ERRATA SHEET FOR THE Ph.D. THESIS
"CONTRIBUTIONS TO THE ECONOMETRICS OF CROSS-SECTIONS"
BY
CARLOS M. JARQUE

FOOTNOTES TO BE ADDED:

PAGE 89 , Line 8

"* Note that - given our definition of $\Omega$ - we have trace $\Omega = \sigma_1^2/\sigma^2 + ... + \sigma_N^2/\sigma^2$, so that, under homoscedasticity, trace $\Omega = N$."

The reference to this footnote should be made after "not proportional to $I_N$". To avoid renumbering of other footnotes a * symbol is used. So, in addition to the footnote, the page would read "not proportional to $I_N$.*"

PAGE 116 , Line 7

"* See also the paper by Koenker published in the Journal of Econometrics 1981, 107-112".

Reference to footnote after "(i.e., robust tests for H).*"

PAGE 323 , Line 8

"* Recall that, in our notation, $E_{\varepsilon/1}[\cdot]$ takes the expectation over the continuous distribution of $\varepsilon$ given a value of $z_1$; whereas $E_{1}[\cdot]$ takes the expectation over the discrete distribution of $z_1$ - which may take as possible values one of $z_1, ... , z_N$. So, $\bar{z}$ and $\Sigma$ are population parameters rather than sample values."  

Reference to this footnote should appear after "From (G.1) we see*".
PAGE 332, line 1

"* To obtain our result we also take into consideration the definition of $S_1$ and $S$."

Reference to this footnote should appear after "given that*".

CORRECTION

PAGE 154, line 15

Definition of $a_i^+$ should read:

$$a_i^+ = -\frac{1}{\sigma^2} \left[ (x_i\tilde{\beta})\tilde{F}_i - \frac{\tilde{\sigma}^2}{\sigma_i^2} \frac{\tilde{F}_i}{1-\tilde{F}_i} - \tilde{F}_i \right]$$
Contributions

to

The Econometrics

of

Cross-Sections

Carlos M. Jarque-Uribe

A thesis submitted to the
Australian National University
for the degree of
Doctor of Philosophy,
DECLARATION

Except where otherwise acknowledged
the work described here is my own.

C.M. JARQUE
I have many people to thank after a number of years of post-graduate education.

During the initial period of two years study at The London School of Economics and Political Science, thanks are due to Professors J. Durbin, D. Hendry, G. Mizon and D. Sargan for my introduction into econometrics; also, to Professor A. Stuart who guided me through many areas of statistics - particularly survey methodology - and who remained interested in my work for the whole of my Ph.D. course.

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This thesis is dedicated with love to Coral and Rodrigo, and with great affection to my parents and the numerous members of my family. Also - in a very special way - to the memory of Marta Acosta.

C.M. Jarque

Canberra
January, 1982
(iv)

ABSTRACT

This thesis is concerned with problems that arise in the statistical analysis of economic models, when using cross-sectional data. Its contents may be conveniently divided into five parts.

Part I comprises Chapter 1, and contains introductory remarks, motivation and overview of the thesis.

Part II, formed by Chapters 2 and 3, relates to the data gathering stage of the analysis. Chapter 2 studies the problem of optimal sample design within an econometric (rather than a purely statistical) framework. Chapter 3 is concerned with the problem of efficient aggregation of data that has been gathered through a census or sample survey. In these two Chapters, several design and aggregation criteria are suggested and the use of clustering techniques is indicated.

Part III, consisting of Chapters 4, 5, 6, 7 and 8, relates to inferential and estimation aspects of the analysis once the data is available. Regarding inference, a contribution is the suggestion of a procedure for the construction of efficient and computationally simple econometric specification tests. This is based on the application of the Lagrange Multiplier test to the Pearson and other families of disturbance distributions (rather than to families of transformations). Illustrations of the procedure are given in Chapters 4, 5 and 6, where normality and/or homoscedasticity tests for the Multiple Regression Model, the Truncated Model and the Tobit Model are derived. Tests for normality, homoscedasticity and parameter variation in Simultaneous
Equations Models are suggested in Chapter 8. Regarding estimation, a two stage procedure is presented in Chapter 7. In the first stage, the observations would be classified into groups of homogeneous parameter values, by the use of a classification criterion suggested. In the second stage, the econometric estimation of the parameters would follow.

Part IV of the thesis comprises Chapter 9, and contains an empirical illustration of some of the tests and estimation procedures suggested. These are used for the study of Household Consumption and Saving behaviour in México.

Part V, formed by Chapter 10, indicates extensions of some of the results of Part III and highlights the fact that many of these results are equally applicable in time-series studies. It also contains some concluding remarks.
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CHAPTER 1

THE ECONOMETRICS OF CROSS-SECTIONS

"As Economics pushes beyond 'statics'
it becomes less like Science and more
like History"

Sir John Hicks
Causality in Economics

1.1 INTRODUCTION

For the statistical analysis of economic models, three types of data are used: cross-sectional, time-series and panel data.

Cross-sectional data consists of a set of observations on a group of entities from a defined population. These entities are typically micro decision-making units in an economy (e.g., consumers). The observations are obtained by 'interviewing', at a particular time, the totality of the given population (i.e., by carrying out a census) or by 'interviewing' only a subgroup of it (i.e., by carrying out a sample survey).

Time-series data comprises a set of observations during a sequence of time periods on a given entity. This is often a macro entity (e.g., a country). The time period can be a
year, a quarter, a month or a week. Here the observations are usually successive and equally spaced in time.

Panel data consists of time-series observations on a set of cross-sectional entities (e.g., a time-series of $T$ years of observations on production, capital and labour for a cross-section of $N$ industries).

Various problems that arise in econometric analysis are specific to the kind of data we are to use. For example, if we are to use cross-sectional data, then we may have some choice as to the way in which this data is to be obtained (i.e., choice of sample design); also, we may have to deal with a problem of data confidentiality. In addition, for the estimation stage of the model, we usually may assume serially independent disturbances. This is in contrast with the problems that arise when using time-series, where the data is typically given and where we would often have to deal with serially correlated disturbances.

In this thesis we concentrate on some problems that arise in the analysis of econometric models when using cross-sectional data. The choice of this research topic was motivated by three considerations.

The first relates to data availability. In many (primarily developing) countries the data available for econometric studies is cross-sectional data obtained, or to be obtained, through a census or a sample survey. These data-gathering techniques allow us to have a large amount of information on economic variables in a short period of time. The econometric methodology employed in these situations would therefore be that of cross-sections.

1 The non-existence of sufficiently long historical series on micro or macroeconomic variables in these countries, would not allow efficient time-series estimation of economic relationships.
The second consideration arises from the recent concern to study more closely the behaviour of micro decision-making units, even when carrying out macroeconomic inferences. The appropriateness of 'macro-studies' (i.e., econometric studies aiming to describe the behaviour of macrovariables through the exclusive use of macrorelations) in both developing and developed countries has been criticized by various authors. Arguments are presented on the grounds that relationships among economic variables should be motivated from economic theory; this theory is often derived for microvariables and it has been noted that results may not necessarily hold for their macro-counterparts [e.g., for a discussion of this point in relation to demand and production function studies see, respectively, Muellbauer (1975) and Fisher (1969); see also Theil (1955) and Green (1964)]. It has also been argued that, even if we were willing to regard the theory as appropriate for the macrovariables, the exclusive use of macrorelations would not give answers to important questions (e.g., see Orcutt (1962, p.231)). As a result, some economists have emphasized the need to develop microanalytic models of economies (e.g., see Orcutt (1962), Goldberger and Lee (1962) and Orcutt, Watts and Edwards (1968)). In this activity we would use extensively, although not exclusively, the econometrics of cross-sections.

---

2 Cases do exist where the use of macrorelations has advantages over the use of microrelations. (e.g., see Grunfeld and Griliches (1960) and Aigner and Goldfeld (1974)).

3 Cross-sectional studies do not provide information on the important question of *dynamics*; for this we have to look at time-series or panel data studies.
The third consideration refers to the view expressed by some economists regarding the 'scientific validity' of time-series econometric studies (e.g., see Hicks' (1980) quotation in the beginning of this Chapter). Indeed, to some - in our opinion pessimistic - economists, 'valid econometrics' would need to be limited to the econometrics of cross-sections.

These three considerations led us to value the study of problems that arise in the econometric analysis of cross-sectional data. In the next section we give necessary definitions, introduce the model to be considered initially and state some assumptions that are often made in the analysis of this model. In Section 1.3 we give an overview of the thesis.

1.2 THE MODEL AND ITS ASSUMPTIONS

A more precise definition of cross-sectional data is now given and some notation introduced. We say we possess cross-sectional data or a cross-section, when we have available a set of, say N, observations on the K+1 variables Y, X₁,...,Xₖ, obtained at a given point in time and which relate to a fixed date or period. These observations correspond to N units or entities which belong to a defined population. For example, the entities may be households and the defined population may consist of the households in a geographical area. Similarly, we may have employees in a given industry, farms in a region or banks in a particular country. We denote by yᵢ the observation on variable Y corresponding to the i'th entity and by xᵢ = (X₁ᵢ,...,Xₖᵢ) the

Hicks (1980) also questions the 'scientific validity' of cross-sectional econometric studies. We have no intention here of debating Hicks' points (for this see Hendry (1980, p.402) and Sims (1981)).
1 by \( K \) vector representing the observation on \( X_1, \ldots, X_K \), also corresponding to the \( i \)'th entity.

We assume that - for the population under study - economic theory suggests a relationship between \( Y \) and \( X_1, \ldots, X_K \), and that this relationship may be written in the form

\[
y_i = x_i'\beta + u_i,
\]

where \( u_i \) is the \( i \)'th unobservable disturbance and \( \beta \) is a vector of unknown parameters that is to be estimated.

Equation (1.1) is linear in parameters and is sufficiently general to allow the study of a wide range of models. Particular cases are polynomial (e.g. see Theil (1971, p.155)), log-linear (e.g. see Desai (1976, p.12)), semi-log (e.g. see Maddala (1977, p.6)), and spline regression models (e.g. see Poirier (1976, p.10)). For instance, in equation (1.1) we could have \( K = 3 \) with \( y_i = \log Q_i, \ X_{i1} = 1, \ X_{i2} = \log C_i \) and \( X_{i3} = \log L_i \), where \( Q_i, C_i \) and \( L_i \) are - respectively - production, capital and labour for the \( i \)'th entity, say, \( i \)'th industry. In this example, \( Q_i, C_i \) and \( L_i \) would be physically observed variables and \( y_i \) and \( x_i' \) the result of some mathematical transformation applied to the observed variables. For simplicity in terminology, throughout the thesis, \( y_i \) and \( x_i' \) and not the originally observed variables - will be referred to as our observations. Also, without loss of generality, in our discussion we will refer to the population units or entities as individuals.

To provide a simple framework for our exposition, we now state assumptions that are commonly made for the econometric analysis of the
model given by equation (1.1).

**ASSUMPTION [1]:** The cross-sectional data to be used, consisting of the $N$ vectors $(y_1^1, x_1^1), \ldots, (y_N^1, x_N^1)$, is given to the researcher (i.e., there is no choice regarding which $N$ individuals are interviewed).

**ASSUMPTION [2]:** The cross-sectional data is not in aggregated form (i.e., each observation refers to an individual).

**ASSUMPTION [3]:** The disturbances $u_i$ are normally distributed.

**ASSUMPTION [4]:** The disturbances $u_i$ are homoscedastic (i.e., $\sigma_1^2 = \ldots = \sigma_N^2$, where $\sigma_i^2$ denotes the variance of $u_i$).

**ASSUMPTION [5]:** The dependent variable $Y$ may take any value (i.e., $-\infty < y_i < \infty$).

**ASSUMPTION [6]:** There is no parameter variation in the population under study (so, in (1.1) the vector $\beta$ is the same for all $i = 1, \ldots, N$).

**ASSUMPTION [7]:** The variables $X_1, \ldots, X_K$ are fixed (i.e., there are no current endogenous variables among $X_1, \ldots, X_K$) and no other model exists with disturbance correlated with $u_i$. 
Each of these seven assumptions provides a topic for discussion in the thesis. The topics are discussed under four maintained hypotheses which we now state.

A first maintained hypothesis is that the model is correctly specified, in the sense that there are no omitted deterministic influences. This assumption is commonly made in the development of econometric methodology. Tests for 'model misspecification' - which is an important inferential problem in econometrics - have been suggested by Ramsey (1969) and, more recently, by Hausman (1978) and White (1980a). Additional results are indicated in Chapter 10.

A second maintained hypothesis is serial independence of disturbances. In most cross-sectional studies this would be a reasonable assumption and we shall not deal with it here. We note, however, that a specific situation in cross-sectional studies, where autocorrelated disturbances may arise, is in the analysis of geographical data. The problem has come to be known as spatial correlation and a description of this is given by Cliff and Ord (1973).

A third maintained hypothesis is that no relation exists between \( \beta \) and the unknown parameters that define the distribution of the disturbances. In particular, we assume that \( x_1 \) is measured without error.

Lastly, a fourth maintained hypothesis is that the rank of \( X \) is equal to \( K \), where \( X = (x_1, \ldots, x_N)' \).
1.3 OVERVIEW OF THE THESIS

Under the assumptions stated in Section 1.2, the model given by equation (1.1) could be 'optimally' estimated by Ordinary Least Squares and inferences about the parameters could be carried out using standard \( t \) and \( F \) tests. This is - of course - treated in detail in elementary econometrics textbooks. Here we are concerned with the study of econometric models, when one or more of assumptions [1] to [7] have 'questionable validity'. Basically, in each Chapter we consider a particular assumption and note the possible consequences when this is invalidly made; we then discuss the problem of testing its validity and, finally, we comment on the analysis of the model when there is evidence that the assumption does not hold. More specifically, the structure of the thesis - highlighting our main results - is as follows.

In Chapter 2, we concentrate on assumption [1]. We discuss the estimation of the regression model when using census and given survey data and study the properties of estimators - taking explicit account of the probabilities of selection of the individuals in the sample. We then discuss the question of sample design to obtain efficient estimators of the regression model, and indicate the use of experimental design results and clustering algorithms in the present setting.

In Chapter 3, we consider assumption [2] and discuss the problem of efficient aggregation or grouping of observations in regression analysis. We suggest various grouping criteria that lead to efficient estimators of the model. We also show the usefulness of Ward's and the K-Means clustering algorithms in the computation of an optimal aggregation.
In Chapter 4, we concentrate on assumption [3]. A main result here is the suggestion of a test for disturbance normality, which is simple to compute and has maximum asymptotic local power. The finite sample properties of this test are also studied and it is found that it performs with very good power, relative to other existing tests - most of which are considerably more difficult to calculate. Our normality test is obtained by applying the Lagrange Multiplier principle to a family of disturbance distributions (e.g., the Pearson Family). This suggested approach can be used in a very wide range of inferential problems and is applied in various sections of the thesis, illustrating its use in inferential problems of the econometrics of cross-sections.

In Chapter 5, we consider assumption [4]. We suggest a fully non-constructive test for homoscedasticity, which may be used when there is weak a-priori knowledge as to the nature or the form of the possible heteroscedasticity. We also apply the procedure suggested in Chapter 4, to obtain a joint test for disturbance normality and homoscedasticity. The power of our tests is studied and found to be good relative to other existing procedures.

In Chapter 6, we discuss assumption [5] and consider two limited dependent variable (LDV) models: the Truncated model and the Tobit model. We comment on the consequences of disturbance non-normality and heteroscedasticity in the usual maximum likelihood estimators of these LDV models. Also, we suggest tests for disturbance normality and/or homoscedasticity in LDV situations.

In Chapter 7, we concentrate on assumption [6] and present a two stage estimation procedure for models with systematic parameter variation. In the first stage, the individuals would be classified into groups of homogeneous parameter values (regimes) by the use of a
clustering criterion suggested; the second stage would consist of the econometric estimation of the regimes.

In Chapter 8, we consider assumption [7] and extend some of our inferential results to simultaneous equations models. We suggest tests for multivariate normality, multivariate homoscedasticity and for parameter variation.

In Chapter 9, we present an illustration of some of the results obtained in previous Chapters. We estimate the Extended Linear Expenditure System, using data from a 1975 Income-Expenditure Household Survey for México. We apply clustering algorithms to form groups of households of homogeneous demand behaviour. Also, we take into consideration the fact that expenditures are non-negative and therefore use LDV models. Additionally, we carry out a comprehensive statistical analysis of the disturbances. To end, we discuss the patterns of household consumption and saving behaviour that emerge.

In Chapter 10, we comment on the maintained hypothesis of correct specification of the deterministic part of the model, and indicate extensions of our work which provide a general specification test. We also highlight the usefulness of our results in time-series studies and present some concluding remarks.

Two statistical techniques are largely used throughout; these are Cluster Analysis (particularly in Chapters 2, 3 and 7), and the Lagrange Multiplier Test (particularly in Chapters 4, 5, 6 and 8). Indeed, the thesis is concerned with the application of these techniques in problems that arise in the econometrics of cross-sections.
CHAPTER 2

THE PROBLEM OF OPTIMAL SAMPLE DESIGN

"I have no satisfaction in formulas unless I feel their numerical magnitude"

Lord Kelvin

Life by Sylvanus

2.1 INTRODUCTION

In Chapter 1 we introduced the regression model, and presented a set of assumptions that are often made for the econometric analysis of this model. Among these was assumption [1], which refers to the data and states that there is no choice regarding the individuals that are to form the cross-section. In this Chapter we discuss the efficient estimation of $\beta$, firstly, for the case when the data is given (i.e., assumption [1] is 'valid') and, secondly, when there is some choice regarding the individuals to be interviewed (i.e., assumption [1] is 'invalid'). The structure of the Chapter is as follows. In Section 2.2 we state results on 'best' estimation of the model when using given census data. Section 2.3 considers the same problem but when the given data was obtained by the use of a sample survey. We then discuss, in Section 2.4, the question of choice of sample design. Some concluding remarks are made in Section 2.5.
2.2 BEST ESTIMATION USING CENSUS DATA

Suppose the total number of individuals in the population under study is \( N^* \). We have used \( N \) to denote the number of individuals for which data is available. If \( N = N^* \) then we can consider having census data, in the sense that all individuals in the population were (or are to be) interviewed and their values of \( Y \) and \( X_1, \ldots, X_K \) recorded.

In Chapter 1 we defined a cross-section as a set of observations obtained at a specific point in time. Therefore, it would be physically impossible to have a cross-section of more than \( N^* \) observations. If, to achieve more observations, re-interview was attempted, some time—however small—would have to pass between the initial interview and the re-interview, and by doing this the cross-sectional data would become panel data. We shall assume, then, that \( N^* \) is an upper limit to the number of observations \( N \), i.e., that census data gives the maximum number of observations in cross-sectional studies.

We should note some survey statisticians would argue that—with census data—the 'regression parameters' would be known exactly and given by the 'non-stochastic vector'

\[
B = (X^* X^*)^{-1} X^* y^* , \tag{2.1}
\]

where \( y^* = (y_1, \ldots, y_{N^*})' \) and \( X^* = (x_1, \ldots, x_{N^*})' \) (e.g., see the discussion in Konijn (1962), Brewer and Mellor (1973), Jönrup and Rennermalm (1976) and Holt, Smith and Winter (1980)).

Here we do not regard the \( \beta \) vector in (1.1) as a characteristic in the finite population of size \( N^* \) (like the mean or the total of a given variable). Rather, we proceed—as is usual in econometrics—
and treat $\beta$ as a parameter underlying the economic model. We interpret (2.1) as an estimator of $\beta$. In other words, our view is that even if all $N^*$ individuals were interviewed, there would still be randomness in the vector $\mathbf{B}$. This is because we recognize the general existence of two random processes or sampling stages that give rise to cross-sectional data:

The first refers to the choice of the $N$ individuals to be interviewed;

The second arises from the departure in the measured quantity $y_i$ from its corresponding expected response, namely, $x_i^T \beta$.

By setting $N = N^*$ we would suppress the randomness from the first process but not from the second.

In this section we assume $N = N^*$ and, therefore, we only have to consider the second stage. We define $u^* = (u_1, \ldots, u_{N^*})'$ and assume $u^*$ has zero expectation. Also, we denote the variance-covariance matrix (VCM) of $u^*$ by $\Omega^*$, where $\sigma^2$ is a finite scalar and $\Omega^*$ is an $N^*$ by $N^*$ diagonal matrix with finite diagonal elements $\sigma_1^2 / \sigma^2, \ldots, \sigma_{N^*}^2 / \sigma^2$. These values may or may not all be equal. Then, using our definitions, it is easily shown that - for census data and under assumptions [2], [6] and [7] - the best linear unbiased estimator (BLUE) of $\beta$ is the Generalized Least Squares (GLS) estimator.

---

1 In Section 2.3 we deal with survey data, i.e., with the case $N < N^*$. There we take account of both stages.

2 Throughout the chapter we concentrate on linear estimators of $\beta$. 
\[ \hat{\beta}_* = (x_*'n_*^{-1}y_*)^{-1}x_*'n_*^{-1}y_* \]  

which has VCM given by\(^3\)

\[ V[\hat{\beta}_*] = \sigma^2(x_*'n_*^{-1}x_*)^{-1}. \]  

Often, the population under study is so large that it would be very costly to interview all \( N^* \) individuals. In these cases, only \( N < N^* \) observations (obtained by the use of some sampling design) may be available for econometric analysis. The best estimation when using survey data is discussed in the next two sections.

### 2.3 BEST ESTIMATION USING GIVEN SURVEY DATA

We now suppose there are observations corresponding to \( N \) individuals, where \( N < N^* \). These individuals are assumed to have been selected by, say, a statistical agency, with the use of a specified (perhaps complex) sampling design. This constitutes the first stage of the two stage sampling procedure recognized in Section 2.2, which gives rise to cross-sectional data. Let \( q \) be the total number of possible samples (sets of \( N \) individuals) under the design employed, and \( p_s \) the probability of selecting sample \( s \) (with \( s = 1, \ldots, q \)).\(^4\) We shall use \( E_s[\cdot] \) and \( V_s[\cdot] \) to denote the expectation and variance operators referring to this stage.

---

\(^3\) To interpret the properties of \( \hat{\beta}_* \) one may suppose that, if the \( N^* \) individuals were 'instantaneously' re-interviewed (giving a new set of values \( u_1^*, \ldots, u_{N^*}^* \)), the distribution of the estimators \( \hat{\beta}_* \) – obtained from the various realizations – would have mean \( \beta \) and VCM defined by (2.3) for a large number of realizations. A similar interpretation may be given to the properties of other estimators – and inference procedures – discussed in the thesis.

\(^4\) The use of \( s \) to denote a sample (set of individuals) is not entirely satisfactory; yet it is used because, for our purposes, it is a simple and convenient notation.
The second and final stage of the procedure that gives rise to cross-sectional data, consists of 'sampling' a \( u_i \) for each of the \( N \) individuals selected in the first stage. We use \( E_{u/s}[\cdot] \) and \( V_{u/s}[\cdot] \) to denote the expectation and variance operators referring to the \( u \)'s for a given sample \( s \). (Observe it is not unreasonable to assume that sampling is independent at the different stages, e.g., the statistical agency selects individuals and these, in turn and independently 'select their \( u \)'s').

In the presentation of econometric methodology and its applications, most textbooks implicitly treat cross-sectional data as if it were obtained by means of a census. This neglect of the first (sampling) stage is also typical of published work in econometric journals. Here we shall see how the expectation and variance of the familiar least squares estimator of \( \beta \) are modified, when one recognizes the existence of random selection of individuals.\(^5\) Our discussion is based on Porter (1973) but generalizing his findings to the unequal disturbance variance case.\(^6\)

Denote by \( X^{(j)} \) the \( N \) by \( K \) matrix with rows given by the subset of rows in \( X^* \) corresponding to the individuals that form sample \( j \) in the design utilized by the statistical agency. Apply an equivalent definition to \( y^{(j)} \), \( \Omega^{(j)} \) and \( u^{(j)} \) with respect to \( y^* \), \( \Omega^* \) and \( u^* \).

\(^5\) White (1980b) considers the estimation of nonlinear regression models with cross-sectional data. White obtains conditions which ensure the consistency and asymptotic normality of the weighted least squares estimators, when the regressors - although independent of the disturbances - are not identically distributed. Our concern here is with linear models, and we look at the finite sample properties of the GLS estimator.

\(^6\) Porter (1973) also deals with varying parameter models. We discuss these models in Chapter 7.
The cross-sectional data available would only consist of one sample, say sample $s$, and would be given by $(y(s), X(s))$. We make assumptions [6] and [7] and note that, for this cross-section, the model to be estimated (given by equation (1.1)) could be written in the form

$$y(s) = X(s)\beta + u(s),$$

where, by our maintained assumptions, we would have $E_{u/s}[u(s)] = 0$ and $E_{u/s}[u(s)u'(s)] = \sigma^2\Omega(s)$.

The usual GLS estimator of $\beta$ is

$$\hat{\beta} = (X'(s)\Omega^{-1}(s)X(s))^{-1}X'(s)\Omega^{-1}(s)y(s). \quad (2.4)$$

We note that, due to our assumptions,

$$E_{u/s}[\hat{\beta}] = \beta \quad (2.5)$$

and

$$V_{u/s}[\hat{\beta}] = \sigma^2(X'(s)\Omega^{-1}(s)X(s))^{-1}. \quad (2.6)$$

Equations (2.5) and (2.6) represent conditional moments of $\hat{\beta}$ on a given sample $s$. We shall take into consideration the first stage and examine the unconditional properties of $\hat{\beta}$.

Using a result on expectations in multi-stage sampling procedures (e.g., see Kendall and Stuart (1966, p.191)) we obtain

$$E[\hat{\beta}] = E_s E_{u/s}[\hat{\beta}],$$

i.e.,

$$E[\hat{\beta}] = \sum_{s=1}^{q} p_s E_{u/s}[\hat{\beta}] \quad (2.7)$$
After substitution of (2.5) into (2.7) and noting that \( \sum_{s=1}^{q} p_s = 1 \) we obtain \( E[\hat{\beta}] = \beta \). Therefore, we see that \( \hat{\beta} \) is a conditionally (conditional on sample \( s \)) and an unconditionally unbiased estimator of \( \beta \).

We now use a result for variances in multi-stage procedures (e.g., see Kendall and Stuart (1966, p.191)), namely,

\[
V[\hat{\beta}] = E_s V_{u/s}[\hat{\beta}] + V_s E_{u/s}[\hat{\beta}].
\]

Since \( E_{u/s}[\hat{\beta}] = \beta \), we have \( V_s E_{u/s}[\hat{\beta}] = 0 \) and, using (2.6), \( V[\hat{\beta}] \) reduces to

\[
V[\hat{\beta}] = \sum_{s=1}^{q} p_s \sigma^2(X'(s)\Omega^{-1}(s)X(s))^{-1} . \tag{2.8}
\]

Furthermore, we can easily show that, if we consider any other linear estimator, say \( \tilde{\beta} = \hat{\beta} + D(s)y(s) \), such that \( \tilde{\beta} \) is unbiased in the sense that \( E_{u/s}[\tilde{\beta}] = \beta \), then

\[
V[\tilde{\beta}] - V[\hat{\beta}] = \sum_{s=1}^{q} p_s \sigma^2 D(s)\Omega(s)D'(s),
\]

where \( D(s) \) is an \( N \) by \( N \) given matrix. Therefore, we obtain that \( V[\tilde{\beta}] - V[\hat{\beta}] \) is a positive semidefinite matrix.

From this it follows that - for given survey data - the estimator \( \tilde{\beta} \) (see (2.4)), with VCM defined by (2.8), is best among the class of linear unbiased estimators \( \tilde{\beta} \).
For the computation of \( \hat{\beta} \) one does not require the values of \( p_1, \ldots, p_q \). To obtain its VCM, however, one would require not only \( p_1, \ldots, p_q \) but also knowledge of the \( N \) individuals that form each one of the \( q \) possible samples, and of \( X^* \) (see (2.8)). In practice, one may be able to obtain information on the sampling design employed; also, \( X^* \) may perhaps be known or obtainable (say, using a proxy matrix) from a previous recent census. Yet, even with this knowledge, the identification of the \( q \) possible samples, determination of \( p_1, \ldots, p_q \), and the computations to obtain (2.8) would require — in general — a formidable amount of effort. An example is presented to illustrate this point.

**EXAMPLE:** Assume the \( N \) individuals were selected by the use of 'simple random sampling'. Then we would have that each possible sample is given by one of the \( q = q^* = \frac{N^*!}{(N^!(N^*-N)!)} \) combinations of \( N \) individuals out of \( N^* \). We would also have \( p_1 = \ldots = p_{q^*} = 1/q^* \). We note that, for this sampling scheme, even for small \( N^* \) and \( N \), the value of \( q^* \) would be substantial. For instance, if \( N^* = 100 \) and \( N = 20 \) we would have \( q^* \) greater than \( 535 \times 10^{18} \). Under the simplifying assumption that \( \Omega(1) = \ldots = \Omega(q^*) = I_N \), where \( I_N \) is an \( N \) by \( N \) identity matrix, we would still have to invert \( q^* \) matrices of dimension \( K \) by \( K \) and to carry out \( q^* \) products and summations to compute (2.8). In practice, \( N^* \) will generally be much larger than one hundred and for this design, the expected computational load would be even heavier.

Whatever the sample design, \( q^* \) will be an upper limit to the possible number of samples \( q \). Typically, the design will be more complex than 'simple random sampling' and, although some of these \( q^* \)
combinations would have zero probability of selection, the computation of (2.8) would still be troublesome. However, it is interesting to consider two sampling designs which would justify the use of (2.6) as the unconditional VCM of $\hat{\beta}$.

In the first we take a situation where the agency decided to interview, with probability one, the last $N$ (or in fact any specified $N$) in a given list of the $N^*$ individuals. Identify this as being sample $q$; then it would follow that $p_q = 1; p_s = 0$ (for $s = 1, \ldots, q-1$); and (2.8) would be simplified to

$$V[\hat{\beta}] = \sigma^2 (X'(q)\Omega(q)^{-1}X(q))^{-1}.$$ 

Hence, under this design rule (which we refer to as a 'probability-one scheme') the VCM of $\hat{\beta}$ would be equal to the commonly used VCM of the GLS estimator of $\beta$, and the computational difficulties mentioned previously would not exist.

A second sampling design of interest is 'cluster sampling'. With this sampling scheme, clusters are formed so that they are homogeneous across clusters, i.e., heterogeneous within. [This is in contrast with, for example, a 'stratified design' where strata are formed so that they are heterogeneous between themselves, i.e., homogeneous within (e.g., see Moser and Kalton (1971, p.105))]. Assume that - in the formation of the clusters - all $K$ regressors were considered, and the sampling rule consisted of selecting one cluster out of those $q$ formed. Then, under the assumption that $\Omega(1) = \ldots = \Omega(q) = I_N$; we would expect

$$(X'(q)\Omega(q)^{-1}X(q))^{-1} \approx (X'(s)\Omega(s)^{-1}X(s))^{-1}, \quad (2.9)$$

where $\approx$ denotes approximate equality, $q$ is the cluster selected and
s = 1, ..., q - 1. Using (2.9) in (2.8) and noting that \( \sum_{s=1}^{q} p_s = 1 \), we would have

\[
\mathbb{V}[\hat{\beta}] = \sigma^2 (X'_{(q)} \Sigma^{-1}_{(q)} X_{(q)})^{-1} .
\]

Hence, for this sampling design also, we could be justified in applying the commonly used VCM for \( \hat{\beta} \) and interpret this as an unconditional VCM. □

In Section 2.2 we discussed the problem of estimation when one has, or is to carry out, a census and presented the BLUE of \( \beta \), \( \hat{\beta}_* \), which has VCM (2.3). In this section we have considered estimation with given survey data. In this case the BLUE of \( \beta \) is \( \hat{\beta} \), defined by (2.4), which has VCM (2.8).

It can be shown that the difference between (2.8) and (2.3) is a positive semidefinite matrix - for any sampling design (note both \( \hat{\beta}_* \) and \( \hat{\beta} \) are linear unbiased estimators in \( y^* \), and that \( \hat{\beta}_* \) is efficient among this class). Hence, as expected, the GLS estimator using census data, \( \hat{\beta}_* \), would be more efficient than the GLS estimator based on survey data, \( \hat{\beta} \). For a given sample size \( N \), a natural question that arises is what choice of sampling design will make the loss of efficiency minimum. This problem is discussed in the next section. Our results there should be of value to econometricians who are able to design their surveys.
2.4 **OPTIMAL SAMPLE DESIGN**

We now consider a situation where the sample of individuals, which is to form the cross-section, is not yet taken and we have some choice on how this is done. Survey data may be obtained for a variety of purposes and not just for the estimation of a regression model. A sample design is determined by the type of analysis for which the data is to be used and, therefore, many elements come into play in its definition (e.g., efficiency gain of specific parameters, domains of study of interest, etc.). For instance, a 'stratified sampling design' may be appropriate when interest resides in obtaining efficient estimators of the population and strata means of certain variables (see Cochran (1963, p.87)). Cost factors may also need to be considered. Here we shall only be concerned with the *choice of sample design* for the purpose of obtaining *efficient* estimators of $\beta$.

We assume that, prior to the survey, we know (or have a proxy for) the matrix $X^* = (x_1^*, ..., x_N^*)'$ but not $y^*$ (knowledge of $y^*$ would imply that there is no justification for sampling since $\beta$ could be estimated optimally by (2.2)). We suppose that cost and/or time limitations and/or other considerations, force us to use a sample of $N (< N^*)$ individuals. We make the additional assumption that disturbances are homoscedastic, i.e., that $\Omega(1) = ... = \Omega(q) = I_N$.

Some criterion or objective function that serves as an *efficiency measure* has to be chosen. For instance, we may consider obtaining, for a given $N$, the sample design that minimizes the generalized variance of $\hat{\beta}$, i.e.,

$$
\text{det} \{V[\hat{\beta}]\} = \text{det} \left\{ \sum_{s=1}^{q} p_s \sigma^2 (x'_s X'_s)^{-1} \right\}.
$$
Unfortunately, this criterion is not amenable to easy computations and it would be difficult to apply in practice. In fact, many criteria share this computational difficulty and, for results that are of practical use, we have to limit the choice of efficiency measures to those that are computationally manageable. One of these measures is now discussed.

We shall assume that, for a given $N$, we want to find the sample design that minimizes $\psi(\cdot)$, where $\psi(\cdot)$ is a weighted sum of the variances of the estimated parameters, i.e., a weighted sum of the diagonal elements in (2.8). More formally we define

$$\psi(p_1, \ldots, p_q) = E[(\hat{\beta} - \beta)'D(\hat{\beta} - \beta)]$$

i.e.,

$$\psi(p_1, \ldots, p_q) = \text{trace}(DV[\hat{\beta}])$$

where $D$ denotes a $K$ by $K$ diagonal matrix with diagonal elements $D_{kk}$ being the weights given to each estimated parameter.

The choice of $D$ is not easy and indeed the solution may be very sensitive to this (see Kiefer (1959)). In general, $D$ should be such that $\psi(\cdot)$ is scale invariant. Hence, reasonable choices - when all parameters are thought to be equally important - are $D_{kk}^{-1} = 1/\bar{x}_k^2$ or $1/V[X_k]$, where $\bar{x}_k$ and $V[X_k]$ are, respectively, the mean and variance of the $k$'th regressor. In what follows, we assume that a value of $D$ has been agreed upon. Using (2.8) with $\Omega(s) = I_N$ and

---

7 Computation of the sample design that minimizes $\det(V[\hat{\beta}])$ by total enumeration of alternatives, would be impractical - if not impossible - due to the magnitude of $q$ in most applications.
using the additivity of the trace operator we have

\[ \psi(p_1, \ldots, p_q) = \sum_{s=1}^{q} p_s \sigma^2 \text{trace}(D(X'_{(s)}X_{(s)})^{-1}) \ . \quad (2.10) \]

It is important to note that, if

\[ \text{trace}(D(X'_{(j)}X_{(j)})^{-1}) = \min_{s=1, \ldots, q} \text{trace}(D(X'_{(s)}X_{(s)})^{-1}) \ , \]

then, for all choices of \( p_1, \ldots, p_q \),

\[ \psi(p_1, \ldots, p_q) \geq \psi(p_1^*, \ldots, p_q^*) \ , \]

with \( p_j^* = 1 \) and \( p_s^* = 0 \) for \( s = 1, \ldots, j-1, j+1, \ldots, q \).

It follows that the optimal sample design is that which selects sample \( j \) with probability 1. In this case, the 'best' sample survey estimator of \( \beta \) would be

\[ \hat{\beta} = (X'_{(j)}\Omega_{(j)}^{-1}X_{(j)})^{-1}X'_{(j)}\Omega_{(j)}^{-1}Y_{(j)} \ , \]

which is unbiased and has unconditional VCM given by

\[ V[\hat{\beta}] = \sigma^2(X'_{(j)}\Omega_{(j)}^{-1}X_{(j)})^{-1} \text{ and where, by assumption, } \Omega_{(j)} = I_N \ . \]

The remaining problem is a numerical one, namely, finding sample \( j \) such that

\[ \phi(X_{(j)}) = \min_{s=1, \ldots, q} \phi(X_{(s)}) \ , \quad (2.11) \]

where
\[ \phi(X^{(s)}) = \text{trace}(D(X^{(s)}X^{(s)})^{-1}) \]

This numerical problem could be solved if all possible matrices \( X^{(s)} \) were considered and \( \phi(X^{(s)}) \) evaluated. The total number of matrices \( X^{(s)} \) is equal to \( N^*!/(N!(N^*-N)!) \) [the number of sets of \( N \) rows out of the possible \( N^* \) rows in \( X^* \)]. In practice this is so large that some computationally simpler approaches are required. We shall present two of these:

1. the **experimental design approach** (based on Conlisk and Watts (1979)), and
2. the **finite selection model approach** (based on Morris (1979)).

### 2.4.1 Experimental Design Approach

Define the standardized data matrix \( X^+ = X^*D^* \), where \( D^* \) is a \( K \) by \( K \) diagonal matrix with \( k' \)th diagonal element being

\[ D_k^* = 1/(V[X_k])^{1/2}. \]

Say we apply, on \( X^+ \), a clustering procedure such as the K-Means algorithm (e.g., see Hartigan (1975, Chapter 4) or Subsection 3.4.2 in this thesis) to form \( m \) groups of individuals that are 'homogeneous' with respect to the \( K \) regressors. Denote by \( \bar{x}_h \) the vector of dimension \( K \) by \( 1 \) which contains the \( K \) regressor means of group \( h \), with \( h = 1, \ldots, m \). For our discussion we draw on some results available in the experimental design literature, so we may conveniently refer to \( \bar{x}_1, \ldots, \bar{x}_m \) as 'design points'.

We may assume that \( x_i \approx \bar{x}_h \) (i.e., that \( x_i \) is approximately equal to the \( h' \)th design point) if - in the clustering of the \( N^* \) individuals - the \( i' \)th individual resulted in group \( h \). This allows us to write, for
sample \( s \),

\[
X'(s)X(s) = \sum_{h=1}^{m} N_h \bar{x}_h \bar{x}'_h,
\]

where \( N_{hs} \) denotes the number of individuals in group \( h \) out of the \( N \) in sample \( s \). We shall assume the inverse of \( \sum_{h=1}^{m} N_h \bar{x}_h \bar{x}'_h \) exists for all \( s = 1, \ldots, q \). Substituting (2.12) into

\[
\phi(X(s)) = \text{trace}(D(X'(s)X(s))^{-1})
\]

gives

\[
\phi(X(s)) = \phi'(N_{1s}, \ldots, N_{ms})
\]

with

\[
\phi'(N_{1s}, \ldots, N_{ms}) = \text{trace}\left(D \left( \sum_{h=1}^{m} N_h \bar{x}_h \bar{x}'_h \right)^{-1} \right).
\]

For computational convenience, in this approach we use \( \phi'(\cdot) \) rather than \( \phi(\cdot) \), as our objective function, and define sample \( j \) as that which minimizes \( \phi'(\cdot) \). To determine this, we proceed as in Conlisk and Watts (1979). We use their experimental design results to obtain the values \( N_1, \ldots, N_m \) as the solution to the mathematical programming problem

\[
\text{Min } \phi'(N_1, \ldots, N_m) = \text{trace}\left(D \left( \sum_{h=1}^{m} N_h \bar{x}_h \bar{x}'_h \right)^{-1} \right)
\]

subject to

\[
N_1 + \ldots + N_m = N \tag{2.13}
\]

and

\[
N_1 > 0, \ldots, N_m > 0.
\]
The solution may be found by an iterative procedure (see Conlisk and Watts (1979, p.32)).

If we let $N_{1j} = N_1, N_{2j} = N_2, \ldots, N_{mj} = N_m$, it would follow that the 'optimal design' is that which selects sample $j$ with probability 1, where sample $j$ consists of interviewing any $N_{1j}$ individuals from group 1, $N_{2j}$ from group 2 and so on. As pointed out by Conlisk and Watts (1979, p.32), the optimal $N_1, \ldots, N_m$ may not be unique. If we had several sets of these, we could evaluate $\phi(X_{(s)})$ and choose from them the one with smallest value $\phi(X_{(s)})$.

In the implementation of this approach an important aspect is choice of design points. Often, $N^*$ will be extremely large (say a few hundred thousand) and heavy calculations would result when computing the design points by the clustering algorithm. For these situations, a reasonable suggestion is to apply the algorithm using a random sample of the $N^*$ individuals. The number of individuals in this sample would need to be sufficiently larger than $N$, so the optimal sample design can be carried out, i.e., so that we guarantee the identification of enough individuals per group.

A final point to note is that we have assumed $N$ to be given. Instead, say we had an upper limit $C$ on the cost of the survey, and that we were able to specify a cost $c_h$ for each individual from group $h$ who was to be interviewed. Then we could replace, in the mathematical programming problem defined in (2.13), the restriction $N_1 + \ldots + N_m = N$ by

$$c_1 N_1 + \ldots + c_m N_m \leq C,$$

and proceed using the results of Conlisk and Watts (1979).
2.4.2 Finite Selection Model Approach

The second approach for the minimization of $\phi(X^{(s)})$ is based on the finite selection model, and uses the hierarchical algorithm described in Morris (1979).

At the $n$'th stage of this procedure (i) there are $n$ rows from $X^*$ that have already been selected, and which are written collectively in the $n$ by $K$ matrix $X^{(n)}$; and (ii) an additional row $x'$ (which is not in $X^{(n)}$ but is in $X^*$) is considered for inclusion in $X^{(n)}$.

We define $S_n = (X^{(n)}'X^{(n)})^{-1}$ and note that, if a row $x'$ is included in $X^{(n)}$, forming $X^{(n+1)}$, then

$$S_{n+1} = S_n - S_n x' x'^{\prime} S_n / (1 + x'^{\prime} S_n x) .$$

Premultiplying this by $D$ and taking the trace operator it follows that, at any given stage,

$$\phi(X^{(n+1)}) = \phi(X^{(n)}) - x'^{\prime} S_n D x / (1 + x'^{\prime} S_n x) .$$

The algorithm then consists of choosing, at each stage, $x' = x^*_j$ (from the remaining rows in $X^*$) to minimize $\phi(X^{(n+1)})$, i.e., to maximize $\Delta = x'^{\prime} S_n D x / (1 + x'^{\prime} S_n x)$. After a choice has been made, we would update $S_n$ and proceed to maximize the new $\Delta$. When $N$ individuals have been selected the procedure would stop, and the corresponding $x_j^*$ (i.e., $X^{(N)}$) would define $X(j)$ (i.e., sample $j$).

Of course, an initial choice of $K$ individuals is required for this hierarchical algorithm to start. Although the solution will depend on this choice the dependence will, hopefully, not be a very strong one (see Morris (1979, p.47)). In practice several choices may be made.
before the solution is taken as final. Other modifications of the algorithm that attempt to give solutions closer to the optimum are described in Morris (1979, pp.47-48). Also, cost factors may be incorporated into the discussion (see Morris (1979, p.44)).

Before concluding this section we shall make three brief comments:

Firstly, we note the problem of optimal sample design was discussed assuming the goal was to minimize $\psi(\cdot)$, a weighted sum of the variances of the estimated parameters. The results presented have more general application, and may be used when interest resides in minimizing a weighted sum of the variances of a set of linear combinations of $\hat{\beta}$, say $R\hat{\beta}$, where $R$ is an $r$ by $K$ matrix of given constants. In this case the criterion would be to minimize $\text{trace}(D_R RV[\hat{\beta}]R') = \text{trace}((R'D_RR)V[\hat{\beta}])$, where $D_R$ is a diagonal $r$ by $r$ matrix whose diagonal elements indicate the 'importance' given to each of the elements in $R\hat{\beta}$. Then, all the results would apply when replacing $D$ in $\psi(\cdot)$ by $R'D_RR$.

Secondly, note we assumed a value for $N$, or upper limit to the cost of the study, $C$, was given. In practice, this value of $N$ (or $C$) may not be appropriate in the sense that with this sample size (or upper cost), no significant test statistics are obtained. In these cases either $N$ (or $C$) should be increased, or the study abandoned. This problem has been considered, in a different context, by Keeley and Robins (1980) (see also Spiegelman and West (1976)).

Thirdly, we assumed $X^*$ to be a known matrix of given numbers. If the cross-sectional study consisted of a controlled experiment (in the
sense that we could specify the values of $X_1, \ldots, X_K$ to be 'assigned' to the individuals) then, for the efficient estimation of the regression model, we could apply (more directly) the experimental design results contained in Aigner and Morris (1979).

2.5 **CONCLUDING REMARKS**

Results of the last three sections show there are cases where application of the usual VCM of the least squares estimator (see (2.6)) would be appropriate, and could be interpreted as an unconditional VCM. This is the case when one has census data, or when one is able (and proceeds) to design the survey as described in Section 2.4. Also, in the (perhaps not very likely) cases where the given survey data has come from either of the two special designs mentioned in Section 2.3 ('probability - one scheme' and 'cluster sampling').

For survey data, it appears that - unfortunately all too often - one would not be able to compute the true VCM of $\hat{\beta}$ (see (2.8)). For instance, when

(i) information on the sample design is not complete
(so that we cannot identify the $q$ possible samples and $p_1, \ldots, p_q$), and/or

(ii) $X^*$ is not available, and/or

(iii) the computational resources are insufficient to carry out extremely extensive calculations.

A way around this problem, at the expense of theoretical rigour, is to proceed by regarding 'repeated sampling' of the $u$'s as being applied to the individuals in the cross-section only. This amounts to
treated the cross-section as coming from a 'probability - one scheme'.

We would then regard expression (2.6) as the VCM of \( \hat{\beta} \). This is the approach presently taken by econometricians and we shall also use this throughout the thesis. Due to this consideration, in what follows we shall drop the subscript \( s \) in \( y(s), X(s), u(s) \) and \( \Omega(s) \), and the operators \( E_{u/s}[\cdot] \) and \( V_{u/s}[\cdot] \) shall be written - simply - as \( E[\cdot] \) and \( V[\cdot] \).

In this Chapter we concentrated on the efficient estimation of \( \beta \). Special attention was given to the VCM of \( \hat{\beta} \) defined in equation (2.8), and the difficulty in computing this was noted. We did not fully specify the distribution of the disturbances, and neglected inferential aspects in most of our discussion. It is important to note that, with given survey data, due to the random selection of individuals, problems would exist when applying the standard \( t \) and \( F \)-tests (e.g., see Schmidt (1976, p.96)). Yet, like in estimation, these problems would disappear when assuming the cross-section comes from a 'probability-one scheme'.
CHAPTER 3

THE PROBLEM OF DATA AGGREGATION*

"It is for the collectors of data to know the fact and for the mathematicians to establish the reason"

Aristotle
The Organon

3.1 INTRODUCTION

As mentioned in Chapter 2, cross-sectional data is obtained when interviewing individuals by means of a census or sample survey. At times, the observations on the individuals interviewed are not made available to the econometrician and, for econometric analysis, only partially aggregated data is supplied. This situation arises, for instance, when there is a need to preserve the confidentiality of the individual's data (due to the existence of data protection legislation)\(^1\), or when statistical agencies carry out partial aggregation of the observations to achieve economy of presentation. In these cases, assumption [2] stated in Section 1.2 would be 'invalid'.

Various authors have considered the problem of partial aggregation or grouping of observations in regression analysis, and several results

* This Chapter is to be published as Jarque (1981b).

\(^1\) Many countries now have data protection legislation aiming, among other things, to preserve the confidentiality of individual's data. According to Dalenius (1979, p.286), the first nation in the world to adopt a Data Act was Sweden in 1973.
are available. For instance, Prais and Aitchison (1954) note that, for a given grouping, the OLS estimators of the regression model will be unbiased. Also, that there is a loss of information due to grouping, and that groups should be formed in order to make this as small as possible. In the single regressor case they consider the variance of the estimator obtained from the grouped data and note that in order to minimize this, the grouping should minimize the within group sum of squares of the regressor. Referring to multiple regression they write (p.2) that

"it is not possible to give such a simple rule ... but the general principle of minimizing the within groups variance suggests that it is desirable to group together observations which are as homogeneous as possible with respect to the determining variables".

Cramer (1964) also notes, for the single regressor case, that in order to maximize the efficiency of the grouped estimator (that obtained from the partially aggregated data) or equivalently to maximize the correlation between the grouped and ungrouped estimator, the groups should be formed so as to minimize the within group sum of squares of the regressor, and points out that (p.238) "the case of multiple regression is much harder to deal with".

Indeed very little work is found on grouping in the multiple regression case. A valuable study on the estimation of regression models from tabulated data is Haitovsky (1973). In a small section on the efficiency of grouped estimators, Haitovsky (1973, p.11) presents Cramer's result for grouping in the single regressor case and concludes that for the multiple regression case
"if one wishes to get reasonably efficient estimates, one should use data which are cross-classified by all explanatory variables appearing in the hypothesized model".

It is clear that, in order to find a grouping that provides reasonably efficient estimates, one should use the information available on all the explanatory variables. However, there are many procedures for grouping and forming the groups by making a cross-classification (e.g., see Haitovsky (1973, p.12)), although computationally inexpensive, may not be the most appropriate of these.

Another work that should be mentioned is that of Feige and Watts (1972). They propose two indexes to evaluate the consequences of alternative specific groupings. Unfortunately these indexes are not numerically manageable, so their applicability as grouping criteria would be limited. For a discussion on this see Feige and Watts (1972, p.349).

In this Chapter we present additional results on the grouping of observations in regression analysis. In Section 3.2 we introduce notation and prove that estimation using partially aggregated data, rather than the original data, causes a loss of efficiency. In Section 3.3 we see that, for the single regressor case, this loss is minimized when using the grouping of the observations that minimizes the within group sum of squares of the regressor. The problem of grouping to minimize the within group sum of squares of a given variable also arises in future chapters. Therefore, we devote Section 3.4 to the discussion of this numerical problem presenting computationally simple procedures for its solution.
In Section 3.5 we prove that, unlike the single regressor case, in the general linear model there does not exist an optimal grouping of the observations in the sense that, for any other grouping, the difference between the variance-covariance matrices of the corresponding estimators is a positive semidefinite matrix. We then propose, in Section 3.6, a computationally feasible criterion to obtain a grouping that provides reasonably efficient estimators. In Section 3.7 we present a numerical exercise to compare the consequences of several grouping procedures at three levels of aggregation. Finally, in Section 3.8 we make some concluding remarks.

3.2 BEST ESTIMATION USING AGGREGATED DATA

The model we consider is given by equation (1.1) and we proceed under assumptions [4], [6] and [7] stated in Section 1.2. For convenience we shall now write this model as

\[ y_i = \beta_0 + (x_i - \bar{x})'\beta + u_i , \]

where \( y_i \) is an observation on \( Y \); \( x_i \) is a \( K \) by 1 fixed vector of values on \( X_1, \ldots, X_K \) (constant not included in these); \( \bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i/N \) is a \( K \) by 1 vector with elements being the means of \( X_1, \ldots, X_K \); \( N \) is the total number of observations; \( \beta \) is a vector which together with the scalar \( \beta_0 \) constitute unknown parameters, and \( u_i \) is a disturbance with mean zero and unknown variance \( \sigma^2 \).

When writing our model in matrix form we have

\[ y = \beta_0 N + \bar{x}\beta + u , \quad (3.1) \]
where \( y = (y_1, \ldots, y_N)' \); \( 1_N \) is an \( N \) by \( 1 \) vector of ones; \( \tilde{X} = MX \) with \( M = I_N - 1_N (1_N 1_N)'^{-1} 1_N' \) and \( X = (x_1, \ldots, x_N)' \); and
\( u = (u_1, \ldots, u_N)' \). It is a standard result that the BLUE of \( \beta \) is given by
\[
\hat{\beta} = (\tilde{X}'\tilde{X})^{-1} \tilde{X}'y
\]
with VCM equal to
\[
V[\hat{\beta}] = \sigma^2 (\tilde{X}'\tilde{X})^{-1} .
\]

The estimator \( \hat{\beta} \) uses all \( N \) observations. We shall now discuss best estimation of \( \beta \) when only partially aggregated data is available. We define an \( N \) by \( N \) fixed transformation matrix \( T_G \) as
\[
T_G = T_G^+ (T_G^+ T_G^+)^{-1} T_G^+ , \tag{3.2}
\]
where \( T_G^+ \) is an \( L \) by \( N \) grouping matrix (e.g., see Prais and Aitchison (1954, p.9)) with elements \( \left[ T_G^+ \right]_{hi} = \frac{1}{N_h} \) if observation \( i \) belongs to group \( h \) and \( \left[ T_G^+ \right]_{hi} = 0 \) if not; \( L \) denotes the number of groups, and \( N_h \) is the number of observations in group \( h \) with \( N_1 + \ldots + N_L = N \). We write the matrices \( T_G \) and \( T_G^+ \) with a suffix \( G \) to emphasize these depend on a particular grouping \( G \). We shall assume \( N > L > K+1 \).

By premultiplying \( y \) and \( \tilde{X} \) by \( T_G^+ \), we would 'replace' the original observations by their corresponding group means. More specifically, we would 'replace' \( y = (y_1, \ldots, y_N)' \) and \( \tilde{X} = (x_1-\bar{x}, \ldots, x_N-\bar{x})' \) respectively by \( T_G^+ y = (\bar{y}_1, \ldots, \bar{y}_L)' \) and \( T_G^+ \tilde{X} = (\bar{x}_1-\bar{x}, \ldots, \bar{x}_L-\bar{x})' \); with

2. \( \left[ A \right]_{ij} \) denotes the \((i,j)'th\) element of the matrix \( A \).
\[ \tilde{y}_h = \sum_{i=1}^{N_h} \frac{y_{hi}}{N_h} \quad \text{and} \quad \tilde{x}_h = \sum_{i=1}^{N_h} \frac{x_{hi}}{N_h} \]
and where \( y_{hi} \) denotes the value of \( Y \) and \( x_{hi} \) denotes the \( K \) by \( 1 \) vector of values on \( X_1, \ldots, X_K \) for the \( i \)'th individual in group \( h \) (with \( h = 1, \ldots, L \)). The data \((T^+_G y, T^+_G x)\) is referred to as *grouped data*.

If *grouped data* is supplied for regression analysis (i.e., if \( y \) and \( \tilde{x} \) are premultiplied by some matrix \( T^+_G \)) the implied transformation on (3.1) would alter the variance-covariance matrix of the disturbances and, to obtain efficient estimates of \( \beta \) using this data, we would need to apply GLS. Using equation (3.2) the BLUE that we obtain in this situation is

\[
\hat{\beta}_G = (\tilde{x}'T_G \tilde{x})^{-1}\tilde{x}'T_G y
\]

with VCM equal to

\[
V[\hat{\beta}_G] = \sigma^2(\tilde{x}'T_G \tilde{x})^{-1}. \quad ^3, ^4
\]

We now introduce further notation required for our discussion. A known identity is\(^5\)

---

\(^3\) The variances of the grouped and ungrouped estimators of \( \beta_0 \) are the same and equal to \( \sigma^2/N \). Given that grouping has no effect on the variance of this particular parameter, we neglect it and concentrate on \( V[\hat{\beta}_G] \).

\(^4\) We regard \( X \) and \( T_G \) as fixed over 'repeated samples' of \( y \). On the question of bias when the regressors are stochastic and \( T_G \) is not independent of the disturbance term see Feige and Watts (1972, p.346).

\(^5\) To obtain this identity, simply add and subtract \( \tilde{x}_h \) from inside each bracket of the left hand side of (3.3).
\[ L \sum_{h=1}^{N_h} \sum_{i=1}^{N_h} (x_{hi} - \bar{x})(x_{hi} - \bar{x})' = \sum_{h=1}^{L} \sum_{i=1}^{N_h} (\bar{x}_h - \bar{x})(\bar{x}_h - \bar{x})', \]  
\hspace{2cm} \text{(3.3)}

where \( \bar{x}_h = \frac{1}{N_h} \sum x_{hi} \). Note that given our definitions of \( \tilde{X}, M \) and \( T_G \), we have

\[ \tilde{X}'\tilde{X} = \sum_{h=1}^{L} \sum_{i=1}^{N_h} (\bar{x}_h - \bar{x})(\bar{x}_h - \bar{x})', \]

and

\[ \tilde{X}'T_G\tilde{X} = \sum_{h=1}^{L} \sum_{i=1}^{N_h} (\bar{x}_h - \bar{x})(\bar{x}_h - \bar{x})'. \]

We now define \( T_G = I_N - T_G \), from which it is immediate that

\[ \tilde{X}'\tilde{X} = \tilde{X}'T_G\tilde{X} + \tilde{X}'T_G\tilde{X}. \]  
\hspace{2cm} \text{(3.4)}

Using the expressions for \( \tilde{X}'\tilde{X} \) and \( \tilde{X}'T_G\tilde{X} \) in (3.4) we readily obtain from (3.3) the result that

\[ \tilde{X}'T_G\tilde{X} = \sum_{h=1}^{L} \sum_{i=1}^{N_h} (x_{hi} - \bar{x}_h)(x_{hi} - \bar{x}_h)', \]

Define the diagonal elements of the previous matrix by \( W_k(G) = [\tilde{X}'T_G\tilde{X}]_{kk} \) and observe that, for \( k = 1, \ldots, K \), we may write

\[ W_k(G) = \sum_{h=1}^{L} \sum_{i=1}^{N_h} (x_{hi}^{(G)} - \bar{x}_h^{(G)})^2, \]  
\hspace{2cm} \text{(3.5)}
where $x_{hik}(G)$ is the value of $X_k$ in the i'th observation of group $h$; $\bar{x}_{hk}(G)$ is the mean of variable $X_k$ in group $h$; and $N_h(G)$ is the number of observations in group $h$. Note we have included the argument $G$ in the previous quantities to emphasize these depend on a given grouping $G$. □

As mentioned previously, the variance-covariance matrices of $\hat{\beta}$ and $\hat{\beta}_G$ are given by $V[\hat{\beta}] = \sigma^2 (\bar{X}'\bar{X})^{-1}$ and $V[\hat{\beta}_G] = \sigma^2 (\bar{X}' \bar{X}_G)^{-1}$. Remembering $A^{-1} - B^{-1}$ is positive semidefinite (p.s.d.) if $B - A$ is p.s.d., and using (3.4) one obtains that $V[\hat{\beta}_G] - V[\hat{\beta}]$ must be p.s.d.. This implies that

$$\text{diagonal}\{V[\hat{\beta}_G]\} \geq \text{diagonal}\{V[\hat{\beta}]\}.$$ 

An additional simple proof of this last result consists of observing that both $\hat{\beta}_G$ and $\hat{\beta}$ are linear unbiased estimators of $\beta$ and noting that, among this class, $\hat{\beta}$ is the estimator with minimum variance. This proves the result of Prais and Aitchison (1954, p.8) that

"the variances of the grouped estimators can never be smaller than those of the estimators based on all the original (ungrouped) observations".

In the following sections we shall consider the problem of grouping observations so that the loss in efficiency is minimized.

3.3 OPTIMAL GROUPING: SINGLE REGRESSOR MODEL

We now comment on the $K = 1$ case, i.e., the single regressor case (apart from the constant). Here we have that $V[\hat{\beta}_G]$, now a scalar, would be equal to $V[\hat{\beta}_G] = \sigma^2/(\sum_{i=1}^{N}(x_i - \bar{x})^2 - W(G))$ (given that $K = 1$
we have omitted the subindex $k$ in $W^k(G)$. This says that to minimize $V[\hat{\beta}_G]$, one should use the grouping that minimizes the within group sum of squares of the regressor, $W(G)$, which is Prais and Aitchison's (1954) finding. Grouping in order to minimize the within group sum of squares also arises in the problem of optimum stratification in the uniparametric estimation case (see Dalenius (1957)). In fact, Connor (1972) describes six different problems in which this mathematical problem has arisen which suggests, apparently, that various univariate grouping problems have implicitly the same objective (see also Aigner, Goldberger and Kalton (1975)).

In the next section we present several procedures available for minimizing the within group sum of squares of a given variable. In Sections 3.5 and 3.6 we deal with the problem of optimal grouping of observations for the multiple regressor ($K > 1$) case.

3.4 MINIMIZING THE WITHIN GROUP SUM OF SQUARES

Consider having $N$ observations, say $v_1, \ldots, v_N$, on a variable $V$. For a grouping $G$ of these observations into $L$ groups, denote by $\phi(G)$ the within group sum of squares, i.e., let

$$\phi(G) = \sum_{h=1}^{L} \sum_{i=1}^{N_h(G)} (v_{hi(G)} - \bar{v}_h(G))^2$$

(3.6)

with $\bar{v}_h(G) = \sum_{i=1}^{N_h(G)} v_{hi(G)}/N_h(G)$ and where $v_{hi(G)}$ denotes the $i$'th observation in group $h$ and $N_h(G)$ is the number of observations in group $h$ when the grouping is $G$. 

For a given value \( L \), a classification problem is finding the grouping \( G^* \) that minimizes \( \phi(G) \). This problem comes up in several sections in the thesis, and it is now convenient to describe four procedures that are presently available for its solution. These are

1. Total enumeration of alternatives,
2. Application of clustering algorithms,
3. Dalenius' stratification approach, and

### 3.4.1 Total Enumeration of Alternatives

One may attempt finding \( G^* \) by total enumeration of alternatives. The number of ways in which \( N \) observations may be classified into \( L \) groups (i.e., the number of possible groupings) is given by Stirlings Numbers of the Second Kind (see Jensen (1968)). This may be surprisingly large even for a relatively small value of \( N \) and \( L \). Gower (1967) calculates that with \( N = 41 \) and \( L = 2 \), evaluating \( \phi(G) \) for all groupings, would require approximately 54000 years on a computer with a 5\( \mu \)-second access time. He states (p.628) that

"even with the fastest projected computers these times could only be decreased by a factor of 100, so the method is impracticable even for small values of \( N \)."

Other procedures, therefore, have to be considered.

### 3.4.2 Application of Clustering Algorithms

A more convenient procedure for the computation of \( G^* \) consists of using clustering algorithms, such as the K-Means algorithm [see Sparks
(1973), MacQueen (1967) or Hartigan (1975, Chapter 4)] or Ward's (1963) procedure. These were devised for the solution of multivariate classification problems. More formally, these algorithms attempt to find the grouping that minimizes

\[ \Phi(G) = \sum_{j=1}^{J} \sum_{h=1}^{L} \sum_{i=1}^{N_h(G)} (v_{hij}(G) - \bar{v}_{hj}(G))^2, \]

(3.7)

where \( v_{hij}(G) \) represents the value of variable \( V_j \) in the \( i \)'th observation of group \( h \) and \( \bar{v}_{hj}(G) = \frac{1}{N_h(G)} \sum_{i=1}^{N_h(G)} v_{hij}(G) \) is the \( h \)'th group mean of variable \( V_j \), when the grouping is \( G \); and where \( J \) is the total number of variables of interest in the classification problem.

The K-Means algorithm requires an initial grouping of the observations, and the general rule is to search for a grouping that minimizes \( \Phi(G) \) by moving observations from one group to another. The algorithm provides local optima and, in practice, should be used with several choices of the initial grouping.

Ward's (1963) procedure is a hierarchical one which considers, initially, the \( N \) observations as \( N \) single-element groups. At each step of the grouping procedure, two groups – either single or multiple membered – are joined to form a new group. When \( L \) groups are formed the procedure would stop. The criterion is to join the pair of groups so that the new groups make the least possible increment to \( \Phi(G) \).

Obviously, these algorithms may be used for the solution of our univariate classification problem posed at the beginning of this section. The grouping \( G^* \) is obtained when setting \( J = 1 \) in the algorithms. (Note in this case \( \Phi(G) \) reduces to \( \Phi(G) \)).
3.4.3 Dalenius' Stratification Approach

Our grouping problem may also be stated as finding the group boundaries \( v^{(1)}, v^{(2)}, \ldots, v^{(L-1)} \); subject to
\[
\quad v^{(0)} < v^{(1)} < \ldots < v^{(L-1)} < v^{(L)},
\]
so that (3.6) is minimized, where
\[
v^{(0)} = \min_{i=1, \ldots, N} \{ v_i \} \quad \text{and} \quad v^{(L)} = \max_{i=1, \ldots, N} \{ v_i \} + \delta, \quad \text{with} \quad \delta > 0.
\]
Having obtained the optimal \( v^{(1)}, \ldots, v^{(L-1)} \), group \( h \) in \( G^* \) would consist of the set of observations \( i \) such that the corresponding \( v_i \) satisfy
\[
v^{(h-1)} < v_i < v^{(h)}, \quad \text{for} \quad h = 1, \ldots, L.
\]

As mentioned previously, this problem also arises in the context of strata formation (when using proportional sample allocation), and Dalenius (1957) has obtained necessary conditions that the optimal \( v^{(h)} \) satisfy. Note first we can write \( \phi(G) \) in (3.6) as
\[
\phi(G) = N \sum_{h=1}^{L} P_h S_h^2,
\]
where \( P_h = N_h(G)/N \) is the proportion of observations in group \( h \), and
\[
S_h^2 = \sum_{i=1}^{N_h(G)} (v_{hi}(G) - \bar{v}_h(G))^2 / N_h(G)
\]
is the variance of the observations in group \( h \), when the grouping is \( G \).

Following Dalenius (1957), we make the simplifying assumption that the distribution of \( V \) is absolutely continuous, with probability density function \( f(v) \). Then, \( P_h \) and \( S_h^2 \) would be given (for large samples) by
\[
P_h = \int_{v^{(h-1)}}^{v^{(h)}} f(v) \, dv
\]
and
\[
P_h = \int_{v^{(h-1)}}^{v^{(h)}} f(v) \, dv.
\]
where
\[ S_h^2 = \int_{v(h-1)}^{v(h)} v^2 \frac{f(v)}{p_h} \, dv - \mu_h^2, \quad (3.9b) \]

By substitution of (3.9) into (3.8), and after some computations, Dalenius (1957) shows that the optimum \( v(1), \ldots, v(L-1) \) must satisfy the equations
\[ v(h) = \frac{1}{2} (\mu_h + \mu_{h+1}), \quad (3.10) \]
for \( h = 1, \ldots, L-1 \). Dalenius (1957, p.165) then suggests the solution to these equations be found by iteration.

3.4.4 Cubic-root Procedure

We now propose a fourth and final procedure for minimizing \( \phi(G) \). The derivation of this is analogous to that employed by Dalenius and Hodges (1957), when computing the optimum univariate stratification with Neyman-Tschuprov sample allocation.

Consider \( f(v) \) to be approximately uniform within \( v^{(h-1)} \) and \( v^{(h)} \) and denote its mean value within this range by \( f_h \). Then \( p_h \) and \( S_h^2 \) in equations (3.9) are such that
\[ p_h = (v^{(h)} - v^{(h-1)})f_h \]
and
\[ S_h^2 = \frac{1}{12} (v^{(h)} - v^{(h-1)})^2. \]
(Recall the variance of a uniform random variable is the range squared over 12). Therefore, (3.8) may be approximated as follows

$$\phi(G) = \frac{N}{12} \sum_{h=1}^{L} \left[ \frac{(v(h) - v(h-1))}{f_h} \right]^{1/3}.$$  

By writing

$$\xi_h - \xi_{h-1} = \int_{v(h-1)}^{v(h)} f^{1/3}(v) dv,$$

we note that

$$\xi_h - \xi_{h-1} = (v(h) - v(h-1)) f_h^{1/3},$$

and hence

$$\phi(G) = \frac{N}{12} \sum_{h=1}^{L} [\xi_h - \xi_{h-1}]^{3/3}.$$  

From this we see that $\phi(G)$ is minimized when $\xi_h - \xi_{h-1} = \xi_{h+1} - \xi_h$, i.e., when $\xi_h - \xi_{h-1}$ is made approximately constant. Therefore, a rule to compute the group boundaries is the following:

(i) divide the range between $v(0)$ and $v(L)$, e.g., into tenths;

(ii) count the number of $v_i$ that lie within each tenth;

(iii) compute the cumulative of the cubic-root of these frequencies and divide it into $L$ equal intervals; and

(iv) take as the values $v(1), \ldots, v(L-1)$ the $v$-points where the cumulative is divided (these do not
necessarily have to coincide with one of $v_1, \ldots, v_N$ and interpolation may be needed.

To evaluate the appropriateness of this cubic-root of the density procedure in minimizing $\phi(G)$, the necessary condition (3.10) was considered. The differences between $v^{(h)}$ and $(\mu_h + \mu_{h+1})/2$ (with $h = 1, \ldots, L-1$) were computed for the nine data sets considered in Jarque (1981a) and the data in Cochran's bank loans example (see Cochran (1963, p.131)). Overall, the procedure gave values $v^{(h)}$ which were close to $(\mu_h + \mu_{h+1})/2$. In addition, for comparison, we used the Dalenius and Hodges (1957) square-root of the density procedure. This was suggested for minimizing $\sum_{h=1}^{L} P_h S_h$ and not $\phi(G)$. The obtained differences between $v^{(h)}$ and $(\mu_h + \mu_{h+1})/2$ were slightly larger in this case; also, our results suggested that small departures from the optimal group boundaries have little effect on the grouping $G^*$. In all, the simplicity of the cubic-root procedure make it an attractive alternative in minimizing $\phi(G)$.

3.5 OPTIMAL GROUPING: MULTIPLE REGRESSOR MODEL

Now we return to the problem of estimation of our regression model with partially aggregated data. In Section 3.2, we saw that OLS estimation, using the ungrouped data, provides the ideal estimation procedure, in the sense that it is BLUE and uses all the information in the sample. We noted that GLS estimation applied to the grouped data also gives us unbiased estimators, but there was a loss of efficiency with respect

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6 Singh (1975) has also suggested this procedure, but his derivation is different to ours, and proceeds along the lines of Ekman (1959) using Taylor series expansion about $v^{(h-1)}$ and $v^{(h)}$. 
to the ungrouped one. This loss of efficiency depends on the grouping used. We also saw that, for \( K = 1 \), the grouping that minimized the within group sum of squares of the regressor gave the smallest variance of the estimator. Here we shall consider the more general case where \( K > 1 \). □

For a given \( L \), we define the fully efficient or optimal grouping as the grouping \( G^* \) such that, for any grouping \( G \), the difference between the variance-covariance matrices of the corresponding estimators is a positive semidefinite (p.s.d.) matrix. More formally, we define \( G^* \) as the grouping such that \( V[\hat{\beta}_G] - V[\hat{\beta}_{G^*}] \) is p.s.d. for all \( G \). \(^7\)

Given \( \sigma^2 \) is positive and that \( A^{-1} - B^{-1} \) is p.s.d. if \( B - A \) is p.s.d., it follows that the previous condition will hold when

\[
\bar{X}'T_{G^*}\bar{X} - \bar{X}'T_G\bar{X} \text{ is p.s.d. .}
\]

Using (3.4) we can see this will be true when

\[
\bar{X}'T\bar{X} - \bar{X}'T_{G^*}\bar{X} \text{ is p.s.d. .}
\]

A necessary condition for this to hold is that the diagonal elements of the previous matrix be non-negative, i.e., that

\[
W_k(G) \geq W_k(G^*) \quad (3.11)
\]

for \( k = 1, \ldots, K \), where we have used the notation introduced in (3.5).

Our aim now is to find \( G^* \) such that, for any other \( G \), condition (3.11) is satisfied for all \( k = 1, \ldots, K \).

\(^7\) Note that if this is true, then the generalized variance of \( \hat{\beta}_{G^*} \) will be no greater than that of \( \hat{\beta}_G \), since this implies

\[
\det(V[\hat{\beta}_G]) \geq \det(V[\hat{\beta}_{G^*}]) .
\]
Assume we proceed to minimize the within group sum of squares of a particular variable $X_j$ as if it were the only regressor. Then we would obtain a grouping – that we denote by $G^*_j$ – which would satisfy, by definition, the condition that, for all $G$,

$$W_j(G) \geq W_j(G^*_j) \quad (3.12)$$

Hence, whatever other grouping $G$ is used, the value $W_j(G^*_j)$ would provide a lower bound to $W_j(G)$. This procedure may be carried out for all variables $X_1, \ldots, X_k$; giving $K$ groupings $G^*_1, \ldots, G^*_K$ and $K$ lower bounds $W^*_1, \ldots, W^*_K$, where we have defined $W^*_k = W_k(G^*_k)$ for $k = 1, \ldots, K$.

We want to find $G^*$ such that for any other $G$ it satisfies (3.11). From (3.12) we know $G^*$ (like any $G$) satisfies

$$W_j(G^*) \geq W_j(G^*_j) \quad (3.12')$$

So, $G^*$ cannot give a smaller within group sum of squares for $X_j$ than $G^*_j$. Therefore, for the $j$'th element of the necessary condition (3.11) to be satisfied for all $G$, we require $G^* = G^*_j$ and then

$$W_j(G) \geq W_j(G^*) = W_j(G^*_j) \quad (3.12')$$

However, the condition needs to be satisfied for every $k = 1, 2, \ldots, K$, and not just one $k = j$. This means that for (3.11) to be true, we would need to have $G^* = G^*_1 = \ldots = G^*_K$.

A grouping $G^*$ satisfying (3.11) for all $G$ would exist in the particular case where (after computing $G^*_1, \ldots, G^*_K$) one finds that

8 We should note, however, that $W_k(G^*_j) > W_k(G^*_j)$.
\( G_1^* = \ldots = G_K^* \). (Then, \( G^* \) would be given by \( G_1^* = G_1^* \) and would attain simultaneously all \( K \) lower bounds). In general, however, not all the groupings \( G_1^*, \ldots, G_K^* \) would be equal \(^9\) and hence (3.11) would not be satisfied. This proves the general non-existence of \( G^* \) for \( K > 1 \).

3.6 VARIOUS GROUPING CRITERIA

In Section 3.5 we proved the general non-existence of a fully efficient, or optimal grouping, in the multiple regressor case. This result raises the question of definition of a criterion, that would provide a grouping which yields reasonably efficient estimators. This is discussed in the present section.

First we define

\[ e_k(G) = \frac{W_k(G)}{W_k^*} \]

with \( k = 1, \ldots, K \). Then, by definition of \( W_k^* \) we have that \( e_k(G) \geq 1 \), for \( k = 1,2,\ldots,K \). Each \( e_k(G) \) may be thought of as the reciprocal of the 'efficiency of a grouping' \( G \) in the \( k' \)th dimension, in the sense that it measures the closeness between \( G \) and \( G_k^* \). The necessary condition for optimality of \( G^* \) given by (3.11) implies the condition \( e_k(G^*) = 1 \), for \( k = 1,2,\ldots,K \). So it is reasonable to say that groupings with low values of \( e_1(G), \ldots, e_K(G) \) (i.e., with values close to one) would be appealing. This motivates the definition of our first criterion which consists of finding \( G \) such that

\(^9\) We have computed \( G_k^* \) for an extensive set of variables and have obtained empirical support to the intuitive idea that (in practice) the event \( G_1^* = \ldots = G_K^* \) has low probability.
\[ D(G) = \sum_{k=1}^{K} e_k(G) \]

is minimized. Writing out explicitly \( D(G) \) we obtain

\[
D(G) = \sum_{k=1}^{K} \sum_{h=1}^{L} \sum_{i=1}^{N_h(G)} \left( \frac{(x_{hik}(G) - \bar{x}_{hik}(G))^2}{\sqrt{w_k}} \right). 
\]

We note that \( D(G) \) has desirable properties as a grouping criterion.

**Firstly**, it is scale invariant.

**Secondly**, it penalizes variables not having a grouping that gives small within group sum of squares.

**Thirdly**, it is a numerically manageable function and, in fact, there are readily available clustering algorithms that may be used in the determination of the desired grouping, e.g., one may use the K-Means Algorithm or Ward's (1963) procedure described in Section 3.4.2.\(^{11}\)

Note that if we define \( W \) as the diagonal matrix with diagonal elements \( W_k \), then we may write \( D(G) \) as \( \text{trace}(W^{-1}(X'TGX)) \), so \( D(G) \) is a 'trace-criterion'.

An alternative criterion is to minimize the square of the Euclidean distance of the vector of the reciprocals of the efficiencies of the

\(^{10}\) The more general form \( \sum_{k=1}^{K} \phi_k e_k(G) \) could be considered, where \( \phi_1, \ldots, \phi_K \) are given weights. This poses no additional complications in terms of computing \( G \).

\(^{11}\) Before using these algorithms the data would have to be appropriately standardized by dividing each value of \( X_k \) by \( \sqrt{W_k} \).
grouping, to the 'ideal' vector of reciprocals of efficiencies $1_k$, i.e., $\sum_{k=1}^{K} (e_k(G) - 1)^2$. (Other functional forms using $e_1(G), \ldots, e_K(G)$ could be considered). Due to computational ease we prefer $D(G)$. In several exercises we have found the $G$ that minimized $D(G)$ also minimized $\sum_{k=1}^{K} (e_k(G) - 1)^2$; however, there are particular cases where this may not be so.

Yet another criterion is to minimize the Euclidian distance measure defined by

$$F(G) = \text{trace}\{\Delta^{-1}(\tilde{\mathbf{X}}' \tilde{\mathbf{X}})\},$$

where $\Delta$ denotes a diagonal matrix with diagonal elements $[\tilde{\mathbf{X}}' \tilde{\mathbf{X}}]_{kk}$. Writing this out more explicitly we obtain

$$F(G) = \sum_{k=1}^{K} \sum_{h=1}^{L} \sum_{i=1}^{N_h(G)} \frac{(x_{hi}(G) - \bar{x}_{hk}(G))^2}{[\tilde{\mathbf{X}}' \tilde{\mathbf{X}}]_{kk}}$$

Note that $F(G)$ is scale invariant. But in this case standardization has been achieved by the division of each variable by its standard deviation (apart from a $\sqrt{N}$ factor). It is thought this is not entirely desirable since here, as in most classification problems, those variables with 'greatest variances' are the ones we would be most concerned with. Once we have standardized in this manner, all the new variables become equally important in terms of variance.

Additional grouping criteria may readily come to mind. For example, one may consider finding $G$ such that it minimizes $\text{trace}\{V[\hat{\mathbf{G}}]\}$ or $\text{trace}\{\psi V[\hat{\mathbf{G}}]\}$, where $\psi$ is a diagonal matrix of fixed weights.
Similarly, one might use the index proposed by Feige and Watts (1972) for comparing the efficiency of specific groupings, i.e.,

\[ \text{trace}((\bar{X}'\bar{X})^{-1}\bar{X}'\bar{T}_G\bar{X}) \]. In addition, one may consider a generalized variance criterion and attempt to minimize

\[ \det(V[\hat{\beta}_G]) = a^2 / \det(\bar{X}'\bar{X} - \bar{X}'\bar{T}_G\bar{X}) \].

All these criteria do not reduce to simple grouping rules and, as we noted in Section 3.4, having to compute the desired grouping by total enumeration of alternatives would be computationally troublesome.\(^{12}\)

This highlights the convenience of having a numerically manageable criterion and our preference for \(D(G)\) or \(F(G)\).

In the next section, we consider a particular data set, and compare the efficiency of the estimators obtained through several grouping procedures, with respect to the ungrouped OLS estimators. We also analyse the consequences of grouping at three different levels of aggregation.

### 3.7 Numerical Exercise

In this numerical exercise, we compare the variance of the estimators obtained by the use of seven grouping procedures for a particular data set. We consider a regression model with three regressors apart from the constant term. Our variable \(X_1\) takes the integer values from 1 to \(N\); \(X_2\) is bimodal with modes at 1 and 2.5 and \(X_3\) is bimodal with modes at 100 and 400. They have coefficients of variation respectively equal to 1.7, 2.2 and 1.6. We also have \(\rho_{12} = -.12\), \(\rho_{23} = -.13\) and \(\rho_{13} = -.14\), where \(\rho_{ij}\) is the correlation between

\(^{12}\) Although in general \(\det(\bar{X}'\bar{X} - \bar{X}'\bar{T}_G\bar{X}) \neq \det(\bar{X}'\bar{T}_G\bar{X})\), the expression for \(\det(V[\hat{\beta}_G])\) suggests that an additional appealing criterion would be to minimize \(\det(\bar{X}'\bar{T}_G\bar{X})\). An algorithm for this is described in Marriot (1971). As Marriot (1971, p.501) states "the computations involved in minimizing" this criterion "are heavy".
variables $X_i$ and $X_j$. We have set $\beta_0 = 10$, $\beta_1 = 1$, $\beta_2 = -2$ and $\beta_3 = .05$ and generated $u_i$ from a Normal distribution with mean zero and variance $\sigma^2 = 1$.

When using partially aggregated data in regression analysis, the closer the number of groups $L$ is to the number of observations $N$, the more efficient one expects the grouped estimators to be, as compared with the ungrouped OLS estimators. In this exercise, we have $N = 60$ observations and consider three levels of aggregation $L = 5$, $L = 10$ and $L = 20$.

The groupings considered for the purpose of our comparison are the following: the grouping $G_D$ obtained by minimizing $D(G)$ using Ward's clustering algorithm (see Subsection 3.4.2); $G_F$ obtained by minimizing $F(G)$ using Ward's clustering algorithm; univariate groupings $G_k$ obtained by minimizing the within group sum of squares of variable $X_k$ ($k = 1, 2, 3$) using the cubic-root procedure presented in Subsection 3.4.4; $G_Y$ obtained by minimizing the within group sum of squares of our sample observations on the dependent variable $Y$ using the cubic-root procedure; $G_R$ obtained by the random assignment of the observations to groups of equal size.

The ratios of the variance of the $k$'th element of the grouped estimator $\hat{\beta}_G$ to the variance of the $k$'th element of the OLS estimator $\hat{\beta}$, are given in Table 3.1 for $L = 5, 10$ and $20$ and for the previous

---

13 For this exercise we used the CLUSTAN package on a UNIVAC 1100/42. On average, it took 2.8 seconds of execution time for each computation of $G_D$ and $G_F$.

14 $G_Y$ is found from a particular vector of observations $y$, and once computed it would be regarded as fixed for future realizations of $y$. This is required in order to have a fixed transformation matrix so that $\Psi[\hat{\beta}_G] = \sigma^2 (\tilde{x}'G\tilde{x})^{-1}$ applies.
choices of \( G \). In this Table we note firstly that, particularly when the number of groups is small, a considerable gain in efficiency is obtained by the use of clustering algorithms (groupings \( G_D \) and \( G_F \)) rather than other grouping rules (e.g., see variance ratios for \( L = 5 \)). Also, we observe the univariate groupings \( G_1^*, G_2^*, G_3^* \) and \( G_Y \) may provide variances that are large even for a large value of \( L \) (e.g., for \( L = 20, \text{V}[\hat{\beta}_{30}^*] = 7.30\text{V}[\hat{\beta}_3] \)). By comparing the random grouping \( G_R \) with the grouping \( G_D \), we note - not surprisingly - that the use of \( G_D \) is substantially better for any number of groups \( L \). In this particular exercise, we may consider \( G_D \) to be marginally better than \( G_F \) for \( L = 5 \) and 10 (this of course depends on the importance given to each parameter), and that for \( L = 20 \) the estimators obtained using \( G_D \) or \( G_F \) are nearly as efficient as the ungrouped OLS estimators.

3.8 CONCLUDING REMARKS

We have proved the general non-existence of a fully efficient or optimal grouping of the observations in the multiple regressor model. This result raised the problem of defining numerically manageable grouping criteria, in order to obtain groupings that provide reasonably efficient estimates.

In Section 3.6 the criterion \( D(G) \) was suggested; this consists of finding the grouping that minimizes the sum of the reciprocals of the 'efficiencies of the grouping'. When \( K = 1 \), minimizing \( D(G) \) is equivalent to minimizing the within group sum of squares which, as noted in Section 3.3, is the criterion obtained by Prais and Aitchison (1954) for the single regressor case. The criterion \( D(G) \) is intuitively appealing, and it seems it may be used in the multivariate extensions of
the problems described in Connor (1972, p.604) and Aigner et al. (1975, p.503). In particular, we have suggested the use of $D(G)$ to obtain an optimal stratification of first stage sampling units in multivariate sampling problems (see Jarque (1981a)).

TABLE 3.1
Variance Ratios of Grouped to Ungrouped Estimators

<table>
<thead>
<tr>
<th></th>
<th>$G_D$</th>
<th>$G_F$</th>
<th>$G_1^*$</th>
<th>$G_2^*$</th>
<th>$G_3^*$</th>
<th>$G_Y$</th>
<th>$G_R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L = 5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V[\hat{\beta}_{1G}] / V[\hat{\beta}_1]$</td>
<td>1.90</td>
<td>2.80</td>
<td>7.40</td>
<td>16.00</td>
<td>21.30</td>
<td>1.92</td>
<td>103.97</td>
</tr>
<tr>
<td>$V[\hat{\beta}_{2G}] / V[\hat{\beta}_2]$</td>
<td>1.99</td>
<td>1.20</td>
<td>312.60</td>
<td>1.82</td>
<td>25.90</td>
<td>20.53</td>
<td>60.21</td>
</tr>
<tr>
<td>$V[\hat{\beta}_{3G}] / [\hat{\beta}_3]$</td>
<td>1.06</td>
<td>1.06</td>
<td>99.30</td>
<td>42.81</td>
<td>2.27</td>
<td>27.94</td>
<td>35.96</td>
</tr>
<tr>
<td>$L = 10$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V[\hat{\beta}_{1G}] / V[\hat{\beta}_1]$</td>
<td>1.21</td>
<td>1.20</td>
<td>1.80</td>
<td>6.60</td>
<td>5.41</td>
<td>1.89</td>
<td>9.72</td>
</tr>
<tr>
<td>$V[\hat{\beta}_{2G}] / V[\hat{\beta}_2]$</td>
<td>1.14</td>
<td>1.15</td>
<td>34.10</td>
<td>1.51</td>
<td>7.52</td>
<td>16.51</td>
<td>4.56</td>
</tr>
<tr>
<td>$V[\hat{\beta}_{3G}] / [\hat{\beta}_3]$</td>
<td>1.01</td>
<td>1.02</td>
<td>19.50</td>
<td>19.40</td>
<td>1.28</td>
<td>18.42</td>
<td>7.73</td>
</tr>
<tr>
<td>$L = 20$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V[\hat{\beta}_{1G}] / V[\hat{\beta}_1]$</td>
<td>1.09</td>
<td>1.05</td>
<td>1.12</td>
<td>3.60</td>
<td>3.01</td>
<td>1.33</td>
<td>4.43</td>
</tr>
<tr>
<td>$V[\hat{\beta}_{2G}] / V[\hat{\beta}_2]$</td>
<td>1.01</td>
<td>1.06</td>
<td>4.60</td>
<td>1.18</td>
<td>2.53</td>
<td>13.75</td>
<td>2.94</td>
</tr>
<tr>
<td>$V[\hat{\beta}_{3G}] / [\hat{\beta}_3]$</td>
<td>1.01</td>
<td>1.01</td>
<td>2.40</td>
<td>7.30</td>
<td>1.10</td>
<td>3.52</td>
<td>4.02</td>
</tr>
</tbody>
</table>

In this survey sampling context, $D(G)$ is equal to a sum of ratios, with each ratio being formed by taking the variance of the estimator of a population parameter - when using stratified sampling with proportional allocation - and dividing this by the minimum possible variance.
In practice, grouping of the original observations may be undertaken by statistical agencies to achieve economy of presentation. Grouping may also be carried out to preserve the confidentiality of the data. When confidentiality of the data is required, groups of size one may not be permitted. Fortunately, in the clustering algorithms that we have referred to for the computation of the grouping that minimizes \( D(G) \), a minimum group size may be specified so this restriction poses no additional complications.

In our discussion we assumed the number of groups \( L \) to be given; in general, the aim should be to have \( L \) as large as possible. We note that Ward's (1963) clustering algorithm gives us the increase in \( D(G) \) due to a reduction in the number of groups. This information may be used in the determination of \( L \). We could choose \( L \) as the value that, when reduced, would give a significant increase in \( D(G) \).

Through a numerical exercise, we have seen that Ward's (1963) clustering algorithm (applied to appropriately standardized data) provides groupings that are substantially more efficient than groupings obtained by other grouping rules - particularly random grouping - even for slight levels of aggregation (i.e., large number of groups). This result suggests the use of our grouping procedures may lead to the supply of better partially aggregated data useful for regression analysis. Through this we may improve our knowledge of the behaviour of the microeconomic decision-making units in our economy.

At this point we finish our discussion of topics relating to the 'data-gathering stage' of the econometrics of cross-sections. In the remainder of the thesis we shall assume cross-sectional data is in our possession. This may have been collected by us - as described in
Chapter 2 - or given to us (e.g., supplied by some statistical agency). Further, we shall assume the data is not in aggregated form. Having the data, we would proceed to the estimation of the model and to carry out inferences about the parameters. Some results regarding estimation have already been noted. To apply inferential procedures, we require further assumptions on the distribution of the regression disturbances. This topic is studied in the next Chapter.
Chapter 4

The Problem of Non-Normal Disturbances*

"To say that 'errors' must obey the normal law means taking away the right of the free-born to make any 'error' he damn well pleases!"

Sir Arthur Eddington
Cambridge Lecture

4.1 INTRODUCTION

We have presented 'best' estimators of the parameters in the linear regression model for various kinds of cross-sectional data (e.g., census, survey and partially aggregated data). After estimation, we would typically want to carry out inferences about the model. For this we have to make some assumptions - in addition to the specification of first and second order moments - regarding the distribution of the disturbances.

We denote the probability density function (p.d.f.) of the i'th disturbance $u_i$ by $f(u_i)$, and proceed under the maintained hypothesis that - apart from scale differentials - $f(u_i)$ is the same for all $i = 1, \ldots, N$. Two additional assumptions frequently made are that disturbances are homoscedastic and that $f(u_i)$ is the normal p.d.f.. We study the homoscedasticity assumption in Chapter 5. For now, we

* Sections 4.3, 4.4 and 4.5 contain results obtained with Anil K. Bera and are based on the paper Jarque and Bera (1981b).
assume its validity, and devote our attention to the normality assumption. The model is given in (1.1) and, apart from the stated maintained hypotheses, we make assumptions [2], [5], [6] and [7] described in Section 1.2.

Under disturbance normality, i.e., under assumption [3], one may justify the use of the OLS estimator for $\beta$ noting that, by the Rao-Blackwell Theorem, it is efficient (e.g., see Schmidt (1976, p.14)). Also, one may apply the usual $t$ and $F$-tests of restrictions on $\beta$, and one may choose from several tests for homoscedasticity which are derived under normality (e.g., see Goldfeld and Quandt (1965) and Harrison and McCabe (1979)). In addition, one may easily obtain confidence intervals for the dependent variable and arrive at particular conclusions about the economic phenomena being studied; for example, Lillard and Willis (1978) investigate earning mobility, and use disturbance normality to make probability statements about the dependent variable (an individual's earnings) given an observation on the regressors (e.g., job history and education). The assumption also plays an important role in Bayesian procedures (e.g., see Zellner (1971, Chapter 3)).

The consequences of violation of the normality assumption have been studied by various authors. In estimation, for instance, the OLS estimator $\hat{\beta} = (X'X)^{-1}X'y$ is known to be very sensitive to long-tailed distributions (e.g., see Hogg (1979)). Regarding inferential procedures, Box and Watson (1962) consider the usual $t$ and $F$-tests, and demonstrate that sensitivity to non-normality is determined by the numerical values of the regressors. They show that, to obtain the desired significance level, some adjustment in the degrees of freedom of these tests may be
required. Similarly, Arnold (1980) studies the asymptotic distribution of $s^2 = (y-Xb)'(y-Xb)/N$ and shows the significance level of the usual $\chi^2$ test of the hypothesis $\sigma^2 = \sigma_0^2$ (or the confidence interval for $\sigma^2$) is not asymptotically valid in the presence of non-normality. Also, the significance level and power of several homoscedasticity tests (suggested for normal disturbances) is studied in Chapter 5, and it is found that these tests may result in incorrect conclusions under non-normal disturbances. In all, violation of the normality assumption may lead to

(i) The use of sub-optimal estimators;

(ii) Invalid inferential statements; and to

(iii) Inaccurate conclusions.

These consequences highlight the importance of testing the validity of the assumption. □

In Section 4.2, we present a procedure for the construction of efficient and computationally simple econometric specification tests. This procedure is used in Section 4.3 to obtain a test for the normality of observations, and in Section 4.4 to obtain a test for the normality of (unobserved) regression disturbances. We then present—in Section 4.5—an extensive simulation study to compare the power of these tests with that of other existing tests. In Section 4.6, we comment on possible estimation methods to follow if the hypothesis of disturbance normality has been rejected. Finally, in Section 4.7, we make some concluding remarks.
4.2 THE LM TEST AND AN INFERENTIAL PROCEDURE

We now present a procedure for the construction of specification tests. This consists of the use of the Lagrange Multiplier, or Rao's score test, on a 'General Family of Distributions'. First, some remarks about the Lagrange Multiplier (LM) test.

The LM test is fully described elsewhere (e.g., see Rao (1948), Aitchison and Silvey (1960), Breusch (1978) and Engle (1981)). So here we shall only introduce notation, and state required results. Consider a random variable $u$ with probability density function (p.d.f.) $f(u)$. For a given set of $N$ independent observations on $u$, say $u_1, \ldots, u_N$, denote by $\ell(\theta) = \sum_{i=1}^{N} \ell_i(\theta)$ the logarithm of the likelihood function, where $\ell_i(\theta) = \log f(u_i)$, $\theta = (\theta_1, \theta_2)'$ is the vector of parameters (of finite dimension), and $\theta_2$ is of dimension $r$ by 1. Assume we are interested in testing the hypothesis $H_0: \theta_2 = 0$.

Define

$$d_j = \frac{\partial \ell(\theta)}{\partial \theta_j}$$

$$= \sum_{i=1}^{N} \frac{\partial \ell_i(\theta)}{\partial \theta_j}$$

and

$$I_{jk} = E\left\{ -\frac{\partial^2 \ell(\theta)}{\partial \theta_j \partial \theta_k} \right\}$$

$$= E\left[ \sum_{i=1}^{N} \left( \frac{\partial \ell_i(\theta)}{\partial \theta_j} \right) \left( \frac{\partial \ell_i(\theta)}{\partial \theta_k} \right) \right]$$

for $j = 1, 2$ and $k = 1, 2$. Let $\hat{d}_j$ and $\hat{I}_{jk}$ denote $d_j$ and $I_{jk}$ evaluated at the restricted (obtained by imposing the restriction
\( \theta_2 = 0 \) maximum likelihood estimator of \( \theta \), say \( \hat{\theta} \).

It may be shown that under general conditions, which are satisfied in all the applications we discuss, the statistic defined by

\[
LM = \frac{d_2'}{d_2} (I_{22} - \hat{I}_{21} \hat{I}_{11}^{-1} \hat{I}_{12})^{-1} d_2 \tag{4.1}
\]

is, under \( H_0: \theta_2 = 0 \), asymptotically distributed as a \( \chi^2 \) with \( r \) degrees of freedom, say \( \chi^2_{(r)} \) (e.g., see Breusch and Pagan (1980, p.241)). A test of \( H_0: \theta_2 = 0 \), based on (4.1), will be referred to as an LM test. Two aspects of this test are worth noting.

Firstly, that it is asymptotically equivalent to the likelihood ratio (LR) test, provided the maximum likelihood estimators under the alternative hypothesis are well defined. We shall assume that the true value of \( \theta, \theta^0 \), is an interior point of \( \Omega \), where \( \Omega \) is the subset of \( \mathbb{R}^m \) for which maximum likelihood estimation is well defined and \( m \) is the dimension of \( \theta \). This implies the LM test has the same asymptotic power characteristics as the LR test, including maximum local asymptotic power, i.e., asymptotic efficiency. This is a most desirable feature since, with large samples, any reasonable test can be expected to have high power for alternatives far away from \( \theta_2 = 0 \), and it is only for alternatives where \( \theta_2 \) is near the zero vector that asymptotic power has relevance.  

\[1 \] The LM test is also known to have optimal small sample power properties in some cases, e.g., see King and Hillier (1980).
A second aspect of this test is that - to compute it - we only require estimation under the null hypothesis of the parameters in the model. In all the inferential problems studied here estimation under $H_0: \theta_2 = 0$ is easily carried out. This makes the LM test computationally attractive, as compared to other asymptotically equivalent tests (i.e., the LR test and the Wald test).  

For these two reasons - good power properties and computational ease - we use the LM test, rather than others, in our inferential procedure.  

The LM test has been recently applied in many econometric inferential problems (e.g., see Byron (1970), Godfrey (1978a,b,c) and Breusch and Pagan (1979,1980)). Our procedure for the construction of specification tests also uses the LM principle, but has its distinct feature in the formulation of $\mathcal{L}(\theta)$. Rather than assuming a 'particular' p.d.f. for $u_1$ (or transformation of $u_1$), we assume that the true p.d.f. for $u_1$ belongs to a 'General Family' (e.g., the Pearson Family), of which the distribution under $H_0$ is a particular member. We then use the LM principle to test $H_0$ within this 'General Family of Distributions'. [Of course, it may be argued that any application of the LM test (e.g., testing for homoscedasticity) specifies a 'General Family'. Here we use this term to refer to Families of p.d.f.'s in the 'statistical' sense (e.g., see Kendall and

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2 Cases exist when LM, LR and Wald tests have identical power for all sample sizes; and choice of test in these cases is based exclusively on computational ease. This occurs when testing linear restrictions in the single equation model (see Evans and Savin (1980)), and in some simultaneous equations situations, e.g., when testing homogeneity restrictions in demand systems (see Bera, Byron and Jarque (1981)).
Stuart (1969, Chapter 6)). Our later discussion will help make this point clearer. We also note the tests obtained are known to have optimal large sample power properties for members of the 'General Family' specified, and that this does not imply they will not have good power properties for non-member distributions. Indeed, for the cases studied, we found that the tests performed with extremely good power for distributions not belonging to the 'General Family' used in our derivations.

The suggested approach for the development of specification tests can be applied in a wide range of statistical and econometric inferential problems. In this thesis we confine ourselves to those applications of the procedure that are of direct interest to the econometrics of cross-sections. The first two applications are presented in the next two sections, where we obtain tests for normality of observations and regression disturbances.

4.3 A TEST FOR NORMALITY OF OBSERVATIONS

Statisticians' interest in fitting curves to data goes a long way back. As noted by Ord (1972, p.1) — although towards the end of the nineteenth century — "not all were convinced of the need for curves other than the normal" (see K. Pearson (1905)), "by the turn of the century most informed opinion had accepted that populations might be non-normal" (some historical accounts may be found in E.S. Pearson (1965)). This naturally led to the development of tests for the

3 For example, as pointed out in Jarque and Bera (1981c), this can be used to test if observations come from a particular truncated distribution, providing an alternative to the Kolmogorov-Smirnov test with fixed truncation point. Another example is given in Lee (1981), who has used our approach to test distributional assumptions in accelerated failure time models.
normality of observations. Interest in this area is still very much alive, and recent contributions to the literature are the skewness, kurtosis and omnibus tests proposed by D'Agostino and Pearson (1973), Bowman and Shenton (1975) and Pearson, D'Agostino and Bowman (1977). Other approaches include: the Analysis of Variance tests of Shapiro and Wilk (1965), and Shapiro and Francia (1972); LR tests based on specific alternatives such as power transformations; goodness of fit tests such as the $X^2$-test and the Kolmogorov-Smirnov test; and graphical methods like normal probability plots. In this section we consider the Pearson Family of distributions, and make use of the LM principle to derive an additional test for the normality of observations. This test is simple to compute and asymptotically efficient.

Before proceeding to our derivations, we note that testing the normality of observations has constituted an important and central statistical problem in the Natural Sciences. Furthermore, this is also a relevant problem in the Social Sciences, e.g., Carlson (1975) considers testing the normality of price expectations of a group of economists. Therefore, the results of this section have potential application in many fields of research.

We now present our derivations. Consider having a set of $N$ independent observations on a random variable $v$, say $v_1, \ldots, v_N$, and assume we are interested in testing the normality of $v$. Denote the unknown population mean of $v_1$ by $\mu = E[v_1]$ and, for convenience, write $v_1 = \mu + u_1$. It follows that $E[u_1] = 0$ and that - apart from location - the p.d.f. of $v_1$ is equal to the p.d.f. of $u_1$. Assume the p.d.f. of $u_1$, $f(u_1)$, has a single mode and "smooth contact with the $u_1$-axis at the extremities". More specifically, assume $f(u_1)$ is a member of the Pearson Family. This is not very restrictive, due to
the wide range of distributions that are encompassed in it (e.g.,
particular members are the Normal, Beta, Gamma, Students' t and F
distributions). This means we can write (see Kendall and Stuart (1969,
p.148))

\[
df(u_i)/du_i = (c_1-u_i)f(u_i)/(c_o-c_1u_1+c_2u_1^2)
\]

or

\[
f(u_1) = \frac{\exp\left[ \int_{-\infty}^{\infty} \frac{c_1-u_i}{c_o-c_1u_1+c_2u_1^2} du_1 \right]}{\int_{-\infty}^{\infty} \exp\left[ \int_{-\infty}^{\infty} \frac{c_1-u_i}{c_o-c_1u_1+c_2u_1^2} du_1 \right] du_1}, \tag{4.2}
\]

with \(-\infty < u_1 < \infty\), and where the denominator in equation (4.2) makes

\(f(u_1)\) a proper p.d.f.

It follows that the logarithm of the likelihood function (or log-
likelihood) of our \(N\) observations \(v_1, \ldots, v_N\) may be written as

\[
f(u, c_0, c_1, c_2) = -N \log \left[ \int_{-\infty}^{\infty} \exp\left[ \int_{-\infty}^{\infty} \frac{c_1-u_i}{c_o-c_1u_1+c_2u_1^2} du_1 \right] du_1 \right]
+ \sum_{i=1}^{N} \left[ \int \frac{c_1-u_i}{c_o-c_1u_i+c_2u_i^2} du_i \right]. \tag{4.3}
\]

Although our procedure is more general, our interest here is to test
the hypothesis of normality, which means, from our expression for \(f(u_1)\),
that we want to test \(H_0: c_1 = c_2 = 0\). Let \(\theta_1 = (u, c_0)'\), \(\theta_2 = (c_1, c_2)'
and \(\theta = (\theta_1, \theta_2)'\). Using these, and the definitions of Section 4.2, we
can show that - for this problem - the LM test statistic is given by
(see Proposition 1 in Appendix A in page 274)
\[ LM = N[(v_{b_1})^2 / 6 + (b_2 - 3)^2 / 24] , \]  

(4.4)

where \( v_{b_1} = \frac{\mu_3}{\mu_2^{3/2}}; \) \( b_2 = \frac{\mu_4}{\mu_2^2}; \) \( \mu_j = \frac{N}{i=1} (v_{i} - \bar{v})^j/N \) and \( \bar{v} = \frac{N}{i=1} v_i/N. \)

(Note \( v_{b_1} \) and \( b_2 \) are, respectively, the skewness of kurtosis sample coefficients). From the results stated in Section 4.2 we know that, under \( H_0: c_1 = c_2 = 0, \) \( LM \) is asymptotically distributed as \( \chi^2(2) \), and that a test based on (4.4) is asymptotically locally most powerful. \( H_0 \) is rejected, for large samples, if the computed value of (4.4) is greater than the appropriate significance point of a \( \chi^2(2). \)

Several tests for normality of observations are available. For example, there are tests based on either of the quantities \( v_{b_1} \) or \( b_2. \) These have optimal properties, for large samples, if the departure from normality is due to either skewness or kurtosis (see Geary (1947)). In addition, there are omnibus tests based on the joint use of \( v_{b_1} \) and \( b_2. \) One example is the R test suggested by Pearson, D'Agostino and Bowman (1977) (see also D'Agostino and Pearson (1973, p.620)).

It is interesting to note that equation (4.4) is the test suggested by Bowman and Shenton (1975). Bowman and Shenton (1975, p.243) only stated the expression of the statistic, and noted it was asymptotically distributed as \( \chi^2(2) \) under normality; they did not study its large or finite sample properties. We have shown expression (4.4) is an LM test statistic. Therefore, we have uncovered a principle that proves its asymptotic efficiency. This finding encourages the study of its finite sample properties. For finite \( N, \) the distributions of \( v_{b_1} \) and \( b_2, \) under \( H_0, \) are still unknown. The problem has engaged statisticians for a number of years, and only approximations to the true distributions are available (e.g., for \( v_{b_1} \) see D'Agostino and Tietjen.
(1973); and for $b_2$ see D'Agostino and Pearson (1973)). This highlights, together with the fact that $\sqrt{b_1}$ and $b_2$ are not independent (e.g., see Pearson, D'Agostino and Bowman (1977, p.233)), the difficulty of analytically obtaining the finite sample distribution of (4.4) under $H_0$.

An alternative is to resort to computer simulation. We see that LM is invariant to the scale parameter, i.e., that the value of LM is the same if computed with $\nu_1/\sigma$ rather than $\nu_1$ (for all finite $\sigma > 0$). Therefore, we may assume $V[\nu_1^2] = 1$, and generate $n$ sets of $N$ pseudo-random variates from a $N(0,1)$. Then, for each of these $n$ sets, LM would be computed, giving $n$ values of LM under $H_0$. By choosing $n$ large enough, we may obtain as good an approximation as desired to the distribution of LM and, so, determine the critical point of the test for a given significance level $\alpha$, or the probability of a Type I error for the computed value of LM from a particular set of observations. Computer simulation is used in Subsection 4.5.1. There, we present a study comparing the finite sample power of LM with that of other existing tests for normality; and a Table of significance points for $\alpha = .10$ and .05.

To finalize this section, we note the procedure utilized here may be applied in a similar way to other families of distributions. We have used the Gram-Charlier (type A) Family (e.g., see Kendall and Stuart (1969, p.156) or Cramer (1946, p.229)), and derived the LM normality test, obtaining the same expression as for the Pearson Family, i.e., equation (4.4). Our approach may also be used to test the hypothesis that $f(u)$ is any particular member of, say, the Pearson Family. This may be done by forming $H_0$ with the appropriate values of $c_0$, $c_1$ and $c_2$ that define the desired distribution, e.g., to test
if \( f(u) \) is a Gamma distribution we would test \( H_0: c_1 = 0 \) (see Kendall and Stuart (1969, p.152)). In some cases this may involve testing non-linear inequalities in \( c_0, c_1 \) and \( c_2 \). For example, to test if \( f(u) \) is a Pearson Type IV we would test \( H_0: c_1^2 - 4c_0c_2 < 0 \). This requires the development of the Lagrange multiplier procedure to test non-linear inequalities, and should be an important area for further research.

4.4 A TEST FOR NORMALITY OF DISTURBANCES

Now we consider the regression model given by equation (1.1). We note that - by our maintained hypotheses - the regression disturbances \( u_1, \ldots, u_N \) are assumed to be independent and identically distributed with population mean equal zero. In addition, we now assume the p.d.f. of \( u_i, f(u_i) \), is a member of the Pearson Family (the same result is obtained if we use the Gram-Charlier (type A) Family). This means we can write \( f(u_i) \) as in (4.2) and the log-likelihood of our \( N \) observations \( y_1, \ldots, y_N \) as in (4.3), where now the parameters, i.e. the arguments in \( \ell(\cdot) \), are \( \beta, c_0, c_1 \) and \( c_2 \), and \( u_i = y_i - x_i'\beta \).

We define \( \theta_1 = (\beta', c_0)' \) and \( \theta_2 = (c_1, c_2)' \), and note that we want to test the normality of the disturbances. This is equivalent to testing \( H_0: \theta_2 = 0 \). It is shown in Proposition 2 in Appendix A (see page 275), that - in this case - the LM test statistic becomes

\[
\text{LM}_N = N\left[ \frac{\hat{\mu}_3^2}{6\hat{\sigma}_2^3} + \left( \frac{\hat{\mu}_4}{\hat{\sigma}_2^4} - 3 \right)^2 / 24 \right] + N\left[ \frac{3\hat{\mu}_2^2}{2\hat{\sigma}_2^2} - \frac{\hat{\mu}_2^2}{\hat{\sigma}_2^2} \right],
\]

(4.5)

where \( \hat{\mu}_j = \frac{1}{N} \sum_{i=1}^{N} \hat{u}_i^j/N \), and the \( \hat{u}_i \) are the OLS residuals, i.e.,
\(\hat{u}_i = y_i - x_i'b\). We have written the resulting test statistic with a suffix \(N\) to indicate this refers to a disturbance normality test. Using the results of Section 4.2, we know \(LM_N\) is, under \(H_0\), asymptotically distributed as \(\chi^2_{(2)}\), and that it is asymptotically efficient. Obtaining the finite sample distribution of \(LM_N\) by analytical procedures appears to be intractable. For a given matrix \(X\), we may resort to computer simulation, generating \(u_i\) from a \(N(0,1)\) (e.g., see Section 4.3 and note \(LM_N\) is invariant to the scale parameter \(\sigma^2\)). In Subsection 4.5.2 we use computer simulation to study the finite sample power of \(LM_N\).

To finalize, we recall that - in linear models with a constant term - OLS residuals satisfy the condition \(\hat{u}_1 + \ldots + \hat{u}_N = 0\). In these cases we have \(\hat{u}_1 = 0\) and, therefore, (4.5) would reduce to

\[
LM_N = N[(\sqrt{\hat{b}_1})^2/6 + (\hat{b}_2 - 3)^2/24]
\]

where \(\hat{b}_1 = \hat{\mu}_3/\hat{\mu}_2^2\) and \(\hat{b}_2 = \hat{\mu}_4/\hat{\mu}_2^2\).

4.5 POWER OF NORMALITY TESTS

In this section we present results of a Monte Carlo study. This was done to compare the power of various tests for normality of observations and regression disturbances.

Not all cross-sectional studies have large samples. There are situations where the sample may be small due to splitting of the original data set (e.g., see Chapter 7), or because of a small cross-section to start with (e.g., observations on a group of countries). Keeping this in mind, we carried out simulations for small and moderate sample sizes. More specifically, we used \(N = 20, 35, 50, 100, 200\) and 300.
We consider four distributions members of the Pearson Family: the Normal, Gamma (2,1), Beta (3,2) and Students t with 5 degrees of freedom; and one distribution which is a non-member of the Pearson Family: the Lognormal. These distributions were chosen because they cover a wide range of values of third and fourth standardized moments (see Shapiro, Wilk and Chen (1968, p.1346)). To generate pseudo-random variates \( u_1 \), from these and other distributions considered throughout the study, we used the subroutines described in Naylor et al. (1966) on a UNIVAC 1100/42. Each of the five variates mentioned above was standardized so as to have zero mean.

4.5.1 Testing for Normality of Observations

We first note that since \( \mu = 0 \), we have \( v_1 = u_1 \) (see Section 4.3 for notation). The tests we consider for the normality of the observations \( u_1 \) are the following:

1. Skewness measure test [with this we would reject normality, i.e. \( H_0 \), if \( v_{1L} \) is outside the interval \( (v_{1L}, v_{1U}) \). For the definition of \( v_{1L}, ... \) etc. see below];

2. Kurtosis measure test [reject \( H_0 \) if \( b_2 \) is outside \( (b_{2L}, b_{2U}) \)];

3. D'Agostino (1971) \( D^* \) test [reject \( H_0 \) if \( D^* = \left[ \sum i/N - (N+1)/(2N^2) \right] e_i^o / \mu_2^{b_2/-(2/\pi)^{-1}} \) is outside \( (D^*_{L}, D^*_{U}) \), where \( e_i^o \) is the i'th order statistic of \( u_1, ..., u_N \)];

4. Pearson, D'Agostino and Bowman (1977) \( R \) test [reject \( H_0 \) if either \( v_{1L} \) is outside \( (R_{1L}, R_{1U}) \) or \( b_2 \) is outside \( (R_{2L}, R_{2U}) \)];
5. Shapiro and Wilk (1965) W test [reject $H_0$ if
$W = \frac{\sum a_i e_i^2}{N \sum e_i^2}$ is less than $W_L$, where the
$a_i$ are coefficients tabulated in Pearson and
Hartley (1972, p.218)];

6. Shapiro and Francia (1972) $W'$ test [reject $H_0$
if $W' = \frac{\sum a'_i e_i^2}{N \sum e_i^2}$ is less than $W'_L$, where
the $a'_i$ are coefficients that may be computed
using the tables in Harter (1961)]; and

7. LM test [reject $H_0$ if $LM > LM^*_U$].

We did not include distance tests (such as the Kolmogorov-Smirnov
test, Cramer-Von Mises test, weighted Cramer-Von Mises test and the
Durbin test) because it has previously been reported that, for a wide
range of alternative distributions, the W test – considered here –
was superior to these (see Shapiro, Wilk and Chen (1968)).

The values $b_{1L}, b_{1U}, b_{2L}, b_{2U}, D^*, D^*_L, D^*_U, R_{1L}, R_{1U}, R_{2L}, R_{2U}, W_L,$ $W'_L$ and $LM^*_U$ are appropriate significance points. We considered a
10 per cent significance level, i.e., we set $\alpha = .10$. All the points
we used are summarized in Table 4.1. For $N = 20, 35, 50$ and $100$
and tests $b_{1L}, b_{1U}, D^*$ and $R$, the points are as given in White and
MacDonald (1980, p.20). For $N = 200$ and $300$, significance points
for $b_{1L}, b_{2L}$ and $b^*$ were obtained respectively from Pearson and
Hartley (1962, p.183), Pearson and Hartley (1962, p.184) and D'Agostino
(1971, p.343); and for the $R$ test we extrapolated the points for
$N \leq 100$. For $W, W'$ and $LM$ we computed the significance points by
simulation using 250 replications so that the empirical $\alpha$, say
$\hat{\alpha}$, was equal to .10. For example, for a given $N$, we set $W_L = W(25)$,
where $W(25)$ was the 25'th largest of the values of $W$ in the 250
TABLE 4.1
Significance points for normality tests (\( \alpha = .10 \))

<table>
<thead>
<tr>
<th></th>
<th>20</th>
<th>35</th>
<th>50</th>
<th>100</th>
<th>200</th>
<th>300</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sqrt{b_{1L}} )</td>
<td>-.769</td>
<td>-.621</td>
<td>-.534</td>
<td>-.389</td>
<td>-.280</td>
<td>-.230</td>
</tr>
<tr>
<td>( \sqrt{b_{1U}} )</td>
<td>.769</td>
<td>.621</td>
<td>.534</td>
<td>.389</td>
<td>.280</td>
<td>.230</td>
</tr>
<tr>
<td>( b_{2L} )</td>
<td>1.82</td>
<td>2.03</td>
<td>2.15</td>
<td>2.35</td>
<td>2.51</td>
<td>2.59</td>
</tr>
<tr>
<td>( b_{2U} )</td>
<td>4.17</td>
<td>4.10</td>
<td>3.99</td>
<td>3.77</td>
<td>3.57</td>
<td>3.47</td>
</tr>
<tr>
<td>( D_{L}^* )</td>
<td>-2.440</td>
<td>-2.295</td>
<td>-2.210</td>
<td>-2.070</td>
<td>-1.960</td>
<td>-1.906</td>
</tr>
<tr>
<td>( D_{U}^* )</td>
<td>.565</td>
<td>.805</td>
<td>.937</td>
<td>1.140</td>
<td>1.290</td>
<td>1.357</td>
</tr>
<tr>
<td>( R_{1L} )</td>
<td>-.891</td>
<td>-.722</td>
<td>-.624</td>
<td>-.457</td>
<td>-.332</td>
<td>-.285</td>
</tr>
<tr>
<td>( R_{1U} )</td>
<td>.891</td>
<td>.722</td>
<td>.624</td>
<td>.457</td>
<td>.332</td>
<td>.285</td>
</tr>
<tr>
<td>( R_{2L} )</td>
<td>1.762</td>
<td>1.973</td>
<td>2.078</td>
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<td>2.450</td>
<td>2.500</td>
</tr>
<tr>
<td>( R_{2U} )</td>
<td>4.530</td>
<td>4.415</td>
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<td>3.955</td>
<td>3.650</td>
<td>3.500</td>
</tr>
<tr>
<td>( W_L )</td>
<td>.925</td>
<td>.945</td>
<td>.957</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( W_{L}' )</td>
<td>.933</td>
<td>.946</td>
<td>.967</td>
<td>.980</td>
<td>.989</td>
<td>.991</td>
</tr>
<tr>
<td>( L_{MU} )</td>
<td>2.18</td>
<td>2.56</td>
<td>2.63</td>
<td>3.36</td>
<td>3.71</td>
<td>4.29</td>
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</table>
replications under Normal observations. Similarly for $W'$. For LM we set $LM_y = LM(225)$. [Initially we used, for $W$, the points from Shapiro and Wilk (1965, p.605); for $W'$ from Weisberg (1974, p.645) and Shapiro and Francia (1972, p.216); and for LM from the values of Table 4.3. With $\alpha = 0.10$, easier power comparisons among the one-sided tests $W, W'$ and LM can be made. Note that $b_1, b_2$ and $D^*$ are two-sided tests and that $R$ is a four-sided test, and hence, for these it is troublesome to adjust the significance points so that $\alpha = 0.10$.]

Every experiment in this simulation study consists of generating $N$ pseudo random variates from a given distribution; computing the values of $b_1, b_2, D^*, W, W'$ and LM and seeing whether $H_0$ is rejected by each individual test. We carried out 250 replications. The estimated power of each test (obtained by dividing the number of times $H_0$ was rejected by 250) for each of the 5 distributions and 6 sample sizes considered are given in Table 4.2, except for $W$ which cannot be computed for $N > 50$ because of the unavailability of the coefficients $a_{1N}$. The power for the Lognormal and $N = 50, 100, 200$ and 300 is not reported; this was equal to 1 for all tests. In the Table, the highest power is underlined for each distribution and sample size, except when three or more tests have this power.

If we have large samples, and we are considering members of the Pearson Family, the theoretical results of Section 4.3 justify the use of the LM test. For finite sample performance we resort to Table 4.2. For $N = 20$, the preferred test would probably be the $W$ test, followed by LM and $W'$. For $N = 35$, tests $W$ and LM may be considered best, and we find these are followed by $W'$. For $N = 50$, perhaps LM would be preferred, followed by the $W$ and $W'$ tests. LM has highest power
### Table 4.2

**Normality of Observations**

*Estimated power with 250 replications (α = .10)*

<table>
<thead>
<tr>
<th></th>
<th>$\sqrt{b_1}$</th>
<th>$b_2$</th>
<th>$D^*$</th>
<th>R</th>
<th>W</th>
<th>W'</th>
<th>LM</th>
</tr>
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<td>.084</td>
<td>.100</td>
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<td>.100</td>
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<tr>
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<td>.120</td>
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<td>.116</td>
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<tr>
<td>Students t</td>
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<td>.252</td>
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<td>.300</td>
<td>.340</td>
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<td>.772</td>
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<td>.996</td>
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<td>.100</td>
<td>.100</td>
</tr>
<tr>
<td>Beta</td>
<td>.776</td>
<td>.940</td>
<td>.804</td>
<td>.972</td>
<td>.996</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>Students t</td>
<td>.560</td>
<td>.984</td>
<td>.988</td>
<td>.980</td>
<td>.964</td>
<td>.992</td>
<td>.992</td>
</tr>
<tr>
<td>Gamma</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
</tbody>
</table>
for all distributions and $N = 100, 200$ and $300$, but the differences in power compared with the $W'$ test are small. We should also note that LM may have good relative power even when the distribution is not a member of the Pearson Family (e.g., see power for Lognormal in Table 4.2). Overall, LM is preferred, followed by $W$ and $W'$, which in turn dominate the other four tests. [This uniformly good relative performance of $W$ and $W'$, is in contrast with the findings of White and MacDonald (1980, p.22). The differences in the results may be due to our use of a one-sided rejection region and their use, apparently, as pointed out by Weisberg (1980, p.30), of a two-sided rejection region for the one-sided tests $W$ and $W'$].

Apart from power considerations, LM has an advantage over $W$ (and $W'$) in that, for its computation, one requires neither ordered observations (which may be expensive to obtain for large $N$) nor expectations and variances and covariances of standard normal order statistics (which may not be available for a particular $N$, e.g., as noted previously $W$ cannot be computed for $N > 50$ because of the unavailability of $a_{1N}$).¹

These results - together with its asymptotic properties - suggest the LM test may be the preferred test in many situations. Therefore, it appeared worthwhile to carry out extensive simulations to obtain - under normality - finite sample significance points for LM. Using expression (4.4) we carried out 10 000 replications and present, in Table 4.3, significance points for $\alpha = .10$ and .05

¹ LM has an additional advantage in providing a convenient framework in which simultaneous specification tests may be derived (e.g., see Chapters 5, 6 and 10).
for a range of sample sizes. The Table suggests that, for large samples, a test of approximately the desired level may be carried out using the asymptotic distribution of LM, i.e. a $\chi^2_{(2)}$, in the choice of the significance point.

### TABLE 4.3

Normality of Observations
Significance points for LM normality test
(10000 replications)

<table>
<thead>
<tr>
<th>N</th>
<th>$\alpha = .10$</th>
<th>$\alpha = .05$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>2.13</td>
<td>3.26</td>
</tr>
<tr>
<td>30</td>
<td>2.49</td>
<td>3.71</td>
</tr>
<tr>
<td>40</td>
<td>2.70</td>
<td>3.99</td>
</tr>
<tr>
<td>50</td>
<td>2.90</td>
<td>4.26</td>
</tr>
<tr>
<td>75</td>
<td>3.09</td>
<td>4.27</td>
</tr>
<tr>
<td>100</td>
<td>3.14</td>
<td>4.29</td>
</tr>
<tr>
<td>125</td>
<td>3.31</td>
<td>4.34</td>
</tr>
<tr>
<td>150</td>
<td>3.43</td>
<td>4.39</td>
</tr>
<tr>
<td>200</td>
<td>3.48</td>
<td>4.43</td>
</tr>
<tr>
<td>250</td>
<td>3.54</td>
<td>4.51</td>
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<td>3.68</td>
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<tr>
<td>400</td>
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<tr>
<td>500</td>
<td>3.91</td>
<td>4.82</td>
</tr>
<tr>
<td>800</td>
<td>4.32</td>
<td>5.46</td>
</tr>
<tr>
<td>$\infty$</td>
<td>4.61</td>
<td>5.99</td>
</tr>
</tbody>
</table>

#### 4.5.2 Testing for Normality of Regression Disturbances

In the present subsection, we study the power of tests for normality of (unobserved) regression disturbances. The tests we consider are the same as those described in Subsection 4.5.1, but we computed them with estimated regression residuals rather than the true disturbances $u_i$. We denote these by $\sqrt{\hat{b}_1}, \hat{b}_2, \hat{D}^*, \hat{R}, \hat{W}, \hat{W}$ and $\text{LM}_N$. The first six are the modified large-sample tests discussed in White and MacDonald (1980). The seventh test is the LM test suggested in
Section 4.4. The modified Shapiro-Wilk test, $\hat{W}$, has been reported to be superior to modified distance tests, so these were excluded (see Huang and Bolch (1974, p.334)).

We consider a linear model with a constant term and three additional regressors, i.e. with $K = 4$, and utilize the OLS residuals $\hat{u}_1$ to compute the modified tests. (Huang and Bolch (1974) and Ramsey (1974, p.36) have found that the power of modified normality tests, computed using OLS residuals, is higher than when using Theil's (1971, p.202) BLUS residuals.) To obtain $\hat{u}_1$ we use the same $u_1$'s as those generated in Subsection 4.5.1. For comparison purposes, our regressors $X_1, \ldots, X_K$ are defined as in White and MacDonald (1980, p.20), i.e., we set $X_{i1} = 1 \, (i = 1, \ldots, N)$ and generate $X_2, X_3$ and $X_4$ from a Uniform distribution. [The specific values of the means and variances of these regressors have no effect on the simulation results. This invariance property follows from the fact that, for a linear model with regressor matrix $X = (x_1, \ldots, x_N)'$, the OLS residuals are the same as those of a linear model with regressor matrix $XR$, where $R$ is any $K$ by $K$ non-singular matrix of constants (see Weisberg (1980, p.29))]}. For $N = 20$ we use the first 20 of the 300 (generated) observations $x_i$. Similarly for $N = 35, 50, 100$ and 200.

For this part of the study we utilize the same significance points as those of Subsection 4.5.1, except for $\hat{W}, \hat{W}'$ and $LM_N$, for which we use the points corresponding to $\hat{\alpha} = .10$ (e.g., as significance point of $\hat{W}$ we use $\hat{W}(25)$, where $\hat{W}(25)$ is the 25'th largest of the values of $\hat{W}$ in the 250 replications under Normal disturbances). The estimated power of each test is given in Table 4.4.
<table>
<thead>
<tr>
<th>N</th>
<th>Normal</th>
<th>Beta</th>
<th>Students t</th>
<th>Gamma</th>
<th>Lognormal</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.084</td>
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<td>0.140</td>
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<tr>
<td></td>
<td>0.068</td>
<td>0.108</td>
<td>0.096</td>
<td>0.100</td>
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<tr>
<td></td>
<td>0.224</td>
<td>0.192</td>
<td>0.188</td>
<td>0.204</td>
<td>0.168</td>
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<tr>
<td></td>
<td>0.640</td>
<td>0.356</td>
<td>0.416</td>
<td>0.572</td>
<td>0.644</td>
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<tr>
<td></td>
<td>0.920</td>
<td>0.844</td>
<td>0.904</td>
<td>0.912</td>
<td>0.924</td>
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<tr>
<td>35</td>
<td>0.108</td>
<td>0.092</td>
<td>0.120</td>
<td>0.120</td>
<td>0.100</td>
</tr>
<tr>
<td></td>
<td>0.128</td>
<td>0.164</td>
<td>0.124</td>
<td>0.184</td>
<td>0.216</td>
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<tr>
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<td>0.292</td>
<td>0.340</td>
<td>0.332</td>
<td>0.324</td>
<td>0.236</td>
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<tr>
<td></td>
<td>0.804</td>
<td>0.544</td>
<td>0.708</td>
<td>0.856</td>
<td>0.872</td>
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<tr>
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<td>0.968</td>
<td>0.988</td>
<td>1.000</td>
<td>0.996</td>
</tr>
<tr>
<td>50</td>
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<td>0.084</td>
<td>0.088</td>
<td>0.084</td>
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<tr>
<td></td>
<td>0.160</td>
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<tr>
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<tr>
<td></td>
<td>0.984</td>
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<td>0.856</td>
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<td>0.988</td>
</tr>
<tr>
<td>100</td>
<td>0.100</td>
<td>0.096</td>
<td>0.108</td>
<td>0.108</td>
<td>0.100</td>
</tr>
<tr>
<td></td>
<td>0.244</td>
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<td>0.296</td>
<td>0.512</td>
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</tr>
<tr>
<td></td>
<td>0.444</td>
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<td>0.664</td>
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</tr>
<tr>
<td></td>
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<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>200</td>
<td>0.088</td>
<td>0.128</td>
<td>0.132</td>
<td>0.112</td>
<td>0.100</td>
</tr>
<tr>
<td></td>
<td>0.520</td>
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</tr>
<tr>
<td></td>
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<tr>
<td></td>
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<tr>
<td>300</td>
<td>0.108</td>
<td>0.116</td>
<td>0.124</td>
<td>0.088</td>
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<td>0.540</td>
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<td>0.984</td>
<td>0.980</td>
<td>0.972</td>
</tr>
<tr>
<td></td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
</tbody>
</table>
For $N = 20$ we find that probably the best tests are $\text{LM}_N^*$ and $\hat{W}$, followed by $\hat{W}'$ and $\sqrt{b_1}$. For $N = 35$ and $50$ we obtain that $\text{LM}_N^*$ and $\hat{W}$ are the best, and that these are followed by $\hat{W}'$. For $N = 100, 200$ and $300$, $\text{LM}_N^*$ has highest power for all distributions and we see that the $\hat{W}'$ test performs quite well also. Our results agree with those of White and MacDonald (1980) in that — in almost all the cases — the modified tests give, correspondingly, lower powers than those using the original disturbances; these power differences diminish as $N$ increases (compare Tables 4.2 and 4.4). We also find that, for a given $N$ and a given distribution, the ranking of the tests in Table 4.2 is approximately the same as that of Table 4.4. To obtain a measure of closeness between the true and modified statistics we computed their correlation. The numerical results are given in Table 4.5. Our findings agree with those of White and MacDonald (1980, p.22): $\sqrt{b_1}$ appears to be closer to $b_1'$; and $\hat{b}_2$ appears to be closer to $b_2'$, than the other modified statistics. In our study, these would be followed by $(D^*, \hat{D}^*)$ and then by $(\text{LM}, \text{LM}_N^*)$. We would then have $(W', \hat{W}')$ and, lastly, $(W, \hat{W})$. □

A further comment is required. It is clear that the OLS residuals $\hat{u} = (I-Q_X)u$ are a linear transformation (defined by $Q_X$) of the unobserved disturbances $u$, where $\hat{u} = (\hat{u}_1, \ldots, \hat{u}_N)'$, $u = (u_1, \ldots, u_N)'$ and $Q_X$ is an $N$ by $N$ matrix defined by $Q_X = X(X'X)^{-1}X'$. As noted by White and MacDonald (1980) and Weisberg (1980), simulation results studying the relative power of tests for the normality of $u$ — computed using $\hat{u}$ — depend on the particular form of $Q_X$. If one is to carry out a Monte Carlo study then, to have a less restrictive result, one should consider various forms of $Q_X$. Different forms may arise due to changes in $N$; due to variations in the way the regressors $X_1, \ldots, X_K$
### TABLE 4.5

Estimated correlations between true and modified statistics

(K=4) (Regressors: \(X_1 = 1; X_2, X_3, X_4 \sim \text{uniform}\) (N varies)

\[
\begin{array}{ccccccc}
\text{(} & \text{vb}_1, \hat{\text{vb}}_1 & \text{ (} & \text{b}_2, \hat{\text{b}}_2 & \text{ (} & \text{D}, \hat{\text{D}} & \text{ (} & \text{W}, \hat{\text{W}} & \text{ (} & \text{W'}, \hat{\text{W'}} & \text{ (} & \text{LM, LM}_N \\
\text{N = 20}
\text{Normal} & .754 & .718 & .669 & .532 & .594 & .726 \\
\text{Beta} & .679 & .580 & .556 & .283 & .326 & .321 \\
\text{Students t} & .874 & .829 & .794 & .740 & .778 & .821 \\
\text{Gamma} & .803 & .862 & .796 & .674 & .724 & .787 \\
\text{Lognormal} & .816 & .894 & .764 & .688 & .728 & .816 \\
\text{N = 35}
\text{Normal} & .893 & .821 & .833 & .704 & .768 & .786 \\
\text{Beta} & .839 & .819 & .804 & .630 & .664 & .782 \\
\text{Students t} & .950 & .944 & .925 & .904 & .921 & .953 \\
\text{Gamma} & .925 & .959 & .916 & .845 & .877 & .946 \\
\text{Lognormal} & .957 & .974 & .870 & .834 & .859 & .961 \\
\text{N = 50}
\text{Normal} & .902 & .837 & .844 & .747 & .790 & .881 \\
\text{Beta} & .888 & .829 & .842 & .726 & .782 & .790 \\
\text{Students t} & .969 & .974 & .958 & .952 & .961 & .994 \\
\text{Gamma} & .942 & .968 & .932 & .862 & .892 & .969 \\
\text{Lognormal} & .980 & .987 & .922 & .892 & .910 & .982 \\
\text{N = 100}
\text{Normal} & .956 & .929 & .932 & .844 & .907 \\
\text{Beta} & .944 & .924 & .927 & .841 & .864 \\
\text{Students t} & .989 & .989 & .979 & .984 & .983 \\
\text{Gamma} & .980 & .990 & .962 & .936 & .991 \\
\text{Lognormal} & .995 & .997 & .954 & .951 & .995 \\
\text{N = 200}
\text{Normal} & .976 & .972 & .972 & .926 & .935 \\
\text{Beta} & .968 & .934 & .955 & .933 & .949 \\
\text{Students t} & .996 & .997 & .992 & .995 & .997 \\
\text{Gamma} & .991 & .996 & .981 & .961 & .998 \\
\text{Lognormal} & .998 & .999 & .971 & .973 & .999 \\
\text{N = 300}
\text{Normal} & .980 & .973 & .975 & .926 & .941 \\
\text{Beta} & .968 & .957 & .964 & .954 & .958 \\
\text{Students t} & .997 & .998 & .995 & .996 & .999 \\
\text{Gamma} & .994 & .997 & .987 & .966 & .998 \\
\text{Lognormal} & .999 & .999 & .978 & .980 & .999 \\
\end{array}
\]
are generated; and/or due to changes in the number of regressors K.

So far we have studied the power of the tests for different values of N, using K = 4 and generating the regressors as in White and MacDonald (1980). This was done to compare our results with theirs. In addition, we have repeated our experiments but generating the regressors in a different way. We set $X_{i1} = 1$ ($i = 1, \ldots, N$) and generated $X_2$ from a Normal, $X_3$ from a Uniform and $X_4$ from a $\chi^2_{10}$. These regressor-distributions are of interest because they are commonly found in cross-sectional studies (we shall also use this regressor set in future parts of the thesis). The numerical results are presented in Tables 4.6 and 4.7 (Tables 4.6 to 4.13 are in Appendix B, page 280).

Our findings do not vary substantially from those stated for the White and MacDonald regressor set. $LM_N^*$ is a preferred test (together with $\hat{W}$ and $\hat{W}'$) for $N \leq 50$ and is preferable to all tests for $N \geq 100$ (see Table 4.6). The conclusions from the analysis of the correlations between the true and modified statistics are also the same (see Table 4.7).

As a final exercise, we carried out our experiment fixing $N = 20$ and using the three regressor Data Sets reported in Weisberg (1980, p.29). Following Weisberg, we varied K, for each Data Set, using K = 4, 6, 8 and 10. The numerical results are summarized in Tables 4.8-4.13. Weisberg found that the power of the $\hat{W}'$ test may vary as K and/or the regressors are changed. We find this to be the case for all the tests considered. For example, for Data Set 1 (see Table 4.8), we obtain that for the Lognormal the power of $\sqrt{b_{11}}$, say $P(\sqrt{b_{11}})$, is equal to .636, for K = 10, and .956 for K = 4, i.e.,

$.636 \leq P(\sqrt{b_{11}}) \leq .956$. Similarly we obtain that $.572 \leq P(\hat{b}_2) \leq .812$; $.608 \leq P(\hat{b}_3) \leq .888$; $.632 \leq P(\hat{R}) \leq .940$; $.592 \leq P(\hat{W}) \leq .928$;
.636 \leq P(\hat{W}') \leq .944 \text{ and } .616 \leq P(LM_N) \leq .952. \text{ We also find that for Data Sets 1 and 2 (see Tables 4.8 and 4.9), the empirical significance level } \tilde{\alpha} \text{ is close to } .10 \text{ for all statistics and all } K. \text{ For Data Set 3, however, } \tilde{\alpha} \text{ increased considerably as } K \text{ increased (e.g., see Table 4.10 and note that for } \sqrt{\hat{b}_1}, \tilde{\alpha} = .088, .104, .200 \text{ and } .216 \text{ for } K = 4, 6, 8 \text{ and } 10 \text{ respectively). This shows that the power and the level of a test may depend on the specific form of } Q_X. \text{ Nevertheless, when comparing the relative power, it is interesting to note that, for all } K \text{ and all three regressor Data Sets, } LM_N, \hat{W}, \hat{W}' \text{ and } \sqrt{\hat{b}_1} \text{ are the preferred tests (as it was found in our earlier 2 sets of experiments with } N = 20). \text{ Regarding the correlations between the true and modified statistics, we observe that (for each Data Set) as } K \text{ increases, all correlations decrease (e.g., see Table 4.11 and note that for Data Set 1, the correlation between } D^* \text{ and } \hat{D}^*, \text{ for the Normal was equal to } .640 \text{ when } K = 4 \text{ and to } .329 \text{ when } K = 10). \text{ However, the ranking among the tests remains the same, i.e., from high to low correlations the order remains being } (\sqrt{\hat{b}_1}, \sqrt{\hat{b}_1}), (b_2, b_2), (b^*, \hat{D}^*), (LM, LM_N), (W', \hat{W}') \text{ and } (\hat{W}, \hat{W}) \text{ (see Tables 4.11, 4.12 and 4.13).}

Our main result of Subsection 4.5.2 is that, for all the forms of matrices } Q_X \text{ that we have studied, } LM_N \text{ performed with good relative power. This was true for both small } N \text{ (e.g., } N = 20) \text{ and large } N \text{ (e.g., } N = 300). \text{ The above findings encourage the use of } LM_N \text{ in testing for the normality of } u_1. \text{ The statistic } LM_N \text{ is simple to compute and, in any regression problem, we may easily obtain an approximation to its finite sample distribution, under } H_0, \text{ by computer simulation. This should not represent a serious problem, particularly with the fast speed and increased availability of modern computers.}
4.6 ANALYSIS OF MODELS WITH NON-NORMAL DISTURBANCES

In Section 4.4 we suggested an asymptotically efficient test for the normality of disturbances, and in Section 4.5 we noted its good power - relative to other existing tests - even for very small sample sizes. If the hypothesis that disturbances are normally distributed \((H_0)\) is accepted, classical econometric analysis may be carried out. In this section, we make a very brief comment on possible procedures to follow if \(H_0\) is rejected.

Some econometricians consider that disturbances represent the sum of effects of omitted variables, and justify the normality assumption by making an appeal to a central limit theorem (e.g., see Johnston (1972, p.11)). Others regard normality as an assumption made for computational convenience, arguing it gives a quasi-likelihood function. Whatever the motivations underlying the assumption, typically, if \(H_0\) is rejected, no alternative disturbance distribution would exist in an econometricians' mind. Under these circumstances one may consider - as an alternative to Least Squares estimation - the use of

1. Robust estimation on the linear model,
2. Estimation within a Family of transformations, or
3. Estimation within a Family of distributions.

We shall - very briefly - describe these.

5 In particular, we could regard OLS estimators as MLE. It is difficult to determine the properties of estimators after a disturbance specification test has been carried out. Apparently this problem has not been dealt with in the econometric and statistical literature. As it is common practice (e.g., see Malinvaud (1980, footnote in pg.292)) we do not pursue this point, and proceed with our presentation disregarding the effect of the pretest.

6 If one had a specific alternative p.d.f. for \(u_i\), one could write down the log-likelihood, obtain MLE's of \(\beta\), and use the general theory of MLE to obtain approximate significance tests. (See Subsection 4.6.3).
4.6.1 Robust Estimation

Consider a situation where, after rejection of $H_0$, we decided to proceed analysing the linear model, but regarding the disturbances as non-normal. If $u_1$ is non-normal, but its variance is finite, the OLS estimator $b$ would still be BLUE. Furthermore, under fairly general conditions, we could apply — for large samples — the usual $t$ and $F$ tests (e.g., see Arnold (1980)). Then it would appear that — particularly for large samples — there is strong reason for the use of OLS estimates even when disturbance normality is rejected. Yet, some statisticians question the 'appropriateness' of the OLS estimator by arguing the class of linear estimators is too restrictive, and the unbiasedness property as being of doubtful value (e.g., see Hampel (1973, p.90)). Others question the use of the OLS estimator because of the finding that it may be quite sensitive to outliers and long-tailed distributions (e.g., see Hogg (1979)). These considerations — among others — have contributed to the development of alternative estimation methods.

An important class of these alternative methods goes under the rubric of robust regression. As stated by Hill and Holland (1977, p.828),

"the emphasis in robust regression is on methods which are not sensitive to deviations from normal distributions and to the effects of outliers in the data".

Work in this area is extensive and the literature is voluminous. This is evident from the five-part article of H. Leon Harter entitled 'The method of Least Squares and some alternatives', which includes some historical accounts and a survey of recent developments (see Harter (1974,1975)). A good, brief illustration of some of these techniques
is found in the textbooks of Maddala (1977, p.308) and Malinvaud (1980, p.318).

4.6.2 Estimation within a Family of Transformation

Analysis of linear models with non-normal disturbances is complicated. This has led people to consider the use of *transformations* applied to the measured variables $y_1$ so that—in particular—disturbance normality is better suited. More formally, if $u_i^o$ denotes the $i$'th disturbance from a transformed model; then we would want the transformation to be such that $u_i^o$ is normally distributed and, at the same time, to have (as was assumed for $u_i$) $E[u_i^o|x_i^o] = 0$, and $E[u_i^{o2}] = \sigma_o^2$ for all $i = 1, \ldots, N$.

Several transformations exist that aim to achieve this, and a popular one is the Box and Cox (1964) transformation. This defines $y_i^{(\lambda)} = (y_i^{\lambda} - 1)/\lambda$ for $\lambda \neq 0$; and $y_i^{(\lambda)} = \log y_i$ for $\lambda \rightarrow 0$, where $\lambda$ is an unknown parameter. If the Box-Cox transformation is applied, the transformed model would be

$$y^{(\lambda)} = x\beta^o + u^o,$$

where $y^{(\lambda)} = (y_1^{(\lambda)}, \ldots, y_N^{(\lambda)})$ is the vector of the $N$ transformed observations; $\beta^o$ is a $K$ by 1 vector of unknown parameters associated with $y^{(\lambda)}$, and $u^o$ is a vector that contains the $N$ disturbances $u_i^o$.

Data transformation is considered here as a statistical device to achieve disturbance normality, so one may consider applying transformations to the dependent variable only. Transformations may also be applied to the variables $x_i$ (e.g., see Box and Tidwell (1962)), and are a tool used in the choice of regression functional form (e.g., see Zarembka (1968, 1974)). In addition, transformations provide a convenient framework for the derivation of specification tests (see Savin and White (1978), Godfrey and Wickens (1981b), and Bera and Jarque (1981b)).
A fundamental assumption of the Box-Cox transformation is that, for some $\lambda$, the disturbances $u_1^0,...,u_N^0$ 'can be treated as' $N$ normally distributed variables with mean zero and finite variance $\sigma_o^2$. This allows writing the log-likelihood, from which MLE's for $\lambda$, $\beta^0$ and $\sigma_o^2$, and corresponding standard errors can be obtained. Before concluding we note that - in econometrics - our usual interest is to explain $Y$ (the 'measured variable'), and not some function of $Y$. In this case, as stated by Box and Cox (1964, p.214)

"we either analyse linearly the untransformed data or, if we do apply a transformation in order to make a more efficient and valid analysis, we convert the conclusions back to the original scale".

In particular, interest could reside in computing elasticities, and for this we may proceed as in Savin and White (1978, p.3).

4.6.3 Estimation within a Family of Distributions

A final alternative to OLS estimation, that we shall briefly comment on, is MLE under a particular family of distributions. For example, an attempt may be made to obtain the MLE's of $\beta$, $c_0$, $c_1$ and $c_2$ for the Pearson Family - unfortunately this is difficult. Indeed, no results are presently available (even for the non-regression case), and this is an area that requires further study (Pearson distributions are usually fitted by the method of moments, e.g., see Kendall and Stuart (1969, p.152)). However, other computationally more manageable

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8 The Box-Cox transformation is well defined for positive values. This implies limits on $y_i^{(1)}$, which means - strictly speaking - that $u_1^0$ cannot be normal (see Poirier (1978)). In the various applications of the transformation, this 'truncation problem' has been typically neglected. This approach is not entirely incorrect when truncation is not severe.
families may be considered. For instance, Goldfeld and Quandt (1980) derive maximum likelihood estimates under the Sargan Family of distributions (see also Zeckhauser and Thompson (1970) and Anscombe (1967)). An advantage of this approach (as the one described in Subsection 4.6.2) is that the general theory of MLE could be used to obtain approximate significance tests.

4.7 CONCLUDING REMARKS

Throughout this Chapter we have discussed normality under the assumption that the variance of the disturbances was constant. In Chapter 5 we study the problem of heteroscedasticity, firstly, under normal disturbances, and then we consider both problems jointly. In particular, we extend the procedure of Section 4.4 to derive a joint test for disturbance normality and homoscedasticity.

We end by noting there are cases in applied econometrics where we would not carry out a disturbance normality test because, by the nature of the model, normality would not hold. This is the case, for example, when estimating a frontier production function (e.g., see Maddala (1977, p.317)). Another example arises when \( y_1 \) is restricted, say, to non-negative values, i.e., \( y_1 = x_1^\prime \beta + u_1 \geq 0 \). Then the range of \( u_1 \) would be restricted to \( u_1 \geq -x_1^\prime \beta \) and, therefore, disturbance normality could not hold. These types of limited dependent variable models are studied in Chapter 6.
CHAPTER 5

THE PROBLEM OF HETEROSCEDASTIC DISTURBANCES*

"If our hypothesis is about anything and not about some one or more particular things, then our deductions constitute mathematics"

Bertrand Russell

5.1 INTRODUCTION

Another assumption typically made for the statistical analysis of economic models is that disturbances are homoscedastic. Indeed, in our previous Chapter, we dealt with linear models with homoscedastic disturbances. Now we shall study the consequences, tests and estimation of models with heteroscedastic disturbances. The model is defined in (1.1) and our discussion in this Chapter will be done under assumptions [2], [5], [6] and [7], which are stated in Section 1.2.

As previously noted, under homoscedasticity and provided \( \sigma^2 \) is finite, the OLS estimator of \( \beta, b \), would have VCM equal to

\[
V[b] = \sigma^2(X'X)^{-1}
\]

and would be BLUE. Further, an unbiased estimator \( \sigma^2 \) would be

\[
s^2 = \hat{u}'\hat{u}/(N-K), \quad \text{where} \quad \hat{u} = (\hat{u}_1, \ldots, \hat{u}_N)'
\]

It is also well known that under heteroscedasticity say

\[
E[u'u'] = \sigma^2 \Omega, \quad V[b] \quad \text{would be given by}
\]

* Sections 5.3 and 5.4 are based on the paper Jarque (1980a). The constructive joint test suggested in Section 5.5 has been published in the form Jarque and Bera (1980).
and we would have (e.g., see Goldfeld and Quandt (1972, p.81))

\[ E[s^2] = \sigma^2 + \sigma^2 \text{trace}\{(X'X)^{-1}X'(I_N-\Omega)X\}/(N-K) \]

and

\[ E[s^2(X'X)^{-1}] = V[b] + \sigma^2(X'X)^{-1}X'(I_N-\Omega)X(X'X)^{-1} \]
\[ + \sigma^2 \text{trace}\{(X'X)^{-1}X'(I_N-\Omega)X(X'X)^{-1}\}/(N-K) , \]

(5.2)

where \( \Omega \) is (as before) a diagonal matrix with finite valued elements \( \alpha^2 \sigma_1^2, \ldots, \alpha^2 \sigma_n^2, \) and not proportional to \( I_N. \) In this case, \( b \) would not be efficient. Rather, the BLUE of \( \beta \) would be the GLS estimator \( \hat{\beta} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y, \) which has VCM given by

\[ V[\hat{\beta}] = \sigma^2(X'\Omega^{-1}X)^{-1} . \]

(5.3)

We see that if we incorrectly assume homoscedasticity and therefore use \( b \) as an estimate of \( \beta, \) and \( s^2(X'X)^{-1} \) as an estimate of \( V[b], \) we would obtain

Firstly, suboptimal estimators in terms of variance; secondly, a biased estimate of the true VCM of \( b; \) and thirdly, invalid inferences about \( \beta \) when using the usual t and F tests. In addition, a fourth consequence would be inappropriate inferences when using disturbance specification tests as, for example, the disturbance normality tests considered in Chapter 4.

The seriousness of these four consequences depends on the form of \( X \) and \( \Omega. \) Analytical results for the general linear model are
difficult to derive, so various authors have proceeded to evaluate the consequences for particular cases.

A good illustration regarding the study of the first two of these aspects is given by Goldfeld and Quandt (1972, p.83). They consider the model $y_i = X_{i1}\beta_1 + X_{i2}\beta_2 + u_i$, with $\alpha_i^2 = \alpha_1 + \alpha_2 X_{i1} + \alpha_3 X_{i2}^2$ and where $X_{i1} = 1$ and $X_{i2}$ is a scalar generated either from a Uniform or a Lognormal distribution (several mean-variance combinations were specified for $X_{i2}$). Their results are invariant to the choice of $\beta_1$ and $\beta_2$, so these need not be specified. Also, they set $\alpha_1 = 20$ and consider various values for $\alpha_2$ and $\alpha_3$.

To study the loss of efficiency by using $b$ rather than $\hat{\beta}$, they computed (5.1) and (5.3), and obtained the ratio of the square root of corresponding diagonal elements. Similarly, to study the effect of the use of the biased estimate of $V[b]$, they computed (5.2) and (5.1), and calculated the ratio of the square root of corresponding diagonal elements.

They summarize the main findings as follows (see Goldfeld and Quandt (1972, pp.83-84)):

(i) Heteroscedasticity can produce gross inefficiencies if one uses $b$ rather than $\hat{\beta}$;

(ii) The classical VCM estimator may considerably understate the true variances of $b$;

(iii) For a given mean and variance of the independent variable, a more skewed distribution of $X_2$ produces greater biases and inefficiencies; and
(iv) Variations in the parameters which underlie the variance of the disturbances can markedly influence the consequences of heteroscedasticity.

They conclude that "heteroscedasticity may be a severe problem". Similar results have also been obtained by other authors [e.g., see Geary (1966), Johnston (1972, pp.215-217) and Kmenta (1971, pp.255-256)].

Regarding the third aspect (i.e., the properties of inferential procedures) Box (1954), Scheffé (1959) and Ito and Schull (1964) have noted that heteroscedasticity may seriously affect the significance level and the power of $t$ and $F$ tests. In addition, Schmidt and Sickles (1977) studied Chow's $F$ test for the identity of regressions and found that heteroscedasticity may vary substantially the assumed significance level. Finally, regarding the fourth aspect, it will be shown in Section 5.6 that the use of disturbance normality tests in the presence of heteroscedasticity may lead to incorrect conclusions.

Overall, these consequences highlight the importance of testing the validity of the homoscedasticity assumption.

The next section presents a classification of several existing tests for homoscedasticity. In Section 5.3 we assume disturbance normality, and suggest a test for homoscedasticity when there is uncertainty about its nature and form. The relative power of this test is studied in Section 5.4. In Section 5.5 we relax the normality assumption and extend the procedure suggested in Section 4.4, in order to derive a joint test for disturbance normality and homoscedasticity. Section 5.6 contains a Monte Carlo experiment to study the power of various tests for normality and/or homoscedasticity, under one and two-directional
departures from the null hypothesis that $u_i$ is normal and homoscedastic. Finally, in Section 5.7 some comments are made on the analysis of models with non-normal and/or heteroscedastic disturbances.

5.2 TESTS FOR HOMOSCEDASTICITY

There are a number of homoscedasticity tests available and, in any particular problem, the choice of test may depend on the a-priori information one has regarding the disturbance variances. One may classify these tests into four classes.

A first class of tests may be defined as those for which one specifies the nature of the variances, i.e. a functional relationship between the variances and observed variables, and proceeds to the estimation of the heteroscedasticity and $\beta$ parameters in the model. [For example, say one specifies $\sigma_i^2 = \sigma^2 + z_i^*\alpha^*$, where $z_i^*$ is a vector of fixed variables, $\sigma^2$ is an unknown scalar, and $\alpha^*$ is a vector of unknown heteroscedasticity parameters. In addition, assume disturbance normality. Then one could write the likelihood of the model; maximize this with respect to $\sigma^2, \alpha^*$ and $\beta$; and use the asymptotic distribution of the resulting estimator of $\alpha^*$ (e.g., see Amemiya (1977, p.368)) or the likelihood ratio (e.g., see Goldfeld and Quandt (1972, pp.95-96) or Harvey (1976)) to test for homoscedasticity, i.e., $H_0: \alpha^* = 0.$] When using these tests, if $H_0$ is rejected, estimates of $\beta$ and $\sigma_i^2$ would be available without further computations. In accordance with the Goldfeld and Quandt (1972, p.86) nomenclature, one may refer to tests in this class as 'FULLY-CONSTRUCTIVE TESTS'.

A second class consists of the 'CONSTRUCTIVE TESTS' (this term is due to Goldfeld and Quandt (1972, p.86)). These also require the
specification of a functional relationship between $\sigma_1^2$ and observed variables. Here, however, the heteroscedasticity and $\beta$ parameters would not be estimated jointly. For these tests, the heteroscedasticity related parameters would be estimated after obtaining $b$ and the OLS residuals, i.e., in a second stage. The estimated values would then be used to test $H_0: \sigma_1^2 = \ldots = \sigma_N^2$ and, if this is rejected, a given heteroscedastic alternative would be available for computing GLS (or maximum likelihood) estimators. Examples of constructive tests are those suggested by Glejser (1969) and Park (1966). The Lagrange Multiplier test of Godfrey (1978c) and Breusch and Pagan (1979) may also be included in this class.

A third class of tests arises when the a-priori information only allows one to order the observations according to increasing values of disturbance variance. Goldfeld and Quandt (1965, p.85) call these 'NON-CONSTRUCTIVE TESTS'. Since no specific heteroscedastic parameterization would be available, these tests ignore the estimation of the heteroscedasticity parameters and their objective is to establish the presence, or absence, of heteroscedasticity. Examples of non-constructive tests are the F-test based on OLS residuals suggested by Goldfeld and Quandt (1965); the F-test based on BLUS residuals suggested by Theil (1971, pp.214-215); the one using recursive residuals of Harvey and Phillips (1974); the bounds F-test suggested by Harrison and McCabe (1979), and the bounds tests of Szroeter (1978).

Recently, an additional test for homoscedasticity has been suggested by White (1980c). This test does not require the specification of the nature of the heteroscedasticity (as those in the first and second classes) nor knowledge of the ordering of the variances (as those in the third class). Therefore, one may think of this test as forming part
of a fourth class which, in accord with the Goldfeld and Quandt (1972, Chapter 3) nomenclature, could be referred to as a class of 'FULLY-NON-CONSTRUCTIVE TESTS' (these may also be referred to as 'pure-significance tests'). In Section 5.3 we suggest a test that also forms part of this class. However, before describing it we shall summarize - for future use - White's result.

In addition to our assumptions, White (1980c) assumes

\[ a_1: \quad E[|u_i^4|^{(1+\delta)}] < \Delta \text{ for some } \Delta > 0 ; \]

\[ a_2: \quad \det \left\{ \frac{1}{N} \sum_{i=1}^{N} (x_i^{(v)} - \bar{x}^{(v)})(x_i^{(v)} - \bar{x}^{(v)})' E[(u_i^2 - \sigma_i^2)^2] \right\} > \delta > 0 ; \text{ and} \]

\[ a_3: \quad \frac{1}{N} \sum_{i=1}^{N} E[(u_i^2 - \sigma_i^2)^2] > \delta > 0 , \]

where \( x_i^{(v)} = \text{Vech}(x_i x_i') \) and \( \bar{x}^{(v)} = \frac{1}{N} \sum_{i=1}^{N} x_i^{(v)}/N \) are vectors of dimension \( K(K+1)/2 \) by 1. Define

\[ \hat{d}_W = \frac{1}{N} \sum_{i=1}^{N} x_i^{(v)} (\hat{\sigma}_i^2 - \sigma_i^2)/N \]

and

\[ \hat{D}_W = \frac{1}{N} \sum_{i=1}^{N} (x_i^{(v)} - \bar{x}^{(v)})(x_i^{(v)} - \bar{x}^{(v)})' (\hat{\sigma}_i^2 - \sigma_i^2)^2/N . \]

Then, White (1980c, p.823) shows that, under the assumptions stated

\[ \text{1 The operator } \text{Vech}\{\cdot\} \text{ is defined such that, if } A \text{ is a } K \times K \text{ matrix, then } \text{Vech}\{A\} \text{ is a } K(K+1)/2 \times 1 \text{ vector containing the elements of the 'lower triangle' of } A. \text{ For example, if } A = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}, \text{ then } \text{Vech}\{A\} = (1,3,4)' . \]
previously, the statistic

\[ HW = Nd W D W^{-1} \]  \hspace{1cm} (5.5) 

would be asymptotically distributed as \( \chi^2_{(K+1)/2} \) under

\( H_0: \sigma_1^2 = \ldots = \sigma_N^2 \). \( H_0 \) would be rejected if the computed value for \( HW \) was greater than the chosen significance point. The power of \( HW \) has not been previously studied and some results on this are presented in Section 5.4.

5.3 A FULLY-NON-CONSTRUCTIVE TEST

Breusch and Pagan (1979) assume normal disturbances, and consider heteroscedasticity of the form \( \sigma_i^2 = g(\sigma^2 + z_i^\alpha \alpha^\star) \), where \( g(\cdot) \) is a twice differentiable positive function; \( \sigma^2 \) is an unknown scalar; \( \alpha^\star = (\alpha_2, \ldots, \alpha_p)' \) is a vector of unknown parameters and \( z_i^\star = (z_{i2}, \ldots, z_{ip})' \) is a \( p \times 1 \) vector of fixed variables (see also Godfrey (1978c)). They show the LM test for homoscedasticity, i.e., for \( H_0: \alpha^\star = 0 \), is given by

\[ LM_H = \frac{1}{2 \sigma^2} \varepsilon' Z (Z'Z)^{-1} Z' \varepsilon \]  \hspace{1cm} (5.6) 

with \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_N)' \), \( \varepsilon_j = e_j - \bar{e} \), \( e_j = u_j^2 \) and \( \bar{e} = \frac{N}{j=1} e_j / N \); and where \( Z \) is an \( N \) by \( p \) matrix with first column equal to a vector of ones, and with the rest of the elements being \( z_{ih} \) (\( i = 1, \ldots, N; h = 2, \ldots, p \)).

In (5.6) we write \( LM \) with a suffix \( H \) to indicate this refers to a homoscedasticity test. It may be shown that another form of writing \( LM_H \) is
\[ LM_H = \frac{1}{2} \hat{\beta}' \hat{\epsilon} (Z^* \hat{\epsilon} M Z^*)^{-1} Z^* \hat{\epsilon} , \quad (5.7) \]

where \( Z^* = (z_1^*, \ldots, z_N^*)' \), \( \hat{\beta} = (\hat{\beta}_1, \ldots, \hat{\beta}_N)' \), \( \hat{\epsilon}_i = (\hat{u}_i / \hat{\sigma}) - 1 \) and
\[
M = I_N - \hat{\epsilon}_N (\hat{\epsilon}_N \hat{\beta}_N)'^{-1} \hat{\epsilon}_N , \quad \text{with} \quad \hat{\epsilon}_N \quad \text{being an} \quad N \quad \text{by} \quad 1 \quad \text{vector of ones}.
\]

We start our discussion by considering normally distributed disturbances, and assuming that we are able to classify the \( N \) observations into \( L < N \) mutually exclusive groups so the groups are homoscedastic.

Say group \( h \) has \( N_h \) observations and note \( N_1 + N_2 + \ldots + N_L = N \).

Then we could write
\[
E[u_i^2] = \sigma_i^2 = \gamma_1 + \gamma_2 z_{i2} + \ldots + \gamma_L z_{iL} , \quad (5.8)
\]

where \( z_{ih} \) is equal to 1 if the i'th observation is in group \( h \) and zero if not. We see the disturbance variance for group 1 would be given by \( \sigma_1^2 = \gamma_1 \) and that for group \( h \) by \( \sigma_h^2 = \gamma_1 + \gamma_h \) with \( h = 2, \ldots, L \). The problem of testing the equality of \( \sigma_1^2, \ldots, \sigma_L^2 \) would then reduce to testing the equality of \( \gamma_1, \ldots, \gamma_L \), i.e., to testing
\[ H_0: \gamma_2 = \ldots = \gamma_L = 0. \]

This is a particular case of the formulation employed by Breusch and Pagan (1979) and, hence, we may define \( p = L, \alpha^* = (\gamma_2, \ldots, \gamma_L)' \) and \( z_{ih} \) as in (5.8), and use (5.6) to test \( H_0 \). We may show that in this case \( Z'Z \) is equal to
\[
Z'Z = \begin{bmatrix}
N & N_2 & N_3 & \ldots & N_L \\
N_2 & N_2 & 0 & \ldots & 0 \\
N_3 & 0 & N_3 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
N_L & 0 & 0 & \ldots & N_L
\end{bmatrix}
\]
and its inverse is given by

\[(Z'Z)^{-1} = \begin{bmatrix}
\frac{1}{N_1} & -\frac{1}{N_1} & -\frac{1}{N_1} & \ldots & -\frac{1}{N_1} \\
-\frac{1}{N_1} & \frac{1}{N_2} + \frac{1}{N_1} & \frac{1}{N_1} & \ldots & \frac{1}{N_1} \\
-\frac{1}{N_1} & \frac{1}{N_1} & \frac{1}{N_3} + \frac{1}{N_1} & \ldots & \frac{1}{N_1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\frac{1}{N_1} & \frac{1}{N_1} & \frac{1}{N_1} & \ldots & \frac{1}{N_L + 1/N_1}
\end{bmatrix} \]

Also that

\[Z'\epsilon = \left( \frac{L}{\Sigma_{h=1}^{N_h} \Sigma_{i=1}^{N_h} \epsilon_{hi}}, \frac{N_2}{\Sigma_{i=1}^{N_2} \epsilon_{2i}}, \ldots, \frac{N_L}{\Sigma_{i=1}^{N_L} \epsilon_{Li}} \right)' \]

where - for convenience - \(\epsilon_{hi}\) denotes the value of \(\epsilon\) for the \(i^{th}\) observation of group \(h\) (the equivalent definition will apply to \(\epsilon_{hi}'\), \(y_{hi}'\) and \(x_{hi}'\)). Using these relations in expression (5.6), and after some algebra, we obtain

\[LM_H = \frac{1}{2\epsilon^2} \left[ \frac{L}{\Sigma_{h=1}^{N_h} \left( \Sigma_{i=1}^{N_h} \epsilon_{hi} \right)^2} \right] = \frac{1}{2\epsilon^2} \left[ \frac{L}{\Sigma_{h=1}^{N_h} (\bar{\epsilon}_h - \bar{\epsilon})^2} \right] \quad (5.9)\]

where \(\bar{\epsilon}_h = \frac{1}{N_h} \Sigma_{i=1}^{N_h} \epsilon_{hi}/N_h\).

Then, if we are able to classify the observations into exactly homoscedastic groups, the use of (5.9) would be justified. This, being an LM test statistic, would be asymptotically distributed as \(\chi^2_{(L-1)}\) under \(H_0: \gamma_2 = \ldots = \gamma_L = 0\), and would yield an asymptotically efficient test. The finite sample distribution of (5.9) could be obtained by computer simulation (e.g., see Breusch and Pagan (1979, p.1290)).
A more general problem arises when the quantities \( \sigma_1^2, \ldots, \sigma_N^2 \) are all numerically different. Then, for \( L < N \), no exact homoscedastic grouping would exist. In this case, as a matter of practical compromise and at the expense of losing power, we may assume that these \( N \) quantities take, 'effectively', \( L(<N) \) different values. In other words we may suppose that, if an appropriate choice of \( L(<N) \) is made, each \( \sigma_i^2 \) may be conveniently approximated by one of \( L \) (unknown) values, say \( \sigma_{(1)}^2, \ldots, \sigma_{(L)}^2 \). We could then assume that (5.8) holds in an approximate sense and justify the use of (5.9) if we are able to classify the observations into 'nearly homoscedastic groups'.

Our interest is testing for homoscedasticity when we have no a-priori knowledge as to what an 'exact or nearly homoscedastic' grouping may be. Our proposal (see below) is motivated from maximum likelihood considerations which we now present.

Under the assumption of normal disturbances the log-likelihood is

\[
\ell(\beta, \sigma_1^2, \ldots, \sigma_N^2) = -\frac{N}{2} \log(2\pi) - \frac{1}{2} \sum_{i=1}^{N} \log \sigma_i^2 - \frac{1}{2} \sum_{i=1}^{N} \frac{1}{\sigma_i^2} (y_i - x_i' \beta)^2.
\]

(5.10)

For a given grouping \( G \), which achieves 'nearly homoscedastic groups', we have that \( \ell(\beta, \sigma_1^2, \ldots, \sigma_N^2) \) would be approximately equal to

\[ \ell(\beta, \sigma_{(1)}^2, \ldots, \sigma_{(L)}^2 | G) \],

where

We may of course use (5.9) with any specified grouping. However, if this - arbitrarily or otherwise - specified grouping does not achieve 'exact or near homoscedasticity' within groups, then the power of the test may be very small (some simulation results are reported in the next section).
\[
\ell(\beta, \sigma^2_1, \ldots, \sigma^2_L | G) = -\frac{N}{2} \log(2\pi) - \frac{1}{2} \sum_{h=1}^{L} N_h \log \sigma^2_h \\
- \frac{1}{2} \sum_{h=1}^{L} \frac{L}{\sigma^2_h} \sum_{i=1}^{N_h} (y_{hi} - \hat{x}'_h \beta)^2.
\]

(5.11)

If we cannot specify an 'exact or near homoscedastic grouping', then a test may be formed by computing the grouping \( G^* \) that maximizes \( \ell(\beta, \sigma^2_1, \ldots, \sigma^2_L | G) \) and using this to evaluate (5.9). This procedure is appealing since it treats \( G \) as an 'unknown parameter' in the approximation to (5.10). Yet, in (5.11), \( \beta \) and \( \sigma^2_1, \ldots, \sigma^2_L \) are also unknown and, to compute \( G^* \), we shall replace these, respectively, by the OLS estimator \( \hat{\beta} \) and the estimated group variances based on OLS residuals, namely, \( \hat{\sigma}^2_h = \bar{e}_h \) for \( h = 1, \ldots, L \). Then, (5.11) would reduce to

\[
\ell(\hat{\beta}, \hat{\sigma}^2_1, \ldots, \hat{\sigma}^2_L | G) = -\frac{N}{2} \log(2\pi) - \frac{1}{2} \sum_{h=1}^{L} N_h \log \bar{e}_h - \frac{N}{2}.
\]

(5.12)

In (5.12) the only 'unknown parameter' is \( G \), and we now find the grouping \( G^* \) such that this is maximized.

We note that in (5.12), \( -\frac{N}{2} \log(2\pi) \) and \( -\frac{N}{2} \) do not depend on \( G \), and that to maximize \( \ell(\hat{\beta}, \hat{\sigma}^2_1, \ldots, \hat{\sigma}^2_L | G) \) we should minimize

\[
\psi = \sum_{h=1}^{L} N_h \log \bar{e}_h.
\]

It appears that, to minimize \( \psi \), we would need to proceed by total enumeration of groupings. As it is evident from the combinatorics of Subsection 3.4.1 this is impractical. So, we present further derivations that will lead us to a computationally simpler procedure.
By taking a Taylor series expansion of \( \log \bar{e}_h \) around \( \bar{e} \), we obtain
\[
\log \bar{e}_h = \log \bar{e} + (\bar{e}_h - \bar{e})/\bar{e} - (\bar{e}_h - \bar{e})^2/(2\bar{e}^2) + R_h,
\]
where
\[
R_h = (\bar{e}_h - \bar{e})^3/(3\xi_h^3) \quad \text{with} \quad \xi_h \in (\bar{e}_h, \bar{e}).
\]
Using this expression for \( \log \bar{e}_h \) in \( \psi \) we obtain
\[
\psi = \sum_{h=1}^L N_h \log \bar{e}_h = N \log \bar{e} - [LM_H] + R,
\]
where \( LM_H \) is given by (5.9) and
\[
R = \sum_{h=1}^L N_h R_h = \sum_{h=1}^L N_h (\bar{e}_h - \bar{e})^3/(3\xi_h^3).
\]

We now look at the \( R \) term. We recall that \( \bar{e}_h \) and \( \bar{e} \) are averages of OLS residuals squared and therefore non-negative. Hence we see that \( N_h R_h = N_h (\bar{e}_h - \bar{e})^3/(3\xi_h^3) \) will be negative for \( \bar{e}_h < \bar{e} \), and positive for \( \bar{e}_h > \bar{e} \). So, \( R \) contains positive and negative terms \( (N_h R_h) \) of the same order of magnitude, which will 'offset' each other when added. Therefore, in \( \psi \) we expect \( R \) to be unimportant in relative magnitude, and conclude that the grouping minimizing \( \psi \) (i.e., maximizing \( \ell(b, \sigma^2_1, \ldots, \sigma^2_L | G) \)) is approximately the same as that which maximizes \( LM_H \).

This result motivates our proposal to use the maximum of \( LM_H \) (i.e., the \( LM_H \) statistic evaluated at the grouping that maximizes its value) to be denoted by \( LM^*_H \), as a test statistic for homoscedasticity. We can write \( LM_H \) in the form

\[ LM_H = \sum_{h=1}^L N_h \log \bar{e}_h - [LM_H] + R. \]

Our approach is similar to that used by Quandt (1958,1960) in testing for a shift in \( \beta \) when using time-series data. Quandt suggests partitioning the sample at the observation which maximizes the likelihood, and evaluating the corresponding likelihood-ratio test at this sample partition.
\[ \text{LM}_H = \frac{1}{2e^2} [\phi_1 - \phi_2] \]

where \( \phi_1 = \sum_{h=1}^{L} \sum_{i=1}^{N_h} (e_{hi} - \bar{e})^2 \) and \( \phi_2 = \sum_{h=1}^{L} \sum_{i=1}^{N_h} (e_{hi} - \bar{e})^2 \). To compute \( \text{LM}_H^* \) we need to find the grouping maximizing \( \text{LM}_H \). For this purpose \( \phi_1 \) and \( \bar{e} \) are fixed, so \( \text{LM}_H \) will be maximized when \( \phi_2 \) (the within group sum of squares of \( e_{hi} \)) is minimized. This grouping problem was discussed in Section 3.4, and we may use - for example - the results of Subsection 3.4.4.

In all, the computation of \( \text{LM}_H^* \) would require simple calculations. First, we would square the OLS residuals. Then, the 'group boundaries', say \( e^{(h)} \), could be computed as in Subsection 3.4.4; and the grouping sought would be given by defining group \( h \) as the set of observations \( i \), such that \( e_i \) is between \( e^{(h-1)} \) and \( e^{(h)} \) for \( i = 1, \ldots, N \), and \( h = 1, \ldots, L \). Having found this grouping, \( \text{LM}_H^* \) would be calculated from (5.9).

Obtaining the distribution of \( \text{LM}_H^* \) under \( H_0 \) by analytical methods, appears to be intractable. However, for a given set of regressors, this can be easily found by computer simulation. For each of \( n \) sets of \( N \) observations on a \( N(0,1) \) variable \( u_1 \), the procedure for computation of \( \text{LM}_H^* \) described in this section may be carried out (the distribution of \( \text{LM}_H^* \) is independent of \( \gamma_1 \), so one may set \( \gamma_1 = 1 \)). By choosing \( n \) large enough one may obtain as good an approximation as desired to the distribution of \( \text{LM}_H^* \), and hence determine the critical point of the test for a given significance level, or the probability of a Type I error for the computed value of \( \text{LM}_H^* \) from a

---

4 In the simulation study presented in Section 5.4, the \( \text{LM}_H^* \) statistic was also computed using the 'group boundaries' obtained from the square-root of the density procedure (see Subsection 3.4.4). This resulted in marginally lower power than when using the cubic-root procedure.
particular set of observations. In the next section the power of $LM^*_H$ is studied.

5.4 POWER OF FULLY-NON-CONSTRUCTIVE TESTS

Monte Carlo studies are available which consider the power of various fully-constructive, constructive and non-constructive tests for homoscedasticity (e.g., see Goldfeld and Quandt (1972, Chapter 3) and Buse (1980)). These studies have shown the merits of particular procedures and may guide us in the choice of test in a given regression problem. Frequently, we may not have sufficient a-priori information about the disturbance variances and, to test for homoscedasticity, we may need to use a fully-non-constructive test. In this Section we present a Monte Carlo study in an attempt to provide some insight on the relative power of the two fully-non-constructive tests $HW$ and $LM^*_H$.

We consider a linear model with four regressors, i.e., with $K = 4$. We set $X_{1i} = 1$ ($i = 1, \ldots, N$) and generated $X_2$ from a Normal, $X_3$ from a Uniform and $X_4$ from a $X^2_{10}$. (This model was also used in the simulation study of Section 4.5). Here, the regressors $X_2$, $X_3$ and $X_4$ were standardized to have population mean equal to 10 and variance equal to 25. The number of observations $N$ was set equal to 50.

So far, a maintained assumption in this Chapter has been disturbance normality. Therefore, in this study all disturbances were generated as pseudo-random normal variates (non-normal disturbances are considered in Section 5.6). Disturbances under the null hypothesis of
homoscedasticity, $H_0$, were generated with $\sigma^2_i = 25$ ($i = 1, \ldots, 50$).

Also, five alternative hypotheses, $H_a$, were considered. More specifically we have $H_0: u_i \sim N(0, 25)$ (i.e., $u \sim NH$), and $H_a: u_i \sim N(0, \sigma^2_i)$ (i.e., $u \sim NH$) with heteroscedasticity $\overline{H}$ given by

- $\overline{H1}: \sigma^2_i = 25$ ($i = 1, \ldots, 20$); $50$ ($i = 21, \ldots, 40$); $75$ ($i = 41, \ldots, 50$),

- $\overline{H2}: \sigma^2_i = 25$ ($i = 1, \ldots, 20$); $75$ ($i = 21, \ldots, 40$); $100$ ($i = 41, \ldots, 50$),

- $\overline{H3}: \sigma^2_i = 25 + .5x_{12}$ ($i = 1, \ldots, 50$)

- $\overline{H4}: \sigma^2_i = 25 + .4x_{13}$ ($i = 1, \ldots, 50$) and

- $\overline{H5}: \sigma^2_i = 25 + 25x_{14}$ ($i = 1, \ldots, 50$).

In each replication of the experiment we generated $N = 50$ variates $\nu_i$ from a $N(0,1)$ (as in Section 4.5 we used the subroutines described in Naylor et al. (1966)). To obtain disturbances under $H_0$ we set $u_i = 5\nu_i$. To obtain heteroscedastic disturbances we set $u_i = \sigma_i \nu_i$, where $\sigma_i$ was given, in turn, by one of the five alternatives $\overline{H1}, \ldots, \overline{H5}$. For each set of $u$'s we calculated the OLS residuals, and proceeded to evaluate all the tests statistics considered (see below). We repeated the experiment 250 times and - as in Section 4.5 - we computed the significance points empirically. The estimated power of each test was calculated by counting the number of times the value of the test statistic was greater than the corresponding ten per cent significance point, and dividing this by 250.

Our main interest is to study the power of fully-non-constructive tests. However, for comparison, we included in our simulations several constructive and non-constructive tests. These consist of various forms of the LM test (described in (i), (ii) and (iii) below), and
of the Goldfeld and Quandt (1965) test \(^5\) (described in (iv)). More specifically, we computed:

(i) The **constructive** LM test statistic as given in (5.9) with \( L = 3, N_1 = 20, N_2 = 20 \) and \( N_3 = 10 \); and with group 1 defined by the first 20 observations, group 2 by the next 20, and group 3 by the last 10. We denote this by \( \text{LM}_C \). This is asymptotically efficient for heteroscedastic structures such as \( \overline{H}_1 \) and \( \overline{H}_2 \).

(ii) We also computed the **constructive** LM test statistic as given in (5.6) with \( p = 2 \) and three definitions of \( z_{12} \). We denote by \( \text{LM}_{X^2}, \text{LM}_{X^3} \) and \( \text{LM}_{X^4} \) the statistics that correspond to using \( z_{12} = X^2_{12} \), \( z_{12} = X^3_{12} \), and \( z_{12} = X^4_{12} \). These are, respectively, asymptotically efficient for heteroscedastic structures such as \( \overline{H}_3 \), \( \overline{H}_4 \) and \( \overline{H}_5 \).

(iii) Another test, which may be thought of as a **non-constructive** LM test, was obtained by ordering the observations according to \( X^2_{12} \) and computing (5.9) with \( L = 10, N_1 = \ldots = N_{10} = 5 \); and with group 1 consisting of the observations with the 5 smallest values of \( X^2_{12} \), group 2 consisting of the observations with the next 5 smallest values - and so on. We

---

\(^5\) The Goldfeld and Quandt (1965) test has been reported by Buse (1980) to be "unambiguously best", among a series of tests. We thought it interesting to see how this performed under an incorrect ordering of the variances; and how the various forms of the LM test and the fully-non-constructive tests compared to it in terms of power.
denote this by $\text{LM}_{G H 2}$ and observe that this would be appealing for structures such as $\overline{H 3}$. We also ordered the observations with respect to $X_{13}^3$ (and $X_{14}$) and proceeded as above to form 10 groups. We then computed (5.9). In this case the resulting statistics are denoted by $\text{LM}_{G H 3}$ and $\text{LM}_{G H 4}$ which are, respectively, appealing for structures such as $\overline{H 4}$ and $\overline{H 5}$.

(iv) All the previous tests are based on the LM procedure. In addition, we included the non-constructive Goldfeld and Quandt (1965) F-test denoted by $GQ_{X^2}$. For this we ordered the observations according to $X_{12}^2$ and deleted the middle 10 observations. This test would be appealing for structures such as $\overline{H 3}$. Similarly, we denote by $GQ_{X^3}$ and $GQ_{X^4}$ the tests obtained when ordering the observations with respect to $X_{13}^3$ and $X_{14}$.

\[ \square \]

The estimated power of these ten constructive and non-constructive tests is given in Table 5.1. In the Table, we have underlined the estimated power of each test in the alternative(s) for which it is supposed to have a good performance (i.e., for the cases in which a test is computed under a correctly specified $\overline{H}$). A reason for producing this Table is to present the underlined quantities .556, .784, .416, .332 and .320; which may be thought of as 'upper bounds' to the power of the fully-non-constructive test $\text{LM}^*_H$, respectively for $\overline{H 1}$, $\overline{H 2}$, $\overline{H 3}$, $\overline{H 4}$ and $\overline{H 5}$. This observation comes from the presumption that $\text{LM}^*_H$ has less power than other more constructive tests, motivated from the same LM principle, when these are based on a heteroscedastic
TABLE 5.1

Estimated power using 250 replications

(K=4) (Regressors: \(X_1 = 1; X_2 \sim \text{normal}; X_3 \sim \text{uniform}; X_4 \sim X^2_{10}\)) (N=50)

<table>
<thead>
<tr>
<th></th>
<th>(LM_G)</th>
<th>(LM_{X2})</th>
<th>(LM_{X3})</th>
<th>(LM_{X4})</th>
<th>(LM_{GX2})</th>
<th>(LM_{GX3})</th>
<th>(LM_{GX4})</th>
<th>(GQ_{X2})</th>
<th>(GQ_{X3})</th>
<th>(GQ_{X4})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(H_0: \mu \sim NH):</td>
<td>.100</td>
<td>.100</td>
<td>.100</td>
<td>.100</td>
<td>.100</td>
<td>.100</td>
<td>.100</td>
<td>.100</td>
<td>.100</td>
<td>.100</td>
</tr>
<tr>
<td>(H_a: \mu \sim NH):</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\overline{H}_1)</td>
<td>.556</td>
<td>.072</td>
<td>.124</td>
<td>.116</td>
<td>.160</td>
<td>.208</td>
<td>.220</td>
<td>.096</td>
<td>.144</td>
<td>.068</td>
</tr>
<tr>
<td>(\overline{H}_2)</td>
<td>.784</td>
<td>.084</td>
<td>.116</td>
<td>.144</td>
<td>.184</td>
<td>.244</td>
<td>.254</td>
<td>.100</td>
<td>.124</td>
<td>.072</td>
</tr>
<tr>
<td>(\overline{H}_3)</td>
<td>.136</td>
<td>.732</td>
<td>.120</td>
<td>.188</td>
<td>.416</td>
<td>.192</td>
<td>.256</td>
<td>.828</td>
<td>.076</td>
<td>.044</td>
</tr>
<tr>
<td>(\overline{H}_4)</td>
<td>.180</td>
<td>.120</td>
<td>.544</td>
<td>.092</td>
<td>.196</td>
<td>.332</td>
<td>.192</td>
<td>.080</td>
<td>.620</td>
<td>.064</td>
</tr>
<tr>
<td>(\overline{H}_5)</td>
<td>.100</td>
<td>.096</td>
<td>.112</td>
<td>.548</td>
<td>.156</td>
<td>.140</td>
<td>.320</td>
<td>.100</td>
<td>.064</td>
<td>.604</td>
</tr>
</tbody>
</table>

structure that is correctly specified. As pointed out, the objective here is to study the power of fully-non-constructive tests. Nevertheless, from Table 5.1 interesting observations arise and we briefly comment on these.

Considering the non-underlined quantities in the Table, we note that all tests have relatively low power under a misspecified \(\overline{H}\) (e.g., for \(LM_G\) the power is .136, .180 and .100 respectively for \(\overline{H}_3, \overline{H}_4\) and \(\overline{H}_5\)). This means that when using these tests with incorrect a-priori information on \(\overline{H}\), there is a low probability of rejecting \(H_0\).

Comparing \(LM_{GX2}, LM_{GX3}\) and \(LM_{GX4}\) respectively with \(LM_{X2}, LM_{X3}\) and \(LM_{X4}\), for \(\overline{H}_3, \overline{H}_4\) and \(\overline{H}_5\), we note that the former only
have approximately half the power of the latter (compare .416, .332 and .320 respectively with .732, .544 and .548). This highlights the possible power gains, when using the $L^2_{\text{H}}$ test, of being able to correctly specify the Z-variables that determine $C_1^2$, rather than just an appropriate ordering of the variances.

The good performance of $GQ_{X2}$, $GQ_{X3}$ and $GQ_{X4}$ under correct a-priori information is interesting. These use, correspondingly, the same information as $LM_{GX2}$, $LM_{GX3}$ and $LM_{GX4}$, namely an ordering of the observations according to their variance. The results in Table 5.1 would then suggest that, under correct a-priori information, it is suboptimal to use tests such as $LM_{GX2}$, $LM_{GX3}$ or $LM_{GX4}$ (i.e. based on (5.9)) since, with the same a-priori information, the Goldfeld and Quandt (1965) GQ test would be substantially more powerful. In fact, $GQ_{X2}$, $GQ_{X3}$ and $GQ_{X4}$ gave, respectively, even higher power (.828, .620 and .604) than the LM tests $LM_{X2}$, $LM_{X3}$ and $LM_{X4}$ (.732, .544 and .548), which are 'more constructive' than GQ. An additional advantage of the GQ test over constructive LM tests is, of course, that its finite sample distribution is known to be an F.  

We now describe the fully-non-constructive tests considered. We computed $L^k_H$ with values $L = 2, 5$ and 10. This was done in order to evaluate the power effects of the number of groups chosen. We denote the resulting test statistics by $L^k_H(2)$, $L^k_H(5)$ and $L^k_H(10)$.

---

6 When using the theoretical ten per cent significance point for $GQ_{X2}$, $GQ_{X3}$ and $GQ_{X4}$, namely $F_{16,16}(.9) = 1.935$, we obtained - for $H_0$ - estimated power (significance level) respectively equal to .104, .094 and .100. This result is just one check for the appropriateness of the data generation process used in our simulation.
We also included the HW test (see (5.5)). The estimated power of these four tests is given in Table 5.2.

**TABLE 5.2**

Estimated power using 250 replications

(K=4) (Regressors: X₁ = 1; X₂ ~ normal; X₃ ~ uniform; X₄ ~ X₂²) (N=50)

<table>
<thead>
<tr>
<th>H₀: u ~ NH:</th>
<th>LM₉*(2)</th>
<th>LM₉*(5)</th>
<th>LM₉*(10)</th>
<th>HW</th>
</tr>
</thead>
<tbody>
<tr>
<td>H₁</td>
<td>.100</td>
<td>.100</td>
<td>.100</td>
<td>.100</td>
</tr>
<tr>
<td>H₂</td>
<td>.276</td>
<td>.280</td>
<td>.280</td>
<td>.128</td>
</tr>
<tr>
<td>H₃</td>
<td>.384</td>
<td>.364</td>
<td>.352</td>
<td>.132</td>
</tr>
<tr>
<td>H₄</td>
<td>.320</td>
<td>.332</td>
<td>.336</td>
<td>.304</td>
</tr>
<tr>
<td>H₅</td>
<td>.274</td>
<td>.276</td>
<td>.274</td>
<td>.272</td>
</tr>
<tr>
<td></td>
<td>.200</td>
<td>.212</td>
<td>.204</td>
<td>.244</td>
</tr>
</tbody>
</table>

By looking at the first three columns in Table 5.2 we see the choice of L does not seem to have a significant effect on the power of LM₉*. So, in practice, for more rapid computations we may set L = 2. We also note that LM₉* (with either L = 2, 5 or 10) has higher estimated power than HW for H₁, H₂, H₃ and H₄, and lower for H₅. In the light of these results it may appear that LM₉* is preferable. However, our view is to consider LM₉* as an alternative to HW; this is because for H₃ and H₄ both tests had similar power and because it may be argued that H₁ and H₂ do not commonly occur in practice.
The tests $\text{HW}$ and $\text{LM}_H^\times$ are equivalent in terms of computational load. $\text{HW}$ requires the inversion of a matrix of dimension $K(K+1)/2$, and $\text{LM}_H^\times$ requires ordered residuals. Yet, with the increased availability of modern computers, the computations to carry out these tests should not represent any real problem.

Comparing Tables 5.1 and 5.2 several interesting results arise.

We note that the power of $\text{LM}_H^\times$ (and $\text{HW}$) lies approximately between one half and one third of the power of the four constructive LM tests $\text{LM}_G^\times$, $\text{LM}_{X2}^\times$, $\text{LM}_{X3}^\times$ and $\text{LM}_{X4}^\times$, when the latter correctly specify $\bar{H}$ (compare the power of $\text{LM}_H^\times$ (or $\text{HW}$) corresponding to $\bar{H}_1$, $\bar{H}_2$, $\bar{H}_3$, $\bar{H}_4$ and $\bar{H}_5$ respectively with .556, .784, .732, .544 and .548). This highlights the value of being able to correctly specify an alternative $\bar{H}$ and hence use a constructive LM test. Considering $\text{LM}_H^\times$ and the three non-constructive LM tests $\text{LM}_{G_{X2}}^\times$, $\text{LM}_{G_{X3}}^\times$ and $\text{LM}_{G_{X4}}^\times$ (or the three non-constructive GQ tests $\text{GQ}_{X2}^\times$, $\text{GQ}_{X3}^\times$ and $\text{GQ}_{X4}^\times$), for $\bar{H}_3$, $\bar{H}_4$ and $\bar{H}_5$, we note that the fully-non-constructive test has, as expected, less power than the latter, when these use a correct ordering of the variances (as pointed out previously, upper bounds to the power of $\text{LM}_H^\times$ for $\bar{H}_3$, $\bar{H}_4$ and $\bar{H}_5$ are, respectively, .416, .332 and .320).

By definition, $\text{LM}_G^\times$ and $\text{LM}_H^\times$ are based on expression (5.9) (as are $\text{LM}_{G_{X2}}^\times$, $\text{LM}_{G_{X3}}^\times$ and $\text{LM}_{G_{X4}}^\times$). $\text{LM}_G^\times$ is computed with a grouping $G$ that achieves 'near' (in fact exact) within group homoscedasticity, when $\bar{H}$ is $\bar{H}_1$ or $\bar{H}_2$; and which may be regarded as an arbitrary
grouping for $\overline{H}_3$, $\overline{H}_4$ and $\overline{H}_5$. $LM^*_H$ uses the grouping that maximizes (5.9), say $G^*$. We observe that, for $\overline{H}_3$, $\overline{H}_4$ and $\overline{H}_5$, the power of $LM_G$ is .136, .180 and .100 whereas that of, say, $LM^*_H(2)$ is respectively .320, .274 and .200. So, for these three alternatives, there is a considerable power gain when using (5.9) with $G^*$, rather than an arbitrary $G$. The latter result shows the advantage of computing $LM^*_H$, when we have weak a-priori knowledge as to what an exact or near homoscedastic grouping may be. A similar result is obtained when comparing $LM^*_H$ with either $LM_{GX^2}$, $LM_{GX^3}$ or $LM_{GX^4}$.

Finally, it is interesting to note that, when using the constructive tests $LM_{X^2}$, $LM_{X^3}$ and $LM_{X^4}$ or the non-constructive tests $GQ_{X^2}$, $GQ_{X^3}$ and $GQ_{X^4}$ under incorrect a-priori information on $H$, there would be a considerable power loss relative to the use of the fully-non-constructive tests. For example, the power for $LM^*_H(2)$ would be .276, .384, .320 and .274 respectively for $\overline{H}_1$, $\overline{H}_2$, $\overline{H}_3$ and $\overline{H}_4$ (see Table 5.2); whereas that for, say $GQ_{X^4}$, would be .068, .072, .044 and .064 (see Table 5.1).

The above findings are specific to our choice of $K$, $N$ and regressors. To have a less partial result, we carried out our computations with variations in all these directions, and found that the qualitative conclusions did not vary from those stated here. The results of some of these additional simulations are given in Appendix C (see page 289). The findings of our studies on the power of homoscedasticity tests (under normal disturbances) can be summarized as follows.
(1) When using the constructive and non-constructive tests LM\(_H\) and GQ with incorrect a-priori information on \(\overline{H}\), there is a low probability of rejecting \(H_0\).

(2) When using a constructive or non-constructive test under correct a-priori information on \(\overline{H}\), the preferred test is the GQ test.

(3) When using a constructive or non-constructive test under correct a-priori information on \(\overline{H}\), there is a considerable power gain relative to the use of the fully-non-constructive tests.

(4) When using a constructive or non-constructive test under incorrect a-priori information on \(\overline{H}\), there is a considerable power loss relative to the use of the fully-non-constructive tests.

5.5 JOINT TESTS FOR NORMALITY AND HOMOSCEDASTICITY

In Chapter 4 we suggested a test for disturbance normality, derived under the maintained hypothesis of homoscedasticity. In fact, all the disturbance normality tests mentioned in that Chapter were developed under this maintained hypothesis. Similarly, in Section 5.3, we have suggested a test for homoscedasticity which - like many other homoscedasticity tests - was proposed under the maintained hypothesis of normality.

As will be shown in the next section, the use of normality tests under the incorrect assumption of homoscedasticity (\(H\)), or the use of homoscedasticity tests under the incorrect assumption of normality
may lead the investigator to inappropriate conclusions. Therefore, unless our a-priori information is sufficiently strong so that we are confident to assume \( H \) and hence test for \( N \) (e.g., as in Section 4.4), or to assume \( N \) and hence test for \( H \) under the maintained hypothesis of normality (e.g., with the Goldfeld and Quandt (1965) test) we should consider using either

(I) Joint tests for disturbance \( N \) and \( H \), or

(II) Robust tests for \( H \) (i.e., tests that do not require \( N \)).

Regarding (I), we now extend the procedure used in Sections 4.3 and 4.4 to derive a joint test for \( N \) and \( H \). (We comment on (II) at the end of this section). As in Section 4.4 we assume \( f(u_i^) \) to be a member of the Pearson Family. Now, however, we allow \( u_1, \ldots, u_N \) to have differing variances \( \sigma_1^2, \ldots, \sigma_N^2 \); so, we define \( f(u_i) \) in terms of the three parameters \( c_{oi}, c_{li} \) and \( c_{2i} \). More formally, we set \( f(u_i) \) equal to

\[
\begin{align*}
f(u_i) &= \exp \left[ \frac{c_{li}-u_i}{c_{oi}-c_{li}u_i+c_{2i}u_i^2} \right] \frac{du_i}{\int_{-\infty}^{\infty} \exp \left[ \frac{c_{li}-u_i}{c_{oi}-c_{li}u_i+c_{2i}u_i^2} \right] du_i} \\
&= \frac{\exp \left[ \frac{c_{li}-u_i}{c_{oi}-c_{li}u_i+c_{2i}u_i^2} \right] du_i}{\int_{-\infty}^{\infty} \exp \left[ \frac{c_{li}-u_i}{c_{oi}-c_{li}u_i+c_{2i}u_i^2} \right] du_i}, \quad (5.13)
\end{align*}
\]

where we have \( -\infty < u_i < \infty \). (Note equation (4.2) and allow \( c_o, c_l \) and \( c_2 \) to vary for each \( i \)).

We know - from the properties of the Pearson Family - that \( f(u_i) \) has a single mode and this occurs at \( c_{li} \). For our purposes we can set \( c_{li} = c_l \) for \( i = 1, \ldots, N \). Also, we may show that
E[u_1^2] = \sigma_i^2 = \frac{c_{oi}}{(1-3c_{21})} \quad (e.g., \text{see Kendall and Stuart (1969, p.149)}),

and that \( f(u_i) \) reduces to a Normal with mean zero and variance equal to \( c_{oi} \) when \( c_1 = c_{21} = 0 \). For convenience, we shall parameterize our density \( f(u_i) \) so that \( c_{21} = c_2 \), and allow \( c_{oi} \) to vary with \( i \) in order to introduce disturbance heteroscedasticity. In addition, we shall assume, for now, that we have a-priori information specifying that \( \sigma_i^2 \) is linearly related to a set of \( p-1 \) fixed variables \( Z_2, \ldots, Z_p \) (satisfying the conditions set out by Amemiya (1977, p.366)). We can then write \( c_{oi} = \sigma^2 + z_i^*a^* \) for \( i = 1, \ldots, N \), where \( z_i^* \) is a 1 by \( p-1 \) vector of observations on \( Z_2, \ldots, Z_p \); \( a^* \) is a vector of unknown parameters; and \( \sigma^2 \) is an unknown scalar.

After substitution of \( c_{oi} = \sigma^2 + z_i^*a^* \), \( c_{i1} = c_1 \) and \( c_{i2} = c_2 \) into (5.13), the log-likelihood for our \( N \) observations would be given by

\[
\ell(\beta, \sigma^2, c_1, c_2, a^*) = -\sum_{i=1}^{N} \log \left[ \int_{-\infty}^{\infty} \exp \left( -\frac{c_{i} - u_1}{\sigma^2 + z_i^*a^* - c_1u_1 + c_2u_1^2} \right) du_1 \right]
+ \sum_{i=1}^{N} \left[ -\frac{c_{i} - u_1}{\sigma^2 + z_i^*a^* - c_1u_1 + c_2u_1^2} \right]. \tag{5.14}
\]

To derive a joint test for disturbance \( N \) and \( \sigma^2 \), say \( NH \), we again use the Lagrange multiplier procedure described in Section 4.2. We now set \( \theta_1 = (\beta', \sigma^2)' \) and \( \theta_2 = (c_1, c_2, a^*)' \), and note our interest is to test \( H_0: \theta_2 = (c_1, c_2, a^*)' = 0 \).

It is shown in Proposition 1 in Appendix D (see page 294) that in this case the LM test statistic is given by
\[ \text{LM}_{NH} = \left\{ N \left[ \frac{\hat{\mu}_3^2}{(6\hat{\mu}_2^3)} + \left( \frac{(\hat{\mu}_4^2/\hat{\mu}_2^3)-3}{24} \right) \right] \right. \\
+ \left. N \left[ 3\hat{\mu}_1^2/(2\hat{\mu}_2^3) - \hat{\mu}_3^2/\hat{\mu}_2^2 \right] \right\} \\
+ \left\{ \frac{1}{2} \left[ \hat{f}_1^2 (Z^* MZ^*)^{-1} \right] \right\}, \]

where, as before \( \hat{u}_j = \frac{1}{N} \sum_{i=1}^N u_{ij}/N \), and \( \hat{f} = (\hat{f}_1, \ldots, \hat{f}_N)' \) with \( \hat{f}_1 = (\hat{u}_1^2/\hat{\mu}_2^2) - 1 \). We have written \( \text{LM}_{NH} \) with the suffix \( NH \) to indicate this refers to a disturbance normality and homoscedasticity test. Now the dimension of \( \theta_2 \) is \( p+1 \) so, by the results of Section 4.2, we have that \( \text{LM}_{NH} \) is asymptotically distributed \( \chi^2 \) as \( \chi^2(p+1) \) under \( H_0 \).

In (5.15) the first term in \{ \} equals the LM disturbance normality test \( \text{LM}_N \) suggested in Section 4.4 (see (4.5)), which is \( \rho \) as \( \chi^2_2 \) under \( H_0 \). The second term in \{ \} is identical to the LM homoscedasticity test \( \text{LM}_H \) suggested by Godfrey (1978c) and Breusch and Pagan (1979) (see (5.7)), which is \( \rho \) as \( \chi^2(p-1) \) under \( H_0 \). Therefore, it is interesting that these may be combined, as above, to obtain an asymptotically efficient test for disturbance \( NH \).

Furthermore, because we have \( \text{LM}_{NH} = \text{LM}_N + \text{LM}_H \) and because (under \( H_0 \)) \( \text{LM}_{NH} \) is \( \rho \) as \( \chi^2 \) with degrees of freedom equal to the sum of the degrees of freedom of the asymptotic \( \chi^2 \) distributions of \( \text{LM}_N \) and \( \text{LM}_H \), it follows that our result proves the asymptotic independence (under \( H_0 \)) of the LM normality and homoscedasticity tests.

7 In fact, this independence of the 'individual parts' of a joint LM test, extends to the serial independence and functional form components (see Jarque and Bera (1980) and Bera and Jarque (1981b)).
8 An interesting interpretation of the 'additivity' property of \( \text{LM}_{NH} \) is obtained by considering the Locally Equivalent Alternative models of Godfrey and Wickens (1981a) (on this point see Bera and Jarque (1981b)).
9 This finding may not apply to other combinations of normality and homoscedasticity tests.
To obtain large sample significance points for (5.15), we would look at tables of the \( \chi^2 \) distribution. For determination of finite sample significance points we could resort to computer simulation, generating disturbances under \( H_0 \) from \( N(0,1) \). (Note that neither \( L_{NH}^M \) nor \( L_{NH}^H \) depend on the value of \( \sigma^2 \), so we may set \( \sigma^2 = 1 \) in the simulations). In the derivation of (5.15) it was assumed that we were able to specify \( \sigma_i^2 = \sigma^2 + z_i^* \alpha_i^* \). This means we can think of \( L_{NH}^M \) as a Constructive Joint Test for disturbance \( NH \).

In applied work we may not have sufficient a-priori information to specify a heteroscedastic structure. Nonetheless, we can use the results of Section 5.3 to obtain a Fully-Non-Constructive Joint Test for \( NH \). We proceed by defining \( \sigma_i^2 \) as in (5.8) and observe that, in this case, \( L_{NH}^M \) defined in (5.15) would reduce to

\[
L_{NH}^M = L_N^M + \frac{1}{2e} \left[ \sum_{h=1}^{L} \frac{N_h}{h} (\bar{e}_h - \bar{e})^2 \right]. \tag{5.16}
\]

(Note the second term arises because \( L_{NH}^H \) would now be given by (5.9)). By the argument presented in Section 5.3, under uncertainty of a 'nearly homoscedastic grouping' \( G \), we would find the grouping \( G^* \) that maximizes \( L_{NH}^M \) as defined in (5.16). The 'conditional statistic' \( L_{NH}^M \) (conditional on a given \( G \)) evaluated at the grouping \( G^* \) is denoted by \( L_{NH}^{M*} \). It is interesting to see that \( L_N^M \) does not depend on \( G \) (i.e., whatever \( G \) is, \( L_N^M \) remains the same) and so

\[
L_{NH}^{M*} = L_N^M + L_H^M, \tag{5.17}
\]

where \( L_H^M \) is defined and may be computed as in Section 5.3. Obtaining the finite sample distribution of \( L_{NH}^{M*} \) by analytical methods appears
to be intractable (as is also the case for $LM_N$ and $LM^*_H$). Once more our suggestion is to resort to computer simulation. For this we see that neither $LM_N$ nor $LM^*_H$ depend on $\sigma^2$ and, so, we may generate pseudo-random variates from a $N(0,1)$.

To end this section, we comment on the use of two recently suggested tests for $H$ that do not require disturbance $N$ (i.e., robust tests for $H$). These are the tests of White (1980c) and Payen (1980). The White (1980c) test, $HW$, is a fully-non-constractive test. This was summarized in Section 5.2 (see (5.5)), and its power for normal disturbances was studied in Section 5.4. Under $H$, $HW$ would be asymptotically distributed as $\chi^2_{(K(K+1)/2)}$. The Payen (1980) test is a constructive test, specifying $\sigma^2_1 = g(\sigma^2 + \alpha^*\alpha^*)$ and is given by

$$JP = \frac{\hat{\mu}_2^2}{\hat{\mu}_4 - \hat{\mu}_2^2} LM_H,$$  \hspace{1cm} (5.18)

where $LM_H$ is the test suggested by Godfrey (1978c) and Breusch and Pagan (1979) (see (5.6)). Under $H_0$: $\alpha^* = 0$, $JP$ would be asymptotically distributed as $\chi^2_{(p-1)}$.

For both $HW$ and $JP$, only the asymptotic distribution under homoscedasticity is known. Of course, we may use asymptotic distributions to obtain significance points. However, this may lead to tests whose size is substantially different from the presupposed. An alternative is to use computer simulation. For this, we need to specify

\footnote{In the simulation study of Section 5.6 we have $N = 50$ and we obtained, under homoscedasticity, estimated significance levels for $HW$ equal to .189, .372, .132, .980 and .696, respectively for disturbances generated from a Normal, Students-t, Beta, Lognormal and Gamma distributions, when using the asymptotic 10 per cent $\chi^2$ significance point. For $JP$ the values were .067, .161, .121, .080 and .029.}
a disturbance distribution and, once this is done, the distribution chosen becomes a part of the null hypothesis. Therefore, if we generated disturbances as pseudo-random normal variates, we would obtain, when using HW, a fully-non-constructive joint test for NH and, when using JP, a constructive joint test for NH. The power of HW, JP and other one and two-directional (i.e. joint) tests is studied in the next section for both normal and non-normal disturbances.

5.6 POWER OF NORMALITY AND HOMOSCEDASTICITY TESTS

In Section 4.5 we reported results on the power of normality tests under the maintained hypothesis of homoscedasticity. Similarly, in Section 5.4 we reported results on the power of homoscedasticity tests under the maintained hypothesis of normality. In this section we present results on the power of normality, homoscedasticity and joint tests under a wider range of disturbance distributional assumptions.

We consider (as in Section 5.4) a linear model with \( N = 50, K = 4, X_1 = 1 \), and generate \( X_2 \) from a Normal, \( X_3 \) from a Uniform and \( X_4 \) from a \( \chi^2_{10} \). The regressors \( X_2, X_3 \) and \( X_4 \) were standardized to have population mean equal to 10 and variance equal to 25.

We are concerned with the null hypothesis \( H_0: u \sim NH \). Departures from \( H_0 \) may arise because \( u \) is non-normal but homoscedastic, say \( u \sim NH \); because \( u \) is normal but heteroscedastic, say \( u \sim NH \); or because \( u \) is both non-normal and heteroscedastic, i.e., \( u \sim NH \). Regarding non-normal alternatives we consider four distributions: Students \( t(5) \), Beta \((3,2)\), Lognormal \((0,1)\) and Gamma \((2,1)\) (say \( t, Be, Log \) and \( Ga \) for short), which cover a wide range of skewness and
kurtosis measures (see Shapiro, Wilk and Chen (1968, p.1346)). These four distributions were standardized to have population mean equal to zero and variance equal to one. Regarding heteroscedasticity, we use the five structures $H_1, H_2, H_3, H_4$ and $H_5$ defined in Section 5.4. In all, we generate disturbances from 29 alternative distributions consisting of 9 one-directional (4 of the form $NH$ and 5 of the form $NH$) and 20 two-directional departures from $H_0$ (see first column in Table 5.3). Disturbances from a $N(0,25)$ were also generated in order to compute the empirical ten per cent significance point for each test.

The tests considered are the following:

(i) The two normality tests $LM_N$ and $\hat{W}$ (see Sections 4.4 and 4.5);

(ii) The homoscedasticity tests $LM_G, LM_{X2}, LM_{X3}$ and $GQ_{X4}$ described in Section 5.4;

(iii) The constructive joint test based on (5.18) and defined by $JP_{X2} = (2\hat{\mu}_2/(\hat{\mu}_4 - \hat{\mu}_2))LM_{X2}$, and those based on (5.15) and defined by $LM_{NG} = LM_N + LM_G$,

$$LM_{NX2} = LM_N + LM_{X2} \quad \text{and} \quad LM_{NX3} = LM_N + LM_{X3}; \quad \text{and, finally,}$$

(iv) The fully-non-constructive joint tests $HW$ (see (5.5)) and that based on (5.17) which is defined by $LM^*_{NH} = LM_N + LM^*_H(5)$, where $LM^*_H(5)$ is the statistic $LM^*_H$ with $L = 5$ (see Section 5.4).

In every experiment, we generate $N = 50$ pseudo-random variates $v_i$ from a given standardized distribution (i.e., with population mean equal zero and variance equal 1); we compute $u_i = \sigma_i v_i$ (where $\sigma_i$
is either equal to 5 or takes the value from one of $H_1, \ldots, H_5$; obtain the OLS residuals; compute the twelve test statistics considered; and see whether $H_0$ is rejected by each test. We carried out 250 replications. The estimated power (obtained by counting the number of times $H_0$ was rejected and dividing this by 250) for everyone of the 29 distributions is given in Tables 5.3 and 5.4.

In Table 5.3 we present the estimated power for the one-directional tests, i.e., tests for either $N$ or $F$ (see categories (i) and (ii)). In this Table we have underlined, for each test, the alternative(s) for which it is supposed to have a good performance. We also note that we have starred the quantities where the tests would be applied under an invalid 'maintained assumption'. More specifically, the power of tests for $N$ is starred when $u$ is $N$ but $H$; and the power of tests for $H$ is starred when $u$ is $H$ but $N$ or when the specified $H$ is incorrect.

We first consider the performance of the normality tests $LM_N$ and $\hat{W}'$. As found in Section 4.5 these are (for $N = 50$) equivalent in power for the alternatives $N \bar{H}$. Here we observe that both have undesirably high power for $N \bar{H}$ when $\bar{H}$ is $H_2$, say $N \bar{H}(H_2)$. For example, the power of $LM_N$ in this case is .236 which may lead one to conclude that $u$ is $\bar{N}$, when in fact it is $N$ but $\bar{H}$. This highlights the possible consequences of using a normality test under the incorrect maintained assumption of homoscedasticity. It is interesting to note that, for a given alternative $\bar{N}$, the power of $LM_N$ and $\hat{W}'$ under $H$, is approximately the same as under $\bar{H}$. For
TABLE 5.3

Estimated power using 250 replications

(K=4) (Regressors: $X_1 = 1$; $X_2 \sim \text{normal}$; $X_3 \sim \text{uniform}$, $X_4 \sim X_1^2$) (N=50)

<table>
<thead>
<tr>
<th></th>
<th>$LM_N$</th>
<th>$\wedge'$</th>
<th>$LM_G$</th>
<th>$LM_{X2}$</th>
<th>$LM_{X3}$</th>
<th>$GQ_{X4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NH</td>
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<td>.100</td>
<td>.100</td>
<td>.100</td>
<td>.100</td>
<td>.100</td>
</tr>
<tr>
<td>One-directional</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>NH: $t$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Be</td>
<td>.424</td>
<td>.408</td>
<td>.212*</td>
<td>.204*</td>
<td>.168*</td>
<td>.156*</td>
</tr>
<tr>
<td>Log</td>
<td>.132</td>
<td>.096</td>
<td>.040*</td>
<td>.084*</td>
<td>.052*</td>
<td>.088*</td>
</tr>
<tr>
<td>Ga</td>
<td>1.000</td>
<td>1.000</td>
<td>.860*</td>
<td>.600*</td>
<td>.576*</td>
<td>.396*</td>
</tr>
<tr>
<td>NH: $H_1$</td>
<td>.972</td>
<td>.972</td>
<td>.420*</td>
<td>.304*</td>
<td>.340*</td>
<td>.220*</td>
</tr>
<tr>
<td>$H_2$</td>
<td>.192*</td>
<td>.180*</td>
<td>.556</td>
<td>.072*</td>
<td>.124*</td>
<td>.068*</td>
</tr>
<tr>
<td>$H_3$</td>
<td>.236*</td>
<td>.240*</td>
<td>.784</td>
<td>.084*</td>
<td>.116*</td>
<td>.072*</td>
</tr>
<tr>
<td>$H_4$</td>
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<td>.164*</td>
<td>.136*</td>
<td>.732</td>
<td>.120*</td>
<td>.044*</td>
</tr>
<tr>
<td>$H_5$</td>
<td>.136*</td>
<td>.132*</td>
<td>.180</td>
<td>.120*</td>
<td>.544</td>
<td>.064*</td>
</tr>
</tbody>
</table>

| Two-directional|        |           |        |           |           |           |
| NH: $t$, $H_1$ |        |           |        |           |           |           |
| $t$, $H_2$     | .500   | .528      | .580   | .208*     | .200*     | .120*     |
| $t$, $H_3$     | .560   | .580      | .712   | .204*     | .196*     | .120*     |
| $t$, $H_4$     | .516   | .524      | .276*  | .632      | .240*     | .060*     |
| $t$, $H_5$     | .496   | .504      | .308*  | .252*     | .500      | .084*     |

| Be, $H_1$      | .468   | .480      | .240*  | .216*     | .200*     | .508      |
| Be, $H_2$      | .108   | .124      | .504   | .060*     | .076*     | .064*     |
| Be, $H_3$      | .132   | .116      | .756   | .060*     | .080*     | .052*     |
| Be, $H_4$      | .124   | .116      | .080*  | .752      | .088*     | .020*     |
| Be, $H_5$      | .104   | .112      | .100*  | .128*     | .516      | .056*     |

| Log, $H_1$     | .084   | .128      | .048*  | .104*     | .064*     | .608      |
| Log, $H_2$     | 1.000  | 1.000     | .864   | .604*     | .580*     | .352*     |
| Log, $H_3$     | .996   | .996      | .864   | .616*     | .556*     | .344*     |
| Log, $H_4$     | 1.000  | 1.000     | .868*  | .632      | .584*     | .328*     |
| Log, $H_5$     | 1.000  | 1.000     | .856*  | .596*     | .644      | .376*     |

| Ga, $H_1$      | .952   | .980      | .620   | .304*     | .336*     | .180*     |
| Ga, $H_2$      | .944   | .972      | .716   | .316*     | .348*     | .184*     |
| Ga, $H_3$      | .976   | .976      | .456*  | .572      | .376*     | .132*     |
| Ga, $H_4$      | .968   | .984      | .460*  | .316*     | .496      | .180*     |
| Ga, $H_5$      | .968   | .980      | .456*  | .268*     | .352*     | .584*     |
example, with $LM_N$ power is .972 for $\bar{NH}(Ga)$ and .952, .944, .976, .968 and .968 respectively for $\bar{NH}(Ga,H1)$, $\bar{NH}(Ga,H2)$, $\bar{NH}(Ga,H3)$, $\bar{NH}(Ga,H4)$ and $\bar{NH}(Ga,H5)$.

Now we consider the performance of the homoscedasticity tests $LM_G$, $LM_{X2}$, $LM_{X3}$ and $GQ_{X4}$.

We first look at one-directional departures of the form $\bar{NH}$ (Departures of the form $\bar{NH}$ were already discussed in Section 5.4).

We observe, for example, that under $\bar{NH}(Log)$ the power is .860, .600, .576 and .396, respectively for $LM_G$, $LM_{X2}$, $LM_{X3}$ and $GQ_{X4}$.

This large undesirable power may lead one to infer - with high probability - that $u$ is $\bar{H}$ when in fact it is $H$ but $\bar{N}$; showing the possible effects of using a homoscedasticity test under the incorrect maintained assumption of normality.

Considering two-directional departures, we find that the power of these tests, under $N$, is very nearly the same as under $\bar{N}$, provided the $H$ structure is correctly specified. To see this we compare, for each test, the corresponding underlined quantities. For example, with $\bar{H}4$ the power of $LM_{X3}$ is .544 under $N$ and .500, .516, .644 and .496 respectively under $t$, Be, Log and Ga. This result does not hold if the $H$ structure is incorrectly specified. For example, with $\bar{H}1$ the power of $LM_{X3}$ is .124 under $N$ and .200, .076, .580 and .336 respectively under $t$, Be, Log and Ga.

In all, the results show that, although the power of normality tests may be robust in the presence of $\bar{H}$, their use may lead to inappropriate conclusions if $u$ is $\bar{NH}$. A similar consideration applies to
homoscedasticity tests. In addition, for homoscedasticity tests, if 
$u$ is $\bar{NH}$, then the power of the tests may be very small if used under 
an incorrectly specified $\bar{H}$. □

We now look at Table 5.4 where we present the power of the included two-
directional tests (see categories (iii) and (iv)). First we discuss 
the performance of the constructive joint tests $LM_{NG}$, $LM_{NX2}$, $JP_{X2}$ and 
$LM_{NX3}$.

For one-directional departures from $H_0$ of the form $\bar{NH}$, we 
observed the joint tests based on the LM principle have approxi-
ately the same power and that, in these alternatives, $JP_{X2}$ 
has comparatively low power (e.g., for $\bar{NH}(t)$ the power is .432, 
.422, and .428 respectively for $LM_{NG}$, $LM_{NX2}$ and $LM_{NX3}$; and 
.132 for $JP_{X2}$).

For $\bar{HH}$, we see that the use of $LM_{NG}$, $LM_{NX2}$, $JP_{X2}$ and $LM_{NX3}$ 
under an incorrectly specified $\bar{H}$ structure would result in 
little power; particularly for $JP_{X2}$ (e.g., when $u$ is $\bar{NH}(H3)$ 
the power is equal to .112, .124, .088 and .104 correspondingly for 
$LM_{NG}$, $LM_{NX2}$, $JP_{X2}$ and $LM_{NX3}$).

For $\bar{NH}$ we see that - as expected - each of $LM_{NG}$, $LM_{NX2}$ and 
$LM_{NX3}$ have (in turn) higher power than the other two when using 
a correctly specified $\bar{H}$. For instance, the power for $LM_{NX2}$ is 
.704 when $u$ is $\bar{NH}(t,H3)$, and that for $LM_{NG}$ and $LM_{NX3}$ is, 
respectively, .528 and .524. Also, we find that $JP_{X2}$ has con-
siderably less power than the LM constructive joint tests. In 
particular, comparing $JP_{X2}$ with $LM_{NX2}$ we obtain that for the 
'correctly specified' $\bar{H}$, i.e. $H3$, the power of $JP_{X2}$ may be
**TABLE 5.4**

Estimated power using 250 replications

(K=4) (Regressor: \(X_1 = 1; X_2 \sim \text{normal}; X_3 \sim \text{uniform}; X_4 \sim \chi^2_{10}\)) (N=50)

<table>
<thead>
<tr>
<th></th>
<th>(\text{LM}_{NG})</th>
<th>(\text{LM}_{NX2})</th>
<th>(\text{JP}_{X2})</th>
<th>(\text{LM}_{NX3})</th>
<th>(\text{HW})</th>
<th>(\text{LM}_{NP})</th>
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<td>.100</td>
<td>.100</td>
<td>.100</td>
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<tr>
<td><strong>One-directional</strong></td>
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<td></td>
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<td></td>
</tr>
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<td>.428</td>
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<td>.576</td>
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<td>1.000</td>
<td>.956</td>
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</tr>
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<td>.252</td>
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<td>H2</td>
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<td>.036*</td>
<td>.236*</td>
<td>.132</td>
<td>.344</td>
</tr>
<tr>
<td>H3</td>
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<td>.696</td>
<td>.704</td>
<td>.180*</td>
<td>.304</td>
<td>.324</td>
</tr>
<tr>
<td>H4</td>
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<td>.180*</td>
<td>.096*</td>
<td>.424</td>
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considerably lower (e.g., when \( u = \overline{NH}(G_{\alpha}, H_3) \) the powers are equal to .988 and .392 respectively for \( LM_{NX2} \) and \( JP_{X2} \). This result would encourage the use of the constructive joint LM tests rather than the Payen (1980) test.

Now we compare the power of one and two-directional LM tests.

For example, we look at the power of \( LM_{NX2} \), and compare this with that of \( LM_N \) and \( LM_{X2} \). We observe that under \( \overline{NH} \), say \( \overline{NH}(t) \), the power of \( LM_{NX2} \) is .422 (see Table 5.4) and that of \( LM_N \) is .424 (see Table 5.3). Similarly, under \( \overline{NH} \), say \( \overline{NH}(H_3) \), power of \( LM_{NX2} \) is .696 and that of \( LM_{X2} \) is .732. Therefore, we see that if the departure from \( H_0 \) is one-directional, and we used the joint test \( LM_{NX2} \) (i.e. 'over-specified' the testing procedure) we would have very little power loss with respect to the use of the optimal LM one-directional test. If the departure was two-directional, then a considerable power gain may be obtained when using \( LM_{NX2} \) rather than any of \( LM_N \) or \( LM_{X2} \).

For instance, under \( \overline{NH} \), say \( \overline{NH}(t,H_3) \), the power of \( LM_{NX2} \) is .704 and that of \( LM_N \) and \( LM_{X2} \) is, respectively, .516 and .632.

(The same results are found for \( LM_{NG} \) and \( LM_{NX3} \).)

Finally, we look at the fully-non-constructive joint tests \( HW \) and \( LM^*_{NH} \).

\[11\] See Subsection 5.7.3 for some comments on a multiple comparison procedure based on the simultaneous use of the one-directional tests \( LM_N \) and \( LM_H \).
We find that $\text{LM}_{{NH}}^k$ has higher power than HW in all except 4
$[\overline{NH}(H4), \overline{NH}(H5), \overline{NH}(Be_3H_4), \overline{NH}(Be_5H_5)]$ of the 29 alternatives (see last two columns in Table 5.4). This is because $\text{LM}_{{NH}}^k$ has a component that efficiently tests for normality. Also, comparing the power of $\text{LM}_{{NH}}^k$ with the 'more constructive' joint tests, we observe this is quite good, with the exception of the Beta alternatives. To see this compare, for a given row, the power of $\text{LM}_{{NH}}^k$ with the maximum value in that row.

We carried out additional simulation studies (varying K, N and the choice of regressors) and found that the qualitative findings did not vary substantially from those stated here. (Some of the results are reported in Appendix E in page 300). On the basis of our studies we can conclude as follows. In testing for disturbance normality and homoscedasticity:

(1) If a-priori information strongly suggests that disturbances are homoscedastic, the recommended normality test is $\text{LM}_N^k$. However, if the a-priori information is not strong, then the use of a joint test is recommended.

(2) If a-priori information strongly suggests that disturbances are normal, then the recommended homoscedasticity tests are the GQ test, and the fully-non-constructive tests $\text{LM}_H^k$ and HW (depending on information available on $\bar{H}$). However, if the a-priori information regarding disturbance normality is weak, then the use of a joint test is recommended.
(3) When using a two-directional test, the use of the constructive joint test $\text{LM}^\text{NH}$ is supported when good a-priori information on $H$ is available.

(4) When using a two-directional test, the use of $\text{LM}^\text{NH}$ is supported when faced with uncertain knowledge as to the possible heteroscedastic structure.

5.7 ANALYSIS OF MODELS WITH NON-NORMAL AND/OR HETEROSCEDASTIC DISTURBANCES

We have described several disturbance normality and homoscedasticity tests. Also, we have provided some evidence on their power properties. In each case (under the appropriate application of a given test) acceptance of a null hypothesis means that classical econometric analysis may follow.\textsuperscript{12} We now briefly comment on procedures for the analysis of the model, that may be applied when a null hypothesis has been rejected. In Subsection 5.7.1, we consider analysis after the use of a one-directional test; and, in Subsection 5.7.2, analysis after the application of two-directional tests. To end, in Subsection 5.7.3, we discuss analysis after the simultaneous use of one-directional tests.

5.7.1 Analysis After Using a One-directional Test

When we have strong a-priori information that $u$ is $H$, we may decide to test for $N$ under the maintained hypothesis that $u$ is $H$. In this case, if $N$ is rejected then we may proceed as described in Section 4.6.

\textsuperscript{12} See footnote 5 in Chapter 4.
Similarly, if strong a-priori information states that \( u \) is \( N \), we may test for \( H \) under the maintained hypothesis that \( u \) is \( N \). In this case, the subsequent analysis of the model would depend on the type of homoscedasticity test carried out:

(i) If a fully-constructive test was used, estimates of all parameters would be available, and large-sample \( t \) and \( F \) tests for \( \beta \) could be readily applied;

(ii) If a constructive test was used, then we may re-estimate the model by MLE and obtain large-sample \( t \) and \( F \) tests;

(iii) If a non-constructive test was used, we may divide the observations by the square root of the variable used to order the variances (assuming it to be non-negative) and re-estimate the model with this transformed data; and finally,

(iv) If a fully-non-constructive test was used, then we may estimate \( \beta \) by the OLS estimator \( b \) and use as an estimate of its VCM the heteroscedasticity-consistent VCM estimator suggested by White (1980c, p.820), namely,

\[
\hat{\Sigma}[b] = (X'X)^{-1} \left( \sum_{i=1}^{N} \frac{u_i^2}{x_i'x_i} \right) (X'X)^{-1} .
\] (5.19)

From this, large sample \( t \) and \( F \) tests for \( \beta \) may be obtained.

---

13 We may view any constructive or non-constructive test as a 'pure-significance test', in the sense that - if a test rejects \( H_0 \) - we will not necessarily adopt \( H_a \). Then, alike fully-non-constructive tests, rejection of \( H_0 \) would be interpreted as giving evidence of 'some sort' of heteroscedasticity, and adjustment of the model could be as in (iv).
5.7.2 Analysis After Using a Two-directional Test

If we cannot justify the use of a one-directional test, then we may test jointly for $NH$. If $NH$ is rejected the type of subsequent analysis would also depend on the type of test used.

(i) If a constructive joint test was used, then we may proceed to adjust for the possible presence of both $N$ and $H$. For this we may consider the Box-Cox transformation under heteroscedasticity (see Gaudry and Dagenais (1979)). The log-likelihood would be given by

$$
\ell(\beta^0, \alpha, \lambda) = \frac{-N}{2} \log(2\pi) - \frac{1}{2} \sum_{i=1}^{N} \log(z_i^1 a) \\
- \frac{1}{2} \sum_{i=1}^{N} \left( y_i^{(\lambda)} - \text{x}_i^1 \beta^0 \right)^2 / (z_i^1 a) + (\lambda - 1) \sum_{i=1}^{N} \log y_i 
$$

from which MLE's of the parameters, and large sample $t$ and $F$ tests, could be obtained (for definition of $\beta^0$ and $y_i^{(\lambda)}$ see Subsection 4.6.2).

(ii) If a fully-non-constructive test was used, then we could estimate $\beta$ by $b$ and the VCM of $b$ by (5.19). We could also compute large sample tests for the parameters as in White (1980c) (these tests do not require disturbance normality).
5.7.3 Analysis After Using One-directional Tests Simultaneously

The use of two-directional tests (e.g., $LM_{NH}^-$) provide what may be called *direct tests* for disturbance $NH$. In Subsection 5.7.2 we assumed that - with these - if $NH$ was rejected, we would 'adjust' for both $N$ and $H$. In practice, we may be interested in identifying the nature of the departure(s) from $NH$. Then, rather than applying a *direct test*, we could consider using a multiple comparison procedure (see Savin (1980)), based on the simultaneous application of the one-directional tests $LM_N$ and $LM_H$. With this procedure, analysis of the model would depend on the dimension(s) in which there is 'evidence' that $NH$ is rejected. More formally the procedure would be as follows:

We would adjust for $N$ if $LM_N > a_1$ and $LM_H < a_2$, where $a_1$ and $a_2$ are, respectively, appropriate significance points for $LM_N$ and $LM_H$. To carry out this 'adjustment' we could proceed as described in Section 4.6.

We would adjust for $H$ if $LM_N < a_1$ and $LM_H > a_2$. Here, the model could be re-estimated by MLE using the specified $H$ structure (see Amemiya (1977)). Large sample $t$ and $F$ tests would be readily available.

14 Under weak a-priori information on $H$ we could use $LM_H^*$ rather than $LM_H$. Other one-directional tests for $N$ and $H$ may be used in the multiple comparison procedure. We prefer LM based one-directional tests because these are known to have optimal power properties under one-directional departures from $NH$ (e.g., $LM_N$ is asymptotically efficient under $NH$).
Similarly, we would adjust for both $\overline{N}$ and $\overline{H}$ if $\text{LM}_N > a_1$ and $\text{LM}_H > a_2$. For example, using the Box-Cox transformation under $\overline{H}$ (see (5.20)).

In turn, if $\text{LM}_N < a_1$ and $\text{LM}_H < a_2$, then no adjustment would be made.

Three important points to note on the use of this multiple comparison procedure are the following.

Firstly that, for a particular combination of values for $\text{LM}_N$ and $\text{LM}_H$, it may be that $H_0: u \sim NH$ would be rejected if we applied the direct test $\text{LM}_{NH}$, and accepted if we used the multiple comparison procedure. This is due to the differing critical regions in the two testing procedures. Viewing $(\text{LM}_N, \text{LM}_H)$ as a point in $\mathbb{R}^2$, we have that $\text{LM}_{NH}$ rejects $H_0$ for points above the line $a = \text{LM}_N + \text{LM}_H$ (where $a$ is the significance point for $\text{LM}_{NH}$), whereas the multiple comparison procedure rejects $H_0$ for points outside the rectangular region $\text{LM}_N < a_1$ and $\text{LM}_H < a_2$ (where $a_1$ and $a_2$ are as above).

Secondly, that the significance level of this multiple comparison procedure, say $\alpha$, would need to be determined. As pointed out previously, for finite samples the distributions of $\text{LM}_N$ and $\text{LM}_H$ are not presently known, and these variables would not be independent. So, for finite $N$, determination of $\alpha$ by analytical methods would be intractable. A bound on $\alpha$ may be obtained by using the Bonferroni inequality (see Savin (1980, p.257)), which states that $\alpha < \alpha_1 + \alpha_2$, where $\alpha_1$ and $\alpha_2$ are, respectively, the 'marginal' significance levels corresponding to the significance points $a_1$ and $a_2$ used for $\text{LM}_N$ and $\text{LM}_H$. For large samples, $\text{LM}_N$ and $\text{LM}_H$ would be independent and
we would have \( \alpha = \alpha_1 + \alpha_2 - \alpha_1 \alpha_2 \). In practice, there may be doubts as to whether the sample size is large enough to give \( \alpha = \alpha_1 + \alpha_2 - \alpha_1 \alpha_2 \). In these situations, for given values of \( \alpha_1 \) and \( \alpha_2 \), \( \alpha \) may be determined by computer simulation; note \( \alpha \) may be estimated by the proportion of simulated values \( (LM_N, LM_H) \) under \( NH \) that fell in the rejection region of \( \mathbb{R}^2 \) (i.e., such that \( LM_N > a_1 \) and/or \( LM_H > a_2 \)). A further question that arises regarding the significance level \( \alpha \), is how to choose the 'marginal' significance levels \( \alpha_1 \) and \( \alpha_2 \), so that \( \alpha \) is close to a desired value and so that high power is obtained.

Thirdly, we point out that cases exist, e.g. when \( u \sim \overline{NH}(Log) \), where with high probability, both \( LM_N \) and \( LM_H \) would be large (see fourth row in Table 5.3), and the use of the multiple comparison procedure would lead us to adjust for both \( \overline{N} \) and \( \overline{H} \), despite the fact that only adjustment for \( \overline{N} \) is required. This is an additional clearly undesirable feature of this procedure.

For the direct test \( LM_{NH} \), determination of the critical point - for a desired significance level - is simple. In addition, this test is known to be asymptotically efficient; but, when it rejects \( NH \) we are left with uncertainty about the nature of the departure(s). In contrast, the multiple comparison procedure has problems associated with the determination of the significance level and may lead to 'improper adjustments'. Yet, until powerful and robust one-directional tests become available (with known finite sample distributions) it appears that, to identify the nature of the departure(s) from \( NH \), we are confined to the use of this type of multiple comparison procedures.

\[15\] This is why, if \( LM_{NH} \) rejects \( NH \), our suggestion is to adjust for both \( \overline{N} \) and \( \overline{H} \).
"What do you mean, less than nothing?" replied Wilbur. "I don't think there is any such thing as less than nothing. Nothing is absolutely the limit of nothingness. It's the lowest you can go. It's the end of the line. How can something be less than nothing? If there were something that was less than nothing then nothing would not be nothing, it would be something - even though it's just a very little bit of something. But if nothing is nothing, then nothing has nothing that is less than it is."

E.B. White

*Charlotte's Web*

(Quoted by Tobin (1958))

6.1 INTRODUCTION

In this Chapter we deal with assumption [5] - described in Section 1.2 - which states the range of the dependent variable $y_1$ is not restricted or limited in any way. There are cases when this is not a valid assumption and this has led people to consider the use of limited dependent variable (LDV) models. Here we discuss aspects of two such models, namely, a Truncated model - considered in Sections 6.2, 6.3 and 6.4 - and the Tobit model - considered in Sections 6.5 and 6.6. Throughout these sections we make assumptions [6] and [7] stated in Section 1.2. The Chapter ends with Section 6.7, where some concluding remarks are made.

* Results of Sections 6.6 and 6.7 are based on the papers Jarque (1981c) and Jarque and Bera (1981c).
6.2 THE TRUNCATED MODEL

In Chapter 4 we discussed the normality assumption and noted an obvious case in applied econometrics where disturbance normality would not hold. This is when assumption [5] is not valid, i.e., when the dependent variable \( y_i \) is restricted or limited, say, to non-negative values, so that we have \( y_i = x_i^\prime \beta + u_i \geq 0 \). Then, the range of \( u_i \) would be restricted to \( u_i \geq -x_i^\prime \beta \), and the p.d.f. of \( u_i \), \( f(u_i) \), would need to be some truncated density.

In economics, many variables are non-negative (e.g. demand for and supply of a given good), and yet one finds models explaining the behaviour of these non-negative economic variables while assuming disturbance normality. The normality assumption in these limited dependent variable situations may be justified on the grounds that, if \( x_i^\prime \beta \) is large relative to \( \sigma_i^2 \), the probability of having \( u_i < -x_i^\prime \beta \) under a \( N(0,\sigma_i^2) \) is so small that it may be neglected for the purpose of analysis. This may well be the case when one is dealing with macroeconomic variables. However, in cross-sectional studies one often deals with microeconomic data for which \( x_i^\prime \beta \) is not large relative to \( \sigma_i^2 \) and - as will become apparent - a preferable analysis is carried out by taking into account the truncated nature of the disturbance distribution.

When specifying the disturbance distribution in these LDV cases, a distribution that readily comes to mind is the truncated-normal with range \( u_i \geq -x_i^\prime \beta \). This has density

\[
f(u_i) = \frac{1}{F(x_i^\prime \beta, \sigma_i^2)} \frac{1}{\sqrt{2\pi \sigma_i^2}} \exp\left(-\frac{u_i^2}{2\sigma_i^2}\right)
\]

\(^1\) We concentrate first on this form of restriction on \( y_i \) and comment on other forms in Section 6.4.
for \( u_i \geq -x_1^t \beta \), where

\[
F(x_1^t \beta, \sigma_1^2) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \int_{-x_1^t \beta}^{\infty} \exp\left(-\frac{u_1^2}{2\sigma_1^2}\right) du_1. \quad (6.1)
\]

For future use we also define

\[
f(x_1^t \beta, \sigma_1^2) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{(x_1^t \beta)^2}{2\sigma_1^2}\right). \quad (6.2)
\]

We now make assumption [4], i.e., we assume \( \sigma_1^2 = \ldots = \sigma_N^2 = \sigma^2 \) and refer to the resulting model as a Truncated-Normal-Homoscedastic model. This model has been previously studied by various authors. In particular, Greene (1981) gives a precise characterization of the bias of the OLS estimator when the regressors are normally distributed. Amemiya (1973, p.1015), considering maximum likelihood estimation, provides a consistent estimator of \( \beta \); proves its asymptotic normality and notes the asymptotic efficiency of the second-round estimator in the method of Newton. In turn, Olsen (1980) has suggested a simple approximation to the MLE which is based on a method of moments due to Pearson and Lee (1908).

For a similar LDV model, Fair (1977) has proposed a procedure for solving the first order conditions of the MLE. As we now illustrate, Fair's procedure may also be used in the model presently considered. It is easy to see that the log-likelihood of the Truncated-Normal-Homoscedastic-model is

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2 In general, the inconsistency of the OLS estimator \( (X'X)^{-1}X'y \) is observed from equation (6.5) noting that the probability limit of \( X'A/N \) is different from zero.
\[ \lambda(\beta, \sigma^2) = -\frac{N}{2} \log(2\pi \sigma^2) - \sum_{i=1}^{N} \log F_i - \frac{1}{2\sigma^2} \sum_{i=1}^{N} (y_i - x_i'\beta)^2, \quad (6.3) \]

where \( F_i = F(x_i'\beta, \sigma^2) \). We also define \( f_i = f(x_i'\beta, \sigma^2) \). When maximizing (6.3) with respect to \( \beta \) and \( \sigma^2 \) we obtain that the maximum likelihood estimators, say \( \hat{\beta} \) and \( \hat{\sigma}^2 \), must satisfy the equations

\[ \hat{\sigma}^2 = \frac{y' (y - X\hat{\beta})}{N} \quad (6.4) \]

and

\[ \hat{\beta} = (X'X)^{-1}X'y - \hat{\sigma}(X'X)^{-1}X'\Lambda, \quad (6.5) \]

where \( \Lambda = (\eta_1, \ldots, \eta_N)' \) and \( \eta_i = \frac{\hat{\sigma}^2 f_i}{F_i}, \) with \( \hat{f}_i = f(x_i'\hat{\beta}, \hat{\sigma}^2) \) and \( \hat{F}_i = F(x_i'\hat{\beta}, \hat{\sigma}^2) \) (for proof see Proposition 1 in Appendix F - page 309). So, to compute \( \hat{\beta} \) and \( \hat{\sigma}^2 \), the procedure described in Fair (1977, p.1724) could be used, replacing his \( R, \bar{X} \) and \( \bar{y} \) respectively by our \( N, X \) and \( \Lambda \). Additionally, by an argument similar to that presented in Olsen (1978) we may show that (6.3) has a single maximum. Therefore, if the above iterative procedure converges - it converges to the MLE. After reaching convergence, as shown in Proposition 2 in Appendix F (page 310), the finite sample density of \( (\hat{\beta}', \hat{\sigma}^2) \) may be approximated by a Normal with mean \( (\beta', \sigma^2) \) and VCM equal to \( V^{-1} \), where

\[ V = \begin{bmatrix} N \sum a_i x_i'x_i & N \\ N \sum b_i x_i' & i=1 \\ N \sum c_i' & i=1 \end{bmatrix}, \quad (6.6) \]

with
\[ a_i = \left[ \frac{1}{\sigma^2} - \frac{\hat{\beta}^2 f_i}{\hat{F}_1} - \frac{(x_i^\prime \hat{\beta}) f_i}{\sigma^2 \hat{F}_1} \right] , \]

\[ b_i = \frac{1}{2\sigma^2} \left[ \frac{(x_i^\prime \hat{\beta})^2 f_i}{\sigma^2 \hat{F}_1} + \frac{\hat{f}_1}{\hat{F}_1} + \frac{(x_i^\prime \hat{\beta})^2 f_i}{\sigma^2 \hat{F}_1} \right] \]

and

\[ c_i = \frac{1}{4\sigma^4} \left[ 2 - \frac{(x_i^\prime \hat{\beta})^2 \hat{f}_1}{\sigma^2 \hat{F}_1} - \frac{(x_i^\prime \hat{\beta}) \hat{f}_1}{\hat{F}_1} - \frac{(x_i^\prime \hat{\beta})^2 \hat{f}_1}{\sigma^2 \hat{F}_1} \right] . \]

Then, large-sample \( t \) and \( F \) tests could be readily computed.

The results presented in this section require disturbance (truncated) normality and homoscedasticity. In Section 6.3 we comment on the normality assumption and in Section 6.4 we discuss the problem of heteroscedasticity in relation to this LDV model.

### 6.3 NON-NORMALITY IN THE TRUNCATED MODEL

In the analysis of the Truncated-Normal-Homoscedastic model, misspecification of the disturbance density, i.e., making the incorrect assumption of disturbance truncated-normality (rather than the true truncated p.d.f.) may lead to inconsistency of the MLE based on (6.3), i.e., \( \hat{\beta} \). A rigorous analytical proof of this is difficult, but some insight is obtained by considering equation (6.5) written - after substitution for \( y \) and noting the definitions for \( \hat{f}_1 \) and \( \hat{F}_1 \) - as

\[ \hat{\beta} = \beta + \left( X'X \right)^{-1} \left[ \frac{1}{N} \sum_{i=1}^{N} x_i u_i - \frac{1}{N} \sum_{i=1}^{N} x_i \hat{\sigma}^2 f(x_i^\prime \hat{\beta}, \hat{\sigma}^2) / F(x_i^\prime \hat{\beta}, \hat{\sigma}^2) \right] . \]
For \( \hat{\beta} \) to be consistent we need the second term on the right hand side (RHS) to have zero probability limit (plim). If the sample size goes to infinity, say, in multiples of \( N \), then

\[
P_1 = \lim_{r \to \infty} \frac{1}{N_r} \sum_{i=1}^{N_r} x_i^1 \hat{u}_i = \frac{1}{N} \sum_{i=1}^{N} x_i E[\hat{u}_i],
\]

which is not necessarily equal to

\[
P_2 = \lim_{r \to \infty} \frac{1}{N_r} \sum_{i=1}^{N_r} x_i \hat{\sigma}^2 f(x_i^1 \hat{\beta}, \hat{\sigma}^2) / F(x_i^1 \hat{\beta}, \hat{\sigma}^2).
\]

To see this, we proceed as in Robinson (1981) and consider a particular model. Suppose that \( u_i \) is uniformly distributed in the range \((-\sqrt{3} \sigma, \sqrt{3} \sigma)\). Also, that \( K = 1, x_i = 1 \) and \( \beta > \sqrt{3} \sigma \), thus having \( y_i > 0 \) for all \( i \). In addition, assume \( \sigma^2 \) is known and denote the true value of \( \beta \) by \( \beta_* \). A necessary condition for consistency of \( \hat{\beta} \) is that, at \( \beta_* \), \( P_1 - P_2 = 0 \). In this case we have \( E[u_1] = 0 \), so \( P_1 = 0 \). We can also show that \( P_2 \) at \( \beta_* \) is given by

\[
P_2^* = \frac{\sigma^2 f(\beta_*, \sigma^2)}{F(\beta_*, \sigma^2)} = \frac{\sigma^2 \left[ (2\pi \sigma^2)^{-\frac{1}{2}} \exp(-\beta_*^2/(2\sigma^2)) \right]}{\int_{-\beta_*}^{\infty} (2\pi \sigma^2)^{-\frac{1}{2}} \exp(-\xi^2/(2\sigma^2)) d\xi}.
\]

Thus we observe \( P_1 - P_2^* = -P_2^* \) which is different from zero (in fact, it is negative) and hence \( \hat{\beta} \) would be inconsistent. \(^3\)

\(^3\) Returning to the general \( K > 1 \) case note that, under truncated-normality, \( E[u_1] = \sigma^2 f(x_1^1 \beta, \sigma^2) / F(x_1^1 \beta, \sigma^2) \). Then, by definition of \( \hat{\sigma}^2 \) and \( \hat{\beta} \), both plim's would be equal (see the expressions for \( P_1 \) and \( P_2 \)).
For a limited dependent variable model similar to the Truncated model, White (1979) conducted a Monte Carlo experiment to study the effect of 'non-normality' on the MLE. His findings are indicative of the inconsistencies that may arise when misspecifying the disturbance distribution in LDV situations. Hence the importance of testing the validity of the 'normality assumption' in LDV models and, in particular, in the Truncated model introduced in Section 6.2. As a result of truncation, the normality tests discussed in Chapter 4 are not directly applicable. A solution to this problem is obtained by considering a family of truncated distributions (containing the truncated-normal as a particular case) and applying the approach presented in Section 4.2 to derive a truncated-normality test. Here we use the Truncated-Pearson Family of distributions which we now introduce.

Assume the p.d.f. of the i'th disturbance $u^i$, say $\tilde{f}(u^i)$, is a member (or is well approximated by a member) of the Truncated-Pearson Family defined by

$$
\tilde{f}(u^i) = \frac{\exp \left[ \int \frac{c_1 u^i}{c_0 - c_1 u^i + c_2 u^i} \, du^i \right]}{\int_{-\infty}^{\infty} \exp \left[ \int \frac{c_1 u^i}{c_0 - c_1 u^i + c_2 u^i} \, du^i \right] \, du^i}, \quad (6.7)
$$

for $u^i \geq -x^i\beta$. From (6.7) it is easy to see that the log-likelihood
would be given by

$$
\lambda(\beta, c_0, c_1, c_2) = -\sum_{i=1}^{N} \log \left[ \int_{-x_i^i}^{\infty} \exp \left( \int \frac{c_i - u_i}{c_0 - c_1 u_i + c_2 u_i^2} \, du_i \right) \, du_i \right] + \sum_{i=1}^{N} \int \frac{c_i - u_i}{c_0 - c_1 u_i + c_2 u_i^2} \, du_i.
$$

(6.8)

Our interest is testing the truncated-normality of $u_i$ which is equivalent to testing $H_0: c_1 = c_2 = 0$. For this we use the LM procedure presented in Section 4.2. Now we define $\theta_1 = (\beta', \sigma^2)'$ and $\theta_2 = (c_1, c_2)'$. As shown in Proposition 3 in Appendix F (see page 312) in this case the LM test statistic becomes

$$
LM_{N(TRUN)} = \hat{d}_2' (\hat{I}_{22} - \hat{I}_{21} \hat{I}_{12})^{-1} \hat{d}_2,
$$

(6.9)

where

$$
\hat{d}_2 = \frac{1}{\hat{\sigma}^2} (\hat{\mu}_1 - \hat{\mu}(1)) - \frac{1}{3\hat{\sigma}^4} (\hat{\mu}_3 - \hat{\mu}(3)) - \frac{1}{4\hat{\sigma}^4} (\hat{\mu}_4 - \hat{\mu}(4))
$$

(6.10)

and

$$
\hat{I} = \begin{bmatrix}
\hat{I}_{11} & \hat{I}_{12} \\
\hat{I}_{21} & \hat{I}_{22}
\end{bmatrix} = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{12} & a_{22} & a_{23} & a_{24} \\
a_{13} & a_{23} & a_{33} & a_{34} \\
a_{14} & a_{24} & a_{34} & a_{44}
\end{bmatrix}
$$

(6.11)

with
\[ a_{11} = \frac{1}{\sigma^4} \sum_{i=1}^{N} x_i x_i^* (\hat{u}_{i(2)} - \hat{u}_{i(1)}) \]

\[ a_{12} = \frac{1}{2\sigma^6} \sum_{i=1}^{N} x_i (\hat{u}_{i(3)} - \hat{u}_{i(1)})^2 \]

\[ a_{13} = \frac{1}{\sigma^4} \sum_{i=1}^{N} x_i (\hat{u}_{i(2)} - \hat{u}_{i(1)}) - \frac{1}{3\sigma^6} \sum_{i=1}^{N} x_i (\hat{u}_{i(4)} - \hat{u}_{i(1)})^2 \]

\[ a_{14} = \frac{1}{4\sigma^6} \sum_{i=1}^{N} x_i (\hat{u}_{i(5)} - \hat{u}_{i(1)}) \]

\[ a_{22} = \frac{1}{2\sigma^8} \sum_{i=1}^{N} (\hat{u}_{i(4)} - \hat{u}_{i(2)})^2 \]

\[ a_{23} = \frac{1}{2\sigma^6} \sum_{i=1}^{N} (\hat{u}_{i(3)} - \hat{u}_{i(2)})^2 + \frac{1}{6\sigma^8} \sum_{i=1}^{N} (\hat{u}_{i(5)} - \hat{u}_{i(1)})^2 \]

\[ a_{24} = \frac{1}{8\sigma^8} \sum_{i=1}^{N} (\hat{u}_{i(6)} - \hat{u}_{i(2)}) \]

\[ a_{33} = \frac{1}{\sigma^4} \sum_{i=1}^{N} (\hat{u}_{i(2)} - \hat{u}_{i(1)})^2 + \frac{1}{3\sigma^6} \sum_{i=1}^{N} (\hat{u}_{i(5)} - \hat{u}_{i(1)})^2 \]

\[ a_{34} = \frac{1}{4\sigma^6} \sum_{i=1}^{N} (\hat{u}_{i(5)} - \hat{u}_{i(1)}) \]

\[ a_{44} = \frac{1}{16\sigma^8} \sum_{i=1}^{N} (\hat{u}_{i(8)} - \hat{u}_{i(4)})^2 \]

and where \( \hat{u}_j = \sum_{i=1}^{N} u_i j / N, \hat{y}_i = y_i - x_i \hat{\beta} \) and \( \hat{\beta} \) and \( \sigma^2 \) are the MLE's.
of $\beta$ and $\sigma^2$ under $H_0$, i.e., the solution to equations (6.4) and (6.5). We have also used $\hat{\mu}_i(j) = \hat{\mu}_i(j)/N$, where $\hat{\mu}_i(j)$ denotes the predicted $j$'th moment about the origin of a truncated-normal random variable $u_i$ [e.g. see equations (F.11) in Appendix F – page 313 – and evaluate these at $\beta = \hat{\beta}$ and $\sigma^2 = \hat{\sigma}^2$]. More specifically, we have

\[
\hat{u}_i(1) = \sigma^2 \frac{F_i}{\hat{F}_i} 
\]

\[
\hat{u}_i(2) = \sigma^2 [1 - (x_i^\beta)^2] \frac{F_i}{\hat{F}_i} 
\]

\[
\hat{u}_i(3) = \sigma^2 [2\sigma^2 + (x_i^\beta)^2] \frac{F_i}{\hat{F}_i} 
\]

\[
\hat{u}_i(4) = \sigma^2 [3\sigma^2 - 3\sigma^2 (x_i^\beta)^2 \frac{F_i}{\hat{F}_i} - (x_i^\beta)^2] \frac{F_i}{\hat{F}_i} 
\]

\[
\hat{u}_i(5) = \sigma^2 [8\sigma^4 + 4\sigma^2 (x_i^\beta)^2 + (x_i^\beta)^4] \frac{F_i}{\hat{F}_i} 
\]

\[
\hat{u}_i(6) = \sigma^2 [15\sigma^4 - 15\sigma^4 (x_i^\beta)^2 \frac{F_i}{\hat{F}_i} - 5\sigma^2 (x_i^\beta)^2 \frac{F_i}{\hat{F}_i} - (x_i^\beta)^2] \frac{F_i}{\hat{F}_i} 
\]

\[
\hat{u}_i(7) = \sigma^2 [48\sigma^6 + 24\sigma^4 (x_i^\beta)^2 + 6\sigma^2 (x_i^\beta)^4 + (x_i^\beta)^6] \frac{F_i}{\hat{F}_i} 
\]

and

\[
\hat{u}_i(8) = \sigma^2 [105\sigma^6 - 105\sigma^6 (x_i^\beta)^2 \frac{F_i}{\hat{F}_i} - 35\sigma^4 (x_i^\beta)^2 \frac{F_i}{\hat{F}_i} - 7\sigma^2 (x_i^\beta)^2 \frac{F_i}{\hat{F}_i} - (x_i^\beta)^2] \frac{F_i}{\hat{F}_i} 
\]
The test statistic (6.9) has been written with the suffix \( N(\text{TRUN}) \) to indicate it relates to a disturbance truncated-normality test.

Under \( H_0: c_1 = c_2 = 0 \), \( \text{LM}_{N(\text{TRUN})} \) would be asymptotically distributed as \( \chi^2_{(2)} \) and we would reject truncated-normality if the computed value of \( \text{LM}_{N(\text{TRUN})} \) exceeded the appropriate critical point. The lengthy expressions that appear in \( \text{LM}_{N(\text{TRUN})} \) are not indicative of computational difficulty. The values \( \hat{f}_1 \) and \( \hat{F}_1 \) are easily computed (a subroutine is available in most FORTRAN compilers for the computation of \( \hat{F}_1 \)). Straightforward calculations would give us \( \hat{u}_i(j) \) (\( i = 1, \ldots, N; j = 1, \ldots, 8 \)) and, in turn, \( a_{kr} \) (\( k, r = 1, \ldots, 4 \)). Having found \( a_{kr} \), we would invert one \((K+1)\) by \((K+1)\) matrix and a \(2\) by \(2\) matrix, obtaining \( \text{LM}_{N(\text{TRUN})} \) after simple products and summations.

Derivation of the finite sample distribution of \( \text{LM}_{N(\text{TRUN})} \) by analytical procedures appears to be intractable. This is not surprising since the Truncated model is a nonlinear model and - for these - even the finite sample distribution of the estimators of the parameters are generally unknown. A further complication here is that computer simulation is not directly applicable because the range of \( u_i \) depends on \( \beta \) [Recall that for the tests suggested in Chapters 4 and 5 we could use computer simulation to derive finite sample significance points]. The difficulty in obtaining finite sample distributions - by analytical procedures or computer simulation - arises in all the tests we discuss in the present Chapter. This limits the applicability of our findings when one has small samples. Fortunately, in the various applications of LDV models, large samples have been typically available and - in these cases - one may use the asymptotic \( \chi^2 \) distribution to obtain approximate significance points [e.g., in the study by Tobin (1958) on household durable expenditure \( N = 735 \); in a study by Quester and Greene (1978) on labour supply \( N = 2798 \); and in the study by Fair...
If $H_0: c_1 = c_2 = 0$ is accepted, we could use the results presented in Section 6.2 to derive significance tests on the parameters $\beta$. If $H_0$ is rejected, we could consider applying transformations so that truncated-normality is better suited. For instance, we could use the Box-Cox transformation. In this case the log-likelihood of the transformed truncated observations would be

$$
\ell(\beta^o, \sigma^2_o, \lambda) = - \frac{N}{2} \log(2\pi) - \frac{N}{2} \log \sigma^2_o + \frac{1}{2} \sum_{i=1}^{N} \frac{(y_i^{(\lambda)} - x_i^\prime \beta^o)^2}{\sigma^2_o} \\
+ (\lambda-1) \sum_{i=1}^{N} \log y_i - \sum_{i=1}^{N} \log \left[ \int_0^{\infty} \left( \frac{1}{(1/\lambda + x_i^\prime \beta^o)} \right)^{\lambda} \exp \left( -\frac{(u_i^o)^2}{2\sigma^2_o} \right) du_i^o \right].
$$

Numerical procedures could be applied to $\ell(\beta^o, \sigma^2_o, \lambda)$ to obtain MLE's of $\beta^o$ and $\sigma^2_o$ (e.g., see Poirier (1978)) and asymptotic results from maximum likelihood estimation could be used to carry out tests of hypotheses.

Before proceeding to our discussion of the problem of heteroscedasticity in the Truncated model, we make three final comments.

Firstly, we note that when $x_i^\prime \beta$ is large relative to $\sigma^2$, we have $\hat{f}_i = 0$ and $\hat{F}_i = 1$ which imply, from (6.12), that

---

4 $\ell(\beta^o, \sigma^2_o, \lambda)$ refers to $\lambda > 0$. An equivalent log-likelihood may be written for $\lambda < 0$. (See Amemiya and Powell (1980, p.5)).

5 Our test $\text{LM}_{N(\text{TRUN})}$ has been derived within a 'Family of Distributions'. An alternative test for truncated-normality may be obtained within a 'Family of Transformations'; for example, using the Box-Cox transformation, the null hypothesis would be $H_0: \lambda = 1$. This asks if a transformation is required to achieve truncated-normality. We could then obtain the LR test or the computationally simpler LM test for $\lambda = 1$. 
\[ \hat{u}_1(j) = 0 \text{ for } j \text{ odd}, \text{ and } \hat{u}_1(j) = 1.3 \cdots (j-1) \hat{\sigma}^j \text{ for } j \text{ even} \]

[i.e., \( \hat{u}_1(j) = 0 \text{ for } j \text{ odd}, \text{ and } \hat{u}_1(j) = 1.3 \cdots (j-1) \hat{\sigma}^j \text{ for } j \text{ even} \)]. When we replace \( \hat{u}_1(j) \) and \( \hat{\nu}(j) \) in equations (6.9), (6.10) and (6.11) by their approximate values under \( \hat{f}_1 = 0 \) and \( \hat{F}_1 = 1 \) [i.e., accordingly zero or \( 1.3 \cdots (j-1) \hat{\sigma}^j \)] we have that \( LM_{N(TRUN)} \) reduces to \( LM_N \) (see equation (4.5)). This shows we can apply the test statistic \( LM_N \) when testing for disturbance normality in the Truncated model if truncation is not a 'serious problem', i.e., if \( x_i^8 \) is large relative to \( \sigma^2 \).

Secondly, we can show that if the model contains a constant term, then we have \( \hat{\mu}_1 = \hat{\mu}_1(1) \). [A proof of this is easily obtained by premultiplying equation (6.5) by \(-X\); adding the vector \( y \) in both sides; and premultiplying the resulting vector by \( 1' \)]. So, in this case, the first element in \( \hat{d}_2 \) (see (6.10)) reduces to 
\[ -(\hat{\mu}_3 - \hat{\mu}_1(1))/(3\hat{\sigma}^4) \]
In general, the test statistic \( LM_{N(TRUN)} \) is intuitively appealing since [apart from the 'location term' \( (\hat{\mu}_1 - \hat{\mu}_1(1))/\hat{\sigma}^2 \)] it compares the sample third and fourth moments \( \hat{\mu}_3 \) and \( \hat{\mu}_4 \) with the hypothesized moments \( \hat{\mu}(3) \) and \( \hat{\mu}(4) \) (see equation (6.10)).

Thirdly, we note that Statisticians have recently been interested in testing if observations come from a particular truncated distribution. To our knowledge, the only available test for this is the Truncated Kolmogorov-Smirnov test (e.g. see Barr and Davidson (1973) and Koziol and Byar (1975)). The test statistic \( LM_{N(TRUN)} \) has been devised for testing the truncated-normality of regression disturbances. Yet, if we wanted to test observations for truncated-normality, \( LM_{N(TRUN)} \) could be used by setting \( K = 1 \).
and defining the 'regressor' \( x_i = 1 \) for \( i = 1, \ldots, N \). Our test - therefore - provides a solution to this statistical problem.

### 6.4 HETEROSCEDASTICITY IN THE TRUNCATED MODEL

In Section 6.3 we studied the problem of non-normality for the Truncated model under the maintained assumption of homoscedasticity. We now relax this maintained assumption. Initially, we discuss the problem of heteroscedasticity given disturbance truncated-normality and then we discuss non-normality and heteroscedasticity jointly.

A specific form of the Truncated-Normal model has been considered by Hurd (1979). Through a Monte Carlo study, Hurd (1979) shows the inconsistency of the MLE of \( \beta \) when this is obtained under the incorrect assumption of homoscedasticity. A rigorous analytical proof of this is difficult, but we may see why - in general - we would obtain inconsistent estimators by looking at equation (6.5). As noted in Section 6.3 this may be written as

\[
\hat{\beta} = \beta + \left( \frac{X'X}{N} \right)^{-1} \left[ -\frac{1}{N} \sum_{i=1}^{N} x_i u_i - \frac{1}{N} \sum_{i=1}^{N} x_i \sigma_i^2 f(x_i \hat{\beta}, \sigma_i^2) / F(x_i \hat{\beta}, \sigma_i^2) \right] .
\]

For \( \hat{\beta} \) to be consistent we need the second term in the RHS to have zero \( \text{plim} \). Yet, we know that if the sample goes to infinity (say in multiples of \( N \)) then

\[
\text{plim}_{r \to \infty} \frac{1}{N^r} \sum_{i=1}^{N^r} x_i u_i = \frac{1}{N} \sum_{i=1}^{N} x_i \sigma_i^2 f(x_i \hat{\beta}, \sigma_i^2) / F(x_i \hat{\beta}, \sigma_i^2)
\]

which is not necessarily equal to
In his Monte Carlo study, Hurd (1979) considers heteroscedasticity "in the range to be expected in empirical work", and the results lead him to conclude that "heteroscedasticity may be a serious empirical problem in truncated-sample models". This highlights the importance of testing for homoscedasticity in the Truncated-Normal model.

Because of truncation, the homoscedasticity tests reviewed and suggested in Chapter 5 are not applicable. We present an LM test for this purpose. First we postulate \( \sigma_i^2 = \sigma^2 + z_i^* \alpha^*_i = z_i \alpha \), where, as in Section 5.3, \( \sigma^2 \) is an unknown scalar; \( \alpha^* = (\alpha_2, \ldots, \alpha_p)' \) is a vector of unknown parameters; \( z_i^* = (z_{i2}, \ldots, z_{ip})' \) is a \( p-1 \) by 1 vector of known fixed finite variables; \( z_i' = (1, z_i^*) \) and \( \alpha = (\sigma^2, \alpha^*)' \). Then, the log-likelihood of our model would be

\[
\ell(\beta, \sigma^2, \alpha^*) = -\frac{1}{2} \sum_{i=1}^{N} \log(2\pi\sigma_i^2) - \sum_{i=1}^{N} \log F_i
- \frac{1}{2} \sum_{i=1}^{N} \frac{1}{\sigma_i^2} (y_i' - x_i'\beta)^2
\]

(see equation (6.3) and 'replace' \( \sigma^2 \) by \( \sigma_i^2 \).

We now define \( \theta_1 = (\beta', \sigma^2)' \) and \( \theta_2 = \alpha^* \) and note that our interest is to test \( H_0: \alpha^* = 0 \). Then, using the LM procedure we obtain (see Proposition 4 in Appendix F - page 312)

\[
LM_{H(TRUN)} = \frac{\sigma^2}{4} \phi_5' \psi_4 \phi_5^{-1} \phi_5' \phi_5 \quad (6.14)
\]

where \( \phi_5 \) is an \( N \) by 1 vector with i'th element equal
\[ \phi_4 = \phi_1 - \phi_2 X (X' \phi_3 X)^{-1} X' \phi_2, \]

and where \( \phi_1, \phi_2 \) and \( \phi_3 \) are \( N \times N \) diagonal matrices given by

\[ \phi_1 = \text{diag}(\hat{\sigma}^6 c_1, \ldots, \hat{\sigma}^6 c_N), \]
\[ \phi_2 = \text{diag}(\hat{\sigma}^4 b_1, \ldots, \hat{\sigma}^4 b_N) \]

and

\[ \phi_3 = \text{diag}(\hat{\sigma}^2 a_1, \ldots, \hat{\sigma}^2 a_N); \]

with \( c_1, b_1 \) and \( a_1 \) defined in (6.6), \( \hat{u}_1 = y_1 - x_1 \hat{\beta}, Z = (z_1, \ldots, z_N)' \), and \( \hat{\beta} \) and \( \hat{\sigma}^2 \) denote the solution to equations (6.4) and (6.5).

The suffix \( H(\text{TRUN}) \) in \( \text{LM}_{H(\text{TRUN})} \) indicates a homoscedasticity test in the Truncated-Normal model. Under \( H_0: \alpha^* = 0 \), \( \text{LM}_{H(\text{TRUN})} \) would be asymptotically distributed as \( \chi^2_{(p-1)} \). We would reject homoscedasticity, for large samples, if the computed value of \( \text{LM}_{H(\text{TRUN})} \) exceeded the appropriate significance point on a \( \chi^2_{(p-1)} \). If \( H_0 \) is accepted, analysis of the model could proceed as in Section 6.2. If \( H_0 \) is rejected, then we may obtain MLE's of \( \beta, \sigma^2 \) and \( \alpha^* \) by maximizing (6.13). Numerical procedures for this need to be investigated. We note that if we set \( \hat{f}_i = 0 \), \( \text{LM}_{H(\text{TRUN})} \) reduces to the homoscedasticity test \( \text{LM}_H \) (see equation (5.6)).

In Sections 5.5 and 5.6, we argued in favour of joint tests for disturbance normality and homoscedasticity in the regression model.
In this section we have, so far, suggested a test for homoscedasticity in the Truncated model under the assumption of disturbance truncated-normality. The framework presented in Section 6.3 may be extended to obtain a joint test for homoscedasticity and truncated-normality. Heteroscedasticity may be incorporated into the log-likelihood (6.8) by setting $c_q = \sigma^2 + z_q'\alpha$. We would then define $\theta_1 = (\beta', \sigma^2)'$ and $\theta_2 = (c_1, c_2, \alpha^*)'$, and test $H_0: c_1 = c_2 = 0; \alpha^* = 0$. As shown in Proposition 5 in Appendix F (see page 312) in this case the LM test statistic is

$$\text{LM}_{NH(TRUN)} = \hat{d}_2'(\hat{I}_{22} - \hat{I}_{21}\hat{I}_{11}^{-1}\hat{I}_{12})^{-1}\hat{d}_2,$$  \hspace{1cm} (6.15)$$

with

$$\hat{d}_2 = N \left[ \frac{1}{\sigma^2} (\hat{\mu}_{1} - \hat{\mu}_{(1)}) - \frac{1}{3\sigma^4} (\hat{\mu}_{3} - \hat{\mu}_{(3)}) \right]$$

$$\frac{1}{4\sigma^4} (\hat{\mu}_{4} - \hat{\mu}_{(4)})$$

$$\sum_{i=1}^{N} \frac{1}{2\sigma^4} (\hat{\mu}_{i} - \hat{\mu}_{(i)})^2 \hat{z}_q^2$$

and

$$\hat{I} = \begin{bmatrix} \hat{I}_{11} & \hat{I}_{12} \\ \hat{I}_{21} & \hat{I}_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{12} & a_{22} & a_{23} & a_{24} & a_{25} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{13} & a_{23} & a_{33} & a_{34} & a_{35} \\ a_{14} & a_{24} & a_{34} & a_{44} & a_{45} \\ a_{15} & a_{25} & a_{35} & a_{45} & a_{55} \end{bmatrix},$$  \hspace{1cm} (6.17)$$

and where the $a_{ij}$ are defined as in (6.11) for $i,j = 1, \ldots, 4$ — and for $j = 5$ we have
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\[ a_{15} = \frac{1}{2a_6} \sum_{i=1}^{N} x_{i}^{2} (\hat{u}_{1}(3) - \hat{u}_{1}(1) - \hat{u}_{1}(2)) \]

\[ a_{25} = \frac{1}{4a_8} \sum_{i=1}^{N} z_{i}^{2} (\hat{u}_{1}(4) - \hat{u}_{1}(2)) \]

\[ a_{35} = \frac{1}{2a_6} \sum_{i=1}^{N} z_{i}^{2} (\hat{u}_{1}(3) - \hat{u}_{1}(1) - \hat{u}_{1}(2)) - \frac{1}{6a_8} \sum_{i=1}^{N} z_{i}^{2} (\hat{u}_{1}(5) - \hat{u}_{1}(2) - \hat{u}_{1}(3)) \]

\[ a_{45} = \frac{1}{8a_8} \sum_{i=1}^{N} z_{i}^{2} (\hat{u}_{1}(6) - \hat{u}_{1}(2) - \hat{u}_{1}(4)) \]

and

\[ a_{55} = \frac{1}{4a_8} \sum_{i=1}^{N} z_{i}^{2} (\hat{u}_{1}(4) - \hat{u}_{1}(2)) \]

Under \( H_0 : \alpha_c = 0 \); \( \alpha^* = 0 \), \( LM_{NH(TRUN)} \) would be asymptotically distributed as \( \chi^2(p+1) \); and we would reject this hypothesis for large values of \( LM_{NH(TRUN)} \).

If \( H_o \) is accepted, analysis of the model could proceed as in Section 6.2. If \( H_o \) is rejected, then we could adjust jointly for \( \bar{N} \) and \( \bar{\alpha} \) by using, say, a Box-Cox transformation under \( \bar{H} \). In this case the log-likelihood would be

\[ \ell(\beta^0,\alpha,\lambda) = -\frac{N}{2} \log(2\pi) - \frac{1}{2} \sum_{i=1}^{N} \log(z_{i}^{1}\alpha) - \frac{1}{2} \sum_{i=1}^{N} (y_{i}^{(\lambda)} - x_{i}^{1}\beta^0)^2/(z_{i}^{1}\alpha) \]

\[ + (\lambda-1) \sum_{i=1}^{N} \log y_{i} - \sum_{i=1}^{N} \log \left[ \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi(z_{i}^{1}\alpha)}} \exp \left[ -\frac{(u_{i}^{0})^2}{2(z_{i}^{1}\alpha)} \right] du_{i} \right] , \]

if \( \lambda > 0 \) (a similar expression exists for \( \lambda < 0 \)). We could obtain MLE's of \( \beta^0, \alpha \) and \( \lambda \) by using numerical methods. Procedures
for this remain to be investigated.

The use of the joint test $L_{NH}^{(TRUN)}$ provides a direct test for $H_0: c_1 = c_2 = 0; \alpha^* = 0$. Above, it was noted that if $H_0$ is rejected one could, in principle, adjust for both $\bar{N}$ and $\bar{H}$. In practice, one may be interested in identifying the possible source of departure(s) from $H_0$. For this - as an alternative to the direct test - one might consider a multiple comparison procedure based on the simultaneous use of the one-directional tests $L_{N}^{(TRUN)}$ and $L_{H}^{(TRUN)}$ [applied in the same fashion as the procedure discussed in Subsection 5.7.3 for the tests $L_{N}$ and $L_{H}$]. In the present setting, however, the use of a multiple comparison procedure has certain difficulties. First of all, determination of the significance level in finite samples would be extremely difficult (as for $L_{NH}^{(TRUN)}$ computer simulation is not applicable because of the $\beta$-dependence of the range of $u_i$).

Additionally, for this model the joint $LM$ test is not the sum of the one-directional tests [unlike the regression model for which $L_{NH} = L_{N} + L_{H}$], so we do not have asymptotic independence. This means that even with large samples we cannot say the significance level of the multiple comparison procedure will be $\alpha_1 + \alpha_2 - \alpha_1 \alpha_2$, where $\alpha_1$ and $\alpha_2$ are, respectively, the significance levels used for $L_{N}^{(TRUN)}$ and $L_{H}^{(TRUN)}$. Furthermore, little is known about the power properties of the procedure. All of the above are difficult questions and need to be further studied before this multiple comparison procedure can be successfully employed.

6 In Chapter 9 we report values of $L_{N}^{(TRUN)}$, $L_{H}^{(TRUN)}$ and $L_{NH}^{(TRUN)}$ computed from several data sets. These show that $L_{NH}^{(TRUN)}$ may be substantially different from the quantity $L_{N}^{(TRUN)} + L_{H}^{(TRUN)}$. 
A final point to note is that in this section and in Sections 6.2 and 6.3, we have discussed a model where the dependent variable $y_i$ is restricted to the interval $0 \leq y_i < \infty$. Yet, our findings have more general use. For instance, if the interval for $y_i$ had a lower non-zero limit $\gamma_1$, so that $\gamma_1 \leq y_i < \infty$, our results would be directly applicable after redefining $y_i$ and the intercept term, say $\beta_1$, respectively as $y_i - \gamma_1$ and $\beta_1 - \gamma_1$. Similarly, our results may be modified to deal with situations where $\gamma_1$ varies with each observation, and in situations where the upper limit for $y_i$ is finite. □

This concludes our discussion of the Truncated model. In the remainder of this Chapter we deal with another LDV model, the Tobit model.

6.5 THE TOBIT MODEL

Under the assumption that disturbances are normal (as discussed in Chapter 4) or truncated-normal (as discussed in Section 6.2), we would have a 'zero probability' of obtaining several observations $y_i$ which are identical. This is because both distributions are continuous. In applied work, situations have been found where the dependent variable $y_i$ takes the same value for a large number of the $N$ individuals. For example, in demand studies it is not uncommon to find that many individuals report zero-expenditures on some commodities. In these cases, analysis under the previous disturbance distributions (normal or truncated-normal) or - in fact - any continuous disturbance distribution, would be inappropriate. In order to have a 'valid' model specification we would need to allow the dependent variable to take the 'common value' with a non-zero probability.
This problem was first discussed in the econometric literature by Tobin (1958) while analysing household durable expenditure. The model Tobin used has come to be known as the Tobit model and - in this - the 'common value' of $y_i$ is zero. More formally, the Tobit model is defined as

$$y_i = x_i'\beta + u_i \quad \text{if} \quad \text{RHS} > 0$$

$$= 0 \quad \text{if} \quad \text{RHS} \leq 0.$$

Various authors have studied this and several results are now available. For instance, under the assumption of disturbance normality and homoscedasticity, Amemiya (1973) proves consistency and asymptotic normality of the MLE, and Olsen (1978) shows the uniqueness of the maximum of the likelihood function. In the remainder of this Chapter we discuss further aspects of this model. First we introduce necessary definitions and state results required for future sections.

As in Amemiya (1973, p.1004) we define an 'indicator variable' $\omega_i$ such that $\omega_i = 1$ if $y_i > 0$, and $\omega_i = 0$ if $y_i = 0$. We observe that given $x_i$ and under $u_i \sim NH$, the probability of a positive observation would be $P[y_i > 0] = P[\omega_i = 1] = F(x_i'\beta, \sigma^2) = F_i$ (see (6.1)) and the probability of a zero observation would be $P[y_i = 0] = P[\omega_i = 0] = 1 - F_i$. [Recall that in both the Truncated model and the usual (untruncated) regression model we had $P[y_i = 0] = 0$, whereas now $P[y_i = 0] > 0$.

We define $f_{1i} = (2\pi \sigma^2)^{-\frac{1}{2}} \exp[-(y_i - x_i'\beta)^2/(2\sigma^2)]$ and $f_{2i} = 1 - F_i$,

and note that, for the Tobit model, the p.d.f. of $y_i$ given $x_i$ is $f_Y(y_i) = f_{1i}$ if $y_i > 0$, and $f_Y(y_i) = f_{2i}$ if $y_i = 0$. Then, we may

Generalizations of this model may be dealt with easily by appropriate redefinition of terms (see Amemiya (1973, p.997)).
write the likelihood for the i'th observation as

\[ L_i = \frac{\omega_i}{f_{1i} f_{2i} (1 - \omega_i)} \]

From this, we have that the likelihood function for the N observations is

\[ L(\beta, \sigma^2) = \prod_{i=1}^{N} L_i = \prod_{i=1}^{N} \frac{\omega_i}{f_{1i} f_{2i} (1 - \omega_i)} \]

i.e.,

\[ L(\beta, \sigma^2) = \prod_{i=1}^{N} \left\{ \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[ -\frac{(y_i - x_i \beta)^2}{2\sigma^2} \right] \right\}^{\omega_i} (1 - \omega_i) \{1 - F_i\} \quad . \quad (6.18) \]

We denote by m the number of positive observations. Hence, N - m would be the number of zero observations. We assume the N observations are ordered so the first m are the positive ones. Then we would have that, for our particular realization, \( \omega_1 = \ldots = \omega_m = 1 \) and \( \omega_{m+1} = \ldots = \omega_N = 0 \). For these values of \( \omega_i \), \( L(\beta, \sigma^2) \) would reduce to

\[ \left\{ \prod_{i=1}^{m} \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[ -\frac{(y_i - x_i \beta)^2}{2\sigma^2} \right] \right\}^{\omega_i} \left\{ \prod_{i=m+1}^{N} (1 - F_i) \right\} \quad . \quad (6.19) \]

By maximizing (6.19), we obtain the MLE's of \( \beta \) and \( \sigma^2 \) which we denote by \( \hat{\beta} \) and \( \hat{\sigma}^2 \). On top of \( \beta \) and \( \sigma^2 \) we now use ~ rather than \( \hat{\ } \Hat{} \).
than to avoid confusion with the MLE's of the Truncated model discussed in Sections 6.2, 6.3 and 6.4. The values $\tilde{\beta}$ and $\tilde{\sigma}^2$ must satisfy the equations (see Fair (1977))

$$\tilde{\sigma}^2 = \frac{Y_1'(Y_1 - X_1\tilde{\beta})}{m} \quad (6.20)$$

and

$$\tilde{\beta} = \left(X_1'X_1\right)^{-1}X_1'Y_1 - \tilde{\sigma}(X_1'X_1)^{-1}X_1'\Lambda_2, \quad (6.21)$$

where $Y_1 = (y_1, \ldots, y_m)'$, $X_1 = (x_1, \ldots, x_m)'$, $X_2 = (x_{m+1}, \ldots, x_N)'$ and $\Lambda_2 = (\delta_{m+1}, \ldots, \delta_N)'$ with $\delta_i = \tilde{\sigma}f_i/(1-F_i)$, $\tilde{f}_i = f(x_1^i\tilde{\beta}, \tilde{\sigma}^2)$ and $\tilde{F}_i = F(x_1^i\tilde{\beta}, \tilde{\sigma}^2)$.

For the solution of the non-linear equations (6.20) and (6.21), Fair (1977) has proposed an iterative procedure and we may use this. After reaching convergence, the VCM of $(\tilde{\beta}', \tilde{\sigma}^2)$ would be estimated by $V^{-1}$, where $V$ is defined as in (6.6) but replacing $a_i$, $b_i$ and $c_i$ respectively by $\hat{a}_i$, $\hat{b}_i$ and $\hat{c}_i$, with

$$a_i^+ = -\frac{1}{\tilde{\sigma}^2} \left[(x_1^i\tilde{\beta})\tilde{f}_i - \frac{\tilde{\sigma}^2\tilde{f}_i}{1-F_i}\right],$$

$$b_i^+ = \frac{1}{2\tilde{\sigma}^4} \left[(x_1^i\tilde{\beta})^2\tilde{f}_i + \tilde{\sigma}^2\tilde{f}_i - \frac{\tilde{\sigma}^2(x_1^i\tilde{\beta})^2\tilde{f}_i}{1-F_i}\right],$$

and

$$c_i^+ = -\frac{1}{4\tilde{\sigma}^4} \left[(x_1^i\tilde{\beta})^3\tilde{f}_i + (x_1^i\tilde{\beta})\tilde{f}_i - \frac{(x_1^i\tilde{\beta})^2\tilde{f}_i}{1-F_i} - 2\tilde{F}_i\right].$$

(see Fair (1977, p.1725)). From this we may obtain large sample $t$ and $F$ tests.
It is important to note that the use of (6.19) (regarding m as fixed) to derive the information matrix would not be appropriate. In fact, to obtain the above estimate of the VCM of \((\beta', \sigma^2)\), i.e. the matrix \(V^{-1}\), the 'assignment' of the observations into zero and non-zero values was regarded as random. In other words, derivations were based on the likelihood function (6.18) in which the 'indicator variables' \(\omega_i\) appear explicitly. This is the correct procedure to follow and makes analysis of the Tobit model somewhat complicated, as may be seen from the derivations in Amemiya (1973, p.1004), or the proofs of the results reported in the next section.

6.6 NON-NORMALITY AND HETEROSCEDASTICITY IN THE TOBIT MODEL

The results presented in Section 6.5 for the Tobit model were derived under the assumption of disturbance normality and homoscedasticity. Now we comment on the effect of violation of these assumptions. Let us denote the p.d.f. of \(u_i\) by \(f^+(u_i)\), which may be non-normal and/or heteroscedastic. For this, we would have

\[
P[\omega_i = 1] = \int_{-x_i^{-1}}^{\infty} f^+(u_i) du_i = F_i^+ \text{ say,}
\]

and

\[
P[\omega_i = 0] = 1 - F_i^+ .
\]

Given \(x_i\), it follows that

\[
E[\omega_i] = 1 \cdot P[\omega_i = 1] + 0 \cdot P[\omega_i = 0] = F_i^+ \quad (6.22)
\]

and

\[
E[u_i \omega_i] = E[u_i \omega_i | \omega_i = 1] P[\omega_i = 1] + E[u_i \omega_i | \omega_i = 0] P[\omega_i = 0] ,
\]
i.e.,
\[ E[u_i \omega_i] = E[u_i | y_i > 0] F_i^+ \]  \hspace{1cm} (6.23)

Incorporating the 'indicator variables' \( \omega_i \) into (6.21) and after substitution for \( y_i = x_i \beta + u_i \) where \( u_i = (u_{1i}, \ldots, u_{m_i})' \), we would have
\[
\beta = \beta + \left( \frac{1}{N} \sum_{i=1}^{N} x_i x_i' \omega_i \right)^{-1} \left[ \frac{1}{N} \sum_{i=1}^{N} x_i u_i \omega_i - \frac{1}{N} \sum_{i=1}^{N} x_i \sigma^2 f_i (1-\omega_i)/(1-F_i) \right].
\]  \hspace{1cm} (6.24)

For \( \tilde{\beta} \) to be consistent we would need the second term on the RHS to have zero \( \text{plim} \). Assuming the sample size goes to infinity in multiples of \( N \), we have, using (6.22)
\[
\text{plim} \left( \frac{1}{N} \sum_{i=1}^{N} x_i x_i' \omega_i \right) = \frac{1}{N} \sum_{i=1}^{N} x_i x_i' F_i^+,
\]  \hspace{1cm}
which is a positive-definite matrix. So, for \( \text{plim} \tilde{\beta} = \beta \) we need the \( \text{plim} \) of the term in \( [ \ ] \) in (6.24) to be zero. Define
\[
P_1^T = \text{plim} \left( \frac{1}{N} \sum_{i=1}^{N} x_i u_i \omega_i \right)
\]
and note that, using (6.23),
\[
P_1^T = \frac{1}{N} \sum_{i=1}^{N} x_i E[u_i | y_i > 0] F_i^+.
\]

Also, define
\[
P_2^T = \text{plim} \left( \frac{1}{N} \sum_{i=1}^{N} x_i \sigma^2 f_i (1-\omega_i)/(1-F_i) \right).
\]
From these expressions we see that there is no reason why we should expect $P_1^T - P_2^T = 0$, i.e., $\text{plim } \tilde{\beta} = \beta$, unless $u_1$ is normal and homoscedastic. [Note that under disturbance $NH$, $E[u_1 | y_1 > 0]F_1^+$ becomes $\sigma^2 f_1$ in $P_1^T$, and $\sigma^2 f_1$ replaces $\tilde{\sigma}^2 f_1(1-\omega_1)/(1-\tilde{\omega}_1)$ in $P_2^T$, giving $P_1^T - P_2^T = 0$.]

The difficulty of a rigorous proof of the inconsistency of $\tilde{\beta}$ has been noted by White (1979) for the special case when $u \sim NH$ (see also Robinson (1981)). For insight on the problem of inconsistency, White (1979) carried out a number of sampling experiments under various non-normal distributions; concluding that "misspecification of (the disturbance) distribution of the Tobit model can have unfortunate consequences for estimation and inference". An additional finding is that of Warner (1976). He considered normal disturbances, and carried out a Monte Carlo experiment to study the sensitivity of three estimators of the Tobit model (including $\tilde{\beta}$) under the presence of heteroscedasticity; concluding that "for all three techniques heteroscedasticity presents a serious problem".

This evidence is sufficient to indicate the importance of testing for disturbance normality and homoscedasticity in the Tobit model. Because of the specific form of this model, the normality and/or homoscedasticity tests presented and suggested in Chapters 4 and 5 (or indeed, those of Sections 6.3 and 6.4) are not applicable. Here we use the same approach of Section 6.4 to derive a joint test for disturbance $NH$ in the Tobit model.

We begin by assuming that the p.d.f. of $u_1$, say $f(u_1)$, can be written as in (5.13), i.e. that it is a member (or is well approximated by a member) of the Pearson Family. Also, we assume that we are able
to specify $\sigma^2_i = \sigma^2 + z^*_1 \alpha^*$, where $z^*_1$ and $\alpha^*$ are defined as before.

Then we can write the likelihood for the $N$ observations as

$$L(\beta, \sigma^2, c_1, c_2, \alpha^*) = \prod_{i=1}^N \{ f(u_i) \} \left\{ 1 - \int_{-\infty}^{\infty} f(u_i) du_i \right\}^{(1-w_i)}, \quad (6.25)$$

where

$$f(u_i) = \frac{\exp \left[ \frac{c_1 - u_i}{\sigma^2 + z^*_1 \alpha^* - c_1 u_i + c_2 u_i^2} \right]}{\int_{-\infty}^{\infty} \exp \left[ \frac{c_1 - u_i}{\sigma^2 + z^*_1 \alpha^* - c_1 u_i + c_2 u_i^2} \right] du_i}$$

for $-\infty < u_i < \infty$ [see paragraph below equation (5.13)].

Our interest is to test $H_0: c_1 = c_2 = 0; \alpha^* = 0$, which implies disturbance normality and homoscedasticity. In this case the LM test statistic is given by (the proof of this result is lengthy and is outlined in Proposition 6 in Appendix F - page 318)

$$LM_{NH(TOBIT)} = \hat{d}_2^2 (\hat{F}_{22} - \hat{F}_{21} - \hat{F}_{11})^{-1} \hat{F}_{12}^2, \quad (6.26)$$

with

$$\hat{d}_2^2 = \left[ \begin{array}{c}
\Sigma \left( \frac{u_i}{\sigma^2} - \frac{u_i^3}{3\sigma^4} \right) - \Sigma \frac{\tilde{F}_i}{i=m+1 (1-\tilde{F}_i)} \left( \frac{\tilde{u}_i(1)}{\sigma^2} - \frac{\tilde{u}_i(3)}{3\sigma^4} \right) \\
\Sigma \left( \frac{3}{4} + \frac{u_i}{4\sigma^4} \right) - \Sigma \frac{\tilde{F}_i}{i=m+1 (1-\tilde{F}_i)} \left( \frac{3}{4} + \frac{\tilde{u}_i(4)}{4\sigma^4} \right) \\
\Sigma \left( -\frac{1}{2\sigma^2} + \frac{z^*_1}{2\sigma^4} \right) - \Sigma \frac{\tilde{F}_i}{i=m+1 (1-\tilde{F}_i)} \left( -\frac{1}{2\sigma^2} + \frac{\tilde{u}_i(2)}{2\sigma^4} \right) z^*_1 \end{array} \right] \quad (6.27)$$
and

\[
\hat{I} = \begin{bmatrix}
\hat{I}_{11} & \hat{I}_{12} \\
\hat{I}_{21} & \hat{I}_{22}
\end{bmatrix} = \begin{bmatrix}
b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\
b_{12} & b_{22} & b_{23} & b_{24} & b_{25} \\
b_{13} & b_{23} & b_{33} & b_{34} & b_{35} \\
b_{14} & b_{24} & b_{34} & b_{44} & b_{45} \\
b_{15} & b_{25} & b_{35} & b_{45} & b_{55}
\end{bmatrix},
\]

where

\[
b_{11} = \frac{1}{\sigma^4} \sum_{i=1}^{N} x_i x_i' \left[ \tilde{F}_i \left( \frac{\tilde{u}_i(1)}{2\sigma^4} + \frac{\tilde{F}_i}{1-F_i} \right) \right]
\]

\[
b_{12} = \sum_{i=1}^{N} x_i \left[ \frac{\tilde{u}_i(1)}{\sigma^4} - \frac{\tilde{u}_i(3)}{3\sigma^6} \right] + \frac{\tilde{F}_i^2}{1-F_i} \left[ \frac{\tilde{u}_i(1)}{2\sigma^4} - \frac{\tilde{u}_i(3)}{3\sigma^6} \right]
\]

\[
b_{13} = \sum_{i=1}^{N} x_i \left[ \frac{3\tilde{u}_i(1)}{4\sigma^2} - \frac{\tilde{u}_i(5)}{4\sigma^6} \right] + \frac{\tilde{F}_i^2}{1-F_i} \left[ \frac{3\tilde{u}_i(1)}{4\sigma^2} + \frac{\tilde{u}_i(5)}{4\sigma^6} \right]
\]

\[
b_{14} = \sum_{i=1}^{N} x_i \left[ \frac{\tilde{u}_i(1)}{2\sigma^4} + \frac{\tilde{F}_i}{1-F_i} \right] \left[ \frac{1}{2\sigma^4} + \frac{\tilde{u}_i(3)}{2\sigma^6} \right] + \frac{\tilde{F}_i^2}{1-F_i} \left[ \frac{1}{2\sigma^4} + \frac{\tilde{u}_i(3)}{2\sigma^6} \right]
\]

\[
b_{15} = \sum_{i=1}^{N} x_i z_i' \left[ \frac{1}{4\sigma^4} - \frac{\tilde{u}_i(2)}{2\sigma^6} + \frac{\tilde{F}_i}{1-F_i} \right] \left[ \frac{1}{4\sigma^4} - \frac{\tilde{u}_i(2)}{2\sigma^6} + \frac{\tilde{F}_i}{1-F_i} \right]
\]

\[
b_{22} = \sum_{i=1}^{N} \left[ \frac{1}{4\sigma^4} - \frac{\tilde{u}_i(2)}{2\sigma^6} + \frac{\tilde{F}_i}{1-F_i} \right] \left[ \frac{1}{4\sigma^4} - \frac{\tilde{u}_i(2)}{2\sigma^6} + \frac{\tilde{F}_i}{1-F_i} \right]
\]
\[ b_{23} = \sum_{i=1}^{N} \left[ F_{i} \left( -\frac{u_{i}(1)}{2\sigma^{4}} + \frac{u_{i}(3)}{6\sigma^{6}} + \frac{u_{i}(5)}{2\sigma^{6}} - \frac{u_{i}(7)}{6\sigma^{8}} \right) \right] \]

\[ + \frac{F_{i}^{2}}{(1-F_{i})} \left( -\frac{u_{i}(1)}{2\sigma^{4}} + \frac{u_{i}(3)}{6\sigma^{6}} + \frac{u_{i}(5)}{2\sigma^{6}} - \frac{u_{i}(7)}{6\sigma^{8}} \right) \]

\[ b_{24} = \sum_{i=1}^{N} \left[ F_{i} \left( -\frac{3u_{i}(2)}{8\sigma^{2}} - \frac{u_{i}(4)}{8\sigma^{6}} + \frac{u_{i}(6)}{8\sigma^{8}} \right) \right] \]

\[ + \frac{F_{i}^{2}}{(1-F_{i})} \left( -\frac{3u_{i}(2)}{8\sigma^{2}} - \frac{u_{i}(4)}{8\sigma^{6}} + \frac{u_{i}(6)}{8\sigma^{8}} \right) \]

\[ b_{25} = \sum_{i=1}^{N} z_{i}^{k} \left[ F_{i} \left( \frac{1}{4\sigma^{4}} - \frac{u_{i}(2)}{2\sigma^{6}} + \frac{u_{i}(4)}{4\sigma^{8}} \right) \right] \]

\[ + \frac{F_{i}^{2}}{(1-F_{i})} \left( \frac{1}{4\sigma^{4}} - \frac{u_{i}(2)}{2\sigma^{6}} + \frac{u_{i}(4)}{4\sigma^{8}} \right) \]

\[ b_{33} = \sum_{i=1}^{N} \left[ F_{i} \left( -\frac{2u_{i}(4)}{3\sigma^{6}} + \frac{u_{i}(6)}{9\sigma^{8}} \right) \right] \]

\[ + \frac{F_{i}^{2}}{(1-F_{i})} \left( -\frac{2u_{i}(4)}{3\sigma^{6}} + \frac{u_{i}(6)}{9\sigma^{8}} \right) \]

\[ b_{34} = \sum_{i=1}^{N} \left[ F_{i} \left( -\frac{3u_{i}(1)}{4\sigma^{2}} + \frac{u_{i}(3)}{4\sigma^{4}} + \frac{u_{i}(5)}{4\sigma^{6}} - \frac{u_{i}(7)}{12\sigma^{8}} \right) \right] \]

\[ + \frac{F_{i}^{2}}{(1-F_{i})} \left( -\frac{3u_{i}(1)}{4\sigma^{2}} + \frac{u_{i}(3)}{4\sigma^{4}} + \frac{u_{i}(5)}{4\sigma^{6}} - \frac{u_{i}(7)}{12\sigma^{8}} \right) \]

\[ b_{35} = \sum_{i=1}^{N} z_{i}^{k} \left[ F_{i} \left( -\frac{u_{i}(1)}{2\sigma^{4}} + \frac{u_{i}(3)}{6\sigma^{6}} + \frac{u_{i}(5)}{2\sigma^{6}} - \frac{u_{i}(7)}{6\sigma^{8}} \right) \right] \]

\[ + \frac{F_{i}^{2}}{(1-F_{i})} \left( -\frac{u_{i}(1)}{2\sigma^{4}} + \frac{u_{i}(3)}{6\sigma^{6}} + \frac{u_{i}(5)}{2\sigma^{6}} - \frac{u_{i}(7)}{6\sigma^{8}} \right) \]
\[ b_{44} = \sum_{i=1}^{N} \left[ \frac{1}{16} - \frac{3\tilde{u}_i(4) + \tilde{u}_i(8)}{16\sigma^8} \right] + \frac{\tilde{F}_i^2}{(1-F_1)} \left[ \frac{9}{16} - \frac{3\tilde{u}_i(4) + \tilde{u}_i(4)}{16\sigma^8} \right] \]

\[ b_{45} = \sum_{i=1}^{N} \tilde{z}_i \left[ \frac{3}{8\sigma^2} - \frac{3\tilde{u}_i(2) - \tilde{u}_i(4) + \tilde{u}_i(6)}{8\sigma^4} - \frac{3\tilde{u}_i(2) - \tilde{u}_i(4) + \tilde{u}_i(6)}{8\sigma^6} \right] + \frac{\tilde{F}_i^2}{(1-F_1)} \left[ \frac{3}{8\sigma^2} - \frac{3\tilde{u}_i(2) - \tilde{u}_i(4) + \tilde{u}_i(6)}{8\sigma^4} - \frac{3\tilde{u}_i(2) - \tilde{u}_i(4) + \tilde{u}_i(6)}{8\sigma^6} \right] \]

and

\[ b_{55} = \sum_{i=1}^{N} \tilde{z}_i \tilde{z}_i \left[ \frac{1}{4\sigma^4} - \frac{\tilde{u}_i(2)}{2\sigma^6} + \frac{\tilde{u}_i(4)}{4\sigma^8} \right] + \frac{\tilde{F}_i^2}{(1-F_1)} \left[ \frac{1}{4\sigma^4} - \frac{\tilde{u}_i(2)}{2\sigma^6} + \frac{\tilde{u}_i(4)}{4\sigma^8} \right] \]

and where \( \tilde{u}_i = y_i - x_1 \tilde{\beta} \), and \( \tilde{u}_i(j) \) is defined as \( \tilde{u}_i(j) \) in (6.12) but using \( \tilde{\beta} \) and \( \tilde{\sigma}^2 \) [i.e., the solution to equations (6.20) and (6.21)] in place of \( \hat{\beta} \) and \( \hat{\sigma}^2 \).

The test statistic (6.26) has been written with the suffix \( NH(TOBIT) \), to indicate this refers to a normality and homoscedasticity test in the Tobit model. Under \( H_0: c_1 = c_2 = 0; \alpha^* = 0 \), \( LM_{NH(TOBIT)} \) would be asymptotically distributed as \( \chi^2(p+1) \) and we would reject \( H_0 \) for large values of \( LM_{NH(TOBIT)} \).

It is interesting to note some 'particular cases' of \( LM_{NH(TOBIT)} \). The first refers to a normality test under the assumption of homoscedasticity. Here we would test \( H_0: c_1 = c_2 = 0 \). Application of the LM procedure in this case would give a test statistic, denoted by \( LM_{N(TOBIT)} \), which would be defined as \( LM_{NH(TOBIT)} \), except that the row corresponding to \( \alpha^* \) in \( \hat{d}_2 \) and the fifth row and column in \( \hat{I} \), would be omitted. Given homoscedasticity and under \( H_0: c_1 = c_2 = 0 \), \( LM_{N(TOBIT)} \) would be asymptotically distributed as \( \chi^2(2) \).
The second 'particular case' of \( LM_{NH(TOBIT)} \) refers to a homoscedasticity test under the assumption of normality. Here we would test \( H_0: \alpha^* = 0 \). Application of the LM procedure would give a test statistic, denoted by \( LM_{H(TOBIT)} \) and defined as \( LM_{NH(TOBIT)} \) but removing the rows corresponding to \( c_1 \) and \( c_2 \) in \( d_2 \) and the 'third' and 'fourth' rows and columns in \( \hat{I} \). This is the test suggested in Jarque (1981c), and may be written explicitly in the form

\[
LM_{H(TOBIT)} = \frac{\hat{\sigma}^2}{4} \phi_5'Z(Z'\phi_4Z)^{-1}Z'\phi_5,
\]

where \( \phi_5 \) is a 1 by \( N \) vector defined by

\[
\phi_5 = [\frac{\hat{\sigma}^2}{\sigma^2}, \ldots, \frac{\hat{\sigma}^2}{\sigma^2}, \frac{(u_{m+1}^+ - \sigma^2)}{\sigma^2}, \ldots, \frac{(u_N^+ - \sigma^2)}{\sigma^2}]
\]

with \( u_1^+ = \frac{\hat{\sigma}^2}{\sigma^2}(1 + x'_{1} \hat{\beta}_{1}/(1 - \hat{\tau}_1)) \) and

\[
\phi_4 = \phi_1 - \phi_2 X(X'\phi_3 X)^{-1}X'\phi_2,
\]

and where \( \phi_1, \phi_2 \) and \( \phi_3 \) are \( N \) by \( N \) diagonal matrices given by

\[
\phi_1 = \text{diag}(\hat{\sigma}^6 c_1^+, \ldots, \hat{\sigma}^6 c_N^+)
\]

\[
\phi_2 = \text{diag}(\hat{\sigma}^4 b_1^+, \ldots, \hat{\sigma}^4 b_N^+)
\]

and

\[
\phi_3 = \text{diag}(\hat{\sigma}^2 a_1^+, \ldots, \hat{\sigma}^2 a_N^+)
\]

with \( c_i^+, b_i^+ \) and \( a_i^+ \) defined below equation (6.21) in Section 6.5.

Under \( H_0: \alpha^* = 0 \), \( LM_{H(TOBIT)} \) would be asymptotically distributed as \( \chi^2(p-1) \).
We should also point out that - as expected - if we set \( \hat{f}_i = 0; \) \( m = N; \) and allow \( \hat{F}_i \) to tend to unity, the disturbance specification tests for the Tobit model would reduce to the corresponding tests for the usual (untruncated) regression model (e.g., \( \text{LM}^*_{H(\text{Tobit})} \) would reduce to \( \text{LM}^*_{H} \) given in equation (5.6)).

When using the test statistic \( \text{LM}^*_{NH(\text{Tobit})} \) [or \( \text{LM}^*_{H(\text{Tobit})} \)] under the maintained assumption of \( H; \) or \( \text{LM}^*_{H(\text{Tobit})} \) under the maintained assumption of \( N \) acceptance of the null hypothesis would mean one may proceed with the analysis of the model as in Section 6.5. If \( \text{LM}^*_{NH(\text{Tobit})} \) [or \( \text{LM}^*_{H(\text{Tobit})} \)] rejected disturbance \( NH, \) one could consider estimating the model using MLE on the likelihood under \( \sigma_i^2 = \sigma^2 + z_i^2 \hat{a}_i \). Regarding 'adjustments' for the use of transformations so that normality is better suited needs further investigation. Problems arise here (as for the Truncated model) when attempting to apply a multiple comparison procedure based on \( \text{LM}^*_{N(\text{Tobit})} \) and \( \text{LM}^*_{H(\text{Tobit})}. \) This is also an area that requires additional study.

We finalize this section noting that, under the assumption of homoscedastic disturbances, White (1979) has suggested normality tests for the Tobit model. These consist of computing the normality tests \( \sqrt{\hat{\beta}_1}, \hat{\beta}_2 \) and \( \hat{R} \) (see Section 4.5) with transformed residuals corresponding to the non-zero observations only. Indeed, these residuals may also be used on the \( \text{LM}^*_{N} \) test (see equation (4.5)). It seems that our test \( \text{LM}^*_{N(\text{Tobit})} \) would be preferable to tests based on transformed residuals since, firstly, it uses all the information in the sample and secondly, because - being an LM test for this situation - it has optimal large-sample power properties. Another test for the Tobit model is suggested in Nelson (1981). This is a general specification test and is based on
the comparison of the sample and hypothesized moments of $x_i y_i$.
Nothing is known about its power properties. In contrast, our tests
are devised for testing specific distributional assumptions and, as
mentioned earlier, are known to have optimal power properties.

6.7 CONCLUDING REMARKS

Two limited dependent variable models were analyzed in this Chapter
and, for both models, disturbance specification tests were suggested.
These tests require computation of the MLE's under the null hypothesis,
which may be obtained by iterative procedures. Other estimators for
these models, which are consistent - but not MLE - are also available.
By appropriate modifications, our tests may be applied when using these
estimators.

In a general context, Neyman (1959) considered this problem and
proposed a pseudo - LM test. This is denoted by $C(\alpha)$ and is defined
by

$$C(\alpha) = (\hat{d}_2 - \hat{I}_{21} \hat{I}_{11} \hat{d}_1)' (\hat{I}_{22} - \hat{I}_{21} \hat{I}_{11} \hat{I}_{12})^{-1} (\hat{d}_2 - \hat{I}_{21} \hat{I}_{11} \hat{d}_1),$$

where all quantities are as in Section 4.2, except that the $\hat{\cdot}$ denotes
quantities evaluated at $\theta = \hat{\theta}$, where $\hat{\theta}$ is now a consistent estimator
of $\theta$ under $H_0: \theta_2 = 0$. (When $\hat{\theta}$ is the MLE note that $\hat{d}_1 = 0$,
so $LM = C(\alpha)$). The asymptotic properties of $C(\alpha)$ are identical to
those of $LM$ and hence, under $H_0: \theta_2 = 0$, it would be asymptotically
distributed as $\chi^2$, with degrees of freedom equal to the dimension of
$\theta_2$. We would accept $H_0$ if the computed value of $C(\alpha)$ was less than
the appropriate upper point of this $\chi^2$ distribution.
To illustrate the use of the \( C(\alpha) \) test we consider the Tobit model. For this, Amemiya (1973) has proposed a consistent computationally simple estimator, say \( \tilde{\beta} \) and \( \tilde{\sigma}^2 \). If we were interested in testing for homoscedasticity (under the maintained hypothesis of disturbance normality) using \( \tilde{\beta} \) and \( \tilde{\sigma}^2 \) then the \( C(\alpha) \) test would be given by

\[
C(\alpha) = \frac{\tilde{\sigma}^2}{4} \phi_7' Z (Z' \phi_4 Z)^{-1} Z' \phi_7,
\]

where \( \phi_7 = (\phi_5', \phi_6') W, \phi_6 = [\tilde{u}_1, \ldots, \tilde{u}_{m}, \tilde{u}_{m+1}/(1-F_{m+1}), \ldots, \tilde{u}_{N}/(1-F_N)] \)
and \( W' = [I_N, -2\phi_3' X (X' \phi_3 X)^{-1} X'/\tilde{\sigma}^2] \), and where all quantities are as defined in Section 6.6 but should be evaluated at \( (\tilde{\beta}', \tilde{\sigma}^2) \) rather than \( (\beta', \sigma^2) \).

The specification tests suggested in this Chapter are used in Chapter 9. There we apply both the Truncated model and the Tobit model in a study of Family Budgets in México.
CHAPTER 7

THE PROBLEM OF PARAMETER VARIATION*

"Divide et impera"

Louis XI

7.1 INTRODUCTION

In Chapters 2 to 6 we discussed various aspects of model (1.1) under assumption [6], i.e., assuming the parameter vector \( \beta \) was the same for all the individuals in the cross-section. Under this assumption (and provided no functional relation exists between the regressors) the \( k' \)th element in \( \beta \), say \( \beta_k \), would be interpreted as the partial derivative of \( Y \) with respect to \( X_k \) - irrespective of \( i \). In practice, there may be reasons to believe the increase in \( Y \), due to a unit increase in \( X_k \), is not the same for all the individuals in the cross-section. Furthermore, it may even be thought that each individual reacts in its own particular way to an increase in \( X_k \), i.e., that each has 'its own value' of \( \beta_k \). To account for this parameter variation, \( \beta \) could be replaced by \( \beta_i \) in (1.1) giving

\[
y_i = x_i' \beta_i + u_i \quad i = 1, \ldots, N
\]

(7.1)

* This Chapter overlaps considerably with Jarque (1980b).
Without additional assumptions, it is not possible to proceed any further due to the fact that - in (7.1) - there are NK parameters to be estimated (apart from those related to disturbance terms) and only N observations. Various assumptions can be made to overcome this problem. For instance, one may assume parameter variation only occurs in the coefficient associated with the intercept term, and therefore introduce variation through the use of dummy variables. Other approaches include Random Coefficient Models (e.g. see Hildreth and Houck (1968) and Swamy (1971)); Switching Regressions (e.g. see Goldfeld and Quandt (1973,1976)); Segmented Polynomial Regressions (e.g. see Hudson (1966) and Gallant and Fuller (1973)); Piecewise Regressions (e.g. see McGee and Carleton (1970)) and Spline Regression Models (e.g. see Poirier (1976)).

Here we assume we can specify a set of p variables Z₁,...,Zₚ that affect the value of the vectors ßᵢ.¹ Also, that zᵢ = (z₁ᵢ,...,zₚᵢ) is known for all the N individuals in the cross-section, where zⱼᵢ is the value of Zⱼ for individual i. Further, we assume to have

\[ ßᵢ = F(zᵢ) + εᵢ \]

i = 1,...,N , (7.2)

where \( F(zᵢ) = (F₁(zᵢ),...,Fₚ(zᵢ))' \); \( Fₖ(zᵢ) \) denotes a non-stochastic function which is equal to the expectation of the k'th element of ßᵢ given \( zᵢ \); and \( εᵢ = (ε₁ᵢ,...,εₚᵢ)' \) is a K by 1 random vector with zero expectation and VCM given by \( E[εᵢεᵢ'] = Ω \) if \( i = j \) and 0 otherwise. We refer to parameter variation of the kind specified in (7.2) as Systematic Parameter Variation (SPV).

¹ These Z-variables may contain expressions involving the regressors, e.g., if parameter variation is thought to be due to having misspecified the functional form in (1.1).
In this Chapter we concentrate on the SPV model given by equations (7.1) and (7.2). In Section 7.2 we comment on a test for SPV, and in Section 7.3 we suggest a two stage estimation procedure that may be used when there is evidence of parameter variation. The first stage of our estimation procedure is presented in Section 7.4. The second stage is discussed in Section 7.5. A numerical exercise is included in Section 7.6, comparing three estimation procedures under various forms of SPV. Other possible approaches to the problem of parameter variation and some concluding remarks are found in Section 7.7.

7.2 TESTING FOR SYSTEMATIC PARAMETER VARIATION

We note the existence of $\epsilon_1$ in (7.2) makes $\beta_1$ a random vector; also, that if an element in $x_1$ is constant for all $i$, $u_1$ would not be distinguishable from the varying intercept and it could be subsumed into the latter. We assume our regression model contains an intercept and (without loss of generality) omit the term $u_1$ in (7.1). We also make assumption [7] stated in Section 1.2.

If $F(z_1)$ were known, substitution of (7.2) into (7.1) would yield an equation amenable to econometric analysis; and its estimation could be carried out – for example – by the use of nonlinear procedures. Unfortunately, in general $F(z_1)$ would be unknown. If we estimated equation (1.1) – neglecting equation (7.2) – problems would arise because of functional misspecification. We may, therefore, be interested in testing the existence of SPV. We have noted $F(z_1)$ is in general unknown. So, to derive a test for SPV we proceed under the presumption that existence of SPV may be detected (hopefully in many cases) by assuming linearity, i.e., by setting
and testing $H_0: \gamma_1^t = \ldots = \gamma_K^t = 0$, where $\gamma_K$ is a scalar and $\gamma_k^t = (\gamma_{k1}, \ldots, \gamma_{kp})$ is a 1 by $p$ vector of coefficients, for $k = 1, \ldots, K$.

When we substitute (7.3) into our model we obtain

$$y_i = x_i^t \gamma + u_{i+}$$  \hspace{1cm} (7.4)$$

where $x_i^t = (x_i^t, (z_i^t \otimes x_i^t))$, $\gamma = \text{Vec}(\Gamma)$, $u_{i+} = x_i^t \varepsilon_i$, $\otimes$ denotes Kronecker product, and $\text{Vec}(\cdot)$ is the vector operator such that, if $A$ is an $n$ by $r$ matrix given by $A = (a_1, \ldots, a_r)$, $\text{Vec}(A)$ is an $nr$ by 1 vector equal to $(a_1, \ldots, a_r)'$.

The disturbances $u_{i+}$ in (7.4) are heteroscedastic but, nevertheless, we can easily test for SPV by using the heteroscedasticity-consistent test suggested by White (1980c, p.820). For this problem the test statistic is

$$F_{SPV} = \hat{\gamma}' \hat{R}' [R(X_+X_+)^{-1}(\sum_{i=1}^N \hat{u}_{i+}^2 x_i^t x_i^t)' (X_+X_+)^{-1} R']^{-1} \hat{R} \hat{\gamma},$$

where $\hat{\gamma} = (X_+X_+)^{-1} X_+ y$, $X_+$ is an $N$ by $(K+Kp)$ matrix with $i$'th row given by $x_i^t$; $R = [0; I_{Kp}]$ is a $Kp$ by $(K+Kp)$ matrix, and $\hat{u}_{i+} = y_i - x_i^t \hat{\gamma}$. Under $H_0: \gamma_1 = \ldots = \gamma_K = 0$ (and provided regular conditions are satisfied) $F_{SPV}$ would be asymptotically distributed as $\chi^2_{(Kp)}$. $H_0$ would be rejected for large values of $F_{SPV}$. If $H_0$
were accepted, we could say there is lack of evidence of SPV and use results of the usual regression model, or results of the random coefficient model (e.g., see Hildreth and Houck (1968)). If $H_0$ is rejected, we may follow two approaches: Firstly, we may suppose $\Gamma(1,z_1')$' is a reasonably good approximation to $F(z_1)$, and regard $x_1 \hat{y}$ as the estimated model; Secondly, we may use the estimation procedure suggested in the next Section. (In Section 7.6 this suggested procedure is found to perform better than the first procedure in terms of goodness-of-fit).

7.3 A TWO STAGE ESTIMATION PROCEDURE

We assume each element of $F(z)$ is a 'smooth function' over the region of interest in the loose sense that, for values of $z = (Z_1, \ldots, Z_p)'$ that are 'close', the values of $F(z)$ would also be 'close'. The motivation for the approach of this section is the idea that if the $N$ individuals are classified into say $L$ groups, so that within a group $h$ the values of $z_1$ are 'close', then - by smoothness of $F(z)$ - the values $F(z_1)$ would be 'close' for members of that group and could be approximated by the group mean.

Now we introduce necessary definitions. Let $I_h$ be the subset of the set of integers $\{1,2,\ldots,N\}$ that defines group $h$ for a given classification; $D_{ih}$ be a dummy variable that takes the value 1 if $i \in I_h$ and 0 if not; and $N_h$ be the number of individuals in group $h$, for $h = 1, \ldots, L$. In addition, observe (7.2) and define

$$\beta(h) = \sum_{i \in I_h} F(z_i)/N_h \quad (7.5)$$
Similarly, define $\bar{\beta} = \sum_{i=1}^{N} F(z_i)/N$ and note that, using (7.5), this is

$$\bar{\beta} = \sum_{h=1}^{L} \frac{N}{N} \beta(h) .$$  (7.6)

We refer to the vectors $\beta(1), \ldots, \beta(L)$ as regimes, and to $\bar{\beta}$ as the macroparameter.

In terms of $\beta(h)$ our original model (given by (7.2) and (7.1) without $u_i$) may be written as

$$y_i = x_i^T \beta_i$$  (7.7)

with

$$\beta_i = \beta(h) + v_{ih} \quad i \in I_h; \ h = 1, \ldots, L ,$$  (7.8)

where $v_{ih} = F(z_i) - \beta(h) + \varepsilon_i$. It may be shown that $E[v_{ih}'] = E_E [v_{ih}'] = 0$ for all $i, h$, and we use $\Delta_h$ to denote $E[v_{ih}v'_{jh}]$ for $i = j$. It is also interesting to note that equation (7.8) may be written as

$$\beta_i = \sum_{h=1}^{L} \beta(h)D_{ih} + v_i \quad i = 1, \ldots, N ,$$  (7.9)

where $v_i = \sum_{h=1}^{L} v_{ih}D_{ih}$.

The first term in the RHS of (7.9) is a vector containing K step-functions, given by
and may be regarded as an approximation to \( F(z^1) \). Our proposal is to estimate the model using \( F^S(z^1) \), i.e., to use (7.9) rather than (7.2). This estimation problem may be more specifically stated as - how to classify the \( N \) individuals into \( L \) groups, and - how to estimate the regimes - so the resulting step functions \( F^S(z^1) \) are the 'best' approximation to the elements in \( F(z^1) \). To estimate \( F^S(z^1) \) we may proceed in two stages. In the \textit{first stage}, \( L \) and \( I_1, I_2, \ldots, I_L \) (and hence \( D_{ih} \)) would be determined by the use of an appropriate classification or clustering criterion. This is discussed in Section 7.4. In the \textit{second stage} the parameter vectors \( \beta(1), \ldots, \beta(L) \) and \( \bar{\beta} \) (i.e., the regimes and the macroparameter) would be estimated by the use of existing econometric procedures. This is illustrated in Section 7.5.

Before concluding this Section, we note the problem of classification of individuals has been referred to in the econometric literature as a 'sample separation problem'. Various authors - although in perhaps different contexts - have commented on this. For instance, Kooyman (1976, p.127) states that observations should be divided into groups "that are homogeneous in respect of value of the parameters" and notes that - unfortunately - subdivision "is in most cases subjective". Similarly, Poirier (1976, p.155) considers the choice of sample partition, and notes the 'difficulty of the problem'. In turn, Chenery and Syrquin (1975, p.162) state that

"splitting the sample and estimating separate patterns for the subgroups may contribute to a better analysis". They note the classification "should rely as much as possible on theoretical arguments". Also that "clustering techniques may be useful in suggesting ways to quantify theory-based group factors, and (that) its applicability to this problem should be further studied".
The approach presented in the next Section (which uses clustering techniques) provides a 'less-subjective' solution to the problem of econometric sample separation. We hope our results lead to the solution of similar problems (e.g., determination of knot location in spline functions).

7.4 FIRST STAGE: Clustering of Individuals

The first stage of our estimation procedure deals with a classification problem, and for this it may seem natural to use Cluster Analysis. Cluster Analysis is a generic term applied to a set of classification techniques. A classification, as generally understood, allocates individuals or entities to initially undefined groups or clusters, so that entities in a cluster are in some sense close to one another.

In the previous Section it was said that if \( z_i \) was 'close' to \( z_j \), it would be assumed the conditional expectation of \( \beta_i \), given \( z_i \), would be 'close' to that of \( \beta_j \) given \( z_j \). The term 'close' was left undefined. Of course a definition is required, and for it a distance measure has to be given. There are many of these. For example, Cormack (1971) presents ten different ones which have been proposed by several authors. It is not the intention to review these here. The important point to note is that various distance measures are used as optimizing criteria in existing clustering algorithms (e.g., see Bolshev (1969), Everitt (1974), Ball (1971) or Hartigan (1975)) and that these criteria have arisen in many fields (e.g. Biology, Psychology, Anthropology and Physics). In general, different clustering

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2 Other names for Cluster Analysis are Q-Analysis, Typology, Grouping, Clumping, Numerical Taxonomy and Unsupervised Pattern Recognition.
criteria would provide different classifications. Although many might seem appealing for the purpose of classification, to this stage it is not clear how these criteria relate to estimation aims of the models here considered.

In Subsection 7.4.1 a clustering criterion is suggested which is derived within an econometric estimation framework. This is obtained by maximizing the "Overall Relative Explanatory Power" of \( F^S(z_i) \) to the conditional expectation of \( \beta_i \) given \( z_i \), i.e., \( F(z_i) \).

In Subsection 7.4.2 several indicators are given for the determination of the number of groups in which the individuals should be classified.

7.4.1 Determination of \( I_1, I_2, \ldots, I_L \)

For the purpose of obtaining our clustering criterion we shall use equation (7.3), i.e., we shall set \( F(z_i) \) equal to \( r(1, z_i) \). This amounts to taking a Taylor-series expansion of \( F_k(z_i) \), and neglecting all the nonlinear terms in the derivation of the clustering criterion. It has been assumed that \( F_k(z_i) \) is a smooth function, and our use of equation (7.3) is based on the presumption that the 'optimum classification' should not be too sensitive to departures from linearity.

\[ \square \]

3 The approach presented was motivated by the work of Aigner, Goldberger and Kalton (1975). They study the explanatory power of dummy variables in a regression equation where one independent variable is categorized. Here the dependent variable is the vector \( \beta_i \) and the more general case where \( p \) variables are to be 'categorized' is considered.

4 Throughout this Subsection it is assumed that \( L \) is given. The determination of \( L \) is discussed in Subsection 7.4.2.
We first consider the case $p = 1$, i.e., the case where $\beta_1$ depends on a single variable, say $Z_1$. Then, $\beta_1$ would be given by (set $p = 1$ in (7.3) and substitute the result in (7.2))

$$\beta_1 = \begin{bmatrix} Y_{10} & Y_{11} \\ Y_{20} & Y_{21} \\ \vdots & \vdots \\ Y_{K0} & Y_{K1} \end{bmatrix} \begin{bmatrix} 1 \\ z_{i1} \end{bmatrix} + \varepsilon_i,$$

(7.10)

and the $k$'th element of $\beta_1$ would be given by

$$\beta_{1k} = F_k(z_{i1}) + \varepsilon_{ik},$$

(7.11)

where $F_k(z_{i1}) = \gamma_{ko} + \gamma_{kl} z_{i1}$, for $i = 1, \ldots, N$ and $k = 1, \ldots, K$. It may be shown the variance explained by the regression of $\beta_{1k}$ on $F_k(z_{i1})$ is equal to

$$R^2_{(1,k)} = \frac{\gamma_{kl}^2 V[Z_1]}{V[\beta_{1k}]}.$$

(7.12)

where $V[Z_1] = \sum_{i=1}^{N} (z_{i1} - \bar{z}_1)^2 / N$ and $\bar{z}_1 = \sum_{i=1}^{N} z_{i1} / N$. (For proof see Proposition 1 in Appendix G in page 323 and set $p = 1$).

Now we assume the individuals are classified into $L$ groups, and that $F_k(z_{i1})$ is approximated by a step function $F^S_k(z_{i1})$ with value $\beta_k(h)$ for all the $N_h$ individuals in group $h$. In this case we may write (see (7.9))

$$\beta_{1k} = F^S_k(z_{i1}) + v_{ik},$$

(7.13)
where $F_k(z_{i1}) = \beta_k(1)D_{i1} + \ldots + \beta_k(L)D_{iL}$ for $i = 1, 2, \ldots, N$ and $k = 1, \ldots, K$. It may be shown (see Proposition 2 in Appendix G in page 324 and set $p = 1$) that for (7.13) the explained variation of the regression, say $R^2(2,k)$, is given by

$$R^2(2,k) = \frac{\sum_{h=1}^{L} N_h \left( \overline{z}_{h1} - \overline{z}_1 \right)^2}{\text{V}[\beta_{ik}]} ,$$

(7.14)

where $\overline{z}_{h1}$ is the h'th group mean of $Z_1$.

We take the ratio of $R^2(2,k)$ to $R^2(1,k)$ as a measure of "Relative Explanatory Power" (this term is used by Aigner, Goldberger and Kalton (1975)). The measure refers to the explanatory power of the step function approximation $F_k^S(z_{i1})$ made to $F_k(z_{i1})$, and is more formally defined as

$$\delta^2_k = \frac{R^2(2,k)}{R^2(1,k)} = \frac{\sum_{h=1}^{L} N_h \left( \overline{z}_{h1} - \overline{z}_1 \right)^2}{\text{V}[Z_1]} ,$$

(7.15)

We may see $\delta^2_k$ is also the squared 'correlation coefficient' between $F_k^S(z_{i1})$ and $F_k(z_{i1})$, and the complement of 'information loss' due to a step function approximation when (7.11) is true. An alternative expression for $\delta^2_k$ is obtained by using the identity

$$\sum_{h=1}^{L} N_h \left( \overline{z}_{h1} - \overline{z}_1 \right)^2 = \text{V}[Z_1] - \frac{1}{N} \sum_{h=1}^{L} \sum_{i=1}^{N} (z_{hil} - \overline{z}_{h1})^2 ,$$

(7.16)

where $z_{hil}$ is the value of $Z_1$ for the i'th individual in group $h$, giving...
\[ \hat{\alpha}_k^2 = 1 - \frac{D}{V[Z_1]} \quad (7.17) \]

with
\[ D = \frac{1}{N} \sum_{h=1}^{L} \sum_{i=1}^{N_h} (z_{hil} - \bar{z}_{hl})^2. \quad (7.18) \]

The average of the "Relative Explanatory Power" coefficients, i.e., the average of the K squared 'correlation coefficients' \( \hat{\alpha}_1^2, \ldots, \hat{\alpha}_K^2 \), may be taken as a measure of "Overall Relative Explanatory Power". This is
\[ \hat{\alpha}^2 = \frac{1}{K} \sum_{k=1}^{K} \hat{\alpha}_k^2 = 1 - \frac{D}{V[Z_1]} \quad (7.19) \]

and a criterion suggested for the classification of the individuals is to find \( I_1, \ldots, I_L \) such that \( \hat{\alpha}^2 \) is maximized. In this case, \( \hat{\alpha}^2 \) is independent of \( \Gamma \) and given that \( V[Z_1] \) is fixed, the "Overall Relative Explanatory Power" will be maximized when \( D \) is minimized. (In fact, minimizing \( D \) implies maximizing \( \hat{\alpha}_k^2 \) for each \( k = 1, \ldots, K \)).

This is equivalent to minimizing the within group sum of squares of \( Z_1 \), and we may therefore use the procedures presented in Section 3.4. The resulting classification shall be denoted by \( C^* \) and referred to as the optimum classification. \( \Box \)

Now we consider the more general case where \( p > 1 \). Here we have
\[ \beta_i = \Gamma(1, z_i')' + \epsilon_i, \]
and the \( k \)'th element of \( \beta_i \) would be given by
\[ \beta_{ik} = F_k(z_i) + \epsilon_{ik} \quad (7.20) \]

where \( F_k(z_i) = \gamma_{k0} + \gamma_{k1}'z_i \) for \( i = 1, \ldots, N \) and \( k = 1, \ldots, K \). It can

\( ^5 \) In Section 7.7 we comment on the use of more general definitions.
be shown (see Proposition 1 in Appendix G in page 323) that in this case

\[ R^2_{(1,k)} = \frac{\gamma_k' \Sigma \gamma_k}{\nu[\beta_{1k}]} , \]

where

\[ \Sigma = \sum_{i=1}^{N} (z_i - \bar{z})(z_i - \bar{z})' / N \]

is the VCM of the Z-variables and \( \bar{z} = \frac{1}{N} \sum_{i=1}^{N} z_i / N \). Similarly, (see Proposition 2 in Appendix G on page 324) the regression of \( \beta_{1k} \) on

\[ \beta_{k}(z_i) = \beta_k(1)D_{i1} + \ldots + \beta_k(L)D_{iL} \]

gives

\[ R^2_{(2,k)} = \frac{\gamma_k' B \gamma_k}{\nu[\beta_{1k}]} , \]

where

\[ B = \sum_{h=1}^{L} (N_h / N)(\bar{z}_h - \bar{z})(\bar{z}_h - \bar{z})' \]

and

\[ \bar{z}_h = \sum_{i \in I_h} z_i / N_h . \]

Therefore, the "Relative Explanatory Power" referring to the k'th element of \( \beta_1 \) is equal to

\[ \rho^2_{k} = \frac{R^2_{(2,k)}}{R^2_{(1,k)}} = \frac{\gamma_k' B \gamma_k}{\gamma_k' \Sigma \gamma_k} . \]  

(7.21)

The quantities \( \rho^2_{k} \), \( B \), \( N_h \) and \( \bar{z}_h \) depend on a given classification \( C \), and to emphasize this in what follows we shall write them as \( \rho^2_{k}(C) \), \( B(C) \),
Using this notation we obtain that the "Overall Relative Explanatory Power" is given by:

\[ R^2(C) = \frac{1}{K} \sum_{k=1}^{K} \delta_k^2(C) = \frac{1}{K} \sum_{k=1}^{K} \frac{\gamma_k^T B(C) \gamma_k}{\gamma_k^T \Sigma \gamma_k} \]  

(7.22)

As for the \( p = 1 \) case, the clustering criterion would be to find the classification \( C^* \) that maximizes \( R^2(C) \).

First, it is interesting to note that if \( K = p \) and each element in \( \beta_i \) is determined by only one \( Z \)-variable, i.e., \( \Gamma \) is given by

\[
\Gamma_D = \begin{bmatrix}
\gamma_{10} & \gamma_{11} & 0 & \cdots & 0 \\
\gamma_{20} & 0 & \gamma_{22} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\gamma_{p0} & 0 & 0 & \cdots & \gamma_{pp}
\end{bmatrix},
\]

then \( R^2(C) \) reduces to a form which is independent of \( \Gamma \) and equal to

\[
R^2(C) = \frac{1}{p} \sum_{j=1}^{p} \frac{[B(C)]_{jj}}{[\Sigma]_{jj}},
\]

where \([A]_{ij}\) denotes the \( i,j \)'th element of the matrix \( A \). The use of identity (7.16) for each variable \( Z_j \), in the expression for \( R^2(C) \),

\[ R^2(C) = (1/K)\Sigma \nu_k \delta_k^2(C), \]

where \( \nu_k \) are given weights. The use of this does not represent any additional problems in terms of computation of the optimum classification. A reasonable choice for \( \nu_k \) is the ratio of the standard deviation of the predictions \( \hat{\beta}_{ik} \) to the absolute value of their mean, with \( \hat{\beta}_i = \hat{\gamma}(1, z_i)' \) and where \( \hat{\gamma} \) is obtained by applying OLS to (7.4). For the \( p = 1 \) case the values of \( \nu_1, \ldots, \nu_K \) do not matter since \( \delta_k^2(C) \) is independent of \( k \).
reduces this to \( \hat{R}^2(C) = 1 - (1/p) \Phi(C) \), with

\[
\Phi(C) = \frac{1}{N} \sum_{j=1}^{p} \sum_{h=1}^{L} \sum_{i=1}^{N_h(C)} \frac{(z_{hij}(C) - \bar{z}_{hj}(C))^2}{V[Z_j]},
\]

(7.23)

and where \( z_{hij}(C) \) and \( \bar{z}_{hj}(C) \) denote, respectively, the value of \( Z_j \) for the \( i \)'th individual in group \( h \), and the \( h \)'th group mean of \( Z_j \), when the classification is \( C \). Therefore, we see that maximizing \( \hat{R}^2(C) \) is equivalent to minimizing \( \Phi(C) \). As noted in Subsection 3.4.2, Ward (1963) and MacQueen (1967) provide algorithms for the minimization of functions such as \( \Phi(C) \). These algorithms may clearly be used for the computation of the optimum classification \( C^* \) simply by setting, in the notation of Subsection 3.4.2, \( J = p \) and

\[
v_{hij}(C) = z_{hij}(C)/(V[Z_j])^{1/2}
\]

(compare equations (7.23) and (3.7)).

When we try to extend the results to a more general form for \( \Gamma \), problems are encountered since \( \hat{R}^2(C) \) would now depend on the last \( p \) columns in \( \Gamma \), i.e., the vectors \( \gamma_1', \ldots, \gamma_K' \) (see (7.3)). However, these may be estimated by the use of OLS on the equation resulting from substitution of (7.20) into \( y_1 = x_1' \beta_1 \). The estimator we obtain is \( \hat{\gamma} = \text{Vec}(\hat{\Gamma}) = \text{Vec}((\hat{\gamma}_1', \ldots, \hat{\gamma}_K')') = (X_+X_+')^{-1}X_+y \), where \( X_+ \) is defined in Section 7.2.

It seems natural to proceed to find \( C^* \) such that \( \hat{R}^2(C) \) is maximized, where \( \hat{R}^2(C) \) is equal to \( \hat{R}^2(C) \) (see (7.22)) but replacing \( \gamma_k \) by \( \hat{\gamma}_k \), i.e.,

\[
\hat{R}^2(C) = \frac{1}{K} \sum_{k=1}^{K} \hat{\gamma}_k \hat{\gamma}_k' \sum_{k=1}^{K} \hat{\gamma}_k \cdot \hat{\gamma}_k' \hat{\gamma}_k \cdot \hat{\gamma}_k
\]

(7.24)
Define

$$W(C) = \frac{1}{N} \sum_{h=1}^{L} \sum_{i=1}^{N_h(C)} (z_{hi}(C) - \bar{z}_h(C))(z_{hi}(C) - \bar{z}_h(C))'$$, \hspace{1cm} (7.25)

and recall that $B(C) = \Sigma - W(C)$ (see equation (3.3) and footnote 5 in Chapter 3). Now substitute this last expression for $B(C)$ into (7.24), obtaining

$$\delta^2(C) = 1 - \frac{1}{NK} \left\{ \sum_{k=1}^{K} \sum_{h=1}^{L} \sum_{i=1}^{N_h(C)} \left( \frac{\hat{\beta}_{hik}(C) - \bar{\beta}_{hk}(C)}{\sqrt{V[\hat{\beta}_k]}} \right)^2 \right\}$$, \hspace{1cm} (7.26)

where $\hat{\beta}_{hik}(C) = \hat{\gamma}_{ko} + \hat{\gamma}_k z_{hi}(C)$,

$\bar{\beta}_{hk}(C) = \gamma_{ko} + \gamma_k \bar{z}_h(C)$

and

$$V[\hat{\beta}_k] = \gamma_k^2 \sum \hat{\gamma}_k^2$$.

We therefore see that $\delta^2(C)$ is maximized when the term enclosed in {} in (7.26) is minimized; and, for this, we may again use the clustering algorithms described in Subsection 3.4.2, but - this time - setting $J = K$ and $\nu_{hi}(C) = \hat{\beta}_{hij}(C)/(\sqrt{V[\hat{\beta}_j]})^{1/2}$.

In summary, to maximize the "Overall Relative Explanatory Power" of the approximation $F_s(z_1)$ made to $F(z_1)$, the $N$ individuals should be classified such that (7.18) is minimized when $p = 1$, and (7.23) is minimized when $\Gamma = \Gamma_D$. For the more general case of a "non-diagonal" $\Gamma$, the criterion is to maximize (7.24). Having found

\[7\] Further considerations leading to a clustering procedure that avoids the estimation of $\Gamma$ are given in Jarque (1980b).
7.4.2 Determination of $L$

In Subsection 7.4.1 it was assumed $L$ was known. The proper choice of $L$ is important given that it will partly determine how good the approximation to $F(z_1)$ is. The number of observations $N$ will restrict the value of $L$, due to a requirement on the minimum number of observations per group in order to estimate the regimes. In general, without consideration of degrees of freedom per group, the higher $L$ the better the approximation will be. However, there may be a value beyond which no 'significant improvement' is made, and it would be desirable to find this.

For example, if $p = 1$ and $Z_1$ has a uniform $(a_1, a_2)$ distribution, $\hat{a}^2(C)$ using $L$ groups, say $\hat{a}^2(C:L)$, would be given by $\hat{a}^2(C:L) = 1 - (1/L^2)$. [See (7.19) and note the variance of a uniformly distributed random variable is simply $1/12$ of the square of the range, i.e., $V[Z_1] = (a_2 - a_1)^2/12$; in this case, within each group, the distribution would be uniform and the range equal to $(a_2 - a_1)/L$ so, $D = (a_2 - a_1)^2/(12L^2)]$. The values of $\hat{a}^2(C:L)$ for $L = 2, 3, 4, 5, 6$ and $7$ are respectively $.750, .889, .938, .960, .972$ and $.979$. Hence beyond $L = 7$ little gain in $\hat{a}^2(C)$ would be obtained. In general, a procedure for determining the number of groups is to compute $\hat{a}^2(C^*)$ (or $\hat{a}^2(C^*)$ if $p = 1$) for different values of $L$, and to choose that beyond which there is no substantial increase in $\hat{a}^2(C^*)$ (or $\hat{a}^2(C^*)$ if $p = 1$). It is interesting to note that if $Z_1, \ldots, Z_p$ are all qualitative variables, so that $Z_j$ can only take one of $n_j$ values, then, by setting $L = \prod_{j=1}^{p} n_j$, we would have $\hat{a}^2(C^*) = 1$ (or $\hat{a}^2(C^*) = 1$).
if \( p = 1 \). In this case, \( C^* \) would be the classification of the individuals with each group consisting of individuals whose \( z_i \) are equal.

The determination of \( L \) may also be carried out within a Cluster Analysis framework. For instance, we could use Ward's (1963) clustering algorithm on the data \( z_{hij}(C)/(V[Z_{ij}])^{1/2} \); and note that if the individuals group 'appropriately' into \( L \) groups, then it is sensible to approximate \( F(z_i) \) by a step function of \( L \) pieces. Several indicators for this are found in the literature. For example, Beale (1969a) suggests the use of

\[
F(p(L_2-L_1), p(N-L_2)) = \frac{b_{L_1} - b_{L_2}}{b_{L_2}} \left[ \frac{N-L_1}{L_2} (\frac{L_2}{L_1})^{2/p} - 1 \right]
\]

where \( b_L = \frac{N-L}{p} \text{trace}(B(C)) \). Using an F-Distribution, a significant result would mean that a subdivision into \( L_2 \) groups is significantly better than into a smaller number of groups \( L_1 \). Calinsky and Harabasz (1971) propose the use of \( \lambda = \frac{\text{trace}(B(C))/(L-1)}{\text{trace}(W(C))/(N-L)} \), where \( W(C) = \Sigma - B(C) \) is the matrix of the within groups sums of squares (see (7.25)). Here, if \( \lambda \) has its maximum value at \( L^* \), we would set \( L = L^* \). Yet another criterion is to use the \( L \) which maximizes \( L^2 \text{det}(W(C)) \), as suggested by Marriot (1971). All of these indicators require the computation of \( B(C) \) (therefore \( W(C) \) would be easily obtainable) and hence, in practice, several of these may be calculated before reaching a final decision on the number of groups to use.
7.5 SECOND STAGE: Estimation of Regimes and Macroparameter

The second stage of our procedure refers to the econometric estimation of the regimes \( \beta(1), \ldots, \beta(L) \), and the macroparameter \( \bar{\beta} \) (see (7.5) and (7.6)). In this section alternative estimators are presented. The results described are conditional on a given optimum classification defined by \( I_1, \ldots, I_L \).

Two general approaches may be taken for the estimation of the regimes. The first uses information on the variables \( Z_1, \ldots, Z_p \) and treats the model as one with systematic parameter variation. This approach is discussed in Subsection 7.5.1. The second approach ignores the information on \( Z_1, \ldots, Z_p \) and estimates the regimes using random coefficient regression methods. This is treated in Subsection 7.5.2. Finally, in Subsection 7.5.3 the estimation of the macroparameter \( \bar{\beta} \) is discussed.

7.5.1 Systematic Parameter Variation Approach

It has been assumed that \( \beta_i \) is given by \( \beta_i = F(z_i) + \varepsilon_i \), where \( F(z_i) = (F_1(z_i), \ldots, F_K(z_i))' \) and \( F_k(z_i) \) is an 'unknown smooth function'. For derivation of the clustering criterion it was further assumed that \( F_k(z_i) \) was approximated by a step function. However, once the individuals have been grouped, it may seem appropriate to approximate \( F_k(z_i) \) by a linear function within each group \( h \), for \( h = 1, 2, \ldots, L \). One may write

\[
\beta_i = \Gamma(h) \begin{bmatrix} 1 \\ \varepsilon_{ih} \end{bmatrix} + z_i \quad \text{for } i \in I_h, \quad (7.27)
\]

The estimators we discuss in this section are not new, and are fully described in the sources cited. Our aim here is to illustrate their use in the present setting.
and assume that $E[\varepsilon_{ih}] = 0$ for all $i,h$ and that $E[\varepsilon_{1h} \varepsilon_{jk}']$ is equal to $\Omega_h$ for $i = j$ and $h = k$ and 0 otherwise. The aim now is to estimate $\Gamma(h)$.

Substitution of (7.27) into the model $y_i = x_i'\beta_i$ gives the relation

$$y_i = ((1,z_i') \otimes x_i')vec(\Gamma(h)) + x_i'\varepsilon_{ih} \quad i \in I_h .$$

Let $\gamma(h) = vec(\Gamma(h))$, $y_h = (y_1,y_2,\ldots,y_N)'$, $\varepsilon_h = (\varepsilon_{1h},\varepsilon_{2h},\ldots,\varepsilon_{Nh})'$, $X_{dh} = diag(x_1',\ldots,x_i')$, $\varepsilon^* = X_{dh}\varepsilon_h$, and $Z_h$ be the $N_h$ by $(p+1)K$ matrix with $i$'th row equal to $((1,z_i') \otimes x_i')$. Assume the rank of $Z_h$ is $(p+1)K$. Also let $X_h = (x_1',\ldots,x_{N_h}')'$. Then the above relation may be written, in matrix form, as

$$y_h = Z_h\gamma(h) + \varepsilon_h^* . \quad (7.28)$$

Equation (7.28) is amenable to econometric estimation. For example, one may assume $\Omega_h$ to be diagonal and proceed à la Hildreth and Houck (1968); estimating $\gamma(h)$ by

$$\hat{\gamma}(h) = (Z_h^*Z_h)^{-1}Z_h^*y_h , \quad (7.29)$$

where $\hat{\gamma}(h)$ is an estimator of the VCM of $\varepsilon_h^*$ [e.g., one may use $[\hat{\gamma}_h]_{jk}$ equal $[\hat{X}_h\alpha_h]_j$, for $j = k$ and equal to zero for $j \neq k$, where $[\alpha_h]_j$ is the maximum between zero and $[\hat{X}_h'\hat{W}_h'\hat{X}_h]^{-1}\hat{X}_h'\hat{W}_h'\hat{Y}_h$], and where $M_h = I_{N_h} - Z_h(Z_h'Z_h)^{-1}Z_h'$, $n_h = M_hy_h$ and $A = [a_{ij}]$ if $A = [a_{ij}]$. From $\hat{\gamma}(h)$ one obtains $\hat{r}(h)$; then one may predict $\beta_i$ by $\hat{T}(h)[1 z_i]'$.

\[9\] In this section subindexes of $y_i$ and $x_i$ are in progressive order within each group (e.g., $y_1',\ldots,y_{N_h}'$). Also note that whenever $y$ has a subscript $h$ (i.e., $y_h$) then it refers to an $N_h$ by 1 vector.
and estimate \( \beta(h) \) by

\[
\hat{\beta}(h) = \tilde{f}(h) \begin{bmatrix} 1 \\ z_h \end{bmatrix}.
\] (7.30)

The assumption that \( \Omega_h \) is diagonal may be avoided by following Swamy and Mehta (1975). They consider a prior distribution for \( \gamma(h) \) with mean \( \theta(h) \) and VCM \( \phi_h \), and suggest using the approximation to the minimum average risk estimator given by

\[
\tilde{y}(h) = (z_h^t \hat{\Sigma}_h^{-1} z_h + \phi_h^{-1})^{-1}(z_h^t \hat{\Sigma}_h^{-1} y_h + \phi_h^{-1} \theta(h)),
\] (7.31)

where \( \hat{\Sigma}_h = X_{dh} (I_{N_h} \hat{\Omega}_h^{-1}) X_{dh}^t \) and where \( \hat{\Omega}_h \) is an estimator of \( \Omega_h \) (see Swamy and Mehta (1975, p.596)).

### 7.5.2 Random Coefficient Regression Approach

A second approach to the estimation of the regimes may be to consider the model as written out in (7.7) and (7.8), and proceed to estimate \( \beta(h) \) and \( \Delta_h \) as in a random coefficient regression model using the data corresponding to \( I_h \). Although this approach neglects the information available in \( z_1, z_2, \ldots, z_N_h \), it provides an alternative estimation procedure that would be particularly useful when the number of observations in a group is small. (In order to use (7.29) one requires that \( N_h \) be greater or equal to \( (p+1)K \)).

Define \( v_h = (v_1^h, \ldots, v_{N_h}^h)' \) and \( \xi_h = X_{dh} v_h \). Then, in matrix form, (7.7) and (7.8) may be written as

\[ y_h = X_h \beta(h) + \xi_h \quad h = 1, 2, \ldots, L. \]

Assuming \( \Delta_h \) to be diagonal one may follow Hildreth and Houck (1968)
and estimate $\beta(h)$ by

$$
\beta^+(h) = (X_h ' \hat{\Sigma}_h^{-1} X_h)^{-1} X_h ' \hat{\Sigma}_h^{-1} y_h ,
$$

(7.32)

where $\hat{\Sigma}_h$ is an estimator of the variance-covariance matrix of $\xi_h$ (see Hildreth and Houck (1968, p.589)).

Alternatively, one may avoid this assumption and follow Swamy and Mehta (1975) and use as estimator of $\beta(h)$

$$
\tilde{\beta}(h) = (X_h ' \hat{\Sigma}_h^{-1} X_h + \psi_h^{-1})^{-1} (X_h ' \hat{\Sigma}_h^{-1} y_h + \psi_h^{-1} r(h)) ,
$$

(7.33)

where $r(h)$ is the prior mean and $\psi_h$ the prior VCM of $\beta(h)$, $\hat{\Sigma}_h = X_d h (I_{N_h} \theta \hat{\Lambda}_h)^' X_d h$ and $\hat{\Lambda}_h$ is an estimator of $\Lambda_h$ (see Swamy and Mehta (1975, p.596)).

7.5.3 Estimation of the Macroparameter

So far various estimators for the regimes have been presented. In practice, one may also be interested in estimating the macroparameter $\bar{\beta}$ (see (7.6)). For this one may consider the estimators of the regimes and use $E(N_h / N) \bar{\beta}(h)$ (see Subsection 7.5.1), or $\Sigma(N_h / N) \beta^+(h)$ or $\Sigma(N_h / N) \tilde{\beta}(h)$ (see Subsection 7.5.2). Alternatively, one may regard the $\beta(h)$ as a sample of independent identically distributed vector random variables with mean $\bar{\beta}$ and VCM $\Delta$, and choose to estimate $\bar{\beta}$ and $\Delta, \Delta_1, \ldots, \Delta_L$. For this, one may follow Swamy and Mehta (1975, Section 3). They discuss the estimation of a random coefficient regression model from panel data, but their results are also applicable in a

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Another procedure is obtained by assuming prior exchangeability and following Lindley and Smith (1972). In addition, the analysis in this subsection may be extended to incorporate information about the $z_i$'s as in Swamy and Tinsley (1980).
purely cross-sectional framework. This is now illustrated.

Firstly note that estimates of $\Delta_1, \ldots, \Delta_L$ may be obtained from estimation within $I_h$ for $h = 1, 2, \ldots, L$ (see comments below (7.33)).

Now define $X = (X_1', X_2', \ldots, X_L')'$, $y = (y_1', y_2', \ldots, y_L')$,

$$b(h) = (X_h^t\Sigma^{-1}_h X_h)^{-1}X_h^t\Sigma^{-1}_h y_h$$

and $S = \Sigma b(h)b'(h) - (1/L)(\Sigma b(h))(\Sigma b'(h))$.

Swamy and Mehta (1975, p.600) suggest estimating $\Delta$ by

$$\hat{\Delta} = \frac{S}{L-1} \sum_{h=1}^L (X_h^t\Sigma^{-1}_h X_h)^{-1}$$

and assuming that the prior mean of $\bar{b}$ is $r$ and the prior VCM is $\psi$, they suggest estimating $\bar{b}$ by

$$\hat{\bar{b}} = (X^t\Sigma^{-1}X + \psi^{-1})^{-1}(X^t\Sigma^{-1}y + \psi^{-1}r),$$

where $\Sigma = \text{diag}(\Delta X_1^t + \Sigma_1, \ldots, \Delta X_L^t + \Sigma_L)$. It may be shown that $\hat{\bar{b}}$ is an approximation to the minimum average risk estimator of $\bar{b}$. Also, that under diffuse prior information (i.e. when setting $\psi^{-1}$ equal to zero) this would reduce to a weighted sum of the estimators of the regimes, $b(h)$, and given by

$$\hat{\bar{b}} = \sum_{h=1}^L \tau_h b(h)$$

where

$$\tau_h = \left[ \sum_{j=1}^L [\hat{\Delta}(X_j^t\Sigma^{-1}_j X_j)^{-1}]^{-1} \right]^{-1} [\hat{\Delta}(X_h^t\Sigma^{-1}_h X_h)^{-1}]^{-1}.$$
7.6 NUMERICAL EXERCISE

In this section, a numerical exercise is presented for the comparison of three estimation procedures under alternative forms of parameter variation. For the study, \( N = 100, K = 2, p = 1 \) and \( L = 5 \). Variable \( X_{i1} \) is equal to one for all \( i = 1,2,...,N \) and the observations on \( X_2 \) are generated from a Normal \((10,1)\). The disturbance terms \( u_1,...,u_N \) are generated from a Normal \((0,\sigma^2)\); and the observations on \( Z \) are generated from a Lognormal such that \( \log(Z) \) is distributed as a Normal \((3,1)\) (the subroutine used is described in Naylor et al. (1966)).

Four models are considered, defined by \( y_i = x_i' \beta_i + u_i \), for \( i = 1,...,N \) and where \( \beta_i \) is non-random and given by the following expressions:

(1) \[
\beta_i = \begin{bmatrix}
\theta_1 \\
\theta_2
\end{bmatrix}
\]

(2) \[
\beta_i = \begin{bmatrix}
\theta_3 + \theta_4 z_i \\
\theta_5 + \theta_6 z_i
\end{bmatrix}
\]

(3) \[
\beta_i = \begin{bmatrix}
\theta_7 + \frac{1}{\theta_8} (z_i)^{\theta_9} \\
\theta_{10} + \frac{1}{\theta_{11}} (z_i)^{\theta_{12}}
\end{bmatrix}
\]

(4) \[
\beta_i = \begin{bmatrix}
\theta_{13}/(\theta_{14} + \theta_{15} (z_i)^{\theta_{16}}) \\
\theta_{17} \exp(z_i/\theta_{18})
\end{bmatrix}
\]
with

\[(\theta_1, \theta_2, \sigma) = (10, 1, 1),\]
\[(\theta_3, \theta_4, \theta_5, \theta_6, \sigma) = (60, -2, 15, -1, 70),\]
\[(\theta_7, \theta_8, \theta_9, \theta_{10}, \theta_{11}, \theta_{12}, \sigma) = (1, 250, 2, 2, 7000, 3, 10),\]
\[(\theta_{13}, \theta_{14}, \theta_{15}, \theta_{16}, \theta_{17}, \theta_{18}, \sigma) = (200, 10, 1, 1, 1, 20, 10).\]

Each model is estimated using three procedures which are as follows.

(i) Estimating \( \beta_i \) by \( \hat{\beta}_i = \Gamma(1, z_i)' \) where

\[\text{Vec} \{ \Gamma \} = (X'_+ X'_+)^{-1} X'_+ y\]
and \( X_+ \) is an \( N \) by 4 matrix

with \( i \)'th row equal to \((1, x_{i1}' z_i, x_{i2}' z_i)\) and

\( y = (y_1, \ldots, y_N)' \). This procedure is equivalent to

taking a linear approximation to \( F_k(z_i) \) and in what

remains is referred to as linear parameter variation -

LPV.

(ii) The second procedure estimates \( \beta_i \) by \( \hat{\beta}_i = \beta^+(h) \) for

\( i \in I_h \), where \( \beta^+(h) \) is defined by (7.32) with

\( \hat{\lambda}_h = \hat{I}_{N_h} \), and \( I_h \) is determined by minimizing (7.18)

using the cubic-root procedure described in Subsection

3.4.4. It should be clear that this is a two stage

estimation procedure (equivalent to OLS estimation

within the optimum groups) and that it approximates

\( F_k(z_i) \) by a step function. The procedure is referred
to as 2S-OLS.
(iii) The third procedure estimates \( \beta_i \) by
\[
\hat{\beta}_i = \hat{\Gamma}(h)(1 \ z_i)'
\]
for \( i \in I_h \), where \( \text{Vec}(\hat{\Gamma}(h)) \) is given by (7.29) with
\[
\hat{\Theta}_h = I_{N_h}
\]
and where \( I_h \) is the same as for (ii). This is also a two stage estimation procedure and approximates \( F_k(z_i) \) by a piecewise linear function. The procedure is referred to as 2S-LPV.

As a goodness-of-fit measure we used \( R^2 \) adjusted for degrees of freedom, denoted by \( \tilde{R}^2 \). The values obtained for each of the four models, using the three estimators described, are reported in Table 7.1.

<table>
<thead>
<tr>
<th>TABLE 7.1</th>
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</thead>
<tbody>
<tr>
<td><strong>Values of ( \tilde{R}^2 ) for Four Models</strong></td>
</tr>
<tr>
<td>MODEL</td>
</tr>
<tr>
<td>(i) LPV</td>
</tr>
<tr>
<td>(ii) 2S-OLS</td>
</tr>
<tr>
<td>(iii) 2S-LPV</td>
</tr>
</tbody>
</table>

The Table shows that for Model 1 (i.e. when parameter variation does not exist) the three procedures give approximately the same \( \tilde{R}^2 \) (in fact, these are close to the one obtained using OLS, which is \( \tilde{R}^2 = .5607 \)). Comparing LPV and 2S-OLS, it is observed that 2S-OLS performs better when parameter variation departs from linearity.

\[ \tilde{R}^2 \text{ is given by } \tilde{R}^2 = 1 - \left( \frac{\sum_{i=1}^{N} \left( \hat{y}_i - \bar{y} \right)^2}{(N-1) \sigma_y^2} \right) \text{, where} \]
\[ \sigma_y^2 = \frac{N}{N-1} \sum_{i=1}^{N} (y_i - \bar{y})^2 \text{,} \quad \bar{y} = \frac{\sum_{i=1}^{N} y_i}{N} \text{,} \quad \hat{y}_i = y_i - \hat{x}_i \beta_1 \text{ and } n \text{ is the number of parameters estimated. For (i) } n = 4, \text{ for (ii) } n = 10 \text{ and for (iii) } n = 20. \] Of course, alternative goodness-of-fit measures could have been used.
(e.g., for Model 3 - where parameter variation is quadratic for the intercept and cubic for the slope - the increase in $R^2$ is approximately 14 per cent). Comparing 2S-OLS and 2S-LPV, one notes that 2S-OLS gives a lower $R^2$ for all four models. This is reasonable, since piecewise linear functions approximate better than step functions. Nevertheless, 2S-OLS should not be discarded given that it may be a useful estimation procedure when there are groups of small size (as mentioned before, 2S-LPV requires at least $(p+1)K$ observations per group in order for the econometric estimation to be possible). Other groupings of the observations were used to compute the estimator 2S-LPV and evaluate the effect of the optimum classification $C^*$. On average, the $R^2$'s obtained were approximately 10 per cent lower than those computed with the optimum classification. Overall, these results indicate the preference of the two stage procedures, particularly 2S-LPV, and provide evidence that the increase in goodness-of-fit over LPV may be substantial (e.g., for Model 4 the increase in $R^2$ is approximately 20 per cent when using 2S-LPV rather than LPV).

7.7 CONCLUDING REMARKS

In this Chapter a two stage procedure for the estimation of systematic varying parameter models has been discussed. In the first stage, the individuals would be classified into groups by the use of a clustering criterion suggested. The second stage refers to the econometric estimation of the regimes, and several estimators for this were described in Section 7.5.

In our estimation procedure the clustering criterion is to maximize the "Overall Relative Explanatory Power", $R^2(C)$, of a step
function approximation to the conditional expectation of $\beta_i$. We defined $R^2(C)$ as

$$R^2(C) = \frac{1}{K} \sum_{k=1}^{K} \hat{\rho}_k^2(C) = \frac{1}{K} \sum_{k=1}^{K} \frac{R^2_{(2,k)}}{R^2_{(1,k)}},$$

where $\hat{\rho}_k(C)$ is a correlation coefficient, and $R^2_{(2,k)}$ and $R^2_{(1,k)}$ are goodness-of-fit measures of particular regressions. When the VCM of $e_i$ in (7.2), i.e. $\Omega$, is not diagonal, the equations defined by (7.2) are a system of seemingly unrelated regressions, and better measures of goodness-of-fit exist for such models (e.g., see Buse (1979)). An alternative definition of "Overall Relative Explanatory Power" is the ratio of Buse's goodness-of-fit measures for the systems (7.11) and (7.13). Unfortunately, this ratio depends on unknown quantities, such as $\Omega$ and, more importantly, for our problem this criterion is not numerically manageable. For these reasons we limited our discussion to the goodness-of-fit measure $R^2(C)$.

Other approaches to the estimation problem may be considered. One may be to take the model as written in (7.7) and (7.8), and assume $v_{ih}$ is normally distributed. We could then impose $N_h > K$, and maximize the likelihood function with respect to the classification and the parameters. Another may be to consider finding the classification that optimizes a function of the second order moments of some estimator of the regimes. An inconvenience with these approaches is that, unlike the two stage procedure that uses $R^2(C)$, we may end up with a numerically unmanageable clustering criterion; and searching for the optimum classification by total enumeration of alternatives would be computationally inefficient due to the large number of these (see Subsection 3.4.1).
An *illustration* of the two stage estimation procedure discussed here is presented in Chapter 9. There we use data from a Mexican Income-Expenditure Household Survey, and apply the clustering criterion $\sigma^2(c)$ to form groups of 'homogeneous consumers'. We then estimate demand functions for the various resulting groups.
CHAPTER 8

THE PROBLEM OF SIMULTANEITY*

"One by one, or all at once"

W.S. Gilbert
The Yeomen of the Guard

8.1 INTRODUCTION

In our previous Chapters we discussed various aspects of the econometrics of cross-sections, under assumption [7] (this states there are no endogenous variables among the regressors - see Section 1.2). It is not uncommon to find cross-sectional studies in which assumption [7] is not valid. To cite one example, the number of hours worked by a husband and wife are interdependent and - to explain these - we would use a simultaneous equations model (e.g., see Kmenta (1978)). Many results are now available on the estimation of these models. By comparison, few results exist regarding inferential procedures. Here we will concentrate on specific inferential problems that relate to some of our previous discussions for the single equation model. First, we introduce necessary definitions.

* Sections 8.3 and 8.4 are based on Jarque (1981d).
We consider a linear simultaneous equations model, which we write in the form

\[ B y_i + \Gamma x_i = u_i \]  

(8.1)

for \( i = 1, \ldots, N \), where \( B \) is an \( n \) by \( n \) non-singular matrix of fixed parameters with diagonal elements equal to \(-1\); \( y_i \) is (now) an \( n \) by 1 vector representing the \( i \)'th observation on the \( n \) endogenous variables \( Y_1, \ldots, Y_n \); \( \Gamma \) is an \( n \) by \( K \) matrix of fixed parameters; \( x_i \) is (as before) a \( K \) by 1 vector representing the \( i \)'th observation on \( K \) predetermined variables \( X_1, \ldots, X_K \) (which may include lagged endogenous and exogenous variables); and \( u_i \) is (now) an \( n \) by 1 vector representing the \( i \)'th unobservable disturbance.

It is assumed:

(1) that sufficient prior zero-restrictions on \( B \) and \( \Gamma \) exist so every parameter in (8.1) is identified;

(2) that the model is linear and correctly specified so there are no omitted deterministic influences, hence \( E[u_i] = 0 \);

(3) that apart from possible scale differences, the p.d.f. of \( u_i, f(u_i) \), is the same for all \( i = 1, \ldots, N \) and independent of \( B \) and \( \Gamma \); and

(4) that the range of \( Y_1, \ldots, Y_n \) is unrestricted.

Typical additional assumptions in the applications of (8.1) are that \( f(u_i) \) is the multivariate normal density, and that the variance-covariance matrices \( \Sigma_i = E[u_i u_i'] \) are equal to \( \Sigma \) for all \( i = 1, \ldots, N \).
This allows writing the likelihood function for the $N$ multivariate observations $y_1, \ldots, y_N$ and, in turn, enables derivation of the first order conditions. It is easily shown that the quantities $\Sigma$, $B$ and $\Gamma$ satisfying the a-priori restrictions and maximizing the likelihood, are

$$\hat{\Sigma} = \frac{1}{N} \sum_{i=1}^{N} \hat{u}_i \hat{u}_i' / N,$$

and

$$[(B^{-1}:0) - \hat{\Sigma}^{-1}(B:\Gamma)(W'W/N)] \hat{u} = 0,$$

where $\hat{u}_i = \hat{B}y_i + \hat{\Gamma}x_i$, $W = (w_1, \ldots, w_N)'$ with $w_i' = (y_i', x_i')$, and $\cong$ denotes that equality holds for the a-priori unrestricted structural parameters (e.g., see Hendry (1976, p.53)). $\hat{B}$, $\hat{\Gamma}$ and $\hat{\Sigma}$ are referred to as the FIML estimates of model (8.1).

Having introduced notation, we now state the contents of the Chapter. The first problem we consider relates to testing the multivariate normality of the disturbance $u_i$. This is discussed in Section 8.2. In Section 8.3, we deal with the problem of testing for disturbance constant VCM. Section 8.4 illustrates the use of the results of Section 8.3, in testing a form of random coefficient variation in simultaneous equations models. Some concluding remarks are made in Section 8.5.
8.2 A TEST FOR MULTIVARIATE NORMALITY

8.2.1 The Problem of Multivariate Non-Normality

Econometric models are usually constructed (a) to obtain information about structural and/or reduced form coefficients, and (b) to make predictions of the endogenous variables $Y_1, \ldots, Y_n$, given certain assumptions about the predetermined variables $X_1, \ldots, X_K$. Now we comment on the relevance of the disturbance normality assumption in relation to both these items.

First consider item (a). In Section 8.1 FIML estimators of the simultaneous equations model were introduced. Of course, as described in most econometrics textbooks, many estimators exist for this model (e.g., 2SLS, 3SLS, LIML and GIVE to name a few) and their asymptotic properties - under mild assumptions - are known. Current research in this field concentrates on derivation of finite sample properties of estimators, and some (rather specific) results are available under multivariate normally distributed disturbances (e.g., see Mariano (1977), Wegge (1971) and Anderson and Sawa (1973)). Due to the manageability of the multivariate normal p.d.f., one also expects future results to be developed under this assumption - at least initially. In relation to item (b), normality has traditionally been assumed when determining confidence intervals for a predicted vector of endogenous variables (e.g., see Hooper and Zellner (1961) and Hymans (1968)).

Some consequences of incorrectly assuming normality in the single equation model were noted in Section 4.1, concluding that violation of this assumption may lead to

(i) The use of sub-optimal estimators;

(ii) Invalid inferential statements; and to
(iii) Inaccurate conclusions.

Apparently, no studies exist about the consequences of non-normality in the simultaneous equations model. In general, one expects the problems for the single equation model to remain in the simultaneous equations case. Although the magnitudes of specific consequences may not be fully known, to apply confidently available results [e.g., on finite sample properties of estimators and/or confidence intervals for predictions of endogenous variables] one should test for disturbance normality.

To our knowledge, no work has been done on testing the multivariate normality of disturbances in the simultaneous equations model. One may be tempted to look at the equations in the model one-by-one and apply, to each, a univariate normality test (e.g., $LM_n$). The presence of multivariate non-normality will be reflected, except in rare cases, on the marginal distributions; so rejection of any of the $n$ marginal normality hypotheses would lead one to conclude the multivariate distribution is non-normal. This approach is - however - limited because marginal normality does not imply multivariate normality (e.g. see Anderson (1958, p.37) or Papoulis (1970, p.184)) and if all $n$ marginal normality hypotheses were accepted, one could not conclude the joint distribution is normal. Due to this we proceeded to search for a more conclusive multivariate normality test, which exploits the multivariate nature of the disturbances.

Our first attempt in deriving such a test, was to generalize the results of Chapter 4. This approach was almost immediately abandoned due to the difficulty in defining the multivariate analogue of the Pearson Family (see equation (4.2)).

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1 Apparently, this problem has not yet been tackled or, at least, no fruitful results have been reported in the statistical literature.
the simpler problem of testing observations for normality. Our findings are summarized in Subsection 8.2.2. These results are later used, in Subsection 8.2.3, to obtain a disturbance normality test.

8.2.2 A Test for Multivariate Normality of Observations

For a scalar random variable \( v \), the population skewness measure is

\[
\mathcal{E}_1 = \frac{(E[(v-\mu)^3])^2}{(E[(v-\mu)^2])^3}, \tag{8.4}
\]

and the kurtosis measure is

\[
\mathcal{E}_2 = \frac{E[(v-\mu)^4]}{(E[(v-\mu)^2])^2}, \tag{8.5}
\]

where \( \mu = E[v] \). For a set of \( N \) observations on \( v \), say \( v_1, \ldots, v_N \), we showed in Section 4.3 that the LM normality test statistic was equal to

\[
LM = N[b_1/6 + (b_2 - 3)^2/24], \tag{8.6}
\]

where \( b_1 \) and \( b_2 \) are, respectively, the sample skewness and kurtosis coefficients. More formally, we have

\[
b_1 = \frac{\mu_3}{\mu_2^3} \quad \text{and} \quad b_2 = \frac{\mu_4}{\mu_2^2},
\]

with \( \mu_j = \sum_{i=1}^{N} (v_i - \bar{v})^j/N \) and \( \bar{v} = \frac{1}{N} \sum_{i=1}^{N} v_i/N \).

For the multivariate problem, Mardia (1970, 1974) has suggested measures of skewness and kurtosis that contain \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) as special cases. These are, respectively,

\[
\mathcal{E}_{1,n} = E[\{(v - \mu)'\Sigma^{-1}_v(v - \mu)^3\}]
\]
and

$$B_{2,n} = E[(((v - \mu)')\Sigma_v^{-1}(v - \mu))^2]$$

where $v$ and $v_o$ are independent and identically distributed random $n$ by 1 vectors, with $\mu = E[v]$ and $\Sigma_v = E[(v - \mu)(v - \mu)']$ (arguments motivating these measures are given in Mardia (1970,1974)). For a set of $N$ observations on $v$, say $v_1, \ldots, v_N$, $B_{1,n}$ and $B_{2,n}$ may be estimated by their sample counterparts, namely, by

$$b_{1,n} = \frac{1}{N^2} \Sigma_{i=1}^{N} \Sigma_{j=1}^{N} [(v_i - \bar{v})'\Sigma_N^{-1}(v_j - \bar{v})]^3$$

and

$$b_{2,n} = \frac{1}{N} \Sigma_{i=1}^{N} [(v_i - \bar{v})'\Sigma_N^{-1}(v_i - \bar{v})]^2$$

where $\bar{v} = \Sigma_{i=1}^{N} v_i / N$ and $\Sigma_N = \Sigma_{i=1}^{N} (v_i - \bar{v})(v_i - \bar{v})' / N$.  2

Mardia (1970,1974) shows that under $H_0: v \sim N(\mu, \Sigma)$,

$$B_{1,n} = 0$$

$$B_{2,n} = n(n+2)$$

$$\varphi_1 = Nb_{1,n}/6 \sim \chi^2_{(n+1)(n+2)/6}$$

and

$$\varphi_2 = N^{1/2}[b_{2,n} - n(n+2)]/[8n(n+2)]^{1/2} \sim N(0,1)$$

and then suggests using $\varphi_1$ as a test for multivariate skewness; $\varphi_2$ as

---

2 A simple algorithm for the computation of $b_{1,n}$ and $b_{2,n}$ is given in Mardia and Zemroch (1975).
a test for multivariate kurtosis; and applying both $D_1$ and $D_2$ to test for multivariate normality. The latter procedure may be seen as the multivariate version of the $R$ test of Pearson, D'Agostino and Bowman (1977). (See Subsection 4.5.1).

Our results for the univariate testing problem (proving that a combination of $b_1$ and $b_2$ produced an asymptotically efficient test) motivate the use of a combination of $D_1$ and $D_2$ in testing observations for their multivariate normality. An obvious choice is the test statistic denoted by $D_n$ and defined as $D_n = D_1 + D_2^2$, i.e.,

$$D_n = N[b_1,n/6 + (b_2,n-n(n+2))^2/(8n(n+2))] . \quad (8.7)$$

It may be shown that, under $H_0: v \sim N(\mu, \Sigma)$, $D_n$ would be asymptotically distributed as $\chi^2(r)$ with $r = (n(n+1)(n+2)/6) + 1$ degrees of freedom. A test based on $D_n$ would seem appealing, but we have not shown it satisfies any optimality criterion (e.g., we have not shown it to be an LM or LR test). This means we cannot say anything yet about its power properties.

Two interesting points about $D_n$ are (I) that when we set $n = 1$ in (8.7) this reduces to (8.6), and (II) that it is invariant to linear transformations [this is reasonable since we are interested in testing the shape of the distribution, and not location or scale]. Because of (II), we may obtain the finite sample distribution of $D_n$ by computer simulation, generating observations $v_i$ from, say, a $N(0, I_n)$. In fact, this was done by Mardia (1974, pp.124-126) to obtain finite sample significance points for $b_{1,n}$ and $b_{2,n}$, with $n = 2$. 
8.2.3 A Test for Multivariate Normality of Disturbances

Now we return to the simultaneous equations model given in (8.1). We can write the reduced form of the model as

\[ y_i = \Pi x_i + v_i \]

where \( \Pi = -B^{-1} \Gamma \) and \( v_i = B^{-1} u_i \). Testing the multivariate normality of \( u_i \) is equivalent to testing that of \( v_i \), and – for convenience – we will approach the problem through the latter.

If the reduced form disturbances \( v_i \) were observed, one could compute \( D_n \) – as defined in (8.7) – to test for their multivariate normality. Of course, the \( v_i \) are unobservable. Yet, these may be 'estimated' by \( \hat{v}_i = y_i - \hat{\Pi} x_i \) [where \( \hat{\Pi}' = (X'X)^{-1}X'Y \), \( x = (x_1, \ldots, x_n)' \) and \( Y = (y_1, \ldots, y_N)' \)]. It may be shown that \( D_n \) computed with \( \hat{v}_1, \ldots, \hat{v}_N \), say \( \hat{D}_n = D(\hat{v}_1, \ldots, \hat{v}_N) \), has the same asymptotic distribution as \( D(v_1, \ldots, v_N) \) under \( H_0: u_i \sim N(0, \Sigma) \), i.e., a \( \chi^2 \) with \( r = n(n+1)(n+2)/6 + 1 \). Also, that if \( X \) contains a constant and \( n = 1 \), then (as expected) \( \hat{D}_n \) would reduce to the LM disturbance normality test for the single equation model (see (4.6)).

The \( \chi^2(\nu) \) distribution may be used, in large samples, to obtain significance points. For small samples, computer simulation can be employed as now illustrated. It is easy to see that \( \hat{V} = MV \), where \( M = I_N - X(X'X)^{-1}X' \), \( \hat{V} = (\hat{v}_1, \ldots, \hat{v}_N)' \) and \( V = (v_1, \ldots, v_N)' \). Now recall that, by assumption, \( v_i \) has zero expectation, i.e., \( E[v_i] = 0 \). Also that, because of the invariance of (8.7) to linear transformations, one

3 For our purposes all that we need is an 'estimate' of \( v_i \) based on a consistent estimate of \( \Pi \). \( \hat{\Pi} \) is the unrestricted reduced form estimate. We could alternatively use the restricted reduced form estimate \( -\hat{B}^{-1} \hat{\Gamma} \), where \( \hat{B} \) and \( \hat{\Gamma} \) are either the solutions to (8.2) and (8.3) or other consistent estimates of the structural parameters.
may assume the VCM of \( v_i \) is the identity, i.e., \( E[v_i v_i'] = I_n \). So, given a sample size \( N \) and matrix \( X \), one may generate \( N \) disturbances \( v_i \) from a \( N(0, I_n) \) and compute \( \hat{V} \) and \( \hat{D}_n \). This could be repeated as many times as desired, obtaining approximate finite sample significance points, or an estimate of the probability of a Type I error for a given value of \( \hat{D}_n \).

If \( H_0 \) is accepted, then one should not hesitate to use available results that require multivariate normality. If \( H_0 \) is rejected, one could consider the use of transformations so that multivariate normality is better suited; this, however, needs further investigation. [In particular, problems of identification may arise, e.g., see Zarembka (1974, p.102)].

8.3 A TEST FOR MULTIVARIATE HOMOSCEDASTICITY

In Section 5.1 we noted evidence that violation of the homoscedasticity assumption - in the single equation model - may lead to

(i) The use of sub-optimal estimators; and to

(ii) Inappropriate inferences.

These consequences also occur in the simultaneous equations model, so it becomes important to test for homoscedasticity in this setting.

In contrast with the variety of tests for homoscedasticity that exist for the single equation model, relatively little work has been done in the development of tests in the simultaneous equations model. A reference in this area is the paper by Harvey and Phillips (1981). They provide tests for constant disturbance variance in an equation from a simultaneous equations model (or system), which are analogous to their
ordinary regression model counterparts. These tests consider each equation in the model separately, and test for heteroscedasticity within a limited information setting (i.e. neglecting some of the restrictions in the system). In this section, we suggest a test for heteroscedasticity derived within a full information setting — that allows us to test more general hypotheses about the second order moments of the disturbances.

The problem studied is testing for disturbance constant VCM, which we refer to as multivariate homoscedasticity. First we introduce necessary definitions. We again consider the model (8.1), but assume $u_i$ is distributed as $N(0,\Sigma_i)$, where $\Sigma_i$ is an $n$ by $n$ non-singular VCM. For now, we take a general formulation that allows variation of every element in the VCM of the disturbances, and consider additive heteroscedasticity which we write in the form

$$\Sigma_i = \begin{bmatrix} a_{11}z_i & \cdots & a_{1n}z_i \\ \vdots & \ddots & \vdots \\ a_{in}z_i & \cdots & a_{nn}z_i \end{bmatrix},$$

where $a_{jk}^i$ is a 1 by $p$ vector of fixed parameters (functionally unrelated to the parameters in $B$ and $\Gamma$), and $z_i$ is a $p$ by 1 vector representing the i'th observation on a set of $p$ fixed and finite variables, whose first component is equal to 1 and is such that $Z'Z$

---

4 In this - our initial formulation - the covariance between the j'th and k'th elements in $u_i$, say $u_{ij}$ and $u_{ik}$, is $\sigma_{jk}(i) = a_{jk}^i z_i$, and varies with $i$. This is not unreasonable. Recall that $\sigma_{jk}(i) = \rho_{jk}(i)\sigma_{jj}(i)\sigma_{kk}(i)$, where $\rho_{jk}(i)$ is the correlation between $u_{ij}$ and $u_{ik}$. Even if $\rho_{jk}(i)$ was the same for all $i = 1, \ldots, N$, variation of either $\sigma_{jj}(i)$, or $\sigma_{kk}(i)$, would lead to non-constant covariances.
is non-singular, where \( Z = (z_1, \ldots, z_n)' \). We assume \( \alpha_j' z_i \) is bounded away from zero for all \( j = 1, \ldots, n \) and \( i = 1, \ldots, N \). We denote the first element in the vector \( \alpha_j' \) by \( \sigma_{jk} \), and write \( \alpha_j' = (\sigma_{jk}, \alpha_{jk}' \) and \( z_i = (1, z_i^*\)'\). Recalling \( \Theta \) denotes the Kronecker product, we can show

\[
\Sigma_i = A(I_n \otimes z_i) = \Sigma + A^*(I_n \otimes z_i^*) ,
\]

where

\[
A = \begin{bmatrix}
\alpha_{11}' & \ldots & \alpha_{1n}' \\
\vdots & \ddots & \vdots \\
\alpha_{ln}' & \ldots & \alpha_{kn}'
\end{bmatrix}, \quad \Sigma = \begin{bmatrix}
\sigma_{11} & \ldots & \sigma_{1n} \\
\vdots & \ddots & \vdots \\
\sigma_{ln} & \ldots & \sigma_{nn}
\end{bmatrix} \quad \text{and} \quad A^* = \begin{bmatrix}
\alpha_{11}' & \ldots & \alpha_{1n}' \\
\vdots & \ddots & \vdots \\
\alpha_{ln}' & \ldots & \alpha_{kn}'
\end{bmatrix}.
\]

It is easy to see the likelihood function of the model is

\[
L(B, \Gamma, A) = \prod_{i=1}^{N} \frac{1}{(2\pi)^{n/2}} \left| \Sigma_i \right|^{-1/2} \left| B \right| \exp\left( -\frac{1}{2} (y_i' B' + x_i' \Gamma') \Sigma_i^{-1} (B y_i + \Gamma x_i) \right) ,
\]

where \( |M| \) denotes the absolute value of \( M \), and \( |B| \) the determinant of \( B \). Therefore the log-likelihood function may be written as

\[
\ell(B, \Gamma, A) = -\frac{Nn}{2} \log(2\pi) + \frac{1}{2} \sum_{i=1}^{N} \log \left| \Sigma_i^{-1} \right| + N \log |B| + \frac{1}{2} \sum_{i=1}^{N} \text{tr}(\Sigma_i^{-1} u_i u_i') ,
\]

where \( \text{tr} \) denotes the trace operator. As in Section 8.1, we assume enough prior restrictions on \( B \) and \( \Gamma \) exist to ensure the identification of the model.
To obtain our test we use the LM procedure, which was described in Section 4.2. Now, for computational convenience, we use a slightly different expression for LM. Consider a log-likelihood function \( \ell(\theta) \), where \( \theta \) is an \( m \) by 1 vector of unknown parameters which we write in partitioned form as \( \theta = (\theta_1', \theta_2') \), where \( \theta_2 \) is of dimension \( m_2 \) by 1. Assume we are interested in testing a set of \( r \) restrictions specified by the hypothesis \( H_0 : h_j(\theta_2) = 0 \) for \( j = 1, 2, \ldots, r \). Let \( d_2 \) denote the \( m_2 \) by 1 vector defined by \( d_2 = \theta \partial \ell(\theta) / \partial \theta_2 \), \( I_{22} \) denote the \( m_2 \) by \( m_2 \) matrix given by \( I_{22} = -E[\partial^2 \ell(\theta) / \partial \theta_2 \partial \theta_2'] \), and \( \hat{\theta} \) denote the maximum likelihood estimator of \( \theta \) under \( H_0 \). Further, let \( \hat{d}_2 \) and \( \hat{I}_{22} \) denote respectively the vector \( d_2 \) and matrix \( I_{22} \) evaluated at \( \theta = \hat{\theta} \). Then it can be shown that if \( I_{22} = E[-\partial^2 \ell(\theta) / \partial \theta_2 \partial \theta_2'] = 0 \), the statistic

\[
LM = \hat{d}_2' \hat{I}_{22}^{-1} \hat{d}_2
\]  

(8.9)
is distributed asymptotically as \( \chi^2_r \) under \( H_0 \). This result also holds when \( \hat{\theta} \) is only a root-N consistent estimator of \( \theta \) under \( H_0 \).

We now define \( \theta_1 \) as the vector with elements being the unknown parameters in \( B \) and \( T \). By construction, \( \Sigma_1 \) is symmetric, so we have that \( a'_{jk} = a'_{kj} \) for all \( j, k = 1, \ldots, n \). We define \( \theta_2 \) as the vector of dimension \( n(n+1)p/2 \) by 1, that contains all the non-identical elements in \( A \), and is given by

\[
\theta_2 = (a'_11, a'_12, \ldots, a'_1n, a'_21, \ldots, a'_2n, \ldots, a'_nn, \ldots, a'_nn)'.
\]
The hypothesis we are interested in testing is a constant disturbance VCM, i.e., \( H_0 : \Sigma_1 = \Sigma \) for \( i = 1, 2, \ldots, N \). This is equivalent to testing \( H_0 : A^* = 0 \) which, due to the requirement that \( a'_{jk} = a'_{kj} \), imposes \( r = n(n+1)(p-1)/2 \) independent restrictions on \( \theta_2 \). To obtain (8.9) we first compute the quantities \( d_2 \) and \( I_{22} \) based on (8.8).
A complication in the derivation we are about to present, arises because of the symmetry of the VCM's $\Sigma_i$, which needs to be taken into account. This has been typically neglected in the econometric literature when obtaining derivatives with respect to VCM's. For example, in the homoscedastic case, most econometrics textbooks (e.g., see Maddala (1977, p.487) and Schmidt (1976, p.217); an exception is Phillips and Wickens (1978, p.332)) neglect the symmetry of $\Sigma$ when obtaining first order conditions of the MLE. As it turns out, this has no effect when the (algebraically incorrect) derivatives are equated to zero.

Computation of the information matrix from these derivatives would - however - be incorrect (see Richard (1975)). For our problem symmetry does matter, because we require the information matrix for the computation of the test statistic. We take symmetry into account which makes our derivation complicated and lengthy. Here we present the results leaving detailed proofs to an Appendix.

As shown in Propositions 5, 7, 8 and 9 in Appendix H (see page 328) if $S'$ is a matrix of dimension $n^2 p$ by $n(n+1)p/2$, that maps $\theta_2$ into $a = \text{Vec} A$ (i.e. such that $a = S' \theta_2$), then

$$\frac{\partial \log |\Sigma_i^{-1}|}{\partial \theta_2} = -S[(I_n \otimes z_i) \otimes I_n] \text{Vec} [\Sigma_i^{-1}] ,$$

$$\frac{\partial \text{tr} (\Sigma_i^{-1} u_i u_i')}{{\partial \theta_2}} = -S[(I_n \otimes z_i) \otimes \Sigma_i^{-1} u_i u_i'] \text{Vec} [\Sigma_i^{-1}] ,$$

$$\frac{\partial^2 \log |\Sigma_i^{-1}|}{{\partial \theta_2 \partial \theta_2'}} = S[\Sigma_i^{-1} \otimes z_i z_i' \otimes \Sigma_i^{-1}] S' ,$$

and
Using these relations we obtain

\[
d_2 = -\frac{1}{2} \sum_{i=1}^{N} \left( (I_n \otimes z_i' \otimes (I_n - \Sigma_1^{-1} u_i' u_i')) \right) \text{vec} \left( \Sigma_1^{-1} \right)
\]

and

\[
I_{22} = \frac{1}{2} \sum_{i=1}^{N} \left( \Sigma_1^{-1} \otimes z_i' z_i' \otimes \Sigma_1^{-1} \right) S' .
\]

If we estimate the model under $H_0$ (i.e. under the assumption that $\Sigma_1 = \Sigma$ for $i = 1, 2, \ldots, N$) we find that the quantities $\Sigma$, $B$ and $\Gamma$ which satisfy the a-priori restrictions and maximize $L(\theta)$ are the FIML estimates $\hat{\Sigma}$, $\hat{B}$ and $\hat{\Gamma}$ defined in equations (8.2) and (8.3). To obtain $\hat{d}_2$ and $\hat{I}_{22}$ we need to replace $\Sigma_1$ and $u_i$ respectively by $\hat{\Sigma}$ and $\hat{u}_i$ in $d_2$ and $I_{22}$. The result of this is

\[
\hat{d}_2 = -\frac{1}{2} \sum_{i=1}^{N} (I_n \otimes \hat{z}_i' \text{vec} (\hat{\Sigma}^{-1})
\]

and

\[
\hat{I}_{22} = \frac{1}{2} \sum_{i=1}^{N} \left( \hat{\Sigma}^{-1} \otimes \hat{z}_i' \hat{z}_i' \otimes \hat{\Sigma}^{-1} \right) S' ,
\]

where

\[
\hat{z}_i' = \sum_{i=1}^{N} \left( z_i' \otimes (I_n - \hat{\Sigma}^{-1} \hat{u}_i' \hat{u}_i') \right) .
\]

We can show that $I_{21} = 0$ (see Proposition 10 in Appendix H, page 331), so it follows from (8.9) that, under $H_0$: $\Sigma_1 = \Sigma$, the statistic
\[ LM = \frac{1}{2} (\text{Vec}[\Sigma^{-1}])' [I_n \otimes \Gamma] S' [S[\Sigma^{-1} \otimes \hat{\Sigma}^{-1}] S']^{-1} S [I_n \otimes \Gamma] (\text{Vec}[\Sigma^{-1}]) \]  

(8.10)

is asymptotically distributed as \( \chi^2_{(r)} \). Therefore, for large samples we would reject \( H_0 \) if the value of \( LM \) exceeds the appropriate upper point of the \( \chi^2_{(r)} \) distribution. The statistic (8.10) is simple to compute, requiring only estimation of the parameters of the model under homoscedasticity. In fact, given that our result is an asymptotic one, for its computation we do not have to calculate \( \hat{u}_i \) and \( \hat{\Sigma} \) using FIML estimators. Rather, we may use root-N consistent estimators under \( H_0 \) without affecting its asymptotic distribution.

We should note that setting \( n = 1 \) in (8.10) (i.e., setting the number of equations in the system equal to one) reduces \( LM \) to the test obtained by Breusch and Pagan (1979) for the ordinary regression model (see also Godfrey (1978c)). So, our result may be seen as the simultaneous equations generalization of their result, although they do not require heteroscedasticity to be necessarily additive.

We should also point out that we have considered a general formulation of heteroscedasticity, allowing each element in the variance-covariance matrix to vary. Nevertheless, by a simple modification, our results may be applied when testing a more specific form of heteroscedasticity. For example, we may assume the covariances are constant, and test only for non-constant variances. For this we note the off-diagonal elements in \( \Sigma_i \), given by \( a'_{jk} z_i = \sigma_{jk} + a_{jk} z_i^2 \) for \( j \neq k = 1, \ldots, n \) and \( i = 1, \ldots, N \), become constant when setting \( a'_{jk} = 0 \).

---

5 A computationally simpler expression for the inverse of \( S[\Sigma^{-1} \otimes \hat{\Sigma}^{-1}] S' \) is \( Q[\Sigma \otimes (Z'Z)^{-1} \hat{\Sigma}] Q' \), where \( Q \) is the Moore-Penrose inverse of \( S' \), i.e., a matrix such that \( QS' = I \) (see Richard (1975, p.59)).
This implies a set of zero restrictions in our \( A \) matrix. In a similar way we could 'impose' constant variance in some of the equations in the system. Recall \( \theta_2 \) denoted the vector of all non-identical elements in \( A \). If some elements in \( A \) are set equal to zero a-priori, then the vector of unknown parameters could be written as \( S_0 \theta_2 \), where \( S_0 \) is a selection matrix that collects the non-zero elements in \( \theta_2 \). In this case, to test for non-constant VCM, we would replace in LM, \( S \) by \( S_0 S \). The resulting statistic would be asymptotically distributed as \( \chi^2_{(r-r_0)} \) under \( H_0 \), where \( r_0 \) is the number of zero elements in \( \theta_2 \).

In every case, acceptance of \( H_0 \) would imply we may proceed under the assumption of constant disturbance VCM. If \( H_0 \) is rejected, we could estimate the parameters \( B \) and \( \Gamma \), by applying the 'simultaneous equations equivalent' of one of the procedures described in Amemiya (1977) (these were suggested for the single equation heteroscedastic model). For example, we may use the Generalized Two Stage Least Squares estimator of Raj, Srivastava and Ullah (1980), replacing their \( \Omega_i \) by, say, \( a_{ij}^t z_i \) when estimating equation \( j \).

8.4 A TEST FOR PARAMETER VARIATION

The results of Section 8.3 may be used to test a form of parameter variation in simultaneous equations models. To see this consider the model

\[
B y_i + \Gamma_i x_i = u_i ,
\]

(8.11)

where \( \Gamma_i \) is a matrix of random coefficients given by

\[
\Gamma_i = \Gamma + \gamma_i \quad \text{for} \quad i = 1,2,\ldots,N ,
\]
and where $\Gamma$ is an $n \times K$ matrix of fixed parameters and $\gamma_i$ is a random term of the same dimension such that $\eta_i = \text{Vec}[\gamma_i'] \sim N(0, \Omega)$.

We write

$$\Omega = \begin{bmatrix}
\Omega_{11} & \Omega_{12} & \cdots & \Omega_{1n} \\
\Omega_{12} & \Omega_{22} & \cdots & \Omega_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\Omega_{1n} & \Omega_{2n} & \cdots & \Omega_{nn}
\end{bmatrix},$$

where the $\Omega_{jk}$ are $K \times K$ matrices.

The identification of model (8.11) is ensured provided the usual conditions for identification of $B$ and $\Gamma$ (for the non-random case) are valid. This is because - in (8.11) - variation only occurs in the parameters associated with the predetermined variables; so Kelejian's (1974) reducibility condition immediately holds.

Note if $x_i$ contains an element which is constant for all $i$, then $u_i$ would not be distinguishable from the varying intercept terms and it could be subsumed into the latter. We assume this to be the case and therefore write our model as

$$By_i + \Gamma x_i = \epsilon_i,$$

where $\epsilon_i = -\gamma_i x_i = \text{Vec}[\epsilon_i'] = -(I_n \otimes x_i') \eta_i$. From this relation it may be seen that $E[\epsilon_i] = 0$ and, after some algebra, that

$$E[\epsilon_i \epsilon_i'] = \Sigma_i = (I_n \otimes x_i')\Omega(I_n \otimes x_i) = A(I_n \otimes z_i),$$

where

$$A = \begin{bmatrix}
(\text{Vec } \Omega_{11})' & \cdots & (\text{Vec } \Omega_{1n})' \\
\vdots & \ddots & \vdots \\
(\text{Vec } \Omega_{1n})' & \cdots & (\text{Vec } \Omega_{nn})'
\end{bmatrix}.$$
and \( z_i = \text{Vec}[x_i x_i'] \). We note that testing random coefficient variation, i.e., \( H_0: \Gamma_i = \Gamma \) for all \( i = 1,2,\ldots,N \), is equivalent to testing the hypothesis that some of the elements in the variance-covariance matrix of \( \eta_i \) are zero, i.e., \( H_0: A^* = 0 \), where \( A^* \) is the appropriate sub-matrix of \( A \). We readily see this is a special case of the problem studied in Section 8.4 with \( z_i \) given by the elements in \( x_i x_i' \). If \( H_o \) is accepted, then analysis can proceed under the assumption of constant parameters. In turn, if \( H_o \) is rejected, we may estimate the model as suggested by Raj, Srivastava and Ullah (1980).

8.5 CONCLUDING REMARKS

In this Chapter, our intention has been to extend some of the results of previous Chapters to the simultaneous equations model. More specifically, we suggested, in Section 8.2, a test for multivariate disturbance normality. Also, in Section 8.3, we derived a test for constant VCM, and noted, in Section 8.4, its use in testing a form of random coefficient variation.

Clearly, there is a lot of scope for further work in the problems we have dealt with. For instance, the relative performance of the limited information homoscedasticity tests of Harvey and Phillips (1981) – and the full information LM test given in (8.10) – needs investigation. Harvey and Phillips (1981) have already studied the performance of their tests when heteroscedasticity occurs only in one equation in the system; it is interesting to see how all tests behave under more general forms of heteroscedasticity.
9.1 INTRODUCTION

In this Chapter we present a cross-sectional study on consumption behaviour at the household level. Household consumption studies are — of course — not new. In fact, the estimation of demand functions in family budget studies is one of the oldest exercises in applied econometrics (some historical accounts are given in Brown and Deaton (1972)). However, we concentrate on México for which — as far as we are aware — the only study in this area is that of Lluch, Powell and Williams (1977); hereafter referred to as LPW (1977). LPW (1977) use data from the 1968 Income-Expenditure Household Survey carried out by the Banco de México. Here we use data from another survey, namely, the 1975 Income-Expenditure Household Survey carried out by the Centro Nacional de Información y Estadísticas del Trabajo, to provide an independent additional source of information on the determinants of household demand for various commodities.¹ The main interest in this

¹ The absence of Mexican national accounts data on commodity expenditures has ruled out any time-series estimation of demand systems. Indeed, the existing econometric models for México estimate a single consumption function. So, our results should be of value for planning purposes.
Chapter is to illustrate the use of some of the procedures suggested in previous Chapters, so our econometric methodology differs in some respects from that of LPW (1977).

Firstly, we apply clustering algorithms, as described in Chapter 7, to form groups of 'homogeneous consumers'.

Secondly, we take into consideration the fact that expenditures are non-negative and therefore use limited dependent variable (LDV) models.

Thirdly, we carry out a more comprehensive statistical analysis of the disturbances, e.g., we apply the normality and homoscedasticity tests for LDV models suggested in Chapter 6.

The structure of the Chapter is as follows. In Section 9.2 we present the model considered and, in Section 9.3, the econometric methodology employed. In Section 9.4 we describe the data used for estimation. The main numerical results are presented in Section 9.5. The Chapter ends with Section 9.6, where we give a summary of our principal findings and make some concluding remarks.

9.2 THE EXTENDED LINEAR EXPENDITURE SYSTEM

The Extended Linear Expenditure System (ELES) results from recent developments in Demand Theory and provides a convenient analytical framework for our purposes. Absence of price data rules out the use of other demand systems, such as Deaton and Muellbauer's (1980) Almost Ideal Demand System.
Consider a given household $i$ and assume that, at a given point in time, it is faced with the decision of allocating its current income $x^*$ among a set of $n$ goods or commodities. (In fact, these will be groups of commodities, e.g., food, clothing, etc.). The quantities that are purchased for each good are denoted by $q_1, \ldots, q_n$, and the price per unit of good $j$ is denoted by $p_j$, for $j = 1, \ldots, n$. Therefore we have that, in the period considered, the household would save or dissave an amount equal to $s = x^* - p'q$, where $p = (p_1, \ldots, p_n)'$ and $q = (q_1, \ldots, q_n)'$.

We postulate that the household makes its choice of $q_1, \ldots, q_n$ and hence $s$, so that it maximizes its utility. We proceed as did Howe (1975) and consider a Stone-Geary utility function $v(\cdot)$, treating saving as an endogenous variable with 'subsistence or committed quantity' equal to zero. More formally, we state the $i$'th household's decision problem as choosing $q_1, \ldots, q_n$ so as to maximize

$$v(q,s) = \sum_{j=1}^{n+1} \theta_j^* \log(q_j - \phi_j),$$

subject to $x^* = s + p'q; \theta_j^* > 0$ and $q_j - \phi_j > 0$ for $j = 1, \ldots, n$; $\phi_{n+1} = 0$; and $\sum_{j=1}^{n+1} \phi_j = 1$; where $s = p_{n+1}'q_{n+1}$ and where $\phi_j$ denotes the committed consumption or subsistence quantity of commodity $j$.

For the solution of this maximization problem we form the Lagrangian

$$\Lambda = v(q,s) + \lambda(x^* - \sum_{j=1}^{n+1} p_j q_j),$$

From this we obtain the first order conditions

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3 In our discussion we use the terms 'household' and 'family' indistinctly.
\[ \frac{\partial \lambda}{\partial q_j} = \frac{\partial \theta_j}{\partial q_j}(q_j - \phi_j) - \lambda p_j = 0 \quad j = 1, \ldots, n+1 \]  
(9.1)

and

\[ \frac{\partial \lambda}{\partial \lambda} = x^* - \sum_{j=1}^{n+1} p_j q_j = 0 \quad . \]  
(9.2)

Equation (9.1) gives

\[ \theta_j = \lambda p_j (q_j - \phi_j) \quad . \]  
(9.3)

Summing (9.3) from \( j = 1 \) to \( n+1 \), and using (9.2) and \( \sum_{j=1}^{n+1} \theta_j^* = 1 \), we obtain

\[ \lambda = 1/(x^* - \sum_{j=1}^{n+1} p_j \phi_j) \quad . \]  
(9.4)

By substitution of (9.4) into (9.3), and noting \( \phi_{n+1} = 0 \) and \( s = p_{n+1} q_{n+1} \), we obtain the ELES, namely,

\[ p_j q_j = p_j \phi_j + \theta_j^* (x^* - p'\phi) \quad j = 1, \ldots, n \]  
(9.5)

and

\[ s = \theta_{n+1}^* (x^* - p'\phi) \quad , \]  
(9.6)

where \( \phi = (\phi_1, \ldots, \phi_n)' \).

\[ \square \]

Demand and Savings Responses:

We note that \( p'q \) represents total consumption of the household. Also, that from (9.6) we have \( \partial s/\partial x^* = \theta_{n+1}^* \) and, therefore, \( \theta_{n+1}^* \) is the marginal propensity to save and \( \mu = 1 - \theta_{n+1}^* \) is the marginal

\[ 4 \text{ It may be shown that second order conditions are satisfied.} \]
propensity to consume. Furthermore, we note that
\[
\frac{\partial p_j q_j}{\partial (p'q)} = \frac{(\partial p_j q_j/\partial x^*)/(\partial (p'q)/\partial x^*)}{0_j^*/\mu}, \text{ and hence } 0_j = 0_j^*/\mu
\]
is the marginal budget share. From this it follows that \(0_j^* = 0_j\mu\) and that equations (9.5) and (9.6) may be written as

\[
p_j q_j = p_j \phi_j + \theta_j \mu (x^*-p'\phi) \quad j = 1,...,n \tag{9.7}
\]

and

\[
s = (1-\mu)(x^*-p'\phi) \tag{9.8}
\]

The parameters \(0_1^*,...,0_n^*\) satisfy the restriction \(\sum_{j=1}^{n} 0_j = 1\), and our aim is to use data on \(p_1 q_1,...,p_n q_n\) and \(x^*\) to estimate the vector \(\theta = (\phi_1,...,\phi_n, 0_1^*,...,0_n^*, \mu)\) subject to this restriction.

**Elasticities of the ELES:**

The elements in \(\theta\) represent household demand and saving responses and, in addition to these, we will consider the following elasticities of the ELES [see LPW (1977, pp.16-20) for a detailed description of these]:

(i) Elasticity of demand for commodity \(j\) with respect to:

- total expenditure: \(\eta_{jt} = \theta_j p'q/(p_j q_j)\)
- own-price: \(\eta_{jj} = (1-\mu \theta_j) p_j \phi_j/(p_j q_j) - 1\)
- cross-price: \(\eta_{jk} = -\mu \theta_j p_k \phi_k/(p_j q_j)\) \tag{9.9}

(ii) Elasticity of total expenditure with respect to income:

\(\eta_{tx} = ux^*/p'q\).
(iii) Elasticity of saving with respect to:

\[ \eta_{s*} = \frac{x^*}{(x^*-p')f} \]

price of commodity j: \[ \eta_{sj} = \frac{-p_j/\bar{f}_j}{(x^*-p')f} \]

(iv) Frisch Parameter:

\[ \hat{\eta} = \frac{-p'q/(p'q-p')}{\eta} \]

Before proceeding to the discussion of estimation of \( \theta \) and the above elasticities, we point out that equations (9.5) and (9.6) have been derived from an atemporal maximization problem. However, these equations may also be obtained by an intertemporal maximization problem (see Lluch (1973)). In this case we would maximize the present value of utility at the beginning of a consumption plan, and \( x^* \) would be permanent rather than current income. As noted by LPW (1977, p.14) either of these set-ups of the problem may be chosen or, in fact, neither. In the latter case, the ELES may be considered as "just a descriptive device to organize data on household saving and expenditure allocation".

9.3 ECONOMETRIC METHODOLOGY

9.3.1 Definitions

Assume that for the estimation of the ELES we have data on a cross-section of \( N \) households, and that this data comes from a region where all the households are exposed to the same prices \( p_1, \ldots, p_n \). In addition, assume that we have classified these \( N \) households into

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5 The available data for our study refers to a particular city, so this is not an unreasonable assumption. A detailed description of this data is given in Section 9.4.
L groups so the parameters of the ELES are approximately the same for households within a given group (how these groups are formed is described in Subsection 9.3.4). Since the parameters are approximately the same, we say that each group contains households of 'homogeneous behaviour'.

In this subsection and in Subsections 9.3.2 and 9.3.3, the discussion will refer to a given group \( h \) containing, say, \( N_h \) households. To avoid unnecessary notation we will not introduce a subscript \( h \) (in \( \theta \) nor in other quantities) to denote group dependencies. Following this point we use \( M = N_h \). Also, we define (see (9.7) and (9.8))

\[
\begin{align*}
\beta_{1j} &= p_j \phi_j - \theta_j \mu \psi \quad j = 1, \ldots, n, \\
\beta_{2j} &= \theta_j \mu \quad j = 1, \ldots, n, \\
\beta_{1(n+1)} &= -(1-\mu) \psi \\
\beta_{2(n+1)} &= 1 - \mu
\end{align*}
\]

Using (9.10) in equations (9.7) and (9.8), and introducing stochastic terms, we may write the model (for the data on the \( M \) households) as follows

\[
y_{ij} = x_i^t \beta_j + \varepsilon_{ij}
\]

with \( j = 1, \ldots, n+1; i = 1, \ldots, M; \) and where \( x_i^t = (1, x_i^k); x_i^k \) is current income of the \( i \)'th household; \( y_{ij} \) is expenditure of the \( i \)'th household on commodity \( j(j = 1, \ldots, n); y_{(n+1)i} = s_i; \beta_j = (\beta_{1j}, \beta_{2j}); \) and \( \varepsilon_{ij} \) is the disturbance corresponding to the \( i \)'th household for the \( j \)'th commodity.
9.3.2 Identification of the ELES

We now consider the identification of the model. The ELES, as defined in equations (9.7) and (9.8), contains the parameters

\[ \theta = (\phi_1, \ldots, \phi_n, \theta_1, \ldots, \theta_n, \mu) \]

We know that \( \theta_1, \ldots, \theta_n \) must satisfy the restriction \( \sum_{j=1}^{n} \theta_j = 1 \) and so there are, in \( \theta \), a total of \( 2n \) independent parameters to be estimated. With observations on \( y_j \) and \( x^* \) we can estimate, from the first \( n \) equations in (9.11), \( 2n \) parameters. Hence, as noted by Howe (1975), the system is perfectly identified. This means that for estimated values of the \( \beta \)'s we can obtain unique estimates for \( \theta \). We also note that, by construction, there are restrictions that the vectors \( \beta_1, \ldots, \beta_{(n+1)} \) must satisfy. This may be readily seen from (9.10) and the restriction on the \( \theta_j \) - from which it follows that

\[ \beta_{1(n+1)} = - \sum_{j=1}^{n} \beta_{1j} \]

and

\[ \beta_{2(n+1)} = 1 - \sum_{j=1}^{n} \beta_{2j} \]

(9.12)

9.3.3 Estimation of the ELES

We now consider the estimation of the model. Define

\[ \epsilon_i = (\epsilon_{1i}, \ldots, \epsilon_{ni})' \]

and denote its VCM by \( \Omega \). Also, define

\[ \epsilon_i^+ = (\epsilon_i^+, \epsilon_{(n+1)i})' \]

and denote its VCM by \( \Omega^+ \). Given we are in a cross-sectional context, we can assume the covariances between \( \epsilon_i^+ \) and \( \epsilon_k^+ \) are zero for \( i \neq k \). Let \( 1_r' \) denote a vector of ones, of dimension 1 by \( r \). Note that, because \( x_i^* = s_i + \sum_{j=1}^{n} y_{ji} \) it follows the \( n+1 \) disturbances \( \epsilon_{1i}, \ldots, \epsilon_{ni}, \epsilon_{(n+1)i} \) are interdependent. More specifically
we have \( (n+1) \varepsilon_i^+ = 0 \), i.e., \( \varepsilon_{(n+1)i} = -1 \varepsilon_i^+ \). Therefore, we can write
\[ \varepsilon_i^+ = (\varepsilon_i^', -1 \varepsilon_i) = H \varepsilon_i \], where \( H = [I_n, -1_n]^\prime \). From this we observe that
\[
\Omega^+ = H \Omega H' = \begin{bmatrix}
\Omega & -\Omega l_n \\
-1\Omega & l\Omega 1_n
\end{bmatrix}
\]  \hspace{1cm} (9.13)

It is clear, then, that the rank of \( \Omega^+ \) cannot exceed that of \( \Omega \), and hence \( \Omega^+ \) is singular. We assume, however, that \( \Omega \) has full rank, so the rank of \( \Omega^+ \) is \( n \). Having made this point, we proceed to the distributional assumptions regarding \( \varepsilon_i^+ \).

(i) The Normal Model

We start with the simplest distributional assumption, which consists of assuming normality, therefore neglecting the non-negativeness of the dependent variable (comments are made later on the estimation of the model under a more appropriate stochastic specification). Given that the matrix \( \Omega^+ \) is singular, we cannot state \( \varepsilon_i^+ \sim N(0, \Omega^+) \) and attempt to maximize the likelihood function subject to the parameter restrictions. An appropriate solution to this problem is to drop one of the equations, and estimate the \( n \) remaining equations by assuming normality. This subsystem would have a non-singular VCM. It can be shown that the estimators would be invariant to the choice of equation being omitted (see Powell (1974, p.48) for various references). For example, we may proceed as in Powell (1973) and drop the equation relating to saving (i.e., equation \( n+1 \) in (9.11)), and obtain estimates for \( \hat{\beta}_1, \ldots, \hat{\beta}_n \), say \( \hat{\beta}_1, \ldots, \hat{\beta}_n \). Then, we may obtain estimates of \( \hat{\beta}_{(n+1)} \) from the relations given in (9.12). This approach is now used to derive the explicit form of the Maximum Likelihood Estimators (MLE) under normality.
Define $y_j = (y_{j1}, \ldots, y_{jM})'$, $X = (x_1, \ldots, x_M)'$ and $e_j = (e_{j1}, \ldots, e_{jM})'$ for $j = 1, \ldots, n$. Also, define $Y = (y_1, \ldots, y_n)$, $B = (\beta_1, \ldots, \beta_n)$ and $E = (e_1, \ldots, e_n)$. Then we can write the first $n$ equations in (9.11) as

$$y = (I_n \otimes X)\beta + \varepsilon,$$  

(9.14)

where $y = \text{Vec}(Y)$, $\beta = \text{Vec}(B)$ and $\varepsilon = \text{Vec}(E)$, and where $\otimes$ denotes the Kronecker product. In turn, our normality assumption, together with our VCM specification for the disturbances, may be written as $\varepsilon \sim N(0, \Omega \otimes I_M)$. Therefore, the logarithm of the likelihood function is given by

$$\ell(\beta, \Omega) = -\frac{Mn}{2} \log(2\pi) + \frac{1}{2} \log|\Omega| - \frac{1}{2} \varepsilon' [\Omega^{-1} \otimes I_M] \varepsilon.$$  

(9.15)

From the maximization of this with respect to $\beta$ and $\Omega$ we obtain

$$\hat{\beta} = [(I_n \otimes X')(\hat{\Omega}^{-1} \otimes I_M)(I_n \otimes X)]^{-1}[(I_n \otimes X')(\hat{\Omega}^{-1} \otimes I_M)y],$$  

which reduces to

$$\hat{\beta}_j = (X'X)^{-1}X'y_j;$$  

and

$$[\hat{\Omega}]_{jk} = e_j' e_k / M,$$

where $e_j = y_j - X\hat{\beta}_j$ for $j = 1, \ldots, n$. Then it is clear that, in this situation, Maximum Likelihood Estimation of the parameters in the system (9.14) gives the same result as Ordinary Least Squares (OLS) estimation in each of the $n$ equations separately. We also note that, using (9.12), we can estimate $\beta_{(n+1)}$ by

$$\hat{\beta}_{(n+1)} = [- \sum_{j=1}^n \hat{\beta}_{1j}, 1 - \sum_{j=1}^n \hat{\beta}_{2j}]'.$$  

(9.16)
Further, it is easy to show that \( \hat{\beta}_{(n+1)} \) obtained in this way is the same as OLS estimation applied to equation \( n+1 \) in (9.11).

So, OLS estimation on each of the \( n+1 \) equations in (9.11) provides efficient estimators which satisfy the parameter restrictions given in (9.12). It also follows that we can use, on each equation separately, the standard t and F tests.

For some groups of households the normality assumption may not be entirely inappropriate and, for these, we could use the estimation procedure just described. For other groups, however, this may not be the case and further considerations regarding the distribution of the disturbances may be required. This is now discussed in greater detail.

(ii) The Truncated Model

All but one of the dependent variables in our model are expenditures. Hence we know these must be non-negative, i.e., that \( y_j^{i*} \) is restricted to satisfy \( y_j^{i*} \geq 0 \) for \( j = 1, \ldots, n \). So far we have assumed disturbance normality. Under this, we would be allowing for a positive probability of having negative values in the variables \( y_1^{i*}, \ldots, y_n^{i*} \).

As mentioned in Chapter 6, this (clearly undesirable) feature of the normality assumption is often disregarded, on the grounds that the above probability is so small that there is little point in complicating the analysis, by restricting dependent variables to non-negative values. This is a valid consideration when the conditional expectations of \( y_1^{i*}, \ldots, y_n^{i*} \) are 'large'. For this study, however, we have that, particularly for some groups of households (e.g., those with low incomes), the conditional expected expenditures on some commodities (e.g., clothing) is 'close' to zero for the majority of the households. In these cases the latter probability would not be 'negligible' and it would
be inappropriate to proceed under the normal model. (These probabilities are presented in Section 9.5). An alternative is to use the Truncated model introduced in Section 6.3. Yet, this approach is also not 'problem-free' since, in order to have a computationally manageable likelihood, we need to assume certain covariances in $\Omega^+$ are zero. This point is now illustrated.

Consider the $n+1$ equations in (9.11), and restrict the range of $\varepsilon_{1j}$ to $\varepsilon_{1j} \geq -x_1^i\beta_j$ for $j = 1, \ldots, n$. Assume $\varepsilon_{1j}$ is truncated normal with truncation point $-x_1^i\beta_j$ (then, $y_{1j}$ would be non-negative with probability one, for $j = 1, \ldots, n$). For expository convenience, assume we have a group where most households have 'large' conditional expenditure $x_1^i\beta_j$ on $n-1$ of the $n$ commodities. Say these are the commodities in equations $j = 2, \ldots, n$. Also, assume that a considerable number of households in the group have 'small' conditional expenditure on commodity 1. Regarding the VCM of the disturbances we now write equation (9.13) in the form

$$
\Omega^+ = \begin{bmatrix} \sigma^2 & \delta_1 \\
\delta_1^t & \Omega_1^+ \end{bmatrix},
$$

(9.17)

where $\sigma^2$ denotes the variance of $\varepsilon_{11}$; $\delta_1 = (\delta_1^t, \delta_2); \delta$ is a 1 by $n-1$ vector; $\delta_1^t$ is a scalar; and $\Omega_1^+$ is an $n$ by $n$ matrix. Assume that $\delta = 0$. This implies that $\Omega_1^+$ is given by

$$
\Omega_1^+ = \begin{bmatrix} \Omega_1 & -\Omega_1^{-1}(n-1) \\
-\Omega_1^{-1}(n-1)& \sigma^2 + \Omega_1^{-1}(n-1)\Omega_1^{-1}(n-1) \end{bmatrix},
$$
where $\Omega_1$ is an $n-1$ by $n-1$ matrix of full rank [The specific form of $\Omega_1$ arises from the fact that $1'(n+1)e_1^* = 0$, which may be written as $e_{(n+1)i} = e_{1i} - 1'(n-1)e_i^*$, and because the covariances between $e_{1i}$ and $e_i^*$, i.e. $\delta'$, are assumed to be zero, where $e_i^* = (e_{2i}, \ldots, e_{ni})'$].

Now consider equations 2 to $n+1$. We can drop one of these, say equation $n+1$, and treat the disturbances in the remaining equations as being jointly normal with mean zero and VCM equal $\Omega_1$. For these equations the probability under normality of having $e_{1j} < -x!\beta_j$ would be so small (since $x_i^j \beta_j$ would be large) that the normality assumption would not be entirely inappropriate. Regarding equation 1 note that, because of our truncated normality assumption, we have

$$p.d.f.(e_{1i}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2} e_{1i}^2\right), \quad (9.18)$$

where $e_{1i} > -x_i^j \beta_j$ and

$$F_i = F(x_i^j \beta_j, \sigma^2)$$

$$= \int_{-\infty}^{x_i^j \beta_j} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\lambda^2}{2\sigma^2}\right) d\lambda.$$  

[Note that if $x_i^j \beta_j$ were large, then $F_i \approx 1$. In this case the p.d.f. of $e_{1i}$ would be approximately normal, and one could use the results for the normal model]. It follows that the likelihood function would be given by

$$L(\beta, \sigma^2, \Omega_1) = L_1(\beta_1, \sigma^2)L_2(\beta_2, \ldots, \beta_n, \Omega_1), \quad (9.19)$$

where
\[
L_1(\beta_1, \sigma^2) = \prod_{i=1}^{M} \exp \left( -\frac{1}{2\sigma^2} \epsilon_{1i}^2 \right)
\]

and

\[
L_2(\beta_2, \ldots, \beta_n, \Omega_1) = \prod_{i=1}^{M} \frac{1}{(\sqrt{2\pi})^{n-1} |\Omega_1|^{-\frac{1}{2}}} \exp \left( -\frac{1}{2} \epsilon_i^* \Omega_1^{-1} \epsilon_i^* \right).
\]

From (9.19) we see that maximizing \( L(\cdot) \) is equivalent to maximizing \( L_1(\cdot) \) and \( L_2(\cdot) \) separately. So, the MLE of \( \beta_1 \) and \( \sigma^2 \) satisfy the first order conditions for the Truncated model (see equations (6.4) and (6.5)); and we can use the statistic \( \text{LM}_{NH(\text{TRUN})} \) to test for disturbance \( NH \) (see equation (6.15)). In turn - from \( L_2(\cdot) \) - the MLE of \( \beta_2, \ldots, \beta_n \) and \( \Omega_1 \) will be equal to the OLS estimators. Equation (9.19) arises - of course - due to our assumption that \( \delta = 0 \) in (9.17). If \( \delta \neq 0 \), then computational difficulties are encountered.

Comparing OLS estimators with MLE on the Truncated model, we observe the following:

If the truncation problem is severe, in the sense that probabilities under normality of obtaining negative observations are 'large', then OLS estimators would not be consistent (see footnote 1 in Chapter 6 for details). Here, MLE on the Truncated model of each commodity would be consistent and asymptotically efficient, provided the covariances in \( \Omega \) are close to zero (see (9.13)).

If the truncation problem is not severe, then (9.15) would be a 'good' approximation to the likelihood of the system and OLS estimators would be close to the ML estimators, whatever the values of the covariances. Yet, in this situation, MLE on the Truncated model
of each commodity would be approximately equal to the respective OLS estimator, so OLS would be no better.

We then see that, in both cases (severe and not severe truncation problem), MLE on the Truncated model of each commodity has advantages or does no worse than OLS estimators. So, provided one has appropriate computer programmes, the use of MLE on the Truncated model would be appealing. In Section 9.5 we present both ML estimators using the Truncated model and OLS estimators.

A final comment is required. Unlike the normal model, in the Truncated model the expectation of $\epsilon_{ji}$ would not be zero. This is due to the nature of the truncated normal distribution. For instance, take commodity 1 with disturbance p.d.f. given by (9.18). Here we would have

$$E[\epsilon_{1i}] = \sigma^2_f_i / F_i,$$

from which it follows that

$$E[y_{1i}] = x_1^* \beta_1 + \sigma^2_f_i / F_i,$$

where $f_i = f(x_1^* \beta_1, \sigma^2)$ and $f(\sigma^2) = (2\pi\sigma^2)^{-1/2}\exp(-\omega^2/(2\sigma^2))$.

In (9.10) we defined $\beta_{1j} = p_j \phi_j - \theta_j u \phi'$ and $\beta_{2j} = \theta_j u$. From (9.7), namely, $p_j q_j = (p_j \phi_j - \theta_j u \phi') + (\theta_j u)x^*$, we see that $\beta_{1j}$ and $\beta_{2j}$ were intended to represent, respectively, the expected expenditure on commodity $j$ when $x^* = 0$; and the derivative of expected expenditure on commodity $j$ with respect to $x^*$. As shown by McDonald and Moffitt (1980) (see also Poirier and Melino (1978)), for the Truncated model we have
\[ \delta E[y_{11}] / \delta x^* = \beta_{21} [1 - x_1^i \delta_{11} f_1 / F_1 - \sigma^2 \delta_{11}^2 / F_1^2] \]

and

\[ E[y_{11} | x^* = 0] = \beta_{11} + \sigma^2 f(\beta_{11}, \sigma^2) / F(\beta_{11}, \sigma^2) . \]

It follows that - before computing estimates of \( \theta \) based on estimates of the \( \beta \)'s - we have to adjust these so they represent the desired demand responses. More formally, after computing the MLE of \( \beta_1 \) and \( \sigma^2 \) for the Truncated model, say \( \hat{\beta}_1 = (\hat{\beta}_{11}, \hat{\beta}_{21})' \) and \( \hat{\sigma}^2 \), we would set

\[ \hat{\beta}_{11} = \hat{\beta}_{11} + \sigma^2 f(\beta_{11}, \sigma^2) / F(\beta_{11}, \sigma^2) , \]

and

\[ \hat{\beta}_{21} = \hat{\beta}_{21} [1 - \bar{x}' \beta_{11} f(\bar{x}' \beta_{11}, \sigma^2) / F(\bar{x}' \beta_{11}, \sigma^2) - \sigma^2 f^2(\bar{x}' \beta_{11}, \sigma^2) / F^2(\bar{x}' \beta_{11}, \sigma^2)] , \]

where \( \bar{x}' = (1, \bar{x}^*) \) and \( \bar{x}^* = (x_1^* + \ldots + x_M^*) / M. \)

Similar considerations apply to other commodities for which the Truncated model is used.

(iii) The Tobit Model

Under the assumption that disturbances are normal or truncated normal, we would have a 'zero probability' of obtaining several observations \( y_{ji} \) which are identically zero. This is because both distributions are continuous. These distributional assumptions are appropriate when few or no households report zero expenditures. In the present study, however, we have that, particularly for some groups of households (e.g., those with low incomes), a considerable number of the reported expenditures on some commodities (e.g., durables) are zero. Then, in order to have a 'valid' model specification, we would need to allow the
dependent variables to take the limiting zero value with a non-zero probability. To achieve this we make use of the Tobit model, which was fully described in Sections 6.5 and 6.6.

Here we proceed as for the Truncated model. Again, to keep the problem numerically manageable, we have to assume that certain covariances in $\Omega$ are zero. For this model OLS estimators are inconsistent, so the use of Tobit estimators is appealing. The analysis of each commodity for which the Tobit model is used is done as described in Chapter 6 (in particular, we would use the statistic $\text{LM}_{NH(TOBIT)}$ to test for disturbance normality and homoscedasticity (see (6.26)).

In the Tobit model, as in the truncated model, we have $E[\varepsilon_{11}] \neq 0$. For instance, if the Tobit model were used for commodity 1 we would have (see McDonald and Moffitt (1980) and Poirier and Melino (1978))

$$E[y_{1i}] = x_i'\beta_{11} + \sigma_1^2 f_1$$

In this case,

$$\partial E[y_{1i}]/\partial x^* = \beta_{21} f_1$$

So, by similar arguments to those presented for the Truncated model, having computed the MLE's of the Tobit model, say $\hat{\beta} = (\hat{\beta}_{11}, \hat{\beta}_{21})'$ and $\hat{\sigma}^2$, we would define

$$\hat{\beta}_{11} = F(\hat{\beta}_{11}, \hat{\sigma}^2) + \sigma^2 f(\hat{\beta}_{11}, \hat{\sigma}^2)$$

and

$$\hat{\beta}_{21} = F(\hat{\beta}_{21}, \hat{\sigma}^2)$$

(9.21)
To summarize this subsection we note that, given a group of 'homogeneous households', our choice of distributional assumptions - and hence model - is made on the basis of the reported expenditures. For a given commodity, if there are reported zero expenditures, then we use the Tobit model; otherwise, we use the Truncated model. OLS estimators are also obtained - in all cases - for comparison. Having computed MLE's of the Tobit and Truncated models we would adjust these to yield \( \hat{\beta}_1, \ldots, \hat{\beta}_n \); and would obtain \( \hat{\beta}_{(n+1)} \) as in (9.16). With the values \( \hat{\beta}_1, \ldots, \hat{\beta}_{(n+1)} \), we would estimate the elements of \( \theta = (\phi_1, \ldots, \phi_n, \theta_1, \ldots, \theta_n, \mu) \) as follows (see (9.10)):

\[
\hat{\mu} = 1 - \hat{\beta}_{2(n+1)} \\
\hat{\theta}_j = \hat{\beta}_{2j}/(1-\hat{\beta}_{2(n+1)}) \tag{9.22}
\]

and

\[
p_j\hat{\phi}_j = (\hat{\beta}_{1j} - \hat{\beta}_{2j}\hat{\beta}_1(n+1)/(\hat{\beta}_{2(n+1)}))
\]

Then, \( \hat{\mu}, \hat{\theta}_j \) and \( p_j\hat{\phi}_j \) would be used for the estimation of the elasticities defined in equations (9.9). All these quantities are required, since they provide the basis by which we analyze household consumption and saving behaviour.

9.3.4 Classification of Households

The data used in this Chapter consists of a cross-section of \( N \) households. In our presentation in Subsections 9.3.1, 9.3.2 and 9.3.3 we assumed the households were classified into \( L \) groups, such that the parameters of the ELES were approximately the same for households within each group. Here we discuss how these groups may be formed.
Empirical evidence exists which shows the consumption behaviour of households (analyzed in terms of the $\beta$-values) depends on variables such as age and occupation of the head of the household, and number of members and income of the household (e.g., see LPW (1977)). We shall assume that we have data on $p$ of such socioeconomic variables. There are various ways of incorporating the effects of socioeconomic variables into demand systems (e.g., see Pollak and Walles (1980) and Williams (1977)). Here we will proceed as in Chapter 7 making use of our clustering criterion $\hat{\alpha}^2(C)$ (see (7.26)).

We first consider equation $j$ in (9.11). Application of the 'single-equation' arguments presented in Section 7.4 would give the clustering criterion

$$\hat{\alpha}^2_j(C) = 1 - \frac{1}{NK} \left\{ \sum_{k=1}^{K} \sum_{l=1}^{L} \sum_{i=1}^{N_h(C)} \left( \frac{(\hat{\beta}_{hikj}(C) - \bar{\beta}_{hkj}(C))^2}{V[\hat{\beta}_{kj}]} \right) \right\},$$

where $\hat{\beta}_{hikj}(C)$, $\bar{\beta}_{hkj}(C)$ and $V[\hat{\beta}_{kj}]$ are defined as $\hat{\beta}_{hik}(C)$, $\bar{\beta}_{hk}(C)$ and $V[\hat{\beta}_{kj}]$ in (7.26), but referring to equation $j$, and where $K = 2$. The quantity $\hat{\alpha}^2_j(C)$ is the 'Overall Relative Explanatory Power' of the equation for expenditures on commodity $j$. Unlike the discussion in Chapter 7, here we have more than one equations to be estimated; more precisely, we have $n$ equations. Yet, in this case we can combine the quantities $\hat{\alpha}^2_1(C), \ldots, \hat{\alpha}^2_n(C)$ to obtain a more comprehensive classification criterion. Our suggestion now is to find the classification of the $N$ households, say $C^*$, that maximizes the 'Average of the Overall Relative Explanatory Powers', i.e.,

$$\hat{\alpha}^2_+(C) = \frac{n}{\sum_{j=1}^{n} \hat{\alpha}^2_j(C)/n}. $$
After substitution for $\hat{\delta}_{ji}^2(C)$, we obtain

$$\hat{\delta}_{ji}^2(C) = 1 - \frac{1}{NKn} \left\{ \sum_{j=1}^{n} \sum_{k=1}^{K} \sum_{l=1}^{L} \sum_{h=1}^{H} \frac{(\hat{\beta}_{hikj}(C) - \hat{\beta}_{hkj}(C))^2}{V(\hat{\beta}_{kj})} \right\} . \quad (9.23)$$

To compute $C^*$, we can use the K-Means or Ward's clustering algorithms described in Subsection 3.4.2. Now we would have $J = 2n$ variables, namely, $\frac{\hat{\beta}_{kj}}{V(\hat{\beta}_{kj})}$ (with $k = 1, 2$ and $j = 1, \ldots, n$), and $N$ observations on each of these.

9.4 THE DATA

The data used for the estimation of the ELES comes from the 1975 Income-Expenditure Household Survey, carried out by the Centro Nacional de Información y Estadísticas del Trabajo, of the Secretaría del Trabajo y Previsión Social. The survey covered the whole of México but, for our study, only the information from the 521 interviewed México City households was made available. This meant that we were not able to study the rural-urban breakdown as LPW (1977) did.

The data supplied divided consumption into $n = 7$ commodities, namely,

1. Food
2. Clothing
3. Housing
4. Durables
5. Education
6. Medical Services and
7. Other.

In particular, the 'Food' category included beverages and tobacco;

---

6 In this survey, a stratified multi-stage sampling design was used. A full description of the survey characteristics is given in CENIET (1977).
'Housing' included payments for electricity and telephone; 'Durables' included furniture, radios, television sets and automobiles; and 'Other' included transport and domestic services. The regressor used is family income (see footnote 3), which excludes payments received by domestic workers; and includes salaries, interest receipts, income from renting or selling agricultural and non-agricultural goods, and receipts from lotteries and inheritances. All the consumption and income variables were given in pesos per month, and — as in LPW (1977) — we expressed these in per capita terms for estimation.

The data on socioeconomic variables contained the family size and age and occupation of the head of the household. This last variable was divided into four categories: Unemployed, Worker, Entrepreneur and Technocrat. Worker includes skilled, unskilled, domestic services and drivers. Technocrat includes professional, technical and administrative employees. The empirical results are given in the next section.

9.5 EMPIRICAL RESULTS

9.5.1 Testing for Systematic Parameter Variation; Empirical Results

We first consider all 521 households in our sample, and define

\[ y_{ij}, \beta_j, x_{i1}^t \text{ and } e_{ij} \text{ as in (9.11). Now we have } n = 7 \text{ and } N = M = 521. \]

We also define \( z_i = (z_{i1}, z_{i2}, z_{i3}, z_{i4}) \), where \( z_{i1} \) is the occupation of the head of the \( i \)'th household (0 if Unemployed; 1 if Technocrat; 2 if Entrepreneur; and 3 if Worker); \( z_{i2} \) is the \( i \)'th family income squared; \( z_{i3} \) is the \( i \)'th family size; and \( z_{i4} \) is age of head of the

\[ \text{We use income squared } (x_{i1}^2) \text{ rather than income } (x_{i1}^t) \text{ to avoid multicollinearity in (9.24). Recall } x_{i1} = (1, x_{i1}^t). \]
i'th household (in years).

We take a given equation, say equation \( j \), i.e., \( y^i = x^i \beta_j + \epsilon_{ji} \), and postulate that the \( \beta_j \) for the i'th household is related to \( z_i \). We proceed as in Section 7.2, formulating a linear functional relation between \( \beta_j \) and \( z_i \). So, we end with the 'augmented regression model'

\[
y_{ji} = (x'_i, (z_i' \Theta x'_i)) \gamma_j + u_{ji+},
\]

where \( u_{ji+} = x'_i \epsilon_{ji} \). [This model is the same as the one given in (7.4), only that now we have an additional subscript, \( j \), indicating the equation refers to commodity \( j \)].

We estimated equation (9.24) by OLS and, to test the hypothesis of systematic parameter variation, we computed the statistic \( F_{SPV} \) (see Section 7.2). The results are summarized in Table 9.1 for each commodity \( j \) (\( j = 1, \ldots, 7 \)). The numbers in brackets below each estimated parameter represent t-statistic, obtained using the heteroscedasticity consistent VCM of the OLS estimate of \( \gamma_j \) (Note \( u_{ji+} \) is heteroscedastic so we have to use (5.19) to get the proper VCM).

Looking at the row for \( F_{SPV} \), we see that, except for the regression for 'Medical Services', there is evidence of existence of systematic parameter variation [All values of \( F_{SPV} \) except that for 'Medical Services' exceed the point 13.4, corresponding to the 5 per cent critical point on a \( \chi^2_8 \)]. This suggests that estimation of (9.11) using all \( N = 521 \) observations and neglecting the effect of the socio-economic variables \( z_i \), would lead to misspecifications.

One could stop at this stage, and base the analysis on the results presented in Table 9.1. We will not confine ourselves to the linear form of parameter variation used above. Rather, we will use a two stage
TABLE 9.1

OLS Estimates of Augmented Regression Models

<table>
<thead>
<tr>
<th>Regressor</th>
<th>Food</th>
<th>Clothing</th>
<th>Housing</th>
<th>Durables</th>
<th>Education</th>
<th>Medical S.</th>
<th>Other</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_1 )</td>
<td>438.9</td>
<td>102.9</td>
<td>-268.0</td>
<td>-62.3</td>
<td>28.9</td>
<td>20.9</td>
<td>-362.3</td>
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<tr>
<td></td>
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<td>(2.26)</td>
<td>(-1.72)</td>
<td>(-2.90)</td>
<td>(.78)</td>
<td>(.76)</td>
<td>(-2.63)</td>
</tr>
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<td>-19.99</td>
<td>63.62</td>
<td>1.46</td>
<td>-5.15</td>
<td>7.00</td>
<td>-2.40</td>
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<td></td>
<td>(-3.67)</td>
<td>(-2.25)</td>
<td>(2.09)</td>
<td>(.34)</td>
<td>(-.72)</td>
<td>(1.30)</td>
<td>(-.08)</td>
</tr>
<tr>
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<td>-9312</td>
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<td>-0.0703</td>
<td>0.009</td>
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<td>(-6.44)</td>
<td>(-1.11)</td>
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<td>(-5.08)</td>
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<td>(1.51)</td>
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<td>.027</td>
<td>.020</td>
<td>.837</td>
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<td>(4.80)</td>
<td>(5.51)</td>
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<td>(.92)</td>
<td>(7.68)</td>
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<td>.00002</td>
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<td>-.0062</td>
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<td>(-5.85)</td>
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<td>(.43)</td>
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<td>.00032</td>
<td>-.00317</td>
<td>-.00095</td>
<td>.00000</td>
<td>.00032</td>
<td>-.00335</td>
</tr>
<tr>
<td></td>
<td>(4.24)</td>
<td>(.96)</td>
<td>(-2.83)</td>
<td>(-6.16)</td>
<td>(.011)</td>
<td>(1.59)</td>
<td>(-3.38)</td>
</tr>
<tr>
<td>( SD_y )</td>
<td>263.4</td>
<td>160.6</td>
<td>631.4</td>
<td>77.9</td>
<td>104.6</td>
<td>82.7</td>
<td>534.5</td>
</tr>
<tr>
<td>( SD_u )</td>
<td>188.6</td>
<td>128.6</td>
<td>440.1</td>
<td>60.8</td>
<td>103.7</td>
<td>77.7</td>
<td>389.0</td>
</tr>
<tr>
<td>( R^2 )</td>
<td>.487</td>
<td>.358</td>
<td>.514</td>
<td>.391</td>
<td>.017</td>
<td>.117</td>
<td>.470</td>
</tr>
<tr>
<td>( F_{SPV} )</td>
<td>118.84</td>
<td>69.49</td>
<td>55.20</td>
<td>31.49</td>
<td>13.74</td>
<td>8.93</td>
<td>19.07</td>
</tr>
</tbody>
</table>

\( SD_y \): Denotes standard deviation of \( y_i \); \( SD_u \) denotes standard deviation of estimated residuals; \( X_1 = 1 \); \( X_2 \) is Income; \( Z_1 \) is Occupation; \( Z_2 = X_2^2 \); \( Z_3 \) is Family Size and \( Z_4 \) is Age of head.
The first stage in this procedure is the classification of the households into homogeneous groups as described in Subsection 9.3.4 (Our empirical results on this stage are given in Subsection 9.5.2). The second stage refers to the estimation of the ELES as described in Subsection 9.3.3 (The empirical results on this stage are presented in Subsection 9.5.3).

9.5.2 FIRST STAGE: Classification of Households; Empirical Results

The first step in the classification of the households is the determination of the number of groups in which they are to be classified. For this, we computed $\hat{\theta}_+^2(C^*)$, i.e., we evaluated the criterion $\hat{\theta}_+^2(C)$ (see (9.23)) at the classification $C^*$ that maximizes its value, for a wide range of choices of $L$ (namely, $L = 2, 3, ..., 20$) [At this point we note that, for all the calculations required for the Chapter, we wrote our own FORTRAN computer programmes; and that for computation of $C^*$ we incorporated into our programmes a subroutine containing the algorithm of Sparks (1973), which is based on the K-Means procedure of Beale (1969b)]. The values of $\hat{\theta}_+^2(C^*)$ for $L = 2$ up to $L = 20$ are given in Table 9.2. Although the choice of $L$ is somewhat arbitrary, we decided to set $L = 14$ because it gives a value of $\hat{\theta}_+^2(C^*)$ over .9, and because - beyond this - little increase in $\hat{\theta}_+^2(C^*)$ is obtained. [For example, the increase in $\hat{\theta}_+^2(C^*)$ from $L = 13$ to $L = 14$ would be .015; similarly, the increase from $L = 14$ to $L = 15$ would be .001.]
The classification $C^*$ corresponding to $L = 14$ was taken, and some characteristics of the groups forming this were computed. The results are presented in Table 9.3. In the column relating to occupation we have written, for each of the 14 groups, the occupational category having most frequent occurrence, together with the corresponding frequency. For example, group 2 is formed by 24 households from which 23 have the head of the household unemployed (U). (We use W, E and T to denote, respectively, Worker, Entrepreneur and Technocrat). Other variables included in the Table are mean family income, mean family size, and mean average age of the head of the household. A striking feature of the clustering is the marked separation of households by occupational categories, more than by income classes (e.g., groups 1 and 3 have similar values for family income, size and age of head; and differ because group 1 is formed by Unemployed whereas group 3 is formed by Worker households). This indicates, apparently, that occupation exerts one of the main influences in the determination of consumption behaviour (more evidence of this is given later).

### TABLE 9.2

<table>
<thead>
<tr>
<th>$L$</th>
<th>$\delta^2_{L}(C^*)$</th>
<th>$\Delta\delta^2_{L}(C^*)$</th>
<th>$L$</th>
<th>$\delta^2_{L}(C^*)$</th>
<th>$\Delta\delta^2_{L}(C^*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.000</td>
<td>-</td>
<td>11</td>
<td>.871</td>
<td>.010</td>
</tr>
<tr>
<td>2</td>
<td>.453</td>
<td>.453</td>
<td>12</td>
<td>.884</td>
<td>.013</td>
</tr>
<tr>
<td>3</td>
<td>.618</td>
<td>.165</td>
<td>13</td>
<td>.895</td>
<td>.011</td>
</tr>
<tr>
<td>4</td>
<td>.699</td>
<td>.081</td>
<td>14</td>
<td>.910</td>
<td>.015*</td>
</tr>
<tr>
<td>5</td>
<td>.740</td>
<td>.041</td>
<td>15</td>
<td>.911</td>
<td>.001</td>
</tr>
<tr>
<td>6</td>
<td>.780</td>
<td>.040</td>
<td>16</td>
<td>.912</td>
<td>.001</td>
</tr>
<tr>
<td>7</td>
<td>.827</td>
<td>.047</td>
<td>17</td>
<td>.913</td>
<td>.001</td>
</tr>
<tr>
<td>8</td>
<td>.843</td>
<td>.016</td>
<td>18</td>
<td>.916</td>
<td>.003</td>
</tr>
<tr>
<td>9</td>
<td>.857</td>
<td>.014</td>
<td>19</td>
<td>.919</td>
<td>.003</td>
</tr>
<tr>
<td>10</td>
<td>.861</td>
<td>.004</td>
<td>20</td>
<td>.919</td>
<td>.000</td>
</tr>
</tbody>
</table>
TABLE 9.3
Characteristics of Groups from Cluster Analysis

<table>
<thead>
<tr>
<th>Group</th>
<th>Occupation</th>
<th>Income</th>
<th>Family Size</th>
<th>Age</th>
<th>Number in Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>U(24/24)</td>
<td>715.</td>
<td>8.0</td>
<td>55.7</td>
<td>24</td>
</tr>
<tr>
<td>2</td>
<td>U(23/24)</td>
<td>1101.</td>
<td>3.3</td>
<td>65.9</td>
<td>24</td>
</tr>
<tr>
<td>3</td>
<td>W(41/42)</td>
<td>540.</td>
<td>9.8</td>
<td>49.0</td>
<td>42</td>
</tr>
<tr>
<td>4</td>
<td>W(95/95)</td>
<td>464.</td>
<td>7.1</td>
<td>38.3</td>
<td>95</td>
</tr>
<tr>
<td>5</td>
<td>W(68/69)</td>
<td>835.</td>
<td>3.6</td>
<td>31.8</td>
<td>69</td>
</tr>
<tr>
<td>6</td>
<td>W(47/47)</td>
<td>859.</td>
<td>4.4</td>
<td>54.6</td>
<td>47</td>
</tr>
<tr>
<td>7</td>
<td>E(13/28)</td>
<td>668.</td>
<td>10.4</td>
<td>45.9</td>
<td>28</td>
</tr>
<tr>
<td>8</td>
<td>E(41/41)</td>
<td>911.</td>
<td>5.4</td>
<td>42.2</td>
<td>41</td>
</tr>
<tr>
<td>9</td>
<td>T(62/65)</td>
<td>1154.</td>
<td>5.8</td>
<td>38.9</td>
<td>65</td>
</tr>
<tr>
<td>10</td>
<td>T(37/41)</td>
<td>1701.</td>
<td>3.1</td>
<td>31.9</td>
<td>41</td>
</tr>
<tr>
<td>11</td>
<td>T(21/26)</td>
<td>3534.</td>
<td>3.6</td>
<td>39.4</td>
<td>26</td>
</tr>
<tr>
<td>12</td>
<td>T(6/10)</td>
<td>5538.</td>
<td>3.1</td>
<td>49.3</td>
<td>10</td>
</tr>
<tr>
<td>13</td>
<td>T(5/5)</td>
<td>7196.</td>
<td>3.0</td>
<td>44.8</td>
<td>5</td>
</tr>
<tr>
<td>14</td>
<td>T(4/4)</td>
<td>9614.</td>
<td>4.0</td>
<td>41.0</td>
<td>4</td>
</tr>
</tbody>
</table>

We may, of course, estimate the ELES for each of these groups of households of 'homogeneous behaviour'. In some of these groups, however, we have few observations; for example, in groups 14 and 15 we have, respectively, 5 and 4 households (see last column in Table 9.3). Also, in some groups we have two or more occupational categories; for instance, group 11 has 21 Technocrats; 2 Entrepreneurs; 2 Workers and 1 Unemployed. Based on the results of the Cluster Analysis, we decided to redefine the groups to have

(i) sufficient number of observations for estimation; and
(ii) readily identifiable domains of study.

8 The values of the socioeconomic variables (which include occupational category) for each of the households in a given group, are in Appendix I in page 333. Apart from presenting the resulting groups, the purpose of this Appendix is to make our data accessible to other researchers.
This required merging some groups, and group-reassignment of a few households. As pointed out previously, the most immediate split of the households is by occupational category.

**Unemployed Households**

Regarding the *Unemployed*, the natural further break-up is by family size, with splitting value 6 (see the 'mostly Unemployed groups', i.e., groups 1 and 2 in Appendix I in page 335, and note group 1 has households of size basically greater or equal to 6 and group 2 less than 6. Also note there are no clear income nor age differentials between these groups). We define large (L) households as those with family size greater or equal to 6, and small (S) households as those with family size less than 6. So, we divided Unemployed households into 2 groups: L and S; hereafter referred to as UL and US.

**Worker Households**

Regarding *Workers*, we have that groups 3 and 4 are large (L) size households; in group 3 most heads are over 45 years of age, and in group 4 most are under 45. We define households where the head is over 45 years of age as old (O) and those where the head is under 45 years of age as young (Y). [It is interesting to note that LPW (1977, p.122) also used this breaking point for age classification]. We also have that groups 5 and 6 are small (S) size Worker households; with group 5 being formed by young (Y) households and group 6 by old (O) households. [In general, groups 3 and 4 contained households with lower income than groups 5 and 6, reflecting that small households have higher incomes]. Because of these features, we decided to divide Worker households into 4 groups: LY, LO, SY and SO; hereafter referred to as WLY, WLO, WSY and WSO.
Entrepreneur Households

Regarding Entrepreneurs, we have that group 7 is formed by large (L) households and group 8 by mainly small (S) households. [Income and age differentials are not very significant among these groups, except for the fact that small size households tend to have slightly higher incomes]. So, we divided Entrepreneur households into 2 groups: L and S; hereafter referred to as EL and ES.

Technocrat Households

Finally, regarding Technocrats, we observe that groups 9 and 10 are basically 'low-income' (relative to other Technocrat) households, with incomes below 3000 pesos per capita per month. Households with incomes below this level are qualified by the symbol Il. Between groups 9 and 10 no clear age differentials are apparent; but group 9 is formed basically by large (L) size households and group 10 by small (S) households. [Again there is a tendency for smaller households to have higher incomes]. We then have group 11, consisting of 'middle-income' households, having incomes between 3000 and 5000 pesos. Households with income in this interval are qualified by the symbol I2 [It is interesting to note that, out of all the Unemployed, Worker or Entrepreneur households, only 11 had incomes within this interval, and none higher than 6230 pesos]. Finally, we have 'high-income' households in groups 12, 13 and 14; these have incomes over 5000 pesos and are qualified with the symbol I3. So, we formed 4 groups (note we previously had 6) of households of Technocrats, namely: I1L, I1S, I2 and I3; hereafter referred to as T1L, T1S, T12 and T13.
In all, we ended with 12 groups - or domains of study - containing different types of households. The average income, family size and age of the head of the household, together with the number of households in each group are given in Table 9.4. Average income for large households is lower than for small households (other attributes equal). Also,

We have divided income into 3 categories; family size into 2, and age into 2. We also have 4 occupational categories. We could, of course, attempt to analyze all possible 48 groups (3x2x2x4). Yet, with the available data, many of these would have insufficient (or not have any) observations for econometric estimation, and our analysis is limited to the above 12 groups.

To compute the averages we excluded the household data points marked with a star (*) in Appendix I in page 333. These points were excluded in our subsequent analysis because of their questionable reliability (e.g., in some cases reported expenditures on a single commodity exceeded twice the total household income).
average incomes for Unemployed, Workers and Entrepreneurs are smaller than for Technocrats. Large households (UL, WLY, WLO, EL and TL11) have an average of approximately 8 members, and small households (US, WSY, WSO, ES and TL1S) have an average of approximately 4 members.

TABLE 9.5
Average Budget Shares

<table>
<thead>
<tr>
<th>Type of Household</th>
<th>Food</th>
<th>Clothing</th>
<th>Housing</th>
<th>Durables</th>
<th>Education</th>
<th>Medical S.</th>
<th>Other</th>
<th>Savings</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unemployed</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>L</td>
<td>.448</td>
<td>.104</td>
<td>.160</td>
<td>.026</td>
<td>.010</td>
<td>.018</td>
<td>.223</td>
<td>.006</td>
</tr>
<tr>
<td>S</td>
<td>.265</td>
<td>.079</td>
<td>.340</td>
<td>.006</td>
<td>.008</td>
<td>.037</td>
<td>.161</td>
<td>.100</td>
</tr>
<tr>
<td>Worker</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>L,Y</td>
<td>.632</td>
<td>.148</td>
<td>.199</td>
<td>.025</td>
<td>.042</td>
<td>.024</td>
<td>.169</td>
<td>-.244</td>
</tr>
<tr>
<td>L,O</td>
<td>.449</td>
<td>.124</td>
<td>.180</td>
<td>.017</td>
<td>.024</td>
<td>.014</td>
<td>.224</td>
<td>.013</td>
</tr>
<tr>
<td>S,Y</td>
<td>.464</td>
<td>.123</td>
<td>.208</td>
<td>.020</td>
<td>.012</td>
<td>.017</td>
<td>.191</td>
<td>-.018</td>
</tr>
<tr>
<td>S,O</td>
<td>.505</td>
<td>.096</td>
<td>.225</td>
<td>.012</td>
<td>.030</td>
<td>.019</td>
<td>.174</td>
<td>-.063</td>
</tr>
<tr>
<td>Entrepreneur</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>L</td>
<td>.399</td>
<td>.124</td>
<td>.207</td>
<td>.017</td>
<td>.029</td>
<td>.010</td>
<td>.198</td>
<td>.012</td>
</tr>
<tr>
<td>S</td>
<td>.410</td>
<td>.121</td>
<td>.225</td>
<td>.024</td>
<td>.014</td>
<td>.022</td>
<td>.195</td>
<td>-.014</td>
</tr>
<tr>
<td>Technocrat</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>II,L</td>
<td>.374</td>
<td>.110</td>
<td>.226</td>
<td>.032</td>
<td>.036</td>
<td>.016</td>
<td>.234</td>
<td>-.031</td>
</tr>
<tr>
<td>II,S</td>
<td>.364</td>
<td>.135</td>
<td>.230</td>
<td>.032</td>
<td>.024</td>
<td>.027</td>
<td>.253</td>
<td>-.067</td>
</tr>
<tr>
<td>I2</td>
<td>.201</td>
<td>.091</td>
<td>.143</td>
<td>.015</td>
<td>.012</td>
<td>.015</td>
<td>.250</td>
<td>.270</td>
</tr>
<tr>
<td>I3</td>
<td>.130</td>
<td>.038</td>
<td>.203</td>
<td>.009</td>
<td>.002</td>
<td>.013</td>
<td>.169</td>
<td>.433</td>
</tr>
</tbody>
</table>

The average budget shares for each of these groups are presented in Table 9.5. Comparing this Table with that of LPW (1977, p.125) - which refers to 1968 data - we observe that the average budget shares for 'Food' tend to be higher for this 1975 data [e.g., for WLY, WLO, WSY and WSO they obtained, respectively, .472, .423, .420 and .376 -
whereas for the 1975 data we obtain .632, .449, .464 and .505]. No more systematic differences are encountered for other commodities. We now look at the last column in Table 9.5, referring to savings, and observe that significant positive savings ratios (above 10 per cent) are only found in 'highly' paid Technocrats (above 3000 pesos). These ratios are .270 and .433 respectively for TI2 and TI3. The next step was to estimate the ELES for each of the groups.

9.5.3 SECOND STAGE: Estimation of the ELES; Empirical Results

To estimate the ELES, for each of the 12 groups, we considered Ordinary Regression and LDV models. Regarding LDV models we used, for commodities 'Food', 'Clothing', 'Housing' and 'Other' the Truncated model; for 'Durables', 'Education' and 'Medical Services' we had reported zero expenditures so we used the Tobit model. The results of the estimated regressions are summarized in Table 9.6 and Tables 9.18 to 9.28. All these tables seem to reflect the same main features, so we only need to discuss one of these in detail. Here we concentrate on Table 9.6, which presents the results for the WSO households (Tables 9.18 to 9.28, containing the results for the other 11 types of households, are given in Appendix J; see page 359).

OLS Estimates

The first half of Table 9.6 contains the results using OLS estimation on the ordinary regression or Normal model (see Subsection 9.3.3 (i)). For each commodity, the numbers in brackets below the OLS estimates $\hat{\beta}_1(\text{OLS})$ and $\hat{\beta}_2(\text{OLS})$, refer to t-statistics obtained by using the heteroscedasticity consistent OLS - VCM (see (5.19)). $SD_u$ contains the square root of the sum of the estimated residuals squared over the
sample size. The row corresponding to $\text{LM}_N$, contains the values of the LM test statistic for normality ($N$) of disturbances (see (4.6)).

Given homoscedasticity ($H$), $\text{LM}_N$ would be asymptotically distributed as $\chi^2$ under $N$. In the Table, $\text{LM}_H$ is the homoscedasticity test (see (5.7)), calculated under the assumption that the disturbance variance is a linear function of income. Given $N$, $\text{LM}_H$ would be asymptotically distributed as $\chi^2$ under $H$. In turn, $\text{LM}_{NH} = \text{LM}_N + \text{LM}_H$ is the LM test statistic for disturbance $NH$. Under $NH$, $\text{LM}_{NH}$ would be asymptotically distributed as $\chi^2$.

With a 1 per cent significance level we have $\chi^2(0.99) = 6.63$, $\chi^2(0.99) = 9.21$ and $\chi^2(0.99) = 11.3$. We use these significance points, and have marked significant test statistics ($\text{LM}_N$, $\text{LM}_H$ and $\text{LM}_{NH}$) with a star (*). So, looking at the row for $\text{LM}_{NH}$, we find that disturbance $NH$ is rejected for 'Clothing', 'Durables', 'Education' and 'Medical Services'. It is interesting to note these are the commodities for which the probabilities of obtaining a negative observation under $NH$ are greater (see the row corresponding to $F(x'^2, s^2)$), and observe the probabilities are .153, .291, .284 and .254, respectively for 'Clothing', 'Durables', 'Education' and 'Medical Services'). Apart from 'Clothing', the more significant contribution to $\text{LM}_{NH}$ comes from $\text{LM}_N$. For example, for 'Durables' we have $\text{LM}_{NH} = 111.67$, with $\text{LM}_N = 110.29$ and $\text{LM}_H = 1.38$. For 'Durables', 'Education' and 'Medical Services' approximately half of the households reported zero expenditures, so it is not surprising that (under the assumption that $H$ holds) normality would be rejected. Indeed, for these commodities it is clear that a Tobit-type model should be used.
\[
\begin{array}{cccccccc}
\hline
& \text{Food} & \text{Clothing} & \text{Housing} & \text{Durables} & \text{Education} & \text{Medical S.} & \text{Other} \\
\hline
\beta_1 (\text{OLS}) & 178.0 & -25.7 & 148.8 & -0.88 & -10.0 & -9.8 & -65.3 \\
 & (2.65) & (-0.75) & (4.65) & (-0.19) & (-0.78) & (-1.35) & (-2.53) \\
\beta_2 (\text{OLS}) & 0.300 & 0.125 & 0.053 & 0.013 & 0.041 & 0.030 & 0.249 \\
 & (3.63) & (2.71) & (2.57) & (2.74) & (3.62) & (3.46) & (7.37) \\
SD_u & 184.4 & 81.6 & 137.8 & 19.7 & 46.3 & 25.0 & 92.6 \\
LM_N & 0.62 & 0.27 & 7.87 & 110.29* & 183.96* & 23.34* & 2.38 \\
LM_H & 7.03 & 27.87* & 1.10 & 1.38 & 0.10 & 4.21 & 7.93 \\
LM_{NH} & 7.65 & 28.14* & 7.98 & 111.67* & 183.97* & 27.55* & 10.32 \\
F(\hat{x}'\hat{\beta}, \hat{\sigma}^2) & 0.008 & 0.153 & 0.077 & 0.291 & 0.284 & 0.254 & 0.051 \\
\hline
\end{array}
\]

Table 9.6
Estimated Regressions: Worker, Small, Old

Results using Ordinary Regression Model

\[
\begin{array}{cccccccc}
\beta_1 (\text{LDV}) & 154.2 & -204.3 & -5.1 & -12.9 & -31.6 & -14.7 & -173.8 \\
 & (2.56) & (-1.65) & (-0.03) & (-1.44) & (-1.55) & (-1.55) & (-2.73) \\
\beta_2 (\text{LDV}) & 0.312 & 0.193 & 0.093 & 0.017 & 0.045 & 0.030 & 0.299 \\
 & (6.18) & (3.96) & (1.37) & (2.34) & (2.76) & (3.79) & (8.19) \\
SD_u & 184.5 & 92.2 & 140.3 & 19.8 & 46.4 & 25.8 & 97.5 \\
LM_N(\cdot) & 0.02 & 3.22 & 3.15 & 130.06* & 229.69* & 1776.03* & 2.20 \\
LM_H(\cdot) & 5.35 & 0.02 & 0.16 & 4.10* & 15.63* & 55.10* & 0.25 \\
LM_{NH}(\cdot) & 6.29 & 4.93 & 4.88 & 137.94* & 246.04* & 1777.27* & 2.27 \\
\hat{\beta}_1 & 223.7 & 43.0 & 160.4 & 5.6 & 12.2 & 6.0 & 46.5 \\
\hat{\beta}_2 & 0.288 & 0.057 & 0.042 & 0.009 & 0.025 & 0.019 & 0.170 \\
\end{array}
\]
LDV Estimates

We now look at the second half of Table 9.6. This presents the results on the MLE of the parameters using LDV models. For each commodity $j$, the values $\hat{\beta}_1^{(LDV)}$ and $\hat{\beta}_2^{(LDV)}$ denote the MLE of $\beta_1^j$ and $\beta_2^j$. For 'Food', 'Clothing', 'Housing' and 'Other' these are based on the Truncated model likelihood function; for 'Durables', 'Education' and 'Medical Services' these are based on the Tobit model likelihood function. In both cases, we used Fair's algorithm; this procedure is described - for the Truncated model - in the paragraph below (6.5); and for the Tobit model in Fair (1977, p.1724). Large sample t-test statistics are given in brackets below the values of $\hat{\beta}_1^{(LDV)}$ and $\hat{\beta}_2^{(LDV)}$. $S_D^u$ denotes the square root of the sample second moment about the mean of the estimated residuals. $LM_{N(\cdot)}$, $LM_{H(\cdot)}$ and $LM_{NH(\cdot)}$ denote test statistics for disturbance $N$ and/or $H$ in LDV models, i.e., these are $LM_{N(TRUN)}$, $LM_{H(TRUN)}$ and $LM_{NH(TRUN)}$, or $LM_{N(TOBIT)}$, $LM_{H(TOBIT)}$ and $LM_{NH(TOBIT)}$ (see Chapter 6) according to which model is used. (Again, for the homoscedasticity test, we specified the variance was a linear function of income). Values with a star (*) denote significance test statistics at the 1 per cent level.

First consider the estimates for the 'Food' expenditure equation. For this equation we see that $F(x'\hat{\beta},\hat{\sigma}^2)$ is small (equal to .008); so in the specification of the likelihood, neglect of the fact that expenditure is non-negative, is not expected to have serious effects. This is indeed the case, and OLS results are quite similar to LDV-ML estimation results. To see this, compare the $\beta$, $S_D^u$ and $LM$ values for both estimation techniques. In all 12 types of households we have that, for 'Food', $F(x'\hat{\beta},\hat{\sigma}^2) \leq .025$ and find that OLS and LDV-ML estimation
results are similar (see Appendix J in page 359); so, for this commodity, we can conclude that 'truncation' is not a 'severe' problem. Now we look at the 'Clothing' expenditure equation. Here we see that $P(\hat{\beta}, \sigma^2)$ is .153. For this commodity we had a significant value of $LM_{NH}$; but using the Truncated model we would accept (truncated)$NH$. Similarly, the estimated equation for 'Housing' gives no evidence of disturbance $\overline{N}$ and/or $\overline{H}$.

We now look at the three commodities for which the Tobit model was used, i.e., 'Durables', 'Education' and 'Medical Services'. Here we find that, in the three cases, $LM_{NH(TOBIT)}$ would reject disturbance $NH$ (see row for $LM_{NH(.)}$ in Table 9.6). Two points are worth noting in relation to this:

(i) The first point refers to finite sample properties. We noted in Chapter 5 that - for the ordinary regression model - finite sample significance points may be obtained for $LM_{NH}$ by computer simulation. This means that, for any sample size, we may carry out tests with significance levels close to the desired. By contrast, for the tests $LM_{NH(TRUN)}$ and $LM_{NH(TOBIT)}$, only the asymptotic distribution is known; and basing tests on asymptotic significance points may lead to actual significance levels that are considerably different from the presupposed. (The need for 'size adjustment' of the LM test has been discussed - in a different context - by Bera, Byron and Jarque (1981)). In our present study - due to sample partitioning to achieve homogeneous groups of households - our sample sizes vary from 15 to 88 (see last column in Table 9.4); and it may very well be that these are not sufficiently large for us to make a valid appeal to the asymptotic distribution. The present state of the art confines ourselves to the use of large
sample tests. Hopefully, future research will lead to results on the finite sample distribution of these test statistics.

(ii) We should also point out that, in some cases, we obtained 'extremely large' values of the test statistics, e.g., we obtained

$$LM_{NH(TOBIT)} = 1777.27$$

for 'Medical Services' (see Table 9.6).\(^{11,12}\)

One could argue that these values, even after adjustment of the asymptotic significance points, would lead to the rejection of disturbance $NH$. In these cases one could re-estimate the equation using MLE on the corresponding (LDV model) likelihood that incorporates $\bar{N}$ (e.g., through the use of a family of transformations) and/or $\bar{H}$. In Chapter 6 we mentioned this area as one that required further study, and no attempt was made at this stage to tackle the problem.

The presence of $\bar{N}$ and $\bar{H}$ makes the estimates $\hat{\beta}_1(LDV)$ and $\hat{\beta}_2(LDV)$ inconsistent. In our study, strong evidence of violation of $NH$ (i.e. a very large $LM_{NH(\cdot)}$ value) was usually found in the equations for expenditures on 'Durables', 'Education' and 'Medical Services'. The appropriateness of subsequent analysis will depend on the magnitude of the inconsistency. Strictly speaking, results for these commodities should be taken with caution. Given the structure of the ELES, the

\(^{11}\) These large values were completely unexpected. To make sure that our computer programmes were not faulty, we generated data under $NH$, with $N = 100$, and computed the test statistic $LM_{NH(TOBIT)}$.

The values obtained were of reasonable magnitude (relative to the corresponding $\chi^2$ asymptotic distribution). Indeed, when using the asymptotic critical point, we found that the percentage of rejections of $NH$ was very close to the theoretical significance level.

\(^{12}\) The extremely large test statistic values occurred in cases where many 'high income' households reported zero expenditures, which is an event that the null hypothesis ($NH$) would assign a very small probability.
coefficients calculated for a given commodity will also enter in the calculation of demand responses for other commodities (e.g. see equation (9.12)). It would appear then, that our use of results based on LDV models might come to an end. Fortunately, the coefficients associated with the equations on 'Durables', 'Education' and 'Medical Services' are very small in relative magnitude, and fluctuations in these values have little effect in the estimated demand and savings responses for other commodities [e.g., see Table 9.6 and note that \( \hat{\beta}_2(LDV) \) for 'Durables', 'Education' and 'Medical Services' are, respectively, .017, .045 and .030; whereas for 'Food', 'Clothing', 'Housing' and 'Other' we have .312, .193, .093 and .299]. So, we decided to proceed with our analysis, neglecting the possible presence of inconsistency in the maximum likelihood estimates of the LDV models.

As mentioned in Section 9.3, the estimated coefficients need to be adjusted so they represent 'appropriate responses'. Therefore, we computed – for each commodity j – the values \( \hat{\beta}_{1j} \) and \( \hat{\beta}_{2j} \); using equations (9.20) for 'Food', 'Clothing', 'Housing' and 'Other'; and (9.21) for 'Durables', 'Education' and 'Medical Services'. The values obtained are given in the last two rows of Table 9.6 (for type of household WSO). It is evident that failure to adjust the coefficients could lead to very misleading results, e.g., all but one of the values \( \hat{\beta}_1(LDV) \) are negative; yet – when 'adjusted' – all would become positive (see row corresponding to \( \hat{\beta}_1 \)). The adjusted values were then used for the computation of the elements in \( \theta \) which represent demand and saving responses (see equations (9.22)); and for the calculation of the elasticities defined in equations (9.9). The results are discussed, respectively, in Subsections 9.5.4 and 9.5.5.
9.5.4 Estimated Demand and Saving Responses

Marginal Budget Shares and Marginal Propensity to Consume

The estimated values of the marginal budget shares \( \hat{\theta}_j \) are given in Table 9.7. Again, the first half of the Table contains results using OLS estimation, and the second half includes our findings using maximum likelihood estimation on LDV models (hereafter referred to as LDV-ML estimation). Out of the 84 estimates, we have that 7 OLS and 10 LDV-ML estimates are negative [In LPW (1977) all the estimates were positive - as required]. Apart from TI2, all negative quantities occurred in cases where reported expenditures were very small. We also have that all but one (TI3) of the estimated marginal propensities to consume \( \hat{\mu} \) are less than 1 [see last column in Table 9.7].

With the results reported in Table 9.7 many comparisons are possible. To make our analysis simpler, we follow LPW (1977) and fit a regression of (i) the estimated values of the marginal budget share for each commodity, and (ii) the estimated values of the marginal propensity to consume, on a set of regressors denoting household characteristics. More specifically, we use seven regressors which are: a constant; average income over 1000; average family size; average age \( \text{13} \); and three dummy variables for occupational category (Workers, Entrepreneurs and Technocrats). In all, for each of the 8 regressions (one for every commodity and one for \( \hat{\mu} \)) we have 12 observations (one from each type of household) and 7 coefficients to be estimated. The results are presented in Table 9.8. Although the quantitative findings are different according to the use of OLS or LDV-ML estimation, it is interesting to note that the qualitative results are basically the same. So, we

\( \text{13} \) The values for these regressors are given in Table 9.4.
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<thead>
<tr>
<th>Type of Household</th>
<th>Food</th>
<th>Clothing</th>
<th>Housing</th>
<th>Durables</th>
<th>Education</th>
<th>Medical S.</th>
<th>Other</th>
<th>μ</th>
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<td>.051</td>
<td>.027</td>
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<td><strong>Results using LDV-ML estimation</strong></td>
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</tr>
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<td>.021</td>
<td>.033</td>
<td>-.009</td>
<td>.120</td>
<td>.499</td>
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<td>.013</td>
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<td>.014</td>
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<td>.014</td>
<td>.041</td>
<td>.031</td>
<td>.278</td>
<td>.611</td>
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<td>L</td>
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<td>.157</td>
<td>.105</td>
<td>-.005</td>
<td>.003</td>
<td>-.003</td>
<td>.152</td>
<td>.243</td>
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<td>.160</td>
<td>.170</td>
<td>.037</td>
<td>-.001</td>
<td>.003</td>
<td>.235</td>
<td>.646</td>
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<td><strong>Technocrat</strong></td>
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<td>-.022</td>
<td>.068</td>
<td>.168</td>
<td>.348</td>
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<td>.009</td>
<td>.024</td>
<td>.398</td>
<td>.675</td>
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can concentrate only in the upper half of Table 9.8. Starred (*) quantities in the Table indicate statistically significant values at the 10 per cent level.

We first look at the row for Income. Unlike LPW (1977, p.133), we do not find a significant income effect on marginal budget shares or the marginal propensity to consume. Although the results in Table 9.7 tend to suggest that — for some groups — as income increases, the marginal budget share for 'Food' decreases, e.g., note that the estimated (OLS) marginal budget shares for 'Food' are .474 and .301 for EL and ES, which have — respectively — average incomes of 832.74 and 1159.93 per capita pesos per month (see Table 9.4). We now look at the row for the Family Size variable. Here we have a significant effect for the equations referring to 'Food' and 'Medical Services'. The row corresponding to the Age variable shows no starred values, which is in agreement with the findings of LPW (1977). Looking at the occupational dummy variables, we see that occupational category has a significant effect on 'Food', 'Education', 'Medical Services', 'Other' and μ. In particular, the marginal budget share for 'Food' is significantly lower for Technocrats. Also, the marginal budget shares for 'Education', 'Medical Services' and 'Other' are higher for Technocrats. Finally, we observe that the marginal propensity to consume is significantly larger for Workers.

Subsistence Expenditures and Frisch Parameters

We now consider the estimated subsistence expenditures, i.e., $p_j$ [see (9.22)]; and Frisch parameters, i.e., $\frac{1}{n}$ [see (9.9)]. These are given in Table 9.9. Except for two cases in TI2(OLS), and one in TI3(OLS), all estimated subsistence expenditures are positive.
### TABLE 9.8
Regressions of Estimated Values of Marginal Budget Shares ($\theta_j$) and Marginal Propensity to Consume ($\mu$) on Household Characteristics

<table>
<thead>
<tr>
<th>Regressor</th>
<th>Food</th>
<th>Clothing</th>
<th>Housing</th>
<th>Durables</th>
<th>Education</th>
<th>Medical S.</th>
<th>Other</th>
<th>$\mu$</th>
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<tr>
<td><strong>Results using OLS estimation</strong></td>
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<td></td>
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</tr>
<tr>
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<td>-.041</td>
<td>.064</td>
<td>.082</td>
<td>-.005</td>
<td>.353</td>
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<tr>
<td></td>
<td>(1.61)</td>
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<td>(.92)</td>
<td>(-.69)</td>
<td>(.55)</td>
<td>(.43)</td>
<td>(-.01)</td>
<td>(.62)</td>
</tr>
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<td>.027</td>
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<td>-.001</td>
<td>-.001</td>
<td>-.002</td>
<td>.003</td>
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<tr>
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<td>(.62)</td>
<td>(-.42)</td>
<td>(-.96)</td>
<td>(-.95)</td>
<td>(-.06)</td>
<td>(.69)</td>
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<td>.003</td>
<td>-.008</td>
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<td>(-.60)</td>
<td>(1.08)</td>
<td>(-1.29)</td>
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<td>-.001</td>
<td>.0005</td>
<td>-.000</td>
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<td>.004</td>
<td>.005</td>
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<tr>
<td></td>
<td>(-1.07)</td>
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<td>(-.19)</td>
<td>(.59)</td>
<td>(-.01)</td>
<td>(.48)</td>
<td>(.72)</td>
<td>(.63)</td>
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<td>Dummy W</td>
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<td>-.135</td>
<td>.013</td>
<td>.022</td>
<td>-.013</td>
<td>.143</td>
<td>.248*</td>
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<td></td>
<td>(-.96)</td>
<td>(.54)</td>
<td>(-.79)</td>
<td>(.63)</td>
<td>(.54)</td>
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<td>(.95)</td>
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<td>(.75)</td>
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<td>-.012</td>
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<td>.002</td>
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<td></td>
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<td>(.10)</td>
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TABLE 9.9
Estimated Subsistence Expenditures ($p_j\delta_j$) and Frisch Parameters ($\nu$)

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mean values in pesos per capita per month are

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when using OLS; and

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when using LDV-ML estimation. Again, to identify systematic effects, we regressed the estimated values on the set of regressors used for the analysis of the estimated marginal budget shares. The results are given in Table 9.10. Goodness-of-fit in these regressions varies considerably according to the use of OLS or LDV-ML estimation [for OLS, \( R^2 \) values range from .02 to .92; and for LDV-ML estimation \( R^2 \) values range from .46 to .85 (see rows for \( R^2 \) in Table 9.10)]. Yet, qualitative results are basically the same; so we only look at the top half of the Table.

Observing the row for the Income regressor we see this appears with a significant effect on subsistence expenditures for 'Food', 'Clothing', 'Housing' and 'Durables'; but - apart from 'Housing' - these effects are of the 'incorrect sign'. For 'Housing', the results state that high income families have higher subsistence expenditures. Regarding occupational category, the main feature of the results is the way in which Technocrats are singled out. We observe that Technocrats have significantly larger subsistence expenditures in all but one of the 7 commodities (see row for Technocrat dummy). Also, there is a tendency for 'Food' and Total subsistence expenditure for Workers, to be less
### Table 9.10

Regressions of Estimated Values of Subsistence Expenditures \( p_j \hat{p}_j \)
on Household Characteristics

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**Results using OLS estimation**

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<td>( R^2 )</td>
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than for Entrepreneurs, which in turn is less than for Technocrats (see last column in Table 9.10). Our results support the notion that (as stated by LPW (1977, p.142)) the subsistence expenditures "are a measure of an acceptable minimum standard for households identifying with a given socioeconomic group". Additionally, looking at the row for the Family Size variable, we note the estimates suggest the existence of economies of scale.

Regarding the Frisch parameter, we find that 7 out of the 12 OLS based estimates, and 11 out of the 12 LDV-ML estimates, have the 'incorrect' positive sign (see last column in Table 9.9). This is because subsistence expenditure exceeded actual mean consumption expenditure [Recall $\eta = -p'q/(p'q-p')$].

9.5.5 Estimated Elasticities

Total Expenditure Elasticities

Estimated elasticities of demand with respect to total expenditure, $\eta_{JT}$; and of total expenditure with respect to income, $\eta_{T^k}$ [see (9.9)] are given in Table 9.11. Results in both parts of the Table are similar, so we only look at the top half. Here we find that out of the 84 (12x7) estimated elasticities, 8 have the (incorrect) negative sign. Excluding these we observe, as LPW (1977, p.145), that - for W, E and T - the elasticity $\eta_{JT}$ for 'Food' tends to be the lowest (compare the 'Food' elasticity with other quantities in a given row).

To study the systematic effects of socioeconomic variables, we estimated regressions as before and present the results in Table 9.12. We find that total expenditure elasticity for 'Food' is significantly higher for large families and lower for Technocrats (see first column
TABLE 9.11
Elasticity of Demand with Respect to Total Expenditure ($\eta_{jt}$) and Total Expenditure with Respect to Income ($\eta_{Tt}$)

<table>
<thead>
<tr>
<th>Type of Household</th>
<th>Food</th>
<th>Clothing</th>
<th>Housing</th>
<th>Durables</th>
<th>Education</th>
<th>Medical S.</th>
<th>Other</th>
<th>Income</th>
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Results using LDV-ML estimation

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<th>Housing</th>
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<th>Education</th>
<th>Medical S.</th>
<th>Other</th>
<th>Income</th>
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### TABLE 9.12

Regression of Estimated Values of Total Expenditure Elasticities ($\eta_{jT}$ and $\eta_{T*}$) on Household Characteristics

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<th>Regressor</th>
<th>Food</th>
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<th>Housing</th>
<th>Durables</th>
<th>Education</th>
<th>Medical S.</th>
<th>Other</th>
<th>Income</th>
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<td>(.13)</td>
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<td>.002</td>
<td>-.006</td>
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<td>-.68</td>
<td>-.07</td>
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<td>(-.02)</td>
<td>(-.91)</td>
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<td>(.95)</td>
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**Results using OLS estimation**

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**Results using LDV-ML estimation**
in Table 9.12). Regarding 'Medical Services', large families tend to have lower (and Technocrats higher) elasticities of total expenditure. We also find important socioeconomic effects on the $\eta_{jT}$ for 'Other', which are significantly higher for older households and for Workers, Entrepreneurs and Technocrats. Looking at the last column in Table 9.12, we observe there is a significant Income effect on the total expenditure elasticity with respect to income, with higher income households having higher $\eta_{T*}$.

**Price Elasticities**

We present estimates of the own-price elasticities, $\eta_{jj}$ [see (9.9)], in Table 9.13. We observe that for 'Food' most own-price elasticities are negative (see first column in Table 9.13), i.e., they have the sign required by the utility function. In turn, elasticities for 'Durables', 'Education' and 'Medical Services' have mostly the wrong (positive) sign. The results of regressing the estimated values of $\eta_{jj}$ on household characteristics are given in Table 9.14. As in LPW (1977, p.149) a systematic pattern is observed: "own-price elasticities increase in absolute value with higher income and larger family size"; although here the Family Size effect is only apparent in LDV-ML estimates. There is also an Occupational category effect, with Technocrats having lower elasticities in absolute value.

The estimated values of the 'Food' cross-price elasticities, $\eta_{j1}$ [see (9.9)], are listed in Table 9.15; and the regressions of these values on household characteristics are given in Table 9.16. From Table 9.16 we find a tendency for 'Medical Services' and 'Other' to have a decline in absolute 'Food' cross-price elasticity as income increases (see rows for Income regressor). Another factor showing systematic effects is Family Size, with a decrease in absolute value of $\eta_{j1}$ as
TABLE 9.13
Own Price Elasticities ($\eta_{jj}$)

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TABLE 9.14
Regression of Estimated Values of Own-price Elasticities ($\eta_{jj}$) on Household Characteristics

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Results using OLS estimation

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<th>Clothing</th>
<th>Housing</th>
<th>Durables</th>
<th>Education</th>
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Results using LDV-ML estimation
TABLE 9.15

Food Cross-Price Elasticities ($\eta_{j1}$)

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<td>.03</td>
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<td></td>
</tr>
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<td>$L$</td>
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<td>.03</td>
<td>-.01</td>
<td>.04</td>
<td>-.08</td>
</tr>
<tr>
<td>$S$</td>
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<td>-.26</td>
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<td>.03</td>
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<td>-.41</td>
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<td>Technocrat</td>
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<td></td>
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</tr>
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<td>$I_1,L$</td>
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<td>-.46</td>
<td>-.25</td>
<td>-.38</td>
<td>-.49</td>
<td>-.50</td>
</tr>
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<td>$I_1,S$</td>
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<td>-.18</td>
<td>-.16</td>
<td>.13</td>
<td>-.34</td>
<td>-.09</td>
</tr>
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<td>$I_2$</td>
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<td>.13</td>
<td>-.66</td>
<td>-.53</td>
<td>-.17</td>
</tr>
<tr>
<td>$I_3$</td>
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<td>-.07</td>
<td>-.06</td>
<td>-.20</td>
<td>-.08</td>
<td>-.10</td>
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TABLE 9.16
Regressions of Estimated Values of Food Cross-Price Elasticities ($\eta_{ij}$) on Household Characteristics

<table>
<thead>
<tr>
<th>Regressor</th>
<th>Clothing</th>
<th>Housing</th>
<th>Durables</th>
<th>Education</th>
<th>Medical S.</th>
<th>Other</th>
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</thead>
<tbody>
<tr>
<td><strong>Results using OLS estimation</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Constant</td>
<td>.07</td>
<td>-.64</td>
<td>1.01</td>
<td>-.65</td>
<td>.77</td>
<td>.38</td>
</tr>
<tr>
<td></td>
<td>(.14)</td>
<td>(-1.59)</td>
<td>(2.13)</td>
<td>(-1.10)</td>
<td>(1.72)</td>
<td>(1.14)</td>
</tr>
<tr>
<td>Income/1000</td>
<td>.05</td>
<td>.04</td>
<td>.03</td>
<td>.08</td>
<td>.17*</td>
<td>.08*</td>
</tr>
<tr>
<td></td>
<td>(1.12)</td>
<td>(1.25)</td>
<td>(.90)</td>
<td>(1.06)</td>
<td>(4.22)</td>
<td>(2.63)</td>
</tr>
<tr>
<td>Family Size</td>
<td>.01</td>
<td>.007</td>
<td>-.002</td>
<td>.08*</td>
<td>.12*</td>
<td>.04*</td>
</tr>
<tr>
<td></td>
<td>(.58)</td>
<td>(.32)</td>
<td>(-.07)</td>
<td>(2.49)</td>
<td>(4.90)</td>
<td>(2.22)</td>
</tr>
<tr>
<td>Age</td>
<td>-.006</td>
<td>.007</td>
<td>-.019*</td>
<td>.0004</td>
<td>-.03*</td>
<td>-.01*</td>
</tr>
<tr>
<td></td>
<td>(-.81)</td>
<td>(1.14)</td>
<td>(-.257)</td>
<td>(.04)</td>
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<td>(-2.87)</td>
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<td>Dummy W</td>
<td>-.48*</td>
<td>-.06</td>
<td>-.578*</td>
<td>-.38*</td>
<td>-.51*</td>
<td>-.55*</td>
</tr>
<tr>
<td></td>
<td>(-2.64)</td>
<td>(-.46)</td>
<td>(-3.43)</td>
<td>(-1.81)</td>
<td>(-3.29)</td>
<td>(-4.70)</td>
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<tr>
<td>Dummy E</td>
<td>-.37*</td>
<td>.07</td>
<td>-.469*</td>
<td>-.05</td>
<td>-.32*</td>
<td>-.40*</td>
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<tr>
<td></td>
<td>(-1.84)</td>
<td>(.49)</td>
<td>(-2.47)</td>
<td>(-.25)</td>
<td>(-1.80)</td>
<td>(-3.03)</td>
</tr>
<tr>
<td>Dummy T</td>
<td>-.14</td>
<td>-.04</td>
<td>-.419*</td>
<td>-.49*</td>
<td>-.120*</td>
<td>-.44*</td>
</tr>
<tr>
<td></td>
<td>(-.62)</td>
<td>(-.24)</td>
<td>(-2.03)</td>
<td>(-1.93)</td>
<td>(-6.22)</td>
<td>(-3.09)</td>
</tr>
<tr>
<td><strong>R²</strong></td>
<td>.64</td>
<td>.36</td>
<td>.60</td>
<td>.54</td>
<td>.84</td>
<td>.76</td>
</tr>
<tr>
<td><strong>Results using LDV-ML estimation</strong></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Constant</td>
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<td>.63</td>
<td>-.48</td>
<td>.53</td>
<td>.34</td>
</tr>
<tr>
<td></td>
<td>(-.29)</td>
<td>(-1.25)</td>
<td>(1.37)</td>
<td>(-.62)</td>
<td>(2.61)</td>
<td>(1.07)</td>
</tr>
<tr>
<td>Income/1000</td>
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<td>.05*</td>
<td>.03</td>
<td>.008</td>
<td>.12*</td>
<td>.07*</td>
</tr>
<tr>
<td></td>
<td>(.91)</td>
<td>(1.88)</td>
<td>(.92)</td>
<td>(.11)</td>
<td>(6.97)</td>
<td>(2.50)</td>
</tr>
<tr>
<td>Family Size</td>
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<td>.005</td>
<td>.01</td>
<td>.08*</td>
<td>.03*</td>
</tr>
<tr>
<td></td>
<td>(-.08)</td>
<td>(.33)</td>
<td>(.20)</td>
<td>(.28)</td>
<td>(7.15)</td>
<td>(1.91)</td>
</tr>
<tr>
<td>Age</td>
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<td>.004</td>
<td>-.01*</td>
<td>.001</td>
<td>-.02*</td>
<td>-.01*</td>
</tr>
<tr>
<td></td>
<td>(-.04)</td>
<td>(.86)</td>
<td>(-1.84)</td>
<td>(.12)</td>
<td>(-7.05)</td>
<td>(-2.49)</td>
</tr>
<tr>
<td>Dummy W</td>
<td>-.26*</td>
<td>-.11</td>
<td>-.43*</td>
<td>.04</td>
<td>-.38*</td>
<td>-.45*</td>
</tr>
<tr>
<td></td>
<td>(-1.86)</td>
<td>(-1.08)</td>
<td>(-2.66)</td>
<td>(.17)</td>
<td>(-5.32)</td>
<td>(-4.00)</td>
</tr>
<tr>
<td>Dummy E</td>
<td>-.18</td>
<td>-.03</td>
<td>-.37*</td>
<td>.34</td>
<td>-.20*</td>
<td>-.33*</td>
</tr>
<tr>
<td></td>
<td>(-1.15)</td>
<td>(-.30)</td>
<td>(-2.06)</td>
<td>(1.12)</td>
<td>(-2.48)</td>
<td>(-2.57)</td>
</tr>
<tr>
<td>Dummy T</td>
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<td>-.17</td>
<td>-.32*</td>
<td>.04</td>
<td>-.84*</td>
<td>-.47*</td>
</tr>
<tr>
<td></td>
<td>(-.27)</td>
<td>(-1.30)</td>
<td>(-1.61)</td>
<td>(.13)</td>
<td>(-9.61)</td>
<td>(-3.39)</td>
</tr>
<tr>
<td><strong>R²</strong></td>
<td>.54</td>
<td>.47</td>
<td>.48</td>
<td>.17</td>
<td>.92</td>
<td>.67</td>
</tr>
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</table>
TABLE 9.17
Elasticity of Saving with Respect to Income ($n_{s1}^*$), Price of Food ($n_{s1}^*$) and Price of Housing ($n_{s3}^*$)

<table>
<thead>
<tr>
<th>Type of Household</th>
<th>Income</th>
<th>Food</th>
<th>Housing</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Results using OLS estimation</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Unemployed</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>L</td>
<td>98.66</td>
<td>-44.02</td>
<td>-15.82</td>
</tr>
<tr>
<td>S</td>
<td>3.19</td>
<td>-.77</td>
<td>-.66</td>
</tr>
<tr>
<td>Worker</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>L,Y</td>
<td>-1.49</td>
<td>1.20</td>
<td>.42</td>
</tr>
<tr>
<td>L,O</td>
<td>-10.88</td>
<td>5.03</td>
<td>2.07</td>
</tr>
<tr>
<td>S,Y</td>
<td>-9.85</td>
<td>4.80</td>
<td>2.18</td>
</tr>
<tr>
<td>S,O</td>
<td>-2.91</td>
<td>1.77</td>
<td>.710</td>
</tr>
<tr>
<td>Entrepreneur</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>L</td>
<td>50.64</td>
<td>-20.05</td>
<td>-10.44</td>
</tr>
<tr>
<td>S</td>
<td>-10.98</td>
<td>4.76</td>
<td>2.60</td>
</tr>
<tr>
<td>Technocrat</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>II,L</td>
<td>-5.13</td>
<td>2.07</td>
<td>1.44</td>
</tr>
<tr>
<td>II,S</td>
<td>-9.19</td>
<td>3.43</td>
<td>2.22</td>
</tr>
<tr>
<td>I2</td>
<td>2.58</td>
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</tr>
<tr>
<td>I3</td>
<td>1.38</td>
<td>-.06</td>
<td>-.20</td>
</tr>
<tr>
<td><strong>Results using LDV-ML estimation</strong></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Unemployed</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>L</td>
<td>-15.49</td>
<td>7.52</td>
<td>2.55</td>
</tr>
<tr>
<td>S</td>
<td>10.96</td>
<td>-2.85</td>
<td>-3.56</td>
</tr>
<tr>
<td>Worker</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>L,Y</td>
<td>-1.42</td>
<td>1.18</td>
<td>.31</td>
</tr>
<tr>
<td>L,O</td>
<td>-4.93</td>
<td>2.36</td>
<td>1.03</td>
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<tr>
<td>S,Y</td>
<td>-5.40</td>
<td>2.81</td>
<td>1.26</td>
</tr>
<tr>
<td>S,O</td>
<td>-2.09</td>
<td>1.43</td>
<td>.51</td>
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<tr>
<td>Entrepreneur</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>L</td>
<td>-15.40</td>
<td>6.44</td>
<td>3.54</td>
</tr>
<tr>
<td>S</td>
<td>-3.02</td>
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<td>.83</td>
</tr>
<tr>
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<td></td>
<td></td>
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<tr>
<td>II,L</td>
<td>-1.29</td>
<td>.64</td>
<td>.58</td>
</tr>
<tr>
<td>II,S</td>
<td>-4.71</td>
<td>1.81</td>
<td>1.28</td>
</tr>
<tr>
<td>I2</td>
<td>24.13</td>
<td>-4.93</td>
<td>-3.49</td>
</tr>
<tr>
<td>I3</td>
<td>1.26</td>
<td>-.08</td>
<td>-.02</td>
</tr>
</tbody>
</table>
Family Size increases. We also find that Age and Occupational category have a significant effect on the $\eta_{jl}$ for 'Durables', 'Medical Services' and 'Other'. Although there are some differences in the OLS, LDV-ML estimation results; most of the qualitative findings are essentially the same (compare placement of starred quantities in both top and bottom parts of Table 9.16).

**Saving Elasticities**

To end, we present in Table 9.17 the estimated values of the elasticity of saving with respect to income ($\eta_{sa}$) and with respect to price of 'Food' ($\eta_{s1}$) and 'Housing' ($\eta_{s3}$) [see (9.9)]. Looking at the first column we note that 7 out of the 12 OLS estimates; and 9 out of the 12 LDV-ML estimates, have the incorrect sign. This is due to the high estimated values of the total subsistence expenditures. Similarly, the number of estimates of either $\eta_{s1}$ or $\eta_{s3}$ with incorrect (positive) sign is 7 for OLS, and 9 for LDV-ML estimates.

An overview of our main findings is given in the next section.

### 9.6 CONCLUDING REMARKS

In this Chapter we have estimated the ELES, to study the pattern of consumption and saving behaviour in México City. The data used was obtained from a 1975 Income-Expenditure Household Survey.

The principal points that emerged from the econometric methodology employed are as follows:
(1) The parameters of the ELES are related to household socioeconomic variables and neglect of this leads to model misspecifications. In particular, estimation of a single ELES with data from all the households is inappropriate.

(2) When forming groups of homogeneous consumers by cluster analysis, some households had to be reassigned, to obtain readily identifiable domains of study.

(3) The results of the cluster analysis show that occupational category is a major factor influencing consumption behaviour.

(4) When the probability of obtaining a negative observation under normality, \( P(x'\beta, \sigma^2) \), is below .025, results from OLS and MLE on the Truncated model are basically the same. So, in these cases, neglect of the fact that expenditures are non-negative and therefore using OLS, does not necessarily lead to 'severe inconsistencies'. In other cases where \( P(x'\hat{\beta}, \hat{\sigma}^2) \) is above .025, we find that significant differences between the estimates can arise.

(5) The test statistic \( \text{LM}_N \), applied to situations where there are reported zero expenditures, almost always leads to the rejection of disturbance normality; identifying the need for the use of the Tobit model [\( \text{LM}_N \) rejected normality in 33 of the 36 cases (12 groups \( \times 3 \) commodities) with reported zero expenditures].

(6) In our study, sample partition results in small sample sizes, making it difficult to assess the significance of the values of the test statistics \( \text{LM}^{\text{NH(Trun)}} \) and \( \text{LM}^{\text{NH(Tobit)}} \). Yet, in some instances, these were extremely large, suggesting the presence of \( \bar{N} \) and/or \( \bar{H} \) in the data. This result signals the need to proceed with caution in applications of LDV models (the presence of \( \bar{N} \) and/or \( \bar{H} \) makes ML estimators inconsistent).
(7) When using LDV models, failure to adjust the ML estimators (so they represent proper responses) can lead to misleading results.

(8) Although the numerical results are different, the qualitative findings on the demand and savings responses based on OLS or LDV-ML estimation, are basically the same.

The summary of the main empirical findings, relating to demand and savings responses for México City households, are as follows:

(9) Average budget shares for 'Food' are higher for the 1975 survey data than for the 1968 survey data used by LPW (1977).

(10) Significant positive savings ratios are observed only in highly paid Technocrats (above 3000 pesos per capita per month).

(11) Within a given occupational category, there is a tendency for the marginal budget shares for 'Food' to decrease with higher income. We also find significant effects of occupational category on marginal budget shares for 'Food', 'Education', 'Medical Services', and 'Other'. In particular, Technocrats have lower marginal budget shares for 'Food' and higher marginal budget shares for 'Education', 'Medical Services' and 'Other'. Additionally, Workers have significantly larger marginal propensities to consume.

(12) Per capita subsistence expenditures are related to occupational category, with Technocrats having significantly larger subsistence expenditures. We also find a tendency for 'Food' and Total subsistence expenditure for Workers to be less than for Entrepreneurs. As in LPW (1977, p.150) our results suggest that subsistence expenditures represent "acceptable minimum standards for households identifying with a given socioeconomic group".
(13) There is a tendency for the elasticity of demand for 'Food' with respect to total expenditure, to be the lowest. This elasticity is also found to be higher for large families and lower for Technocrats.

(14) Own price elasticities increase in absolute value with higher income and larger family size.

(15) 'Food' cross-price elasticities for 'Medical Services' and 'Other' seem to have a decline in absolute value as income increases.

To conclude, we note that some of our estimates on demand and saving responses were - all too often - of the incorrect sign (e.g., Frisch parameters and saving elasticities). To explain this we could, of course, question the quality of the data (Indeed, some weaknesses of this were noted in footnote 10). Also, we might question the LDV-ML estimation technique used. As pointed out previously, this gives inconsistent estimators in cases where disturbance NH does not hold. Yet, in the present study the inconsistency is not expected to be a significant factor, because of the small relative magnitude of the coefficients in which the NH hypothesis was rejected. Additionally, we could question the validity of the ELES as an appropriate description of household demand and saving behaviour. In all, for the available data, our study does not give strong support for the ELES and suggests the need to consider other demand systems. It also suggests that an important area for future research is the analysis of LDV models with non-normal and/or heteroscedastic disturbances. Proper techniques for this could give sufficiently accurate numerical results on which more precise answers to policy questions may be given.
CHAPTER 10
EXTENSIONS AND CONCLUSION

"'Mine is a long and a sad tale!' said the Mouse, turning to Alice, and sighing. 'It is a long tail, certainly, said Alice, looking down with wonder at the Mouse's tail; 'but why do you call it sad?'"

Lewis Carroll
Alice's Adventures in Wonderland

10.1 INTRODUCTION

We have obtained results for the statistical analysis of economic models when using cross-sectional data. Further extensions of our work are indicated in Section 10.2, and comments on the application of our findings when using time-series data are given in Section 10.3. The thesis ends with Section 10.4 where we make some concluding remarks.

10.2 CROSS-SECTIONAL EXTENSIONS

In previous Chapters, we have proceeded under the maintained hypothesis that the deterministic part of the model was correctly specified. In applied econometric work, we may commit specification errors such as exclusion of some relevant variables and/or misspecification of the way in which the included variables enter the model. These types of misspecifications can have serious effects on both estimation and inferential results. [For example, the disturbance normality and/or
homoscedasticity tests suggested in Chapters 4 and 5 were derived under
the maintained hypothesis that the model was correctly specified; and
failure of this maintained hypothesis can lead to statistically
significant test statistics - even when the 'true' disturbances are
normal and homoscedastic \((NH)\). It is therefore important to test for
the specification of the 'functional form' of the model.

This inferential problem has received extensive discussion in the
recent econometric literature, and general specification tests are now
available; notably, those of Hausman (1978) and White (1980a). Also
available are joint tests for functional form and certain disturbance
distributional assumptions. See, for instance, the tests suggested by
Savin and White (1978), Dagenais, Gaudry and Liem (1980), Ghali and
Snow (1981) and Lahiri and Egy (1981). These latter tests are based
on the likelihood ratio principle, so they require MLE under both the
null and the alternative hypotheses.

The LM principle as applied in Section 5.5, can be extended to
obtain a joint test for functional form and disturbance \(NH\). Full
details of this extension are given in Bera and Jarque (1981b). Here
we only note the resulting test statistic is simple to compute and has
optimal asymptotic power properties, so it should prove to be a useful
diagnostic check in applied econometric modelling.

10.3 TIME-SERIES EXTENSIONS

The discussion in this thesis has been presented within a cross-
sectional context. Nevertheless, many of our results are also applic-
able when analysing time-series data. This is now illustrated.
Regarding inferential results, we note that all the normality and/or homoscedasticity tests developed in Chapters 4, 5, 6 and 8, may be directly used with time-series data. These tests are based on the LM principle, which has been previously shown – by Godfrey (1978a,b) and Breusch (1978) – to be particularly attractive when testing for serial independence in time-series regression models. Indeed, our joint test $L_{NH}^{M}$ may be extended to incorporate a serial independence ($T$) component, giving the test statistic $L_{NHT}^{M}$ defined in Jarque and Bera (1980) [Evidence on the power of this test is given in Bera and Jarque (1981a,b)]. Additionally, the LM principle is extremely useful when testing for serial independence in the LDV models that we analyzed in Chapter 6. This latter point is discussed in detail in Jarque and Bera (1981a).

Regarding estimation results, we point out that the two stage procedure suggested in Chapter 7 is also applicable in time-series studies. Here, however, we would need to modify slightly the algorithms used for the classification of the observations (as in McGee and Carleton (1970)) so the resulting groups contain subsequent observations in time. No other changes would be required.

10.4 CONCLUSION

In every Chapter we have made concluding remarks and highlighted areas that required further investigation. Here it is sufficient to say that – in the solution of several econometric problems – we have found Cluster Analysis and the Lagrange Multiplier Test to be useful analytical tools. Hopefully, our application of these techniques will lead to the solution of other econometric problems and – our results – to a better understanding of the relationship among economic variables.
APPENDIX A

DERIVATIONS OF SECTIONS 4.3 AND 4.4

PROPOSITION 1: The LM test statistic for testing the normality of observations is given by equation (4.4).

Proof:

The problem of testing the normality of the observations \( v_1, \ldots, v_N \), is the same as testing the normality of the 'regression disturbances' \( u_1, \ldots, u_N \) in the regression model

\[
y_i = x_i^1 \beta + u_i,
\]

with \( K \) (the dimension of \( x_i^1 \)) equal to one, \( x_i = 1 \), \( \beta = \mu \) and \( y_i = v_i \). So, the proof of this proposition is a particular case of the proof given for Proposition 2 (see below). Note that, in the present case, the MLE of \( \beta \) (i.e., \( \mu \)) under \( H_0: c_1 = c_2 = 0 \) is \( \hat{\beta} = \hat{\mu} = (v_1 + \ldots + v_N)/N = \bar{v} \), which implies that \( \hat{u}_1 = v_1 - \bar{v} \), and that \( \hat{u}_1 = (\hat{u}_1 + \ldots + \hat{u}_N)/N = 0 \).
PROPOSITION 2: The LM test statistic for testing the normality of regression disturbances, \( LM_n \), is given by equation (4.5).

Proof:

Define

\[
\phi(\theta, u_i) = \int \frac{c_1 - u_i}{c_0 - c_1 u_i + c_2 u_i^2} du_i,
\]

\[v_i = c_0 - c_1 u_i + c_2 u_i^2, \quad \theta_1 = (\beta', c_0)', \quad \theta_2 = (c_1, c_2)' \text{ and} \]

\[\theta = (\theta_1', \theta_2')'. \]

Then the log-likelihood for the \( i \)th observation can be written as

\[
\ell_i(\theta) = -\log \left[ \int_{-\infty}^{\infty} \exp[\phi(\theta, u_i)] du_i \right] + \phi(\theta, u_i).
\]

We can show that

\[
\frac{\partial \ell_i(\theta)}{\partial \beta} = - \frac{x_i \int_{-\infty}^{\infty} \exp[\phi(\theta, u_i)] \left[ \frac{v_i - (c_1 - u_i)(c_1 - 2c_2 u_i)}{v_i^2} \right] du_i}{\int_{-\infty}^{\infty} \exp[\phi(\theta, u_i)] du_i} 
+ \frac{x_i \int \frac{v_i - (c_1 - u_i)(c_1 - 2c_2 u_i)}{v_i^2} du_i}{\int_{-\infty}^{\infty} \exp[\phi(\theta, u_i)] du_i}
\]

\[
\frac{\partial \ell_i(\theta)}{\partial c_0} = - \frac{\int_{-\infty}^{\infty} \exp[\phi(\theta, u_i)] \left[ - \frac{c_1 - u_i}{v_i^2} \right] du_i}{\int_{-\infty}^{\infty} \exp[\phi(\theta, u_i)] du_i} 
+ \int - \frac{(c_1 - u_i)}{v_i^2} du_i
\]
\[
\frac{\partial \mathcal{S}_1(\theta)}{\partial c_1} = -\frac{\int_{-\infty}^{\infty} \exp[\phi(\theta, u_1)] \left[ \frac{v_1 - (c_1 - u_1)(-u_1)}{v_1^2} \right] du_1}{\int_{-\infty}^{\infty} \exp[\phi(\theta, u_1)] du_1}
\]

\[
+ \int_{-\infty}^{\infty} \frac{v_1 - (c_1 - u_1)(-u_1)}{v_1^2} du_1
\]

\[
\frac{\partial \mathcal{S}_1(\theta)}{\partial c_2} = -\frac{\int_{-\infty}^{\infty} \exp[\phi(\theta, u_1)] \left[ \frac{-(c_1 - u_1)u_1^2}{v_1^2} \right] du_1}{\int_{-\infty}^{\infty} \exp[\phi(\theta, u_1)] du_1} + \int_{-\infty}^{\infty} \frac{-(c_1 - u_1)u_1^2}{v_1^2} du_1.
\]

Setting \( c_1 = c_2 = 0 \) in the above expressions, and noting that under normality \( E[u_1] = E[u_1^3] = 0 \) and \( E[u_1^4] = 3c_0^2 \), we obtain

\[
\frac{\partial \mathcal{S}_1(\theta)}{\partial \beta} = \frac{x_i u_i}{c_0},
\]

\[
\frac{\partial \mathcal{S}_1(\theta)}{\partial c_0} = -\frac{1}{2c_0} + \frac{u_1^2}{2c_0^2},
\]

(A.1)

\[
\frac{\partial \mathcal{S}_1(\theta)}{\partial c_1} = \frac{u_1}{c_0} - \frac{u_1^3}{3c_0^2}
\]

and

\[
\frac{\partial \mathcal{S}_1(\theta)}{\partial c_2} = -\frac{3}{4} + \frac{u_1^4}{4c_0^2}.
\]

(In particular, note that when \( c_1 = c_2 = 0 \), we have

\[
\int_{-\infty}^{\infty} \exp[\phi(\theta, u_1)] du_1 = \sqrt{(2\pi c_0)}.
\]

Adding (A.1) from \( i = 1 \) to \( N \), and evaluating the resulting
quantities at the MLE of \( \theta \) under \( H_0 : c_1 = c_2 = 0 \) (i.e. setting \( \beta = \hat{\beta} = (X'X)^{-1}X'y \) and \( c_0 = \hat{\sigma}^2 \)) we obtain

\[
\hat{d}_2 = \left[ \sum_{i=1}^N \frac{\hat{\ell}_1(\theta)}{\partial c_1}, \sum_{i=1}^N \frac{\hat{\ell}_1(\theta)}{\partial c_2} \right]'
\]

\[
= N \left[ \frac{\hat{\mu}_1}{\mu_1} - \frac{\hat{\mu}_2}{3\mu_2}, -\frac{3}{4} + \frac{\hat{\mu}_4}{4\mu_2} \right]', \quad \text{(A.2)}
\]

where \( \hat{\mu}_j = \sum_{i=1}^N \frac{u_i^j}{N} \) and \( \hat{u}_i = y_i - x_i'\hat{\beta} \).

Now we use (A.1) to compute \( \psi(\theta) = \Sigma_{i=1}^N \frac{\partial \hat{\ell}_1(\theta)/\partial \theta}{\partial \hat{\ell}_1(\theta)/\partial \theta}' \) obtaining

\[
\psi(\theta) = N \begin{bmatrix}
\psi_{11} & \psi_{12} & \psi_{13} & \psi_{14} \\
\psi_{12} & \psi_{22} & \psi_{23} & \psi_{24} \\
\psi_{13} & \psi_{23} & \psi_{33} & \psi_{34} \\
\psi_{14} & \psi_{24} & \psi_{34} & \psi_{44}
\end{bmatrix},
\]

with

\[
\psi_{11} = \frac{1}{N} \sum_{i=1}^N \frac{x_i x_i' u_i^2}{c_0^2}
\]

\[
\psi_{12} = \frac{1}{N} \sum_{i=1}^N x_i \left\{ \frac{u_i^3}{2c_0} - \frac{u_i^4}{2c_0^3} \right\}
\]

\[
\psi_{13} = \frac{1}{N} \sum_{i=1}^N x_i \left\{ \frac{u_i^3}{2c_0} - \frac{u_i^4}{3c_0^3} \right\}
\]

\[
\psi_{14} = \frac{1}{N} \sum_{i=1}^N x_i \left\{ -\frac{3u_i^1}{4c_0} + \frac{u_i^3}{4c_0^3} \right\}
\]
\[
\psi_{22} = \frac{1}{4c_o^2} + \frac{\mu_4}{4c_o} - \frac{\mu_2}{2c_o^3}
\]

\[
\psi_{23} = -\frac{\mu_1}{2c_o^2} + \frac{2\mu_3}{3c_o^3} - \frac{\mu_5}{6c_o^4}
\]

\[
\psi_{24} = \frac{3}{8c_o} - \frac{\mu_4}{8c_o^3} - \frac{3\mu_2}{8c_o^2} + \frac{\mu_6}{4c_o}
\]

\[
\psi_{33} = \frac{\mu_2}{2c_o^2} - \frac{2\mu_4}{3c_o^3} + \frac{\mu_6}{9c_o^4}
\]

\[
\psi_{34} = \frac{\mu_5}{4c_o^3} + \frac{\mu_3}{4c_o^2} - \frac{\mu_7}{12c_o} - \frac{3\mu_1}{4c_o}
\]

\[
\psi_{44} = \frac{9}{16} + \frac{\mu_8}{16c_o^4} - \frac{6\mu_4}{16c_o^2}
\]

where

\[
\mu_j = \frac{\sum_{i=1}^{N} u_i^j}{N}.
\]

Taking the expectation of \( \psi(\theta) \) (noting that \textit{under normality} -

\[
\]

\[
E[u_1^2] = E[u_2] = c_o, E[u_4] = E[u_4] = 3c_o^2, E[u_6] = 15c_o^3 \text{ and}
\]

\[
E[u_8] = 105c_o^4,
\]

we obtain

\[
I = E[\psi(\theta)] = N
\]

\[
\begin{bmatrix}
\frac{X'X}{c_o} & 0 & 0 & 0 \\
0 & 1/(2c_o^2) & 0 & 3/(2c_o) \\
0 & 0 & 2/(3c_o) & 0 \\
0 & 3/(2c_o) & 0 & 6
\end{bmatrix}
\]
After evaluating $I$ at $\theta = \hat{\theta}$ (i.e. setting $c_o = \hat{\sigma}^2 = \hat{\mu}_2$) we obtain

$$(\hat{I}_{22}^{-1}\hat{I}_{21}^{-1}\hat{I}_{11}^{-1}\hat{I}_{12}^{-1})^{-1} = \frac{1}{N} \begin{bmatrix} 3\hat{\mu}_2/2 & 0 \\ 0 & 2/3 \end{bmatrix}.$$  \hspace{1cm} (A.3)

Using (A.3) and (A.2) in (4.1) we obtain our result. \hfill \square
APPENDIX B

TABLES CONTAINING RESULTS OF SIMULATION STUDY OF SECTION 4.5
TABLE 4.6
Estimated power with 250 replications (α = .10)

(K=4) (Regressors: $X_1 = 1; X_2 \sim \text{normal}; X_3 \sim \text{uniform}; X_4 \sim X^2_{10}$) (N varies)

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### TABLE 4.7

Estimated correlations between true and modified statistics

(K=4) (Regressors: \(X_1 = 1; X_2 \sim \text{normal}; X_3 \sim \text{uniform}; X_4 \sim \chi^2_{10}\)) (N varies)

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### TABLE 4.8

Estimated power with 250 replications ($\alpha = .10$)

(K varies) (Regressors: Weisberg Data Set 1) (N = 20)

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**TABLE 4.9**

Estimated **power** with 250 replications \( (\alpha = .10) \)

(K varies) (Regressors: Weisberg Data Set 2) \( (N = 20) \)

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* Given that for $\hat{b}_1$, $\hat{b}_2$, $\hat{D}^*$ and $\hat{R}$ the estimated $\alpha$ was far from 0.10 for $K = 8$ and 10; we did not adjust $\hat{W}$, $\hat{W}'$ and $\text{LM}_N$ to have $\hat{\alpha} = 0.10$ in these cases. We present the power obtained when using the theoretical significance points, obtained from Shapiro and Wilk (1965, p.605), Weisberg (1974, p.645) and Table 4.3.
TABLE 4.11
Estimated correlations between true and modified statistics
(K varies) (Regressors: Weisberg Data Set 1) (N = 20)

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TABLE 4.12

Estimated correlations between true and modified statistics
(K varies) (Regressors: Weisberg Data Set 2) (N = 20)

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**TABLE 4.13**

Estimated correlations between true and modified statistics

(K varies) (Regression: Weisberg Data Set 3) (N = 20)

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<th>(D, D)</th>
<th>(W, W)</th>
<th>(W', W')</th>
<th>(LM, LM)</th>
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K = 6

| Normal | .618   | .612   | .531   | .304     | .413     | .490     |
| Beta   | .476   | .457   | .379   | .186     | .193     | .369     |
| Students t | .675 | .693   | .658   | .518     | .588     | .761     |
| Gamma  | .652   | .790   | .731   | .579     | .642     | .783     |
| Lognormal | .676 | .774   | .759   | .685     | .708     | .712     |

K = 8

| Normal | .480   | .494   | .426   | .240     | .327     | .399     |
| Beta   | .373   | .313   | .188   | .095     | .096     | .261     |
| Students t | .460 | .572   | .520   | .412     | .470     | .688     |
| Gamma  | .487   | .674   | .598   | .530     | .565     | .728     |
| Lognormal | .456 | .577   | .675   | .648     | .642     | .595     |

K = 10

| Normal | .346   | .334   | .245   | .081     | .143     | .149     |
| Beta   | .278   | .264   | .151   | .002     | .040     | .206     |
| Students t | .406 | .475   | .394   | .339     | .375     | .667     |
| Gamma  | .317   | .485   | .439   | .377     | .411     | .482     |
| Lognormal | .323 | .432   | .526   | .479     | .482     | .433     |
APPENDIX C

TABLES CONTAINING RESULTS OF
SIMULATION STUDY OF SECTION 5.4
TABLE 5.5
Estimated power using 250 replications
(K=2) (Regressors: constant; $x^2$) (N=50)

<table>
<thead>
<tr>
<th></th>
<th>$L^G_M$</th>
<th>$L^X_2$</th>
<th>$L^X_3$</th>
<th>$L^X_4$</th>
<th>$L^X_{G_2}$</th>
<th>$L^X_{G_3}$</th>
<th>$L^X_{G_4}$</th>
<th>$GQ^X_2$</th>
<th>$GQ^X_3$</th>
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<tbody>
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<td>.100</td>
<td>.100</td>
<td>.100</td>
<td>.100</td>
<td>.100</td>
<td>.100</td>
<td>.100</td>
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</tr>
<tr>
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<td>.168</td>
<td>.156</td>
<td>.168</td>
<td>.208</td>
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<td>.076</td>
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<tr>
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<td>.192</td>
<td>.064</td>
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<tr>
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<td>.528</td>
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<td>.596</td>
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<td>.148</td>
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<td>.076</td>
<td>.152</td>
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TABLE 5.6
Estimated power using 250 replications
(K=2) (Regressors: constant; normal) (N=50)

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<td>.100</td>
<td>.100</td>
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<td>.016</td>
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TABLE 5.7

Estimated power using 250 replications
(K=3) (Regressors: constant; normal; uniform) (N=50)

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TABLE 5.8

Estimated power using 250 replications
(K=3) (Regressors: constant; normal; $\chi^2$) (N=50)

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TABLE 5.9
Estimated power using 250 replications
(K=2) (Regressors: constant; $\chi^2$) (N=50)

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<tr>
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<td>$\overline{H}_3$</td>
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</table>

TABLE 5.10
Estimated power using 250 replications
(K=2) (Regressors: constant; normal) (N=50)

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<th>$L_{H}^*(5)$</th>
<th>$L_{H}^*(10)$</th>
<th>HW</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_0: u \sim NH$:</td>
<td>.100</td>
<td>.100</td>
<td>.100</td>
<td>.100</td>
</tr>
<tr>
<td>$H_a: u \sim NH$:</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\overline{H}_1$</td>
<td>.260</td>
<td>.232</td>
<td>.268</td>
<td>.116</td>
</tr>
<tr>
<td>$\overline{H}_2$</td>
<td>.344</td>
<td>.328</td>
<td>.336</td>
<td>.112</td>
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<tr>
<td>$\overline{H}_3$</td>
<td>.344</td>
<td>.336</td>
<td>.364</td>
<td>.428</td>
</tr>
<tr>
<td>$\overline{H}_4$</td>
<td>.240</td>
<td>.216</td>
<td>.232</td>
<td>.096</td>
</tr>
<tr>
<td>$\overline{H}_5$</td>
<td>.188</td>
<td>.212</td>
<td>.228</td>
<td>.112</td>
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</table>
TABLE 5.11

Estimated power using 250 replications
(K=3) (Regressors: constant; normal; uniform) (N=50)

<table>
<thead>
<tr>
<th></th>
<th>$\text{LM}_H^a(2)$</th>
<th>$\text{LM}_H^a(5)$</th>
<th>$\text{LM}_H^a(10)$</th>
<th>$\text{HW}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_0: u \sim NH$:</td>
<td>.100</td>
<td>.100</td>
<td>.100</td>
<td>.100</td>
</tr>
<tr>
<td>$H_a: u \sim \overline{NH}$:</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\overline{H1}$</td>
<td>.204</td>
<td>.220</td>
<td>.240</td>
<td>.128</td>
</tr>
<tr>
<td>$\overline{H2}$</td>
<td>.280</td>
<td>.320</td>
<td>.304</td>
<td>.136</td>
</tr>
<tr>
<td>$\overline{H3}$</td>
<td>.264</td>
<td>.292</td>
<td>.328</td>
<td>.404</td>
</tr>
<tr>
<td>$\overline{H4}$</td>
<td>.188</td>
<td>.192</td>
<td>.208</td>
<td>.408</td>
</tr>
<tr>
<td>$\overline{H5}$</td>
<td>.152</td>
<td>.204</td>
<td>.228</td>
<td>.120</td>
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</table>

TABLE 5.12

Estimated power using 250 replications
(K=3) (Regressors: constant; normal; $\chi^2$) (N=50)

<table>
<thead>
<tr>
<th></th>
<th>$\text{LM}_H^a(2)$</th>
<th>$\text{LM}_H^a(5)$</th>
<th>$\text{LM}_H^a(10)$</th>
<th>$\text{HW}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_0: u \sim NH$:</td>
<td>.100</td>
<td>.100</td>
<td>.100</td>
<td>.100</td>
</tr>
<tr>
<td>$H_a: u \sim \overline{NH}$:</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\overline{H1}$</td>
<td>.232</td>
<td>.256</td>
<td>.264</td>
<td>.124</td>
</tr>
<tr>
<td>$\overline{H2}$</td>
<td>.316</td>
<td>.372</td>
<td>.376</td>
<td>.160</td>
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<tr>
<td>$\overline{H3}$</td>
<td>.328</td>
<td>.388</td>
<td>.380</td>
<td>.404</td>
</tr>
<tr>
<td>$\overline{H4}$</td>
<td>.200</td>
<td>.228</td>
<td>.228</td>
<td>.132</td>
</tr>
<tr>
<td>$\overline{H5}$</td>
<td>.164</td>
<td>.240</td>
<td>.236</td>
<td>.388</td>
</tr>
</tbody>
</table>
Proposition 1: The LM test statistic for disturbance normality and homoscedasticity, $LM_{NH}$, is given by equation (5.15).

Proof:

Define

$$\phi(\theta, u_i) = \int \frac{c_1 - u_i}{\sigma^2 + z^* a^* - c_1 u_i + c_2 u_i^2} du_i,$$

$$v_i = \sigma^2 + z^* a^* - c_1 u_i + c_2 u_i^2, \quad \theta_1 = (\beta', \sigma^2)' \quad \theta_2 = (c_1, c_2, a^*)' \quad \text{and} \quad \theta = (\theta_1', \theta_2').$$

Then the log-likelihood for the i'th observation can be written as

$$\ell_i(\theta) = -\log \left[ \int_{-\infty}^{\infty} \exp[\phi(\theta, u_i)] du_i \right] + \phi(\theta, u_i).$$

We can show that
\[
\frac{\partial x_1(\theta)}{\partial \theta} = - \int_{-\infty}^{\infty} \exp[\phi(\theta, u_1)] \left[ \frac{v_1 - (c_1 - u_1)(c_1 - 2c_2 u_1)}{v_1^2} \right] du_1 
\]

\[
+ x_1 \int_{-\infty}^{\infty} \exp[\phi(\theta, u_1)] \left[ \frac{v_1 - (c_1 - u_1)(c_1 - 2c_2 u_1)}{v_1^2} \right] du_1
\]

\[
\frac{\partial x_1(\theta)}{\partial \sigma^2} = - \int_{-\infty}^{\infty} \exp[\phi(\theta, u_1)] \left[ \frac{-(c_1 - u_1)}{v_1^2} \right] du_1 
\]

\[
+ \int_{-\infty}^{\infty} \exp[\phi(\theta, u_1)] \left[ \frac{v_1 - (c_1 - u_1)(c_1 - 2c_2 u_1)}{v_1^2} \right] du_1
\]

\[
\frac{\partial x_1(\theta)}{\partial c_1} = - \int_{-\infty}^{\infty} \exp[\phi(\theta, u_1)] \left[ \frac{v_1 - (c_1 - u_1)(-u_1)}{v_1^2} \right] du_1 
\]

\[
+ \int_{-\infty}^{\infty} \exp[\phi(\theta, u_1)] \left[ \frac{v_1 - (c_1 - u_1)(-u_1)}{v_1^2} \right] du_1
\]

\[
\frac{\partial x_1(\theta)}{\partial c_2} = - \int_{-\infty}^{\infty} \exp[\phi(\theta, u_1)] \left[ \frac{-(c_1 - u_1)u_1^2}{v_1^2} \right] du_1 
\]

\[
+ \int_{-\infty}^{\infty} \exp[\phi(\theta, u_1)] \left[ \frac{-(c_1 - u_1)u_1^2}{v_1^2} \right] du_1
\]
\[ \frac{\partial \ell_1(\theta)}{\partial a^*} = - \int_{-\infty}^{\infty} \exp[\phi(\theta, u)] \left[ \frac{-(c_1 - u_i)}{v_i} \right] du_i \]

\[ + z_i^* \int_{-\infty}^{\infty} \exp[\phi(\theta, u_i)] du_i . \]

Setting \( c_1 = c_2 = 0 \) and \( a^* = 0 \) in the above expressions, and noting that under normality and homoscedasticity, \( E[u_i] = E[u_i^3] = 0, \)
\( E[u_i^2] = \sigma^2 \) and \( E[u_i^4] = 3\sigma^4 \), we obtain

\[ \frac{\partial \ell_1(\theta)}{\partial \beta} = \frac{x_i u_i}{\sigma^2} , \]

\[ \frac{\partial \ell_1(\theta)}{\partial \sigma^2} = - \frac{1}{2\sigma^2} + \frac{u_i^2}{2\sigma^4} , \]

\[ \frac{\partial \ell_1(\theta)}{\partial c_1} = \frac{u_i}{\sigma^2} - \frac{u_i^3}{3\sigma^4} , \quad (D.1) \]

\[ \frac{\partial \ell_1(\theta)}{\partial c_2} = - \frac{3}{4} + \frac{u_i^4}{4\sigma^4} \quad \text{and} \]

\[ \frac{\partial \ell_1(\theta)}{\partial a^*} = - \frac{z_i^*}{2\sigma^2} + \frac{z_i^* u_i^2}{2\sigma^4} . \]

Adding (D.1) from \( i = 1 \) to \( N \), and evaluating the resulting quantities at the MLE of \( \theta \) under \( H_0: c_1 = c_2 = 0; a^* = 0 \) (i.e., setting \( \beta = \hat{\beta} \) and \( \sigma^2 = \hat{\sigma}^2 = \hat{\mu}_2 \)) we obtain
\[ \hat{d}_2 = \left[ \sum_{i=1}^{N} \frac{\hat{d}_{i1}(\theta)}{\hat{c}_1} \right], \sum_{i=1}^{N} \frac{\hat{d}_{i1}(\theta)}{2\hat{c}_2}, \sum_{i=1}^{N} \frac{\hat{d}_{i1}(\theta)}{2\hat{d}_{i1}(\theta)} \right] \]

\[ = N \left[ \frac{\hat{u}_1}{\mu_2} - \frac{\hat{u}_3}{3\mu_2} - \frac{\hat{u}_4}{4\mu_2^2}, \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\mu_2} \left\{ -\frac{1}{2\mu_2} + \frac{\hat{u}_1^2}{2\mu_2^2} \right\} \right], \quad (D.2) \]

where \( \hat{u}_j = \frac{N}{i=1} u_{j1}^3/N \) and \( \hat{u}_4 = y_4^3 - x_4^3 \).

We now use (D.1) to compute \( \psi(\theta) = \sum_{i=1}^{N} \frac{\partial \hat{d}_{i1}(\theta)}{\partial \theta}(\partial \hat{d}_{i1}(\theta)/\partial \theta) \), obtaining

\[
\psi(\theta) = N \begin{bmatrix}
\psi_{11} & \psi_{12} & \psi_{13} & \psi_{14} & \psi_{15} \\
\psi_{12} & \psi_{22} & \psi_{23} & \psi_{24} & \psi_{25} \\
\psi_{13} & \psi_{23} & \psi_{33} & \psi_{34} & \psi_{35} \\
\psi_{14} & \psi_{24} & \psi_{34} & \psi_{44} & \psi_{45} \\
\psi_{15} & \psi_{25} & \psi_{35} & \psi_{45} & \psi_{55}
\end{bmatrix},
\]

with

\[
\psi_{11} = \frac{1}{N} \sum_{i=1}^{N} \frac{x_i x_i^2 u_{i1}^2}{\sigma^4}
\]

\[
\psi_{12} = \frac{1}{N} \sum_{i=1}^{N} x_i \left\{ -\frac{u_{i1}}{2\sigma^4} + \frac{u_{i1}^3}{2\sigma^6} \right\}
\]

\[
\psi_{13} = \frac{1}{N} \sum_{i=1}^{N} x_i \left\{ \frac{u_{i1}^2}{\sigma^4} - \frac{u_{i1}^4}{3\sigma^6} \right\}
\]

\[
\psi_{14} = \frac{1}{N} \sum_{i=1}^{N} x_i \left\{ -\frac{3u_{i1}}{4\sigma^2} + \frac{u_{i1}^5}{4\sigma^6} \right\}
\]
\[
\psi_{15} = \frac{1}{N} \sum_{i=1}^{N} x_i z_i^{*1} \left( \frac{u_i}{2\sigma^4} + \frac{u_i^3}{2\sigma^6} \right)
\]
\[
\psi_{22} = \frac{1}{4\sigma^4} + \frac{\mu_4}{4\sigma^8} - \frac{\mu_2}{2\sigma^6}
\]
\[
\psi_{23} = -\frac{\mu_1}{2\sigma^4} + \frac{2\mu_3}{3\sigma^6} - \frac{\mu_5}{6\sigma^8}
\]
\[
\psi_{24} = \frac{3}{8\sigma^2} - \frac{\mu_4}{8\sigma^6} - \frac{3\mu_2}{8\sigma^4} + \frac{\mu_6}{8\sigma^8}
\]
\[
\psi_{25} = \frac{1}{N} \sum_{i=1}^{N} z_i^{*1} \left( \frac{1}{4\sigma^4} - \frac{u_i^2}{2\sigma^6} + \frac{u_i^4}{4\sigma^8} \right)
\]
\[
\psi_{33} = \frac{\mu_2}{\sigma^4} - \frac{2\mu_4}{3\sigma^6} + \frac{\mu_6}{9\sigma^8}
\]
\[
\psi_{34} = \frac{\mu_5}{4\sigma^6} + \frac{\mu_3}{4\sigma^4} - \frac{\mu_7}{12\sigma^8} - \frac{3\mu_1}{4\sigma^2}
\]
\[
\psi_{35} = \frac{1}{N} \sum_{i=1}^{N} z_i^{*1} \left( -\frac{u_i^2}{2\sigma^4} + \frac{2u_i^3}{3\sigma^6} - \frac{u_i^5}{6\sigma^8} \right)
\]
\[
\psi_{44} = \frac{9}{16} + \frac{\mu_8}{16\sigma^8} - \frac{6\mu_4}{16\sigma^4}
\]
\[
\psi_{45} = \frac{1}{N} \sum_{i=1}^{N} z_i^{*1} \left( -\frac{u_i^4}{8\sigma^6} + \frac{3}{8\sigma^2} + \frac{u_i^6}{8\sigma^8} - \frac{3u_i^2}{8\sigma^4} \right)
\]
\[
\psi_{55} = \frac{1}{N} \sum_{i=1}^{N} z_i^{*2} z_i^{*1} \left( \frac{1}{4\sigma^4} + \frac{u_i^4}{4\sigma^8} - \frac{u_i^2}{2\sigma^6} \right)
\]

where

\[
\mu_j = \frac{\sum_{i=1}^{N} u_i^j}{N}
\]
Taking the expectation of $\psi(\theta)$ (noting that under normality and homoscedasticity - $E[u_j^1] = E[u_j] = 0$ for odd $j$; and $E[u_j^1] = E[u_j] = 1 \cdot 3 \cdots (j-1)\sigma^j$ for even $j$) we obtain

$$I = E[\psi(\theta)] = \begin{bmatrix}
\frac{X'X}{\sigma^2} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{(2\sigma^4)} & 0 & \frac{3}{(2\sigma^2)} & \frac{1'Z^*}{(2N\sigma^4)} \\
0 & 0 & \frac{2}{(3\sigma^2)} & 0 & 0 \\
0 & \frac{3}{(2\sigma^2)} & 0 & 6 & \frac{3(1'Z^*)}{(2N\sigma^2)} \\
0 & \frac{Z^*1}{(2N\sigma^4)} & 0 & \frac{3Z^*1}{(2N\sigma^2)} & \frac{Z^*Z^*}{(2N\sigma^4)}
\end{bmatrix},$$

where $1'$ is a 1 by $N$ vector of ones.

After evaluating $I$ at $\theta = \hat{\theta}$ (i.e., setting $\sigma^2 = \hat{\sigma}^2 = \hat{\mu}_2$) we obtain

$$\hat{I}_{22}^{-1} = \begin{bmatrix}
\frac{3\hat{\mu}_2/2}{N} & 0 & 0 \\
0 & 2/3 & 0 \\
0 & 0 & (Z^*M^*Z)^{-1}(2N\hat{\mu}_2)
\end{bmatrix}^{-1} = \begin{bmatrix}
3\hat{\mu}_2/2 & 0 & 0 \\
0 & 2/3 & 0 \\
0 & 0 & (Z^*M^*Z)^{-1}(2N\hat{\mu}_2)
\end{bmatrix}^{-1} \hat{I}_{22}^{-1}, \quad (D.3)$$

where $M = I - 1(1'1)^{-1}1'$. Substituting (D.2) and (D.3) into (4.1) we obtain our result. \qed
Appendix E

Tables Containing Results of Simulation Study of Section 5.6
TABLE 5.13

Estimated power using 250 replications

(K=2) (Regressors: constant; \( \chi^2 \)) (N=50)

<table>
<thead>
<tr>
<th></th>
<th>( \hat{w} )</th>
<th>( \hat{w} )</th>
<th>( \hat{w} )</th>
<th>( \hat{w} )</th>
<th>( \hat{w} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{N}H: )</td>
<td>.100</td>
<td>.100</td>
<td>.100</td>
<td>.100</td>
<td>.100</td>
</tr>
<tr>
<td>( \hat{N}H: ) t</td>
<td>.488</td>
<td>.488</td>
<td>.328*</td>
<td>.172*</td>
<td>.268*</td>
</tr>
<tr>
<td>( \hat{N}H: ) Be</td>
<td>.200</td>
<td>.200</td>
<td>.040*</td>
<td>.072*</td>
<td>.112*</td>
</tr>
<tr>
<td>( \hat{N}H: ) Log</td>
<td>1.000</td>
<td>1.000</td>
<td>.928*</td>
<td>.560*</td>
<td>.712*</td>
</tr>
<tr>
<td>( \hat{N}H: ) Ga</td>
<td>.992</td>
<td>.996</td>
<td>.480*</td>
<td>.288*</td>
<td>.356*</td>
</tr>
<tr>
<td>( \hat{N}H: ) H1</td>
<td>.192*</td>
<td>.180*</td>
<td>.564</td>
<td>.088*</td>
<td>.168*</td>
</tr>
<tr>
<td>( \hat{N}H: ) H2</td>
<td>.252*</td>
<td>.224*</td>
<td>.768</td>
<td>.096*</td>
<td>.176*</td>
</tr>
<tr>
<td>( \hat{N}H: ) H3</td>
<td>.276*</td>
<td>.264*</td>
<td>.180*</td>
<td>.788</td>
<td>.180*</td>
</tr>
<tr>
<td>( \hat{N}H: ) H4</td>
<td>.176*</td>
<td>.164*</td>
<td>.160*</td>
<td>.088*</td>
<td>.596</td>
</tr>
<tr>
<td>( \hat{N}H: ) H5</td>
<td>.152*</td>
<td>.148*</td>
<td>.124*</td>
<td>.076*</td>
<td>.152*</td>
</tr>
<tr>
<td>( \hat{N}H: ) Be, H1</td>
<td>.592</td>
<td>.580</td>
<td>.636</td>
<td>.196*</td>
<td>.256*</td>
</tr>
<tr>
<td>( \hat{N}H: ) Be, H2</td>
<td>.640</td>
<td>.612</td>
<td>.764</td>
<td>.180*</td>
<td>.260*</td>
</tr>
<tr>
<td>( \hat{N}H: ) Be, H3</td>
<td>.700</td>
<td>.708</td>
<td>.368*</td>
<td>.684</td>
<td>.320*</td>
</tr>
<tr>
<td>( \hat{N}H: ) Be, H4</td>
<td>.596</td>
<td>.568</td>
<td>.368*</td>
<td>.212*</td>
<td>.648</td>
</tr>
<tr>
<td>( \hat{N}H: ) Be, H5</td>
<td>.564</td>
<td>.568</td>
<td>.316*</td>
<td>.188*</td>
<td>.240*</td>
</tr>
<tr>
<td>( \hat{N}H: ) Log, H1</td>
<td>.180</td>
<td>.176</td>
<td>.520</td>
<td>.040*</td>
<td>.100*</td>
</tr>
<tr>
<td>( \hat{N}H: ) Log, H2</td>
<td>.172</td>
<td>.184</td>
<td>.824</td>
<td>.060*</td>
<td>.100*</td>
</tr>
<tr>
<td>( \hat{N}H: ) Log, H3</td>
<td>.256</td>
<td>.216</td>
<td>.072*</td>
<td>.808</td>
<td>.160*</td>
</tr>
<tr>
<td>( \hat{N}H: ) Log, H4</td>
<td>.188</td>
<td>.172</td>
<td>.112*</td>
<td>.080*</td>
<td>.632</td>
</tr>
<tr>
<td>( \hat{N}H: ) Log, H5</td>
<td>.184</td>
<td>.172</td>
<td>.084*</td>
<td>.072*</td>
<td>.140*</td>
</tr>
<tr>
<td>( \hat{N}H: ) Ga, H1</td>
<td>1.000</td>
<td>1.000</td>
<td>.912</td>
<td>.548*</td>
<td>.672*</td>
</tr>
<tr>
<td>( \hat{N}H: ) Ga, H2</td>
<td>1.000</td>
<td>1.000</td>
<td>.900</td>
<td>.544*</td>
<td>.664*</td>
</tr>
<tr>
<td>( \hat{N}H: ) Ga, H3</td>
<td>1.000</td>
<td>1.000</td>
<td>.908*</td>
<td>.660</td>
<td>.692*</td>
</tr>
<tr>
<td>( \hat{N}H: ) Ga, H4</td>
<td>1.000</td>
<td>1.000</td>
<td>.912*</td>
<td>.548*</td>
<td>.696</td>
</tr>
<tr>
<td>( \hat{N}H: ) Ga, H5</td>
<td>1.000</td>
<td>1.000</td>
<td>.912*</td>
<td>.548*</td>
<td>.700*</td>
</tr>
</tbody>
</table>


**TABLE 5.14**

Estimated power using 250 replications

(K=2) (Regression: constant; normal) (N=50)

<table>
<thead>
<tr>
<th></th>
<th>(L_{M_N}^\prime)</th>
<th>(W'^\prime)</th>
<th>(L_{M_G})</th>
<th>(L_{M_{X2}})</th>
<th>(L_{M_{X3}})</th>
<th>(GQ_{X4})</th>
</tr>
</thead>
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<tr>
<td><strong>NH:</strong></td>
<td>.100</td>
<td>.100</td>
<td>.100</td>
<td>.100</td>
<td>.100</td>
<td>.100</td>
</tr>
<tr>
<td><strong>One-directional</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\overline{NH}: t)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(Be)</td>
<td>.232</td>
<td>.217</td>
<td>.036*</td>
<td>.088*</td>
<td>.100*</td>
<td>.024*</td>
</tr>
<tr>
<td>(Log)</td>
<td>1.000</td>
<td>1.000</td>
<td>.924*</td>
<td>.604*</td>
<td>.700*</td>
<td>.392*</td>
</tr>
<tr>
<td>(Ga)</td>
<td>1.000</td>
<td>1.000</td>
<td>.436*</td>
<td>.312*</td>
<td>.388*</td>
<td>.184*</td>
</tr>
<tr>
<td>(NH: H_1)</td>
<td>.216*</td>
<td>.180*</td>
<td>.572</td>
<td>.088*</td>
<td>.164*</td>
<td>.044*</td>
</tr>
<tr>
<td>(H_2)</td>
<td>.280*</td>
<td>.256*</td>
<td>.800</td>
<td>.104*</td>
<td>.164*</td>
<td>.064*</td>
</tr>
<tr>
<td>(H_3)</td>
<td>.264*</td>
<td>.232*</td>
<td>.136*</td>
<td>.780</td>
<td>.152*</td>
<td>.016*</td>
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### TABLE 5.15

Estimated power using 250 replications

(K=3) (Regressors: constant; normal; uniform) (N=50)

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TABLE 5.16

Estimated power using 250 replications
(K=3) (Regressors: constant; normal; $\chi^2$) (N=50)

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**TABLE 5.17**

Estimated power using 250 replications

(K=2) (Regressors: constant; $\chi^2$) (N=50)

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TABLE 5.18

Estimated power using 250 replications
(K=2) (Regressors: constant; normal) (N=50)

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One-directional

| $NH$: $t$ | .468      | .488       | .072       | .476        | .188 | .516        |
| $Be$     | .044      | .152       | .108       | .116        | .084 | .080        |
| $Log$    | 1.000     | 1.000      | .088       | 1.000       | .660 | 1.000       |
| $Ga$     | .956      | .964       | .096       | .992        | .304 | .860        |

| $NH$: $H_1$ | .484     | .192*      | .072*      | .232*       | .116 | .228        |
| $H_2$     | .708     | .252*      | .056*      | .264*       | .112 | .324        |
| $H_3$     | .224*    | .716       | .704       | .256*       | .428 | .316        |
| $H_4$     | .180*    | .168*      | .056*      | .452        | .096 | .192        |
| $H_5$     | .184*    | .184*      | .056*      | .172*       | .112 | .204        |

Two-directional

| $NH$: $t$, $H_1$ | .732     | .584       | .056       | .572        | .204 | .660        |
| $t$, $H_2$       | .812     | .644       | .056       | .612        | .200 | .728        |
| $t$, $H_3$       | .640     | .808       | .476       | .636        | .576 | .728        |
| $t$, $H_4$       | .548     | .592       | .056       | .708        | .208 | .640        |
| $t$, $H_5$       | .536     | .552       | .052       | .528        | .220 | .628        |

| $Be$, $H_1$     | .484     | .160       | .064       | .112        | .104 | .080        |
| $Be$, $H_2$     | .696     | .164       | .056       | .128        | .120 | .100        |
| $Be$, $H_3$     | .124     | .776       | .812       | .196        | .756 | .160        |
| $Be$, $H_4$     | .120     | .156       | .076       | .504        | .084 | .032        |
| $Be$, $H_5$     | .092     | .148       | .076       | .124        | .104 | .084        |

| $Log$, $H_1$    | 1.000    | 1.000      | .068       | 1.000       | .696 | .996        |
| $Log$, $H_2$    | 1.000    | 1.000      | .072       | 1.000       | .700 | .992        |
| $Log$, $H_3$    | 1.000    | 1.000      | .140       | 1.000       | .744 | 1.000       |
| $Log$, $H_4$    | .996     | 1.000      | .084       | 1.000       | .676 | .996        |
| $Log$, $H_5$    | 1.000    | 1.000      | .056       | 1.000       | .668 | 1.000       |

| $Ga$, $H_1$     | .980     | .960       | .072       | .972        | .316 | .872        |
| $Ga$, $H_2$     | .984     | .968       | .072       | .972        | .316 | .896        |
| $Ga$, $H_3$     | .968     | .992       | .356       | .984        | .580 | .896        |
| $Ga$, $H_4$     | .936     | .964       | .080       | .976        | .336 | .848        |
| $Ga$, $H_5$     | .920     | .956       | .084       | .948        | .328 | .856        |
TABLE 5.19

Estimated power using 250 replications

(K=3) (Regressors: constant; normal; uniform) (N=50)

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<td>.544</td>
<td>.892</td>
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<td>.980</td>
<td>.364</td>
<td>.988</td>
<td>.748</td>
<td>.920</td>
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<td>.964</td>
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<td>.856</td>
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<td>$Ga$, $H_5$</td>
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<td>.944</td>
<td>.084</td>
<td>.952</td>
<td>.596</td>
<td>.860</td>
</tr>
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</table>
APPENDIX F

DERIVATIONS OF SECTIONS 6.2-6.6

PROPOSITION 1: The Maximum Likelihood Estimators of $\beta$ and $\sigma^2$ for the Truncated model satisfy equations (6.4) and (6.5).

Proof:

Using

$$\frac{\partial F_i}{\partial \beta} = f_i x_i \quad (F.1)$$

and

$$\frac{\partial F_i}{\partial \sigma^2} = -\frac{1}{2\sigma^2} x_i' \beta f_i \quad (F.2)$$

we obtain

$$\frac{\partial \ell(\theta)}{\partial \beta} = -\frac{N}{\sigma^2} \sum_{i=1}^{N} \frac{x_i' \beta f_i}{f_i} + \frac{1}{\sigma^2} \sum_{i=1}^{N} (y_i - x_i' \beta) x_i = 0 \quad (F.3)$$

and

$$\frac{\partial \ell(\theta)}{\partial \sigma^2} = -\frac{N}{2\sigma^2} + \frac{1}{2\sigma^2} \sum_{i=1}^{N} \left( x_i' \beta f_i \right)^2 + \frac{1}{2\sigma^4} \sum_{i=1}^{N} \left( y_i - x_i' \beta \right)^2 = 0 \quad (F.4)$$
where \( \ell(\theta) \) is given in (6.3). Premultiplication of (F.3) by \( \beta'/(2\sigma^2) \) gives

\[
- \frac{N}{\sigma^2} \sum_{i=1}^{N} x_i' \beta + \frac{1}{2\sigma^4} \sum_{i=1}^{N} (y_i - x_i' \beta)x_i' \beta = 0 . \tag{F.5}
\]

Adding (F.4) and (F.5), and replacing \( \beta \) and \( \sigma^2 \) by \( \hat{\beta} \) and \( \hat{\sigma}^2 \), we obtain equation (6.4). Premultiplying (F.3) by \( \sigma \) we obtain

\[
- \sum_{i=1}^{N} x_i' \beta + \frac{1}{\sigma} \sum_{i=1}^{N} (y_i - x_i' \beta)x_i = 0 . \tag{F.6}
\]

Replacing \( \beta \) and \( \sigma^2 \) in (F.6) by \( \hat{\beta} \) and \( \hat{\sigma}^2 \), and using matrix notation we have

\[
- X'A + \frac{1}{\sigma} X'(y - X\hat{\beta}) = 0 .
\]

Equation (6.5) readily follows from this last expression. \( \square \)

**PROPOSITION 2:** The variance-covariance matrix of the MLE of \( \beta \) and \( \sigma^2 \) for the Truncated model may be estimated by \( V^{-1} \), where \( V \) is given in equation (6.6).

**Proof:**

We can proceed as in Amemiya (1973) to show that, for finite samples, the density of \( (\hat{\beta}', \hat{\sigma}^2)' \) may be approximated by \( N((\beta', \sigma^2)', \Delta^{-1}) \), where

\[
\Delta = -E\begin{bmatrix}
\frac{\partial^2 \ell(\theta)}{\partial \beta \partial \beta'} & \frac{\partial^2 \ell(\theta)}{\partial \beta \partial \sigma^2} \\
\frac{\partial^2 \ell(\theta)}{\partial \sigma^2 \partial \beta'} & \frac{\partial^2 \ell(\theta)}{\partial \sigma^2 \partial \sigma^2}
\end{bmatrix},
\]
and \( \ell(\theta) \) is given by equation (6.3). Using (F.1) and (F.2), and

\[
\frac{\partial f_i}{\partial \beta} = -\frac{1}{\sigma^2} x_i^2 \beta f_i x_i,
\]

and

\[
\frac{\partial f_i}{\partial \sigma^2} = \frac{1}{2\sigma^2} 
\left[ \frac{(x_i^2 - \bar{x}^2)}{2\sigma^4} \right] f_i,
\]

we can show that, in this case,

\[
-3 \frac{\partial^2 \ell(\theta)}{\partial \beta \partial \beta'} = \sum_{i=1}^{N} \left\{ \frac{1}{\sigma^2} - \frac{F_i(x_i^2 \beta^2)}{\sigma^2 F_i} \right\} x_i^2 x_i,
\]

\[
-3 \frac{\partial^2 \ell(\theta)}{\partial \beta \partial \sigma^2} = \frac{1}{2\sigma^2} \sum_{i=1}^{N} \left[ \frac{1}{\sigma^2} F_i(x_i^2 \beta^2) - F_i + (x_i^2 \beta f_i) \right] x_i + \frac{1}{\sigma^4} \sum_{i=1}^{N} (y_i - x_i^2 \beta) x_i
\]

and

\[
-3 \frac{\partial^2 \ell(\theta)}{\partial \sigma^2 \partial \sigma^2} = -\frac{N}{2\sigma^4} \left\{ \frac{1}{4\sigma^4} \sum_{i=1}^{N} \left[ \frac{F_i(x_i^2 \beta^2)^3}{\sigma^2} - 3F_i(x_i^2 \beta) + (x_i^2 \beta^2 f_i) \right] \right\} + \frac{1}{\sigma^6} \sum_{i=1}^{N} (y_i - x_i^2 \beta)^2.
\]

Taking expectations in (F.9) and (F.10); noting \( E[u_i] = \sigma^2 f_i / F_i \) and \( E[u_i^2] = \sigma^2 (1-(x_i^2 \beta f_i / F_i)) \); and evaluating all the resulting quantities at \( \hat{\beta} \) and \( \hat{\sigma^2} \), we obtain our result. \( \square \)
PROPOSITION 3: The LM normality test statistic for the Truncated model, $LM_{NH(TRUN)}$, is given by equations (6.9) to (6.11).

Proof:

Note that $LM_{NH(TRUN)}$ is a particular case of the test statistic $LM_{NH(TRUN)}$ derived in Proposition 5. To obtain $LM_{NH(TRUN)}$ delete, in the $\hat{d}_2$ vector that defines $LM_{NH(TRUN)}$, the row corresponding to $\alpha^*$; and, in $\hat{I}$, the last row and the last column [see equations (6.16) and (6.17)]. □

PROPOSITION 4: The LM homoscedasticity test statistic for the Truncated model, $LM_{H(TRUN)}$, is given by equation (6.14).

Proof:

Note that $LM_{H(TRUN)}$ is a particular case of the test statistic $LM_{NH(TRUN)}$ derived in Proposition 5. To obtain $LM_{H(TRUN)}$ from $LM_{NH(TRUN)}$ delete, in $\hat{d}_2$ given in (6.16), the first two rows; and the third and fourth row and column in $\hat{I}$ given in (6.17). □

PROPOSITION 5: The LM normality and homoscedasticity test statistic for the Truncated model, $LM_{NH(TRUN)}$, is given by equations (6.15)–(6.17).

Proof:

We first consider a random variable $u_1^*$ with probability density function (p.d.f.)

$$p.d.f.(u_1^*) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{u_1^* - \mu^*_1}{2\sigma^2} \right) \text{ for } u_1^* \geq -x_1^* \beta,$$

where
\[ F_1 = F(x_1', \sigma^2) = \int_{-x_1'}^{\infty} \frac{1}{\sqrt{2\pi}\sigma^2} \exp\left(-\frac{u_1^2}{2\sigma^2}\right) du_1. \]

We note that - using integration by parts - the first eight moments of \( u_1^* \) (about the origin) are:

\[ E[u_1^*] = \sigma^2 \frac{f_1}{F_1} \]

\[ E[u_1^{*2}] = \sigma^2 \left[ 1 - (x_1')^2 \frac{f_1}{F_1} \right] \]

\[ E[u_1^{*3}] = \sigma^2 \frac{f_1}{F_1} \left[ 2\sigma^2 + (x_1')^2 \right] \]

\[ E[u_1^{*4}] = \sigma^2 \left[ 3\sigma^2 - 3\sigma^2(x_1')^2 \frac{f_1}{F_1} - (x_1')^3 \frac{f_1}{F_1} \right] \]

\[ E[u_1^{*5}] = \sigma^2 \frac{f_1}{F_1} \left[ 8\sigma^4 + 4\sigma^2(x_1')^2 + (x_1')^4 \right] \]

\[ E[u_1^{*6}] = \sigma^2 \left[ 15\sigma^6 - 15\sigma^4(x_1')^2 \frac{f_1}{F_1} - 5\sigma^2(x_1')^3 \frac{f_1}{F_1} - (x_1')^5 \frac{f_1}{F_1} \right] \]

\[ E[u_1^{*7}] = \sigma^2 \frac{f_1}{F_1} \left[ 48\sigma^6 + 24\sigma^4(x_1')^2 + 6\sigma^2(x_1')^4 + (x_1')^6 \right] \]

\[ E[u_1^{*8}] = \sigma^2 \left[ 105\sigma^6 - 105\sigma^6(x_1')^2 \frac{f_1}{F_1} - 35\sigma^4(x_1')^3 \frac{f_1}{F_1} \right. \\

\[ - 7\sigma^2(x_1')^5 \frac{f_1}{F_1} - (x_1')^7 \frac{f_1}{F_1} \] .

Define

\[ \phi(\theta, u_1) = \frac{c_{1-u_1}}{\sigma^2 + z_1^{*2}a^*-c_{1-u_1}+c_2u_1^2} du_1, \]
Then the log-likelihood for the i'th observation of the Truncated-Pearson-Heteroscedastic model can be written as

\[ l_i(\theta) = -\log[G_i] + \phi(\theta, u_i) \]

We can show that

\[ \frac{\partial l_i(\theta)}{\partial \beta} = -\frac{1}{G_i} x_1 \exp[\phi(\theta, x_1, \beta)] + x_1 \int \frac{v_i - (c_1 - u_i)(c_1 - 2c_2 u_i)}{v_i^2} du_i \]

\[ \frac{\partial l_i(\theta)}{\partial \sigma^2} = -\frac{1}{G_i} \int_{-x_1, \beta}^{\infty} \exp[\phi(\theta, u_i)] \left[ \int \frac{-(c_1 - u_i)}{v_i^2} du_i \right] du_i \]

\[ + \int \frac{-(c_1 - u_i)}{v_i^2} du_i \]

\[ \frac{\partial l_i(\theta)}{\partial c_1} = -\frac{1}{G_i} \int_{-x_1, \beta}^{\infty} \exp[\phi(\theta, u_i)] \left[ \int \frac{v_i - (c_1 - u_i)(-u_i)}{v_i^2} du_i \right] du_i \]

\[ + \int \frac{v_i - (c_1 - u_i)(-u_i)}{v_i^2} du_i \]

\[ \frac{\partial l_i(\theta)}{\partial c_2} = -\frac{1}{G_i} \int_{-x_1, \beta}^{\infty} \exp[\phi(\theta, u_i)] \left[ \int \frac{-(c_1 - u_i)u_i^2}{v_i^2} du_i \right] du_i \]

\[ + \int \frac{-(c_1 - u_i)u_i^2}{v_i^2} du_i \]
\[
\frac{\partial \xi_i(\theta)}{\partial \alpha} = - \frac{z^*}{G_i} \int_{-x_i^\beta}^{\infty} \exp[\phi(\theta, u_i)] \left[ \int \frac{-(c_1-u_i)}{v_i^2} \, du_i \right] \, du_i
\]

\[
+ z^* \int \frac{-(c_1-u_i)}{v_i^2} \, du_i
\]

Setting \(c_1 = c_2 = 0\) and \(\alpha^* = 0\) in the above expressions, we obtain [note, in particular, that in this case \(G_1\) reduces to \(F_1/\sqrt{2\pi\sigma^2}\)]

\[
\frac{\partial \xi_i(\theta)}{\partial \beta} = - \frac{x_i^d}{F_1} + \frac{x_i u_i}{\sigma^2}
\]

\[
\frac{\partial \xi_i(\theta)}{\partial \sigma^2} = - \int_{-x_i^\beta}^{\infty} \frac{1}{F_1 \sqrt{2\pi\sigma^2}} \exp \left( -\frac{u_i^2}{2\sigma^2} \right) \left[ \frac{u_i^2}{2\sigma^4} \right] \, du_i + \frac{u_i^2}{2\sigma^4}
\]

\[
\frac{\partial \xi_i(\theta)}{\partial c_1} = - \int_{-x_i^\beta}^{\infty} \frac{1}{F_1 \sqrt{2\pi\sigma^2}} \exp \left( -\frac{u_i^2}{2\sigma^2} \right) \left[ \frac{u_i^3}{2\sigma^4} - \frac{u_i^4}{3\sigma^4} \right] \, du_i + \frac{u_i^4}{2\sigma^4}
\]

\[
\frac{\partial \xi_i(\theta)}{\partial c_2} = - \int_{-x_i^\beta}^{\infty} \frac{1}{F_1 \sqrt{2\pi\sigma^2}} \exp \left( -\frac{u_i^2}{2\sigma^2} \right) \left[ \frac{u_i^4}{4\sigma^4} \right] \, du_i + \frac{u_i^4}{4\sigma^4}
\]

\[
\frac{\partial \xi_i(\theta)}{\partial \alpha^*} = - \int_{-x_i^\beta}^{\infty} \frac{1}{F_1 \sqrt{2\pi\sigma^2}} \exp \left( -\frac{u_i^2}{2\sigma^2} \right) \left[ \frac{u_i^2}{2\sigma^4} z^*_i \right] \, du_i + \frac{u_i^2}{2\sigma^4} z^*_i
\]

The integrals in these expressions are moments of a truncated normal random variable and, hence, we can use the first four equations in (F.11) to obtain
\[ \frac{\partial \mathcal{L}_1(\theta)}{\partial \beta} = \frac{x_i}{\sigma^2} (u_i - \mathbb{E}[u_i]) \]

\[ \frac{\partial \mathcal{L}_1(\theta)}{\partial \sigma^2} = \frac{1}{2\sigma^4} (u_i^2 - \mathbb{E}[u_i^2]) \]

\[ \frac{\partial \mathcal{L}_1(\theta)}{\partial c_1} = \frac{1}{\sigma^2} (u_i - \mathbb{E}[u_i]) - \frac{1}{3\sigma^4} (u_i^3 - \mathbb{E}[u_i^3]) \quad (F.12) \]

\[ \frac{\partial \mathcal{L}_1(\theta)}{\partial c_2} = \frac{1}{4\sigma^4} (u_i^4 - \mathbb{E}[u_i^4]) \]

\[ \frac{\partial \mathcal{L}_1(\theta)}{\partial \alpha*} = \frac{1}{2\sigma^4} (u_i^2 - \mathbb{E}[u_i^2]) \alpha_i^* . \]

We obtain (6.16) by adding (F.12) from \( i = 1 \) to \( N \); evaluating the resulting quantities at the MLE of \( \theta \) under

\( H_0: c_1 = c_2 = 0, \alpha* = 0 \); and using \( \hat{\mu}_j = \frac{1}{N} \sum_{i=1}^{N} u_i^{(j)} / N, \)

\( \hat{\nu}(j) = \frac{1}{N} \sum_{i=1}^{N} u_i^{(j)} / N, \hat{u}_i = y_i - x_i \hat{\beta} \) and \( \hat{u}_i^{(j)} \) to denote \( \mathbb{E}[u_i^{(j)}] \)

evaluated at \( \beta = \hat{\beta} \) and \( \sigma^2 = \hat{\sigma}^2 \).

Using (F.12) to compute

\[ I = \mathbb{E}[\psi(\theta)] = \sum_{i=1}^{N} \mathbb{E}[(\partial \mathcal{L}_1(\theta)/\partial \theta)(\partial \mathcal{L}_1(\theta)/\partial \theta)'] \]

we obtain

\[ I = \sum_{i=1}^{N} \begin{bmatrix}
\psi_{11i} & \psi_{12i} & \psi_{13i} & \psi_{14i} & \psi_{15i} \\
\psi_{12i} & \psi_{22i} & \psi_{23i} & \psi_{24i} & \psi_{25i} \\
\psi_{13i} & \psi_{23i} & \psi_{33i} & \psi_{34i} & \psi_{35i} \\
\psi_{14i} & \psi_{24i} & \psi_{34i} & \psi_{44i} & \psi_{45i} \\
\psi_{15i} & \psi_{25i} & \psi_{35i} & \psi_{45i} & \psi_{55i}
\end{bmatrix}, \]

with
\[ \psi_{111} = \frac{1}{\sigma^4} x_1 x_1'(E_{12} - E_{11}) \]

\[ \psi_{121} = \frac{x_1}{2\sigma^6} (E_{13} - E_{11} E_{12}) \]

\[ \psi_{131} = \frac{x_1}{\sigma^4} (E_{12} - E_{11}^2) - \frac{x_1}{3\sigma} (E_{14} - E_{11} E_{13}) \]

\[ \psi_{141} = \frac{x_1}{4\sigma^6} (E_{15} - E_{11} E_{14}) \]

\[ \psi_{151} = \frac{z^*}{2\sigma^6} (E_{13} - E_{11} E_{12}) \]

\[ \psi_{221} = \frac{1}{8\sigma^8} (E_{14} - E_{12}^2) \]

\[ \psi_{231} = \frac{1}{2\sigma^6} (E_{13} - E_{12} E_{11}) - \frac{1}{6\sigma^8} (E_{15} - E_{13} E_{12}) \]

\[ \psi_{241} = \frac{1}{8\sigma^8} (E_{16} - E_{14} E_{12}) \]

\[ \psi_{251} = \frac{z^*}{4\sigma^8} (E_{14} - E_{12}^2) \]

\[ \psi_{331} = \frac{1}{\sigma^4} (E_{12} - E_{11}^2) + \frac{1}{9\sigma^8} (E_{16} - E_{13}^2) - \frac{2}{3\sigma^6} (E_{14} - E_{11} E_{13}) \]

\[ \psi_{341} = \frac{1}{4\sigma^6} (E_{15} - E_{11} E_{14}) - \frac{1}{12\sigma^8} (E_{17} - E_{13} E_{14}) \]

\[ \psi_{351} = \frac{z^*}{2\sigma^6} (E_{13} - E_{11} E_{12}) - \frac{z^*}{6\sigma^8} (E_{15} - E_{12} E_{13}) \]

\[ \psi_{441} = \frac{1}{16\sigma^8} (E_{18} - E_{14}^2) \]
PROPOSITION 6: The LM normality and homoscedasticity test statistic for the Tobit model, $\text{LM}_{\text{NH(TOBIT)}}$, is given by equations (6.26)-(6.28).

Proof:

Define

$$\phi(\theta,u_i) = \int \frac{c_1-u_i}{\sigma^2+z^*\alpha^*c_1u_1+c_2u_1^2} \, du_i$$

$$G_i = \int_{-\infty}^{\infty} \exp[\phi(\theta,u_i)] \, du_i$$

$$B_i = \int_{-\infty}^{\infty} \exp[\phi(\theta,u_i)] \, du_i$$

$$A_i = G_i/B_i$$

$$D_i = -\log[B_i] + \phi(\theta,u_i)$$

$$v_i = \sigma^2 + z^*\alpha^* - c_1u_1 + c_2u_1^2, \quad \theta_1 = (\beta', \sigma^2)', \quad \theta_2 = (c_1, c_2, \alpha^*)'$$

and

$$\theta = (\theta_1', \theta_2').$$

Then the log-likelihood for the $i$'th observation of the Tobit-Heteroscedastic model can be written as (see (6.25))

$$\ell_i(\theta) = \omega_i D_i + (1-\omega_i) \log(1-A_i).$$
We want to find $\frac{\partial \ell_1(\theta)}{\partial \theta}$. For this we shall first concentrate on the term $\log(1-A_1)$. We can show that

$$\frac{\partial \log(1-A_1)}{\partial \beta} = -\frac{x_1}{1-A_1} \left\{ \frac{1}{B_1} \exp[\phi(\theta, x_1^i \beta)] - \frac{G_1}{B_1^2} \int_{-\infty}^{\infty} \exp[\phi(\theta, u_1)] \left[ \frac{v_1^{-1}(c_1-u_1)(c_1-2c_2u_1)}{v_1^2} du_1 \right] du_1 \right\}$$

$$\frac{\partial \log(1-A_1)}{\partial \sigma^2} = -\frac{1}{1-A_1} \left\{ \frac{1}{B_1} \int_{-\infty}^{\infty} \exp[\phi(\theta, u_1)] \left[ \frac{v_1^{-1}(c_1-u_1)}{v_1^2} du_1 \right] du_1 \right\}$$

$$\frac{\partial \log(1-A_1)}{\partial c_1} = -\frac{1}{1-A_1} \left\{ \frac{1}{B_1} \int_{-\infty}^{\infty} \exp[\phi(\theta, u_1)] \left[ \frac{v_1^{-1}(c_1-u_1)(u_1)}{v_1^2} du_1 \right] du_1 \right\}$$

$$\frac{\partial \log(1-A_1)}{\partial c_2} = -\frac{1}{1-A_1} \left\{ \frac{1}{B_1} \int_{-\infty}^{\infty} \exp[\phi(\theta, u_1)] \left[ \frac{v_1^{-1}(c_1-u_1)u_1^2}{v_1^2} du_1 \right] du_1 \right\}$$
\[
\frac{\partial \log(1-A_i)}{\partial \alpha^k} = - \frac{z_i^k}{1-A_i} \left\{ \frac{1}{B_i} \int_{-\infty}^{\infty} \exp[\phi(\theta,u_i)] \left[ -\frac{(c_i-u_i)}{v_i^2} \right] du_i \right\} + \frac{G_i}{B_i^2} \left\{ \frac{(1-\omega_i)}{\sigma^2} \right\}.
\]

We note that \( \partial \log(1-A_i)/\partial \theta = \omega_1 (\partial \pi_1/\partial \theta) + (1-\omega_1)(\partial \log(1-A_i)/\partial \theta) \). We have already computed \( \partial \log(1-A_i)/\partial \theta \). Also, we may note that \( \partial D_1/\partial \theta \) is given in Appendix D (observe that \( D_1 \) is, in fact, \( \ell_1(\theta) \) of Appendix D). It is easy to see that at \( c_1 = c_2 = 0 \) and \( \alpha^* = 0 \), we have \( B_1 = \sqrt{2\pi \sigma^2} \), \( G_1 = \sqrt{2\pi \sigma^2} F_1 \) and \( A_1 = F_1 \), where \( F_1 = F(x_1^*, \sigma^2) \) (see equation (6.1)). Using these results we may show that, at \( c_1 = c_2 = 0 \) and \( \alpha^* = 0 \), we have

\[
\frac{\partial \pi_1(\theta)}{\partial \beta} = \omega_1 \left\{ \frac{x_i u_i}{\sigma^2} \right\} - \frac{(1-\omega_i)}{1-F_1} \frac{E[u^*_i]}{\sigma^2},
\]

\[
\frac{\partial \pi_1(\theta)}{\partial \sigma^2} = \omega_1 \left\{ \frac{1}{2\sigma^2 + \frac{3}{4}u_i^4} \right\} - \frac{(1-\omega_i)}{1-F_1} \frac{E[u^*_i]}{2\sigma^2} - \frac{E[u^*_i]^2}{2\sigma^4},
\]

\[
\frac{\partial \pi_1(\theta)}{\partial c_1} = \omega_1 \left\{ \frac{u_i^3 - u_i^4}{3\sigma^4} \right\} - \frac{(1-\omega_i)}{1-F_1} \frac{E[u^*_i]}{3\sigma^4},
\]

\[
\frac{\partial \pi_1(\theta)}{\partial c_2} = \omega_1 \left\{ \frac{3}{4} + \frac{u_i^4}{4\sigma^4} \right\} - \frac{(1-\omega_i)}{1-F_1} \frac{3}{4 + \frac{E[u^*_i]}{4\sigma^4}},
\]

\[
\frac{\partial \pi_1(\theta)}{\partial \alpha^k} = \omega_1 \left\{ \frac{z_i^k}{2\sigma^2} + \frac{z_i^k u_i^2}{2\sigma^4} \right\} - \frac{(1-\omega_i)}{1-F_1} \frac{z_i^k}{2\sigma^2} - \frac{z_i^k E[u^*_i]}{2\sigma^4},
\]

\[(F.13)\]
where $E[u_{ij}^2]$ is given in (F.11).

We obtain (6.27) by adding, in turn, $\partial \ell_i(\theta) / \partial c_1$, $\partial \ell_i(\theta) / \partial c_2$ and $\partial \ell_i(\theta) / \partial \alpha^*$ from $i = 1$ to $N$; using $\omega_1 = \ldots = \omega_N = 1$ and $\omega_{m+1} = \ldots = \omega_N = 0$; and evaluating the resulting quantities at $\beta = \tilde{\beta}$ and $\sigma^2 = \tilde{\sigma}^2$, where $\tilde{\beta}$ and $\tilde{\sigma}^2$ are the solution to equations (6.20) and (6.21). In particular, we use $\tilde{u}_i = y_i - x_i\tilde{\beta}$ and $\tilde{u}_i(i)$ to denote the value of $E[u_{ij}^2]$ evaluated at $\beta = \tilde{\beta}$ and $\sigma^2 = \tilde{\sigma}^2$.

To compute $\hat{I}$ we proceed as follows. We first obtain the matrix $(\partial \ell_i(\theta) / \partial \theta)(\partial \ell_i(\theta) / \partial \theta)'$, noting that, for all $i$, $\omega_i(1-\omega_i) = 0$ (hence, terms involving $\omega_i(1-\omega_i)$ may be set equal to zero). We then take expectations. For this we note that, under $H_0: c_1 = c_2 = 0; \alpha^* = 0$, we have

$$E[\omega_1 u_1^2] = E[\omega_1] = 0 + 0 = 0$$

where $E[u_{ij}^2]$ is given in (F.11). We also have $E[\omega_1] = F_1$. After computing $\sum_{i=1}^N E[(\partial \ell_i(\theta) / \partial \theta)(\partial \ell_i(\theta) / \partial \theta)']$, and replacing $\beta$ and $\sigma^2$ respectively by $\tilde{\beta}$ and $\tilde{\sigma}^2$ (i.e., the solution to equations (6.20) and (6.21)), we obtain $\hat{I}$ as given in equation (6.28). For example, to get $b_{11}$ we use the first equation in (F.13) and obtain

$$
\begin{bmatrix}
\frac{\partial \ell_i(\theta)}{\partial \beta}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \ell_i(\theta)}{\partial \beta}
\end{bmatrix} '
= x_i x_i' \frac{\omega_i^2 u_{1i}^2}{\sigma^4} + x_i x_i' \frac{F_1^2}{(1-F_1)^2} \frac{(1-\omega_i) \frac{E[u_{ij}^2]}{\sigma^4}}{\sigma^4}.
$$

We note $\omega_1^2 = \omega_i$ and $(1-\omega_i)^2 = 1-\omega_i$, and that (using (F.14))
E[\omega^2] = E[u^2]F_i. It follows that

$$\begin{align*}
E \left[ \left( \frac{\partial \lambda_i(\theta)}{\partial \beta} \right) \left( \frac{\partial \lambda_i(\theta)}{\partial \beta} \right) \right] = x_1 x_1' \frac{F_i E[u^2]}{\sigma^4} + x_1 x_1' \frac{F_i^2 (E[u^2])^2}{\sigma^4}.
\end{align*}$$

Adding from \( i = 1 \) to \( N \), and setting \( \beta \) and \( \sigma^2 \) equal to \( \tilde{\beta} \) and \( \tilde{\sigma}^2 \) (i.e., replacing \( E[u^2], F_i \), \( E[u^2] \) and \( \sigma^4 \) respectively by \( \tilde{u}_i(2), \tilde{F}_i, \tilde{u}_i(1) \) and \( \tilde{\sigma}^4 \)) we obtain

$$b_{11} = \frac{1}{\tilde{\sigma}^4} \sum_{i=1}^{N} x_i x_i' \left[ \tilde{F}_i \tilde{u}_i(2) + \frac{\tilde{F}_i^2}{(1-F_i)} (\tilde{u}_i(1)) \right]$$

as in equation (6.28). \( \square \)
Appendix G

Derivations of Section 7.4

**Proposition 1**: The explained variation of the regression of $\beta_{1k}$ on $F_k(z_i)$ is given by $R^2_{(1,k)}$.

**Proof**:

Equation (7.20) defines

$$
\beta_{ik} = F_k(z_i) + \epsilon_{ik},
$$

where

$$
F_k(z_i) = \gamma_{ko} + \gamma_{k'}z_i.
$$

We know $E[\beta_{1k}] = E_1 E_{i/1}[\beta_{1k}]$. From (G.1) we see

$$
E_{i/1}[\beta_{1k}] = \gamma_{ko} + \gamma_{k'}z_i.
$$

It follows that

$$
E[\beta_{1k}] = \gamma_{ko} + \gamma_{k'}z
$$

where $z = \frac{1}{N} \sum_{i=1}^{N} z_i$. We also have
\[ V[\beta_{ik}] = \sum_{i=1}^{N} \varepsilon_{i} \left( \beta_{ik} + E \varepsilon_{i} \right) + \sum_{i=1}^{N} \varepsilon_{i} \varepsilon_{i} \left( \beta_{ik} \right). \]

But \( E \varepsilon_{i} \left( \beta_{ik} \right) \) is \( \gamma_{k0} + \gamma_{k1} z_i \) and \(\varepsilon_{i} \varepsilon_{i} \left( \beta_{ik} \right) \) is \( V[\varepsilon_{ik}] \), reducing \( V[\beta_{ik}] \) to

\[ V[\beta_{ik}] = \gamma_{k0}^2 + \gamma_{k1}^2 + V[\varepsilon_{ik}] \]

where

\[ \Sigma = \sum_{i=1}^{N} (z_i - \bar{z})(z_i - \bar{z})'/N. \]

Therefore, it is seen that the explained variation of the regression of \( \beta_{ik} \) on \( F_k(z_i) \) is simply \( \gamma_{k0}^2 + \gamma_{k1}^2 \) over \( V[\beta_{ik}] \), which is equal to \( R^2_{(1,k)}. \)

**PROPOSITION 2:** The explained variation of the regression of \( \beta_{ik} \) on \( i^S_k(z_i) \) is given by \( R^2_{(2,k)}. \)

**Proof:**

Our regression equation is now

\[ \beta_{ik} = F^S_k(z_i) + \varepsilon_{ik}, \quad \text{(G.3)} \]

where

\[ F^S_k(z_i) = \sum_{h=1}^{L} \beta_k(h) D_{ih}. \]

By definition, \( \beta(h) = \sum_{i \in I_h} F(z_i)/N_h \) (see (7.5)). We have

\[ F_k(z_i) = \gamma_{k0} + \gamma_{k1} z_i, \] so, \( \beta_k(h) = \gamma_{k0} + \gamma_{k1} z_h \), where \( z_h = \sum_{i \in I_h} z_i/N_h. \)
This means we can rewrite equation (G.3) in the form

\[ \beta_{ik} = \sum_{h=1}^{L} (\gamma_{ik} + \gamma_{ik}^2 \bar{z}_h) D_{ih} + v_{ik} \]  \hspace{1cm} (G.4)

Subtracting (G.2) from (G.4) and noting that \( \sum_{h=1}^{L} D_{ih} = 1 \), we obtain

\[ \beta_{ik} - E[\beta_{ik}] = \gamma_{ik} \sum_{h=1}^{L} (\bar{z}_h - \bar{z}) D_{ih} + v_{ik} \]  \hspace{1cm} (G.5)

From (G.5) we can show that the variance of \( \beta_{ik} \) may be written as

\[ V[\beta_{ik}] = \gamma_{ik}^2 B_{ik} + V[v_{ik}] \]  \hspace{1cm} (G.6)

where \( B = \sum_{h=1}^{L} (N_h / N) (\bar{z}_h - \bar{z}) (\bar{z}_h - \bar{z})' \). (To obtain this result note - in particular - that \( D_{ih}^2 = D_{ih} \), since \( D_{ih} \) is either 1 or 0; \( D_{ih} D_{jk} = 0 \) except when \( i = j \) and \( h = k \); and \( E[D_{ih}] = N_h / N \).)

From (G.6) the explained variation of the regression of \( \beta_{ik} \) on \( E_k(z_i) \) is seen to be \( \gamma_{ik}^2 B_{ik} \) over \( V[\beta_{ik}] \), which is \( R^2(2,k) \). \( \Box \)
In Section 8.4 we defined $S'$ as the $n^2p$ by $n(n+1)p/2$ matrix that maps $\theta_2$ into $a = \text{Vec}[A]$, i.e., such that $a = S'\theta_2$. It follows that $\partial(\text{Vec}[A])'/\partial\theta_2 = S$, and from this it may be seen that the general form for $S$ is

$$S = \begin{bmatrix}
I_p\psi(1,0,0,\ldots,0) & 0 & 0 & \ldots & 0 \\
I_p\psi(0,1,0,\ldots,0) & I_p\psi(1,0,0,\ldots,0) & 0 & \ldots & 0 \\
I_p\psi(0,0,1,\ldots,0) & 0 & I_p\psi(1,0,0,\ldots,0) & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
I_p\psi(0,0,0,\ldots,1) & 0 & 0 & \ldots & I_p\psi(1,0,0,\ldots,0)
\end{bmatrix}$$
In the proofs in this Appendix we proceed as did Richard (1975), neglecting initially the 'symmetry constraints' in \( \mathbf{a} = \text{Vec} \mathbf{A} \), and later incorporating these by premultiplying (or postmultiplying) relevant vectors and matrices by \( \mathbf{S} \) (or \( \mathbf{S}' \)), e.g., we use

\[
\frac{3 \log |\Sigma_1^{-1}|}{\partial \theta_2} = \mathbf{S} \left[ \frac{3 \log |\Sigma_1^{-1}|}{\partial \mathbf{a}} \right],
\]

and

\[
\frac{3^2 \log |\Sigma_1^{-1}|}{\partial \theta_2 \partial \theta_2'} = \mathbf{S} \left[ \frac{3^2 \log |\Sigma_1^{-1}|}{\partial \mathbf{a} \partial \mathbf{a}'} \right] \mathbf{S}',
\]

where \( \frac{3 \log |\Sigma_1^{-1}|}{\partial \mathbf{a}} \) and \( \frac{3^2 \log |\Sigma_1^{-1}|}{\partial \mathbf{a} \partial \mathbf{a}'} \) are obtained neglecting the functional dependence in the elements in \( \mathbf{a} \).

**PROPOSITION 1:** If \( \mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3 \) and \( \mathbf{M}_4 \) are suitably dimensioned matrices, then

(i) \( \text{Vec}[\mathbf{M}_1 \mathbf{M}_2 \mathbf{M}_3] = (\mathbf{M}_3^* \otimes \mathbf{M}_1) \text{Vec} \mathbf{M}_2 \)

(for proof see Dhrymes (1978, p.519)) and

(ii) \( (\mathbf{M}_1 \otimes \mathbf{M}_2)(\mathbf{M}_3 \otimes \mathbf{M}_4) = (\mathbf{M}_1 \mathbf{M}_3 \otimes \mathbf{M}_2 \mathbf{M}_4) \)

(for proof see Dhrymes (1978, pp.460-461)).

**PROPOSITION 2:** If \( \Sigma_1 \) is a non-singular matrix having functionally independent elements, then

\[
\frac{3 \log |\Sigma_1^{-1}|}{\partial \text{Vec} \Sigma_1} = \text{Vec} \Sigma_1,
\]

(for proof see Dhrymes (1978, pp.533-534)).
PROPOSITION 3: If $\Sigma_1$ is a non-singular matrix having functionally independent elements, then

$$\frac{\partial \text{Vec } \Sigma_1^{-1}}{\partial \text{Vec } \Sigma_1} = -(\Sigma_1^{-1} \otimes \Sigma_1^{-1})$$

(for proof see Dhrymes (1978, pp. 538-539)). □

PROPOSITION 4: Neglecting the functional dependence of the elements in $a$ it follows that

$$\frac{\partial \text{Vec } \Sigma_1^{-1}}{\partial a} = -[(I_n \otimes z_i) \otimes I_n](\Sigma_1^{-1} \otimes \Sigma_1^{-1})$$

Proof:

Write

$$\frac{\partial \text{Vec } \Sigma_1^{-1}}{\partial a} = \left[\frac{\partial (\text{Vec } \Sigma_1^{-1})'}{\partial a}\right] \frac{\partial (\text{Vec } \Sigma_1^{-1})'}{\partial \text{Vec } \Sigma_1} \cdot \quad (H.1)$$

By definition, $\Sigma_1 = A(I_n \otimes z_i)$. Using proposition 1(i) this may be written as $\text{Vec } \Sigma_1 = [(I_n \otimes z_i') \otimes I_n]a$. We therefore have

$$\frac{\partial (\text{Vec } \Sigma_1)'}{\partial a} = [(I_n \otimes z_i) \otimes I_n].$$

Using this last relation and proposition 3 in (H.1) we obtain our result. □

PROPOSITION 5: $\frac{\partial \log |\Sigma_1^{-1}|}{\partial \theta_2} = -S[(I_n \otimes z_i) \otimes I_n] \text{Vec } [\Sigma_1^{-1}]$. 
Proof:

Write \[ \frac{\partial \log |\Sigma^{-1}|}{\partial \theta_2} = S \left( \frac{\partial (\text{Vec } \Sigma^{-1})'}{\partial a} \right) \frac{\partial \log |\Sigma^{-1}|}{\partial \text{Vec } \Sigma^{-1}} \] and use propositions 4 and 2. Now use proposition 1(i) in reverse order to obtain our result. \( \square \)

**PROPOSITION 6:** If \( \Sigma \) is a non-singular matrix with functionally independent elements, then

\[ \frac{\partial \text{tr}(\Sigma^{-1} u_i u_i')}{{\partial \theta_2}} = \text{Vec}[u_i u_i'] \]

(for proof see Dhrymes (1978, p. 531)). \( \square \)

**PROPOSITION 7:**

\[ \frac{\partial \text{tr}(\Sigma^{-1} u_i u_i')}{{\partial \theta_2}} = -S[(I_n \Theta z_i) \Theta \Sigma^{-1} u_i u_i'] \text{Vec}[\Sigma^{-1}] . \]

Proof:

Write \[ \frac{\partial \text{tr}(\Sigma^{-1} u_i u_i')}{{\partial \theta_2}} = S \left( \frac{\partial (\text{Vec } \Sigma^{-1})'}{\partial a} \right) \frac{\partial \text{tr}(\Sigma^{-1} u_i u_i')}{{\partial \theta_2}} \] and use propositions 4 and 6. Then, using proposition 1(i), note that

\((\Sigma^{-1} \Theta \Sigma^{-1}) \text{Vec}[u_i u_i'] = \text{Vec}[\Sigma^{-1} u_i u_i' \Sigma^{-1}] = [I_n \Theta \Sigma^{-1} u_i u_i'] \text{Vec}[\Sigma^{-1}] . \)

From this we obtain our result by using proposition 1(ii). \( \square \)

**PROPOSITION 8:**

\[ \frac{\partial^2 \log |\Sigma^{-1}|}{{\partial \theta_2 \partial \theta_2'}} = S[\Sigma^{-1} \Theta z_i' \Theta \Sigma^{-1}] S' . \]
Proof:

Using proposition 5 we note that

\[ \frac{\partial^2 \log |\Sigma_i^{-1}|}{\partial \theta_2 \partial \theta_2'} = -S \left[ \frac{\partial (\operatorname{Vec} \Sigma_i^{-1})}{\partial a} \right] \left[ (I_n \otimes z_i') \otimes I_n \right] S'. \]

Our result is obtained by using propositions 4 and 1(ii). \[ \square \]

PROPOSITION 9:

\[ \frac{\partial^2 \operatorname{tr}(\Sigma_i^{-1} u_i u_i^t)}{\partial \theta_2 \partial \theta_2'} = S \left[ \Sigma_i^{-1} \otimes z_i z_i' \otimes \Sigma_i^{-1} u_i u_i' \Sigma_i^{-1} \right] + \]

\[ + [\Sigma_i^{-1} u_i u_i' \Sigma_i^{-1} \otimes z_i z_i' \otimes \Sigma_i^{-1}] S'. \]

Proof:

Using propositions 7 and 1 we note that

\[ \frac{\partial^2 \operatorname{tr}(\Sigma_i^{-1} u_i u_i^t)}{\partial \theta_2 \partial \theta_2'} = -S \left[ \frac{\partial}{\partial a} (\operatorname{Vec} [\Sigma_i^{-1} u_i u_i' \Sigma_i^{-1} (I_n \otimes z_i')]) \right] S'. \]

Again using proposition 1(i) we can write

\[ (\operatorname{Vec} [\Sigma_i^{-1} u_i u_i' \Sigma_i^{-1} (I_n \otimes z_i')])' = (\operatorname{Vec} \Sigma_i^{-1})' [ (I_n \otimes z_i') \otimes u_i u_i' \Sigma_i^{-1} ] \]

and

\[ (\operatorname{Vec} [\Sigma_i^{-1} u_i u_i' \Sigma_i^{-1} (I_n \otimes z_i')])' = (\operatorname{Vec} \Sigma_i^{-1})' [ u_i u_i' \Sigma_i^{-1} (I_n \otimes z_i') \otimes I_n ] . \]

Therefore we have

\[ \frac{\partial^2 \operatorname{tr}(\Sigma_i^{-1} u_i u_i^t)}{\partial \theta_2 \partial \theta_2'} = -S \left[ \frac{\partial (\operatorname{Vec} \Sigma_i^{-1})'}{\partial a} \right] \left[ [ (I_n \otimes z_i') \otimes u_i u_i' \Sigma_i^{-1} ] + \right. \]

\[ + \left. [ u_i u_i' \Sigma_i^{-1} (I_n \otimes z_i') \otimes I_n ] \right] S'. \]

Our result may be easily obtained from this by using propositions 4 and 1(ii). \[ \square \]
PROPOSITION 10: \[ \mathcal{I}_{21} = E \left[ -\frac{\partial^2 \mathcal{L}(\theta)}{\partial \theta_1 \partial \theta_2} \right] = 0. \]

Proof:

Recall that our model is \( B y + \Gamma x = u \) (see equation (8.1)).

Let \( B = -I_n + B_0 \), \( C = (B_0 ; I) \) and (as before) \( w = (y',x') \).

Then we can write \( y = B y + \Gamma x - u \). Using proposition 1(i) this is \( y = (w^T \Theta I_n) Vec C - Vec u \). Let \( S \) denote a selection matrix that selects the non-zero elements in \( C \). Then we can write our model as

\[ y = w^T \Theta_1 - Vec u, \]  

(H.2)

where \( w = (w^T \Theta I_n) S \) and \( \Theta_1 = S^T Vec C \). From our relation for \( \ell(\theta) \) (see equation (8.8)) we note that

\[
\frac{\partial^2 \mathcal{L}(\theta)}{\partial \theta_1 \partial \theta_2} = \frac{1}{2} \sum_{i=1}^{N} \frac{\partial \log |\Sigma^{-1} u_i|}{\partial \theta_i} - \frac{1}{2} \sum_{i=1}^{N} \frac{\partial \text{tr}(\Sigma^{-1} u_i u_i^T)}{\partial \theta_i}.
\]

Using proposition 5 we readily see that the first term will be zero (we are assuming there is no functional relationship between the parameters in \( \Sigma \) and \( \theta \)). Using propositions 7 and 1(i), and noting that

\[ \text{Vec}[u_i u_i^T] = (u_i \Theta I_n) \text{Vec}[u_i] \quad \text{and} \quad \text{Vec}[u_i u_i^T] = (I_n \Theta u_i) \text{Vec}[u_i], \]

we have that

\[
\frac{\partial^2 \mathcal{L}(\theta)}{\partial \theta_1 \partial \theta_2} = \frac{1}{2} \sum_{i=1}^{N} \frac{\partial \text{Vec}[u_i]}{\partial \theta_i} \left[ (I_n \Theta u_i^T) + (u_i \Theta I_n^T) \right] [\Sigma^{-1} \Theta \Sigma^{-1}] [\Sigma^{-1} \Theta^T u_i^T] S'.
\]
From (H.2) it follows that \( \frac{\partial (\text{Vec } u_i)'}{\partial \theta_1} = w_i' \), and given that

\[ E[u_i] = 0 \]

we have that \( I_{21}' = E \left[ \frac{\partial^2 \ell(\theta)}{\partial \theta_1 \partial \theta_2'} \right] = 0. \]
Appendix I

Results of Cluster Analysis Performed for Family Budget Study of Section 9.5

In this Appendix we present the 14 groups of households that resulted when applying our clustering criterion.

Firstly, we present the values on the socioeconomic variables for the households in each group. In every case, the first quantity is a reference number (between 1 and 521) which serves to identify households. Next we have the values of

$Z_1$: Occupation of the Head of the Household
$Z_1 = 0$ if Unemployed
$Z_1 = 3$ if Worker
$Z_1 = 2$ if Entrepreneur
$Z_1 = 1$ if Technocrat

$Z_2$: Family Income (in per capita 1975 pesos per month)

$Z_3$: Family Size

$Z_4$: Age of the Head of the Household (in years).

Secondly, we present the expenditures for the households in each group. In every case, the first quantity is the reference number;
followed by household expenditures on

\[ Y_1: \text{Food} \]
\[ Y_2: \text{Clothing} \]
\[ Y_3: \text{Housing} \]
\[ Y_4: \text{Durables} \]
\[ Y_5: \text{Education} \]
\[ Y_6: \text{Medical Services} \]
\[ Y_7: \text{Other} \]

All these are expressed in per capita 1975 pesos per month.

In the data contained in this Appendix, 25 households are marked with a star (*). These households were excluded in our econometric analysis because of their questionable reliability. For instance, the household with reference number 88 was excluded because it reported expenditures amounting to approximately five times its total income.
### SOIOECONOMIC VARIABLES:

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Appendix J

Tables Containing Results of Family Budget Study of Section 9,5
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Estimated Regressions: Unemployed, Large

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Results using Ordinary Regression Model

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TABLE 9.20

Estimated Regressions: Worker, Large, Young

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<td>36.94* 151.06*</td>
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<td>.083</td>
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<td>\hat{\beta}_2</td>
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TABLE 9.21
Estimated Regressions: Worker, Large, Old

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|                  |       |          |         |          |           |            |       |
| Results using LDV Model |       |          |         |          |           |            |       |
| $\tilde{\beta}_1(LDV)$ | 210.7 | -117.9   | -7.4    | -8.2     | 18.1      | -2.8       | -177.4|
|                  | (6.47) | (-1.89)  | (-.19)  | (-1.41)  | (2.35)    | (-.52)     | (-2.27)|
| $\tilde{\beta}_2(LDV)$ | .151  | .189     | .146    | .024     | -.002     | .011       | .345  |
|                  | (4.34) | (4.67)   | (4.67)  | (4.01)   | (-.28)    | (2.04)     | (6.23)|
| $SD_u$           | 127.9 | 79.4     | 74.7    | 18.6     | 28.9      | 16.8       | 114.6 |
| $LM_N(\cdot)$    | 17.81*| .77      | 1.62    | 688.92*  | 332.54*   | 56.42*     | 1.44  |
| $LM_H(\cdot)$    | 6.72* | 4.63     | .95     | 10.52*   | .27       | 2.11       | 3.45  |
| $LM_{NH}(\cdot)$ | 25.87*| 20.26*   | 3.84    | 720.03*  | 332.86*   | 56.65*     | 16.52*|
| $\tilde{\beta}_1$ | 224.2 | 36.9     | 69.1    | 4.9      | 23.2      | 6.5        | 56.4  |
| $\tilde{\beta}_2$ | .145  | .081     | .097    | .016     | -.001     | .007       | .179  |
TABLE 9.22

Estimated Regressions: Worker, Small, Young

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<th>Other</th>
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Results using LDV Model

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**Table 9.23**

Estimated Regressions: *Entrepreneur, Large*

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<td>.190</td>
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| **Results using LDV Model** |            |            |            |            |            |            |           |
| $\tilde{\beta}_1$ (LDV) | 146.0      | -21.0      | -3013.2    | 14.2       | 21.9       | 1.3        | -2463.8   |
|                | (2.34)     | (-.33)     | (-.86)     | (2.46)     | (1.54)     | (.17)      | (-.77)    |
| $\tilde{\beta}_2$ (LDV) | .177       | .078       | .432       | -.0001     | .001       | -.002      | .475     |
|                | (5.07)     | (3.70)     | (.92)      | (-.42)     | (.13)      | (-.35)     | (1.00)    |
| $SD_u$         | 166.9      | 70.3       | 464.9      | 21.9       | 56.7       | 20.5       | 468.2     |
| $LM_N(\cdot)$ | 25.58*     | .01        | 5.42       | 52.99*     | 639.07*    | 432.02*    | 8.77      |
| $LM_H(\cdot)$ | 3.12       | 2.27       | 11.83*     | 1.12       | 3.78       | 5.91       | 13.71*    |
| $LM_{NH}(\cdot)$ | 30.17*    | 5.24       | 18.19*     | 53.34*     | 643.39*    | 437.03*    | 14.09*    |
| $\hat{\beta}_1$ | 220.4      | 70.3       | 168.4      | 18.3       | 35.6       | 11.9       | 145.9     |
| $\hat{\beta}_2$ | .143       | .038       | .025       | -.001      | .0009      | -.001      | .037      |
## TABLE 9.24

Estimated Regressions: *Entrepreneur, Small*

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<th>Education</th>
<th>Medical S.</th>
<th>Other</th>
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### Results using Ordinary Regression Model

### Results using LDV Model

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TABLE 9.25
Estimated Regressions: Technocrat, Income group II, Large

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<th>Medical S.</th>
<th>Other</th>
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<td>(1.43)</td>
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<td>(-.68)</td>
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<tr>
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<td>.257</td>
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<td>148.07*</td>
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Results using Ordinary Regression Model

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<th>Education</th>
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<th>Other</th>
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<td>(6.14)</td>
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<td>69.17*</td>
<td>505.78*</td>
<td>993.24*</td>
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<td>.32</td>
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<td>7.11</td>
<td>113.91*</td>
<td>72.52*</td>
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TABLE 9.26

Estimated Regressions: Technocrat, Income group II, Small

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<th>Housing</th>
<th>Durables</th>
<th>Education</th>
<th>Medical S.</th>
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TABLE 9.27

Estimated Regressions: Technocrat, Income group I2

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<th>Medical S.</th>
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<td>.053</td>
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<td>(1.06)</td>
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<td>85.64*</td>
<td>157.19*</td>
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<td>171.03*</td>
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<tr>
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<td>.057</td>
<td>.232</td>
<td>.325</td>
<td>.367</td>
<td>.095</td>
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</table>

Results using Ordinary Regression Model

| \( \hat{\beta}_1 \) (LDV) | 465.2   | 1070.1   | 388.7    | 100.4    | -313.8    | -355.9     | -1319.6|
|                            | (1.63)  | (.98)    | (.48)    | (.73)    | (-1.22)   | (-.78)     | (-.48) |
| \( \hat{\beta}_2 \) (LDV) | .065    | -.269    | .014     | -.012    | .083      | .089       | .460  |
|                            | (.33)   | (-.84)   | (.06)    | (-.34)   | (1.23)    | (.75)      | (.70) |
| SDu            | 382.1   | 305.9    | 335.5    | 76.8     | 103.0     | 174.2      | 716.8 |
| LM\(_N\)(*)    | 15.16*  | 74.36*   | 7.59     | 40.29*   | 65.00*    | 73.50*     | 2.00  |
| LM\(_H\)(*)    | 20.48*  | 4.71     | 6.01     | 13.65*   | 31.83*    | 74.21*     | 17.25*|
| LM\(_{NH}\)(*) | 35.64*  | 77.98*   | 14.06*   | 56.09*   | 114.14*   | 157.32*    | 24.92*|
| \( \hat{\beta}_1 \) | 559.8   | 1074.9   | 501.3    | 104.3    | .5        | 7.2        | 454.2 |
| \( \hat{\beta}_2 \) | .055    | -.109    | .009     | -.009    | .040      | .041       | .209  |

Results using LDV Model
TABLE 9.28
Estimated Regressions: Technocrats, Income group I3

<table>
<thead>
<tr>
<th></th>
<th>Food</th>
<th>Clothing</th>
<th>Housing</th>
<th>Durables</th>
<th>Education</th>
<th>Medical S.</th>
<th>Other</th>
</tr>
</thead>
<tbody>
<tr>
<td>Results using Ordinary Regression Model</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \hat{\beta}_1 ) (OLS)</td>
<td>83.9 (-.56)</td>
<td>-121.7 (-2.43)</td>
<td>-2.2 (-.03)</td>
<td>-.004 (-2.12)</td>
<td>-.051 (-.43)</td>
<td>-.063 (-1.02)</td>
<td>-.702 (-1.02)</td>
</tr>
<tr>
<td>SE</td>
<td>.117 (1.84)</td>
<td>.056 (3.93)</td>
<td>532 (.57)</td>
<td>.0102 (2.78)</td>
<td>.009 (1.06)</td>
<td>.022 (2.77)</td>
<td>.272 (2.77)</td>
</tr>
<tr>
<td>SDu</td>
<td>308.6</td>
<td>156.1</td>
<td>673.2</td>
<td>91.0</td>
<td>17.5</td>
<td>107.8</td>
<td>501.2</td>
</tr>
<tr>
<td>LM( N )</td>
<td>1.14</td>
<td>.98</td>
<td>.71</td>
<td>8.96</td>
<td>7.09</td>
<td>4.18</td>
<td>.92</td>
</tr>
<tr>
<td>LM( H )</td>
<td>.04</td>
<td>1.20</td>
<td>3.09</td>
<td>.52</td>
<td>.00</td>
<td>.15</td>
<td>.02</td>
</tr>
<tr>
<td>LM( NH )</td>
<td>1.17</td>
<td>2.18</td>
<td>3.80</td>
<td>9.48</td>
<td>7.09</td>
<td>4.33</td>
<td>.94</td>
</tr>
<tr>
<td>( F(x'\hat{\beta}, \sigma^2) )</td>
<td>.005</td>
<td>.046</td>
<td>.019</td>
<td>.238</td>
<td>.208</td>
<td>.202</td>
<td>.010</td>
</tr>
<tr>
<td>Results using LDV Model</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \hat{\beta}_1 ) (LDV)</td>
<td>62.0 (-.92)</td>
<td>-301.1 (-2.09)</td>
<td>-2437.6 (-.48)</td>
<td>-84.8 (-2.07)</td>
<td>-117.9 (-.59)</td>
<td>-110.6 (-1.21)</td>
<td>-926.1 (-1.21)</td>
</tr>
<tr>
<td>SE</td>
<td>.120 (1.80)</td>
<td>.075 (2.65)</td>
<td>.405 (.66)</td>
<td>.016 (2.17)</td>
<td>.016 (2.94)</td>
<td>.024 (2.84)</td>
<td>.298 (2.84)</td>
</tr>
<tr>
<td>SDu</td>
<td>308.6</td>
<td>158.2</td>
<td>683.8</td>
<td>91.5</td>
<td>20.1</td>
<td>107.9</td>
<td>502.2</td>
</tr>
<tr>
<td>LM( N(*) )</td>
<td>1.25</td>
<td>.45</td>
<td>318.72*</td>
<td>3.67</td>
<td>223.68*</td>
<td>.72</td>
<td>.77</td>
</tr>
<tr>
<td>LM( H(*) )</td>
<td>.06</td>
<td>1.45</td>
<td>811.09*</td>
<td>1.64</td>
<td>492.90*</td>
<td>.15</td>
<td>.23</td>
</tr>
<tr>
<td>LM( NH(*) )</td>
<td>1.43</td>
<td>4.33</td>
<td>838.57*</td>
<td>4.87</td>
<td>615.57*</td>
<td>.79</td>
<td>1.60</td>
</tr>
<tr>
<td>( \hat{\beta}_1 )</td>
<td>273.5</td>
<td>76.7</td>
<td>-137.5</td>
<td>19.7</td>
<td>.002</td>
<td>17.3</td>
<td>214.3</td>
</tr>
<tr>
<td>( \hat{\beta}_2 )</td>
<td>.117</td>
<td>.050</td>
<td>.205</td>
<td>.009</td>
<td>.006</td>
<td>.016</td>
<td>.269</td>
</tr>
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BIBLIOGRAPHY

Aigner, D.J. and S.M. Goldfeld (1974), "Estimation and prediction from aggregate data when aggregates are measured more accurately than their components", *Econometrica*, 42, 113-134.


Bowman, K.O. and L.R. Shenton (1975), "Omnibus contours for departures from normality based on $\sqrt{b_1}$ and $b_2$", Biometrika, 62, 243-250.


D'Agostino, R.B. and E.S. Pearson (1973), "Tests for departure from normality. Empirical results for the distributions of $b_2$ and $\sqrt{b_1}$", Biometrika, 60, 613-622; Correction, Biometrika (1974), 61, 647.

D'Agostino, R.B. and G.L. Tietjen (1973), "Approaches to the null distribution of $\sqrt{b_1}$", Biometrika, 60, 169-173.


Gallant, A.R. and W.A. Fuller (1973), "Fitting segmented polynomial regression models whose join joints have to be estimated", Journal of the American Statistical Association, 68, 144-147.


Godfrey, L.G. (1978a), "Testing against general autoregressive and moving average error models when the regressors include lagged dependent variables", *Econometrica*, 46, 1293-1301.

Godfrey, L.G. (1978b), "Testing for higher order serial correlation in regression equations when the regressors include lagged dependent variables", *Econometrica*, 46, 1303-1310.


Hudson, D.J. (1966), "Fitting segmented curves whose join points have to be estimated", *Journal of the American Statistical Association*, 61, 1097-1129.


Ito, K. and W.J. Schull (1964), "On the robustness of the $T_0^2$ test in multivariate analysis of variance when variance covariance matrices are not equal", *Biometrika*, 51, 71-82.


