controllability.

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4 Where two signs occur, the top sign refers to a naturally stable system (w > 0); the bottom sign, to a naturally unstable system (w < 0). Hence w = |w| throughout this chapter. This explicit treatment is of interest in the sequel.

Solution of the stabilisation problem with the state representation (8) will determine the instantaneous policy $\hat{g}(t)$ as a function of the state, here the current levels of the target $y(t)$ and the lagged policy variable $g(t)$. But $g(t)$ is determined through the lag operator (6) as a function of past levels of $\hat{g}(t)$, so that current stabilisation policy will be determined by feedbacks from the current target and past policy. It is convenient, however, to consider stabilisation policy as being determined from target feedback alone, as follows. The system (8) of two first-order differential equations reduces to one second-order equation

$$\ddot{y}(t) + (\eta \pm sw)\dot{y}(t) \pm \eta swy(t) = \pm\eta w\hat{g}(t). \tag{11}$$

Applying from (3.32) the canonical transformation,

$$\begin{aligned} y(t) &= z_1(t) \\ \dot{y}(t) &= z_2(t) = \dot{z}_1(t), \end{aligned} \tag{12}$$

to (11) provides the alternative state representation:

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \mp\eta sw & -(\eta \pm sw) \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \pm\eta w \end{bmatrix} \hat{g}(t). \tag{13}$$

Hence the state vector $z = [z_1(t) \quad z_2(t)]^T = [y \quad \dot{y}]^T$ implies that the stabilisation feedback $K(D)$ will be specified as a function of the current level and rate of change of the target.

Coming to the provision of a performance functional for this stabilisation problem, equation (7) and Figure 6.4 focus attention on the target $y(t)$ and potential policy demand $\hat{g}(t)$; so that this functional is taken as

$$\underset{\hat{g}}{\text{MIN}}\ W = \tfrac{1}{2}\int_0^T [y^2(t) + \phi\hat{g}^2(t)]\, dt, \quad \phi > 0. \tag{14}$$

The minimisation of target deviations subject to the constraint that instantaneous stabilisation expenditure is not excessive seeks to ensure, via the lag operator, that lagged or actual stabilisation expenditure is also not excessive. Expressing this performance criterion in terms of the state vector z(t) of (13), the optimal stabilisation problem for the Phillips model with lagged instrument may be summarised as:

Model II

$$\min_{\hat{g}} W = \tfrac{1}{2}\int_0^T [z^T(t)Vz(t) + \phi\hat{g}^2(t)]\,dt, \quad \phi > 0, \tag{15}$$

subject to

$$\dot{z}(t) = Az(t) + b\hat{g}(t), \quad z(0)=z_0, \quad z(T) \text{ free} \tag{16}$$

$$T \text{ fixed}, \; \hat{g}(t) \text{ unconstrained}, \tag{17}$$

where

$$\left\{\begin{array}{l} V = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} 0 & 1 \\ \mp\eta sw & -(\eta\pm sw) \end{bmatrix} \\ b = \begin{bmatrix} 0 \\ \pm\eta w \end{bmatrix}, \; z(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}. \end{array}\right. \tag{18}$$

Model II is a two-state analogue of the fixed time free endpoint regulator (Model I) considered in the previous chapter; and its solution follows in the same manner. Evaluating the canonical equations $\dot{z}(t) = \partial H/\partial p$, $\dot{p}(t) = -\partial H/\partial z$, from the Hamiltonian

$$H(z,p,\hat{g}) \equiv \tfrac{1}{2}z^TVz + \tfrac{1}{2}\phi\hat{g}^2 + p^T(Az+b\hat{g}), \tag{19}$$

where the instrument and costate are connected by

$$\frac{\partial H}{\partial \hat{g}} = 0 \quad \Longleftrightarrow \quad \hat{g}(t) = -\phi^{-1}b^Tp(t), \tag{20}$$

provides the canonical system

$$\begin{bmatrix} \dot{z}(t) \\ \hline \dot{p}(t) \end{bmatrix} = \left[\begin{array}{c|c} A & -\frac{bb^T}{\phi} \\ \hline -V & -A^T \end{array}\right] \begin{bmatrix} z(t) \\ \hline p(t) \end{bmatrix}, \quad \begin{bmatrix} z(0)=z_o \\ \hline p(T)=0 \end{bmatrix}. \tag{21}$$

In the parameters of the model (A, b, V, ϕ), this system is

$$\begin{bmatrix} \dot{z}(t) \\ \hline \dot{p}(t) \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ \mp\eta sw & -(\eta\pm sw) & 0 & -\frac{\eta^2 w^2}{\phi} \\ \hline -1 & 0 & 0 & \pm\eta sw \\ 0 & 0 & -1 & \eta\pm sw \end{array}\right] \begin{bmatrix} z(t) \\ \hline p(t) \end{bmatrix}. \tag{22}$$

Solutions for the optimal target and instrument are obtained by formulating the second-order matrix Riccati equation; just as the first-order Riccati equation (5.20) was derived for model I. Computational difficulties immediately obtrude because this Riccati solution requires the 4x4 transition matrix of (22) for all finite stabilisation horizons. To avoid these difficulties and to conform to the assumption used by Phillips, an infinite horizon will be assumed.

The canonical system (22) for the finite horizon model does contain some interesting qualitative information. The incidence of zero elements in the Hamiltonian matrix of (22) enables the characteristic equation, $|\lambda I-H|=0$, to be found with relative ease as

$$X(\lambda) = \lambda^4 - (\eta^2+s^2w^2)\lambda^2 + \eta^2w^2(s^2+\phi^{-1}). \tag{23}$$

That $X(\lambda)$ is even-powered in (λ) complies with corollary 2.1, which states that all characteristic equations associated with optimal regulator models are purely even-powered. Hence

$$\begin{cases} X(\delta) = \delta^2 - (\eta^2+s^2w^2)\delta + \eta^2w^2(s^2+\phi^{-1}) \\ \lambda = \pm\, \delta^{\frac{1}{2}} \\ \delta = \tfrac{1}{2}(\eta^2+s^2w^2) \pm \tfrac{1}{2}[(\eta^2+s^2w^2)^2 - 4\eta^2w^2(s^2+\phi^{-1})]^{\frac{1}{2}}. \end{cases} \tag{24}$$

Thus if the roots δ are (i) real, they are necessarily positive, and the eigenvalues λ are therefore real and symmetric on the real line; and (ii) complex, then the eigenvalues λ are also complex and symmetric with respect to the real and complex axes. In either case, corollary 2.2, the saddle point property of optimal eigenvalues, is illustrated. As observed with respect to the FST paradox only the negative eigenvalues of (23) survive in the asymptotic model.

Asymptotic solutions ($T=\infty$) for model II are determined as follows. From theorem 1.1, the Hamiltonian system (22) satisfies the matrix Riccati equation

$$\dot{K} = - K(t)A - A^T K(t) + K(t)\frac{bb^T}{\phi} K(t) - V, \tag{25}$$

with boundary condition $K(T)=0$ determined from the transversality condition. Theorem 1.2 then asserts that the real, symmetric, positive definite solution of the matrix equation ($\dot{K}=0$):

$$- \bar{K}A - A^T\bar{K} + \bar{K}\frac{bb^T}{\phi}\bar{K} - V = 0 \tag{26}$$

specifies the optimal feedback coefficients for the asymptotic controller. Utilising this solution and the state-costate relation $p(t)=\bar{K}z(t)$ in equations (16) and (20) subsequently generates the asymptotic control and state solutions

$$\hat{g}(t) = -\phi^{-1}b^T\bar{K}z(t), \tag{27}$$

$$\dot{z}(t) = [A - \frac{bb^T}{\phi}\bar{K}]z(t), \quad z(0)=z_o. \tag{28}$$

The Riccati solution is significantly implanted in the solution structure. In the present two-state model, $\bar{K}$ is a symmetric matrix defined by

$$\bar{K} = \begin{bmatrix} k_o & k_1 \\ k_1 & k_2 \end{bmatrix}. \tag{29}$$

Using this definition (and the model specification) in equations (27) and (28) supplies

$$\hat{g}(t) = \mp\phi^{-1}\eta w[k_1(\pm) \quad k_2(\pm)]z(t), \tag{30}$$

$$\begin{cases} \dot{z}(t) = G_{II}z(t), \quad z(0)=z_o \\ G_{II} = (A-\frac{bb^T}{\phi}\bar{K}) = \begin{bmatrix} 0 & 1 \\ \mp\eta sw - \frac{\eta^2w^2}{\phi}k_1, & -(\eta\pm sw)-\frac{\eta^2w^2}{\phi}k_2 \end{bmatrix}. \end{cases} \tag{31}$$

Thus only the pair (k_1, k_2) of (29) are required; not the triple (k_o, k_1, k_2). From Appendix VIa (pp.270-2 below), these two solutions are:

$$\begin{cases} k_1 = \dfrac{\theta \mp s}{\phi^{-1}\eta w}, & \theta = (s^2+\phi^{-1})^{\frac{1}{2}}; \\ k_2 = \dfrac{\gamma - (\eta\pm sw)}{\phi^{-1}\eta^2w^2}, & \gamma = (\eta^2+s^2w^2+2\eta w\theta)^{\frac{1}{2}}. \end{cases} \tag{32}$$

Hence the optimal control (30) is:

$$\hat{g}(t) = \mp[\ \theta \mp s, \quad \frac{\gamma - (\eta\pm sw)}{\eta w}\ ]z(t), \tag{33}$$

and the optimal closed-loop matrix G_{II} of (31) is:

$$G_{II} = \begin{bmatrix} 0 & 1 \\ -\eta w\theta & -\gamma \end{bmatrix}. \tag{34}$$

With G so defined, an explicit solution of the 2x2 differential system (31) is all that is required to completely specify general solutions. This solution may be readily found but does not contribute to the following analysis, and is not therefore presented.

6.3 INSTANTANEOUS AND LAGGED CONTROLLER STRUCTURES

Given these optimal solutions to model II, this present section investigates (i) the structure of the optimal controller and its relation to that of model I; and (ii) the relation holding between the instantaneous and lagged controllers.

The instantaneous controller (33) may be written more revealingly as

$$\hat{g}(t) = \mp \frac{1}{\eta w} [h(\pm) + m(\pm)D]y(t), \tag{35}$$

where

$$\begin{cases} h(\pm) = \eta w(\theta \mp s) > 0, \\ \\ m(\pm) = \gamma - (\eta \pm sw) > 0. \end{cases} \tag{36}$$

The differential effects of natural stability properties may be noted immediately. From (35), naturally stable systems employ negative feedback; naturally unstable systems, positive feedback. From (36), the levels of feedback are higher for naturally unstable systems. But although the optimal controllers differ in sign and magnitude, they both generate, by (34), identical closed-loop state dynamics. Natural instability simply requires more controller effort to produce the same target response.

The optimal instantaneous controller (35) consists of two components - proportional feedback and derivative feedback - as illustrated in Figure 6.6. To develop the logic of the defining coefficients h and m, it is necessary to consider the stability properties of both the pre-optimal and the optimal structures. For effective dynamic stabilisation, the controller, whatever the method of its design, must be directly related to the stability properties of the stabilisation model. Thus suppose that the policy-maker, in a travesty of Phillips' classic dictums, is content to pursue a dynamic policy of constant expenditure. Then information on the stability of

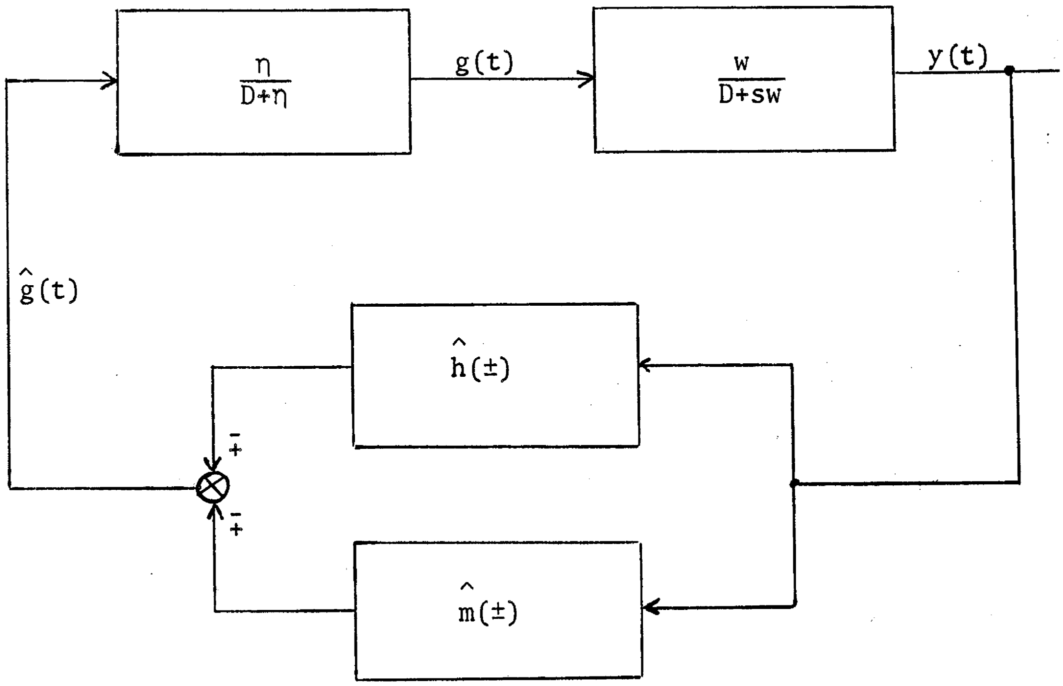

Figure 6.6

Phillips Regulator with Lagged Controller

this pre-optimal system, given by (13) for $\hat{g} = \bar{g}$, i.e.,

$$\mathring{z}(t) = Az(t) + b\bar{g}, \tag{37}$$

is contained in the characteristic equation

$$\nu^2 - (\mathrm{tr}A)\nu + |A| = 0, \tag{38}$$

possessing the eigenvalues

$$\nu_1 = \mp sw, \quad \nu_2 = -\eta. \tag{39}$$

A constant-expenditure policy has no effect on the location of eigenvalues and is ineffectual as a dynamic policy. What this limiting case serves to underline is the pure separation of the stability properties of the pre-optimal system between the natural dynamics ($\mp$sw), the outside lag, and the control lag dynamics ($-\eta$), the inside lag. Since $\eta > 0$, the inside lag does not make a naturally stable system unstable or a naturally unstable system even more unstable. Thus the considerations justifying the necessity for control in the first-order model are also relevant to the second-order model. The significance of the inside lag derives from the introduction of an additional mode of dynamic response. Although the pre-optimal system can never be oscillatory (39), the optimal system can be: a problem receiving attention in the following section.

Moving from this situation of complete impotence to the optimal situation given in (34), optimal stability properties are summarised in the characteristic equation

$$\nu^2 - (\mathrm{tr}G_{II})\nu + |G_{II}| = 0, \tag{40}$$

with eigenvalues[5]

$$\nu_i = -\tfrac{1}{2}\gamma \pm \tfrac{1}{2}(\gamma^2-4\eta w\theta)^{\frac{1}{2}}. \tag{41}$$

Because γ and θ are functions of ϕ^{-1}, the location of eigenvalues is directly controllable - a logical prerequisite for dynamic stabilisation.

Comparison of the two control structures via their characteristic equations (38), (40) establishes

$$\begin{cases} m(\pm) = \mathrm{tr}(A - G_{II}) = -(\eta \pm sw) + \gamma \\ \\ h(\pm) = |G_{II}| - |A| = \eta w\theta \mp \eta sw. \end{cases} \tag{42}$$

Apart from a constant factor $1/\eta w$, the optimal proportional feedback coefficient $h(\pm)$ is determined as the difference between the optimal and the pre-optimal eigenvalue products; and the optimal derivative feedback coefficient $m(\pm)$ is determined as the difference between the pre-optimal and the optimal eigenvalue sums. Proportional feedback operates on the determinant, and derivative feedback operates on the trace, of the system matrix G_{II}. This understanding of feedback design will be used in the following section to design policy to combat oscillations.

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5 The assertion above that the negative eigenvalues (24) are the asymptotic eigenvalues (41) may be verified.

Writing the optimal controller (35) as

$$\begin{cases} \hat{g}(t) = \mp\ [\hat{h}(\pm) + \hat{m}(\pm)D]y(t), \\ \hat{h} = h/\eta w = \theta \mp s; \quad \hat{m} = m/\eta w, \end{cases} \tag{43}$$

the lagged stabilisation model returns an optimal controller which includes a proportional feedback $(\hat{h})$ identical with that of the unlagged model[6]. Thus the emergence of derivative feedback is entirely the product of the inside lag. From equations (32), (36), (43), the distribution of parameters between the two feedback coefficients is

$$\hat{h} = \hat{h}(s, \phi^{-1}); \quad \hat{m} = \hat{m}(s, w, \eta, \hat{h}). \tag{44}$$

There is, therefore, a very specific task orientation in the optimal controller: which accords with the dynamic separation noted in (39). The derivative feedback is explicitly moulded to the dynamic structure of the policy lag, with

$$\lim_{\eta\to\infty} \hat{m}(\pm) = \lim_{\eta\to\infty} \frac{\left(1+\frac{s^2w^2}{\eta^2} + 2\frac{w\theta}{\eta}\right)^{\frac{1}{2}} - 1 \mp \frac{sw}{\eta}}{w} = 0. \tag{45}$$

As the policy lag shortens and finally vanishes, the role of derivative feedback decreases and vanishes also. Alternatively, the faster the policy speed of response (η), the smaller is the required level of derivative feedback.

In summary, the logic of the optimal instantaneous controller is that (i) the proportional feedback of the first-order model is retained to control the outside lag dynamics; and (ii) a derivative feedback is applied to cater for the inside lag dynamics. Coordination of these two feedbacks is achieved through the appearance of the performance parameter ϕ^{-1} in each. That the second-order model tends to the first-order model as the policy speed of reaction becomes infinite is of relevance to the analysis of dynamic adjustment mechanisms given by

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6 See the naturally stable controller (5.23).

Sargan, and exposited by Bergstrom [pp.107-12], and Newman [pp.18-21]. But this is a separate topic and is pursued no further in this thesis.

The optimisation analysis generates an instantaneous controller $\hat{g}(t)$ that is implemented, and therefore observed, as the lagged controller $g(t)$. The block diagram of this lagged controller appears in Figure 6.7. How does the lag operator $\eta/(D+\eta)$ affect $g(t)$? The answer to this question is provided by the solution of the differential equation

$$(D+\eta)g(t) = \eta\hat{g}(t), \tag{46}$$

obtained from (7). Now from (43), the instantaneous controller $\hat{g}(t)$ may be written (taking only the naturally stable case):

$$\begin{aligned}\hat{g}(t) &= -\hat{m}\left(\frac{\hat{h}}{\hat{m}} + D\right)y(t)\\ &= -\frac{\varepsilon}{\eta}(\delta + D)y(t), \quad \varepsilon = \eta\hat{m}, \quad \delta = \hat{h}/\hat{m}.\end{aligned} \tag{47}$$

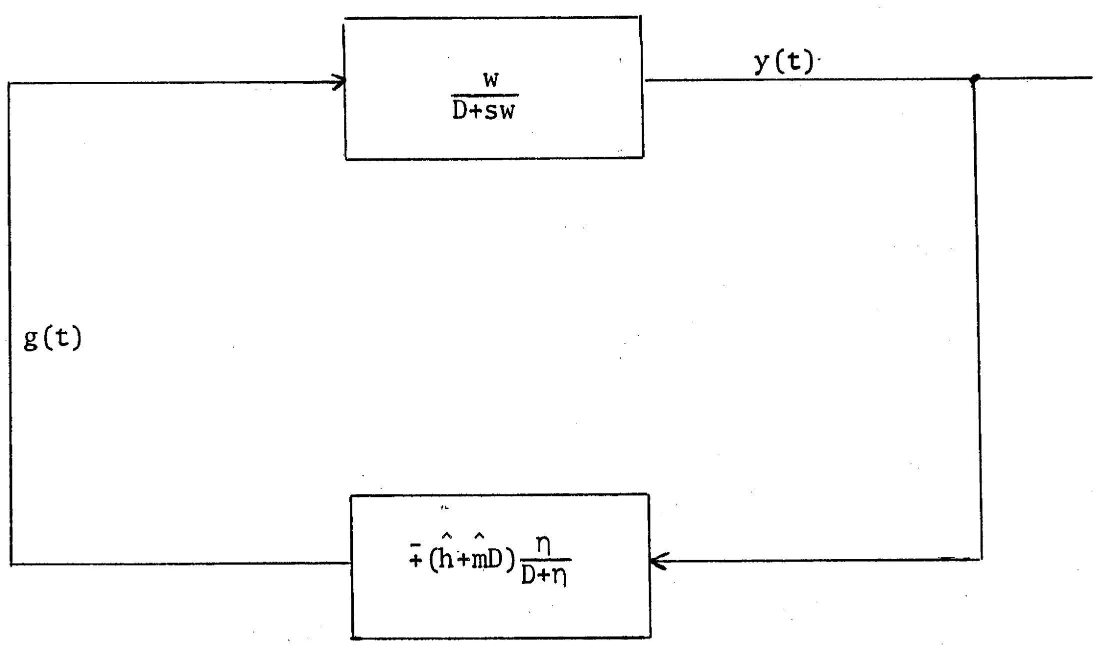

Figure 6.7

Lagged Controller Structure for Phillips Regulator

Hence the optimal lagged controller is the solution $g(t)$ of

$$(D+\eta)g(t) = -\varepsilon(\delta+D)y(t). \tag{48}$$

Since the target variable y(t) is the component $z_1(t)$ of the linear constant system (31), (34), y(t) will possess a solution of the form

$$y(t) = ae^{\alpha t}y(0) + be^{\beta t}\dot{y}(0), \tag{49}$$

where α, β are the optimal eigenvalues (41), and where a,b are coefficients determined from the initial conditions. Thus

$$\dot{y}(t) = \alpha ae^{\alpha t}y(0) + \beta be^{\beta t}\dot{y}(0), \tag{50}$$

and the differential equation (48) becomes

$$(D+\eta)g = -\varepsilon[(\delta+\alpha)ae^{\alpha t}y(0)+(\delta+\beta)be^{\beta t}\dot{y}(0)]. \tag{51}$$

A particular solution to (51) is[7]

$$S = -\varepsilon\{\frac{(\delta+\alpha)}{(\eta+\alpha)}ae^{\alpha t}y(0) + \frac{(\delta+\beta)}{(\eta+\beta)}be^{\beta t}\dot{y}(0)\}, \tag{52}$$

and the complete solution is, for given initial condition g(0),

$$g(t) = (1 - e^{-\eta t})S + e^{-\eta t}g(0). \tag{53}$$

Merely to simplify interpretation of (53), suppose that $\dot{y}(0)=0$, so that

$$\begin{cases} y(t) = ae^{\alpha t}y(0) \\ \hat{g}(t) = -\frac{\varepsilon}{\eta}(\delta+\alpha)y(t) \\ S = -\varepsilon\frac{\delta+\alpha}{\eta+\alpha}y(t) = \frac{\eta}{\eta+\alpha}\hat{g}(t). \end{cases} \tag{54}$$

Given the inside lag, the initial condition g(0) hinges on whether the system was being stabilised prior to the implementation of an optimal policy. If so, some nonzero g(0) will exist; otherwise g(0) may be

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[7] See, for example, Allen [1960, pp.725-33].

assumed zero. Taking the second case, again for simplicity, the explicit relation between the optimal instantaneous control (which optimises with respect to given inside and outside lags) and the optimal lagged control is

$$g(t) = (1 - e^{-\eta t}) \frac{\eta}{\eta+\alpha}\hat{g}(t). \tag{55}$$

Thus the lagged control is obtained from the instantaneous control via a damping factor $(1-e^{-\eta t})$ and an amplitude factor $\eta/(\eta+\alpha)$. The damping factor is the cumulative form of the first-order exponential lag; and approaches unity with increasing time. For example, if $1/\eta = 1$, this damping factor is .63 by the end of the first unit period; and for $t > 2/\eta$, $g(t)$ may be approximated by

$$g(t) \doteqdot \frac{\eta}{\eta+\alpha}\hat{g}(t). \tag{56}$$

Since the optimal system is stable, the optimal speed of response (α) is negative; and this amplitude factor may also be negative. In this case, after an initial adjustment period in which the damping factor works itself out, the lagged policy would appear as a linear function of the instantaneous policy, of the same phase but opposite sign. Note that $\lim_{\eta\to\infty} g(t) = \hat{g}(t)$, as required.

6.4 THE POLICY-CYCLE NEXUS

The limited amplitude fluctuations generally characterising the postwar performance of the developed economies have been the result of policy-makers accepting an increasingly sophisticated stabilisation function. As A.H. Hansen remarks, stabilisation policy has so modified dynamic behaviour that "The rocking chair doesn't rock in quite the old familiar way", [1964, p.609]. Yet this achievement itself fosters a problem of fine-tuning: to what extent have these postwar cycles been induced by stabilisation policy? Thus Heller et al argue [p.16]:

> " In trying to maintain their economies in this position [of full employment], governments have relatively narrow room for manoeuvre between ... an unacceptably low level of employment and ... a pressure of demand that creates an inflationary strain on resources ... [To] operate within this 'narrow band' requires skill, foresight and flexibility. Governments have, to a large extent, succeeded in subduing or overcoming the rhythmic

> fluctuations which used to be called the 'trade cycle'. ...
> But mistakes in policy can easily set up a new sort of
> oscillation in the economy, and large countries which make
> them may still cause serious problems for their smaller
> neighbours."

This present section is concerned, in the theoretical context of the optimal Phillips model, with operating in this narrow band without inducing oscillations attributable to policy.

Designing stable policy is not therefore sufficient; it is also necessary to ensure that such policy does not impart a cyclical movement to the target. Consequently, two subgoals of the stabilisation objective must be recognised. The first, which has been stressed so far, is the attainment of desired rates of convergence in target variables through the control of dynamic speeds of adjustment[8]. The second is the prevention of policy-induced oscillation. Given these two subgoals, the stabilisation problem is, to use Mundell's terminology [pp.313-317], a problem not only of stability but also of hyperstability.

Of the set of all dynamic systems of general order (n), the first-order system is pathological in the sense that oscillatory behaviour is precluded. With a classical controller (and still assuming impulse disturbances), such systems remain first-order systems and are nonoscillatory; and with an optimal controller, the saddle point property also necessitates nonoscillatory behaviour. Thus in first-order models, the efficacy of dynamic control is directly manifested in the rate of target convergence: in model I for example, the optimal speed of response ($w\theta$) clearly exceeds the natural speed of response (sw). Optimal solutions to the lagged Phillips model, model II, do however permit either damped oscillatory or exponentially damped responses. Which type of response actually occurs is governed by the influence of ϕ^{-1} on the location of the optimal eigenvalues. This influence is now investigated as a preliminary to the discretionary control of oscillations.

To establish the conditions demarcating damped oscillatory from exponentially damped responses, consider the following analysis of the

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[8] Horizon and endpoint considerations are facets of this subgoal.

time-form of response. The inequality conditions on the discriminant of the optimal characteristic equation (40):

$$\Delta = tr^2 G_{II} - 4|G_{II}| = \gamma^2 - 4\eta w\theta \gtrless 0, \tag{57}$$

define real and complex eigenvalues respectively - or noncyclical and cyclical responses. After simplification,

$$\begin{cases} \Delta = w^2 Q(\rho), \\ Q(\rho) = \rho^2 - 2\theta\rho + s^2, \ \rho = \eta/w > 0. \end{cases} \tag{58}$$

The polynomial $Q(\rho)$ possesses a minimum at $\rho = \theta$ of $-\phi^{-1}$, and roots at $\rho = \theta \pm \phi^{-\frac{1}{2}}$. Complex eigenvalues occur if $Q(\rho) < 0$, or if

$$\theta - \phi^{-\frac{1}{2}} < \frac{\eta}{w} < \theta + \phi^{-\frac{1}{2}}. \tag{59}$$

The restriction (59) is more conveniently written

$$\frac{s}{\theta+\phi^{-\frac{1}{2}}} < \frac{sw}{\eta} < \frac{s}{\theta-\phi^{-\frac{1}{2}}}$$

$$\Longleftrightarrow \tag{60}$$

$$LB < \frac{sw}{\eta} < UB,$$

where the lower and upper boundaries have the following properties:

$$\begin{cases} \dfrac{\partial}{\partial\phi^{-1}}(LB) < 0, & \dfrac{\partial}{\partial\phi^{-1}}(UB) > 0; \\ \lim\limits_{\phi^{-1}\to 0}(LB) = 1, & \lim\limits_{\phi^{-1}\to 0}(UB) = 1. \end{cases} \tag{61}$$

Figure 6.8 graphs the oscillatory condition (60) as a function of the

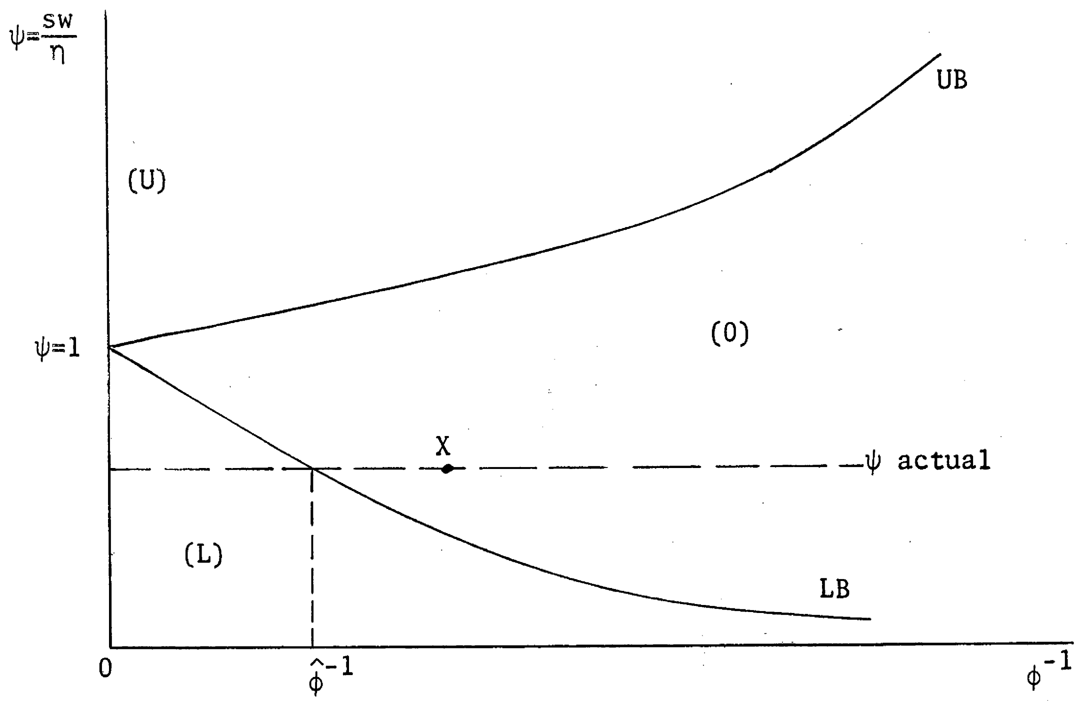

Figure 6.8

Response Regions for Lagged Phillips Regulator

control parameter ϕ^{-1}. With an increasing level of control, there is an ever-widening range of values of the ratio $\psi = sw/\eta$ for which oscillatory target response occurs, as indicated by region (0). In the lagged Phillips model, the probability of oscillatory response therefore varies directly with the level of stabilisation. If oscillatory outcomes are undesirable, this is a perverse result for policy-making.

Figure 6.8 suggests a first method for designing a hyperstable controller. The quotient $\psi = sw/\eta$ is the ratio of the natural speed of response to the policy speed of response (or the ratio of the inside lag to the outside lag). Suppose the parameters of model II are such that ψ has the value appropriate to point X in region (0), implying target oscillation. A sufficient increase (decrease) in policy speed of response would relocate ψ in region (L), (U) - provided ϕ^{-1} is not too large. Variation in policy speed of response is therefore being suggested as a design tool for avoiding policy-induced cycles. But further thought indicates that it is not a useful option, for two reasons. Firstly, given the institutional and informational difficulties inherent in attempts to reduce the policy lag, the brunt of adjustment would have to fall on increases in this lag - the option is therefore asymmetric. Secondly, increasing the policy lag means that nonoscillatory target behaviour can only be achieved at the expense of further delaying policy action.

Yet if this option of moving X vertically out of region (0) is unsatisfactory, so also is the alternative of moving it horizontally out of this region. Since ϕ^{-1} is the strength of control parameter, any reduction in it compromises the rate of convergence to equilibrium. Thus unless the inside and outside lags are fortuitously matched, so that ψ remains outside region (0) for all admissible values of ϕ^{-1}, the problem of oscillatory target behaviour persists. This leads to a rephrasing of the hyperstable design problem. Is it possible to compensate for the occurrence and length of the inside lag so that the optimal response is never oscillatory, whatever the length of the outside lag? Rejecting policy lag variation and performance parameter variation as tools for combatting policy-induced cycles, can an optimal feedback controller be designed to ensure noncyclical response?

Coming to closer grips with this design problem, observe that the optimal system

$$\begin{cases} \dot{z}(t) = G_{II}z(t), \quad z(0)=z_o \\ G_{II} = \begin{bmatrix} 0 & 1 \\ -\eta w\theta & -\gamma \end{bmatrix} \quad \begin{array}{l} \gamma = (\eta^2+s^2w^2+2\eta w\theta)^{\frac{1}{2}}, \\ \theta = (s^2+\phi^{-1})^{\frac{1}{2}}, \end{array} \end{cases} \tag{62}$$

is stable and will therefore be represented by a point in the first quadrant of Figure 6.9, constructed from the characteristic equation (40). Design of a hyperstable controller requires that the optimal response (62) be shifted from point A in the cyclically damped region to some point above the boundary curve $tr^2G = 4|G|$. Although the arrows in Figure 6.9 suggest several ways of achieving this transfer, all are combinations of two basic possibilities. Recall from (42) that proportional feedback $h(\pm)$ operates on the determinant of G_{II} and that

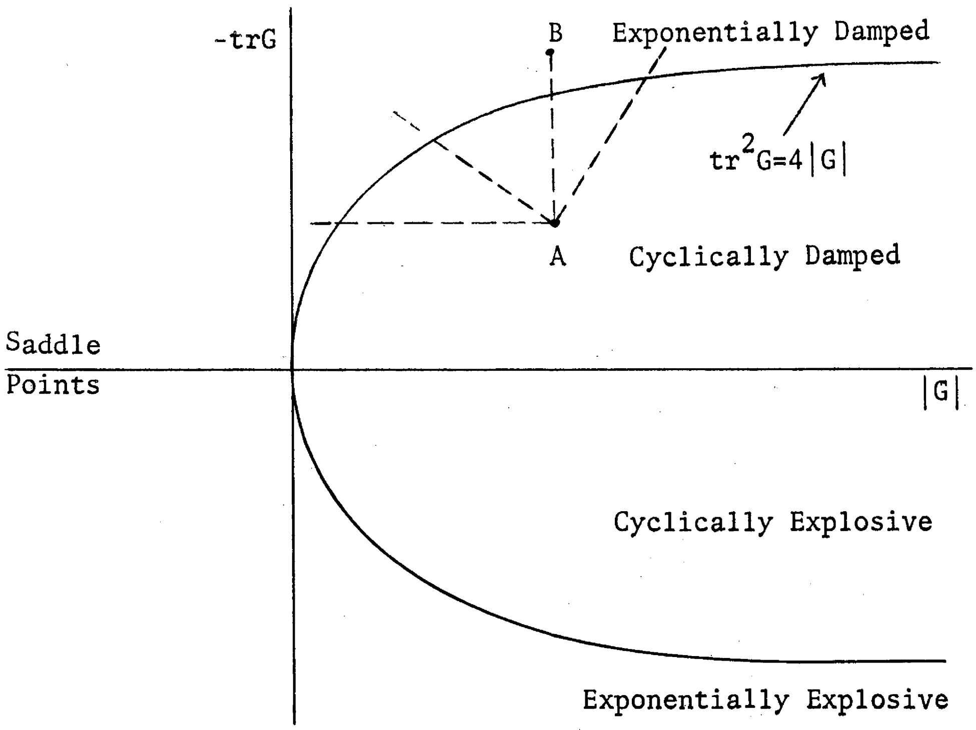

Figure 6.9

Stability and the Policy Design Problem

derivative feedback m(±) operates on the trace of G_{II}. Horizontal movements from point A will therefore require variation in proportional feedback; and vertical movements, variation in derivative feedback.

Variation in proportional feedback results from variation in the weighting of the ratio g/y in the performance functional, as expressed by ϕ^{-1}. But this has just been rejected as a means of avoiding oscillations; therefore, consider the alternative, the use of derivative feedback. In the performance functional (15), the state weighting matrix V attaches a zero weight to the state variable $z_2(t)=\dot{y}(t)$, expressing no concern about the speed at which the target y(t) adjusts under control. Since this rate of adjustment is related to oscillatory response behaviour, an additional term is now incorporated in the performance functional:

$$\underset{\hat{g}}{\text{MIN}}\ W = \tfrac{1}{2}\int_0^{T=\infty}[y^2(t) + \mu^2\dot{y}^2(t) + \phi\hat{g}^2(t)]dt, \qquad (63)$$

$$\mu \geqslant 0, \quad \phi > 0.$$

That is, the problem of hyperstability will be attacked with target derivative weighting, where μ enters (63) quadratically for convenience in later interpretation.

This optimal stabilisation problem may then be expressed in form analogous to Model II as:

Model III

$$\underset{\hat{g}}{\text{MIN}}\ W = \tfrac{1}{2}\int_0^\infty [z^T(t)Vz(t) + \phi\hat{g}^2(t)]\ dt,\ \phi > 0, \tag{64}$$

subject to

$$\dot{z}(t) = Az(t) + b\hat{g}(t),\ z(0)=z_o, \tag{65}$$

where

$$\left\{\begin{aligned} V &= \begin{bmatrix} 1 & 0 \\ 0 & \mu^2 \end{bmatrix},\ A = \begin{bmatrix} 0 & 1 \\ \mp\eta sw & -(\eta\pm sw) \end{bmatrix} \\ b &= \begin{bmatrix} 0 \\ \pm\eta w \end{bmatrix},\ z(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}. \end{aligned}\right. \tag{66}$$

Comparison of models II and III reveals that the only effect of derivative weighting is the appearance of the performance parameter μ^2 on the diagonal of the state weighting matrix V. Since V is the constant term of the matrix Riccati equation (26), it is necessary to rederive the Riccati parameters k_1, k_2. Appendix VIb (p.273 below) demonstrates that the equational results of model II may be retained if the definitional parameter γ is redefined as follows:

$$\hat{\gamma} = (\eta^2 + s^2w^2 + 2\eta w\theta + \phi^{-1}\eta^2w^2\mu^2)^{\frac{1}{2}}. \tag{67}$$

The degree of discretionary control over oscillatory outcomes may now be inferred from the familiar study of the optimal characteristic equation. Modifying (40), this equation is

$$\nu^2 + \hat{\gamma}\nu + \eta w\theta = 0, \tag{68}$$

with discriminant

$$\Delta(\eta) = (1 + \phi^{-1}w^2\mu^2)\eta^2 - (2w\theta)\eta + s^2w^2. \tag{69}$$

The polynomial $\Delta(\eta)$ possesses

(i) a minimum of $w^2\phi^{-1}(s^2w^2\mu^2 - 1)$

at $\dfrac{\eta}{w} = \dfrac{\theta}{1+\phi^{-1}w^2\mu^2}$; and

(ii) roots $\dfrac{\eta}{w} = \dfrac{\theta \pm [\phi^{-1}(1-s^2w^2\mu^2)]^{\frac{1}{2}}}{1 + \phi^{-1}w^2\mu^2}$.

These two factors imply that for all $\mu^2 > 1/s^2w^2$, the polynomial $\Delta(\eta)$ is positive, so that the optimal eigenvalues of model III are real. Thus the condition for nonoscillatory response is

$$\mu > \frac{1}{sw}, \tag{70}$$

providing a simple rule for preventing cyclical policy: choose the target derivative weight to be greater than the outside lag. When the system structure is such that ψ lies outside region (0) in Figure 6.8, no cycles occur, and the appropriate value for the target derivative weight is zero. But once ψ is shifted into that region, for example by increasing the strength of stabilisation policy, the value given in (70) is then appropriate. To summarise :

$$\begin{cases} \mu = 0 \ \forall \ \mathrm{tr}^2\, G > 4|G| \quad <=> \quad \phi^{-1} < \hat{\phi}^{-1}, \\ \\ \mu > \dfrac{1}{sw} \ \forall \ \mathrm{tr}^2\, G \leqslant 4|G| \quad <=> \quad \phi^{-1} \geqslant \hat{\phi}^{-1}, \end{cases} \tag{71}$$

where $\hat{\phi}^{-1}$ is that value of ϕ^{-1} for which ψ first enters region (0). Because the problem of hyperstability is binary - in the sense that the optimal response is either oscillatory or nonoscillatory - the decision rule (71) is also binary.

Analogously to (35), the optimal instantaneous controller associated with model III is

$$\begin{cases} \hat{g}(t) = \mp [\, \hat{h}(\pm) + \bar{m}(\pm)D \,]y(t), \\ \hat{h}(\pm) = \theta \mp s \\ \bar{m}(\pm) = \dfrac{\hat{\gamma} - (\eta \pm sw)}{\eta w} \,. \end{cases} \tag{72}$$

Whereas the level of proportional feedback $\hat{h}(\pm)$ is unchanged, the level of derivative feedback is a function now of two performance parameters, ϕ^{-1}, μ. Referring to Figure 6.9, the use of target derivative weighting for hyperstable design corresponds to the movement from A to B, since only the trace $(-\hat{\gamma})$ of G_{II} is affected by μ. Concern for control-induced oscillation expressed at the derivative level of preferences manifests itself appropriately in derivative feedback. In terms of Figure 6.8, a comparison of the discriminants (58) and (69) demonstrates that target derivative weighting changes the boundary curves LB and UB so that X is no longer in region (0).

Once target derivative weighting becomes necessary, systems can be ranked according to the additional derivative feedback required to prevent oscillations: from (70), the greater the natural speed of response, or the shorter the outside lag, the smaller is the additional feedback. Again, this penalises policy-making in an a priori likely situation - that of a sluggish economy slowly converging to equilibrium. In such an economy, a relatively high level is necessary not only of stabilisation action ϕ^{-1} but also of auxiliary derivative feedback μ.

6.5 THE EFFECTS OF CONTROL LAG

Model II illustrates that the penalty for a lagging policy response is an increase in complexity of the optimal controller. This proposition is valid for more realistic forms of policy lag. The increase in dynamic order from model I to model II is solely the product of control lag; the optimal controller for model II therefore retains the proportional feedback of model I but accompanies it with a new derivative feedback, shown to vanish with the policy lag. Policy lag results in policy complexity.

Policy lag also engenders the possibility of cyclical response as the result of increased dynamic order. Although policy lag is not the only cause of increased order, model II abstracts from other sources, such as increasing complexity in the outside lag structure, and explores the unique relation between policy lag and the cycle. Symbolically, as $\psi = sw/\eta \to 1$, the greater is the probability of policy-induced cycles. The more closely matched are the inside and outside lags, the greater the tendency for oscillations.

These conclusions provoke attempts to sever the policy-cycle nexus. Three such approaches are considered: (i) policy lag variation, (ii) performance parameter variation, and (iii) target derivative weighting. Policy lag variation is a doubtful tool because of institutional inflexibility in the downward direction and convergence speed tradeoff in the upward direction. Performance parameter variation, requiring a reduction in ϕ^{-1}, also compromises the general stabilisation objective by limiting the strength with which policy may be applied. Target derivative weighting provides a simple rule: once oscillations appear, select the magnitude of the target derivative weight at least equal to the length of the outside lag. This does not mean that target derivative weighting is costless. Since the trace of the closed-loop matrix increases with the determinant remaining constant, one system eigenvalue increases and the other decreases, while their product remains constant. One mode now decays more rapidly; the other, less rapidly.

Analysis of the hyperstable design problem is readily unified in terms of the optimal state dynamics. Thus Figure 6.10 plots the loci $\dot{z}(t)=0$ for models II and III, where the optimal closed-loop matrices in $\dot{z} = Gz$ are given by

$$G_{II} = \begin{bmatrix} 0 & 1 \\ -\eta w\theta & -\gamma \end{bmatrix}, \quad G_{III} = \begin{bmatrix} 0 & 1 \\ -\eta w\theta & -\hat{\gamma} \end{bmatrix}, \tag{73}$$

and for which $|G_{II}| = |G_{III}| = |G|$. Since both matrices are sign stable, the state space for each system is globally stable, as required of linear stabilisation policies. The directional vectors

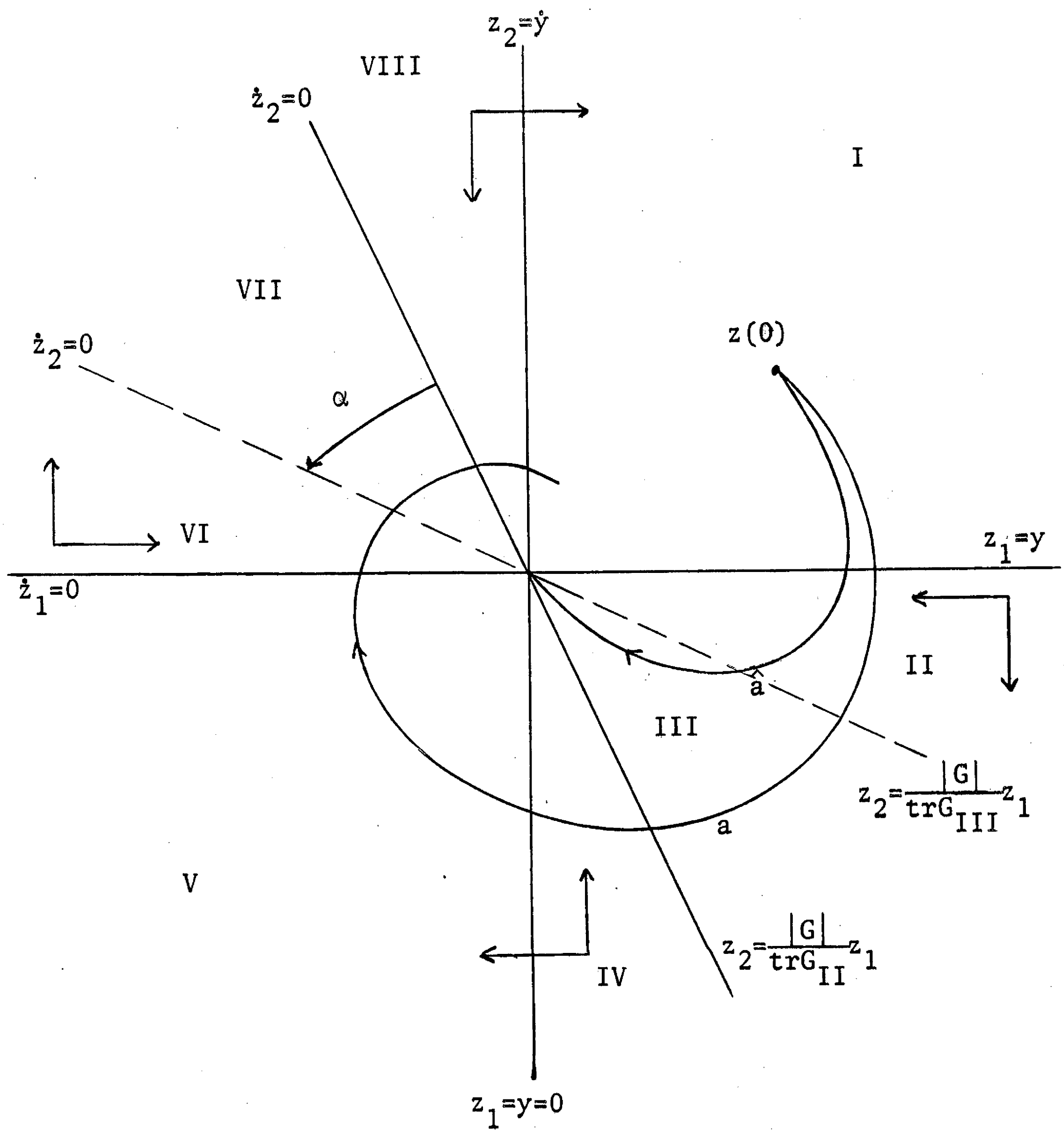

Figure 6.10
Hyperstable Design with a Lagged Instrument

are evaluated from (73) and apply to each model (regions III and VII switching dynamics between models). Both models share the $\dot{z}_1=0$ locus on the z_1 axis.

The vertical axis $z_1=0$ is the full employment locus: regions to the right (I-IV) involve demand inflation; regions to the left (V-VIII), unemployment. Suppose that an optimal stabilisation policy for model II is applied at time t=0, with initial conditions in region I - both the target error and its rate of change being positive. If target behaviour is to be oscillatory, the optimal response must follow a trajectory similar to (a), in which there is at least one sign change in y(t) - from regions IV to V, VIII to I, with possible repetitions. Thus the problem of hyperstability is to confine the optimal response entirely to the vertical half-plane in which the initial condition occurs; or to ensure that the optimal trajectory does not cross the full employment locus.

Beginning from region I, it is only in region IV that the dynamics of model II permit any movement towards the equilibrium point (0,0). But as path (a) depicts, there is no guarantee that the system will actually equilibrate within region IV. The crux of designing a hyperstable controller is to convert this region into a trap such that any trajectory entering it is captured and forced to equilibrate. Target derivative weighting creates this trap by swivelling the locus $\dot{z}_2=0$ anticlockwise through α radians. In model III, region III then belongs, in terms of its dynamics, to region IV, rather than to region II, as in model II. The size of the trap is thereby expanded to a point where it is always sprung. This is shown by considering the factor producing the rotation α. Target derivative weighting introduces the parameter (μ) into the trace of G_{III}, and since the determinant of G_{III} is independent of μ by (73), the negative slope of the $\dot{z}_2=0$ locus is directly lessened by increasing the trace. The optimal angle of rotation is then determined by finding μ such that $\mathrm{tr}^2 G_{III} > 4|G_{III}| = 4|G_{II}|$, this being the necessary and sufficient condition, from Figure 6.9, to inhibit target oscillation.

The performance difference between models II and III, between paths (a) and ($\hat{a}$) in Figure 6.10, aptly illustrates the maxim that "optimal" performance is only as good as the stabilisation objectives actually incorporated in the performance functional. If hyperstability is an objective, this objective must be explicitly defined in the performance functional: otherwise its attainment is a random matter of system structure.

Figure 6.10 has been drawn on the assumption that the current level of stabilisation ϕ^{-1} combines with the outside and inside lag dynamics to permit cyclical target behaviour. In this case, region IV is not a natural trap. If, however, the combined lag dynamics are such that target behaviour is nonoscillatory for all admissible ϕ^{-1}, then region IV is a natural trap, and $\alpha=0$. For some ϕ^{-1}, the trap will exist naturally, for other ϕ^{-1} it will not, and this is the reason for the binary, on-off nature of the target derivative rule. As a means of combatting cyclical fluctuations, Smith also suggests [pp.12-13] inclusion of the target derivative in the performance functional but with a unity weight. This unnecessarily restricts the flexibility of the optimisation analysis, precluding derivation of a binary decision rule comparable to (71).

In defining the power of dynamic feedback for the design of stabilisation policy, Phillips [1954] stressed inter alia the role of proportional and derivative feedbacks. Now optimal stabilisation policies for the second-order Phillips models - models II and III - generate optimal controllers using these feedbacks. Therefore, although the design methods differ, classical and optimal control techniques lead to structurally equivalent controllers. Further, referring to Figure 6.10, the slopes of the $\dot{z}_2=0$ loci are determined, loosely speaking, by the ratio of proportional feedback to derivative feedback. In turn, this ratio is positively related to the strength of control parameter ϕ^{-1} and negatively related to the target derivative parameter μ. And the binary role of μ corresponds precisely to the need defined by Phillips [1957, p.276] that "it is usually necessary to include an element of derivative correction in a stabilisation policy if regulation is to be satisfactory".

Chapter VI therefore reveals a close formal correspondence between the old and the new. The same principles underlie classical and optimal design; qualitatively similar controllers are designed for the same model. Yet the correspondence is not complete. Phillips [1954] also proposed the use of integral feedback for dynamic stabilisation. So far, this type of feedback has not been proposed for optimal stabilisation policy. One of the tasks of chapter VII is to examine this discrepancy.

CHAPTER VII

DEGREES OF FREEDOM IN THE STABILISATION PROBLEM

Even after specifying a quadratic criterion functional with linear dynamics as equality constraint, several degrees of freedom still remain before the optimisation model is fully specified. The procedure adopted in this thesis has been to select a particular configuration of these degrees of freedom, for reasons relating partly to mathematical simplicity and partly to historically-imposed initial conditions. Thus the computational algorithm developed in chapter II relies on the convenient linearity of the Hamiltonian system; and the FST paradox, refuted in chapter V, requires a certain configuration for its enunciation.

The configuration chosen fathers the autonomous, fixed-time, free endpoint regulator possessing a linear, autonomous, homogenous, dynamic equality constraint. The task of this present chapter is to develop the motivation for, and some of the consequences of, the choices embodied in this configuration.

7.1 DEGREES OF FREEDOM

Degrees of freedom exist with respect to the following aspects of the stabilisation problem

(i) time dependence of coefficient structures,
(ii) the stabilisation horizon,
(iii) dynamic instrument usage,
(iv) control constraints,
(v) terminal objectives, and
(vi) nature of disequilibrating disturbances.

Item (i) refers to a decision on the temporal behaviour of system dynamics and preference structure: on whether these evolve or remain constant over time. Section 3.1 argues for constancy as an initial simplification avoiding analysis of evolutionary or nonautonomous dynamics. For stabilisation policy viewed as a shortrun problem, the constancy assumption is reasonable; but in the longrun is questionable. The assumption of a constant preference structure over lengthy horizons ignores the customary economic device of time discounting. The effects of discounting on optimisation are discussed by Arrow; while Kurz [pp.160-66] provides an analysis of discounting applicable to the regulator model. Because there is no substantial literature devoted to controllability, optimality and computability in stabilisation theory, the limitations of this constancy assumption have been accepted throughout this thesis. Its removal awaits the development of optimal models of cyclical growth, comparable to the classical models of Phillips [1961], Bergstrom [chaps. 5, 6], and Allen [1968, chap. 20].

Item (ii) defines whether the stabilisation horizon is to be determined optimally or is preset by the policy-maker. In the first case, the stabilisation horizon is free; in the second, fixed. A fixed horizon assumption is used throughout this thesis. A possible rationale is provided, at least in the finite horizon regulator, by the fixed and finite electoral life of democratic governments. These governments may be expected to maximise their probability of re-election by using stabilisation policy to promote a favourable economic climate immediately prior to an election. Design of such stabilisation policy is therefore undertaken with a fixed horizon measuring remaining electoral life. Another possibility is a fixed horizon corresponding to the budget period; and yet another, some form of intermediate planning, with or without revision.

None of these justifications is applicable to the infinite horizon assumption. This assumption, when used, is used either to simplify the endpoint complications of the finite horizon regulator or to accord with the assumption used in classical stabilisation theory. The simplicity of asymptotic solutions relative to finite horizon solutions is valuable both theoretically and computationally. Theoretically, an infinite horizon assumption provides optimal controllers that are time-invariant. Thus in Figure 5.2 (p.126 above), the finite horizon trajectories are approximately linear except towards the terminal manifold; and the longer the horizon, the later this linearity disappears. A numerical illustration of this phenomenon is provided by Athans & Falb [p.779]. Removal of this endpoint behaviour does not affect the qualitative conclusions drawn about optimal stabilisation policy but considerably facilitates solution and analysis. Computationally, both the resultant time-invariance and dimensional reduction afford considerable savings, as noted with respect to the algorithm of chapter II. More will be said concerning finite and infinite horizon assumptions in section 7.3 below.

An interesting economic problem may emerge from the shortrun policy manipulation alluded to above. To sketch the problem briefly, suppose that the given economic system is dynamically controllable, and that the government desires a specified target vector transferred from a current position to a desired position by a fixed date. Suppose further that a controller minimising control energy, that is, $\underset{u}{\text{MIN}}\ W = \frac{1}{2}\int_0^T u^T R u\, dt$, is designed to achieve the required transfer. Then plotting minimum control energy as a function of the horizon T, it

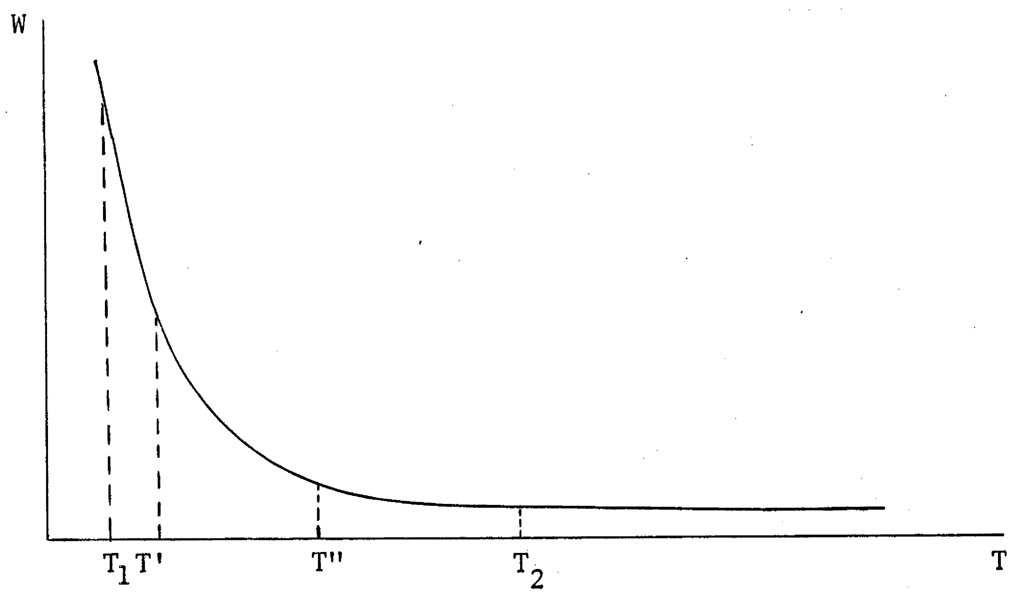

Figure 7.1
Minimum Control Energy

is possible[1], depending on the precise system dynamics, that this relation is similar to that of Figure 7.1.

Figure 7.1 depicts the tradeoff between the time (T) taken to

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1 Cf. Athans & Falb [pp.466-74].

achieve the stated stabilisation objectives and the required control energy (W). For $T < T'$, a small increase in the stipulated horizon permits a much larger decrease in control energy; while for $T > T''$, a significant decrease in control energy is only achievable with a very large increase in the horizon. Thus if the stabilisation horizon is fixed by electoral considerations as T_1, it is clear that the same objectives could be achieved a little later with a significant cost decrease. And by setting the stabilisation horizon at T'' rather than T_2, the same objectives could be achieved considerably faster with a small increase in control expenditure.

If economic dynamics generate a tradeoff comparable to Figure 7.1, questions must be raised about the social cost of stabilisation policies determined by electoral considerations: such policies may be too hasty and too extravagant or too tardy and too parsimonious. Either case is undesirable, given the alternatives. Nor is the possibility dependent on the minimum energy assumption. This merely selects a particular controller: all other controllers will necessarily exhibit a similar tradeoff. Such behaviour is an empirical question that may be worth investigation.

Item (iii) concerns the option of reduced versus regular stabilisation. Necessary and sufficient conditions for existence of reduced models are proposed in section 4.2. But no criteria are presented for selecting a particular reduced system. Reduced stabilisation appears to be an important policy option but its full significance must await further research.

Item (iv) refers to the presence or absence of explicit constraints on the available instruments. Explicit control constraints generate bang-bang controllers for which control operates for some or all of the time on the boundary of the constraint set. Lack of explicit constraints produces the smoothly-continuous controller, for which control is assumed to operate in the interior of some implicit constraint set. The smooth controller is the simpler principle, both theoretically and numerically. It is used in the development of the computational algorithm of chapter II; in the analysis of controllability in chapters III and IV; and in the optimal stabilisation models of chapters V and VI. While it is desirable in some formulations of the stabilisation problem to impose explicit control constraints, this possibility is not treated in this thesis[2]. Section 7.2, however, investigates the way in which an unconstrained regulator model can act as a surrogate for an explicitly constrained regulator.

Item (v) summarises the choice to be made with respect to terminal or endpoint objectives: is the terminal target vector to be fixed or free? The necessary conditions for optimality accommodate either choice but, again for simplicity, the free endpoint assumption has generally been used; although fixed endpoint solutions to the Phillips regulator are briefly identified in chapter V. A strong case may be made for regarding the stabilisation problem as a fixed endpoint problem: as formulated, for instance, in the investigation of controllability in chapter III. Where there does exist specific concern for the target endpoint, section 7.3 proposes that terminal target weighting is a reasonable alternative to the fixed endpoint assumption, while still retaining the simplicity of the free endpoint assumption.

Finally, item (vi) defines a choice about the nature of the disturbances assumed to perturb economic equilibrium. Two classes of disturbance are conveniently recognised: transient and persistent. Transient disturbances are impulse disturbances assumed to have occurred and vanished prior to the implementation of policy to counteract their effects. These effects are therefore captured in the initial conditions, representing displacement from static equilibrium. Persistent disturbances are those disturbances - such as ramp, step, sinusoidal and exponential disturbances - whose effects are maintained and cannot be captured in the initial conditions but must be explicitly introduced into the model's formulation. To this point, the thesis has employed the transient assumption. Section 7.4 therefore investigates the significance of the persistent assumption for the particular class of step disturbances. This leads to the complete unification of classical and optimal stabilisation theory.

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2 A recent paper by Turnovsky illustrates each controller type in an optimal regulator model of a single market.

7.2 DYNAMIC CONSTRAINT INTERNALISATION

The explicit introduction of time into the analysis of stabilisation introduces elements which are exclusively dynamic. One such element is the temporal distribution of control resources over the stabilisation period. Statically, control resources are disbursed at a constant rate; dynamically, the stipulation, for example, of a fixed quantity of resources available for stabilisation over a given horizon in no way constrains the temporal pattern of usage. If there exist flow constraints on the instruments that must be satisfied at every point of time, the problem therefore arises of ensuring their satisfaction. Whether or not this is possible without explicit introduction of these constraints is now considered in the simplified context of the Phillips regulator.

Flow constraints are typically a saturation or magnitude constraint

$$\underline{u} \leqslant u(t) \leqslant \bar{u}, \qquad \underline{u} \neq \bar{u}, \ \forall \ t \in [0,T], \tag{1}$$

positing upper and lower bounds on the level of allowable control usage at every point of time during the control period. For stabilisation models obtained under the equilibrium partition, the saturation constraint

$$-\underline{u} \leqslant u(t) \leqslant \bar{u}, \qquad \underline{u}, \bar{u} > 0, \tag{2}$$

would normally be assumed: that is, actual control $U(t)$ is permitted to deviate positively or negatively from its desired equilibrium level $\bar{U}$, within the band $(-\underline{u}, \bar{u})$.

Two basic approaches to optimal dynamic stabilisation may be defined by the manner in which such control constraints are incorporated. In the first approach, the saturation constraint (2) is explicitly introduced into the optimal formulation:

$$\begin{cases} \underset{u(t)}{\text{MIN}}\ W = \displaystyle\int_0^T J(x)\, dt \\ \underline{\text{subject to}}\ \dot{x} = f(x,u), \quad x(0)=x_o, \quad |u(t)| \leqslant \bar{u}, \end{cases} \tag{3}$$

where, for simplicity, the constraint band is assumed symmetric. In the second approach, the saturation constraint is implicitly introduced via the criterion functional:

$$\begin{cases} \underset{u(t)}{\text{MIN}}\ W = \int_0^T J(x,u)\,dt, \quad J_u,\ J_{uu} > 0, \\ \underline{\text{subject}}\ \underline{\text{to}}\quad \dot{x} = f(x,u),\quad x(0)=x_0. \end{cases} \tag{4}$$

These two classes of model are differentiated according as the control constraint is external (3) or internal (4). In (4), the conditions $J_u > 0$, $J_{uu} > 0$ act as a proxy for $|u(t)| \leqslant \bar{u}$ in (3), with performance cost increasing at an increasing rate with u(t). Examination of whether or not the proxy is sufficient, or can be made so, is therefore of interest.

The possibility of internalising a magnitude constraint will now be demonstrated in the context of the Phillips regulator of chapter V. Specifically, consider the optimisation problem

$$\begin{cases} \underset{g(t)}{\text{MIN}}\ W = \tfrac{1}{2} \int_0^T [y^2(t) + \phi g^2(t)]\,dt, \quad \phi > 0, \\ \underline{\text{subject}}\ \underline{\text{to}}\quad \dot{y} = -swy + wg,\quad y(0)=y_0,\quad y(T)\ \text{free}. \end{cases} \tag{5}$$

Suppose the policy-maker specifies the control magnitude constraint

$$|g(t)| \leqslant g_* \ \forall\ t \in [0,T]. \tag{6}$$

Then given the performance integrand,

$$J' = \alpha y^2(t) + \beta g^2(t), \quad \alpha, \beta > 0, \tag{7}$$

which may be written[3]

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[3] Since optimality with respect to the criterion $\int_0^T J'\,dt$ is preserved up to the linear transformation $\rho J' + \sigma$, for ρ, σ constants.

$$J = \alpha^{-1}J' = y^2(t) + \phi g^2(t), \quad \phi \equiv \beta/\alpha, \tag{8}$$

select a particular $\phi = \phi_*$ such that[4]

$$\begin{cases} \alpha^* = 1/\max y^2(t) = 1/y_*^2 \\ \beta^* = 1/\max g^2(t) = 1/g_*^2. \end{cases} \tag{9}$$

How do the weights α^*, β^* ensure satisfaction of the constraint (6)? Suppose that the maximum deviation of income y from its desired level of zero is

$$\frac{|Y(t)-\bar{Y}|}{\bar{Y}} \leq \gamma \quad \Rightarrow \quad |y_*| = \gamma\bar{Y}. \tag{10}$$

The control constraint (6) also implies

$$\frac{|G(t)-G|}{\bar{G}} \leq \delta \quad \Rightarrow \quad |g_*| = \delta\bar{G}, \tag{11}$$

and it is necessary, given the static income identity, that

$$\bar{G} = \varepsilon\,\bar{Y}, \qquad 0 < \varepsilon < 1. \tag{12}$$

Hence, from (8) through (12),

$$\phi_*^{-1} = \alpha^*/\beta^* = (g_*/y_*)^2 = (\delta\varepsilon/\gamma)^2, \tag{13}$$

where to summarise

$$\varepsilon = \frac{\bar{G}}{\bar{Y}}, \qquad \delta = \max\frac{|G-\bar{G}|}{\bar{G}}, \qquad \gamma = \max\frac{|Y-\bar{Y}|}{\bar{Y}} \quad . \tag{14}$$

The parameter ε defines the optimum size of the public sector with respect to full employment income. It varies between the limits $0 < \varepsilon < 1$, loosely from the invisible hand to socialism, and may be termed the Hayek parameter. Although this parameter is predetermined as a structural characteristic of the dynamic stabilisation problem,

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4 Following a suggestion by Bryson & Ho [p.149].

$1-\varepsilon$ must be assumed large enough to validate the multiplier-accelerator model. The parameter δ may be labelled the Keynes parameter; it specifies the maximum dynamic variation in public sector size for stabilisation purposes. If the macro-adjustment process of an economy with given Hayek parameter is either unstable, or unduly slow though stable, or unduly cyclical but stable, the public sector must modify the system response. The magnitude of the Keynes parameter indicates the scope available in a given economic system for such dynamic remedial action.

According to the Keynes parameter, the size of the public sector is dynamically variable within the limits

$$\begin{gathered}(1-\delta)\bar{G} \leqslant G(t) \leqslant (1+\delta)\bar{G} \\ \Longleftrightarrow \\ -\delta\bar{G} \leqslant G-\bar{G} \leqslant \delta\bar{G},\end{gathered} \tag{15}$$

so that

$$|g(t)| \leqslant \delta\varepsilon\bar{Y} \tag{16}$$

is the allowable range of variation in the dynamic control variable $g(t)$. And from (10), the range of variation in the dynamic target variable $y(t)$ is

$$|y(t)| \leqslant \gamma\bar{Y}. \tag{17}$$

Now from (5.23), the optimal target and instrument solutions, for an infinite horizon, are

$$\begin{cases} y^*(t) = y(0)e^{-w\theta t} \\ g^*(t) = -(\theta-s)y^*(t), \qquad \theta = (s^2+\phi^{-1})^{\frac{1}{2}}. \end{cases} \tag{18}$$

Hence (16) is satisfied if, using (17) and (18),

$$|g^*(t)| = (\theta-s)|y^*(t)| \leqslant (\theta-s)\gamma\bar{Y} \leqslant \delta\varepsilon\bar{Y}, \tag{19}$$

or if,

$$(s^2+\phi^{-1})^{\frac{1}{2}} - s \leq \delta\varepsilon/\gamma = \phi_*^{-\frac{1}{2}}. \tag{20}$$

And a sufficient condition for this is

$$\phi^{-1} \leq \phi_*^{-1} = (\delta\varepsilon/\gamma)^2. \tag{21}$$

From (18), the maximum income deviation is always the initial disequilibrium y(0). Given an estimate, therefore, of the maximum such target error, $y_* = \max y(0)$, that the Phillips regulator is expected to face, the particular value ϕ_*^{-1} of the performance parameter ϕ^{-1} specifies an upper bound to the strength of control parameter. For all ϕ^{-1} satisfying (21), the explicit control constraint (6) is satisfied. This attempt to internalise the magnitude constraint within the structure of the Phillips regulator therefore produces a simple rule: the maximum value of the strength of control parameter is equal to the square of the ratio of the product of the Keynes and Hayek parameters to the maximum target error.

Since the magnitude constraint refers to the absolute value of $|g(t)|$ and since ϕ^{-1} determines the ratio $|g/y|$, ϕ_*^{-1} is larger, and control easier, when disturbances generate relatively small target errors. Similarly, the greater the static and dynamic intrusion of the public sector into economic activity, as expressed in the product $\delta\varepsilon$, the greater is ϕ_*^{-1}. The Hayek parameter ε will be determined by decisions based on the optimum allocation of resources and is therefore external to the stabilisation problem (excluding the problem of intergoal conflict). The Keynes parameter δ is the essence of the control constraint (6), (11). If a magnitude constraint is relevant in formulating the stabilisation problem for the Phillips model, the factors determining δ must be defined and substantiated. This will not be attempted here. It may be noted that the size of δ will condition the efficacy of controllers irrespective of the design technique, raising yet another empirical question: how binding are such constraints on the strength with which control can be applied?

The simplicity of (21) for the Phillips regulator is lost in higher-order models: performance parameters proliferate as target and instrument priorities are established. Extension of the principle behind (9) is, however, immediate.

7.3 TERMINAL WEIGHTING OR FIXED ENDPOINTS?

Another aspect of the degrees of freedom associated with specification of the stabilisation problem is the treatment of the terminal target vector. For example, chapter V identified a knockoff syndrome in the behaviour of the naturally unstable Phillips regulator, the target starting to diverge at some point near the finite horizon T. Now there are two methods available for correcting this behaviour. The first is terminal target weighting, briefly mentioned in chapter V; the second is conversion to a fixed endpoint specification, constraining the target to achieve a specific value at t=T. For a finite horizon, a choice therefore arises between fixed endpoint regulators and terminal weighting regulators. Observing that the knockoff syndrome is merely an illustrative context for this choice, this present section uses the naturally unstable Phillips regulator to define some of the dimensions of this choice problem.

Fixed endpoint solutions for the Phillips regulator are identified in section 5.3 above. Turning therefore to terminal weighting, the optimal free endpoint Phillips regulator problem, equation (5.12), becomes

$$\underset{g(t)}{\text{MIN}}\ W = \tfrac{1}{2}fy^2(T) + \tfrac{1}{2}\int_0^T [y^2(t) + \phi g^2(t)]\ dt, \quad \phi, f > 0, \tag{22}$$

<u>subject to</u>

$$\dot{y} = -swy + wg, \quad y(0)=y_0\neq 0, \quad y(T)\ \text{free.} \tag{23}$$

Introducing the endpoint term into (5.27), the transversality condition for this terminal weighting regulator is

$$[p(t) - fy(t)]\delta y(t) = 0, \quad t = \{{0 \atop T}, \tag{24}$$

where (5.30) is a particular case (f=0) of (24). For a free endpoint, (24) is satisfied by

$$p(T) = fy(T), \tag{25}$$

and the corresponding Riccati boundary condition is thus

$$k(T) = f. \tag{26}$$

Revising Appendix V (pp.268-9 below) to allow for $f > 0$, $w < 0$, the Riccati solution for the naturally unstable Phillips regulator with terminal weighting is

$$k(t) = \left\{\frac{C - \frac{\theta-s}{\theta+s}e^{-2w\theta(T-t)}}{C + e^{-2w\theta(T-t)}}\right\} k^*, \quad w = |w|, \tag{27}$$

where

$$\begin{cases} C = \dfrac{\theta-s+\phi^{-1}wf}{\theta+s-\phi^{-1}wf}, & k^* = \lim\limits_{T\to\infty} k(t) = \phi w^{-1}(\theta+s), \\ \theta = (s^2+\phi^{-1})^{\frac{1}{2}}. & \end{cases} \tag{28}$$

Three special cases of this Riccati solution are of interest, corresponding to $f=0$, $f=k^*$, and $f=\infty$. From (28),

$$\begin{cases} f = 0 & \Rightarrow \quad C = \frac{\theta-s}{\theta+s}, \\ f = k^* & \Rightarrow \quad C^{-1} = 0, \\ f = \infty & \Rightarrow \quad C = -1, \end{cases} \tag{29}$$

so that these three cases are

$$k(t)_{/f=0} = \left\{\frac{1 - e^{-2w\theta(T-t)}}{1 + \frac{\theta+s}{\theta-s}e^{-2w\theta(T-t)}}\right\} k^*, \tag{30}$$

$$k(t)_{/f=k^*} = k^*, \tag{31}$$

$$k(t)_{/f=\infty} = \{ \frac{1 + \frac{\theta-s}{\theta+s}e^{-2w\theta(T-t)}}{1 - e^{-2w\theta(T-t)}} \} k^*. \tag{32}$$

In each case, $k(T)=f$, as required by (26). The bracketed terms of (30) and (32) are approximately reciprocal, and it is readily shown that

$$k(t)_{/f=0} < k(t)_{/f=k^*} < k(t)_{/f=\infty}. \tag{33}$$

Since the Hamiltonian saddle point ($\pm w\theta$) is invariant with respect to f, and since the terminal manifold may be written

$$y(T) = p(T)/k(T) = f^{-1}p(T), \tag{34}$$

Figure 7.2 follows from the naturally unstable portion of Figure 5.2 (p.126 above). Commencing with $f=0$, increased endpoint weighting rotates the terminal manifold (34) clockwise; from $p(T)=0$ for $f=0$, to $y(T)=0$ for $f=\infty$. Now the ranking (33) holds for $t=0$ in particular, so that initial phases are also shifted rightwards on the initial manifold $y=y(0)$. This rightward shift of initial and terminal conditions therefore implies rightward shifts, as f increases, of any trajectory corresponding to a given T. For the free endpoint zero terminal weighting trajectory AB, an illustrative progression is shown as CD, EF, ..., GH, corresponding to f values of grad^{-1} ($\dot{y}=0$), k^*, ..., ∞. Thus the locus BDFH describes the endpoint behaviour of a particular finite horizon solution of the Phillips terminal weighting regulator ($w < 0$) for increasing terminal weights f. As the endpoint phase transcribes this locus, the Phillips regulator tends to discard its free endpoint character and to acquire a fixed endpoint character. In the limit, for $f=\infty$, the free endpoint and fixed endpoint solutions coincide. To illustrate, the optimal target solution for $w < 0$ and $f > 0$ is, by analogy to (5.21),

$$y(t) = y(0) \{ \frac{(\theta-s+\phi^{-1}wf)e^{-w\theta t} + (\theta+s-\phi^{-1}wf)e^{-w\theta(2T-t)}}{(\theta-s+\phi^{-1}wf) + (\theta+s-\phi^{-1}wf)e^{-2w\theta T}} \}. \tag{35}$$

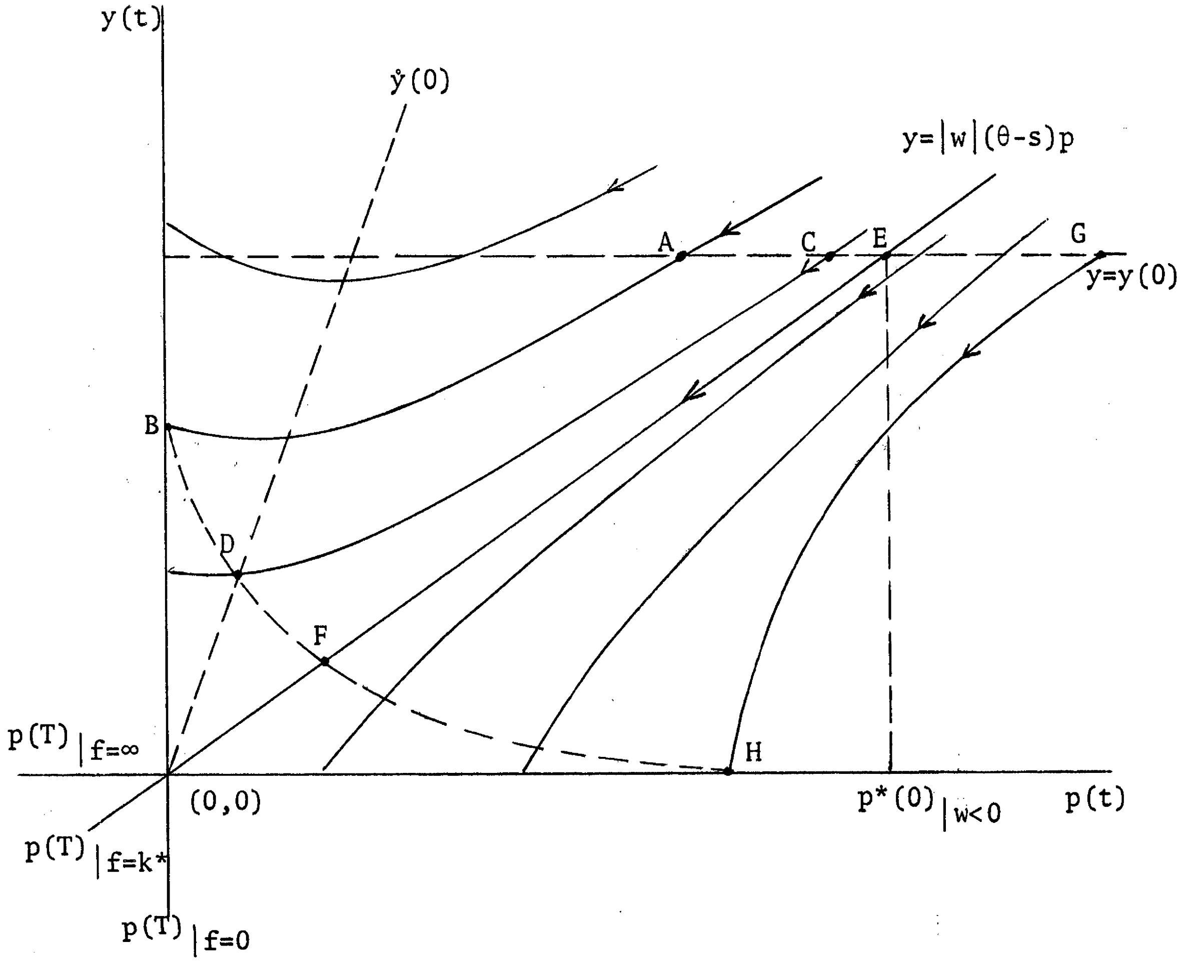

Figure 7.2

Phillips Regulator with Terminal Weighting (w < 0)

Hence

$$\lim_{f\to\infty} y(T) = 0, \tag{36}$$

as required by the trajectory GH of Figure 7.2.

Thus the free endpoint solution f=0 occurs at one end of the locus BDFH; the fixed endpoint solution, at the other. For a significant part of the stabilisation horizon, the trajectories AB, CD, EF, ..., GH are linear. The rightward shift due to increasing terminal weighting therefore results in higher but approximately constant levels of feedback. This constancy is destroyed by endpoint adjustments to the locus BDFH. At the endpoint,

$$\begin{array}{lll} \text{either (i)} & p(T)=0, & y(T)>0, \\ \text{or (ii)} & p(T)>0, & y(T)>0, \\ \text{or (iii)} & p(T)>0, & y(T)=0. \end{array} \tag{37}$$

Since the target and costate never vanish simultaneously, full static equilibrium is not attainable under either regulator specification - fixed endpoint or free endpoint - with a finite horizon. The choice of a particular terminal weight f, $f\varepsilon[0,\infty]$, is therefore a matter of preference concerning the target-instrument tradeoff[5].

Riccati terminal weighting, $f=k^*$, acts, as does the infinite horizon assumption, to filter out the positive eigenvalue by appropriate adjustment of the initial costate p(0). The trajectory EF therefore coincides with the asymptotic trajectory for the entire finite horizon T; and may be interpreted as the balanced stabilisation path, both target and instrument equilibrating at the same rate $\forall\ t\varepsilon[0, T]$. Now the cardinal difference between finite horizon and infinite horizon regulators is that controllers for $T < \infty$ are time-varying whereas controllers for $T = \infty$ are constant. For this reason alone, the asymptotic algorithm of chapter II is considerably simpler than the finite algorithm. Given practical problems of dimension and computation, it is therefore tempting to argue that, even though the stabilisation problem may best be conceived of as a finite horizon problem, the simplicity ensuing from the asymptotic assumption is too valuable to relinquish. The balanced path EF suggests another possibility. If, in the general regulator formulation, the terminal weighting matrix is made identical to the asymptotic Riccati matrix, finite horizon solutions will possess constant feedback controllers; avoiding the disadvantage of time-varying controllers but retaining the logic of the finite horizon context. In this case, the free endpoint regulator may be preferred to the fixed endpoint regulator.

It follows from (27) that

$$\lim_{T\to\infty} k(t)_{/f\geqslant 0} = k^*. \tag{38}$$

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5 By equation (5.14), $g = \mp \phi^{-1}wp$, $w = |w|$, so that the costate variable is a direct proxy for the instrument variable in the single-target, single-instrument Phillips regulator.

And using the limiting procedure of equation (2.56) for $F \neq 0$, the general asymptotic Riccati solution $\bar{K}$ can also readily be shown to be independent of F, the terminal weighting matrix. That is, as the stabilisation horizon is lengthened, terminal weighting becomes increasingly irrelevant; and is totally so for $T = \infty$. Taking $f=0$ to symbolise the pure free endpoint regulator and $f = \infty$ to symbolise the fixed endpoint regulator, (38) argues that the endpoint assumption itself becomes asymptotically irrelevant. This is a logical result because the asymptotic trajectory achieves the desired equilibrium position (0, 0); and this corresponds to the intersection of all free endpoint ($f \geqslant 0$) and fixed endpoint manifolds.

Analysis of the degrees of freedom relating to endpoint specification in the stabilisation problem does not permit a definitive choice of formulation on a priori grounds. It does, however, clarify the basis upon which a subjective choice can be made. In particular, fixed endpoints are just one possibility in the spectrum of possible endpoints under terminal weighting; while Riccati endpoint weighting is the unique method for constructing time-invariant, finite horizon controllers.

7.4 OPTIMAL CONTROLLERS WITH INTEGRAL FEEDBACK

An examination of the models of chapters V and VI establishes that the optimal controllers so far considered utilise proportional and first- and higher-order derivative feedbacks: there is a direct matching of controller dynamics to system dynamics. Thus if the state vector is of dimension 2, and consists of the target variable and its rate of change, as in models II and III, the optimal controller will comprise a proportional and a derivative feedback to match. Now one of the prime weapons used by Phillips [1954] in designing dynamic stabilisation policy is integral feedback; yet the optimal models just referred to make no use of this feedback type. They utilise information on the current state of the system (proportional feedback), and information on the future state through its current rate of change (derivative feedback), but do not accumulate information on past state behaviour (integral feedback).

Why does this disparity exist between classical and optimal controllers? Does the use of the optimal regulator hypothesis imply, because of this neglect, an unnecessary depletion of the weapons available to the policy-maker? This section provides answers to these questions. It will be shown that the issue turns both on the type of disturbance that the regulator is designed to counter and on the informational constraints placed on knowledge of these disturbances.

Under the equilibrium partition, the Phillips multiplier-accelerator model

$$\dot{Y}(t) = -swY(t) + wG(t) + wA, \quad w \neq 0, \quad s \varepsilon (0,1), \tag{39}$$

has been written

$$\dot{y} = -swy + wg, \quad y(0)=y_0 \neq 0, \tag{40}$$

where

$$g(t) \equiv G(t) - \bar{G}, \quad y(t) \equiv Y(t) - \bar{Y}, \tag{41}$$

and where the static equilibrium pair $(\bar{G}, \bar{Y})$ is determined from solution of

$$0 = -s\bar{Y} + \bar{G} + A. \tag{42}$$

Optimal stabilisation policies have been designed for this model by minimising a quadratic performance functional to which (40) is adjoined as a dynamic equality constraint. The need for policy is assumed to arise through a transient fluctuation in autonomous demand disturbing the static equilibrium (42); the effect of this disturbance, subsequent to its disappearance, being captured in the initial condition $y(0) \neq 0$.

The classical approach to stabilisation policy, as developed by Phillips [1954, 1957] and Allen [1960, 1968], introduces an additional element into the policy problem. Suppose that the autonomous demand disturbance persists rather than disappears. In particular, suppose that autonomous demand increases at time t=0 by an amount C, disturbing an established equilibrium (42), and consider the design problem given by[6]

$$\dot{y} = -swy + wg + wC, \quad y(0)=0. \tag{43}$$

Noting that the classical approach invariably assumes an infinite horizon, the design objective is then posed as

$$\dot{y}(t=\infty) = 0, \quad y(t=\infty) = 0 \iff Y(t=\infty) = \bar{Y}. \tag{44}$$

Given the system dynamics (43) and the objectives (44), Phillips (1954] shows that integral feedback is

(i) necessary and sufficient for the removal of any persistent inflationary or deflationary gap; but

(ii) neither necessary nor necessarily sufficient to modify the dynamic stability properties of a given system.

To illustrate (i), observe that if the policy-maker remains inert, so that $g(t) = 0 \; \forall \, t \geq 0$, the equilibrium solution of (43) for assumed natural stability ($w > 0$) is

$$y(\infty) = C/s > 0. \tag{45}$$

The continuing disturbance C causes an inflationary gap whose magnitude is determined by the static multiplier result (45). Neither proportional feedback, derivative feedback, nor higher order derivative feedbacks can remove this gap; but if the integral feedback

$$g(t) = - i \int_0^t y(\tau) \, d\tau, \quad i > 0, \tag{46}$$

is applied to (43), then

$$\ddot{y}(t) + sw\dot{y}(t) + wiy(t) = 0, \quad (w > 0), \tag{47}$$

which possesses the desired equilibrium, $y(\infty) = 0$.

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6 Cf., for example, Allen [1960, pp.69-74].

Integral feedback has not been required when designing optimal policies for (40) because no constant disturbances appear in that equation. Even if a constant step disturbance is observed, one policy option is to offset the disturbance statically for as long as it persists, and to simultaneously implement a dynamic policy to counteract any disequilibrium dynamics following its appearance. In this case, the system dynamics are

$$\begin{cases} \dot{y} = -swy + wg, \quad y(0)\neq 0, \\ \\ g(t) \equiv G(t) - \bar{G} + C. \end{cases} \tag{48}$$

This option is therefore consistent with equation (40), after redefining g(t).

Whether ongoing disturbances are to be treated as a dynamic problem (43) or a static problem (48) creates yet another choice problem. What criteria exist for preferring one approach to the other? A basic distinction can be made between step disturbances that persist for a long time, and those that are more than transient but less than permanent. The first represent an enduring shift in the underlying static equilibrium while the second are closer to transient disturbances of the given equilibrium. It can then be argued that static offsetting is appropriate for the longer-run step disturbance and that integral feedback is appropriate for the shorter-run disturbance.

This dichotomy is based on the expected frequency with which these two types of disturbance occur. Thus the persistent demand disturbance can be measured and offset through static policy; and since such disturbances are likely to be infrequent, this measurement process need not strain policy resources. On the other hand, the temporary demand disturbance is likely to be of much greater frequency; and it is therefore desirable to obviate the need for measurement. And for this purpose, the dynamic option of integral feedback is necessary. Thus, as Phillips argues, [1954, p.297]:

> "Is it clear that with an integral stabilisation policy the final equilibrium position, if it exists, will be one in which the error is completely eliminated, since so long as even the smallest error persists the cumulated error or time integral

of the error must be continuously increasing, and with it the magnitude of the correcting action, so that equilibrium is possible only when the error is zero."

An integral policy (46) applied to (43), and evaluated at the equilibrium (44), provides

$$g(t{=}\infty) = -\, i \int_0^{\infty} y(t)\, dt = C. \tag{49}$$

Policy is activated in terms of the measurable target error y(t) and not in terms of the disturbance C; and continues until this disturbance is completely offset.

If the practical utility of integral feedback for an economy subject to numerous small step disturbances is accepted, the utility of stabilisation policies designed with the regulator hypothesis must be reconsidered. The technique is adequate for impulse disturbances and for longrun demand shifts; its adequacy in response to step disturbances is now investigated.

Define the variables x(t), v(t) such that

$$x(t) = g(t) + C, \quad \dot{x} = \dot{g} = v. \tag{50}$$

Then the system dynamics (43) may be written

$$\begin{cases} \dot{y} = -swy + wx, \quad y(0)=0, \\ \dot{x} = v, \end{cases} \tag{51}$$

or

$$\begin{cases} \dot{z} = A^{*}z + b^{*}v, \\ A^{*} = \begin{bmatrix} -sw & w \\ 0 & 0 \end{bmatrix}, \quad b^{*} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad z = \begin{bmatrix} y \\ x \end{bmatrix}. \end{cases} \tag{52}$$

In this state formulation, the control variable $v(t) = \dot{g}(t)$ is the of the error must be continuously increasing, and with it the magnitude of the correcting action, so that equilibrium is possible only when the error is zero."

An integral policy (46) applied to (43), and evaluated at the equilibrium (44), provides

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Define the variables $x(t)$, $v(t)$ such that

$$x(t) = g(t) + C, \quad \mathring{x} = \mathring{g} = v. \tag{50}$$

Then the system dynamics (43) may be written

$$\begin{cases} \mathring{y} = -swy + wx, \quad y(0)=0, \\ \mathring{x} = v, \end{cases} \tag{51}$$

or

$$\begin{cases} \mathring{z} = A^* z + b^* v, \\ A^* = \begin{bmatrix} -sw & w \\ 0 & 0 \end{bmatrix}, \quad b^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad z = \begin{bmatrix} y \\ x \end{bmatrix}. \end{cases} \tag{52}$$

In this state formulation, the control variable $v(t) = \mathring{g}(t)$ is the model. Thus if $\dot{y} = Ay + bx$ is in the canonical form of the scalar policy model (3.34), the step disturbance state formulation becomes

$$\begin{bmatrix} \dot{y} \\ \hline \dot{x} \end{bmatrix} = \left[\begin{array}{c|c} A & b \\ \hline 0 & 0 \end{array}\right] \begin{bmatrix} y \\ \hline x \end{bmatrix} + \begin{bmatrix} 0 \\ \hline 1 \end{bmatrix} v, \tag{56}$$

or

$$\dot{z} = A^* z + b^* v, \tag{57}$$

where, from (3.34),

$$A_{nxn} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -a_o & & \cdots & & -a_{n-1} \end{bmatrix}, \quad b_{nx1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{58}$$

Now (56) is dynamically controllable iff

$$\rho(Q^*) = \rho[b^* \quad A^* b^* \quad \ldots \quad (A^*)^n b^*] = n+1, \tag{59}$$

where Q^* is the n+1xn+1 controllability matrix. But from (56), and using the canonical structure (58),

$$Q^* = \left[\begin{array}{c|c|c|c|c|c} 0 & b & Ab & A^2b & \ldots & A^{n-1}b \\ \hline 1 & 0 & 0 & 0 & \ldots & 0 \end{array}\right] = \left[\begin{array}{c|c} 0 & Q \\ \hline 1 & 0 \end{array}\right] \tag{60}$$

where Q is the controllability matrix for $\dot{y} = Ay + bx$. Hence

$$\rho(Q^*) = n+1 \quad \text{iff} \quad \rho(Q) = n, \tag{61}$$

and $\rho(Q) = n$ is guaranteed by (3.88). By a theorem of Lee & Markus [p.90], every scalar controllable system $\dot{y} = Ay + bx$ can be written in the canonical form (58). Hence, by (61), dynamic controllability with respect to impulse disturbances is necessary and sufficient for dynamic controllability with respect to step disturbances.

Thus the system (52) is dynamically controllable provided that the first-order Phillips model is dynamically controllable; or provided $w \neq 0$. Returning to the optimisation problem (53), and following, for example, the solution procedure of section 6.2, the optimal controller $v(t)$ is

$$v(t) = \gamma^T z(t), \qquad \gamma = -\phi^{-1}\bar{K}b^*, \tag{62}$$

where

$$\bar{K} = \begin{bmatrix} k_o & k_1 \\ k_1 & k_2 \end{bmatrix}, \tag{63}$$

is the positive definite symmetric solution of

$$(A^*)^T\bar{K} + \bar{K}A^* - \bar{K}\frac{b^* b^{*T}}{\phi}\bar{K} + V^* = 0. \tag{64}$$

From (62), the optimal controller is therefore

$$v(t) = -\phi^{-1}(k_1 y + k_2 x). \tag{65}$$

To express this controller independently of the unknown disturbance C, contained in x, use is made of the system dynamics (51), to give

$$x = w^{-1}(\dot{y} + swy). \tag{66}$$

Thus (65) is

$$v(t) = -\phi^{-1}\{(k_1 + sk_2)y + w^{-1}k_2\dot{y}\}. \tag{67}$$

Substitution of (67) into (55) then provides

$$g(t) = g(0) + \beta y(0) - \alpha\int_0^t y(\tau)d\tau - \beta y(t), \tag{68}$$

where

$$\alpha \equiv \phi^{-1}(k_1+sk_2), \qquad \beta \equiv \phi^{-1}w^{-1}k_2. \tag{69}$$

Apart from the initial conditions, the optimal controller (68) for the Phillips step-regulator comprises constant proportional and integral feedback from the target y(t). To the optimal proportional feedback of the Phillips impulse-regulator of chapter V is added an optimal integral feedback specifically designed to counteract step disturbances. Nor, as required, is there need to measure these disturbances: the controller (68) is independent of C.

With the introduction of optimal integral feedback, the correspondence between classical and optimal policies for the Phillips multiplier-accelerator model is complete. Proportional, integral and derivative feedbacks may occur with either type of design procedure. The tradeoff between static error removal and the incidence of oscillations observed by Phillips to characterise integral feedback will apply equally to the optimal controller (68). Thus there will exist a functional relation between the Riccati coefficients k_1, k_2 demarcating damped oscillatory from exponentially damped responses; the relation between cycles and policy being susceptible to the analysis conducted in chapter VI.

Whether or not integral feedback actually occurs in optimal controllers depends therefore on two factors: firstly, whether step disturbances do or do not occur; and secondly, whether step disturbances, if occurring, are treated as a static or dynamic design problem. Although freedom exists with respect to this design choice, this section argues that integral feedback, by avoiding the need for explicit measurement, is preferable when shortrun step disturbances perturb the system with any frequency.

7.5 CONCLUSIONS

Selection of a particular configuration of the degrees of freedom in regulator specifications of the stabilisation problem must therefore be accompanied by an analysis and appreciation of the costs of that choice. This is clearly stated by Zadeh [p.59]:

> "One of the most serious weaknesses of the current theories of optimal control is that they are predicated on the assumption that the performance of a system [S] can be measured by a single number ... The trouble with this concept of optimality is that, in general, there is more than one consideration that enters into the assessment of performance of S and in most cases these considerations cannot be subsumed under a single scalar-valued criterion. In such cases, a system S may be superior to a system S' in some respects and inferior to S' in others, and the class of systems is not completely ordered."

Zadeh suggests a two-stage design process: (i) determine the set of noninferior systems, and (ii) select one system from this set according to explicit, if subjective, criteria. Alternatively, Waltz proposes an hierarchical procedure in which a primary criterion is optimised; a secondary criterion then being optimised to compromise the primary objective within prescribed tolerances; and so on sequentially.

But whether the scalar-valued approach, the vector-valued approach, or the hierarchical approach is used, the problem remains of adequately specifying all relevant objectives, their priorities, and the sensitivity of design to changes in these priorities. Research in two directions will contribute to understanding of these issues. Empirically, the increasing use of econometric models for analysis of dynamic control will generate a sharper definition of objectives and of the opportunity costs of alternative specifications. Theoretically, an attempt should be made to embed as many of the degrees of freedom as possible within a general specification of the control problem so that the effects of alternative choices may also be assessed theoretically. The optimal regulator model offers no simple panacea for the practical design of stabilisation policy until these specification problems are resolved.

CHAPTER VIII

CONCLUSIONS

This final chapter presents the major conclusions of the thesis together with appropriate qualifications and possible extensions.

Once the Keynesian premise of government intervention for purposes of economic stabilisation is accepted dynamically as well as statically, conditions for which dynamic intervention is effective must be defined. Attention then shifts from the natural stability properties of an economic system to its controlled stability properties - or to controllability. Controllability is shown to unify the analysis of policy existence. Statically, the fundamental existence proposition is Tinbergen's rule that there exist as many independent static instruments as there are independent static targets. This thesis provides a companion rule for dynamic existence: either there exist as many independent dynamic instruments as there are independent dynamic targets; or there exist fewer dynamic instruments than dynamic targets, each instrument being independent of all other instruments and all targets in a fashion defined precisely in theorem 4.4. Dynamic existence need no longer be treated in an intuitive ad hoc manner, but should be tested for in all policy models as an indispensable preliminary to policy design.

That the static and dynamic problems of existence are distinct problems is stressed. Static controllability determines the equilibrium position to which the controlled economic system tends to settle in the absence of disturbances; dynamic controllability specifies the policy ability to regulate the economic system to this particular equilibrium in the presence of disturbances. This distinction is important when considering the implications of reduced stabilisation, a policy option arising if the minimal number of dynamic instruments necessary and sufficient for stabilisation is less than the actual number of instruments available. Surprisingly, if an economic system is dynamically controllable, it will generally be reducibly controllable with just one instrument, provided a certain structural condition - the Jordan chain condition - is satisfied. If the separate functions of static and dynamic controllability are not recognised, the ability to stabilise the system dynamically with a single instrument appears to contradict the necessity of using many instruments to stabilise it statically. The processes of static and dynamic stabilisation occur simultaneously; to use just one instrument for dynamic stabilisation means not that Tinbergen's static rule is irrelevant but that only one of these necessary static instruments need vary dynamically, thus simplifying the complexity of the dynamic design problem.

This static-dynamic dichotomy is also important at the design level. Dynamically, the optimal regulator hypothesis is used to design policy to stabilise the system around the preferred static equilibrium; with the optimal controllers thereby constructed belonging to the class of either automatic or discretionary stabilisers, as discussed, for example, by Pack. Little is said in the thesis concerning the determination of the optimal static equilibrium, beyond noting that the flexible targets approach is a device to select an attainable static equilibrium in the absence of static controllability. Yet this approach is not the only manifestation of the optimality concept in the static literature: a second variant extends the preference ordering to the static instruments, thus assigning them target status. For example, Holt presents an optimal Keynesian multiplier model in which the target level of the instrument, government expenditure, is defined [p.23] "by the need for governmental services not considering the requirements of economic stabilization and growth". Static optimisation in this case refers to the tradeoff between the allocation and stabilisation goals, the type of intergoal conflict resolved by Musgrave in terms of his multiple budgets theory. Although, in examining the dynamic problems of existence and design, the thesis assumes that these static problems are settled, these points emphasise that both controllability and optimality serve different functions statically and dynamically.

Dynamic controllability is analysed in this thesis in the context of a linear dynamic model of a nongrowing economy. An immediate avenue for further research is the role of controllability in cyclical growth models, as presented for example by Phillips [1961] and Bergstrom. Nonlinear dynamics, when occurring in these models, will confine controllability to a local rather than global relevance. The controllability criteria presented also ignore the effects of possible control constraints; investigation of these represents another theoretical extension. In practice, it will be necessary to develop a theory of stochastic controllability applicable to econometric control models. The concept of reduction must also be investigated further. To rationalise the choice of a reduced system, a measure of the degree of controllability possessed by a given system seems useful. Kalman, Ho & Narendra explore this problem in terms of the minimum control energy necessary to send a given state to zero; while Kalman [1969, p.39] suggests, for linear constant systems, the absolute values of the nxn determinants of the controllability matrix. Development of these ideas may ultimately provide policy-makers with guide-lines for selecting the best mix to comprise a minimal set of dynamic instruments.

Resolution of the dynamic existence problem allows full attention to focus on the design problem. Two approaches to the design of dynamic stabilisation policy are recognised: classical and optimal. The optimal approach is ostensibly a powerful generalisation of the classical approach; yet examination of the first major analysis of optimal stabilisation policy implies that the technique is still-born - for according to Fox, Sengupta & Thorbecke, optimal policies are destabilising. It is shown that the relevant contribution to the theory of optimal design is not this impossibility theorem but a sensitivity theorem stating that suboptimal policies are destabilising. Implementation requires precision, but optimal policies may certainly be designed[1].

Turning to theoretical investigation of the properties of optimal stabilisation policies, optimal design with a lagged instrument is considered specifically. The concept of policy lag, as defined by Phillips, appears ambiguous. In reconciliation, it is argued that the distinction between potential and actual policy demands is a distinction between unobservable and observable variables, serving to identify the effect of inside lag. The nature of the lagged design problem is then

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1 The history of the paradox is perplexing: Sengupta, in a subsequent paper, accepts that optimal design is possible but without reference to, or renunciation of, the paradox.

developed. Although inside lag is not destabilising, it tends to induce cycles the more closely it matches in length the outside lag and the more strongly policy is applied. To inhibit these cycles, the target derivative is weighted in the performance functional, and it is demonstrated that choice of the weighting parameter greater than the length of the outside lag is sufficient to prevent cyclical fluctuations.

These conclusions are specific to the model employed and cannot be generalised. But the primary importance of this simplified analysis derives from the understanding gained of the structure of the optimal controller. When compared with a classical controller, the two controllers are seen to possess a common structure involving proportional and derivative feedbacks. And from the analysis of target derivative weighting, one of the roles of derivative feedback in the optimal controller is to dampen oscillations, a prime function of derivative feedback in classical stabilisation policies. Classical controllers also usually contain integral feedback but this is not so for the customary optimal controller. Again, the two design approaches may be reconciled after recognising that optimal policy is optimal not only with respect to a given criterion but also with respect to a given class of disturbances. Optimal controllers are commonly designed to counter impulse disturbances whereas classical controllers are traditionally designed to counter step disturbances. By including step disturbances in the formulation of the optimal design problem, integral feedback is shown to occur naturally in the optimal controller.

Appearance of integral feedback in any controller, classical or optimal, reflects a prior decision about the appropriate treatment of step disturbances. These may be treated as either a static or a dynamic problem. If treated statically, these disturbances must be measured exactly and offset in full by an opposite movement in static instruments; if treated dynamically, these disturbances need not be measured but are ultimately offset in full. The choice between the two approaches rests, therefore, on the tradeoff between design complexity and informational requirements. The static procedure is simpler to design but requires precise disturbance measurement; the dynamic procedure is more complicated to design but avoids the need for measurement. Where step disturbances occur with any frequency, the use of integral feedback appears preferable.

Mundell's principle of effective market classification [chap. 14] also utilises integral feedback (although not explicitly identified as such) to compensate for measurement difficulties. Mundell proposes that even though a system is statically controllable, the static structure may be incompletely known, precluding policy design by direct inversion. By specifying the rate of change of the instrument vector as a linear function of the level of the target deviation vector (that is, an integral feedback policy), where the linear relation is chosen so that the policy solution is stable and uses only known structural information, equilibrium can only be achieved if the target deviation is zero, as desired. This principle of effective market classification is a dynamic procedure for solving a static design problem; and therefore differs from the classical and optimal techniques which refer to the problem of dynamic design.

Thus it is demonstrated that classical and optimal stabilisation, as design methods, are fully reconciled; for an illustrative low order model, qualitatively similar controllers will be designed under each method. Optimal stabilisation is, however, more general than classical stabilisation which employs only proportional, integral, and derivative feedbacks. In the general regulator model, feedback is a function of state so that, depending on the state definition, any order of derivative feedback may occur in optimal controllers. The price of this generality, of matching control dynamics precisely to system dynamics, is that the state is internal to the system and generally unobservable. To institute feedback as a function of state, it is therefore necessary to observe or reconstruct the state from knowledge of the system and of the observable inputs and outputs. Kalman's Principle of Duality - Kalman [1961], Sage [chap. 11] - reveals that this problem of state observation is the mathematical dual of the problem of policy optimisation. Practical application of optimal design techniques will require explicit consideration of state reconstruction.

As a generalisation of classical design, optimal design makes explicit both the design criteria and their relative importance. Since optimality is defined expressly with respect to a given criterion, the onus rests with the analyst to ensure not only that all desired policy objectives are adequately incorporated in the criterion but also that no concealed objectives are included. This problem of appropriately specifying and balancing objectives is the problem of degrees of freedom in performance specification; and considerable work remains to be done in defining policy options in this respect. Further insight may be gained by analysing optimal design as a frequency domain as well as time domain problem, along the lines suggested by Kalman [1964] and Brockett.

Dimensionality is an aspect of dynamic policy design that has not received sufficient attention in the theory of economic policy. This thesis therefore presents a preliminary exploration of computational problems associated with optimal stabilisation. Separate routines are derived for the finite horizon and infinite horizon regulator models. The preliminary nature of this work is stressed; extensions are necessary in at least three directions. Firstly, a particular computational approach is adopted without attempting to justify it as the best approach. The procedure therefore needs to be related to the family of available procedures - considered, for example, by Bryson & Ho [chap. 7] and Falb [pp. 142-60] - and evaluated comparatively. Secondly, the procedure needs to be applied to stabilisation models that are essentially numerical without being unduly complex; in this way, an appreciation of applied problems can be obtained and the effects of increasing dimensional complexity can be assessed. Thirdly, the computational effects of introducing constraints on the state and control spaces need to be considered concurrently with their theoretical introduction. The problem of computation in implementing dynamic stabilisation policy is involved, substantial, and unavoidable.

Looking at dimensionality from a theoretical viewpoint, Peacock & Shaw [p.141] have emphasised the inadequacy of scalar policy models for investigation of what is realistically a multi-dimensional design problem. This criticism is supported in this thesis with respect to both existence and design. The scalar policy model, traditionally used for analyses of dynamic stabilisation, suppresses concern for existence, and also precludes the possibility of reduction, a multi-dimensional option. So far as design is concerned, the simplest case of the scalar policy model, the first-order model, is deficient in several respects: the range of possible dynamic behaviour is limited by the real saddle point requirement, thus inhibiting the problem of hyperstability; the precision loss problem, caused by comparison of nxn matrix blocks, cannot occur because n = 1; and there is no problem of establishing relative target and instrument tradeoffs. For these reasons alone, the effects of dimension must be carefully studied when attempting to generalise theoretical results obtained from low-order models.

Two extensions of the optimality concept in a multi-dimensional framework appear to have particular significance for economic policy: control by aggregation and hierarchical control. Aoki has proposed that large-scale dynamic systems be controlled by aggregating their relations according to certain criteria, and considering the optimal design problem for the aggregated system of reduced state dimension. Aggregated control has clear economic relevance and, since Aoki explicitly uses the aggregation concept developed by economists, incentive exists for pursuing this topic further. Recently, Mesarovic *et al* have developed the notion of hierarchical optimal control. The economic significance of hierarchical control has already been established by Tinbergen's analysis [1954] of centralised and decentralised policy-making; and is confirmed by such topical issues as coordination of Federal-State policy-making and policy harmonisation in customs unions. Integration of the mathematical and economic theory is therefore a natural progression.

To conclude, controllability and optimality, the conceptual expressions for existence and design used to structure this thesis, are united in a basic manner by Kalman [1969, p.49] who states that

> "The possibility of constructing an arbitrarily good control law is limited only by the controllability properties of the plant."

In other words, the deeper significance of controllability is that it is not necessary that the economic system be naturally well-behaved; optimal behaviour can be induced artificially by feedback, provided the controllable model is an accurate representation of the real economic system. Yet for economists, if not for engineers, the proviso of accurate system modelling is a serious constraint. Phillips [1968, p.164] strongly emphasises that the economic control problem is a problem of simultaneous estimation and control, or of adaptive control; but, according to Kalman [1969, p.51],

> "The theory of adaptive systems is much talked about, but very little has been accomplished."

This, then, is the challenge for economists. On the one hand, the magnificent promise of arbitrary adjustment of economic systems to promote stability, locally if not globally; on the other hand, the sobering prospect of accurate system estimation as a precondition. Whether this promise will be realised is presently one of the imponderables; but it is yet too soon for pessimism.

APPENDIX IIa

AN OPTIMAL BERGSTROM REGULATOR

Bergstrom presents [pp.35-8] a Keynesian model of the goods market in an open economy, characterised by dynamic adjustment mechanisms on both the demand side and the supply side. Using the definitions

$$C^{*} \equiv (1-s)(Y-T) + A, \quad 0 < s < 1, \quad A \text{ constant,} \tag{a.1}$$

$$E \equiv C + I + G + X, \tag{a.2}$$

$$P \equiv Y + M, \tag{a.3}$$

where C^{*}, E, and P denote desired consumption, aggregate demand, and aggregate supply respectively, Bergstrom's model is

$$C = \frac{\alpha}{D+\alpha}\, C^{*}, \quad \alpha > 0, \quad D \equiv d/dt, \tag{a.4}$$

$$I = \frac{\gamma}{D+\gamma}\, vDY, \quad v,\gamma > 0, \tag{a.5}$$

$$DY = \mu(S^{*}-S) - \lambda DS, \quad \mu,\lambda > 0, \tag{a.6}$$

$$DS = P - E, \tag{a.7}$$

$$S^{*} = eE + F, \quad e > 0, \; F \text{ constant}, \tag{a.8}$$

$$M = mE, \quad 0 < m < 1, \tag{a.9}$$

$$T = kY - B, \quad 0 < k < 1, \quad B > 0, \tag{a.10}$$

$$X \text{ constant,} \tag{a.11}$$

$$G \text{ to be specified.} \tag{a.12}$$

The variables of the model are:

$C \equiv$ aggregate consumption
$I \equiv$ " investment
$G \equiv$ " government expenditure
$X \equiv$ " exports
$Y \equiv$ " domestic supply
$M \equiv$ " imports
$S \equiv$ " inventories
$S^* \equiv$ " desired inventories
$T \equiv$ " taxation.

For the economics of this specification, see Bergstrom [chap. 3].

Equations (a.1) to (a.12) yield the dynamic reduced form

$$\mathring{Z}(t) = AZ(t) + bG(t) + d, \quad Z(0)=Z_o, \tag{a.13}$$

where

$$\left\{\begin{array}{l} A = \begin{bmatrix} -\alpha & 0 & \alpha w & 0 \\ \gamma v r & \gamma(vr-1) & -\gamma v\lambda & -\gamma v\mu \\ r & r & -\lambda & -\mu \\ -(1-m) & -(1-m) & 1 & 0 \end{bmatrix}, \quad \begin{array}{l} w \equiv (1-s)(1-k), \\ r \equiv \lambda(1-m)+\mu e, \end{array} \\ \\ b = \begin{bmatrix} 0 \\ \alpha v r \\ r \\ -(1-m) \end{bmatrix}, \quad d = \begin{bmatrix} \alpha(1-s)B+\alpha A \\ \gamma v r X+\gamma v\mu F \\ rX+\mu F \\ -(1-m)X \end{bmatrix}, \quad Z(t) = \begin{bmatrix} C(t) \\ I(t) \\ Y(t) \\ S(t) \end{bmatrix}. \end{array}\right. \tag{a.14}$$

Hence there are four state variables and one control variable.

Associated with (a.13) is a desired static equilibrium

$$0 = A\bar{Z} + b\bar{G} + d, \tag{a.15}$$

determined by principles considered in chapter III below. Subtraction of (a.15) from (a.13) provides the dynamic disequilibrium system

$$\dot{z}(t) = Az(t) + bg(t), \quad z(0)=z_0 \neq 0, \tag{a.16}$$

where

$$\begin{cases} z(t) \equiv Z(t) - \bar{Z} \equiv \begin{bmatrix} C(t)-\bar{C} \equiv c(t) \\ I(t)-\bar{I} \equiv i(t) \\ Y(t)-\bar{Y} \equiv y(t) \\ S(t)-\bar{S} \equiv s(t) \end{bmatrix}, \\ \\ g(t) \equiv G(t) - \bar{G}. \end{cases} \tag{a.17}$$

A regulator formulation to illustrate computational problems may be obtained by specifying a criterion functional for the dynamic system (a.16). Thus, taking the traditional targets of internal and external balance, and assuming $\bar{Y}$ in (a.17) is full employment income,

$$\begin{aligned} &y = 0, \\ &B = X - M = 0 \Leftrightarrow b = M - \bar{M} = m(c+i+g) = 0, \end{aligned} \tag{a.18}$$

are used as dynamic stabilisation objectives. Incorporating these objectives in a quadratic criterion functional then gives

$$W = \tfrac{1}{2} \int_0^T [y^2(t) + \theta b^2(t) + \phi g^2(t)]\, dt, \quad \theta \geqslant 0,\ \phi > 0, \tag{a.19}$$

where θ and ϕ weight the costs of external target deviation and dynamic control relative to each other and to the internal target.

Using the expression for b in (a.18), and recasting (a.19) as a quadratic form in the state vector z(t), an optimal Bergstrom regulator is given by:

$$\left\{\begin{array}{l} \underset{g(t)}{\text{MIN}}\ W = \tfrac{1}{2}\displaystyle\int_0^T [z^T(t)Vz(t)+2\kappa^T z(t)g(t)+\pi g^2(t)]dt \\ \qquad\qquad\qquad\qquad\qquad\qquad + \tfrac{1}{2}z^T(T)Fz(T) \\ \\ \underline{\text{subject to}} \\ \\ \dot{z} = Az + bg, \quad z(0)=z_0 \neq 0, \quad z(T)\ \text{free}, \\ \\ T\ \text{fixed}, \quad g(t), \quad t \in [0, T],\ \text{unconstrained}, \end{array}\right. \tag{a.20}$$

where

$$\left\{\begin{array}{l} V = \begin{bmatrix} \theta m^2 & \theta m^2 & 0 & 0 \\ \theta m^2 & \theta m^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \kappa = \begin{bmatrix} \theta m^2 \\ \theta m^2 \\ 0 \\ 0 \end{bmatrix}, \\ \\ \pi = \phi + \theta m^2, \\ \\ F = \begin{bmatrix} f_1 & 0 & 0 & 0 \\ 0 & f_2 & 0 & 0 \\ 0 & 0 & f_3 & 0 \\ 0 & 0 & 0 & f_4 \end{bmatrix}. \end{array}\right. \tag{a.21}$$

The parameters $f_i \geqslant 0$ of the terminal weighting matrix F are terminal weights on the state variables $z_i(T)$.

Chapter II uses this Bergstrom regulator to demonstrate certain problems occurring in the development of a general computational algorithm for the regulator. Although no numerical applications to linear optimal stabilisation are presented in this thesis, the Bergstrom regulator is indicative of a numerical approach. The conceptual problems of the regulator specification are amplified in subsequent chapters.

APPENDIX IIb

SUPPORTING PROOFS FOR COMPUTATIONAL ALGORITHM

The proofs of Lemma 2.1 and Theorem 2.1 are reproduced from Zadeh & Desoer [pp.302-5]. Theorem 2.2 is not used in the final computational algorithm and its proof is therefore omitted.

Lemma 2.1

Since $B(s)$ is the adjoint of $[sI - H]$ and $d(s) = |sI - H|$, the (i,k)-element of $[sI - H]^{-1}$ is $M_{ki}(s)/d(s)$, where $M_{ki}(s)$ is the cofactor of the (k,i) element of $[sI - H]$. Therefore, $M_{ki}(s)$ is of degree $2n-1$ at most in s. If B_{j+1} is the matrix whose (i,k) element is the coefficient of s^{2n-1-j} in $M_{ki}(s)$, $j = 0, 1, \ldots, 2n-1$, then the lemma follows.

...

The proof of Theorem 2.1 requires the following corollary.

Corollary

$$d_{2n-1} = - \sum_{i=1}^{2n} H_{ii} = \operatorname{tr} B(s)\big|_{s=0}, \tag{b.1}$$

where H_{ii} is the cofactor of element h_{ii} of H.

proof

Equation (2.14) implies

$$\lim_{s\to 0} \frac{d}{ds}\{d(s)\} = d_{2n-1} = \lim_{s\to 0} \frac{d}{ds}|sI - H|. \tag{b.2}$$

By differentiation of the determinant and evaluation of the limit,

$$d_{2n-1} = - \sum_{i=1}^{2n} H_{ii}. \tag{b.3}$$

But from (2.15),

$$B(s)\big|_{s=0} = B_{2n} \quad \Rightarrow \quad \text{tr } B(s)\big|_{s=0} = \text{tr } B_{2n}. \tag{b.4}$$

Now $B_{2n}(i,k)$ is the matrix of cofactors of $M_{ki}(s)\big|_{s=0}$ or of $-H$. Hence

$$\text{tr } B(0) = \text{tr } B_{2n} = - \sum_{i=1}^{2n} H_{ii}$$

$$= d_{2n-1}, \tag{b.5}$$

from (b.3).

...

Theorem 2.1

From (2.13),

$$d(s)I = B(s)\ [sI - H], \tag{b.6}$$

or, from (2.14) and (2.15),

$$[s^{2n}+d_1 s^{2n-1}+\ldots+d_{2n-1}s+d_{2n}]I$$

$$= s^{2n}B_1+s^{2n-1}[B_2-B_1H]+s^{2n-2}[B_3-B_2H]+ \ldots$$

$$+s[B_{2n}-B_{2n-1}H]-B_{2n}H. \tag{b.7}$$

Equating matrix coefficients in (b.7):

$$B_1 = I$$

$$B_2 = B_1H + d_1I$$

$$\vdots$$

$$B_{k+1} = B_kH + d_kI \tag{b.8}$$

$$\vdots$$

$$B_{2n} = B_{2n-1}H + d_{2n-1}I$$

$$0 = B_{2n}H + d_{2n}I.$$

This result establishes the first set of (2.16).

The scalar coefficients d_j remain to be determined. Let σ be an arbitrary complex number. Then

$$d(s+\sigma) = |(s+\sigma)I - H|, \tag{b.9}$$

with Taylor expansion

$$d(s+\sigma) = d(\sigma) + sd'(\sigma) + \frac{s^2}{2!}d''(\sigma) + \ldots + \frac{s^{2n}}{(2n)!}d^{(2n)}(\sigma), \tag{b.10}$$

Since

$$\lim_{s\to 0} \frac{d}{ds}\{d(s+\sigma)\} = d'(\sigma) = d_{2n-1}, \tag{b.11}$$

then

$$d'(\sigma) = \text{tr } B(\sigma) \iff d'(s) = \text{tr } B(s), \tag{b.12}$$

by change of variable, and given the corollary above. Evaluating (b.12) from (2.14) and (2.15) provides

$$\sum_{k=o}^{2n} kd_{2n-k}s^{k-1} = \text{tr}[s^{2n-1}B_1+\ldots+sB_{2n-1}+B_{2n}],\ d_o \equiv 1. \tag{b.13}$$

Equating scalar coefficients:

$$\begin{aligned} \text{tr } B_1 &= 2n \\ \text{tr } B_2 &= (2n-1)d_1 \\ &\vdots \\ \text{tr } B_{k+1} &= (2n-k)d_k \\ &\vdots \\ \text{tr } B_{2n} &= 0. \end{aligned} \tag{b.14}$$

And from the trace of the matrix equations (b.8),

$$\begin{aligned}
\text{tr } B_1 &= 2n \\
\text{tr } B_2 &= \text{tr } B_1H + 2nd_1 \\
&\vdots \\
\text{tr } B_{k+1} &= \text{tr } B_kH + 2nd_k \\
&\vdots \\
\text{tr } B_{2n} &= \text{tr } B_{2n-1}H + 2nd_{2n-1} \\
0 &= \text{tr } B_{2n}H + 2nd_{2n}.
\end{aligned} \tag{b.15}$$

The second set of (2.16) follows from solution of (b.14) and (b.15), using $B_1 = I$ from (b.8).

...

Theorem 2.3

To establish that W of (2.49) is simplectic, O'Donnell (i) exhibits the structure of W^{-1} implied by the Hamiltonian saddle point and (ii) demonstrates that W satisfies definition 2.2.

(i) Suppose that $-\lambda$, λ are eigenvalues of the Hamiltonian matrix H with right eigenvectors α, β. Then

$$H\beta = \lambda\beta \quad \Rightarrow \quad JH^TJ\beta = \lambda\beta, \text{ by (2.33)},$$

so that

$$H^TJ\beta = -\lambda J\beta \quad \Rightarrow \quad (J\beta)^TH = -\lambda(J\beta)^T. \tag{b.16}$$

And

$$H\alpha = -\lambda\alpha \quad \Rightarrow \quad JH^TJ\alpha = -\lambda\alpha,$$

so that

$$H^TJ\alpha = \lambda J\alpha \quad \Rightarrow \quad (J\alpha)^TH = \lambda(J\alpha)^T. \tag{b.17}$$

Hence (b.16) and (b.17) imply that $J\beta$ and $J\alpha$ are left eigenvectors corresponding to $-\lambda$, λ.

In order that the left eigenvector matrix be the inverse of the right eigenvector matrix, biorthogonal normalisation is necessary. Now

$$(J\beta)^T\alpha = 1, \qquad (J\alpha)^T\beta = 1 \tag{b.18}$$

are, given (2.33), inconsistent. But by (b.16), $(-J\beta)$ is also a left eigenvector of H corresponding to $-\lambda$, permitting

$$(-J\beta)^T\alpha = 1, \qquad (J\alpha)^T\beta = 1 \tag{b.19}$$

as consistent normalising relations. Hence the right eigenvector matrix W associated with H has the structure

$$\begin{aligned} W_{2n} &= [W_1 \mid W_2] \\ W_1 &= [\alpha_1 \ \dots \ \alpha_n]_{2nxn}, \quad W_2 = [\beta_1 \ \dots \ \beta_n]_{2nxn}, \end{aligned} \tag{b.20}$$

and the normalised left eigenvector matrix W^{-1} has the structure

$$\begin{aligned} W_{2n}^{-1} &= [-JW_2 \mid JW_1]^T \\ &= [-J\beta_1 \ \dots \ -J\beta_n \mid J\alpha_1 \ \dots \ J\alpha_n]^T. \end{aligned} \tag{b.21}$$

(ii) From (b.21) and the definition of J,

$$W^{-1} = [-JW_2 \mid JW_1]^T = \left[\begin{array}{c} W_2^TJ \\ \hline -W_1^TJ \end{array}\right]. \tag{b.22}$$

But

$$JW^TJ = \left[\begin{array}{c|c} 0 & -I \\ \hline I & 0 \end{array}\right]\left[\begin{array}{c} W_1^T \\ \hline W_2^T \end{array}\right] J = \left[\begin{array}{c} -W_2^TJ \\ \hline W_1^TJ \end{array}\right]. \tag{b.23}$$

Hence from (b.22), (b.23),

$$-JW^TJ = W^{-1} \tag{b.24}$$

and W is simplectic by (2.47). Performing the indicated multiplication of the lefthand side of (b.24) yields equation (2.49) for W^{-1} in terms of transposed partitions of W.

...

Equation (2.82)

Using the Laplace transform method (cf. (2.12) and lemma 2.1) to evaluate E(t),

$$[sI-\Lambda_1^*]^{-1} = \begin{bmatrix} [sI+X]^{-1} & & & 0 \\ & [sI+\Sigma]^{-1} & & \\ & & [sI-X]^{-1} & \\ 0 & & & [sI-\Sigma]^{-1} \end{bmatrix}. \tag{b.25}$$

Because of the block diagonality of (b.25), E(t) is then given by

$$E(t) = \mathcal{L}^{-1}[sI-\Lambda_1^*]^{-1} = \begin{bmatrix} \mathcal{L}^{-1}[sI+X]^{-1} & & & 0 \\ & \mathcal{L}^{-1}[sI+\Sigma]^{-1} & & \\ & & \mathcal{L}^{-1}[sI-X]^{-1} & \\ 0 & & & \mathcal{L}^{-1}[sI-\Sigma]^{-1} \end{bmatrix}, \tag{b.26}$$

where $\mathcal{L}^{-1}$ is the inverse Laplace transform operator.

Since $\pm\Sigma$ are, by (2.59), scalar diagonal matrices,

$$\mathcal{L}^{-1}[sI \mp \Sigma]^{-1} = e^{\pm\Sigma t} \tag{b.27}$$

directly. Now, using (2.66),

$$\left\{\begin{aligned} &\mathcal{L}^{-1}[sI \mp X]^{-1} = \mathcal{L}^{-1}\begin{bmatrix} \frac{1}{\Delta_1}\begin{pmatrix} s\mp\sigma_1 & \pm\omega_1 \\ \mp\omega_1 & s\mp\sigma_1 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\Delta_c}\begin{pmatrix} s\mp\sigma_c & \pm\omega_c \\ \mp\omega_c & s\mp\sigma_c \end{pmatrix} \end{bmatrix}, \\ &\Delta_j = (s \mp \sigma_j)^2 + \omega_j^{\,2}, \quad j = 1, \ldots, c. \end{aligned}\right. \tag{b.28}$$

The second part of equation (2.82) follows after evaluation of the inverse Laplace transform in (b.28).

APPENDIX IIc

CODING FOR COMPUTATIONAL ALGORITHMS

Table II.1 depicts the basic structure of the computational programs derived from chapter II. Finite horizon regulator solutions are obtained using the sequence MAIN, [1], [2], [3], [4A], and [5], together with dependent subroutines. Infinite horizon regulator solutions use the same sequence with [4A] replaced by [4B].

Routines belonging to columns A and B are either program control or input/output routines, and are therefore specified with maximum dimensions. Routines belonging to columns C to E are task-oriented and are written with variable dimensions, the exceptions being the output routines ØUTPUT, RØUT, TEST, ATEST. Column E contains routines performing standard tasks of matrix manipulation required at various points.

Coding for the finite horizon program is given on pp. 214-41 below, with the maximum dimensions and input/output routines appropriate for the Bergstrom regulator of Appendix IIa. Output for a single run is attached on pp. 242-9. The subroutine GRAPH, written to handle large quantities of output graphically, is model-dependent and not included here. Also not shown are the double precision IBM library subroutines, DPRQD and MINV.

Member routines of [4B], necessary for the infinite horizon program, and not appearing already in column D, are shown on pp. 250-4 below. Output for a single run, with the same system configuration as the finite horizon run, follows on pp. 255-60.

TABLE II.1

STRUCTURE OF REGULATOR PROGRAMS

A	B	C	D	E
MAIN	[1] PARAMS			
	[2] AFORM			
	[3] PASS1	EIGVAL	CHAREQ RØØTS EIGSRT RØUT	
	[4A] PASS2	FINITE	VECGET └EIGVEC FEEDCØ ØUTPUT TEST PARTP RICCØN TRANEX TIMATS CØSRØG STACØN	MATEQO VMULT TRACE MSAVE MULTM MATADD
	[5] GRAPH			
	[4B] PASS2	ATØTIC	VECGET └EIGVEC FEEDCØ ØUTPUT ATEST PARASP CØST EXMAT STATE STEER	

```
C       ILLUSTRATIVE MAIN PROGRAM FOR FINITE HORIZON REGULATOR.
C       MAXIMUM DIMENSIONS ARE FOR BERGSTROM 4-STATE MODEL.
        IMPLICIT REAL*8(A-H,O-Y)
        REAL *8ZINT
        COMMON /MODELS/ MODEL,NDS,KTIME,KH,NOH,NSIM
        COMMON /PRMS/ M,M2,MK,M1,M3,INCRR,INCRC,IC1,IC3,IC4,IC5
        COMMON /TIME/ TT,TINC,TRIP
        COMMON /GRAPHS/ ZOORDS(5,101,6),ZX(101),ZAP(11,5)
        DIMENSION H(8,8), ENDK(4,4), ZINT(4), RR(8), RI(8), R(4,4), BB(4,4
       1), QQ(4,4)
        NDS=1
        NSIM=1
1       WRITE (3,4)
        CALL PARAMS
        WRITE (3,5) MODEL,NDS
        CALL AFORM (H,M2,M,MK,ENDK,ZINT,R,BB,NSIM,QQ,&1000)
        CALL PASS1 (H,RR,RI)
        CALL PASS2 (H,RR,RI,ENDK,ZINT,R,BB,QQ)
        IF (NSIM.NE.5) GO TO 2
        CALL GRAPH
        NSIM=1
        GO TO 3
2       NSIM=NSIM+1
3       NDS=NDS+1
        GO TO 1
1000    CONTINUE
        CALL EXIT
C
4       FORMAT ('1')
5       FORMAT ('0',15X,'MODEL = ',I3,5X,'DATA SET = ',I2)
        END
        SUBROUTINE PARAMS
C       S/R PARAMS ACQUIRES HORIZON AND DIMENSION INFORMATION.
C       KTIME = 0 IF FINITE HORIZON REGULATOR, KTIME = 1 IF ASYMPTOTIC
C       REGULATOR.  (TT) IS HORIZON SOLUTIONS ARE COMPUTED TO, IN (KH)
C       STEPS OF LENGTH (TINC*TT).  OUTPUT FOR DIFFERENT LENGTH HORIZONS
```

```
C     IS OBTAINABLE BY SPECIFYING THE NUMBER OF HORIZONS (NOH) AND THE
C     CONSTANT INCREMENT (TRIP) TO FIRST HORIZON TT.  M,MK ARE STATE
C     AND CONTROL VECTOR DIMENSIONS.  OTHER INFORMATION IN COMMON BLOCK
C     /PRMS/ IS PROGRAM GENERATED.
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON /TIME/ TT,TINC,TRIP
      COMMON /MODELS/ MODEL,NDS,KTIME,KH,NOH,NSIM
      COMMON /PRMS/ M,M2,MK,M1,M3,INCRR,INCRC,IC1,IC3,IC4,IC5
      MODEL=11
C     HORIZON INFORMATION
      KTIME=0
      TT=10.
      KH=101
      TINC=.01D0
      NOH=1
      TRIP=.0
C     DIMENSION INFORMATION
      M=4
      M2=2*M
      M3=M2+1
      MK=1
      M1=M+1
      RETURN
      END
      SUBROUTINE PASS2 (H,RR,RI,ENDK,ZINT,R,BB,QQ)
C     S/R PASS2 TRANSFERS CONTROL TO FINITE.  MAXIMUM DIMENSIONS ARE
C     USER ADJUSTABLE.
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON /PRMS/ M,M2,MK,M1,M3,INCRR,INCRC,IC1,IC3,IC4,IC5
      DIMENSION H(M2,M2), RR(M2), RI(M2), ENDK(M,M), ZINT(M), R(M,M), BB
     1(M,M), QQ(M,M)
      DIMENSION STA(101,4), CON(101,2), RQ(4,4)
      DIMENSION DVI(8), DVR(8), P(8,8), A(8,8), P11(4,4), P12(4,4), P21(
     14,4), P22(4,4), PT11(4,4), PT12(4,4), PT21(4,4), PT22(4,4), EPV(4,
     24), EMV(4,4), EM(4,4), DR(4,4), RB(4,4), NCOL(8), LAL(8), LAM(8)
      CALL FINITE (RR,RI,DVI,DVR,LAL,LAM,P,P11,P12,P21,P22,PT11,PT12,PT2
```

```
     11,PT22,ENDK,EPV,EMV,EM,STA,CON,DR,RB,A,NCOL,H,ZINT,BB,R,QQ,RQ)
      RETURN
      END
      SUBROUTINE TEST (RR,RI,P,PT11,PT12,PT21,PT22,ENDK,RB,P11,P12,P21,P
     122,A,BB,R)
C     S/R TEST CALCULATES ACCURACY OF EIGENVECTOR MATRIX AS BLOCK
C     DIAGONALISING SIMILARITY TRANSFORM.  FORMAT STATEMENTS ARE USER-
C     DEPENDENT.
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON /PRMS/ M,M2,MK,M1,M3,INCRR,INCRC,IC1,IC3,IC4,IC5
      DIMENSION RR(M2), RI(M2), P(M2,M2), PT11(M,M), PT12(M,M), PT21(M,M
     1), PT22(M,M), ENDK(M,M), RB(MK,M), P11(M,M), P12(M,M), P21(M,M), P
     222(M,M), A(M2,M2), BB(M,M), R(M,M)
      WRITE (3,2)
      WRITE (3,1) ((RB(I,J),J=1,M),I=1,MK)
      WRITE (3,3)
      WRITE (3,4) ((A(I,J),J=1,M2),I=1,M2)
      RETURN
C
1     FORMAT (4F10.4)
2     FORMAT ('1 CONSTANT CONTROL FEEDBACK COEFFICIENT RB')
3     FORMAT ('0 DIAGONAL MATRIX OF EIGENVALUES')
4     FORMAT (8F10.5)
      END
      SUBROUTINE AFORM (A,M,JM,MK,ENDK,ZINT,RA,BBA,NSIM,QQA,*)
      IMPLICIT REAL*8(A-H,O-Y)
      REAL *8ZINT
      COMMON /GRAPHS/ ZOORDS(5,101,6),ZX(101),ZAP(11,5)
      DIMENSION A(M,M), AA(4,4), BBT(4,4), VV(4,4), BCT(4,4), CCT(4,4),
     1X(4,4), W(4,4), XMT(4,4)
      DIMENSION ENDK(JM,JM), ZINT(JM), RA(MK,MK), BBA(JM,MK)
      DIMENSION QQA(JM,MK)
      DIMENSION BB(4), CC(4)
C
C     AFORM CONSTRUCTS THE DESIRED SEQUENCE OF INPUT MATRICES A(I,J).
      ZINT(1)=.75
```

```
      ZINT(2)=.25
      ZINT(3)=1.
      ZINT(4)=.5
C     READ PARAMETERS OCCURRING IN SYSTEM MATRIX.
      READ (1,21,END=20) ALP,S,AK,AM,GA,ALA,AMU,EE,V,PHI,THE
      ZAP(1,NSIM)=ALP
      ZAP(2,NSIM)=S
      ZAP(3,NSIM)=AK
      ZAP(4,NSIM)=AM
      ZAP(5,NSIM)=GA
      ZAP(6,NSIM)=ALA
      ZAP(7,NSIM)=AMU
      ZAP(8,NSIM)=EE
      ZAP(9,NSIM)=V
      ZAP(10,NSIM)=PHI
      ZAP(11,NSIM)=THE
      R=ALA*(1.0-AM)+(AMU*EE)
      AA(1,1)=-ALP
      AA(2,1)=GA*V*R
      AA(3,1)=R
      AA(4,1)=-(1.0-AM)
      AA(1,2)=0.0
      AA(2,2)=GA*(V*R-1.0)
      AA(3,2)=R
      AA(4,2)=-(1.0-AM)
      AA(1,3)=ALP*(1.0-S)*(1.0-AK)
      AA(2,3)=-GA*V*ALA
      AA(3,3)=-ALA
      AA(4,3)=1.0
      AA(1,4)=0.0
      AA(2,4)=-GA*V*AMU
      AA(3,4)=-AMU
      AA(4,4)=0.0
      BB(1)=0.0
      BB(2)=GA*V*R
      BB(3)=R
```

```
      BB(4)=-(1.0-AM)
      CC(1)=THE*(AM**2)
      CC(2)=CC(1)
      CC(3)=0.0
      CC(4)=0.0
      DO 1 I=1,JM
1     QQA(I,1)=CC(I)
      PIE=PHI+(THE*(AM**2))
      DO 3 I=1,4
      DO 2 J=1,4
2     VV(I,J)=0.0
3     CONTINUE
      VV(1,1)=THE*(AM**2)
      VV(2,2)=VV(1,1)
      VV(3,3)=1.0
      VV(2,1)=VV(1,1)
      VV(1,2)=VV(2,1)
      DO 5 I=1,4
      DO 4 J=1,4
4     BBT(I,J)=(BB(I)*BB(J))/(-PIE)
5     CONTINUE
      DO 7 I=1,4
      DO 6 J=1,4
6     BCT(I,J)=(BB(I)*CC(J))/PIE
7     CONTINUE
      DO 9 I=1,4
      DO 8 J=1,4
8     CCT(I,J)=(CC(I)*CC(J))/PIE
9     CONTINUE
      DO 11 I=1,4
      DO 10 J=1,4
10    X(I,J)=AA(I,J)-BCT(I,J)
11    CONTINUE
      DO 13 I=1,4
      DO 12 J=1,4
12    W(I,J)=CCT(I,J)-VV(I,J)
```

```
13    CONTINUE
      DO 15 I=1,4
      DO 14 J=1,4
14    XMT(I,J)=-X(J,I)
15    CONTINUE
      DO 17 I=1,4
      DO 16 J=1,4
      A(I,J)=X(I,J)
      IX=I+4
      A(IX,J)=W(I,J)
      JX=J+4
      A(I,JX)=BBT(I,J)
16    A(IX,JX)=XMT(I,J)
17    CONTINUE
      WRITE (3,22)
      DO 18 I=1,M
      WRITE (3,23) (A(I,J),J=1,M)
18    CONTINUE
      DO 19 I=1,JM
19    BBA(I,1)=BB(I)
      RA(1,1)=PIE
      CALL MATEQO (ENDK,JM)
      WRITE (3,24)
      WRITE (3,25) ((ENDK(I,J),J=1,JM),I=1,JM)
      WRITE (3,26)
      WRITE (3,25) (ZINT(I),I=1,JM)
      WRITE (3,27)
      WRITE (3,28)
      WRITE (3,29) (ZAP(J,NSIM),J=1,11)
      RETURN
20    RETURN 1
C
21    FORMAT (11F6.2)
22    FORMAT ('0',35X,'A ARRAY')
23    FORMAT (8F10.2)
24    FORMAT ('0',10X,'TERMINAL WEIGHTING MATRIX')
```

```
25    FORMAT ('0',4F10.5)
26    FORMAT ('0',10X,'INITIAL STATE VECTOR')
27    FORMAT ('0',10X,'SYSTEM PARAMETERS')
28    FORMAT ('0',T19,'ALPHA',T29,'S',T37,'K',T45,'M',T51,'GAMMA',T58,'L
     1AMBDA',T68,'MU',T77,'E',T85,'V',T92,'PHI',T99,'THETA'/)
29    FORMAT (' ',13X,11F8.3)
      END
      SUBROUTINE OUTPUT (STA,CON,COST)
      IMPLICIT REAL*8(A-H,O-Y)
      COMMON /PRMS/ M,M2,MK,M1,M3,INCRR,INCRC,IC1,IC3,IC4,IC5
      COMMON /TIME/ TT,TINC,TRIP
      COMMON /MODELS/ MODEL,NDS,KTIME,KH,NOH,NSIM
      COMMON /GRAPHS/ ZOORDS(5,101,6),ZX(101),ZAP(11,5)
      DIMENSION ZBOP(101)
      DIMENSION STA(KH,M), CON(KH,MK)
      WRITE (3,8) NDS,TT,(I,I=1,M)
      DO 1 I=1,101
      IF (I.NE.35) GO TO 111
      WRITE (3,222)
222   FORMAT ('1')
111   CONTINUE
      IF (I.NE.71) GO TO 333
      WRITE (3,222)
333   CONTINUE
      WRITE (3,9) (STA(I,J),J=1,M)
1     CONTINUE
      AM=ZAP(4,NSIM)
      DO 2 J=1,KH
2     ZBOP(J)=-AM*(CON(J,1)+STA(J,1)+STA(J,2))
      WRITE (3,10)
      WRITE (3,9) (ZBOP(J),J=1,KH)
      WRITE (3,11)
      DO 3 J=1,MK
      WRITE (3,12) J
      WRITE (3,9) (CON(I,J),I=1,101)
3     CONTINUE
```

```
      WRITE (3,13) COST,MODEL,NDS
      WRITE (3,14) KTIME,TT,TINC,NOH,TRIP
      ZX(1)=0.0
      DO 4 I=2,KH
4     ZX(I)=ZX(I-1)+1.0
      IF (NOH.NE.1) RETURN
      DO 7 J=1,KH
      DO 5 K=1,M
5     ZOORDS(NSIM,J,K)=STA(J,K)
      ZOORDS(NSIM,J,M1)=ZBOP(J)
      DO 6 K=1,MK
      KX=K+M1
6     ZOORDS(NSIM,J,KX)=CON(J,K)
7     CONTINUE
      RETURN
C
8     FORMAT ('1 OUTPUT SET ',I2,3X,'TIME HORIZON = ',F5.2,/'   STATE ',
     1 6(I2,9X))
9     FORMAT (10F11.5)
10    FORMAT ('1',25X,'BERGSTROM TRADE BALANCE'/)
11    FORMAT ('0 CONTROL VARIABLES')
12    FORMAT ('-',30X,'CONTROL VARIABLE = ',I2)
13    FORMAT ('-',10X,'POLICY COST SCALAR = ',F6.2,3X,'MODEL = ',I2,3X,'
     1DATA SET = ',I3)
14    FORMAT ('0 CONTROL TYPE = ',I2,2X,'TIME HORIZON = ',F5.2,2X,'TIME
     1INCREMENT = ',F5.2,2X,'NUMBER OF HORIZONS = ',I2,2X,'INCREMENT TO
     2HORIZON = ',F5.2)
      END
      SUBROUTINE PASS1 (H,RR,RI)
C     S/R PASS1 PASSES CONTROL TO EIGVAL FOR HAMILTONIAN EIGENVALUE
C     COMPUTATION.  MAXIMUM DIMENSIONS ARE USER ADJUSTABLE.
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION H(M2,M2), RR(M2), RI(M2)
      DIMENSION B(8,8), D(8), POL(9), DA(9)
      COMPLEX *16PRO(8),CPRP(4)
      COMMON /PRMS/ M,M2,MK,M1,M3,INCRR,INCRC,IC1,IC3,IC4,IC5
```

```
      CALL EIGVAL (H,RR,RI,B,D,POL,DA,PRO,CPRP)
      RETURN
      END
      SUBROUTINE EIGVAL (H,RR,RI,B,D,POL,DA,PRO,CPRP)
C     S/R EIGVAL CONTROLS HAMILTONIAN EIGENVALUE COMPUTATION.  EIGVAL
C     AND ASSOCIATED S/R'S ARE VARIABLE DIMENSIONED.
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION H(M2,M2), RR(M2), RI(M2), B(M2,M2), D(M2), POL(M3), DA(M
     13)
      COMPLEX *16PRO(M2),CPRP(M)
      COMMON /PRMS/ M,M2,MK,M1,M3,INCRR,INCRC,IC1,IC3,IC4,IC5
      CALL CHAREQ (B,H,POL,D,SUMD)
      CALL ROOTS (RR,RI,D,POL,DA,PRO,SUMD,IR,IER)
      CALL EIGSRT (RR,RI,CPRP,PRO)
      CALL ROUT (DA,POL,IR,IER,B,RR,RI)
      RETURN
      END
      SUBROUTINE CHAREQ (B,H,POL,D,SUMD)
C     S/R CHAREQ CONSTRUCTS HAMILTONIAN CHARACTERISTIC EQUATION
C     ACCORDING TO ZADEH-DESOER ALGORITHM.
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION B(M2,M2), H(M2,M2), POL(M2), D(M2)
      COMMON /PRMS/ M,M2,MK,M1,M3,INCRR,INCRC,IC1,IC3,IC4,IC5
C     CONSTRUCT UNIT MATRIX
      CALL MATEQO (B,M2)
      DO 1 I=1,M2
1     B(I,I)=1.0
C     CALCULATE CHARACTERISTIC COEFFICIENT VECTOR D(K) WITHOUT STORING
C     ADJOINT COEFFICIENT MATRIX B
      DO 3 K=1,M2
      CALL VMULT (B,H,POL,M2)
      CALL TRACE (B,TR,M2)
      D(K)=-TR/(1.0*K)
C     ADD TRACE TO DIAGONAL OF B
      DO 2 I=1,M2
2     B(I,I)=B(I,I)+D(K)
```

```
3     CONTINUE
C     CALCULATE SUM OF ODD COEFFICIENTS
      SUMD=0.0
      MM=M2-1
      DO 4 J=1,MM,2
4     SUMD=SUMD+D(J)
      RETURN
      END
      SUBROUTINE TRACE (C,TR,M)
C     CALCULATE TRACE OF A MATRIX.
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION C(M,M)
      TR=0.0
      DO 1 I=1,M
1     TR=TR+C(I,I)
      RETURN
      END
      SUBROUTINE ROOTS (RE,CO,D,POL,DA,PRO,SUMD,IR,IER)
C     S/R ROOTS CALCULATES EIGENVALUES THROUGH IBM S/R DPRQD, AFTER
C     HALVING CHAR EQN DEGREE FOR VANISHING ODD COEFFICIENTS.
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION RE(M2), CO(M2), D(M2), DA(M3), POL(M3)
      COMPLEX *16PRO(M2)
      COMMON /PRMS/ M,M2,MK,M1,M3,INCRR,INCRC,IC1,IC3,IC4,IC5
      DO 1 I=1,M3
      POL(I)=0.0D0
1     DA(I)=0.0D0
      IF (SUMD.LT.0.01D0.AND.SUMD.GT.-.01D0) GO TO 2
      MX=M2
      JA=1
      GO TO 3
2     MX=M
      JA=2
3     JM1=MX+1
      DO 4 J=1,MX
      JM=M2+JA*(1-J)
```

```
4     DA(J)=D(JM)
      DA(JM1)=1.
      CALL DPRQD (DA,JM1,RE,CO,POL,IR,IER)
      DO 5 J=1,MX
5     PRO(J)=DCMPLX(RE(J),CO(J))
      IF (MX.EQ.M2) GO TO 7
      DO 6 J=1,MX
      JX=J+MX
      PRO(J)=CDSQRT(PRO(J))
6     PRO(JX)=-PRO(J)
7     CONTINUE
      DO 8 J=1,M2
      RE(J)=REAL(PRO(J))
8     CO(J)=AIMAG(PRO(J))
      RETURN
      END
      SUBROUTINE ROUT (DA,POL,IR,IER,B,RR,RI)
C     S/R ROUT PRINTS EIGENVALUE INFORMATION.  FORMAT STATEMENTS ARE
C     USER-DEPENDENT.
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION DA(M3), POL(M3), B(M2,M2), RR(M2), RI(M2)
      COMMON /PRMS/ M,M2,MK,M1,M3,INCRR,INCRC,IC1,IC3,IC4,IC5
      WRITE (3,2) (DA(K),K=1,M3)
      WRITE (3,3) (POL(K),K=1,M3)
      SUMR=0.0
      DO 1 I=1,M2
1     SUMR=SUMR+RR(I)
      WRITE (3,4) SUMR
      WRITE (3,5) (RR(I),I=1,M2)
      WRITE (3,6) (RI(I),I=1,M2)
      WRITE (3,7) INCRR,INCRC
      WRITE (3,8) IR,IER
      WRITE (3,9)
      WRITE (3,10) ((B(I,J),J=1,M2),I=1,M2)
      RETURN
C
```

```
2     FORMAT ('0',15X,'CHARACTERISTIC COEFFICIENTS - ASCENDING ORDER'//(
     112(F10.5)))
3     FORMAT('1',20X,'ESTIMATED COEFFICIENTS'//(12(F10.5)))
4     FORMAT ('-',15X,'SUM OF REAL EIGENVALUES = ',F6.4)
5     FORMAT ('-',15X,'ORDERED REAL-PART EIGENVALUES'//(12(F10.5)))
6     FORMAT ('-',15X,'ORDERED COMPLEX-PART EIGENVALUES'//(12(F10.5)))
7     FORMAT ('-',5X,'NUMBER OF REAL NEGATIVE ROOTS = ',I3,5X,'NUMBER OF
     1 COMPLEX NEGATIVE ROOTS = ',I3)
8     FORMAT ('-',5X,'SSP-DPRQD ERROR MESSAGES',5X,'IR = ',I3,5X,'IER =
     1',I3)
9     FORMAT ('1',10X,'FINAL ADJOINT COEFFICIENT MATRIX NULL')
10    FORMAT ('0',8F10.5)
      END
      SUBROUTINE MATEQO (AAA,IX)
C     SETS A MATRIX NULL.
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION AAA(IX,IX)
      DO 2 I=1,IX
      DO 1 J=1,IX
1     AAA(I,J)=0.0D0
2     CONTINUE
      RETURN
      END
      SUBROUTINE VMULT (AAA,BBB,RR,NX)
C     MULTIPLIES TWO MATRICES AND STORES PRODUCT IN FIRST.
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION AAA(NX,NX), BBB(NX,NX), RR(NX)
      DO 4 I=1,NX
      DO 2 K=1,NX
      RR(K)=0.0D0
      DO 1 J=1,NX
1     RR(K)=RR(K)+AAA(I,J)*BBB(J,K)
2     CONTINUE
      DO 3 L=1,NX
3     AAA(I,L)=RR(L)
4     CONTINUE
```

```
      RETURN
      END
      SUBROUTINE EIGSRT (RR,RI,CPRP,CRMP)
C     S/R EIGSRT SORTS EIGENVALUES TO SATISFY CANONICAL FORM ASSOCIATED
C     WITH MODIFIED NEGATIVE EXPONENTIAL PROCEDURE.
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON /PRMS/ M,M2,MK,M1,M3,INCRR,INCRC,IC1,IC3,IC4,IC5
      COMPLEX *16CPRP(M),CRMP(M2)
      DIMENSION RR(M2), RI(M2)
      INCRC=0
      INCRR=0
      DO 2 I=1,M
      IF (RI(I).EQ.0.0D0) GO TO 1
      INCRC=INCRC+1
      CPRP(INCRC)=DCMPLX(RR(I),RI(I))
      GO TO 2
1     INCRR=INCRR+1
      RR(INCRR)=RR(I)
2     CONTINUE
      IC1=INCRC+1
      DO 3 I=IC1,M
      IX=I-INCRC
      CPRP(I)=DCMPLX(RR(IX),0.0D0)
3     CONTINUE
      DO 4 I=1,M
4     CRMP(I)=-CPRP(I)
      DO 5 I=M1,M2
      IX=I-M
5     CRMP(I)=CPRP(IX)
C     RESTORE SORTED EIGENVALUES TO REAL ARRAYS.
      DO 6 I=1,M2
      RR(I)=REAL(CRMP(I))
6     RI(I)=AIMAG(CRMP(I))
      RETURN
      END
      SUBROUTINE FEEDCO (BB,QQ,R,DR,RB,RQ,LAL,LAM)
```

```
C     S/R FEEDCO CONSTRUCTS CONSTANT FEEDBACK COEFFICIENT VECTORS FOR
C     CONTROL SOLUTION.
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON /PRMS/ M,M2,MK,M1,M3,INCRR,INCRC,IC1,IC3,IC4,IC5
      DIMENSION LAL(MK), LAM(MK)
      DIMENSION BB(M,MK), R(MK,MK), DR(MK,M), RB(MK,M)
      DIMENSION QQ(M,MK), RQ(MK,M)
      IF (MK.EQ.1) GO TO 1
      CALL MTPSE (BB,DR,M,MK)
      CALL MTPSE (QQ,RB,M,MK)
      CALL MINV (R,MK,DET,LAL,LAM)
      CALL MULTM (R,RB,RQ,MK,MK,M,-1)
      CALL MULTM (R,DR,RB,MK,MK,M,-1)
      GO TO 3
1     ZC=-1.0D0/R(1,1)
      DO 2 I=1,M
      RQ(1,I)=QQ(I,1)*ZC
2     RB(1,I)=BB(I,1)*ZC
3     CONTINUE
      RETURN
      END
      SUBROUTINE VECGET (A,H,P,DVR,DVI,NCOL,RR,RI)
C     S/R VECGET OBTAINS HAMILTONIAN EIGENVECTORS THROUGH S/R EIGVEC,
C     STORING IN FORM APPROPRIATE TO MODIFIED NEP.
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON /PRMS/ M,M2,MK,M1,M3,INCRR,INCRC,IC1,IC3,IC4,IC5
      COMMON /MODELS/ MODEL,NDS,KTIME,KH,NOH,NSIM
      DIMENSION A(M2,M2), H(M2,M2), P(M2,M2), DVR(M2), DVI(M2), NCOL(M2)
     1, RR(M2), RI(M2)
      IF (INCRC.EQ.0) GO TO 3
      NPASS=0
      IC3=IC1-2
      IC4=M1+INCRC
      IC5=IC4-2
      DO 2 IVEC=1,IC3,2
      IVX=IVEC+1
```

```
      DSR=RR(IVEC)
      DSI=RI(IVEC)
      CALL EIGVEC (A,H,DSR,DSI,M2,DVR,DVI,NCOL,NPASS)
      DO 1 J=1,M2
      P(J,IVEC)=DVR(J)
1     P(J,IVX)=DVI(J)
2     CONTINUE
      GO TO 5
3     CONTINUE
      IF (KTIME) 4,4,5
4     MT=M2
      GO TO 6
5     MT=M
6     CONTINUE
      NPASS=1
      DO 8 IVEC=IC1,MT
      DSR=RR(IVEC)
      DSI=0.0D0
      CALL EIGVEC (A,H,DSR,DSI,M2,DVR,DVI,NCOL,NPASS)
      DO 7 J=1,M2
7     P(J,IVEC)=DVR(J)
8     CONTINUE
      IF (INCRC.EQ.0) GO TO 14
      IF (KTIME) 9,9,14
9     NPASS=0
      DO 11 IVEC=M1,IC5,2
      IVX=IVEC+1
      DSR=RR(IVEC)
      DSI=RI(IVEC)
      CALL EIGVEC (A,H,DSR,DSI,M2,DVR,DVI,NCOL,NPASS)
      DO 10 J=1,M2
      P(J,IVEC)=DVR(J)
10    P(J,IVX)=DVI(J)
11    CONTINUE
      NPASS=1
      DO 13 IVEC=IC4,M2
```

```
      DSR=RR(IVEC)
      DSI=0.0D0
      CALL EIGVEC (A,H,DSR,DSI,M2,DVR,DVI,NCOL,NPASS)
      DO 12 J=1,M2
12    P(J,IVEC)=DVR(J)
13    CONTINUE
14    CONTINUE
      RETURN
      END
      SUBROUTINE MATADD (AAA,BBB,CCC,IX,JX)
C     ADDS TWO MATRICES AND STORES RESULT IN THIRD.
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION AAA(IX,JX), BBB(IX,JX), CCC(IX,JX)
      DO 2 I=1,IX
      DO 1 J=1,JX
1     CCC(I,J)=AAA(I,J)+BBB(I,J)
2     CONTINUE
      RETURN
      END
      SUBROUTINE MULTM (AAA,BBB,CCC,IX,JX,KX,NX)
C     MULTIPLIES TWO MATRICES STORING IN THIRD AS POSITIVE OR NEGATIVE
C     PRODUCT, DEPENDING ON SIGN NX.
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION AAA(IX,JX), BBB(JX,KX), CCC(IX,KX)
      DO 3 I=1,IX
      DO 2 K=1,KX
      CCC(I,K)=0.0D0
      DO 1 J=1,JX
1     CCC(I,K)=CCC(I,K)+AAA(I,J)*BBB(J,K)
2     CONTINUE
3     CONTINUE
      IF (NX.EQ.1) GO TO 6
      DO 5 I=1,IX
      DO 4 K=1,KX
4     CCC(I,K)=-CCC(I,K)
5     CONTINUE
```

```
6     CONTINUE
      RETURN
      END
      SUBROUTINE EIGVEC (A,H,RR,RI,N,VR,VI,NCOL,NPASS)
C     S/R EIGVEC SLIGHTLY MODIFIED IGVEC5 ROUTINE OF VAUGHAN-BLACKBURN.
C     CALCULATES EIGENVECTOR CORRESPONDING TO GIVEN EIGENVALUE.  USER-
C     DEPENDENT ON LOGICAL UNIT = 11 FOR SOME I/O.
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(N,N), H(N,N), VR(N), VI(N), NCOL(N)
      T=RI
      IF (T.EQ.0.0D0) GO TO 5
      IF (NPASS.GT.0) GO TO 1
      CALL MULTM (H,H,A,N,N,N,1)
      REWIND 11
      WRITE (11) A
      REWIND 11
      NPASS=1
      GO TO 2
1     CONTINUE
      READ (11) A
      REWIND 11
2     CONTINUE
      DO 3 I=1,N
      DO 3 J=1,N
3     A(I,J)=A(I,J)-2.0D0*RR*H(I,J)
      DO 4 I=1,N
4     A(I,I)=A(I,I)+RR*RR+RI*RI
      GO TO 8
5     CONTINUE
      DO 6 I=1,N
      DO 6 J=1,N
6     A(I,J)=H(I,J)
      DO 7 I=1,N
7     A(I,I)=A(I,I)-RR
8     CONTINUE
      BIG=10.0E-25
```

```
      IBIG=0
      JBIG=0
      DO 9 I=1,N
      DO 9 J=1,N
      X=DABS(A(I,J))
      IF (X.LT.BIG) GO TO 9
      BIG=X
      IBIG=I
      JBIG=J
 9    CONTINUE
      TEMP=A(IBIG,JBIG)
      DO 10 I=1,N
      DO 10 J=1,N
 10   A(I,J)=A(I,J)/TEMP
      DO 11 I=1,N
 11   NCOL(I)=I
      N1=N-1
      ICOLX=0
      DO 27 K=1,N
      K1=K-1
      BIG=10.0E-26
      IBIG=0
      DO 13 I=K,N
      VI(I)=A(I,I)
      IF (K.EQ.1) GO TO 13
      DO 12 L=1,K1
 12   VI(I)=VI(I)-A(I,L)*A(L,I)
 13   CONTINUE
      IF (K.EQ.N) GO TO 30
      IF (T.EQ.0.0D0) GO TO 14
      IF (K.LT.N1) GO TO 14
      X=VI(N)-VI(N1)
      X=DABS(X)
      IF (X.LT.10.E-30) GO TO 25
 14   CONTINUE
      DO 15 I=K,N
```

```
      X=DABS(VI(I))
      IF (X.LT.BIG) GO TO 15
      IBIG=I
      BIG=X
15    CONTINUE
      IF (IBIG.EQ.0) GO TO 25
      IF (IBIG.EQ.K) GO TO 18
      I=NCOL(IBIG)
      NCOL(IBIG)=NCOL(K)
      NCOL(K)=I
      DO 16 I=1,N
      TEMP=A(I,K)
      A(I,K)=A(I,IBIG)
16    A(I,IBIG)=TEMP
      DO 17 J=1,N
      TEMP=A(K,J)
      A(K,J)=A(IBIG,J)
17    A(IBIG,J)=TEMP
18    CONTINUE
      IF (K.EQ.1) GO TO 21
      DO 20 I=K,N
      DO 19 L=1,K1
19    A(I,K)=A(I,K)-A(I,L)*A(L,K)
20    CONTINUE
21    CONTINUE
      KP1=K+1
      DO 24 J=KP1,N
      IF (K.EQ.1) GO TO 23
      DO 22 L=1,K1
22    A(K,J)=A(K,J)-A(K,L)*A(L,J)
23    A(K,J)=A(K,J)/A(K,K)
24    CONTINUE
      GO TO 26
25    CONTINUE
      ICOLX=K
      GO TO 28
```

```
26    CONTINUE
27    CONTINUE
28    CONTINUE
      IF (ICOLX.LT.N1) GO TO 39
      IF (ICOLX.GT.N1) GO TO 30
      A(N1,N1)=VI(N1)
      N2=N-2
      DO 29 I=1,N2
      A(N,N1)=A(N,N1)-A(N,I)*A(I,N1)
29    A(N1,N)=A(N1,N)-A(N1,I)*A(I,N)
30    CONTINUE
      A(N,N)=VI(N)
      X=DABS(VI(N))
      IF (X.LT.10.0E-12) GO TO 31
      WRITE (3,40) X,RR,RI
31    CONTINUE
      VI(N)=1.0D0
      DO 33 K=1,N1
      I=N-K
      VI(I)=0.0D0
      I1=I+1
      DO 32 L=I1,N
32    VI(I)=VI(I)-A(I,L)*VI(L)
33    CONTINUE
      DO 34 I=1,N
      J=NCOL(I)
34    VR(J)=VI(I)
      IF (T.EQ.0.0D0) GO TO 37
      DO 36 I=1,N
      VI(I)=-RR*VR(I)
      DO 35 J=1,N
35    VI(I)=VI(I)+H(I,J)*VR(J)
36    VI(I)=-VI(I)/RI
      RETURN
37    CONTINUE
      DO 38 I=1,N
```

```
38    VI(I)=0.0D0
      RETURN
39    CONTINUE
      WRITE (3,41) ICOLX,(VI(I),I=1,N)
      RETURN
C
40    FORMAT (1H0,10X,18HEIGVEC - X,RR,RI ,3E18.8)
41    FORMAT (1H0,//20X,22HEIGVEC ERROR - ICOLX =,I10/(6X,10E12.4))
      END
      SUBROUTINE MSAVE (AA,BB,M)
C     SAVES A MATRIX IN ANOTHER.
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION AA(M,M), BB(M,M)
      DO 1 I=1,M
      DO 1 J=1,M
1     BB(I,J)=AA(I,J)
      RETURN
      END
      SUBROUTINE MTPSE (AAA,BBB,IX,JX)
C     TRANSPOSES A MATRIX AND STORES IN ANOTHER.
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION AAA(IX,JX), BBB(JX,IX)
      DO 2 I=1,IX
      DO 1 J=1,JX
1     BBB(J,I)=AAA(I,J)
2     CONTINUE
      RETURN
      END
      SUBROUTINE FINITE (RR,RI,DVI,DVR,LAL,LAM,P,P11,P12,P21,P22,PT11,PT
     112,PT21,PT22,ENDK,EPV,EMV,EM,STA,CON,DR,RB,A,NCOL,H,ZINT,BB,R,QQ,R
     2Q)
C     S/R FINITE COMPUTES EIGENVECTOR MATRIX, RICCATI SOLUTION, AND
C     OPTIMAL STATE AND CONTROL SOLUTIONS FOR FINITE HORIZON REGULATOR.
C     S/R FINITE AND ASSOCIATED ROUTINES ARE VARIABLE DIMENSIONED.
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION RR(M2), RI(M2), DVI(M2), DVR(M2), P(M2,M2), P11(M,M), P1
```

```
     12(M,M), P21(M,M), P22(M,M), PT11(M,M), PT12(M,M), PT22(M,M), PT21(
     2M,M), ENDK(M,M), EPV(M,M), EMV(M,M), EM(M,M), STA(KH,M), CON(KH,MK
     3), DR(M,M), RB(M,M), A(M2,M2), H(M2,M2), ZINT(M), BB(M,M), R(M,M),
     4 LAL(M2), LAM(M2), NCOL(M2)
      DIMENSION RQ(M,M), QQ(M,M)
      COMMON /PRMS/ M,M2,MK,M1,M3,INCRR,INCRC,IC1,IC3,IC4,IC5
      COMMON /TIME/ TT,TINC,TRIP
      COMMON /MODELS/ MODEL,NDS,KTIME,KH,NOH,NSIM
C     ACQUIRE RIGHT EIGENVECTORS(REAL OR COMPLEX).  STORE IN REAL P.
      CALL VECGET (A,H,P,DVR,DVI,NCOL,RR,RI)
C     PARTITION P MATRIX
      CALL PARTP (P,P11,P12,P21,P22)
C     CONSTRUCT AND PARTITION LEFT MODAL MATRIX
      CALL MSAVE (P,A,M2)
      CALL MINV (A,M2,DET,LAL,LAM)
      CALL PARTP (A,PT22,PT12,PT21,PT11)
C     CONSTRUCT CONSTANT CONTROL FEEDBACK COEFFICIENT RB.
      CALL FEEDCO (BB,QQ,R,DR,RB,RQ,LAL,LAM)
C     CONSTRUCT CONSTANT RICCATI COEFFICIENT ENDK
      CALL RICCON (P11,P12,P21,P22,R,BB,ENDK,LAL,LAM)
C     TEST ACCURACY OF EIGENVECTOR ROUTINE.  A IS THE DIAGONAL MATRIX OF
C     EIGENVALUES A=P(INVERSE)*H*P.
      CALL VMULT (A,H,DVR,M2)
      CALL VMULT (A,P,DVR,M2)
      CALL TEST (RR,RI,P,PT11,PT12,PT21,PT22,ENDK,RB,P11,P12,P21,P22,A,B
     1B,R)
C     BEGIN MULTIPLE HORIZON LOOP.
      DO 5 LTH=1,NOH
      T1=0.0D0
C     ACQUIRE INCREMENTAL TRANSITION PARTITIONS.
      STEP=TINC*TT
      CALL TRANEX (RR,RI,EMV,EPV,STEP)
      CALL MULTM (P11,EMV,BB,M,M,M,1)
      CALL MULTM (BB,PT12,DR,M,M,M,1)
      CALL MULTM (BB,PT22,EMV,M,M,M,1)
      CALL MULTM (P12,EPV,BB,M,M,M,1)
```

```
      CALL MULTM (BB,PT11,EPV,M,M,M,1)
      CALL MULTM (BB,PT21,EM,M,M,M,1)
      CALL MATADD (EMV,EM,EMV,M,M)
      CALL MATADD (DR,EPV,EPV,M,M)
C     BEGIN SPECIFIC HORIZON LOOP.
      DO 4 KTT=1,KH
      T2=TT-T1
C     CALCULATE MATRIX EPONENTIAL
      CALL TIMATS (RR,RI,EM,T2)
C     CONSTRUCT MATRIX EPONENTIAL RICCATI COEFFICIENT R
      CALL MULTM (EM,ENDK,BB,M,M,M,1)
      CALL MULTM (BB,EM,R,M,M,M,1)
C     ACQUIRE TIME/VARYING RICCATI SOLUTION
      CALL MULTM (P22,R,BB,M,M,M,1)
      CALL MULTM (P12,R,DR,M,M,M,1)
      CALL MATADD (P21,BB,BB,M,M)
      CALL MATADD (P11,DR,DR,M,M)
      IF (M.EQ.1) GO TO 1
      CALL MINV (DR,M,DET,LAL,LAM)
      GO TO 2
1     DR(1,1)=1.0D0/DR(1,1)
2     CONTINUE
      CALL MULTM (BB,DR,R,M,M,M,1)
C     CALCULATE POLICY COST
      IF (KTT.NE.1) GO TO 3
      CALL COSROG (ZINT,R,DVR,COST)
3     CONTINUE
C     ACQUIRE STATE AND CONTROL TRAJECTORIES
      CALL STACON (ZINT,EMV,EPV,R,EM,RB,DR,RQ,STA,CON,KTT)
C     CLOSE TIME LOOP
      T1=T1+TINC*TT
4     CONTINUE
C     PASS OUTPUT
      CALL OUTPUT (STA,CON,COST)
      TT=TT+TRIP
5     CONTINUE
```

```
      RETURN
      END
      SUBROUTINE TRANEX (RR,RI,EMV,EPV,T1)
C     S/R TRANEX CALCULATES EIGENVALUE MATRIX EXPONENTIAL IN FORM
C     SUITABLE FOR MODIFIED NEP.
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON /PRMS/ M,M2,MK,M1,M3,INCRR,INCRC,IC1,IC3,IC4,IC5
      DIMENSION RR(M2), RI(M2), EMV(M,M), EPV(M,M)
      CALL MATEQO (EPV,M)
      CALL MATEQO (EMV,M)
      IF (INCRC.EQ.0) GO TO 2
      DO 1 I=1,IC3,2
      IX=I+1
      DSR=RR(I)*T1
      ZP=DEXP(-DSR)
      DSR=DEXP(DSR)
      DSI=RI(I)*T1
      CN1=DCOS(DSI)
      SN1=DSIN(DSI)
      EPV(I,I)=ZP*CN1
      EPV(IX,IX)=EPV(I,I)
      EPV(I,IX)=ZP*SN1
      EPV(IX,I)=-EPV(I,IX)
      EMV(I,I)=DSR*CN1
      EMV(IX,IX)=EMV(I,I)
      EMV(I,IX)=-DSR*SN1
      EMV(IX,I)=-EMV(I,IX)
1     CONTINUE
2     CONTINUE
      DO 3 I=IC1,M
      DSR=RR(I)
      EPV(I,I)=DEXP(-DSR*T1)
3     EMV(I,I)=DEXP(DSR*T1)
      RETURN
      END
      SUBROUTINE TIMATS (RR,RI,EM,T2)
```

```
C     S/R TIMATS CALCULATES TIME-VARYING EXPONENTIAL MATRIX OF MODIFIED
C     RICCATI SOLUTION.
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON /PRMS/ M,M2,MK,M1,M3,INCRR,INCRC,IC1,IC3,IC4,IC5
      DIMENSION RR(M2), RI(M2), EM(M,M)
      CALL MATEQO (EM,M)
      IF (INCRC.EQ.0) GO TO 2
      DO 1 I=1,IC3,2
      IX=I+1
      ZMX=RR(I)*T2
      ZMX=DEXP(ZMX)
      DSX=RI(I)*T2
      CN2=DCOS(DSX)
      SN2=DSIN(DSX)
      EM(I,I)=ZMX*CN2
      EM(IX,IX)=EM(I,I)
      EM(I,IX)=-ZMX*SN2
      EM(IX,I)=-EM(I,IX)
1     CONTINUE
2     CONTINUE
      DO 3 I=IC1,M
      DSR=RR(I)
3     EM(I,I)=DEXP(DSR*T2)
      RETURN
      END
      SUBROUTINE COSROG (ZINT,R,DVR,COST)
C     S/R COSROG COMPUTES MINIMUM POLICY COST
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON /PRMS/ M,M2,MK,M1,M3,INCRR,INCRC,IC1,IC3,IC4,IC5
      DIMENSION ZINT(M), R(M,M), DVR(M)
      DO 2 I=1,M
      ZC=0.0D0
      DO 1 J=1,M
1     ZC=ZC+ZINT(J)*R(J,I)
2     DVR(I)=ZC
      ZC=0.0D0
```

```
      DO 3 I=1,M
3     ZC=ZC+DVR(I)*ZINT(I)
      COST=0.5D0*ZC
      RETURN
      END
      SUBROUTINE STACON (ZINT,EMV,EPV,R,EM,RB,DR,RQ,STA,CON,KTT)
C     S/R STACON CALCULATES STATE AND CONTROL TRAJECTORIES.
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON /PRMS/ M,M2,MK,M1,M3,INCRR,INCRC,IC1,IC3,IC4,IC5
      DIMENSION ZINT(M), EMV(M,M), EPV(M,M), R(M,M), EM(M,M), RB(MK,M),
     1DR(MK,M), RQ(MK,M), STA(KH,M), CON(KH,MK)
      COMMON /MODELS/ MODEL,NDS,KTIME,KH,NOH,NSIM
C     CALCULATE STATE SOLUTIONS.
      IF (KTT.NE.1) GO TO 2
      DO 1 I=1,M
1     STA(KTT,I)=ZINT(I)
      GO TO 5
2     CALL MULTM (EPV,R,EM,M,M,M,1)
      CALL MATADD (EMV,EM,EM,M,M)
      KT1=KTT-1
      DO 4 I=1,M
      ZC=0.0D0
      DO 3 J=1,M
3     ZC=ZC+EM(I,J)*STA(KT1,J)
4     STA(KTT,I)=ZC
C     CALCULATE CONTROL SOLUTIONS.
5     CALL MULTM (RB,R,DR,MK,M,M,1)
      CALL MATADD (RQ,DR,DR,MK,M)
      DO 7 I=1,MK
      ZC=0.0D0
      DO 6 J=1,M
6     ZC=ZC+DR(I,J)*STA(KTT,J)
7     CON(KTT,I)=ZC
      RETURN
      END
      SUBROUTINE PARTP (P,P11,P12,P21,P22)
```

```
C     S/R PARTP PARTITIONS RIGHT EIGENVECTOR MATRIX P.
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION P(M2,M2), P11(M,M), P12(M,M), P21(M,M), P22(M,M)
      COMMON /PRMS/ M,M2,MK,M1,M3,INCRR,INCRC,IC1,IC3,IC4,IC5
      DO 3 I=1,M
      DO 1 J=1,M
1     P11(I,J)=P(I,J)
      DO 2 J=M1,M2
      JX=J-M
2     P12(I,JX)=P(I,J)
3     CONTINUE
      DO 6 I=M1,M2
      IX=I-M
      DO 4 J=1,M
4     P21(IX,J)=P(I,J)
      DO 5 J=M1,M2
      JX=J-M
5     P22(IX,JX)=P(I,J)
6     CONTINUE
      RETURN
      END
      SUBROUTINE RICCON (P11,P12,P21,P22,R,BB,ENDK,LAL,LAM)
C     S/R RICCON CONSTRUCTS CONSTANT COMPONENT OF MOD RICCATI SOLUTION.
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON /PRMS/ M,M2,MK,M1,M3,INCRR,INCRC,IC1,IC3,IC4,IC5
      DIMENSION P11(M,M), P12(M,M), P21(M,M), P22(M,M), ENDK(M,M), BB(M,
     1M), R(M,M)
      DIMENSION LAL(M), LAM(M)
      CALL MULTM (ENDK,P12,BB,M,M,M,-1)
      CALL MULTM (ENDK,P11,R,M,M,M,-1)
      CALL MATADD (P22,BB,BB,M,M)
      CALL MATADD (P21,R,R,M,M)
      IF (M.EQ.1) GO TO 1
      CALL MINV (BB,M,DET,LAL,LAM)
      GO TO 2
1     BB(1,1)=1.0D0/BB(1,1)
```

```
2     CONTINUE
      CALL MULTM (BB,R,ENDK,M,M,M,-1)
      RETURN
      END
```

MODEL = 11 DATA SET = 1

A ARRAY

-0.50	0.0	0.28	0.0	0.0	0.0	0.0	0.0
3.81	3.64	-1.36	-1.36	0.0	-36.25	-26.66	7.62
2.80	2.80	-1.00	-1.00	0.0	-26.66	-19.60	5.60
-0.80	-0.80	1.00	0.0	0.0	7.62	5.60	-1.60
0.0	0.0	0.0	0.0	0.50	-3.81	-2.80	0.80
0.0	0.0	0.0	0.0	0.0	-3.64	-2.80	0.80
0.0	0.0	-1.00	0.0	-0.28	1.36	1.00	-1.00
0.0	0.0	0.0	0.0	0.0	1.36	1.00	0.0

TERMINAL WEIGHTING MATRIX

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

INITIAL STATE VECTOR

0.75000 0.25000 1.00000 0.50000

SYSTEM PARAMETERS

ALPHA	S	K	M	GAMMA	LAMBDA	MU	E	V	PHI	THETA
0.500	0.250	0.250	0.200	0.170	1.000	1.000	2.000	8.000	0.400	0.0

CHARACTERISTIC COEFFICIENTS - ASCENDING ORDER

0.01375 -0.79901 10.62623 -28.22004 1.00000 0.0 0.0 0.0 0.0

```
          ESTIMATED COEFFICIENTS

0.01375  -0.79901  10.62623 -28.22004   1.00000   0.0       0.0       0.0       0.0

          SUM OF REAL EIGENVALUES = 0.0

          ORDERED REAL-PART EIGENVALUES

-0.53535  -5.27630  -0.26298  -0.15783   0.53535   5.27630   0.26298   0.15783

          ORDERED COMPLEX-PART EIGENVALUES

0.0       0.0       0.0       0.0       0.0       0.0       0.0       0.0

   NUMBER OF REAL NEGATIVE ROOTS =   4     NUMBER OF COMPLEX NEGATIVE ROOTS =   0

   SSP-DPRQD ERROR MESSAGES     IR =   4     IER =   0
```

```
      FINAL ADJOINT COEFFICIENT MATRIX NULL
-0.00000   0.00000  -0.00000  -0.00000  -0.00000  -0.00000  -0.00000   0.00000
 0.00000   0.00000   0.00000  -0.00000   0.00000   0.00000   0.00000   0.00000
-0.00000  -0.00000   0.00000   0.00000   0.00000   0.00000   0.00000  -0.00000
 0.00000   0.00000  -0.00000  -0.00000  -0.00000  -0.00000  -0.00000   0.00000
-0.00000  -0.00000   0.00000   0.00000  -0.00000   0.00000   0.00000  -0.00000
-0.00000  -0.00000   0.00000   0.00000   0.00000   0.00000   0.00000  -0.00000
-0.00000  -0.00000  -0.00000   0.00000  -0.00000  -0.00000  -0.00000  -0.00000
 0.00000   0.00000  -0.00000  -0.00000  -0.00000  -0.00000  -0.00000  -0.00000
      EIGVEC - X,RR,RI ,    0.71689846D-09   -0.53534842D 00    0.0
      EIGVEC - X,RR,RI ,    0.72329475D-09   -0.52763033D 01    0.0
      EIGVEC - X,RR,RI ,    0.17455257D-08   -0.26298106D 00    0.0
      EIGVEC - X,RR,RI ,    0.80412285D-09   -0.15783030D 00    0.0
      EIGVEC - X,RR,RI ,    0.71689846D-09    0.53534842D 00    0.0
      EIGVEC - X,RR,RI ,    0.72329475D-09    0.52763033D 01    0.0
      EIGVEC - X,RR,RI ,    0.17455257D-08    0.26298106D 00    0.0
      EIGVEC - X,RR,RI ,    0.80412283D-09    0.15783030D 00    0.0
```

CONSTANT CONTROL FEEDBACK COEFFICIENT RB

0.0	-9.5200	-7.0000	2.0000

DIAGONAL MATRIX OF EIGENVALUES

-0.53535	-0.00000	0.00000	-0.00000	0.00000	0.00000	0.00000	-0.00000
0.00000	-5.27630	-0.00000	-0.00000	0.00000	-0.00000	-0.00000	0.00000
0.00000	-0.00000	-0.26298	-0.00000	0.00000	0.00000	0.00000	-0.00000
0.00000	-0.00000	-0.00000	-0.15783	0.00000	-0.00000	0.00000	0.00000
0.00000	0.00000	-0.00000	0.00000	0.53535	-0.00000	0.00000	0.00000
-0.00000	0.00000	0.00000	0.00000	0.00000	5.27630	0.00000	0.00000
-0.00000	-0.00000	0.00000	-0.00000	0.00000	0.00000	0.26298	-0.00000
0.00000	-0.00000	-0.00000	0.00000	-0.00000	-0.00000	-0.00000	0.15783

OUTPUT SET 1 TIME HORIZON = 10.00

STATE 1	2	3	4
0.75000	0.25000	1.00000	0.50000
0.73469	-0.30651	0.59013	0.65586
0.71143	-0.62524	0.34976	0.73731
0.68421	-0.80396	0.20931	0.77527
0.65536	-0.90032	0.12774	0.78797
0.62619	-0.94834	0.08084	0.78615
0.59747	-0.96808	0.05432	0.77613
0.56960	-0.97138	0.03976	0.76162
0.54279	-0.96518	0.03221	0.74480
0.51715	-0.95361	0.02872	0.72692
0.49269	-0.93906	0.02758	0.70874
0.46942	-0.92296	0.02778	0.69066
0.44730	-0.90614	0.02872	0.67293
0.42629	-0.88906	0.03006	0.65567
0.40635	-0.87202	0.03158	0.63894
0.38742	-0.85517	0.03318	0.62277
0.36946	-0.83860	0.03478	0.60717
0.35241	-0.82234	0.03635	0.59213
0.33624	-0.80643	0.03787	0.57764
0.32090	-0.79086	0.03934	0.56368
0.30635	-0.77564	0.04073	0.55023
0.29255	-0.76077	0.04207	0.53727
0.27945	-0.74623	0.04334	0.52478
0.26703	-0.73201	0.04456	0.51275
0.25524	-0.71811	0.04571	0.50116
0.24407	-0.70452	0.04682	0.49000
0.23346	-0.69122	0.04787	0.47923
0.22340	-0.67821	0.04888	0.46886
0.21386	-0.66547	0.04985	0.45885
0.20481	-0.65301	0.05077	0.44921
0.19623	-0.64080	0.05166	0.43992
0.18809	-0.62885	0.05252	0.43095
0.18037	-0.61714	0.05334	0.42231
0.17304	-0.60566	0.05414	0.41397

0.16610	-0.59442	0.05490	0.40594
0.15952	-0.58340	0.05565	0.39819
0.15327	-0.57259	0.05637	0.39071
0.14735	-0.56199	0.05707	0.38350
0.14174	-0.55159	0.05776	0.37655
0.13642	-0.54139	0.05843	0.36985
0.13138	-0.53139	0.05908	0.36338
0.12660	-0.52156	0.05972	0.35715
0.12208	-0.51192	0.06035	0.35114
0.11779	-0.50245	0.06097	0.34535
0.11372	-0.49316	0.06159	0.33976
0.10988	-0.48403	0.06219	0.33438
0.10623	-0.47506	0.06279	0.32919
0.10278	-0.46625	0.06339	0.32419
0.09952	-0.45759	0.06398	0.31938
0.09643	-0.44908	0.06457	0.31474
0.09350	-0.44072	0.06516	0.31028
0.09074	-0.43249	0.06575	0.30598
0.08812	-0.42441	0.06634	0.30185
0.08565	-0.41646	0.06693	0.29787
0.08332	-0.40864	0.06752	0.29405
0.08112	-0.40094	0.06812	0.29038
0.07904	-0.39338	0.06872	0.28685
0.07708	-0.38593	0.06932	0.28346
0.07523	-0.37861	0.06993	0.28021
0.07349	-0.37140	0.07054	0.27709
0.07185	-0.36431	0.07116	0.27410
0.07030	-0.35732	0.07179	0.27123
0.06885	-0.35045	0.07242	0.26849
0.06749	-0.34368	0.07306	0.26587
0.06621	-0.33702	0.07370	0.26337
0.06501	-0.33046	0.07435	0.26098
0.06389	-0.32400	0.07501	0.25870
0.06284	-0.31764	0.07568	0.25653
0.06186	-0.31137	0.07635	0.25447
0.06095	-0.30521	0.07704	0.25251

0.06010	-0.29913	0.07773	0.25065
0.05931	-0.29315	0.07842	0.24889
0.05858	-0.28725	0.07913	0.24723
0.05790	-0.28145	0.07984	0.24566
0.05728	-0.27573	0.08056	0.24419
0.05671	-0.27010	0.08129	0.24281
0.05618	-0.26456	0.08202	0.24151
0.05570	-0.25910	0.08277	0.24030
0.05527	-0.25373	0.08351	0.23918
0.05487	-0.24844	0.08426	0.23814
0.05452	-0.24324	0.08502	0.23718
0.05420	-0.23812	0.08577	0.23630
0.05392	-0.23309	0.08653	0.23549
0.05368	-0.22815	0.08727	0.23477
0.05346	-0.22332	0.08801	0.23412
0.05328	-0.21860	0.08871	0.23355
0.05312	-0.21403	0.08937	0.23306
0.05299	-0.20964	0.08995	0.23265
0.05288	-0.20552	0.09038	0.23232
0.05279	-0.20179	0.09058	0.23209
0.05269	-0.19867	0.09037	0.23199
0.05259	-0.19653	0.08948	0.23204
0.05246	-0.19598	0.08743	0.23232
0.05225	-0.19809	0.08342	0.23294
0.05190	-0.20464	0.07609	0.23409
0.05129	-0.21866	0.06314	0.23609
0.05024	-0.24530	0.04068	0.23950
0.04842	-0.29324	0.00209	0.24525
0.04528	-0.37724	-0.06382	0.25495
0.03991	-0.52230	-0.17602	0.27130
0.03072	-0.77080	-0.36668	0.29888

BERGSTROM TRADE BALANCE

0.27060	0.13289	0.05236	0.00552	-0.02146	-0.03674	-0.04516	-0.04954	-0.05157	-0.05222
-0.05209	-0.05151	-0.05069	-0.04974	-0.04874	-0.04772	-0.04670	-0.04571	-0.04473	-0.04379
-0.04288	-0.04200	-0.04115	-0.04033	-0.03954	-0.03878	-0.03805	-0.03734	-0.03666	-0.03601
-0.03538	-0.03477	-0.03418	-0.03362	-0.03308	-0.03255	-0.03205	-0.03157	-0.03110	-0.03065
-0.03022	-0.02981	-0.02941	-0.02903	-0.02866	-0.02831	-0.02797	-0.02765	-0.02734	-0.02705
-0.02676	-0.02649	-0.02623	-0.02599	-0.02575	-0.02553	-0.02532	-0.02511	-0.02492	-0.02474
-0.02457	-0.02441	-0.02426	-0.02412	-0.02398	-0.02386	-0.02374	-0.02364	-0.02354	-0.02345
-0.02337	-0.02329	-0.02323	-0.02317	-0.02311	-0.02307	-0.02303	-0.02300	-0.02298	-0.02296
-0.02294	-0.02294	-0.02293	-0.02294	-0.02294	-0.02295	-0.02296	-0.02295	-0.02294	-0.02290
-0.02281	-0.02264	-0.02233	-0.02176	-0.02076	-0.01896	-0.01550	-0.00788	0.01127	0.05741
0.14801									

CONTROL VARIABLES

CONTROL VARIABLE = 1

-2.35298	-1.09265	-0.34798	0.09213	0.35224	0.50587	0.59641	0.64948	0.68022	0.69757
0.70681	0.71110	0.71229	0.71149	0.70938	0.70635	0.70265	0.69846	0.69385	0.68892
0.68370	0.67822	0.67253	0.66665	0.66059	0.65437	0.64801	0.64152	0.63493	0.62824
0.62146	0.61460	0.60769	0.60072	0.59370	0.58665	0.57957	0.57247	0.56536	0.55824
0.55112	0.54401	0.53690	0.52982	0.52275	0.51571	0.50870	0.50173	0.49479	0.48788
0.48103	0.47421	0.46745	0.46074	0.45407	0.44747	0.44092	0.43443	0.42799	0.42162
0.41531	0.40907	0.40289	0.39677	0.39072	0.38474	0.37883	0.37298	0.36721	0.36150
0.35586	0.35030	0.34480	0.33938	0.33403	0.32875	0.32354	0.31840	0.31334	0.30835
0.30344	0.29860	0.29384	0.28916	0.28457	0.28007	0.27568	0.27142	0.26735	0.26351
0.26005	0.25715	0.25515	0.25464	0.25654	0.26216	0.27256	0.28424	0.27560	0.19532
0.00000									

POLICY COST SCALAR = 0.61 MODEL = 11 DATA SET = 1

CONTROL TYPE = 0 TIME HORIZON = 10.00 TIME INCREMENT = 0.01 NUMBER OF HORIZONS = 1

```
      SUBROUTINE ATEST (RB,R,BB,P21,RR,RI)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION RR(M2), RI(M2)
      DIMENSION RB(MK,M), R(M,M), BB(M,M), P21(MK,M)
      COMMON /PRMS/ M,M2,MK,M1,M3,INCRR,INCRC,IC1,IC3,IC4,IC5
      WRITE (3,1)
      WRITE (3,2) ((RB(I,J),J=1,M),I=1,MK)
      WRITE (3,3)
      WRITE (3,2) ((BB(I,J),J=1,4),I=1,4)
      WRITE (3,4)
      WRITE (3,2) ((P21(I,J),J=1,M),I=1,MK)
      RETURN
C
1     FORMAT ('1',10X,'ASYMPTOTIC CONTROL COEFFICIENT RB')
2     FORMAT ('0',4F10.4)
3     FORMAT ('0',10X,'ASYMPTOTIC RICCATI SOLUTION P21P-11')
4     FORMAT ('0',10X,'ASYMPTOTIC FEEDBACK RBP21P-11')
      END
      SUBROUTINE PASS2 (H,RR,RI,ENDK,ZINT,R,BB,QQ)
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON /PRMS/ M,M2,MK,M1,M3,INCRR,INCRC,IC1,IC3,IC4,IC5
      DIMENSION H(M2,M2), RR(M2), RI(M2), ENDK(M,M), ZINT(M), R(MK,MK),
     1BB(M,MK), QQ(M,MK)
      DIMENSION ASTA(101,4), ACON(101,2), RQ(4,4)
      DIMENSION DVI(8), DVR(8), P(8,4), P11(4,4), P21(4,4), A(8,8), RB(4
     1,4), DR(4,4), EM(4,4), LAL(8), LAM(8), NCOL(8)
      CALL ATOTIC (H,RR,RI,ENDK,ZINT,R,BB,ASTA,ACON,DVI,DVR,P,P11,P21,A,
     1RB,DR,EM,LAL,LAM,NCOL,QQ,RQ)
      RETURN
      END
      SUBROUTINE ATOTIC (H,RR,RI,ENDK,ZINT,R,BB,ASTA,ACON,DVI,DVR,P,P11,
     1P21,A,RB,DR,EM,LAL,LAM,NCOL,QQ,RQ)
C     S/R ATOTIC COMPUTES ASYMPTOTIC EIGENVECTOR, RICCATI, STATE AND
C     CONTROL SOLUTIONS.  ATOTIC AND ASSOCIATED ROUTINES ARE
C     VARIABLE DIMENSIONED.
      IMPLICIT REAL*8(A-H,O-Z)
```

```
      DIMENSION H(M2,M2), RR(M2), RI(M2), ENDK(M,M), ZINT(M), R(MK,MK),
     1BB(M,MK)
      DIMENSION ASTA(KH,M), ACON(KH,MK), DVI(M2), DVR(M2), P(M2,M), P11(
     1M,M), P21(M,M), A(M2,M2), RB(M,M), DR(M,M), EM(M,M), LAL(M2), LAM(
     2M2), NCOL(M2), RQ(M,M), QQ(M,M)
      COMMON /PRMS/ M,M2,MK,M1,M3,INCRR,INCRC,IC1,IC3,IC4,IC5
      COMMON /MODELS/ MODEL,NDS,KTIME,KH,NOH,NSIM
      COMMON /TIME/ TT,TINC,TRIP
C     ACQUIRE STABLE RIGHT EIGENVECTORS(REAL OR COMPLEX)
      CALL VECGET (A,H,P,DVR,DVI,NCOL,RR,RI)
C     PARTITION P MATRIX
      CALL PARASP (P,P11,P21)
C     CONSTRUCT ASYMPTOTIC CONTROL FEEDBACK MATRIX
      CALL FEEDCO (BB,QQ,R,DR,RB,RQ,LAL,LAM)
      CALL MSAVE (P11,R,M)
      IF (M.EQ.1) GO TO 1
      CALL MINV (R,M,DET,LAL,LAM)
      GO TO 2
1     R(1,1)=1.0D0/R(1,1)
2     CONTINUE
      CALL MULTM (P21,R,BB,M,M,M,1)
      CALL MULTM (RB,BB,P21,MK,M,M,1)
      CALL MATADD (RQ,P21,P21,MK,M)
      CALL ATEST (RB,R,BB,P21,RR,RI)
C     CALCULATE ASYMPTOTIC COST
      CALL COST (ZINT,BB,DVR,ACOST)
C     BEGIN TIME LOOP
      DO 4 LTH=1,NOH
      T1=0.0D0
      DO 3 KTT=1,KH
C     CALCULATE TIMEVARYING MATRIX EXPONENTIAL
      CALL EXMAT (RR,RI,EM,T1)
C     ACQUIRE STATE SOLUTION
      CALL STATE (P11,EM,R,DR,ZINT,ASTA,KTT)
C     ACQUIRE CONTROL SOLUTION
      CALL STEER (ASTA,P21,ACON,KTT)
```

```
C     CLOSE TIME LOOP
      T1=T1+TINC*TT
3     CONTINUE
C     PASS OUTPUT
      CALL OUTPUT (ASTA,ACON,ACOST)
      TT=TT+TRIP
4     CONTINUE
      RETURN
      END
      SUBROUTINE PARASP (P,P11,P21)
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON /PRMS/ M,M2,MK,M1,M3,INCRR,INCRC,IC1,IC3,IC4,IC5
      DIMENSION P(M2,M), P11(M,M), P21(M,M)
      DO 2 I=1,M
      IX=I+M
      DO 1 J=1,M
      P11(I,J)=P(I,J)
1     P21(I,J)=P(IX,J)
2     CONTINUE
      RETURN
      END
      SUBROUTINE COST (ZINT,BB,DVR,ACOST)
C     S/R COST CALCULATES MINIMUM ASYMPTOTIC POLICY COST.
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON /PRMS/ M,M2,MK,M1,M3,INCRR,INCRC,IC1,IC3,IC4,IC5
      DIMENSION ZINT(M), BB(M,M), DVR(M)
      DO 2 I=1,M
      ZC=0.0D0
      DO 1 J=1,M
1     ZC=ZC+ZINT(J)*BB(J,I)
2     DVR(I)=ZC
      ZC=0.0D0
      DO 3 I=1,M
3     ZC=ZC+DVR(I)*ZINT(I)
      ACOST=0.5D0*ZC
      RETURN
```

```
      END
      SUBROUTINE EXMAT (RR,RI,EM,T1)
C     S/R EXMAT COMPUTES ASYMPTOTIC STATE TRANSITION MATRIX.
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON /PRMS/ M,M2,MK,M1,M3,INCRR,INCRC,IC1,IC3,IC4,IC5
      DIMENSION RR(M), RI(M), EM(M,M)
      CALL MATEQO (EM,M)
      IF (INCRC.EQ.0) GO TO 2
      DO 1 I=1,IC3,2
      IX=I+1
      DSR=RR(I)
      DSR=DEXP(DSR*T1)
      DSI=RI(I)*T1
      CN=DCOS(DSI)
      SN=DSIN(DSI)
      EM(I,I)=DSR*CN
      EM(I,IX)=-DSR*SN
      EM(IX,I)=-EM(I,IX)
1     EM(IX,IX)=EM(I,I)
2     CONTINUE
      DO 3 I=IC1,M
      DSR=RR(I)*T1
3     EM(I,I)=DEXP(DSR)
      RETURN
      END
      SUBROUTINE STATE (P11,EM,R,DR,ZINT,ASTA,KTT)
C     S/R STATE CALCULATES ASYMPTOTIC STATE SOLUTIONS.
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON /PRMS/ M,M2,MK,M1,M3,INCRR,INCRC,IC1,IC3,IC4,IC5
      COMMON /MODELS/ MODEL,NDS,KTIME,KH,NOH,NSIM
      DIMENSION P11(M,M), EM(M,M), R(M,M), ZINT(M), ASTA(KH,M), DR(M,M)
      CALL MULTM (P11,EM,DR,M,M,M,1)
      CALL MULTM (DR,R,EM,M,M,M,1)
      DO 2 I=1,M
      ZC=0.0D0
      DO 1 J=1,M
```

```
1     ZC=ZC+EM(I,J)*ZINT(J)
2     ASTA(KTT,I)=ZC
      RETURN
      END
      SUBROUTINE STEER (ASTA,P21,ACON,KTT)
C     S/R STEER CALCULATES ASYMPTOTIC CONTROL SOLUTIONS.
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON /PRMS/ M,M2,MK,M1,M3,INCRR,INCRC,IC1,IC3,IC4,IC5
      DIMENSION ASTA(101,M), P21(MK,M), ACON(101,MK)
      DO 2 I=1,MK
      ZC=0.0D0
      DO 1 J=1,M
1     ZC=ZC+P21(I,J)*ASTA(KTT,J)
2     ACON(KTT,I)=ZC
      RETURN
      END
```

```
FINAL ADJOINT COEFFICIENT MATRIX NULL
-0.00000   0.00000  -0.00000  -0.00000  -0.00000  -0.00000  -0.00000   0.00000
 0.00000   0.00000   0.00000  -0.00000   0.00000   0.00000   0.00000   0.00000
-0.00000  -0.00000   0.00000   0.00000   0.00000   0.00000   0.00000  -0.00000
 0.00000   0.00000  -0.00000  -0.00000  -0.00000  -0.00000  -0.00000   0.00000
-0.00000  -0.00000   0.00000   0.00000  -0.00000   0.00000   0.00000  -0.00000
-0.00000  -0.00000   0.00000   0.00000   0.00000   0.00000   0.00000  -0.00000
-0.00000  -0.00000  -0.00000   0.00000  -0.00000  -0.00000  -0.00000  -0.00000
 0.00000   0.00000  -0.00000  -0.00000  -0.00000  -0.00000  -0.00000  -0.00000
        EIGVEC - X,RR,RI ,    0.71689846D-09   -0.53534842D 00    0.0
        EIGVEC - X,RR,RI ,    0.72329475D-09   -0.52763033D 01    0.C
        EIGVEC - X,RR,RI ,    0.17455257D-08   -0.26298106D 00    0.0
        EIGVEC - X,RR,RI ,    0.80412285D-09   -0.15783030D 00    0.0
```

ASYMPTOTIC CONTROL COEFFICIENT RB

0.0	-9.5200	-7.0000	2.0000

ASYMPTOTIC RICCATI SOLUTION P21P-11

0.3758	0.5845	-0.6665	-0.1736
0.5845	1.1679	-1.5214	-0.3086
-0.6665	-1.5214	2.3778	0.3839
-0.1736	-0.3086	0.3839	0.0863

ASYMPTOTIC FEEDBACK RBP21P-11

-1.2459	-1.0854	-1.3925	0.4227

OUTPUT SET 1 TIME HORIZON = 10.00

STATE 1	2	3	4
0.75000	0.25000	1.00000	0.50000
0.73457	-0.31690	0.58242	0.65774
0.71104	-0.64176	0.33737	0.73972
0.68346	-0.82409	0.19403	0.77745
0.65419	-0.92260	0.11062	0.78946
0.62460	-0.97190	0.06249	0.78670
0.59543	-0.99242	0.03510	0.77558
0.56713	-0.99619	0.01988	0.75989
0.53989	-0.99030	0.01179	0.74185
0.51382	-0.97894	0.00783	0.72274
0.48895	-0.96455	0.00627	0.70330
0.46527	-0.94858	0.00605	0.68398
0.44275	-0.93186	0.00659	0.66501
0.42135	-0.91490	0.00752	0.64651
0.40102	-0.89797	0.00864	0.62855
0.38172	-0.88123	0.00982	0.61116
0.36339	-0.86478	0.01101	0.59435
0.34598	-0.84865	0.01216	0.57810
0.32946	-0.83287	0.01325	0.56240
0.31377	-0.81745	0.01428	0.54724
0.29887	-0.80238	0.01524	0.53260
0.28472	-0.78766	0.01612	0.51845
0.27129	-0.77329	0.01693	0.50478
0.25854	-0.75925	0.01768	0.49157
0.24642	-0.74553	0.01836	0.47881
0.23492	-0.73212	0.01899	0.46646
0.22399	-0.71902	0.01955	0.45452
0.21361	-0.70622	0.02006	0.44297
0.20375	-0.69369	0.02052	0.43179
0.19438	-0.68145	0.02092	0.42097
0.18548	-0.66947	0.02129	0.41050
0.17702	-0.65775	0.02161	0.40036
0.16898	-0.64629	0.02189	0.39053
0.16135	-0.63507	0.02213	0.38101

0.15409	-0.62408	0.02234	0.37179
0.14719	-0.61332	0.02252	0.36285
0.14063	-0.60279	0.02266	0.35418
0.13439	-0.59248	0.02278	0.34577
0.12847	-0.58237	0.02286	0.33761
0.12283	-0.57247	0.02293	0.32970
0.11747	-0.56277	0.02297	0.32202
0.11237	-0.55326	0.02298	0.31456
0.10752	-0.54394	0.02298	0.30732
0.10291	-0.53480	0.02296	0.30029
0.09852	-0.52584	0.02291	0.29345
0.09434	-0.51706	0.02286	0.28682
0.09036	-0.50844	0.02278	0.28037
0.08658	-0.49999	0.02270	0.27409
0.08298	-0.49169	0.02259	0.26800
0.07955	-0.48356	0.02248	0.26207
0.07629	-0.47558	0.02236	0.25630
0.07318	-0.46774	0.02222	0.25069
0.07022	-0.46005	0.02208	0.24523
0.06739	-0.45250	0.02192	0.23991
0.06471	-0.44510	0.02176	0.23474
0.06215	-0.43782	0.02159	0.22970
0.05971	-0.43068	0.02142	0.22480
0.05738	-0.42366	0.02123	0.22002
0.05516	-0.41678	0.02105	0.21536
0.05304	-0.41001	0.02085	0.21082
0.05103	-0.40337	0.02066	0.20640
0.04910	-0.39684	0.02045	0.20209
0.04727	-0.39043	0.02025	0.19789
0.04551	-0.38413	0.02004	0.19380
0.04384	-0.37794	0.01983	0.18980
0.04224	-0.37185	0.01962	0.18590
0.04072	-0.36588	0.01940	0.18210
0.03926	-0.36000	0.01918	0.17839
0.03787	-0.35423	0.01896	0.17477
0.03654	-0.34856	0.01874	0.17124

0.03527	-0.34298	0.01852	0.16779
0.03405	-0.33750	0.01830	0.16442
0.03289	-0.33211	0.01808	0.16113
0.03178	-0.32681	0.01785	0.15792
0.03072	-0.32160	0.01763	0.15479
0.02970	-0.31648	0.01741	0.15172
0.02873	-0.31145	0.01719	0.14873
0.02779	-0.30649	0.01697	0.14581
0.02690	-0.30163	0.01675	0.14295
0.02604	-0.29684	0.01653	0.14015
0.02522	-0.29213	0.01631	0.13742
0.02444	-0.28750	0.01609	0.13476
0.02369	-0.28294	0.01587	0.13215
0.02296	-0.27847	0.01566	0.12959
0.02227	-0.27406	0.01544	0.12710
0.02160	-0.26973	0.01523	0.12466
0.02097	-0.26546	0.01502	0.12227
0.02035	-0.26127	0.01481	0.11994
0.01976	-0.25715	0.01460	0.11765
0.01920	-0.25309	0.01440	0.11542
0.01865	-0.24910	0.01419	0.11323
0.01813	-0.24517	0.01399	0.11109
0.01763	-0.24131	0.01379	0.10900
0.01714	-0.23751	0.01359	0.10695
0.01668	-0.23377	0.01340	0.10494
0.01623	-0.23009	0.01320	0.10297
0.01580	-0.22647	0.01301	0.10105
0.01538	-0.22291	0.01282	0.09917
0.01498	-0.21941	0.01263	0.09732
0.01459	-0.21596	0.01245	0.09552
0.01422	-0.21257	0.01226	0.09375

BERGSTROM TRADE BALANCE

0.27739	0.13732	0.05543	0.00785	-0.01952	-0.03499	-0.04346	-0.04783	-0.04979	-0.05037
-0.05014	-0.04946	-0.04853	-0.04747	-0.04635	-0.04521	-0.04408	-0.04296	-0.04188	-0.04082
-0.03979	-0.03879	-0.03782	-0.03689	-0.03598	-0.03510	-0.03424	-0.03342	-0.03262	-0.03184
-0.03109	-0.03036	-0.02965	-0.02896	-0.02829	-0.02764	-0.02701	-0.02640	-0.02580	-0.02522
-0.02466	-0.02411	-0.02358	-0.02307	-0.02256	-0.02207	-0.02160	-0.02113	-0.02068	-0.02024
-0.01981	-0.01939	-0.01899	-0.01859	-0.01820	-0.01783	-0.01746	-0.01710	-0.01675	-0.01641
-0.01608	-0.01575	-0.01543	-0.01512	-0.01482	-0.01453	-0.01424	-0.01396	-0.01368	-0.01341
-0.01315	-0.01289	-0.01264	-0.01240	-0.01216	-0.01192	-0.01169	-0.01147	-0.01125	-0.01103
-0.01082	-0.01062	-0.01042	-0.01022	-0.01003	-0.00984	-0.00966	-0.00948	-0.00930	-0.00913
-0.00896	-0.00879	-0.00863	-0.00847	-0.00831	-0.00816	-0.00801	-0.00786	-0.00772	-0.00758
-0.00744									

ONTROL VARIABLES

CONTROL VARIABLE = 1

-2.38696	-1.10427	-0.34645	0.10137	0.36599	0.52224	0.61428	0.66819	0.69939	0.71696
0.72630	0.73059	0.73174	0.73088	0.72869	0.72557	0.72178	0.71749	0.71280	0.70777
0.70245	0.69689	0.69111	0.68514	0.67899	0.67270	0.66626	0.65970	0.65303	0.64627
0.63943	0.63251	0.62553	0.61851	0.61144	0.60434	0.59721	0.59007	0.58291	0.57576
0.56861	0.56146	0.55433	0.54722	0.54014	0.53308	0.52605	0.51907	0.51212	0.50521
0.49835	0.49153	0.48477	0.47806	0.47140	0.46481	0.45827	0.45179	0.44537	0.43901
0.43272	0.42649	0.42033	0.41423	0.40821	0.40224	0.39635	0.39052	0.38477	0.37908
0.37346	0.36791	0.36243	0.35701	0.35167	0.34639	0.34118	0.33604	0.33097	0.32597
0.32103	0.31616	0.31135	0.30661	0.30194	0.29733	0.29278	0.28830	0.28388	0.27953
0.27523	0.27100	0.26683	0.26271	0.25866	0.25467	0.25073	0.24685	0.24303	0.23927
0.23556									

POLICY COST SCALAR = 0.66 MODEL = 11 DATA SET = 1

ONTROL TYPE = 1 TIME HORIZON = 10.00 TIME INCREMENT = 0.01 NUMBER OF HORIZONS = 1

APPENDIX III

PROOF OF THEOREM 3.4

The proof of theorem 3.4 follows a proof given by Ogata [pp.385-7], simplified by the assumption that the structural matrix A has n distinct eigenvalues. For supplementary material, see also Kalman, Ho & Narendra, Athans & Falb [pp.200-20], and Zadeh & Desoer [pp.495-514].

It is necessary to demonstrate that a solution $u(\tau)$ exists satisfying equation (3.45): i.e.

$$x(0) = - \int_0^T e^{-A\tau} Bu(\tau)\, d\tau. \tag{1}$$

Now

$$e^{-A\tau} = \sum_{i=0}^{n-1} \alpha_i(\tau) A^i, \tag{2}$$

where

$$\begin{bmatrix} \alpha_0(\tau) \\ \vdots \\ \alpha_{n-1}(\tau) \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ \vdots & & & & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} e^{\lambda_1 \tau} \\ \vdots \\ e^{\lambda_n \tau} \end{bmatrix}. \tag{3}$$

The reasoning leading to (2) encompasses two concepts: (i) that a convergent infinite matrix series (such as the matrix exponential) of an nxn matrix A with distinct eigenvalues can be expressed as a polynomial in A of degree n-1; and (ii) that this polynomial can be given a unique representation by means of the Sylvester-Lagrange interpolation formula. See Ogata [pp.257-60, 317] and Zadeh & Desoer [pp.607-9].

The result (2) is used to convert (1) into a set of algebraic equations, to which theorem 3.1 is applied to define necessary and sufficient conditions for existence. Thus

$$x(0) = - \sum_{i=o}^{n-1} \int_0^T \alpha_i(\tau) A^i Bu(\tau) d\tau. \tag{4}$$

But

$$u(\tau) = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_k \end{bmatrix} = \sum_{j=1}^{k} u_j(\tau)\varepsilon_j, \tag{5}$$

where ε_j is the jth column of the unit matrix.

Therefore (4) becomes

$$x(0) = - \sum_{i=o}^{n-1} \sum_{j=1}^{k} \{\alpha_i(\tau)u_j(\tau)d\tau\} A^i B_j, \tag{6}$$

where $B_j = B\varepsilon_j$ is the j^{th} column of B.

To simplify the algebraic expression (6), define the scalar time functions

$$\beta_{ij} = - \int_0^T \alpha_i(\tau)u_j(\tau)d\tau. \tag{7}$$

$$\therefore\ x(0) = \sum_{i=o}^{n-1} A^i \{\sum_{j=1}^{k} \beta_{ij}B_j\}. \tag{8}$$

The bracketed expression in (8) is

$$\sum_{j=1}^{k} \beta_{ij}B_j = [B_1\ B_2\ \ldots\ B_k] \begin{bmatrix} \beta_{i1} \\ \beta_{i2} \\ \vdots \\ \beta_{ik} \end{bmatrix} \equiv B.\beta_i. \tag{9}$$

$$\therefore\ x(0) = \sum_{i=o}^{n-1} A^i B\beta_i = B\beta_o + AB\beta_1 + \ldots + A^{n-1}B\beta_{n-1}$$

$$= [B\ AB\ \ldots\ A^{n-1}B] \begin{bmatrix} \beta_o \\ \vdots \\ \beta_{n-1} \end{bmatrix} . \tag{10}$$

Or

$$Q\beta = x(0), \qquad (nxnk.nkx1 = nx1), \tag{11}$$

where Q is the controllability matrix (3.46), and β is the nkx1 vector on the right of (10).

Hence a transfer between $x(0)\neq 0$ and $x(T)=0$ is possible iff the linear algebraic system (11) possesses a solution vector β. And from theorem 3.1, a necessary and sufficient condition for this is that Q possess row rank (n), as stated in (3.47).

APPENDIX IV

PROOF OF THEOREM 4.4

Commencing from equation (4.50), postulate the smooth controller

$$u(t) = F^T(t)\gamma, \qquad (kxn, nx1), \tag{1}$$

such that

$$P^{-1}x(0) = -\{\int_0^T F(t)F^T(t)dt\}\gamma, \tag{2}$$

where

$$F(t) \equiv \Psi^{-1}S(-t)\hat{B}, \qquad \hat{B} \equiv P^{-1}B. \tag{3}$$

Then necessary and sufficient conditions for which the integral term in (2) is nonsingular are necessary and sufficient conditions for dynamic controllability of $\dot{x} = Ax + Bu$.

Now

$$F(t)F^T(t) = \begin{bmatrix} F_{n_1} \\ \hline \vdots \\ \hline F_{n_s} \end{bmatrix} \left[F_{n_1}^T \,\vdots\, \cdots \,\vdots\, F_{n_s}^T \right] = \begin{bmatrix} F_{n_1}F_{n_1}^T & \cdots & F_{n_1}F_{n_s}^T \\ \vdots & & \vdots \\ F_{n_s}F_{n_1}^T & \cdots & F_{n_s}F_{n_s}^T \end{bmatrix}$$

$$n = \sum_{i=1}^{s} n_i, \quad s \leq n, \tag{4}$$

where from (3) and (4.48), (4.51):

$$F_{n_i} = e^{-\lambda_i t} S^{-1}_{n_i} \hat{B}_{n_i} = e^{-\lambda_i t} \left[\begin{array}{c} S^{-1}_{n_i \cdot 1} x \hat{B}_{n_i \cdot 1} \\ \hline \vdots \\ \hline S^{-1}_{n_i \cdot r_i} x \hat{B}_{n_i \cdot r_i} \end{array} \right] \begin{array}{l} n_i \cdot 1 \text{ rows} \\ \\ \\ n_i \cdot r_i \text{ rows} \end{array}$$

$$i = 1, \ldots, s. \qquad\qquad k \text{ columns} \tag{5}$$

Consider a particular case of F_{n_i} in (5), corresponding to the group of Jordan blocks:

$$J_{n_i} = \left[\begin{array}{cc|cc|c} \lambda_i & 1 & & & 0 \\ 0 & \lambda_i & & & \\ \hline & & \lambda_i & 1 & \\ & & 0 & \lambda_i & \\ \hline 0 & & & & \lambda_i \end{array} \right] = \left[\begin{array}{c|c|c} J_1 & & 0 \\ \hline & J_2 & \\ \hline 0 & & J_3 \end{array} \right], \tag{6}$$

where

$$\begin{cases} n_i = 5, & r_i = 3, \quad p_i = 2, \\ n_i \cdot 1 = 2, & n_i \cdot 2 = 2, \quad n_i \cdot 3 = 1. \end{cases} \tag{7}$$

Then for k = 2, (5) becomes

$$F_{n_i=5} = e^{-\lambda_i t} \left[\begin{array}{c} \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \hat{b}_{11} & \hat{b}_{12} \\ \hat{b}_{21} & \hat{b}_{22} \end{pmatrix} \\ \hdashline \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \hat{b}_{31} & \hat{b}_{32} \\ \hat{b}_{41} & \hat{b}_{42} \end{pmatrix} \\ \hdashline \begin{pmatrix} 1 \end{pmatrix} \quad \begin{pmatrix} \hat{b}_{51} & \hat{b}_{52} \end{pmatrix} \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}\right\} \text{2 rows} \\ \left.\vphantom{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}\right\} \text{2 rows} \\ \quad \text{1 row} \end{array} \tag{8}$$

2 columns

Hence,

$$F_5 F_5^T = e^{-2\lambda_i t} \left[\begin{array}{cc} \delta_{11}(t) & \delta_{12}(t) \\ \hat{b}_{21} & \hat{b}_{22} \\ \hdashline \delta_{31}(t) & \delta_{32}(t) \\ \hat{b}_{41} & \hat{b}_{42} \\ \hdashline \hat{b}_{51} & \hat{b}_{52} \end{array}\right] \left[\begin{array}{cc:cc:c} \delta_{11}(t) & \hat{b}_{21} & \delta_{31}(t) & \hat{b}_{41} & \hat{b}_{51} \\ \delta_{12}(t) & \hat{b}_{22} & \delta_{32}(t) & \hat{b}_{42} & \hat{b}_{52} \end{array}\right] \tag{9}$$

$$\delta_{ij}(t) = \hat{b}_{ij} - t\hat{b}_{(i+1)j}.$$

Consider the r_ixk matrix

$$\bar{B}_i = \left[\begin{array}{cc} \hat{b}_{21} & \hat{b}_{22} \\ \hdashline \hat{b}_{41} & \hat{b}_{42} \\ \hdashline \hat{b}_{51} & \hat{b}_{52} \end{array}\right]_{3\text{x}2}, \tag{10}$$

formed from the last rows of each partition of $e^{+\lambda_i t}F_{n_i}$ in (5), for the particular case (8). Since $\hat{B} = P^{-1}B$, and $\rho(B) = k$, r_i-k rows of $\bar{B}_i$ can be written as linear combinations of the remaining k rows. But $\bar{B}_i$ occurs in the product $F_{n_i}F_{n_i}^T$, as in (9), which must therefore contain r_i-k linearly dependent rows, unless

$$\rho(\bar{B}_i) = r_i. \tag{11}$$

Condition (11) is therefore necessary for nonsingularity of the product (4) and (9). It is also sufficient: if, in (9), the constant elements of $\delta_{ij}(t)$ vanish, satisfaction of (11) ensures that rows containing the terms $\delta_{ij}(t)$ are linearly independent.

Regarding (4), if (11) is not satisfied for some i, the $n_i$xn matrix

$$\left[F_{n_i} F_{n_1}^T \;\vdots\; \dots \;\vdots\; F_{n_i} F_{n_s}^T \right], \qquad n = \sum_{i=1}^{s} n_i, \tag{12}$$

must contain linearly dependent rows. Hence

$$\rho(\bar{B}_i) = r_i, \quad i = 1, \dots, s, \quad s \leqslant n, \tag{13}$$

are necessary and sufficient conditions for nonsingularity of $\int_0^T F(t)F^T(t)dt$ in (2), and therefore for dynamic controllability.

APPENDIX V

RICCATI AND TARGET SOLUTIONS FOR PHILLIPS REGULATOR

From Murphy [p.229,(60)], if

$$\mathring{k} = a_2k^2 + a_1k + a_o, \tag{1}$$

then

$$k - k_1 = (k-k_2)\ C\exp\ \{a_2(k_1-k_2)t\} \tag{2}$$

where k_1, k_2 are the distinct roots of the quadratic

$$a_2k^2 + a_1k + a_o = 0. \tag{3}$$

Apply the boundary condition $k(T)=0$ to obtain

$$C = (k_1/k_2)\exp - \{a_2(k_1-k_2)T\}. \tag{4}$$

Solving (2) for k, and using (4);

$$k(t) = \frac{k_1-k_1\exp\{-a_2(k_1-k_2)(T-t)\}}{1-(k_1/k_2)\exp\{-a_2(k_1-k_2)(T-t)\}}. \tag{5}$$

From the text equation (5.19),

$$a_2 = \phi^{-1}w^2;\quad a_1 = 2sw;\quad a_o = -1. \tag{6}$$

Hence

$$k_{1/2} = \frac{\pm\ \theta-s}{\phi^{-1}w},\quad \theta = (s^2+\phi^{-1})^{\frac{1}{2}}. \tag{7}$$

The Riccati solution (5.20) follows readily from (5) and (7).

The differential equation (5.17) in the target has the solution

$$y(t) = -y(0)\exp\{-(swt + \phi^{-1}w^2\int_o^t k(\tau)d\tau)\}, \tag{8}$$

so that

$$\int_o^t k(\tau)d\tau = \int_o^t \frac{A(1-De^{B\tau})}{1+CDe^{B\tau}}\,d\tau, \tag{9}$$

is required, where from (5.20),

$$A = \phi w^{-1}(\theta-s); \quad D = e^{-BT}; \quad B = 2w\theta; \quad C = \frac{\theta-s}{\theta+s}. \tag{10}$$

Now

$$\int_o^t F(e^{B\tau})d\tau = \frac{1}{B}\int_1^{e^{B\tau}} \frac{F(z)}{z}\,dz, \quad z = e^{B\tau}. \tag{11}$$

Hence (9) becomes

$$\begin{aligned} I &= \frac{A}{B}\int_1^{e^{B\tau}} \frac{1 - Dz}{z(1+CDz)}\,dz \\ &= \frac{A}{B}\int_1^{e^{B\tau}} \{\frac{1}{z} - \frac{(1+C)D}{1+CDz}\}\,dz \\ &= [\frac{A}{B}\log z - \frac{A(1+C)}{BC}\log(1+CDz)]_1^{e^{B\tau}} \end{aligned} \tag{12}$$

But $\log z = B\tau$, and $A(1+C)/BC = \phi w^{-2}$, so that

$$I = At - \phi w^{-2}\log\left(\frac{1+CDe^{Bt}}{1+CD}\right). \tag{13}$$

Applying (13) to (8), and using the identity $k=\exp(\log k)$,

$$y(t) = y(0)\,\frac{1+CDe^{Bt}}{1+CD}\exp\{-(sw+\phi^{-1}w^2A)t\}. \tag{14}$$

The solution (5.21) follows after substitution for A,B,C,D.

APPENDIX VIa

SOLUTION OF ASYMPTOTIC RICCATI EQUATION

From (6.26) the solution matrix $\bar{K}$ of

$$-\bar{K}A - A^T\bar{K} + \bar{K}\frac{bb^T}{\phi}\bar{K} = V, \tag{a.1}$$

is required, where from (6.18) and (6.29):

$$\bar{K} = \begin{bmatrix} k_o & k_1 \\ k_1 & k_2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ \mp\eta sw & -(\eta\pm sw) \end{bmatrix}$$

$$\frac{bb^T}{\phi} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{\eta^2 w^2}{\phi} \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{a.2}$$

Equations (a.1), (a.2) yield:

$$\begin{bmatrix} \phi^{-1}\eta^2 w^2 k_1^2 \pm 2\eta swk_1, & \pm\eta swk_2 - k_o + (\eta\pm sw)k_1 + \phi^{-1}\eta^2 w^2 k_1 k_2 \\ \pm\eta swk_2 - k_o + (\eta\pm sw)k_1 + \phi^{-1}\eta^2 w^2 k_1 k_2, & \begin{matrix}\phi^{-1}\eta^2 w^2 k_2^2 + 2(\eta\pm sw)k_2 \\ -2k_1\end{matrix} \end{bmatrix} = V. \tag{a.3}$$

Or

$$\begin{bmatrix} f_{11}(k_1) & f_{12}(k_o, k_1, k_2) \\ f_{12}(k_o, k_1, k_2) & f_{22}(k_1, k_2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{a.4}$$

with the solution sequence $f_{11} \to f_{22} \to f_{12}$, aided by the real positive definite conditions on $\bar{K}$:

$$k_o,\ k_2 > 0$$

$$k_o k_2 > k_1^2 \tag{a.5}$$

$$k_o,\ k_1,\ k_2 \text{ real.}$$

Solution of $f_{11}(k_1) = 1$ provides

$$k_1 = \frac{(\mp s) \pm \theta}{\phi^{-1}\eta w}, \tag{a.6}$$

where

$$\theta = (s^2+\phi^{-1})^{\frac{1}{2}}. \tag{a.7}$$

(The bracketed term ($\pm s$) refers to the stability conditions $w \gtrless 0$, the term $\pm\theta$ to the two solutions of the quadratic.).

Solution of $f_{22}(k_1, k_2) = 0$ provides

$$k_2 = \frac{-(\eta \pm sw) \pm [(\eta \pm sw)^2+2\phi^{-1}\eta^2 w^2 k_1]^{\frac{1}{2}}}{\phi^{-1}\eta^2 w^2}. \tag{a.8}$$

Having regard to conditions (a.5),

$$k_2 \gtrless 0 \text{ as } k_1 \gtrless 0. \tag{a.9}$$

Hence

$$k_1 = \frac{\theta \mp s}{\phi^{-1}\eta w} > 0, \tag{a.10}$$

$$k_2 = \frac{\gamma - (\eta \pm sw)}{\phi^{-1}\eta^2 w^2} > 0, \tag{a.11}$$

where

$$\gamma = [(\eta \pm sw)^2+2\phi^{-1}\eta^2 w^2 k_1]^{\frac{1}{2}} = (\eta^2+s^2w^2+2\eta w\theta)^{\frac{1}{2}}. \tag{a.12}$$

The solution for k_o is only required to show that the positive definite conditions (a.5) are satisfied. The solutions (a.10) and (a.11) can be shown to satisfy (a.5).

APPENDIX VIb

RICCATI SOLUTION WITH TARGET DERIVATIVE WEIGHTING

The state weighting matrix with $\mu \neq 0$ is

$$V = \begin{bmatrix} 1 & 0 \\ 0 & \mu^2 \end{bmatrix}. \qquad \text{(b.1)}$$

A new solution for k_2, obtained from $f_{22}(k_1, k_2) = \mu^2$, is thus required; that is, using (a.3) of Appendix VIa, of

$$\phi^{-1}\eta^2 w^2 k_2^2 + 2(\eta \pm sw)k_2 - (2k_1+\mu^2) = 0. \qquad \text{(b.2)}$$

Hence the term $(2k_1)$ in (a.8) and (a.12) of Appendix VIa must be accompanied by μ^2, so that (a.11) and (a.12) become

$$k_2 = \frac{\hat{\gamma} - (\eta \pm sw)}{\phi^{-1}\eta w} > 0, \qquad \text{(b.3)}$$

where

$$\hat{\gamma} = (\eta^2 + s^2 w^2 + 2\eta w\theta + \phi^{-1}\eta^2 w^2 \mu^2)^{\frac{1}{2}}. \qquad \text{(b.4)}$$

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