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APPLICATION OF SPECTRAL METHODS IN ECONOMIC DATA ANALYSIS

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The results presented in this Thesis are my

own except where otherwise stated

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I BRIEF THEORETICAL BACKGROUND

1.1 Introduction

Because a stationary time series may be represented in spectral (or frequency) terms it has become apparent that certain areas of economic investigation can be effectively performed in this domain. This approach may arise because *a priori* information is most easily expressed in frequency terms or it may be that greater insight results from using spectral methods in conjunction with the more usual procedures developed for the time domain. A large area of empirical economics exists where *data analysis* in the frequency domain is not contemplated because of insufficient data. Nevertheless, where *theoretical explanations* are sought spectral methods may still be very useful.

The essential mathematical reasons for the introduction of spectral methods in data analysis will, it is hoped, appear from the discussion to follow. One misconception should however be mentioned here and dismissed. This is the notion that only where the idea of wave motion is present can these methods be expected to be useful. A typical expression of this viewpoint is: 'It is not the cyclical behaviour of an economy that is of interest to me but only the long term trend and so spectral methods are not relevant'. The reason for the usefulness of spectral methods is that they require only a general assumption of temporal homogeneity, which appears to be adequately fulfilled for most data discussed in this thesis. This assumption leads inexorably to spectral methods and this is no less truly so when the nature of the phenomena is such as to show no very marked oscillation. An examination of a monthly time series of economic data and for example an oceanographic record measured hourly over a period of a few days will not reveal any intrinsic differences which enable one to say one is a wave phenomenon and one is not.

Both series will show certain clear and nearly periodic oscillations (seasonal and tidal effects). Both will also show a large amount of fairly haphazard occasional fluxion. The fact that the term 'wave' occurs naturally in connection with the second phenomenon is a matter for historical explanation. In particular the later discussion will show, it is hoped, how relevant the spectral methods are to the measurement of trend as well as to the clearly oscillatory motion, called seasonal fluctuation.

Before developing applications of spectral methods in economic data analysis a general framework for this approach to time series analysis must be presented. This introductory chapter includes a sketch of underlying theory which is a necessary basis for discussion. In particular the results incorporated in §1.2 - §1.6 are well known - see for example Whittle [57], Grenander and Rosenblatt [18], Yaglom [59] and Hannan [19,20].

1.2 Spectral Representation

A series $x_j(n)$ is termed a second order stationary[1] process if the second order moments depend only on t, i.e.

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[1] It is usual to begin with strictly stationary processes, but as the restriction on the series necessary for this property to hold would be far more wide-reaching, unless the data is Gaussian, only the restriction of second order or wide sense stationarity is imposed. It must be emphasized however that the assumption of second order stationarity is all that is necessary for the development of spectral, prediction and filtering theory but that if distributional results for spectral estimation and regression procedures are to be established then stronger assumptions will be needed.

$$\mathcal{E}\left(x_j(n)x_k(n+\tau)\right) = \gamma_{jk}(\tau) \qquad j = 1,\ldots,p; \quad n = 0,\pm1,\pm2,\ldots \tag{1.2.1}$$

In specifying stationarity it is assumed that $\mathcal{E}(x_j) = 0$. This assumption will be used in presenting the theoretical background but will naturally be relaxed in Chapter II when the actual nature of economic data is given further consideration. If assumption (1.2.1) is valid the following relation may be derived

$$\begin{aligned}\Gamma(\tau) = \left\{\gamma_{jk}(\tau)\right\} &= \left\{\int_{-\pi}^{\pi} e^{i\tau\lambda} dF_{jk}(\lambda)\right\} \\ &= \int_{-\pi}^{\pi} e^{i\tau\lambda} dF(\lambda)\end{aligned} \tag{1.2.2}$$

where $F(\lambda)$ is the spectral distribution matrix and $dF(\lambda)$ a Hermitian, non-negative definite matrix. If $j = k$ then that element of the matrix is referred to as the cumulative power spectrum of the variable j. To develop greater understanding of the nature of the incremental spectral distribution matrix, $dF(\lambda)$, the following properties of this matrix are stated and then employed

$$\begin{aligned}& dF(\lambda) = dF'(-\lambda) = \overline{dF(-\lambda)} \\ \text{i.e.} \quad & dF(-\lambda) = \overline{dF(\lambda)} \text{ or } dF'(-\lambda) = dF^*(\lambda).\end{aligned} \tag{1.2.3}$$

The * on the $F(\lambda)$ matrix signifies the joint operation of transposition and conjugation. If the complex distribution function $F(\lambda)$ is rewritten as

$$F(\lambda) = \tfrac{1}{2}\left\{C(\lambda)-iQ(\lambda)\right\} \tag{1.2.4}$$

where $C(\lambda)$ is a matrix composed of the real part of $F(\lambda)$ and $Q(\lambda)$ a matrix composed of the complex part of $F(\lambda)$, the expression (1.2.2) becomes

$$\Gamma(\tau) = \int_{0}^{\pi} \left(\cos\tau\lambda dC(\lambda)+\sin\tau\lambda dQ(\lambda)\right). \tag{1.2.5}$$

If it is assumed that $F(\lambda)$ is absolutely continuous and therefore

$$dF(\lambda) = f(\lambda)d\lambda$$

then

$$f(\lambda) = \tfrac{1}{2}\left\{c(\lambda)-iq(\lambda)\right\} \tag{1.2.6}$$

where $c(\lambda)$ is the symmetric co-spectral density matrix and $q(\lambda)$ is the skew-symmetric quadrature spectral density matrix which leads to the rewriting of (1.2.5) as

$$\Gamma(\tau) = \int_0^{\pi}\left(c(\lambda)\cos\tau\lambda+q(\lambda)\sin\tau\lambda\right)d\lambda. \tag{1.2.7}$$

In (1.2.7) the second order moments of the series are represented in frequency terms. It is also illuminating to represent the series, rather than their moments, in this domain as follows

$$\begin{aligned} x(n) = \left\{x_j(n)\right\} &= \left\{\int_{-\pi}^{\pi} e^{-in\lambda}dZ_j(\lambda)\right\} \\ &= \int_{-\pi}^{\pi} e^{-in\lambda}dZ(\lambda) \end{aligned} \tag{1.2.8}$$

where $Z(\lambda)$ is a vector such that

$$\begin{aligned} \mathcal{E}\left(dZ(\lambda_1)dZ^*(\lambda_2)\right) &= 0 \qquad \lambda_1 \neq \lambda_2 \\ &= dF(\lambda) \qquad \lambda_1 = \lambda_2 = \lambda. \end{aligned} \tag{1.2.9}$$

The vector $Z(\lambda)$ is rewritten to make explicit its real part $U(\lambda)$ and its complex part $V(\lambda)$ as

$$Z(\lambda) = \tfrac{1}{2}\left\{U(\lambda)+iV(\lambda)\right\}$$

and a more illuminating expression for the series $x(n)$ is given by

$$x(n) = \int_0^{\pi}\left\{\cos\lambda n dU(\lambda)+\sin\lambda n dV(\lambda)\right\} \tag{1.2.10}$$

where the vectors $dU(\lambda)$ and $dV(\lambda)$ are such that

$$\begin{aligned} \mathcal{E}\left\{dU(\lambda_1)dU'(\lambda_2)\right\} &= \mathcal{E}\left\{dV(\lambda_1)dV'(\lambda_2)\right\} = \delta_{\lambda_2}^{\lambda_1}dC(\lambda_1) \\ \mathcal{E}\left\{dU(\lambda_1)dV'(\lambda_2)\right\} &= \delta_{\lambda_2}^{\lambda_1}dQ(\lambda_1)\ . \end{aligned} \tag{1.2.11}$$

The symbol $\delta_{\lambda_2}^{\lambda_1}$ is unity where $\lambda_1 = \lambda_2$ and zero otherwise. The representation of the vector series given in (1.2.10) lends itself most readily to interpretation, for a typical series $x_j(n)$ is a linear superposition of sinusoidal terms with random amplitudes and phases at each λ determined by $U_j(\lambda)$ and $V_j(\lambda)$.

If for the moment only two series $x_j(n)$ and $x_k(n)$ are considered then the correlation between the $n\lambda^{th}$ term for $x_j(n)$ in (1.2.10) and the $(n\lambda+\theta)^{th}$ term for $x_k(n)$ in (1.2.10) is maximized when

$$\theta_{jk}(\lambda) = \arctan \frac{dQ_{jk}(\lambda)}{dC_{jk}(\lambda)} \tag{1.2.12}$$

and the value of this correlation is given by

$$W_{jk}(\lambda) = \left\{ \frac{dC^2_{jk}(\lambda)+dQ^2_{jk}(\lambda)}{dF_{jj}(\lambda)dF_{kk}(\lambda)} \right\}^{\frac{1}{2}} . \tag{1.2.13}$$

The phase, $\theta_{jk}(\lambda)$, and the coherence, $W_{jk}(\lambda)$, are characteristics which measure the dependence of two time series, $x_j(n)$ and $x_k(n)$.

1.3 Linear Filtering

If the relation between two series is that the first series $y(n)$ is produced by the operation of a linear filter[2] on the series $x(n)$ then a specific description of the dependence may be derived. To develop the filtering concept in more detail the transformation of a time series $x(n)$ $\{n = 0,\pm1,\pm2,......\}$ to produce another series $y(n)$ $\{n = 0,\pm1,\pm2,.......\}$ is known as linear digital filtering if

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[2] A linear filter must have two properties. To economically express these requirements one defines L to be the operator which performs the filtering and U^m an operation which translates a variable m periods forward. The requirements of a linear filter are

(i) $L\Sigma\alpha_j x_j(n) = \Sigma\alpha_j Lx_j(n)$

and (ii) $LU^m x_j(n) = U^m Lx_j(n)$.

$$y(n) = \Sigma_{-\infty}^{\infty} b_j x(n-j) \qquad n = 0, \pm 1, \pm 2, \ldots\ldots \tag{1.3.1}$$

The sequence of coefficients b_j, often restricted so that $\Sigma_{j-\infty}^{\infty} |b_j| < \infty$, is the impulse response of the filter. The essential condition that the b_j must satisfy is that $B(\lambda)$, defined below, must be square integrable with respect to $dF(\lambda)$. A more important function, derived from this sequence, is

$$B(\lambda) = \Sigma_{j-\infty}^{\infty} b_j e^{-ij\lambda} \tag{1.3.2}$$

the frequency response function of the filter, which can obviously be complex for a real series b_j. The importance of this response function arises from its use in interpreting the action of the filter on an arbitrary series. It can be interpreted as the way in which an input of a complex harmonic, $e^{i\lambda n}$ will be modified at each frequency to provide an output, since the output procedure is $B(\lambda)e^{i\lambda n}$. This function is also valuable in relating the spectral distributions of the input and output series as follows,

$$dF_y(\lambda) = |B(\lambda)|^2 dF_x(\lambda). \tag{1.3.3}$$

Thus the spectral distribution of the filtered series is produced by multiplying the original spectral distribution function by $|B(\lambda)|^2$, where $|B(\lambda)|$ is known as the gain of the filter. When the input to the filter is a vector time series the relation between the input and output vector is

$$dF_y(\lambda) = B(\lambda) dF_x(\lambda) B^*(\lambda) \tag{1.3.4}$$

where $B(\lambda)$ is now a matrix and is best comprehended by exhibiting its form when the most common form of filtering,

$$\underline{y}(n) = \Sigma_{j-\infty}^{\infty} B_j \underline{x}(n-j), \tag{1.3.5}$$

is used. If this situation, where $\underline{y}$ and $\underline{x}$ are vectors and the B_j are matrices, the matrix function $B(\lambda)$ in (1.3.4) is given by

$$B(\lambda) = \Sigma_{-\infty}^{\infty} B_j e^{ij\lambda}. \tag{1.3.6}$$

Filtering of the above kind will, except for special circumstances, inevitably introduce phase shifts. The relative phase between the j^{th} and k^{th} series depends only upon the cross spectrum and even if $B(\lambda)$ was diagonal the cross-spectrum would be changed. If $B(\lambda)$ is diagonal and real this will not happen. In particular this will be so if $B_j = B_{-j}$ and the filter is therefore symmetric. But if a one sided filter (i.e. $B_j \equiv 0$ for either $j < 0$ or $j > 0$) is used it must introduce phase shifts.

1.4 Spectral Estimation

It would be rather pointless developing a representation of an economic time series vector in the frequency domain if the fundamental quantities in this representation, $f_{jk}(\lambda)$,[3] were not able to be estimated. The general estimation procedures are only briefly outlined as more detailed problems are set aside for further consideration in Chapter II.

In a statistical analysis of a time series $x(n)$ which is to concentrate on sources of variation the finite Fourier transform $w(\lambda)$ plays an important role. The finite Fourier transform is defined as

$$w(\lambda) = \left\{1/\sqrt{2\pi N}\right\} \Sigma_1^N x(n)e^{in\lambda} \tag{1.4.1}$$

and is evaluated at the points $\lambda = \frac{2\pi k}{N}$, for $k = 1,2,\ldots,N$, where N is the number of observations in the realization. The quantity defined in (1.4.1) is very simply related to the periodogram $I(\lambda)$,

$$I(\lambda) = w(\lambda)w^*(\lambda). \tag{1.4.2}$$

The periodogram was the focal point of early studies of the source of variation in time series. The following expectation

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3 For convenience of exposition only it is assumed that $dF(\lambda)$ is absolutely continuous and in future comment in this section $f_{jk}(\lambda)$, the element in the j^{th} row and k^{th} column of $f(\lambda)$, is considered.

$$\mathcal{E}\left(I(\lambda)\right) = \left\{1/2\pi N\right\} \mathcal{E}|\Sigma_1^N x(n)e^{in\lambda}|^2$$

$$= \left\{1/2\pi\right\} \Sigma_{-N+1}^{N-1} \left\{1 - \frac{|n|}{N}\right\} \Gamma(n)e^{in\lambda} \tag{1.4.3}$$

indicates that since $\mathcal{E}(I(\lambda))$ is equal to the Cesaro mean of $f(\lambda)$ it will converge to $f(\lambda)$ as N becomes large, if, for example, $f(\lambda)$ is continuous. It is well known however that $I(\lambda)$ is not a consistent estimate of $f(\lambda)$. This defect does not prevent the periodogram from being of use in the discussion of estimates of spectra and cross-spectra because it reappears in a modified form in the appropriate estimates of spectral quantities.

If in choosing an estimator for $f(\lambda)$ the choice is restricted to a quadratic function of the observations, since the spectrum is itself a quadratic quantity, then the form of the estimator is

$$\hat{f}_{jk}(\lambda) = \Sigma\Sigma_{mp} b_{mp}(\lambda)x_i(m)x_j(p). \tag{1.4.4}$$

It is natural in a context of stationarity to restrict the coefficients to depend only on the lag (m-p) and Grenander and Rosenblatt [18] have vindicated such a choice. Replacing b_{mp} by b_{m-p} (1.4.4) can be rewritten as (see [18, p 123])

$$\hat{f}_{jk}(\lambda) = \frac{1}{2\pi} \Sigma_{-N+1}^{N-1} k_n e^{-in\lambda}(1 - \frac{|n|}{N})c_{jk}(n) \tag{1.4.5}$$

where

$$c_{jk}(n) = \frac{1}{N-n} \Sigma_{1\,m}^{N-n} x_j(m)x_k(m+n).$$

The function k_n is interpreted as a covariance averaging kernel and its Fourier transform is,

$$K_N(\lambda) = \frac{1}{2\pi} \Sigma_{-N+1}^{N-1} k_n e^{-in\lambda}, \tag{1.4.6}$$

known as the spectral window. The use of the spectral window leads to a highly instructive expression for the estimator $\hat{f}_{jk}$, i.e.

$$\hat{f}_{jk}(\lambda) = \frac{\pi}{N}\Sigma_{-N+1}^{N-1} K_N(\lambda - \frac{\pi n}{N}) I_{jk}(\frac{\pi n}{N})$$

$$= \int_{-\pi}^{\pi} K_N(\lambda-\theta) I_{jk}(\theta) d\theta. \quad (1.4.7)$$

It is not obvious that the two expressions in (1.4.7) are equal and it is surprising that the integral should exactly equal its approximate sum. The first relation in (1.4.7) is due to the orthogonality properties of $\phi_k(n) = \exp\{(in\pi k)/N\}$, $n = -N+1,\ldots,N$. The latter is due to the orthogonality properties of $\phi_k(\lambda) = \exp\{in\lambda\}$ as function of λ on $(-\pi,\pi)$. The representation of the estimator given in (1.4.7) clearly points out that estimators of the form chosen consist of a view of the periodogram ordinates. The nature of the view is determined by $K_N(\lambda)$, the spectral window.

1.5 Prediction and Signal Extraction

The techniques developed in this section relate mainly to any real world phenomena which generate an observed series which may be considered as a signal (another series) which is unobservable because it is obscured by a further unobservable (noise) series. The technique is also suitable for predicting the value of the signal series at some time point in the future. The information we do have about the phenomena is the spectral (or equivalent) properties of the signal and noise series.

An introduction to these topics could begin in either the time or frequency domain and ideally both approaches should be presented as they are complementary. As my object is merely to sketch the basis of techniques used later this can be done most economically by presenting the methods as an example of filtering methods.

If a predictor $\hat{x}(n)$ is to be based on either a finite number (say p) of past values of the variable x or the complete past history of x, then a linear predictor is of the form

$$\hat{x}^{(p)}(n) = b_1 x(n-1)+b_2 x(n-2)+\ldots\ldots+b_p x(n-p)$$
$$\hat{x}(n) = b_1 x(n-1)+b_2 x(n-2)+\ldots\ldots . \qquad (1.5.1)^{4}$$

The spectral representation of $\hat{x}(n)$ (a scalar example of (1.2.8)) is

$$\hat{x}(n) = \int_{-\pi}^{\pi} e^{-in\lambda} B(\lambda) dZ(\lambda) \qquad (1.5.2)$$

where

$$B(\lambda) = b_1 e^{i\lambda}+b_2 e^{i2\lambda}+\ldots\ldots$$
$$\text{and} \quad B^{(p)}(\lambda) = b_1 e^{i\lambda}+b_2 e^{i2\lambda}+\ldots\ldots+b_p e^{ip\lambda}. \qquad (1.5.3)$$

The residual or innovation is defined in terms of these quantities as $x(n)-\hat{x}(n)$. The spectral representation of the innovation is

$$x(n)-\hat{x}(n) = \int_{-\pi}^{\pi} e^{-in\lambda} \left\{1-B(\lambda)\right\} dZ_x(\lambda) \qquad (1.5.4)$$

and therefore the mean square prediction error will be

$$\mathcal{E}\left\{\left(x(n)-\hat{x}(n)\right)^2\right\} = \int_{-\pi}^{\pi} |1-B(\lambda)|^2 dF_x(\lambda). \qquad (1.5.5)$$

This minimal mean square prediction error for a linear predictor may be shown to be

$$\mathcal{E}\left\{\left(x(n)-\hat{x}(n)\right)^2\right\} = \exp\left[\left(\frac{1}{2\pi}\right)\int_{-\pi}^{\pi} \log\left(2\pi f_x(\lambda)\right) d\lambda\right] \qquad (1.5.6)$$

and equation (1.5.6) therefore indicates that the prediction error depends only on the absolutely continuous part of the spectrum, for any term contributing to a jump would be perfectly predictable on the basis of the infinite past.

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[4] It is possible that an optimal predictor of $x(n)$ may not be expressible in the form (1.5.1) but instead one may have to define it by a sequence of predictors $\hat{x}^{(p)}(n)$ with p increasing indefinitely and the coefficients of $x(n-j)$ depending on p.

Two loose ends must be tied up. First, how is an appropriate $B(\lambda)$ decided upon? Second, can this formulation suitably handle problems of prediction and signal extraction. So far the predictor of $x(n)$ has been based on either a finite or infinite number of past values of x. The predictor $\hat{x}(n+\nu)$, $\nu \geqq 1$, is a prediction of $x(n+\nu)$ based on the values of x, up to and including N. So we can represent $x(n+\nu)$ as follows

$$\hat{x}(n+\nu) = \int_{-\pi}^{\pi} e^{-in\lambda} B^{(\nu)}(\lambda) dZ(\lambda) \tag{1.5.7}$$

where $B^{(\nu)}(\lambda) = b_o^{(\nu)} + b_1^{(\nu)} e^{i\lambda} + b_2^{(\nu)} e^{i2\lambda} + \ldots\ldots$. Another time series $y(n)$ is introduced and it is assumed that $y(n)$ and $x(n)$ are jointly covariance stationary. For expositional convenience only, it is also assumed that both series have spectral representations containing only an absolutely continuous part. The cross covariance between the two series is therefore given by

$$\mathcal{E}\left(x(m)y(n)\right) = \int_{-\pi}^{\pi} e^{i\lambda(n-m)} f_{xy}(\lambda) d\lambda. \tag{1.5.8}$$

If the series $y(n)$ is just the x series translated ν periods forward, i.e. $y(n) = x(n+\nu)$, then (1.5.8) becomes (see (1.2.2))

$$\mathcal{E}\left(x(m)x(n+\nu)\right) = \int_{-\pi}^{\pi} e^{i\lambda(n-m)} e^{i\nu\lambda} f_{xx}(\lambda) d\lambda. \tag{1.5.9}$$

It is possible however to regard $y(n)$ as the signal series where a predictor of this series is to be obtained from the observed series $x(m)$. The prediction of $y(n)$, on the basis of $x(m)$, $m \leqq n$, is

$$\hat{y}(n) = c_o x(n) + c_1 x(n-1) + \ldots\ldots \tag{1.5.10}$$

and its spectral representation is

$$\hat{y}(n) = \int_{-\pi}^{\pi} e^{-i\lambda n} C(\lambda) dZ_x(\lambda) \tag{1.5.11}$$

with[5]

$$C(\lambda) = \Sigma_{j}{}_{o} c_j e^{ij\lambda}. \qquad (1.5.12)$$

The development of the determination of the c_j coefficients is that given by Whittle [57]. The frequency response (1.5.12) is written in terms of $z = e^{i\lambda}$ so that C(z), often referred to as the transfer function, is

$$C(z) = \Sigma_{j}{}_{o} c_j z^j. \qquad (1.5.13)$$

The covariance generating functions $g_{xx}(z)$, $g_{xy}(z)$ are assumed analytic in a region $\rho < z < \rho^{-1}$ $(0 < \rho < 1)$ and so are represented as

$$g_{xx}(z) = \Sigma_{\tau}{}_{-\infty}^{\infty} z^{\tau} c_{xx}(\tau)$$
$$g_{xy}(z) = \Sigma_{\tau}{}_{-\infty}^{\infty} z^{\tau} c_{xy}(\tau). \qquad (1.5.14)$$

If the c_j coefficients are chosen to minimize the mean square error, $\mathcal{E}(y(n)-\hat{y}(n))^2$, then the following relation between the covariances and the c_j coefficients must hold,

$$\Sigma_{j}{}_{o}^{\infty} c_{xx}(k-j)c_j = c_{xy}(k), \qquad k = 0,1,2,\ldots\,. \qquad (1.5.15)$$

If (1.5.15) is multiplied by z^k and added over all integral k then this expression becomes

$$\Sigma_{k}{}_{o}^{\infty} \Sigma_{j}{}_{o}^{-\infty} c_{xx}(k-j)c_j z^k = \Sigma_{k}{}_{o}^{\infty} c_{xy}(k)z^k$$
$$\therefore \quad \Sigma_{k}{}_{o}^{\infty} \Sigma_{j}{}_{o}^{\infty} c_{xx}(k-j)c_j z^{k-j} z^j = \Sigma_{k}{}_{o}^{\infty} c_{xy}(k)z^k. \qquad (1.5.16)$$

The left hand side of (1.5.16) may be written as $g_{xx}(z)C(z)-h_1(z)$ where $h_1(z)$ involves only negative powers of z and similarly the right hand side can be thought of as $g_{xy}(z)-h_2(z)$ where again $h_2(z)$ contains only negative powers of z and thus (1.5.16) becomes

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5 For economy of presentation the repetition of the finite and infinite case is discontinued. The upper summation limit is therefore left open.

$$g_{xx}(z)C(z) = h(z)+g_{xy}(z) \tag{1.5.17}$$

where $h(z) = h_1(z)-h_2(z)$ is an expression in negative powers of z.

If it is assumed that the prediction error is positive (see (1.5.6)) then $g_{xx}(z)$ may be factorized as follows,

$$g_{xx}(z) = \sigma^2|\theta(z)|^2 = \sigma^2\theta(z)\theta(z^{-1}),$$
$$\theta(z) = 1+\theta_1 z+\theta_2 z^2+\ldots.. , \tag{1.5.18}$$

to provide what is termed a canonical factorization.[6] If (1.5.17) is divided by $\theta(z^{-1})$ then it may be rewritten as

$$\sigma^2\theta(z)C(z) = \frac{h(z)}{\theta(z^{-1})} + \frac{g_{xy}(z)}{\theta(z^{-1})}. \tag{1.5.19}$$

Because the term on the left hand side of (1.5.19) consists only of positive powers of z and because the first term on the right hand side consists only of negative powers then equating of like powers of z produces a solution for $C(z)$ of the form

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6 To uniquely define the canonical factorization it is required that $\theta(z)$ has no zeros for $|z| < 1$ for although

$$\frac{1}{2\pi}\left\{\frac{1}{1+\rho^2-2\rho\cos\lambda}\right\} = \frac{1}{2\pi}\left\{\frac{1}{(1-\rho z)(1-\rho z^{-1})}\right\} = \theta(z)\theta(z^{-1})$$

it is also true that

$$\frac{1}{2\pi}\left\{\frac{1}{1+\rho^2-2\rho\cos\lambda}\right\} = \frac{1}{2\pi}\left\{\frac{1}{\rho^2(1-\rho^{-1}z)(1-\rho^{-1}z^{-1})}\right\} = \frac{1}{2\pi}\left\{\frac{1}{(\rho(1-\rho^{-1}z))(\rho(1-\rho^{-1}z^{-1}}\right.$$

$$= \phi(z)\phi(z^{-1})$$

however the zero of $\phi(z)$ is within the unit circle.

$$C(z) = \frac{1}{\sigma^2\theta(z)}\left[\frac{g_{xy}(z)}{\theta(z^{-1})}\right]_+ . \qquad (1.5.20)$$

As it may be shown [57,pp 67-8] that $\Sigma_{k}^{\infty}{}_{o} \Sigma_{j}^{\infty}{}_{o} c_{xx}(k-j)c_j z^k$ has a valid Laurent representation in an annulus including the unit circle, the symbol $+$ indicates that only the positive terms in a Laurent expansion should be used. The formula given in (1.5.20) may be regarded as a general solution which may be shown to cover the following cases.

(a) Prediction ν steps ahead is handled by setting $y(n) = x(n+\nu)$. This results in a simplification of the covariance generating function $g_{xy}(z)$ so that

$$\begin{aligned} g_{xy}(z) &= z^{\nu} g_{xx}(z) \\ &= z^{\nu}\theta(z)\theta(z^{-1})\sigma^2 . \end{aligned} \qquad (1.5.21)$$

The generating function given in (1.5.20) then becomes

$$C(z) = \frac{1}{\theta(z)}\Big[\theta(z)z^{\nu}\Big]_+ . \qquad (1.5.22)$$

(b) Extraction of a signal series $s(n)$ at time point n, where the observed series $x(n) = s(n)+\epsilon(n)$ and $s(n)$ and $\epsilon(n)$ are independent. If $y(n) = s(n)$ then $g_{xy}(z) = g_{ss}(z)$, since $\epsilon(n)$ and $s(n)$ are independent so that the appropriate generating function is now

$$C(z) = \frac{1}{\sigma^2\theta(z)}\left[\frac{g_{ss}(z)}{\theta(z^{-1})}\right]_+ . \qquad (1.5.23)$$

(c) Prediction $\nu > 0$ or Signal Extraction $\nu \leq 0$ of $s(n)$ in similar circumstances to those given in (b). Now $y(n) = s(n+\nu)$ and therefore $g_{xy}(z) = z^{\nu} g_{ss}(z)$ and the generating function specializes to

$$C(z) = \frac{1}{\sigma^2\theta(z)}\left[\frac{g_{ss}(z)z^{\nu}}{\theta(z^{-1})}\right]_+ . \qquad (1.5.24)$$

In the above examples the formulae for the coefficients of the predictor $\hat{y}(n)$ are established. It is straightforward to attach an expression for the mean square prediction error, which for convenience is related to the general generating function (1.5.20). If $\log g_{xx}(z)$ has a valid Laurent expansion in the annulus $\rho < |z| < \rho^{-1}$ where $0 < \rho < 1$ (see again footnote 6) then $x(n)$ has both a moving average and autoregressive representation. Suppose the predictor is

$$\hat{y}(n) = \Sigma_{j}{}_{o}^{\infty} c_j x(n-j) \qquad (1.5.25)$$

then employing the moving average representation of x which is

$$x(n) = \Sigma_{k}{}_{o}^{\infty} b_k \epsilon(n-k) \qquad (1.5.26)$$

where the $\epsilon(n)$ are independently and identically distributed random variables with zero mean and variance σ^2, (I.I.D $(0,\sigma^2)$), the expression for the predictor in terms of the $\epsilon(n)$ is

$$\begin{aligned}\hat{y}(n) &= C(z)B(z)\epsilon(n) \\ &= q(z)\epsilon(n)\end{aligned} \qquad (1.5.27)$$

with $B(z) = \Sigma_{k}{}_{o}^{\infty} b_k z^k$ and $q(z) = \Sigma_{j}{}_{o}^{\infty} q_j z^j$.

The mean square prediction error is given by

$$\begin{aligned}\mathcal{E}\left(y(n)-\hat{y}(n)\right)^2 &= \text{var}(y(n))+\mathcal{E}\hat{y}^2(n)-2\mathcal{E}\hat{y}(n)y(n) \\ &= \text{var}(y(n))+\Sigma_{j}{}_{o}^{\infty} q_j^2-\Sigma_{j}{}_{o}^{\infty} q_j k_j \\ &= \text{var}(y(n))-\Sigma_{j}{}_{o}^{\infty} k_j^2+\Sigma_{j}{}_{o}^{\infty}(k_j-q_j)^2\end{aligned} \qquad (1.5.28)$$

where $k_j = \text{cov}(y(n)\epsilon(n-j))$.

The expression for the prediction error will be minimized if $q_j = k_j$ and the resulting prediction error is

$$\begin{aligned}\mathcal{E}\left(y(n)-\hat{y}(n)\right)^2 &= \text{var}(y(n))-\Sigma_{o}^{\infty} q_j^2 \\ &= \text{var}(y(n))-\mathcal{A}|q(z)|^2 \\ &= \text{var}(y(n))-\mathcal{A}|C(z)B(z)|^2\end{aligned} \qquad (1.5.29)$$

where

$$\mathfrak{A}|q(z)|^2 = \frac{1}{2\pi i}\oint_{|z|=1} \frac{q(z)q(z^{-1})}{z}\,dz$$

and

$$C(z)B(z) = \frac{1}{\sigma^2}\left[\frac{g_{xy}(z)}{B(z^{-1})}\right]_+ .$$ [7]

The prediction methods are now slightly extended to cover an area of obvious application when the signal is what is termed an accumulated process. Thus the signal is assumed to be a solution of

$$\Delta^p s(n) = \eta(n) \qquad (1.5.30)$$

where $\eta(n)$ is a stationary process with covariance generating function $g_{\eta\eta}(z)$, which is analytic in an annulus including $|z| = 1$. The nature of the process generated in this way is best understood by reference to the following autoregressive process

$$\Sigma_{j=0}^{p} d_j s(n-j) = \eta(n) \qquad (1.5.31)$$

where $\eta(n)$ in this situation is an autocorrelated error term. For the process to be stationary and stable the zeros of $\Sigma d_j z^j$ must be outside the unit circle, thus a zero of order p at z = 1 will result in an evolutive process. The evolution has two sources: a polynomial trend and the variance which is increasing with n.

The observation-signal-noise model is still

$$x(n) = s(n)+\epsilon(n) \qquad (1.5.32)$$

but now

$$\left(1-\phi(z)\right)^p s(n) = \eta(n) \qquad (1.5.33)$$

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7 The expression for $C(z)B(z)$ is easily obtained from (1.5.20) since (1.5.26) implies $g_{xx}(z) = B(z)B(z^{-1})\sigma^2$ so that $B(z) = \theta(z)$.

with $\eta(n)$ having the properties given in (1.5.30) and $-1 < \phi \leq 1$.[8] If we employ the formulae already given for a prediction of $\hat{s}(n+\nu)$ in terms of the $x(n)$'s the coefficients for the predictor are obtained from

$$C(z) = \frac{1}{B(z)}\left[\frac{g_{ss}(z)}{z^{\nu}B(z^{-1})}\right]_{+}. \tag{1.5.34}$$

Application of the operator $(1-\phi z)^p$ to both sides of (1.5.32) gives

$$(1-\phi z)^p x(n) = \eta(n)+(1-\phi z)^p \epsilon(n)$$

$$\text{i.e.} \qquad p(n) = \eta(n)+(1-\phi z)^p\epsilon(n) \tag{1.5.35}$$

making the obvious substitution $p(n) = (1-\phi z)^p x(n)$. The new variable $p(n)$ is assumed to have the following canonical factorization

$$g_{pp}(z) = \sigma^2 D(z)D(z^{-1}) \tag{1.5.36}$$

and because there is the obvious relation

$$g_{xx}(z) = \frac{1}{|1-\phi z|^{2p}}\, g_{pp}(z) \tag{1.5.37}$$

then

$$B(z) = \frac{1}{|1-\phi z|^p}\, D(z), \tag{1.5.38}$$

using the representation of $B(z)$ given in (1.5.38) and using (1.5.33) to obtain an expression for $g_{ss}(z)$ in terms of the formula for the prediction coefficients given in (1.5.34), becomes

$$C(z) = \frac{(1-\phi z)^p}{D(z)}\left[\frac{g_{\eta\eta}(z)}{z^{\nu}(1-\phi z)^p D(z^{-1})}\right]. \tag{1.5.39}$$

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[8] In Whittle [57] it is assumed $|\phi| < 1$ and the accumulated process is handled by allowing ϕ to tend to 1 from below. Hannan has shown [24] that the following presentation may be used for $\phi = 1$.

Two comments on the utilization of (1.5.39) are called for. First the extraction of the canonical factor D(z) is obtained from the relation between spectra or covariance generating functions implied by (1.5.35). Second, Whittle [57,p 93] rewrites the expression for C(z) as follows

$$C(z) = \frac{1}{D(z)}\left\{\Sigma_o^{p-1}\psi_j(1-\phi z)^j + \left[\frac{g_{\eta\eta}(z)}{z^\nu D(z^{-1})}\right]_+\right\} \tag{1.5.40}$$

with the first term in the brackets arising from Taylor's expansion of $\left[\frac{g_{\eta\eta}(z)}{z^\nu D(z^{-1})}\right]_-$ about ϕ^{-1}, i.e. from the representation of $\left[\frac{g_{\eta\eta}(z)}{z^\nu D(z^{-1})}\right]_{-1} = Q_-$ as a partial Taylor's expansion about ϕ^{-1} of the following form,

$$Q_-(z) = \Sigma_{j^o}^{p-1} \frac{Q_-^{(j)}(z)-\phi^{-1})}{j!}(z-\phi^{-1})^j. \tag{1.5.41}$$

This latter representation may facilitate the actual computations of the predictor.

1.6 Regression for Time Series

Often the situation under consideration is one where the signal is actually known, rather than the situation considered in the previous section where the only knowledge available related to second order properties of the signal, i.e. the spectral distribution or covariance generating function. This section presents, without proof, well established results (see [18], [20]) which underpin the consideration of single equation regression procedures when the interdependence of observations at different points of time are essential to the model.

The single equation model is

$$z(n) = y(n)\delta + e(n) \tag{1.6.1}$$

or in matrix notation is

$$z = Y\delta + e \tag{1.6.2}$$

where e is an N dimensional (non-observable) vector arising from a stationary process with zero mean and a covariance matrix, $\mathcal{E}(ee') = \Gamma_N$. Y is composed of r sequences of known constants and δ is an r dimensional vector of unknown constants, which are to be estimated. If the covariance matrix Γ_N is known then the Best Linear Unbiassed Estimator (B.L.U.E.) is

$$\tilde{\delta} = (Y'\Gamma_N^{-1}Y)^{-1}Y'\Gamma_N^{-1}z \tag{1.6.3}$$

with covariance matrix

$$\Gamma_{\tilde{\delta}} = (Y'\Gamma_N^{-1}Y)^{-1}. \tag{1.6.4}$$

If the Least Squares Estimator (L.S.E.) is used in these circumstances then this estimator $\hat{\delta}$ is

$$\hat{\delta} = (Y'Y)^{-1}Y'z \tag{1.6.5}$$

with covariance matrix

$$\Gamma_{\hat{\delta}} = (Y'Y)^{-1}(Y'\Gamma_N Y)(Y'Y)^{-1}. \tag{1.6.6}$$

If Γ_N ($\neq$ I) is unknown then any reasonably efficient procedure will be highly non-linear, involving an estimate of Γ_N before estimating of δ begins. The best one can hope to do then is to obtain some form of asymptotically good estimator and some asymptotic expansion, or perhaps the first term in one, for the limiting distribution. For this reason the $y_j(n)$ which make up the Y matrix, that is the regressor variables, must be invested with certain asymptotic properties. These properties are usually referred to as Grenander's conditions [18], and are:

(i) $\lim_{N\to\infty} \Sigma_1^N y_j^2(n) = \lim_{N\to\infty} d_j^2(N) = \infty$.

This condition ensures that consistent estimates of the parameters, δ, exist for otherwise the variance of $\hat{\delta}$ could not be expected to decrease as N increases. This assumption provides no practical difficulties.

(ii) $$\lim_{N\to\infty} \frac{y_j^2(N)}{d_j^2(N)} = 0.$$

The regressor variables, thus restricted, will not increase too fast, guaranteeing that end effects which are neglected are truly asymptotically negligible. It should be noted that an exponentially increasing (or decreasing) sequence does not satisfy Grenander's conditions. Special techniques would be needed for the exponentially increasing case since then the last few observations will never be negligible.

(iii) $$\lim_{N\to\infty} \left[\frac{\Sigma_{1}^{N} y_j(n) y_k(n+m)}{d_j(N) d_k(N)} \right] = \rho_{jk}(m) \text{ exists.}$$

This condition in conjunction with (ii) ensures that

$$\lim_{N\to\infty} \left[\frac{\Sigma_{1}^{N-m} y_j(n) y_k(n+m)}{d_j(N) d_k(N)} \right] = \rho_{jk}(m).$$

Consider a vector of the form

$$x(n) = \Sigma_{-\infty}^{\infty} A(j) \epsilon(n-j)$$

where the $\epsilon(n)$ are identically and independently distributed random vectors with zero means, finite fourth moments and $\Sigma_j \|A(j)\| < \infty$ where $\|A(j)\|$ is the norm of the matrix $A(j)$ (i.e. the smallest number m such that $|Ax| \leq m|x|$ for all vectors x, where $|x|$ is the length of the vector). Then it may be shown that $x(n)$ is stationary and satisfies (i), (ii), (iii). However, $x(n)$ could be modified by adding $\mu(n)$, where e.g. $\mu(n) = \Sigma \alpha_j n^j$ and the conditions would still hold. Similarly after addition of a finite number of terms of the kind $\alpha n^j \cos(n\theta + \phi)$ the sequence would still satisfy the properties. The sequences may therefore be evolving relatively rapidly but not exponentially. Defining

$$R(m) = \begin{bmatrix} \rho_{11}(m) & \rho_{12}(m) & \cdots & \rho_{1r}(m) \\ \vdots & \vdots & \cdots & \vdots \\ \rho_{r1}(m) & \rho_{r2}(m) & \cdots & \rho_{rr}(m) \end{bmatrix} \tag{1.6.7}$$

the spectral representation of the 'correlation' matrix is

$$R(m) = \int_{-\pi}^{\pi} e^{im\lambda} dM(\lambda) \tag{1.6.8}$$

where $M(\lambda)$ is a matrix function with increments which are Hermitian, non-negative definite matrices, and, moreover, $dM(-\lambda) = \overline{dM(\lambda)}$ (since $\rho_{jk}(m) = \rho_{kj}(-m)$). Thus for the case of a polynomial regression of degree $(q-1)$ the matrix $M(\lambda)$ is composed of elements $m_{jk}(\lambda)$ that are zero up to $\lambda = 0$ and then jump by $(j+k+1)^{-1}\sqrt{(2j+1)(2k+1)}$ remaining at that value thereafter. In another case where $r = 3$, $y_1(n) \equiv 1$, $y_2(n) = \cos\theta n$, $y_3(n) = \sin\theta n$, $\theta \neq 0, \pi$, there are three points of increase for $M(\lambda)$; $-\theta$, 0, θ. The three increments are

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & -{}^{i}/_{2} \\ 0 & {}^{i}/_{2} & \frac{1}{2} \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & {}^{i}/_{2} \\ 0 & -{}^{i}/_{2} & \frac{1}{2} \end{bmatrix}.$$

If there were $2\ell+1$ terms corresponding to the frequencies $0, \theta_1, \ldots, \theta_\ell$, $(\theta_1, \ldots, \theta_\ell \neq 0, \pi)$ then $M(\lambda)$ would have $2\ell+1$ points of increase at $\pm\theta_j$, $j = 1, \ldots, \ell$ and at the origin. The increase at θ_j has zero elements except in the row and column corresponding to $\cos\theta_j$, $\sin\theta_j$ where the submatrix would be of the form of that in the last two rows and columns of the last matrix of increments displayed above. If $\cos\pi n$ is adjoined to the set there would be an additional jump at π which would be of the same nature as the jump at the origin. The allowable regressors do not restrict the model to consideration of only stationary z sequences as it is possible to introduce non-stationarity into the model through the mean.

It is useful to obtain an expression in spectral terms for the covariances of the B.L.U.E. and the L.S.E. when

(a) the regressors satisfy Grenander's conditions

(b) the spectral density of $e(n)$, $f(\lambda)$, is continuous[9]

(c) $f(\lambda) \geq \alpha > 0, \quad \lambda \in [-\pi,\pi]$.

Defining $D_N = \begin{bmatrix} d_1(N) & & 0 \\ & d_2(N) & \\ 0 & & d_r(N) \end{bmatrix}$ the expressions required are

$$\lim_{N\to\infty} \left\{ D_N \mathcal{E} \left\{ (\hat{\delta}-\delta)(\hat{\delta}-\delta)' \right\} D_N \right\} = \left[R^{-1}(o) \left\{ \int_{-\pi}^{\pi} 2\pi f(\lambda) dM(\lambda) \right\} R^{-1}(o) \right]^{-1} \qquad (1.6.9)$$

$$\lim_{N\to\infty} \left\{ D_N \mathcal{E} \left\{ (\tilde{\delta}-\delta)(\tilde{\delta}-\delta)' \right\} D_N \right\} = \left[\int_{-\pi}^{\pi} \{2\pi f(\lambda)\}^{-1} dM(\lambda) \right]^{-1}. \qquad (1.6.10)$$

To discuss the conditions which lead to equality of (1.6.9) and (1.6.10) it is helpful to define

$$N(\lambda) = R^{-\frac{1}{2}}(o) M(\lambda) R^{-\frac{1}{2}}(o) \qquad (1.6.11)$$

so that $N(\lambda)$ is also Hermitian with non-negative increments and $N(-\pi) = 0$, $N(\pi) = I_r$. The set of points in $(-\pi,\pi)$ where $N(\lambda)$ increases is denoted S and is termed the spectrum of the regressor set. S may be maximally decomposed into p ($\leq$ the number of regressors, r) disjoint sets E_i. The increments

$$N(E_i) = [R(o)]^{-\frac{1}{2}} \left\{ \int_{E_i} dM(\lambda) \right\} [R(o)]^{-\frac{1}{2}} \qquad (1.6.12)$$

are Hermitian symmetric with

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9 $f(\lambda)$ could be piecewise continuous providing there are no discontinuities at the 'jumps' in $M(\lambda)$.

$$N(E_i)N(E_j) = 0 \qquad i \neq j$$
$$= N(E_i) \qquad i = j \tag{1.6.13}$$

and

$$\Sigma_1^p N(E_i) = I_r. \tag{1.6.14}$$

The necessary and sufficient condition for $\hat{\delta}$ and $\tilde{\delta}$ to have the same asymptotic covariance matrix is that $f(\lambda)$ is constant on each set E_i (see [18],p 244). To attempt to illustrate the nature of the E_i a particular case, $r = p = 2$, is considered. Roughly speaking $N(\lambda)$ is obtained from $M(\lambda)$ by a linear transformation of the $y_j(n)$. Now, replace the transformed $y_j(n)$ by two new complex linear combinations - the same combinations at each time point n - so that the newly formed regressor sequences, $\tilde{y}_1(n)$ and $\tilde{y}_2(n)$ are not merely incoherent but that their spectra (i.e. sets of points where their $f(\lambda)$ are non-zero) are disjoint. So the transformed sequences constitute, roughly speaking, two signals sent over completely different frequency bands. This situation is spectrally equivalent to that where all the regressors, $y_j(n)$, are solutions of a difference equation, $\Sigma\alpha_k y_j(n-k) = 0$, whose characteristic roots lie on the unit circle. Such solutions are of the general form $n^a \cos\theta_j n$, $n^a \sin\theta_j n$, $0 \leq a \leq m_j-1$, where $\exp(i\theta_j)$ is a root of the equation of multiplicity m_j. Thus a deterministic evolving seasonal pattern would be included in the asymptotically efficient group. In Chapter III comparisons will be made which will contrast actual and asymptotic efficiency.

The framework erected for the discussion of the previous theoretical results is also suitable for the discussion of an efficient estimation procedure when the $\Gamma(n)$ covariance matrix is unknown, but the conditions (a), (b) and (c) hold. To best comprehend the construction of efficient estimates in this situation (see [20]) one can consider a group of mutually exclusive band pass filters, passing bands, such that

$U f_m = (0,\pi)$. The filtered series $z^{(k)}(n), y_1^{(k)}(n), y_2^{(k)}(n), \ldots\ldots, y_r^{(k)}$ and $e^{(k)}(n)$ are thus produced for $k = 1,2,3,\ldots,m$. If the width of the filter bands could be chosen so that the band filtered series now had only minimal variation in power in $e(n)$ within bands then approximately efficient estimates could be made by using a L.S.E. for each band. The next task is to weight together the approximately efficient individual band estimates to obtain an overall estimate with efficiency at least as good as the lowest in any band. Although this method is not exactly available in practice it does indicate what is being striven for in the following calculations.

Spectra are computed at $\frac{\pi k}{m}$, $k = 0,1,2,\ldots,m$ to form estimated quantities $\hat{f}_{zz}(\frac{\pi k}{m})$, $\hat{f}_{yy}(\frac{\pi k}{m})$ and $\hat{f}_{yz}(\frac{\pi k}{m})$. The estimates $\hat{f}_{yz}(\frac{\pi k}{m})$ and $\hat{f}_{yy}(\frac{\pi k}{m})$ may be formed even if $y(n)$ is not stationary. In the single equation situation described in (1.6.2) $\hat{f}_{zz}(\frac{\pi k}{m})$ is a scalar, $\hat{f}_{yy}(\frac{\pi k}{m})$ an $r \times r$ matrix and $\hat{f}_{yz}(\frac{\pi k}{m})$ an $r \times 1$ vector, all of which are evaluated at each k. Before setting out the estimation formulae it is worth digressing to the simple special case when $r = 1$. As in this situation

$$f_{yz}(\frac{\pi k}{m}) = f_{yy}(\frac{\pi k}{m})\beta \qquad (1.6.15)$$

an obvious estimate for each band is

$$b(k) = \hat{f}_{yy}^{-1}(\frac{\pi k}{m})\hat{f}_{yz}(\frac{\pi k}{m}) \qquad (1.6.16)$$

which approximates the ideal of a L.S.E. based on a band of width $\frac{\pi}{m}$ located at $\frac{\pi k}{m}$. To weight together these estimates within bands it is necessary to obtain knowledge of the power of the noise or error term within each band because the optimal weighting entails the ratio of signal to noise power, i.e. the relative power in each band of $y_1^{(k)}(n)$ and $\tilde{e}^{(k)}(n)$. The proposed estimation in the case when $r = 1$ is

$$b = \left[\frac{1}{2m}\Sigma^{m}_{-m+1}\tilde{f}^{-1}_{ee}(\frac{\pi k}{m})\hat{f}_{yz}(\frac{\pi k}{m})\right]\left[\frac{1}{2m}\Sigma^{m}_{-m+1}\tilde{f}^{-1}_{ee}(\frac{\pi k}{m})\hat{f}_{yy}(\frac{\pi k}{m})\right]^{-1}. \quad (1.6.17)$$

The discussion of the situation when $r = 1$ is included only as a means of helping insight and is of course a special case of the following estimation formula when $r > 1$

$$b = \left[\frac{1}{2m}\Sigma^{m}_{-m+1}\tilde{f}^{-1}_{ee}(\frac{\pi k}{m})\hat{F}_{yy}(\frac{-\pi k}{m})\right]\left[\frac{1}{2m}\Sigma^{m}_{-m+1}\tilde{f}^{-1}_{ee}(\frac{\pi k}{m})\hat{f}_{yz}(\frac{\pi k}{m})\right]. \quad (1.6.18)$$

To carry out the computation of the estimates, b, (1.6.18) can be simplified considerably [28]. A tilde is placed on the estimate of $f_{ee}(\frac{\pi k}{m})$ to emphasize that this quantity is not directly measurable and therefore a first estimate must be obtained either from a calculation of the residuals using a L.S.E. or assuming that the y sequences are realizations from a stationary vector process and therefore using

$$\tilde{f}_{ee}(\frac{\pi k}{m}) = \hat{f}_{zz}(\frac{\pi k}{m}) - \hat{f}_{zy}(\frac{\pi k}{m})\hat{f}^{-1}_{yy}(\frac{\pi k}{m})\hat{f}_{yz}(\frac{\pi k}{m}). \quad (1.6.19)$$

This latter method has the appeal that the estimate of $f_{ee}(\frac{\pi k}{m})$ use all effects from y(n) and that vector's lagged values even if the postulated model is incorrect. There is also, of course, no need for a preliminary estimate of δ. However it has the drawback that it uses up more degrees of freedom (roughly one for each band if $r = 1$) and has meaning only when y(n) is stationary. The estimates used in the applications discussed will be based on the first procedure.

This section is completed with an example of the need for caution in claiming asymptotic efficiency for a least squares estimate. The condition for asymptotic efficiency given above may be restated as follows. If $M(\lambda)$ increases at a finite set of points θ_j, then to each θ_j in the set there must also be $-\theta_j$ since R(m) is real. Adding together the increments in $M(\lambda)$ at $+\theta_j$ and $-\theta_j$ the resulting matrices <u>may</u> form a set of $p < r$ orthogonal idempotents, and if this is so, least squares is asymptotically efficient. In the case of the Fourier series, discussed earlier, even before adding $+\theta_j$ and $-\theta_j$ the matrices were orthogonal idempotents. Of course the above condition is not necessary in order for least squares to be efficient even if the e(n) are serially dependent. However, if least squares is to be efficient for <u>any</u> continuous spectral density function of the e(n) sequence then the above condition must be fulfilled. It must be emphasized that the condition may be very nearly fulfilled and the result fail to be true. For example let $e(n) = \rho e(n-1)+\epsilon(n)$ and $y_1(n) = \rho y_1(n-1)+\eta(n)$ where $\epsilon(n)$ and $\eta(n)$ are serially independent sequences with zero mean and unit variance and totally independent of each other. If $y_1(n)$ is the only variable regressed upon then it is shown [31] that the asymptotic efficiency, i.e. ratio of the asymptotic variance of the B.L.U.E. to that of the L.S.E., is $(1-\rho^2)/(1+\rho^2)$, which will be very low for ρ near 1. However since $f_{y_1}(\lambda)$ is

$$dM(\lambda) = f_{y_1}(\lambda)d\lambda = \frac{(1-\rho^2)}{2\pi(1+\rho^2-2\rho\cos\lambda)}\,d\lambda$$

it approximates a delta function at the origin as ρ tends to 1. Thus $m(\lambda)$ is near to a function - indeed arbitrarily near - that jumps only at the origin, but the result is drastically not true. The reason is most easily explained in terms of the efficient regression procedure previously described. If $\delta y_1(n)$ is regarded as an amplitude modulated signal sent by a carrier wave in which each band of frequencies is represented in proportion to the area under $f_{y_1}(\lambda)$ and above the band then the efficient procedure has been stated to be to estimate δ from each of an increasingly large number of bands, and then to recombine each band estimate of δ using the weights $f_{y_1}(\lambda)/f_e(\lambda)$. In the example proposed $f_{y_1}(\lambda)/f_e(\lambda) \equiv 1$ so that all bands should be given equal weight however least squares does not give equal weight but weights according to $f_{y_1}(\lambda)$ only and for large ρ $f_{y_1}(\lambda)$ is very small away from the origin.

1.7 Time Series Regression Procedures in Small Samples

Previous sections have concentrated on time series regression problems for large samples of data; however a much more common situation is for the econometrician to be presented with a set of data which is too small for asymptotic procedures. A traditional approach to the task of estimating economic relationships in this context has been to employ the least squares procedure and then to test the residuals computed from the least squares regression for the presence of serial correlation [29]. To outline the methods of testing for serial correlation in the regression model

$$y(n) = \Sigma_1^q \delta_j x_j(n) + u(n) \qquad n = 1,2,\ldots,N \tag{1.7.1}$$

it is assumed that $x_1(n) \equiv 1$ and for later use y is a vector with $y(n)$ in the n^{th} place. The situation which has received almost all of the attention is that where $x_j(n)$ are totally independent of the $u(n)$ sequence so that the $x_j(n)$ can be treated as fixed sequences of numbers. The symbol $x_j(n)$ will be used for such sequences. When $q = 1$ and $u(n)$ are N.I.D.$(0,\sigma^2)$ the problem was solved by von Neumann [53] who considered the 'von Neumann' ratio

$$v = \frac{\Sigma_2^N\{y(n)-y(n-1)\}^2/N-1}{\Sigma_1^N\{y(n)-\bar{y}\}^2/N} \tag{1.7.2}$$

the significance points of which were tabulated by Hart [32]. Another statistic which has been considered is the 'circular serial correlation'

$$r_1' = \frac{\Sigma_1^N\hat{u}(n)\hat{u}(n-1)}{\Sigma_1^N\{\hat{u}(n)\}^2} \tag{1.7.3}$$

where $\hat{u}(n)$ is the residual from the calculated regression of $y(n)$ on the $x_j(n)$.[10] Anderson [1] obtained the distribution of r_1' when a mean correction only has been made and tabulated the significance points. Another case which has been studied is that of a fitted Fourier series, where

$$x_{2j}(n) = \frac{\cos 2\pi jn}{q}, \quad x_{2j+1}(n) = \frac{\sin 2\pi jn}{q} \quad j = 1,2,\ldots,[q/2] \tag{1.7.4}$$

and the term for $j = q/2$ being omitted for the sine series if $j = q/2$ is even. Anderson and Anderson [2] obtained the distribution of r_1' and tabulated some significance points.

A major work in relation to this problem was that of Durbin and Watson [13]. Replacing $y(n)$ with $\hat{u}(n)$ they considered, under the same assumption for $u(n)$ as previously specified, the statistic $d = \{(N-1)v\}/N$ which may be written as

$$d = u'QA_dQu/u'Qu. \tag{1.7.5}$$

Here $Q = (I-P) = I-X(X'X)^{-1}X'$ where X has $x_j(n)$ in row n and column j, u has $u(n)$ in row n, and the matrix A_d is

$$A_d = \begin{bmatrix} 1 & -1 & 0 & \cdots\cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots\cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots\cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots\cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots\cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots\cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots\cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots\cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots\cdots & -1 & 1 \end{bmatrix}. \tag{1.7.6}$$

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10 To complete the summation from 1 to N in the numerator of (1.7.3) it is necessary to define $\hat{u}(o)$ as equal to $\hat{u}(n)$.

The statistics v and r_1' can, of course, also be written in this form for a suitable matrix. If the non-zero eigenvalues of QA_dQ are μ_j, $j = 1,\ldots,N-1$, while those of A_d are $\lambda_j = 2(1-\cos(\pi j/N))$, $j = 1,\ldots,N-1$, then Durbin and Watson established the following bounds

$$d_\ell = \frac{\Sigma_1^{N-q}\lambda_j\xi_j^2}{\Sigma_1^{N-q}\xi_j^2} \leqq d = \frac{\Sigma_1^{N-1}\mu_j\xi_j^2}{\Sigma_1^{N-1}\xi_j^2} \leqq \frac{\Sigma_1^{N-q}\lambda_{j+q-1}\xi_j^2}{\Sigma_1^{N-q}\xi_j^2} . \tag{1.7.7}$$

They tabulated the significance points for d_ℓ and d_u that are independent of X and that provided bounds for the true significance point to d. Durbin and Watson also showed how the moments of d of arbitrary order could be calculated and thus showed how an arbitrarily good approximation to the significance point for d could be obtained for example by the use of a sequence of beta distributions with the appropriate moments.

The cases of straight mean correction and the fitting of a finite Fourier Series are special in that for these it is possible to choose an appropriate matrix, i.e. one yielding a test having good powers against a simple Markov alternative for the process generating u(n) and which has the vectors x_j (having $x_j(n)$ in the n^{th} place) as eigenvectors. A somewhat similar circumstance was discussed by Hannan [26] who pointed out that to an order of accuracy higher than N^{-1}, which is the magnitude of $d-d_u$ and $d-d_\ell$, the upper bound to the significance point of d was appropriate for the case of <u>certain</u> regressors including that where $x_j(n) = n^j$ (see also McGregor [43]). A very similar observation was made by Theil and Nagar [50] who tabulated <u>approximations</u> to the significance point of d which they observed were close to those of d_u.

Much of the work which follows rests upon a simply proved result due to Grenander [17]. The result relates to W_N, a sequence of matrices, the N^{th} of N rows and columns, with elements $w_{jk}(N)$ satisfying

$$w_{jk}(N) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(j-k)\lambda} w(\lambda)d\lambda \qquad (1.7.8)$$

where $w(\lambda)$ is an even continuous function. Thus all elements down the same diagonal are independent of N. If D_N is a diagonal matrix with $d_j(N)$ in the j^{th} place then Grenander [17] shows

$$\lim_{N\to\infty} D_N^{-1}X'W_N X D_N^{-1} = \int_{-\pi}^{\pi} w(\lambda)dM(\lambda) \qquad (1.7.9)$$

and it is apparent that a special case of (1.7.9) gives $R(o) = \lim_{N\to\infty} D_N^{-1}X'XD_N^{-1}$. As has been already noted when $q = 1$, i.e. only a mean correction is made, the situation is well understood and tabulations (Hart [32]) of v and (Anderson [1]) of r_1' are available. It is worth adding that the statistic r_1, where the j^{th} serial correlation is

$$r_j = \frac{c_j}{c_o}, \quad c_j = N^{-1}\Sigma_1^{N-j}\hat{u}(n)\hat{u}(n+j) \qquad (1.7.10)$$

has on the null hypothesis of serial independence for $u(n)$ a mean $-N^{-1}$ and a variance $(N-2)^2/\{N^2(N-1)\} = (N+3)^{-1}-8/N^3+O(N^{-4})$. Thus to an adequate approximation, when $N \geqq 15$ certainly, (r_1+N^{-1}) has the variance of an ordinary correlation coefficient (with mean corrections) from (N+4) pairs of serially independent Gaussian observations. As suggested by Watson [54] an examination of the significance points of the distribution of selected tabulated quantities shows this approximation to be quite adequate. Thus the statistics

$$t = \frac{(r_1+N^{-1})\sqrt{N+2}}{\sqrt{\{1-(r_1+N^{-1})^2}} \qquad (1.7.11)$$

is Student's t with N+2 degrees of freedom and tests the serial independence of the data. The derivation of the mean and variance is quite straightforward and follows Durbin and Watson [13] but as it is a method that is also employed for $q > 1$ it is outlined in preparation for the later work. If one represents r_1 as

$$r_1 = \frac{y'QW_NQy}{y'Qy} \tag{1.7.12}$$

where $Q = I-N^{-1}11'$ (1 being a vector composed entirely of units) and for r_1 the matrix $2W_N$ has units in the 2 diagonals bordering the main diagonal and zeros elsewhere. Then since r_1 is independent of the denominator (see Pitman [47], Watson [54]) the p^{th} moment about the origin of r_1 is

$$\mathcal{E}(r_1^p) = \frac{\mathcal{E}\{[y'QW_NQy]^p\}}{\mathcal{E}\{[y'Qy]^p\}} . \tag{1.7.13}$$

If we take y to be composed of N.I.D.(0,1) variates and note that, for any symmetric matrix A,

$$\mathcal{E}\left([y'Ay]^p\right) = \mathcal{E}\left\{\Sigma\Sigma\Sigma\ldots\Sigma\alpha_{j_1}\ldots\alpha_{j_p}\xi_{j_1}^2\ldots\xi_{j_p}^2\right\}$$

where the α_j are the eigenvalues of A and ξ_j are N.I.D.(0,1). Following Durbin and Watson [13] and Kendall and Stuart [41,p 68] the relation between the p^{th} moment of the quadratic form y'Ay and its cumulants is given by

$$\mathcal{E}\left\{[y'Ay]^p\right\} = \Sigma a(s,r)k_{s_1}^{r_1}k_{s_2}^{r_2}\ldots\ldots k_{s_m}^{r_m} \tag{1.7.14}$$

where k_j is the j^{th} cumulant of $\Sigma\alpha_j\xi_j^2$; the summation is over all $s_1,s_2,\ldots,s_m$, such that $s_1r_1+s_2r_2+\ldots+s_mr_m = p$, $s_1 < s_2 < s_3 < \ldots < s_m$ and

$$a(s,r) = \frac{p!}{(s_1!)^{r_1}(s_2!)^{r_2}\ldots(s_m!)^{r_m}} \cdot \frac{1}{r_1!r_2!\ldots r_m!} .$$

As the cumulant of the quadratic form y'Ay is $k_j = \{2^{j-1}(j-1)!\}\text{tr}\{A^j\}$ its p^{th} moment can be expressed directly in terms of $\text{tr}A^j$. Since

$$tr(Q^p) = trQ = N-1,$$

$$tr(QW_NQ) = tr(W_NQ) = tr(W_N)-N^{-1}tr11'W_N = -N^{-1}1'W_N1 = -(N-1)/N$$

and

$$\begin{aligned} tr([QW_NQ]^2) &= tr(QW_NQW_N) = trW_N^2-2N^{-1}1'W_N1+N^{-2}(1'W_N1)^2 \\ &= \tfrac{1}{2}(N-1)-N^{-1}(2N-3)+N^{-2}(N-1)^2 \end{aligned} \qquad (1.7.15)$$

one finds moments about the origin,

$$\mathcal{E}(r_1) = -N^{-1}, \quad \mathcal{E}(r_1^2) = (N^2-3N+3)|(N^2(N-1)) \qquad (1.7.16)$$

and so the quoted results for the mean and variance of r_1 hold. The above procedure has repeatedly made use of the fact that the trace of a product of two factors is independent of their order and this will again be used below. The techniques apply in general (see [13]) <u>for any Q</u> and the moments depend only upon the evaluation of $tr[(QW_NQ)^p]$. Here W_N is of the general form discussed above (see (1.7.8)). The trace in question is a linear combination of $tr(W_N^p)$ and expressions of the form $tr(PW_N^{a_1}PW_N^{a_2}.....PW_N^{a_m})$ where as before $P = X(X'X)^{-1}X'$ and $\sum_j a_j = p$. For example if $p = 3$ then in the expression of $[(I-P)W_N(I-P)]^3$ one obtains a term $W_NPW_N^2$, however, $tr[W_NPW_N^2] = tr[PW_N^3]$ which is of the required form. Now, repeatedly using the idempotency of P and the fact that $trAB = trBA$ a general expression,

$$\begin{aligned} tr(PW_N^{a_1}PW_N^{a_2}...PW_N^{a_m}) &= tr(X(X'X)^{-1}X'W_N^{a_1}X...X'W_N^{a_m}) \\ &= tr(D_N(X'X)^{-1}D_ND_N^{-1}X'W_N^{a_1}XD_N^{-1}...D_N^{-1}X'W_N^{a_m}XD_N^{-1}) \\ &= tr\left\{\prod_1^m\left[R^{-1}(o)\int_{-\pi}^{\pi}\{w(\lambda)\}^{a_j}dM(\lambda)\right]\right\} + O(1) \\ &= tr\left\{\prod_1^m\int_{-\pi}^{\pi}w(\lambda)^{a_j}dN(\lambda)\right\} + O(1) \end{aligned} \qquad (1.7.17)$$

where $w(\lambda)$ is the generating function of the matrix W_N. Thus the moments depend only upon the traces of products of matrices of the form $\int_{-\pi}^{\pi} w(\lambda)^p dN(\lambda)$. Since $\mathcal{E}(y'Qy)^p = O(N^p)$, the order of the error in the j^{th} moment, corrected for the mean for $j > 1$ - obtained by inserting the correct expression for $tr(W_N^p)$ and approximations such as (1.7.17) in the other traces - is $O(N^{-j})$.

A special case (where the L.S.E. is efficient) is that where $N(\lambda)$ increases at S points in such a way that when these points are grouped together in pairs, symmetrically placed with respect to the origin, the resulting increments in $N(\lambda)$ - N_j being the sum of increments at points $+\theta_j$ and $-\theta_j$ - are orthogonal idempotents, i.e. $\Sigma N_j = I$, $N_j N_k = \delta_j^k N_j$. There cannot be more than q of these of course. Then in this case the expression in (1.7.16) becomes

$$\prod_1^m \int_{-\pi}^{\pi} \{w(\lambda)\}^{a_j} dN(\lambda) = \prod_1^m \left\{\sum_k \{w(\theta_k)\}^{a_j} N_k\right\}$$
$$= \sum_k w^p(\theta_k) N_k = \int_{-\pi}^{\pi} w^p(\lambda) dN(\lambda) \qquad (1.7.18)$$

and so one may write

$$tr\left\{[QW_N Q]^p\right\} = tr(W_N^p) - tr\left\{\int_{-\pi}^{\pi} \{w(\lambda)\}^p dN(\lambda)\right\} \qquad (1.7.19)$$

since the sum of the coefficients of the expression of $tr\{[QW_N Q]^p\}$ in terms of expressions such as $tr\{PW_N^{a_1} PW_N^{a_2} \ldots PW_N^{a_m}\}$ is evidently zero. The right hand side of (1.7.18) may be written as

$$tr\left\{[QW_N Q]^p\right\} = tr(W_N^p) - \int_{-\pi}^{\pi} w^p(\lambda) dn(\lambda) \qquad (1.7.20)$$

where $tr(N(\lambda)) = n(\lambda)$ and this is thus a function which increases by jumps of integral amounts at points θ_j to a value q. Grenander and Rosenblatt have shown [18,p 103] in a much more general context that

$$N^{-1}\mathrm{tr}\{W_N^p\} = (1/2\pi)\int_{-\pi}^{\pi}\{w(\lambda)\}^p d\lambda + O(1)$$

so that $w(2\pi j/N)$, $j = -[N/2],\ldots,[(N+1)/2]$, are approximately the eigenvalues of W_N. Thus (1.7.20) can be roughly interpreted as the removal from the spectrum of the eigenvalues $w(\theta_j)$ - repeated $\mathrm{tr}(N_j)$ times - so that effect has been as if the q regressor vectors x_j had been eigenvectors of W_N for eigenvalues $w(\theta_j)$. Of course the eigenvalues of W_N may not be repeated, but if N is not small the eigenvalues will be very close together and there will be a number near any one $w(\theta_j)$. If $N(\lambda)$ has a single jump at the origin, then the correction term due to regression in (1.7.19) is $q\{w(o)\}^p$. Since $w(\theta)$ is the function $\cos\theta$, the largest possible eigenvalues have been removed. If instead of r_1, d (see (1.7.5)) had been studied a similar result would be obtained. Although the matrix A_d is not quite of the form required, since the two end elements in the main diagonal differ from those elsewhere in that diagonal, this effect will be of an order of magnitude no larger than those already neglected. Now $w(\theta)$ is $2(1-\cos\theta)$ and the q smallest eigenvalues are being removed. Thus the upper bound to the test statistic will be appropriate to an order of approximation higher than N^{-1}. It has already been pointed out (see [26] and [43]) that this will be the case, for example, when a trend has been eliminated by fitting a polynomial in n. Returning to the general case (no specification that would make the L.S.E. asymptotically efficient) straightforward but somewhat lengthy calculations show that mean and variance of r_1 are

$$\mathcal{E}(r_1) = -\frac{1}{N-q}\int_{-\pi}^{\pi} w(\lambda)dn(\lambda)+O(N^{-1})$$

$$\mathrm{var}(r_1) = \frac{2}{(N-q)(N-q+2)}\left\{\mathrm{tr}W_N^2-2\int_{-\pi}^{\pi} w^2(\lambda)dn(\lambda)\right. \tag{1.7.21}$$

$$\left. + \mathrm{tr}\left[\left(\int_{-\pi}^{\pi} w(\lambda)dN(\lambda)\right)^2\right] + \tfrac{1}{2}\left[\mathrm{tr}\int_{-\pi}^{\pi} w(\lambda)dN(\lambda)\right]\right\} - (\mathcal{E}(r_1))^2.$$

Of course $w(\lambda) = \cos\lambda$ and $\mathrm{tr}(W_N^2) = \frac{1}{2}(N-1)$ for r_1 but (1.7.21) is general in the sense that it applies for all W_N of the form specified in (1.7.8). Higher moments can be similarly expressed, though the expression quickly becomes quite complicated. Consider a stationary $x(n)$ vector with serial correlations satisfying assumption (iii) of §1.6. With economic data it is very likely that all $f_{jj}(\lambda)$ will be very concentrated at the origin (see [16]). If $w(\lambda)$ is fairly smooth near the origin, as is the case when $w(\lambda) = \cos\lambda$ or indeed is the case for any $w(\lambda)$ likely to arise, then the terms in (1.7.21) may be <u>approximated</u> as follows

$$\int_{-\pi}^{\pi} w(\lambda)dn(\lambda) \cong qw(o)$$

$$\mathrm{tr}\left\{\left[\int_{-\pi}^{\pi} w(\lambda)dN(\lambda)\right]^2\right\} = \iint w(\lambda_1)w(\lambda_2)\mathrm{tr}(dN(\lambda_1)dN(\lambda_2)) \tag{1.7.22}$$

$$\cong qw^2(o).$$

So, in this case also, the lower moments again will be close to those which would obtain if the q eigenvalues nearest to $w(o)$ are removed by the regression so that again for the Durbin Watson statistic, the smallest eigenvalues are being removed and the upper bound to the statistic is appropriate.

To summarize the effect of the regression on the significance point of d (or any selected statistic) depends substantially on the cross spectra of the regressor vector $x(n)$, i.e. upon $N(\lambda)$. The most important effect is to reduce the mean by a quantity that is, to order N^{-1}, $-(N-q)^{-1}\int_{-\pi}^{\pi} w(\lambda)dn(\lambda)$. If as is often the case in economics, the spectrum of $x(n)$ is relatively very concentrated at the origin of frequencies the effect will be approximately allowed for by using the significance point for d_u as the true significance point. This procedure will be accurate to order N^{-1}.

1.8 Serial Dependence Under the Null Hypothesis

Serial dependence in the disturbance term $u(n)$ in the relation (1.7.1) is so common-place that a more important problem from the point of view of the economist is that which arises when the null hypothesis is not serial independence for $u(n)$ as it was in §1.7 but rather

$$u(n) = \rho u(n-1)+\epsilon(n) \tag{1.8.1}$$

where the $\epsilon(n)$ are N.I.D.$(0,\sigma_u^2)$ (see [29]). The alternative one has principally in mind is

$$u(n)+\rho_1 u(n-1)+\rho_2 u(n-2) = \epsilon(n) \tag{1.8.2}$$

which becomes of the form (1.8.1) if $\rho_2 = 0$. The hypothesis $\rho_2 = 0$ is appropriately tested by means of a form of partial autocorrelation,

$$r_{02.1} = \frac{r_2-r_1^2}{1-r_1^2} \tag{1.8.3}$$

where $r_j = c_j/c_o$ and the c_j are defined in (1.7.10). If, for notational convenience, $\Sigma_1^q \beta_j x_j(n)$ is set equal to $\mu(n)$ then the regression model (1.7.1) may be written

$$y(n)-\mu(n) = \rho \left\{y(n-1)-\mu(n-1)\right\} + \epsilon(n). \tag{1.8.4}$$

It is as well to distinguish clearly this model (1.8.4) from another often proposed in economics, i.e.

$$y(n) = \rho y(n-1)+\nu(n)+\epsilon(n) \tag{1.8.5}$$

where $\nu(n)$ is some other linear combination of the $x_j(n)$, e.g. $\Sigma\beta_j' x_j(n)$ as it would be very convenient if the treatment which will be proposed for (1.8.4) could also be used for (1.8.5).

The procedures suggested for (1.8.4) will only also be applicable to (1.8.5) if the vectors $x_j(n)$ are of a particular form. The nature of $x_j(n)$ required for this equivalence may be deduced by equating a general form of (1.8.4)

$$\Sigma\rho_j\Big(y(n-j)-\beta'x(n-j)\Big) = u(n) \tag{1.8.6}$$

to the general form of (1.8.5) i.e.

$$\Sigma\rho_j y(n-j) = \beta_1'x(n)+u(n) \tag{1.8.7}$$

where $x_j(n)$ are now treated in vector form. The solution requires that $\Sigma\rho_j\beta'x(n-j) = \beta_1'x(n)$ and therefore will have to be of the form $x(n) = n^u\cos\theta_j n$, since roots which are not on the unit circle are not acceptable. Thus the nature of the $x_j(n)$ vectors which would result in applicability of the methods to both (1.8.4) and (1.8.5) is in fact that the vectors be those for which the L.S.E. is always asymptotically efficient.

The definition of c_j adopted in (1.7.10) ensures that $|r_{02.1}| \leqq 1$. Indeed $\Sigma\Sigma\alpha_j\alpha_k c_{j-k} = N^{-1}\hat{u}'T\hat{u}$ where $\hat{u}$ has $u(n)$ as its n^{th} element and T has $\Sigma\alpha_j^2$ in the main diagonal, $(\alpha_o\alpha_1+\alpha_1\alpha_2)$ in the 2 diagonals adjacent to the main diagonal, and $\alpha_o\alpha_2$ in the next two diagonals. This is the covariance matrix of $\alpha_1\epsilon(n)+\alpha_2\epsilon(n-1)+\alpha_3\epsilon(n-2)$ and thus is positive definite so that $\Sigma\Sigma\alpha_j\alpha_k c_{j-k} \geqq 0$. Thus

$$\begin{bmatrix} 1 & r_1 & r_2 \\ r_1 & 1 & r_1 \\ r_2 & r_1 & 1 \end{bmatrix}$$

is positive definite and this ensures that $(1-r_1^2)(1-r_{02.1}^2) \geqq 0$.

In the case where only a mean correction has been made there has been a detailed investigation of statistics of the type $r_{02.1}$ by Daniels [9], Jenkins [35,36] and Watson [54]. The exact derivations have been based upon the use of a circular definition which replaces c_j by

$$c'_j = N^{-1}\,\Sigma_1^N \hat{u}(n)\hat{u}(n+j) \qquad (1.8.8)$$

where again to complete the summation from 1 to N in c'_j the definition $\hat{u}(n+j) = \hat{u}(j)$ is employed. This means that all the matrices which occur both in c'_j and the quadratic form in the exponent of the likelihood function are circulants and thus commute and may be simultaneously diagonalized. Unfortunately once something other than a mere mean correction, or a regression on the trigonometric functions considered by Anderson and Anderson [2] is made the advantages of this definition fade. If U^j is the circulant matrix which makes $c'_j = N^{-1}\hat{u}'U^j\hat{u}$ and Γ is the circulant covariance matrix of the u(n) then the transition from u to $Qu = \hat{u}$ means that we are concerned with a set of matrices QU^jQ, $Q\Gamma Q$, that no longer necessarily commute and there is little point in adopting the circular definition.

The situation in which one wishes to test whether (1.8.1) or (1.8.2) is the appropriate form for the error in (1.7.1) is the same as testing whether the disturbance $\epsilon(n)$ in (1.8.4) is serially independent. The test statistic used will be the one already mentioned in §1.7, $r_{02.1}$, however the $\hat{u}(n)$ which are used in the definition of the c_j used in $r_{02.1}$ have to be those residuals resulting from the B.L.U.E. under the null hypothesis. A simple computing procedure involves searching over a grid in the range $-1 < \rho < 1$, computing for each grid value of ρ estimates, $\hat{\beta}_j(\rho)$ and $\hat{u}(\rho)$, and choosing that set of $\hat{u}(\rho)$ with minimum sum of squares (see [10]). In fact the method of computation of the $\hat{\beta}_j(\rho)$ and $\hat{u}(\rho)$ was to transform y(n) and the $x_j(n)$ by an N dimensional matrix M',

$$M' = \begin{bmatrix} \sqrt{(1-\rho^2)}, & \rho, & 0 & & & & \\ 0 & , 1, & \rho & & & & \\ 0 & , 0, & 1, & \rho & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & & 1, & \rho \\ & & & & & & 1 \end{bmatrix} \qquad (1.8.9)$$

for each ρ in the grid. The L.S.E. and the associated residuals were found for each transformation of the original data and the $\hat{u}(\rho)$ used in the test statistic was the set with the minimum sum of squares.

An alternative to using the statistic $r_{02.1}$ would be to remove both the term $\mu(n)$ and $y(n-1)$ by regression and to use d for the resulting residuals. This does not seem quite appropriate since it gives a statistic that except for end effects will be $2(1+r_1 r_{02.1})$ and this is nearly 2 if r_2 is near to r_1^2 OR if r_1 is near to zero. Thus d seems inappropriate for testing $\rho_2 = \rho_1^2$ independently of the value of ρ_1.

It has already been established that $|r_{02.1}| \leq 1$ and $r_{02.1}$ is simply expressed as a function of the c_j, i.e.

$$r_{02.1} = \frac{c_o c_2 - c_1^2}{c_o^2 - c_1^2} \,. \qquad (1.8.10)$$

Expanding H as a function of (c_o, c_1, c_2) in a Taylor's series about $(1, \rho, \rho^2)$ and following Cramer [8,pp 354-5] produce the expressions,

$$\mathcal{E}(r_{02.1}) = \Sigma_o^2 H_u \mathcal{E}(c_u - \rho^u) + \tfrac{1}{2}\sum_o^2 \Sigma H_{uv} \mathcal{E}(c_u - \rho^u)(c_v - \rho^v) + O(N^{-1})$$
$$\mathcal{E}(r_{02.1}^2) = \Sigma\Sigma H_u H_v \mathcal{E}\left\{(c_u - \rho^u)(c_v - \rho^v)\right\} + O(N^{-1}) \qquad (1.8.11)$$

where H_u are the first order derivatives evaluated at $(1, \rho, \rho^2)$ and H_{uv} are the second order derivatives similarly evaluated. These results follow from the arguments in the reference just cited, using the fact that the k^{th} moment about the mean of c_v

is $O(N^{-k+1})$, $k > 1$, while the bias is $O(N^{-1})$ so that $\mathcal{E}(c_u-\rho^u)$ is $O(N^{-k+1})$. Now

$$\begin{aligned}\mathcal{E}(c_j) &= N^{-1}\mathcal{E}\left\{u'QW_N^{(j)}Qu\right\} \\ &= N^{-1}\text{tr}\left\{RQW_N^{(j)}Q\right\}\end{aligned} \tag{1.8.12}$$

where $W_N^{(j)}$ is $N \times N$ with $\frac{1}{2}$ in the j^{th} diagonal above and below the principle diagonal and zeros elsewhere, R has $\rho^{|j|}$ everywhere in the j^{th} diagonals above and below the main diagonals and var.u(n) is assumed equal to unity since $r_{02.1}$ is scale free. The elements of R are generated by (see (1.7.8))

$$f_u(\lambda) = \frac{(1-\rho^2)}{2\pi(1+\rho^2-2\rho\cos\lambda)} \tag{1.8.13}$$

and those of $W_N^{(j)}$ are generated by $(\frac{1}{2\pi})\cos j\lambda$. Thus (1.8.12) becomes

$$\begin{aligned}\mathcal{E}(c_j) = N^{-1}\text{tr}\Big\{&RW_N^{(j)}-(X'X)^{-1}X(W_N^{(j)}R+RW_N^{(j)})X' \\ &+(X'X)^{-1}XRX'(X'X)^{-1}XW_N^{(j)}X'\Big\}.\end{aligned} \tag{1.8.14}$$

Since $\text{tr}\{R_N W_N^{(j)}\} = (N-j)\rho^j$ it is easily checked that $\Sigma H_j \text{tr}\{RW_N^{(j)}\} = (N-j)\Sigma H_j \rho^j = 0$ and therefore only the last two expressions in $\mathcal{E}(c_j)$ need be considered. The last term gives - to the order N^{-1} - a contribution to $\Sigma H_u \mathcal{E}(c_u-\rho^u)$ which is

$$N^{-1}\text{tr}\left\{\int_{-\pi}^{\pi}2\pi f(\lambda)dN(\lambda)\int_{-\pi}^{\pi}\sum_j(\cos j\lambda H_j)dN(\lambda)\right\}$$

$$= N^{-1}\text{tr}\left\{\int_{-\pi}^{\pi}\frac{1-\rho^2}{1+\rho^2-2\rho\cos\lambda}dN(\lambda)\int_{-\pi}^{\pi}\frac{(\rho^2-2\rho\cos\lambda+\cos 2\lambda)}{1-\rho^2}dN(\lambda)\right\}. \tag{1.8.15}$$

The second term may be evaluated by replacing $W_N^{(j)}R$ (and $RW_N^{(j)}$) by a matrix $A_N^{(j)}$ whose elements are generated by $2f(\lambda)\cos j\lambda$. For example the second term is not changed if $W_N^{(j)}R$ is replaced by

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \rho & \cdots & \rho^{N-1} \\ \rho & 1 & \cdots & \rho^{N-2} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \rho^{N-1} & \rho^{N-2} & \cdots & 1 \end{bmatrix} \qquad (1.8.16)$$

The matrix (1.8.16) differs from the matrix $A_N^{(j)}$ only because of the elements in the first and last rows. For example the k^{th} element in the last row should have ρ^{N-k+1} added to bring it to the form of $A_N^{(j)}$. The contribution to $X(W_N^{(j)}R+RW_N^{(j)})X'$ from this missing last row is a matrix with $x_j(N)\sum_{n=1}^{N}\rho^{N-n+1}x_k(n)$ in row j column k. Then it may be seen that the contribution to $\mathcal{E}(c_j)$ from this missing row is dominated by const.$N^{-1}\sum_j \frac{|x_j(N)|}{d_j(N)} = O(N^{-1})$.

A similar argument holds for other elements. To the same order of accuracy it is possible to show that

$$\mathcal{E}(c_j-\rho^j)(c_k-\rho^k) = (^2/N^2)\,\text{tr}\left\{RW_N^{(j)}RW_N^{(k)}\right\} + O(N^{-1}) \qquad (1.8.17)$$

and by direct but tedious manipulations to find

$$\tfrac{1}{2}\sum_o^2 \Sigma H_{uv}\,\mathcal{E}\left\{(c_u-\rho^u)(c_v-\rho^v)\right\} = -N^{-1} \qquad (1.8.18)$$

so that the final expressions for the mean and variance are

$$\mathcal{E}(r_{02.1}) = -N^{-1}-2N^{-1}\int_{-\pi}^{\pi}\frac{\rho^2-2\rho\cos\lambda+\cos2\lambda}{1+\rho^2-2\rho\cos\lambda}\,dn(\lambda)$$

$$+N^{-1}\text{tr}\left[\int_{-\pi}^{\pi}\frac{(1-\rho^2)}{(1+\rho^2-2\rho\cos\lambda)}\,dN(\lambda)\int_{-\pi}^{\pi}\frac{\rho^2-2\rho\cos\lambda+\cos2\lambda}{(1-\rho^2)}\,dN(\lambda)\right] + O(N^{-1})$$

$$\text{Var}(r_{02.1}) = N^{-1}+O(N^{-1}) \qquad (1.8.19)$$

and so to order N^{-1} only the mean is affected. It might be better to obtain the variance to a higher order of accuracy but this has not been done because of the labour involved. In case of a straight mean correction the results of Jenkins [36] suggest that neglecting terms of order N^{-3}, the variance should be $1/(N+2)$. In the case where $N(\lambda)$ is a function that jumps only at points which may be put into pairs of symmetrically placed points (with respect to the origin) so that the corresponding sums of the pairs of jumps are orthogonal idempotents then $\mathcal{E}(r_{02.1})$ in (1.8.19) reduces to

$$-N^{-1}\left(1+\int_{-\pi}^{\pi}\frac{\rho^2-2\rho\cos\lambda+\cos 2\lambda}{1+\rho^2-2\rho\cos\lambda}\,dn(\lambda)+o(N^{-1})\right) \tag{1.8.20}$$

and <u>in particular</u> if the only jump is at the origin the expectation becomes $-(q+1)/N$, which to order N^{-1} agrees with known results for a mean correction. If the $x_j(n)$ series have spectra which are relatively very concentrated at the origin this may again be a good approximation. So the test procedure if the $x_j(n)$ are of this nature is to use $(r_{02.1}+(q+1)/N)$ as an ordinary correlation from $(N+2)$ pairs of observations. It may also be advisable to obtain higher order approximations to the lower moments of $r_{02.1}$. Higher order partial correlations need consideration in the same way. For example the next step logically would be to estimate the $\hat{\beta}_j$ and $\hat{u}(n)$ over a grid of values of ρ_1 and ρ_2 that are associated with stationary $u(n)$. The $\hat{u}(\rho_1,\rho_2)$ which were associated with the minimal sum of squares would then be used in forming $r_{03.12}$ to test an alternative hypothesis of a 3rd order autoregression for $u(n)$ against a null hypothesis assuming $u(n)$ is a second order autoregression.

II ESTIMATION AND INTERPRETATION OF SPECTRA ARISING FROM ECONOMIC DATA

2.1 Basic Spectral Estimators

It is again assumed, to simplify the discussion, that the spectral distribution matrix, $F(\lambda)$, is absolutely continuous so that the random vectors employed in (1.2.10), the spectral representation of $x(n)$, are therefore characterized by the matrices $c(\lambda)$ and $q(\lambda)$ defined in (1.2.6). It is apparent from (1.2.6) that an estimate of $f(\lambda)$, say $\hat{f}(\lambda)$, will be composed of estimates of these fundamental quantities, henceforth referred to as $\hat{c}(\lambda)$ and $\hat{q}(\lambda)$. In §1.4, in which the subject of estimation was briefly introduced, an expression was presented for the estimator of element in the j^{th} row and k^{th} column of $f(\lambda)$. It is necessary to expand on the expression (1.4.5) to extract the estimators of $c_{jk}(\lambda)$ and $q_{jk}(\lambda)$. First, we must relax the assumption that $x(n)$ has a zero mean vector and redefine the estimate of the cross covariance between the j^{th} and k^{th} elements of $x(n)$ as

$$c'_{jk}(n) = \frac{1}{N-n}\Sigma_{m}{}_1^{N-n}(x_j(m)-\bar{x}_j)(x_k(m+n)-\bar{x}_k) \qquad (2.1.1)$$

where $\bar{x}_j = \Sigma_m{}_1^N x_j(m)/N$, $j = 1,\ldots,p$, and replace $c_{jk}(n)$ in (1.4.5) by $c'_{jk}(n)$. Now it is possible to represent $c'_{jk}(n)$ as follows,

$$c'_{jk}(n) = O_{jk}(n)+E_{jk}(n) \qquad (2.1.2)$$

where

$$O_{jk}(n) = \tfrac{1}{2}\left\{c'_{jk}(n)-c'_{jk}(-n)\right\}$$
$$E_{jk}(n) = \tfrac{1}{2}\left\{c'_{jk}(n)+c'_{jk}(-n)\right\} \qquad (2.2.3)$$

and since it is apparent from (2.1.1) that $c'_{jk}(n) = c'_{kj}(-n)$ another expression for $O_{jk}(n)$ and $E_{jk}(n)$ is

$$O_{jk}(n) = \tfrac{1}{2}\left\{c'_{jk}(n)-c'_{kj}(n)\right\}$$
$$E_{jk}(n) = \tfrac{1}{2}\left\{c'_{jk}(n)+c'_{kj}(n)\right\}. \qquad (2.1.4)$$

When the expression for $c'_{jk}(n)$, given in (2.1.2), is inserted in (1.4.5) we find

$$\hat{f}_{jk}(\lambda) = \hat{c}_{jk}(\lambda)+\hat{q}_{jk}(\lambda) \tag{2.1.5}$$

with

$$\begin{aligned}\hat{c}_{jk}(\lambda) &= \frac{1}{2\pi}\Sigma_{-N+1}^{N-1}\cos n\lambda k_n(1-\frac{|n|}{N})E_{jk}(n) \\ &= \frac{1}{2\pi}k_o E_{jk}(o)+\frac{1}{\pi}\Sigma_1^{N-1}\cos n\lambda k_n(1-\frac{|n|}{N})E_{jk}(n)\end{aligned} \tag{2.1.6}$$

and

$$\begin{aligned}\hat{q}_{jk}(\lambda) &= \frac{1}{2\pi}\Sigma_{-N+1}^{N-1}\sin n\lambda k_n(1-\frac{|n|}{N})O_{jk}(n) \\ &= \frac{1}{\pi}\Sigma_1^{N-1}\sin n\lambda k_n(1-\frac{|n|}{N})O_{jk}(n).\end{aligned} \tag{2.1.7}$$

The estimates defined in (2.1.6) and (2.1.7) are the basis of a spectral investigation of $x(n)$. From these quantities a number of characteristics are developed to aid in understanding the relation between the elements in the vector. Two of these have already arisen, the coherence $W_{jk}(\lambda)$ and the phase $\theta_{jk}(\lambda)$. The estimate of $W_{jk}(\lambda)$ is

$$\hat{W}_{jk}(\lambda) = \left\{\frac{\hat{c}^2_{jk}(\lambda)+\hat{q}^2_{jk}(\lambda)}{\hat{f}_{jj}(\lambda)\hat{f}_{kk}(\lambda)}\right\}^{\frac{1}{2}} \tag{2.1.8}$$

and the estimate of the phase, $\theta_{jk}(\lambda)$, is defined to avoid any ambiguity as follows,

$$\begin{aligned}\theta_{jk}(\lambda) &= \tan^{-1}\left(\frac{\hat{q}_{jk}(\lambda)}{\hat{c}_{jk}(\lambda)}\right) \text{ if } c_{jk}(\lambda) > 0 \\ &= \left\{\tan^{-1}\frac{\hat{q}_{jk}(\lambda)}{\hat{c}_{jk}(\lambda)} + \pi\,\mathrm{sign}\,\hat{q}_{jk}(\lambda)\right\} \text{ if } c_{jk}(\lambda) < 0.\end{aligned} \tag{2.1.9}$$

The measure of coherence given in (2.1.8) should be treated cautiously for the following reason. If the phase angle between any two series j and k, θ_{jk} is changing rapidly then using (2.1.8) will probably lead to an under-estimate of coherence. The smoothing procedure necessary to reduce the sampling variability means that

$W_{jk}(\lambda)$ is being estimated from averages of the estimators $c_{jk}(\lambda)$, $q_{jk}(\lambda)$, $f_{jj}(\lambda)$ and $f_{kk}(\lambda)$ over a band of frequencies, which will be designated $G(\lambda)$. Thus each coherency estimate $\hat{W}_{jk}(\lambda)$ might be represented as

$$\left| \int_{G(\lambda)} f_{jk}(\omega) e^{i\theta_{jk}(\omega)} d\omega \right| / \left\{ f_{jj}(\lambda) f_{kk}(\lambda) \right\}^{\frac{1}{2}} \tag{2.1.10}$$

where $f_{jj}(.)$ and $f_{kk}(.)$ are assumed to change little over the band $G(\lambda)$ and therefore if $|f_{jk}(\omega)|$ is also almost constant over the band then the expression for $\hat{W}_{jk}(\lambda)$ becomes

$$W_{jk}(\lambda) \left| \int_{G(\lambda)} e^{i\theta_{jk}(\omega)} d\omega \right| \tag{2.1.11}$$

If $\theta_{jk}(\omega)$ is changing rapidly the second factor may well be close to zero and thus the bias in the coherence estimate may not be negligible. Although no attempt has been made in this work to allow for this problem it must be mentioned that because the bias in the phase is negligible it is possible to estimate $\theta_{jk}(\omega)$ and then to approximate it over the band $G(\lambda)$ as $\theta_{jk}(\lambda)+(\omega-\lambda)\theta'_{jk}(\lambda)$ and then make a phase shift to eliminate the 2nd term. The work on proper estimation when the phase is changing rapidly (relative to the band $G(\lambda)$) still appears to be exploratory (see [3], [51]).

If for any two of the elements of the vector $x(n)$ it is thought that $x_j(n)$ may be explained by $x_k(n)$ then this would lead to the investigation of $B_{jk}(\lambda)$, the regression transfer function and $f_{j:k}(\lambda)$, the residual spectral density function. Both these functions may be expressed in terms of spectral estimators previously defined as

$$\begin{aligned} B_{jk}(\lambda) &= \frac{\hat{f}_{jk}(\lambda)}{\hat{f}_{kk}(\lambda)} = \alpha_{jk}(\lambda)+i\beta_{jk}(\lambda) \\ &= \frac{c_{jk}(\lambda)}{f_{kk}(\lambda)} + i\,\frac{q_{jk}(\lambda)}{f_{kk}(\lambda)} \end{aligned} \tag{2.1.12}$$

and

$$f_{j:k}(\lambda) = f_{jj}(\lambda) \left\{ 1-W^2_{jk}(\lambda) \right\}. \tag{2.1.13}$$

The complex regression transfer function defined in (2.1.12) is usually considered in the following polar form

$$B_{jk}(\lambda) = G_{jk}(\lambda)e^{-i\theta_{jk}(\lambda)} . \qquad (2.1.14)$$

The only new quantity in this expression is $G_{jk}(\lambda)$, usually referred to as the gain of the function, which is estimated by

$$\hat{G}_{jk}(\lambda) = \frac{|\hat{f}_{jk}(\lambda)|}{\hat{f}_{kk}(\lambda)}$$

$$= \sqrt{\hat{c}^2_{jk}(\lambda)+\hat{q}^2_{jk}(\lambda)}/f_{kk}(\lambda). \qquad (2.1.15)$$

The form of the covariance averaging kernel which has been used to obtain $\hat{f}_{jj}(\lambda)$, $\hat{c}_{jk}(\lambda)$ and $\hat{q}_{jk}(\lambda)$ and which therefore underlies all quantities based on these estimates is due to Parzen and is

$$k(\tfrac{n}{m}) = 1-6(\tfrac{n}{m})^2+6|\tfrac{n}{m}|^3 \qquad |\tfrac{n}{m}| \leq \tfrac{1}{2}$$

$$= 2(1-|\tfrac{n}{m}|)^3 \qquad \tfrac{1}{2} \leq |\tfrac{n}{m}| \leq 1 \qquad (2.1.16)$$

$$= 0 \qquad |\tfrac{n}{m}| > 1$$

where m is the number of autocovariances or cross-covariances included in the estimates. The spectral window (see (1.4.6)) for this weight function is

$$K_N(\lambda) = \frac{1}{2\pi}\left(\frac{3}{4}\frac{1}{m^3}\right)\left[\frac{\sin(\frac{m\lambda}{4})}{\frac{1}{2}\sin(\frac{\lambda}{2})}\right]^4\left\{1-(\tfrac{2}{3})(\sin(\tfrac{\lambda}{2}))^2\right\} . \qquad (2.1.17)$$

It is instructive also to consider the limiting form of this spectral window, i.e.

$$\lim_{N\to\infty}\left\{\frac{1}{2\pi m}\sum_{n=-N+1}^{N-1} k(\tfrac{n}{m})e^{-i(\frac{n}{m})\lambda}\right\} = K(\lambda)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} k(x)e^{-i\lambda x}dx. \qquad (2.1.18)$$

$K(\lambda)$, for the weights given in (2.1.14), is given by

$$K(\lambda) = \frac{3}{8\pi}\left\{\frac{\sin(\frac{\lambda}{4})}{(\lambda/4)}\right\} . \tag{2.1.19}$$

The choice of the Parzen weight function has not been based on a detailed study of this and competing functions with a view to minimizing mean square error as there appeared to be a rather small pay off in this sort of investigation.

The decision was based on computational ease and the fact that the associated spectral estimator of $f_{jj}(\lambda)$ is never negative.

If it is most appropriate for distributed lag relations to be estimated by spectral methods the bias in the estimates proposed (see [21]) will be minimized for large N if the truncated covariance averaging kernel is used. This weight function is

$$\begin{aligned} k(\tfrac{n}{m}) &= 1 \qquad |\tfrac{n}{m}| \leq 1 \\ &= 0 \qquad |\tfrac{n}{m}| > 1 \end{aligned} \tag{2.1.20}$$

so that the estimator of $f_{jj}(\lambda)$ is

$$\hat{f}_{jj}(\lambda) = \frac{1}{2\pi}\Sigma_{n=-m}^{m}(1-\frac{|n|}{N})c'_{jj}(n)e^{in\lambda} \tag{2.1.21}$$

and the spectral window is

$$K_N(\lambda) = \frac{1}{2\pi}\frac{\sin(\frac{m+1}{2}\lambda)}{\sin(\frac{\lambda}{2})} \tag{2.1.22}$$

with the following limiting form,

$$K(\lambda) = \frac{\sin\lambda}{\pi\lambda} . \tag{2.1.23}$$

2.2 <u>Mean Correction of Covariances</u>

The programs which have been developed for spectral estimation had their starting point in a program proposed by Karreman [40]. As mentioned in 2.1 the spectral estimators computed are to be based on the Parzen weight function and it is therefore desirable that the computation of the mean corrected covariances should be performed so that estimated spectra continue to be positive. To consider this problem in more detail we assume that $j = k$ and make the division N in (2.1.1) so that the mean corrected covariance is

$$c'_{jj}(n) = \frac{1}{N}\Sigma_{m1}^{N-n}\left(x_j(m)-\bar{x}_j\right)\left(x_j(m+n)-\bar{x}_j\right)$$

$$= \frac{1}{N}\left\{\Sigma_{m1}^{N-n}x_j(m)x_j(m+n)-(N-n)(\bar{x}_j\bar{x}_j^{\ell}+\bar{x}_j\bar{x}_j^{u}-\bar{x}_j^2)\right\} \qquad (2.2.1)$$

where $\bar{x}_j^{\ell} = \Sigma_{m1}^{N-n}x_j(m)/(N-n)$ and $\bar{x}_j^{u} = \Sigma_{m1}^{N-n}x_j(m+n)/(N-n)$. If either form of (2.2.1) is used in the computations then there will be no problem with the positive nature of the estimated spectrum. The initial form of (2.2.1) would be used if all the data were passed through a detrending subroutine prior to spectral estimation where one of the detrending options would be the production of mean corrected series. This computational organization means that new series $y_j = x_j(n)-T_j(n)$ are the series which form the input to the spectral computation procedures. Although the latter form of (2.2.1) is susceptible to greater simplification for computing purposes, it is tempting to use the following approximation,

$$c^*_{jj}(n) = \frac{1}{N}\Sigma_{m1}^{N-n}x_j(m)x_j(m+n)-(N-n)\bar{x}_j^2$$

$$= \frac{N-n}{N}\left(c_{jj}(n)-\bar{x}_j^2\right). \qquad (2.2.2)^{11}$$

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11 The approximation suggested in [40] has a correction term $-N\bar{x}_j^2$ in the first form of (2.2.2) which does not appear to be correct and will accentuate the magnitude of possible negative estimators.

The obvious query is whether this approximation could lead to negative spectral estimators. The multiplicative factor $\frac{N-n}{N}$ will affect the nature of the bias and the variance of the spectral estimator but will not affect the presence or absence of negative estimates, only the magnitude of these estimates. The result that must be established is therefore what will be the nature of the estimator $\hat{f}_{jj}(\lambda)$ if $c'_{jj}(n)$ is replaced by $\frac{N}{(N-n)}\, c^*_{jj}(n)$ so that (1.4.5), with j = k, becomes

$$f^*_{jj}(\lambda) = \frac{1}{2\pi}\,\Sigma^m_{\substack{n\\-m}} \left\{1-\frac{|n|}{N}\right\} c'_{jj}e^{in\lambda}k_n - \frac{1}{2\pi}\,\Sigma^m_{\substack{n\\-m}} \left\{1-\frac{|n|}{N}\right\} \bar{x}^2_j e^{in\lambda}k_n . \tag{2.2.3}$$

To evaluate (2.2.3) it is necessary to express the latter factor in terms of $I_N(\lambda)$ (see (1.4.3)) as follows

$$\frac{1}{2\pi}\,\Sigma^m_{-m}(1-\frac{|n|}{N})\bar{x}^2_j e^{in\lambda}k_n = \frac{1}{2\pi}\,\Sigma^{N-1}_{-N+1}c_{jj}(1-\frac{|j|}{N})\left[\Sigma^m_{\substack{n\\-m}} e^{in\lambda}(1-\frac{|n|}{m})k(\frac{n}{m})\right]$$

$$= \frac{1}{2\pi}\,\Sigma^{N-1}_{-N+1}c_{jj}(1-\frac{|j|}{N})\left[2\pi\int_{-\pi}^{\pi} K_N(\lambda-\phi)\,\frac{\sin^2\frac{N\phi}{2}}{2\pi N\sin^2\frac{\phi}{2}}\,d\phi\right] \tag{2.2.4}$$

$$= I_N(o)2\pi\int_{-\pi}^{\pi} K_N(\lambda-\phi)\,\frac{\sin^2(\frac{N\phi}{2})}{2\pi N\sin^2(\frac{\phi}{2})}\,d\phi .$$

If the first term in (2.2.3) is expressed as proposed in (1.4.3) then the estimate is written

$$\hat{f}^*_{jj}(\lambda) = \int_{-\pi}^{\pi}\left\{I_N(\theta)K_N(\lambda-\theta)-I_N(o)\int_{-\pi}^{\pi}K_N(\lambda-\phi)\,\frac{\sin^2(\frac{N\phi}{2})}{2\pi N\sin^2(\frac{\phi}{2})}\,d\phi\right\}d\theta$$

$$= \iint_{-\pi}^{\pi}\left[\left\{I_N(\theta)K_N(\lambda-\theta)-I_N(o)K_N(\lambda-\phi)\right\}\frac{\sin^2\frac{N\phi}{2}}{2\pi N\sin^2(\frac{\phi}{2})}\right]d\theta d\phi \tag{2.2.5}$$

$$= \int_{-\pi}^{\pi}\left\{I_N(\theta)K_N(\lambda-\theta)-I_N(o)K^*_N(\lambda)\right\}d\theta$$

$$= \int_{-\pi}^{\pi} I_N(\theta)K_N(\lambda-\theta)d\theta - I_N(o)K^*_N(\lambda).$$

It must be noted that the expression

$$K_N(\lambda) \neq K_N^*(\lambda) = \int_{-\pi}^{\pi} K_N(\lambda-\phi) \frac{\sin^2 \frac{N}{2}\phi}{2\pi N \sin^2 \frac{1}{2}\phi} d\phi,$$ but that there is

approximate equality if $N \gg m$. It is apparent that if the amplitude of the periodogram at zero frequency is large relative to the amplitude at and around λ and the weight K_N is not small, then it is quite possible if m is not large that negative spectral estimates could occur. To completely avoid the possibility of negative spectral estimates the approximation $c^*_{jj}(n)$ was not used. Although some computing time is lost by this insurance it is possible to express the latter term in (2.2.1) in a more convenient form for computational purposes as follows

$$c'_{jj}(n) = \frac{1}{N}\left\{ \Sigma_{m\,1}^{N-n} x_j(m)x_j(m+n) - (N-n)\bar{x}_j(\bar{x}_j^{\ell} + \bar{x}_j^{u} - \bar{x}) \right\}$$

$$= \frac{1}{N} \Sigma_{m\,1}^{N-n} x_j(m)x_j(m+n) - \frac{(N-n)}{N}\bar{x}\left\{ \frac{S_1+S_2}{(N-n)} + \frac{S_2+S_3}{N-n} - \frac{(S_1+S_2+S_3)}{N} \right\}$$

$$= \frac{1}{N} \Sigma_{m\,1}^{N-n} x_j(m)x_j(m+n) - \frac{(N-n)}{N}\bar{x}\left\{ \frac{NS_2+n(S_1+S_2+S_3)}{N(N-n)} \right\} \qquad (2.2.6)$$

$$= \frac{1}{N} \Sigma_{m\,1}^{N-n} x_j(m)x_j(m+n) - \frac{1}{N}\bar{x}\left\{ NS_2+n\bar{x} \right\}$$

$$= \frac{1}{N}\left\{ \Sigma_{m\,1}^{N-n} x_j(m)x_j(m+n) - \bar{x}\left\{ NS_2+n\bar{x} \right\} \right\}$$

where

$$S_1 = \Sigma_{m\,1}^{n} x(m), \quad S_2 = \Sigma_{m\,n+1}^{N-n} x(m) \quad \text{and} \quad S_3 = \Sigma_{m\,N-n+1}^{N} x(m).$$

2.3 Missing Observations

Granger [15] has suggested a method of treating a single gap in the data when the series has no trend in the mean. It does seem that the treatment of missing observations must be expanded to handle the sort of situation which arises where, for example, prices are recorded on a weekly basis. The series of weekly wool prices (on an aggregated or type or quality basis) is an instance of a time series which is always incomplete for the following reasons. The custom of the wool trade is to close the Australian market at Easter, for 2-3 weeks, at Christmas again, for 3-4 weeks, and during the European summer for 5-6 weeks. The gap at Easter is naturally variable in its calendar location but the other two breaks are reasonably constant in calendar location. It may be tempting to close these gaps in the data, arguing that the market mechanism is not active or perhaps is less than usually active. This does not seem to be justified as the factors determining supply and demand for the commodity are still active in these periods so that while the Australian Wool Market is not registering a price in these periods, there may be in effect changes in the potential price of the commodity. Apart from the underlying feeling that the mechanism which determines prices does not halt during these gaps there is a purely practical difficulty if the gaps are closed in that the period of the oscillation thus obtained cannot be easily interpreted in relation to the normal concept of time. There is now market time and calendar time.

The procedure used was to fill in all missing observations with zeros,[12] thus maintaining an equivalence between market and calendar time. It is obviously necessary to adjust the methods for computing autocovariances and cross-covariances so that the

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12 There were no actual observations with zero value so that all zero observations or cross-products arose from missing observations.

only market generated information will be used in computing the covariances.

To attempt to fill in the theoretical background on this approach to missing observations the analogue of $I_N(\lambda)$, which is the sum of squares of regression of $x(n)$ on $\sin n\lambda$ and $\cos n\lambda$ divided by 4π, is considered. The sum of squares is

$$SS = \frac{1}{4\pi} y' A^{-1} y \tag{2.3.1}$$

where

$$y' = \left(\Sigma' x(n)\cos n\lambda,\ \Sigma' x(n)\sin n\lambda\right)$$

and

$$A = \begin{pmatrix} \Sigma'\cos^2 n\lambda & \Sigma'\cos n\lambda \sin n\lambda \\ \Sigma'\cos n\lambda\sin n\lambda & \Sigma'\sin^2 n\lambda \end{pmatrix}.$$

The symbol ' is to emphasize that the sum includes only those market generated values. Whether the A matrix will be close to orthogonality will depend on the distribution of the missing values over the time sequence 1,...,N. If A is close to orthogonality then

$$SS \cong \frac{1}{2\pi}\frac{1}{N'}\,|\Sigma' x(n)e^{in\lambda}|^2 = I'_N(\lambda) \tag{2.3.2}$$

where N' is the number of market generated observations, i.e. N less the number of missing observations. The quantity that is now to be used in estimating $\hat{f}_{jj}(\lambda)$ is the periodogram computed from data with missing values replaced by zeros, i.e. $I'_N(\lambda)$. To understand the implications in this redefinition it is useful to express $I'_N(\lambda)$ as follows

$$I'_N(\lambda) = \frac{1}{2\pi}\Sigma_{-N'+1}^{N'-1} c'(n)e^{-in\lambda} \tag{2.3.3}$$

and

$$c'(n) = \frac{1}{N'}\Sigma_{m=1}^{N-n} x(m)x(m+n).$$

The estimator proposed will be

$$\hat{f}'_{jj}(\lambda) = \frac{1}{2\pi} \Sigma_{-m}^{m} c'(n)e^{-in\lambda}k(\tfrac{n}{m})$$
$$= \int_{-\pi}^{\pi} K(\lambda-\theta)I'_N(\theta)d\theta \qquad (2.3.4)$$

which will be non-negative for the Parzen spectral window. It is necessary also to introduce a double primed notation which means that in summing or counting between prescribed limits only non-zero cross products are included. The expected value of $I'_N(\lambda)$ is then given by

$$\mathcal{E}\left(I'_N(\lambda)\right) = \frac{1}{2\pi} \Sigma_{-N'+1}^{N'-1} \frac{N''(n)}{N'} \gamma(n)e^{-in\lambda} \qquad (2.3.5)$$

where $N''(n)$ is the number of cross-products entering the definition of each $c'(n)$, so that the expected value of the proposed estimator is given by

$$\mathcal{E}\left(f'_{jj}(\lambda)\right) = \frac{1}{2\pi} \Sigma_{-m}^{m} \frac{N''(n)}{N'} \gamma(n)k(\tfrac{n}{m})e^{-in\lambda}. \qquad (2.3.6)$$

The only situation where there will be much distortion, compared to the normal situation without missing observations, is when $\frac{N''(n)}{N'}$ is small in relation to $\frac{N-n}{N}$, for n such that $\gamma(n)$ is <u>not</u> small (i.e. small n).

The problem of mean correction arises again and as much of the detailed argument would be on lines similar to those given in 2.2 only a brief sketch of the details are given.

The mean correction may be made by a pre-filtering routine which produces new series $y'_j(n) = x'_j(n)-\bar{x}'_j$ where $\bar{x}'_j = \frac{\Sigma' x_j(n)}{N'}$.

Alternatively the covariances for $x_j(n)$ may be obtained from

$$c'_j(n) = \frac{1}{N'} \Sigma_1^{(N-n)''} \left(x_j(m)-\bar{x}'_j\right)\left(x_j(m+n)-\bar{x}'_j\right). \qquad (2.3.7)$$

If the covariances are obtained using (2.3.7) then the most convenient formula for computation is

$$c'_j(n) = \frac{1}{N'} \Sigma'' x_j(m) x_j(m+n) - \bar{x}'_j \left\{ \Sigma'' x_j(m) + \Sigma'' x_j(m+n) - N''_j(n) \bar{x}'_j \right\} \quad (2.3.8)$$

and it is emphasized that the sums included in the second term are also performed only for non-zero cross products.[13] The formula (2.3.8) provides the focal point for an auto-spectral estimation program when missing observations are present. An analogous formula for cross-spectral purposes has also been developed by merely setting j = k in the second bracket in (2.3.7) and simplifying. The programs employ two counting procedures; one automatically totals the non-zero observations N', and the other counts for each covariance or cross-covariance the non-zero cross products, $N''_{jk}(n)$. It may seem even more tempting in these circumstances to use the approximation to $c'_j(n)$ given by

$$\tilde{c}'_j(n) = \frac{1}{N'} \Sigma'' x_j(m) x_j(m+n) - N''_j(n)(\bar{x}'_j)^2 \quad (2.3.9)$$

but <u>certain</u> non-negativity of $\hat{f}_{jj}(\lambda)$ would then be sacrificed. The only extra computing that (2.3.8) requires when compared to (2.3.9) is the upper and lower sums, $\Sigma'' x_j(m)$ and $\Sigma'' x_j(m+n)$.

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13 The operation $\Sigma'' x_j(m)$ means that those elements usually in the lower sum which correspond to a zero cross-product are excluded from the sum. The reason for using the notation in the 1st term on the right hand side of (2.3.8) needs a comment. Obviously it will not matter for the <u>sum of cross-products</u> whether the zero cross-products are added or not, but it does matter when correctly computing the sums in the second term on the right hand side of (2.3.8) and so the Σ" notation is appropriate to (2.3.7) and to (2.3.8).

2.4 Preliminary Transformation of Economic Data

It is standard practice to examine a graph of the series under consideration, both as a means of 'editing' discrepant values and perhaps as a guide to suitable transformations of the data. It is not uncommon for logarithms of the data to be taken. There is little doubt that much economic data so transformed is closer to normality than in its original state. If the observed data is denoted $x(n)$ then it is assumed that $y(n) = \log_e x(n)$ is normally distributed and for expositional convenience that $\mu_y = 0$ and $\sigma_y^2 = 1$. To establish a relation between the spectra of $y(n)$ and $x(n)$ it is necessary on several occasions to use the following result. If $y(n)$ is normally distributed with mean μ_y and variance σ_y^2 (N.D.(μ_y, σ_y^2)) then the expectation of $x(n) = e^{y(n)}$ is given by [15]

$$\mathcal{E}\left(x(n)\right) = \exp\left\{\sigma_y^2/2 + \mu_y\right\}. \tag{2.4.1}$$

The lag covariance function of $x(n)$, $\gamma_x(\tau)$, is

$$\gamma_x(\tau) = \mathcal{E}\left(x(n)x(n+\tau)\right) - \left\{\mathcal{E}\left(x(n)\right)\right\}^2 \tag{2.4.2}$$

and

$$\mathcal{E}\left(x(n)x(n+\tau)\right) = \mathcal{E}\left(e^{y(n)+y(n+\tau)}\right)$$

is derived from (2.4.1) as $y(n)+y(n+\tau)$ is N.D.$(0, 2(\sigma_y^2+\gamma_y(\tau)))$. So the first term in (2.4.2) is

$$\mathcal{E}\left(x(n)x(n+\tau)\right) = \exp\left\{\sigma_y^2 + \gamma_y(\tau)\right\}$$

$$= e.\, e^{\gamma_y(\tau)} \tag{2.4.3}$$

and since $\mathcal{E}(x(n)) = e^{\frac{1}{2}}$ from (2.4.1) the simple expression for $\gamma_x(\tau)$ is

$$\gamma_x(\tau) = e\left(e^{\gamma_y(\tau)} - 1\right). \tag{2.4.4}$$

The variance of x(n) is obtained directly from (2.4.4) with $\tau = 0$, so that the lag correlation function for x(n) is

$$\rho_x(\tau) = \frac{e(e^{\gamma_x(\tau)}-1)}{e(e-1)} = \frac{e^{\gamma_y(\tau)}-1}{e-1}. \qquad (2.4.5)$$

As it has been assumed that $\gamma_y(o) = 1$ the following relation holds between lag correlations

$$\rho_x(\tau) = \frac{e^{\rho_y(\tau)}-1}{e-1}. \qquad (2.4.6)$$

If the exponential terms in (2.4.6) are expanded in power series and then a weighted Fourier Transform is taken of each side of the equation we obtain the following relation between spectra

$$f_x(\lambda) = \Sigma_{j=1}^{\infty} f_y^{*j}(\lambda)/j!/\Sigma_{j=1}^{\infty} (\tfrac{1}{j!}) \qquad (2.4.7)$$

where $f_y^{*1}(\lambda) = f_y(\lambda)$ and $f_y^{*j}(\lambda) = \int_{-\pi}^{\pi} f_y^{*(j-1)}(\lambda-\theta)f_y(\theta)d\theta$ and $f_y^{*j}(\lambda)$ is referred to as the j^{th} convolution of $f_y(\lambda)$.

To interpret (2.4.7) it is best to imagine starting with normal variable y(n), which is exponentiated and then normalized. The spectrum of x(n), in regular cases, will be much smoother than that of $f_y(\lambda)$ as it comprises a weighted average of $f_y(\lambda)$ and its convolutions. Of course if y(n) has a very sharp peak in its spectrum, say at $\pm\theta$, then the convolution $f_y^{*2}(\lambda)$ will have a peak at $\lambda = 0, \pm 2\theta$, $f_y^{*3}(\lambda)$ at $\pm\theta, \pm 3\theta$, and so on. Thus $\Sigma_{j}^{\infty} {}_1 f_y^{*j}(\lambda)$ will tend to have peaks at all harmonics of $\pm\theta$. The fact that the convolution may still be regarded as smoother is less important than the recognition of how the convolving procedure redistributes a sharp peak of power at a particular frequency to its harmonics. To understand what will happen to a spectrum when logarithms are taken one merely has to imagine reversing the above explanation and the spectrum of the logarithm will be much more peaked at points of power.

In econometric work it is often the case that the dependent, independent or even all variables in the relation are in ratio form. It is useful to attempt to obtain some understanding of the nature of the spectrum of this type of variable in terms of the spectra of the numerator and denominator and to do this the typical ratio variable, $r(n) = \frac{c(n)}{y(n)}$ is investigated. As is the case with most economic variables it is assumed that $\log_e c(n) = z(n) \sim N.D.(\mu_z, \sigma_z^2)$ and $\log_e y(n) = u(n) \sim N.D.(\mu_u, \sigma_u^2)$. The spectrum of $\log_e r(n) = w(n)$ is expressed in terms of the spectra of $u(n)$ and $z(n)$ as

$$f_w(\lambda) = f_z(\lambda) + f_u(\lambda) - 2c_{zu}(\lambda) \qquad (2.4.8)$$

where $c_{zu}(\lambda)$ is the co-spectrum between $z(n)$ and $u(n)$ defined in (1.2.11) and estimated by (2.1.6).

If the variables $z(n)$ and $u(n)$ have similar spectral shapes (see [16]) then it would be expected that $f_w(\lambda)$ will contrast strongly with $f_z(\lambda)$ and $f_u(\lambda)$ in that it has much reduced power and is much flatter.

Attention should not be focussed only on the spectrum of $w(n)$, but rather on that of $r(n)$. Since $\log_e r(n) = w(n)$ and $w(n)$ is $N.D.(\mu_z - \mu_u, \sigma_u^2 + \sigma_z^2 - 2\sigma_{uz})$ it is possible as before to exponentiate, normalize and thus express the spectrum of $r(n)$, i.e. $f_r(\lambda)$, as a weighted average of the convolutions of $f_w(\lambda)$. It is therefore clear in principle how the shape of $z(n)$ and $u(n)$ will be modified in $f_w(\lambda)$ and also how it will be further modified by the convolution operations.

The problem as it usually arises in economics is where a monetary value is deflated by a price index. A detailed investigation of this problem has not been considered because the variable in the denominator of the ratio, i.e. $y(n)$, is itself a ratio, with both the numerator and the denominator being sums of products. The development of the spectrum of $y(n)$ itself would therefore need a number of assumptions and an extension of the above approach.

2.5 Spectrum of a Controlled Variable

Examples are common in primary production of an authority being given the task of controlling some aspect of the market for particular commodities. The approach of this section acts on the presumption that the control procedure proposed is successful and no attempt is made to investigate the interaction of the control procedures and the behavioural relations in the market. The aim, therefore, is to see how the spectrum of the uncontrolled variable will be modified when subjected to successful control.

The type of control which is considered is the imposition of an upper and lower limit to the values the variable may take. The upper and lower limits on the original variable, which for expositional purposes will be taken to be a price variable, will be denoted a and b respectively. The situation where only an upper or lower limit is employed may be easily dealt with by using obvious special cases. The uncontrolled variable is assumed normally distributed with mean μ and serial covariances $\gamma(m)$. The controlled variable is then

$$w(n) = \begin{cases} b & z(n) \leqq b \\ z(n) & b \leqq z(n) \leqq a \\ a & z(n) \geqq a \; . \end{cases} \qquad (2.5.1)$$

If the uncontrolled variable $z(n)$ is standardized to form a new variable, $x(n)$ as follows

$$x(n) = \Big(z(n)-\mu\Big) / \gamma(o), \qquad (2.5.2)$$

then the standardized variable is N.D.(0,1) with autocorrelations $\rho(m) = \gamma(m)/\gamma(o)$. New standardized central limits, $\alpha = \frac{(a-\mu)}{\gamma(o)}$ and $\beta = \frac{(b-\mu)}{\gamma(o)}$ are also established. The standardized controlled variable, $y(n)$, can then be defined as

$$y(n) = f\left(x(n)\right) = \begin{cases} \beta & x(n) \leq \beta \\ x(n) & \beta \leq x(n) \leq \alpha \\ \alpha & x(n) \geq \alpha. \end{cases} \qquad (2.5.3)$$

A description of the nature of the controlled variable, $y(n)$, can then be obtained from its spectrum. To establish this function a Taylor's expansion of $R'(m) = \mathcal{E}(y(n)y(n+m))$ the expected value of the lag cross-products of the standardized controlled variable, is required. The Taylor's series expansion is in terms of the autocorrelations of $x(n)$ and is given by

$$R'(m) = \Sigma_{k=0}^{\infty} \frac{\rho^k(m)}{k!} \left\{ \frac{d^k R'(m)}{d\rho(m)^k} \right\}_{\rho(m)=0}. \qquad (2.5.4)$$

To proceed with this evaluation of $R'(m)$ it is necessary to invoke the equality

$$\frac{d^j \mathcal{E}(y(n)y(n+m))}{d\rho(m)^j} = \mathcal{E}\left\{ f^{(j)}(x(n)) f^{(j)}(x(n+m)) \right\} \qquad (2.5.5)$$

given by Price [49]. The expression one obtains for the term $\mathcal{E}\{f^{(j)}(x(n))f^{(j)}(x(n+m))\}$ may be denoted A_j^2. The calculation of the A_j^2 is much simplified by special attention to the nature of the function $f(x(n))$ and its derivatives $f^{(j)}(x(n))$. Referring to (2.5.3) it is apparent that

$$\frac{df(x(n))}{dx(n)} = f^{(1)}(x(n)) = \begin{cases} 0 & x(n) \leq \beta \\ 1 & \beta \leq x(n) \leq \alpha \\ 0 & x(n) \geq \alpha \end{cases} \qquad (2.5.6)$$

and the general derivative $f^{(j)}(x(n))$ is

$$f^{(j)}(x) = \delta^{(j-2)}(x-\beta) - \delta^{(j-2)}(x-\alpha), \qquad j = 2,3,\ldots, \qquad (2.5.7)$$

where δ is the Dirac delta function. Using the above expression for $f^{(j)}$ the following table of A_j^2 can be developed

j	A_j^2
0	$\left\{\beta\,\Phi(\beta) + \int_{\beta}^{\alpha} x\phi(x)dx + \alpha\left(1 - \Phi(\alpha)\right)\right\}^2$
1	$\left\{\Phi(\alpha) - \Phi(\beta)\right\}^2$
2,3,...	$\left\{\left(\phi^{(j-2)}(x)\right)_{x=\alpha} - \left(\phi^{(j-2)}(x)\right)_{x=\beta}\right\}$

(2.5.8)

where $\Phi(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$, $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ and

$\phi^{(j)}(x) = \phi(x)H_j(x)(-1)^j$. The function $H_j(x)$ is the j^{th} Hermite polynomial and is defined (see [8]) by

$$\left(\frac{d}{dx}\right)^j \left\{e^{-\frac{x^2}{2}}\right\} = (-1)^j H_j(x) e^{-\frac{x^2}{2}}, \qquad j = 0,1,2,\ldots . \tag{2.5.9}$$

Another approach to obtaining the expression which is proposed by Grenander and Rosenblatt [18,p 51 ff.] is introduced because it proves to be advantageous when considerations of computation efficiency arise. It is shown in [18] that

$$R'(m) = \Sigma_{j^o}^{\infty} A_j'^2 \rho^j \tag{2.5.10}$$

where the coefficient A_j' are

$$A_j' = \int_{-\infty}^{\infty} \frac{1}{\sqrt{j!}} H_j(x) f(x) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx. \tag{2.5.11}$$

It is also shown in [18] that if $\rho = 1$ then using the Parseval relation it follows immediately that an expression for the sum of the $A_j'^2$ coefficients is

$$\Sigma_{j^o}^{\infty} A_j'^2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^2(x) e^{-\frac{x^2}{2}} dx. \tag{2.5.12}$$

The weighted Fourier Transform of $R'(m)$ will give a spectral representation of $y(n)$, denoted $f'_y(\lambda)$ which consists of the following weighted average of convolutions of $f_x(\lambda)$

$$f'_y(\lambda) = \Sigma_{j^o}^{\infty} A'^{2}_{j} f^{*j}_{x}(\lambda). \qquad (2.5.13)$$

It is usual for the spectrum to be defined in terms of the covariances, so the weighted Fourier Transform should be applied to $R'(m) - (\mathcal{E}(y(n)))^2$. As the mean, $\mathcal{E}(y(n))$, is easily expressed in terms of the A'_j as A'_o, the usually defined spectral density of $y(n)$ is $f_y(\lambda)$, with the prime omitted, and is calculated from

$$f_y(\lambda) = \Sigma_{j^1}^{\infty} A'^{2}_{j} f^{*j}_{x}(\lambda). \qquad (2.5.14)$$

To illustrate how a spectral density will be modified in practice by this sort of control procedure a set of data on monthly price/lb of wool sold at auction, issued by the Council of Wool Selling Brokers, is used. Two general control schemes are proposed; the first has variable upper and lower limits and the second only a variable lower limit. Table 1 below sets out the limits used in each scheme in terms of the original and standardized variables.

TABLE 1

CONTROL SCHEMES

Scheme I - Upper and Lower Restriction

		I_1	I_2	I_3	I_4	I_5	I_6	I_7	I_8	I_9	I_{10}
Lower Limit	Original Variable	55	53	52	51	49	55	53	52	51	49
	Standardized Var.	-.72	-.88	-.97	-1.05	-1.22	-.72	-.88	-.97	-1.05	-1.22
Upper Limit	Original Variable	70	68	66	65	63.6	75	74	77	78	80
	Standardized Var.	.53	.37	.20	.12	0	.95	.87	1.12	1.20	1.36

Scheme II - Only Lower Restriction

		II_1	II_2	II_3	II_4	II_5	II_6	II_7	II_8	II_9	II_{10}
Lower Limit	Original Variable	55	53	52	51	49	57	59	61	62	63.6
	Standardized Var.	-.72	-.88	-.97	-1.05	-1.22	-.55	-.38	-.22	-.13	0
Upper Limit	Original Variable	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
	Standardized Var.	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞

Sample Statistics $N = 168$, $\overline{X} = 63.6$, $S = 12.02$.

The computation of the spectra of the price variable when subjected to the control of upper and lower limits suggested above is based on (2.5.12). The obvious dilemma with the expression for f_y given in that formula is how many terms in the infinite sum should be used. The following resolution of this problem follows the lines suggested by Grenander and Rosenblatt [18] and makes use of the fact that $f_x^{*j} \to \frac{1}{2\pi}$ as j increases. If (2.5.13) is rewritten as

$$f_y(\lambda) = \Sigma_{j1}^{p} A_j'^2 f_x^{*j}(\lambda) + \Sigma_{jp}^{\infty} A_j'^2 f_x^{*j}(\lambda)$$

$$\cong \Sigma_1^p A_j'^2 f_x^{*j}(\lambda) + \frac{1}{2\pi} \sum_{p+1}^{\infty} A_j'^2 \quad (2.5.15)$$

$$\cong \Sigma_1^p A_j'^2 \left(f_x^{*j}(\lambda) - \frac{1}{2\pi} \right) + \frac{1}{2\pi} \Sigma_o^{\infty} A_j'^2 - \frac{A_o^2}{2\pi}$$

then using the expression given in (2.5.12) for $\Sigma A_j'^2$ $f_y(\lambda)$ is expressed in a way which suggests a method of computation, that is

$$f_y(\lambda) \cong \Sigma_1^p A_j'^2 (f_x^{*j} - \frac{1}{2\pi}) + \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} f^2(x) e^{-\frac{1}{2}x^2} dx - A_o'^2/2\pi. \quad (2.5.16)$$

If a small finite number of convolutions is calculated until the last is reasonably close to $\frac{1}{2\pi}$ (in the example discussed p = 10) an estimate of $f_y(\lambda)$ is obtained by adding the latter two terms. To facilitate computations it is necessary to express the terms

$\frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} f^2(x) e^{-\frac{1}{2}x^2} dx$ as follows,

$$\frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} f^2(x) e^{-\frac{1}{2}x^2} dx = \beta^2 \, \Phi(\beta) + \alpha^2 (1 - \Phi(\alpha)) + \Phi(\alpha) - \Phi(\beta) + \beta e^{-\frac{1}{2}\beta^2} - \alpha e^{-\frac{1}{2}\alpha^2}. \quad (2.5.17)$$

The spectra of the controlled variable for each scheme proposed in Table 1 are contrasted with the spectrum of the standardized variable x(n) in Fig.I.

FIG. I

COMPARISON OF SPECTRA OF RESTRICTED AND UNRESTRICTED STANDARDIZED VARIABLES

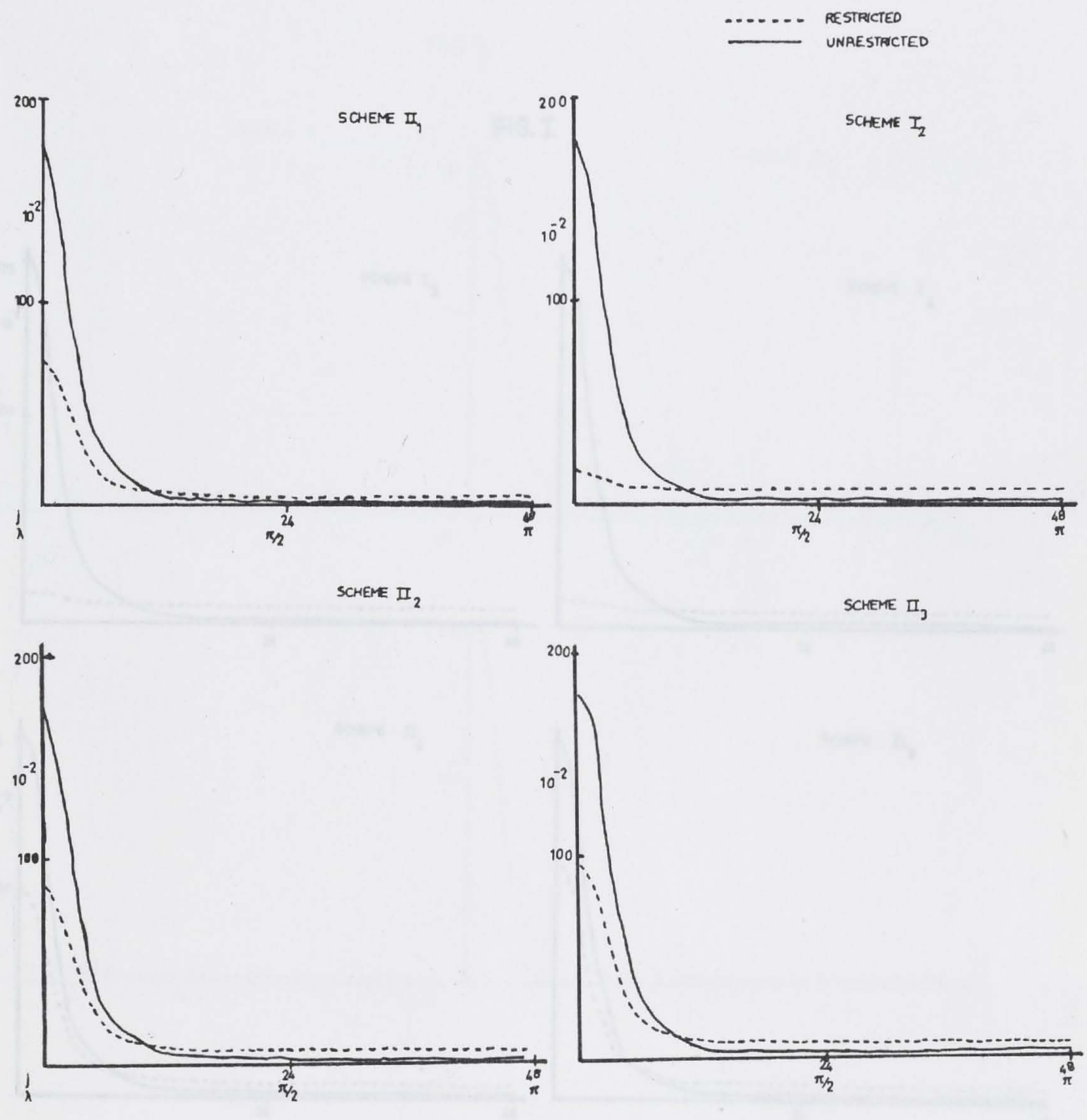


FIG. I

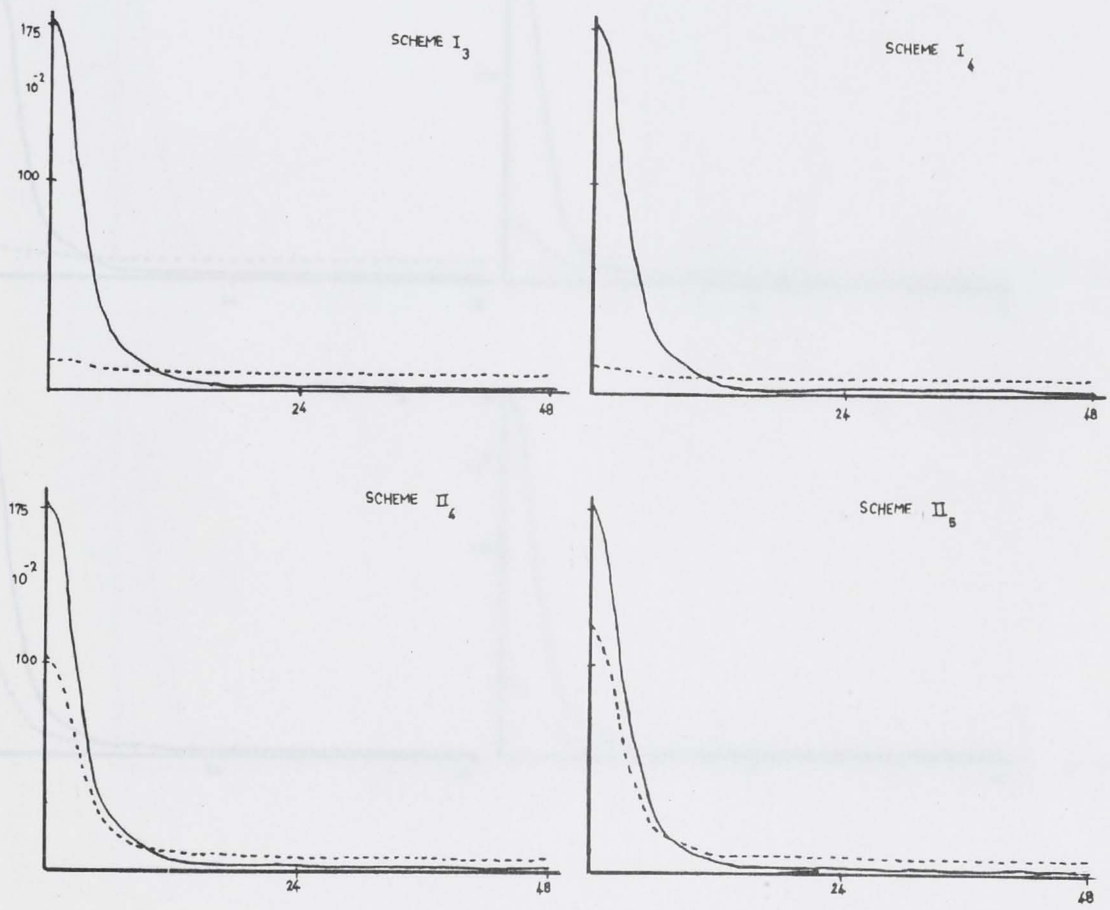

175
10^{-2}
100
SCHEME I$_3$
24
48
SCHEME I$_4$
24
48
175
10^{-2}
100
SCHEME II$_4$
24
48
SCHEME II$_5$
24
48

FIG. I

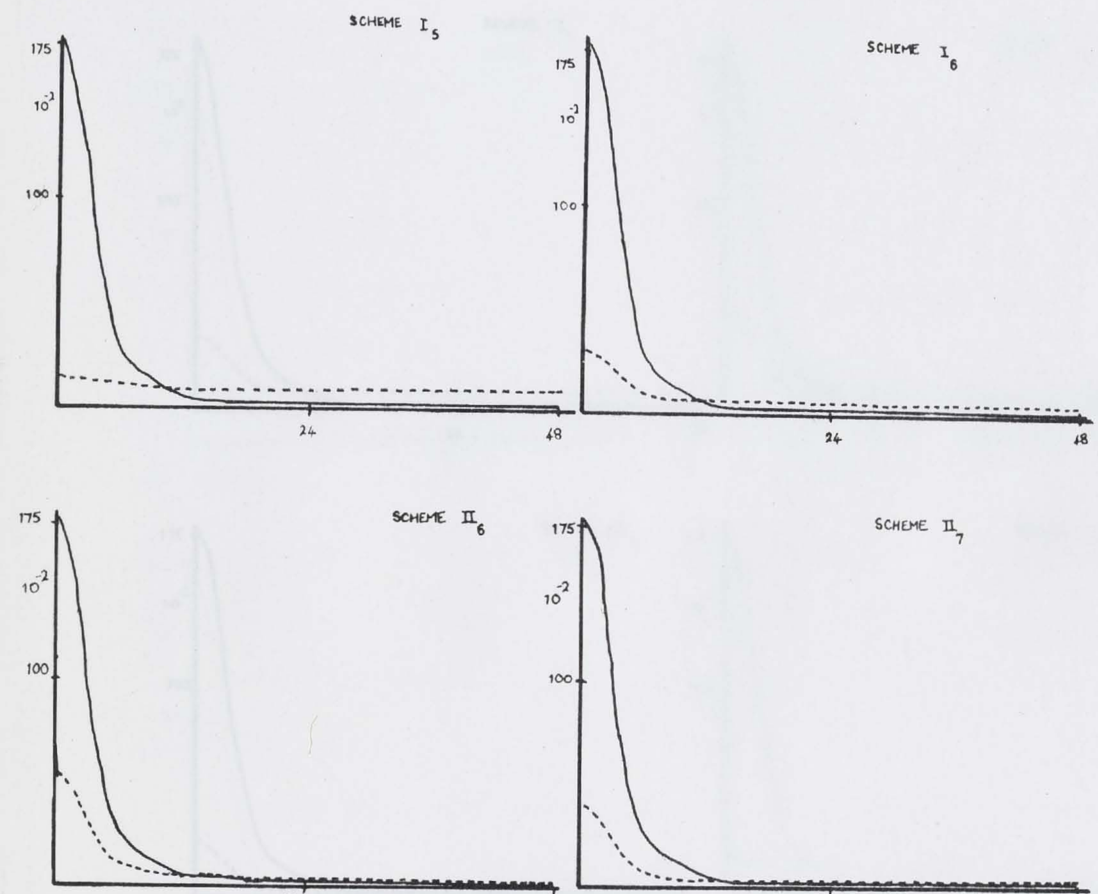

SCHEME I_5
175
10^{-2}
100
24
48
SCHEME I_6
175
10^{-2}
100
24
48
SCHEME II_6
175
10^{-2}
100
24
48
SCHEME II_7
175
10^{-2}
100
24
48

FIG. I

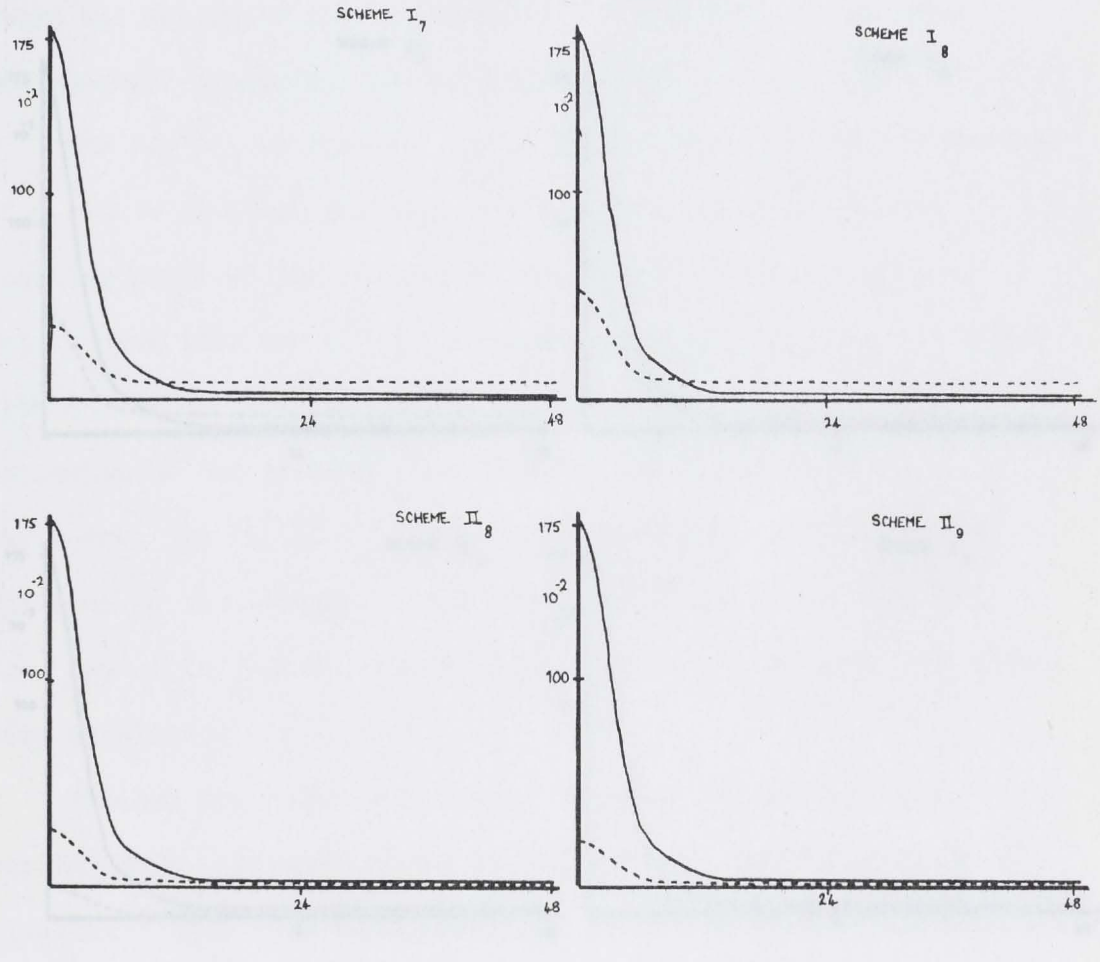
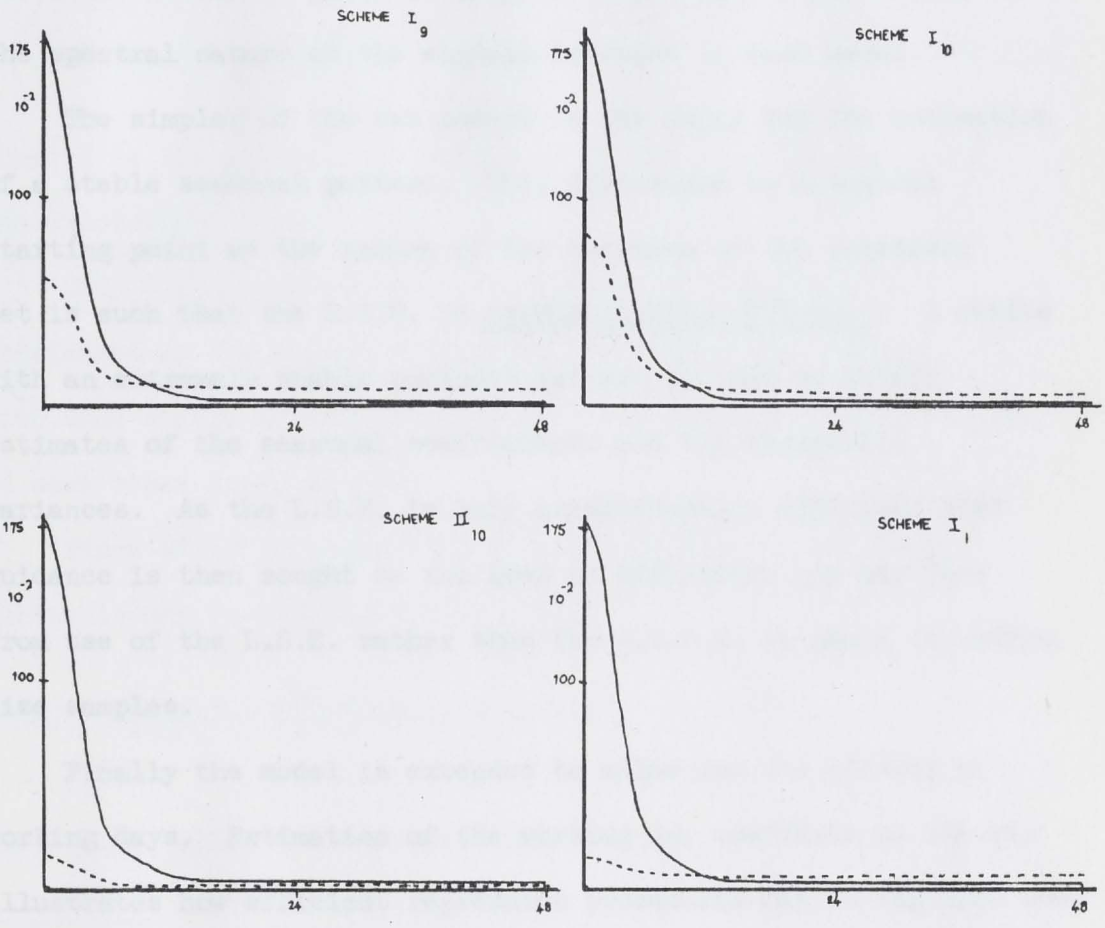

SCHEME I7
175
10^{-2}
100
24
48
SCHEME I8
175
10^{-2}
100
24
48
SCHEME II8
175
10^{-2}
100
24
48
SCHEME II9
175
10^{-2}
100
24
48

FIG. I


SCHEME I 9
175
10^-2
100
24
48
SCHEME I 10
175
10^-2
100
24
48
SCHEME II 10
175
10^-2
100
24
48
SCHEME I 1
175
10^-2
100
24
48

III NARROW BAND SPECTRAL REGRESSION PROCEDURES

3.1 Introduction

This chapter is concerned with two problems, in each of which the model specifies exactly the signal (the regressor). To emphasize the point just made in this Chapter we consider estimation problems where the actual frequency properties of the carrier wave are known and one wishes to detect only the amplitude (or frequency) modulation. In contrast in Chapter IV we will begin considering signal extraction where the only characteristic of the signal which is known *a priori* is the signal's average spectral properties. Although the signal is known the method of analysis adopted will depend a great deal on the spectral nature of the signals included in each model.

The simpler of the two models is the basis for the estimation of a stable seasonal pattern. This discussion is a logical starting point as the nature of the spectrum of the regressor set is such that the L.S.E. is *asymptotically efficient*. A series with an extremely stable periodic pattern is used to obtain estimates of the seasonal coefficients and the asymptotic variances. As the L.S.E. is only asymptotically efficient some guidance is then sought on the loss in efficiency one may face from use of the L.S.E. rather than the B.L.U.E. in small to medium size samples.

Finally the model is extended to allow for the effects of working days. Estimation of the working day coefficients clearly illustrates how efficient regression procedures may be employed when the spectra of the signals (regressors) and the disturbance term are such that the L.S.E. is no longer even asymptotically efficient.

Although most economic time series seem to exhibit evolving seasonal patterns there are occasions when the estimation of a stable seasonal pattern is apposite. This position may prevail for two reasons; because the seasonal pattern is in fact unchanging or more likely because the seasonal evolution is sufficiently slow for a stable pattern to be useful over short periods (say 4-8 years) and therefore the slowly evolving pattern may be estimated by a moving stable regression procedure.

3.2 Stable Seasonal Model

The basis for discussion is the model

$$w(n) = p(n)+s^*(n)+u(n) \tag{3.2.1}$$

where $w(n)$ is the observed series, $p(n)$ is a 'trend' term, $s^*(n)$ is the seasonal component and $u(n)$ is a stationary residual with zero mean. The stationarity of $u(n)$ means that one may write

$$\mathcal{E}\left(u(m)u(m+n)\right) = \gamma_u(n) = \int_{-\pi}^{\pi} e^{i\lambda n} f_u(\lambda)d\lambda \tag{3.2.2}$$

where $\mathcal{E}$ is the expectation operator and the function $f_u(\lambda)$ is the spectrum of $u(n)$.

Only the case where the unit of time is one month is considered as the approach taken can be translated, perhaps with some change of emphasis, to any case where the time interval is some other known period.

Since $s^*(n)$ is assumed to be periodic with period twelve months it may be expressed in the form[14]

$$s^*(n) = \Sigma_1^6 s_j^*(n) = \Sigma_1^6(\alpha_j^*\cos\lambda_j n+\beta_j^*\sin\lambda_j n), \quad \lambda_j = \frac{2\pi j}{12}, \quad j = 1,2,\ldots,6. \tag{3.2.3}$$

The λ_j are called the seasonal frequencies.

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[14] An equivalent alternative formulation for the stable component is

$$s^*(n) = \Sigma_1^{12} a_j e_j(n) , \qquad \Sigma_1^{12} a_j = 0$$

where $e_j(n)$ is unity for $(n-j)$ divisible by 12 and is otherwise zero. The restriction on the a_j coefficients implies $\alpha_0 = 0$. This equivalent approach is fully described in Nerlove [44, pp 451-2].

The distinction that has been made between $p(n)$ and $u(n)$ is arbitrary. It could be argued that the distinction between all three components tends to be arbitrary in practice, but as a stable seasonal pattern is under consideration $s^*(n)$ can be clearly distinguished. Alternatively one could consider

$$p(n)+u(n) = z(n)+\mu \tag{3.2.4}$$

where μ is a constant and $z(n)$ is taken to be stationary but with a very large concentration of power in $f_z(\lambda)$ close to the origin. An example would be $z(n) = \rho z(n-1)+\epsilon(n)$, where the $\epsilon(n)$ are serially independent with variance σ_ϵ^2 and ρ is close to unity. The associated spectrum will be

$$f_z(\lambda) = \frac{\sigma_\epsilon^2}{2\pi\{1+\rho^2-2\rho\cos\lambda\}} \tag{3.2.5}$$

and will be very large close to the origin. Economic phenomena are of course not stationary but are evolving. Nevertheless a stationary process with a spectrum of the type just described would have an appearance which accords with what one would expect from an economic time series, over reasonable periods, and statistical procedures could well be based on such a model. It is however simpler and a little more realistic to work in terms of (3.2.1) although it is convenient on occasions to interpret the results of the following investigations in terms of the alternative just described.

The data is initially filtered by an operator which replaces $w(n)$ by (see (1.3.1))

$$y(n) = \Sigma_{-p}^{q} b_k w(n-k). \tag{3.2.6}$$

The effect on $u(n)$ is to replace it by a new series, $x(n)$, with spectrum (see (1.3.3))

$$|\Sigma_{-p}^{q} b_k e^{i\lambda_k}|^2 f_u(\lambda) = |B(\lambda)|^2 f_u(\lambda) = f_x(\lambda). \tag{3.2.7}$$

The function $B(\lambda)$ is the response function of the filter defined in (1.3.2). So far as $s^*(n)$ is concerned a modified series $s(n)$ is obtained with modified coefficients given by

$$\alpha_j = \alpha_j^* \Sigma_{-p}^{q} b_k \cos\lambda_j k - \beta_j^* \Sigma_{-p}^{q} b_k \sin\lambda_j k$$
$$\beta_j = \alpha_j^* \Sigma_{-p}^{q} b_k \sin\lambda_j k - \beta_j^* \Sigma_{-p}^{q} b_k \cos\lambda_j k. \qquad (3.2.8)$$

The b_k are chosen so that $p(n)$ is made as small as practicable, i.e. so that in a model based on $z(n)$, $|B(\lambda)|^2 f_z(\lambda)$ no longer has a large peak close to the origin. As $(p+q)$ observations are lost in such a filtering process, for simplicity it is assumed that after filtering the number of observations available, N, is an integral multiple of 12. This is usually no restriction since the initial point for the analysis (i.e. the point to which one can return and still regard the seasonal as stable) is somewhat uncertain and may as well be chosen so that N is an integral multiple of 12.

3.3 Possible Regression Procedures

Least Squares Estimates

The L.S.E. is obtained from the regression of $y(n)$ on $\cos n\lambda_j$ and $\sin n\lambda_j$ and provides estimators $\hat{\alpha}_j$, $\hat{\beta}_j$. The equations (3.2.8) with circumflexes inserted throughout to denote estimates, are then solved for $\hat{\alpha}_j^*$, $\hat{\beta}_j^*$. It is not necessary to proceed in this way. A precisely equivalent procedure is to average the $y(n)$ for each month of the year and then adjust these twelve averages to add to zero by subtracting their mean, thus obtaining regression estimates of the filtered a_j (see footnote (14)), which are denoted $\hat{a}_j'$ (see [11]). Now the original coefficients may be recovered from the $\hat{a}_j'$ by employing the following relations

$$\hat{a}_j = \Sigma_{k}^{12} \hat{a}_k' g_{k-j}, \quad g_k = \frac{1}{12} \Sigma_{j}^{11} e^{ik\lambda_j} B^{-1}(\lambda_j) \qquad (3.3.1)$$

assuming that $B(o) = 0$ and $B(\lambda_j) \neq 0$ for all $j \neq 0$. The g_k are defined to be periodic with period 12 and are always real. This procedure is numerically equivalent to the one described earlier in this section. There is nothing new or radical about this technique proposed by Hannan [22]; its virtue is merely that it enables <u>any</u> filter to be used, subject to the proposed restrictions on $B(\lambda)$.

The variance of the estimates $\hat{\alpha}_j$, $\hat{\beta}_j$ satisfy (see [23])

$$\lim_{N\to\infty} \mathrm{var}\hat{\alpha}_j = \lim_{N\to\infty} \mathrm{var}\hat{\beta}_j = \frac{4\pi}{N} f_x(\lambda_j) \qquad j \neq 6$$

$$\lim_{N\to\infty} \mathrm{var}(\hat{\alpha}_6) = \frac{2\pi}{N} f_x(\lambda_6) \tag{3.3.2}$$

and the covariances approach zero as $N \to \infty$. To employ these formulae one requires knowledge of $f_x(\lambda)$, or an estimate of this function. The series, Passenger Airline Reservations (see [7]) appears to be stable over the short period which has been chosen for analysis, i.e. the seven years, 1954-60. This series is used to illustrate how in practice one might obtain an estimate of the asymptotic variances of $\hat{\alpha}_j$, $\hat{\beta}_j$. It is argued in more detail later in the section that it is most unlikely that $f_x(\lambda)$ will be known so it is necessary to make an estimate of the spectrum at each λ_j. To produce this estimate the filtered residuals from the regression of $y(n)$ on $\cos n\lambda_j$ and $\sin n\lambda_j$, is formed using

$$\hat{x}(n) = y(n) - \sum_j (\hat{\alpha}_j \cos n\lambda_j + \hat{\beta}_j \sin n\lambda_j). \tag{3.3.3}$$

The periodogram, $I_{\hat{x}}(\lambda)$, is then calculated for equi-spaced frequency points between zero and π from the formula

$$I_{\hat{x}}(\lambda) = \frac{1}{2\pi N} \left|\sum_{k}^{N} \hat{x}(n) e^{-in\lambda}\right|^2 \tag{3.3.4}$$

$$\lambda = \lambda_k = \frac{2\pi k}{N}, \quad k = 0,1,\ldots,[N/2]$$

which is itself derived directly from (1.4.2). To convert these periodogram ordinates of the residuals to values which are relevant to the coefficients of the original relation, i.e. α_j^*, β_j^*, $I_{\hat{x}}(\lambda)$ must be recoloured by the factor

$$\left\{1 - \frac{\sin 6\lambda \sin\lambda}{24\sin^2\frac{1}{2}\lambda}\right\} \tag{3.3.5}$$

to become $I^R_{\hat{x}}(\lambda)$, an estimate of the periodogram of the residuals in the original relation. It should be immediately mentioned that because of the narrow band regression that has been performed and because the factor enclosed in the curly bracket in (3.3.5) has zeros at all λ_j no meaning can be attached to the value of $I^R_{\hat{x}}(\lambda)$ at the exact points λ_j. Table 2 below presents the values of $I_{\hat{x}}(\lambda)$ and $I^R_{\hat{x}}(\lambda)$ that were obtained for the Airline Passenger Reservation series. The recoloured periodogram, $I^R_{\hat{x}}(\lambda)$, has been smoothed by a simple average to provide an estimate, $\hat{f}_x(\lambda)$, of the spectrum of $x(n)$ (see [38]). Two simple averages were used, a three term and a five term,[15] and the results of this averaging are also included in Table 2 after the frequencies of interest, λ_j. These estimates of $f(\lambda_j)$ are then used in (3.3.2) to provide some indication of the significance of the $\hat{\alpha}^*_j$, $\hat{\beta}^*_j$. Some perspective on the relevance of this asymptotic variance estimate when $N = 72$ will be given in future sections of this chapter. Table 3 below presents the estimates $\hat{\alpha}^*_j$, $\hat{\beta}^*_j$ and the associated asymptotic variance estimates.

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15 The middle term in either average has to be neglected because of the lack of meaning of the ordinate at λ_j so that in fact the average only involves either one or two terms each side of any λ_j.

TABLE 2

j	$\lambda = \frac{\pi j}{36}$	$I_{\hat{x}}(\lambda)$	$I^{R}_{\hat{x}}(\lambda)$	omitting $(\pi j/6)$ 4 term av.	2 term av.
1	$\pi/36$	$.13357\ 10^{2}$	$.39706\ 10^{4}$		
2		$.35159\ 10^{1}$	$.14634\ 10^{3}$		
3		$.30335\ 10^{2}$	$.22522\ 10^{3}$		
4		$.51653\ 10^{2}$	$.13481\ 10^{3}$		
5		$.21556\ 10^{3}$	$.34892\ 10^{3}$		
6	$\pi/6$	$.39762\ 10^{-7}$	$.39762\ 10^{-7}$	$.17237\ 10^{3}$	$.26586\ 10^{3}$
7		$.23923\ 10^{3}$	$.18279\ 10^{3}$		
8		$.32705\ 10^{2}$	$.22941\ 10^{2}$		
9		$.10708\ 10^{2}$	$.74237\ 10^{1}$		
10		$.37151\ 10^{2}$	$.28239\ 10^{2}$		
11		$.42558\ 10^{2}$	$.35820\ 10^{2}$		
12	$\pi/3$	$.61170\ 10^{-7}$	$.61170\ 10^{-7}$	$.55545\ 10^{2}$	$.73345\ 10^{2}$
13		$.97544\ 10^{2}$	$.11087\ 10^{3}$		
14		$.38273\ 10^{2}$	$.47251\ 10^{2}$		
15		$.41520\ 10^{2}$	$.52300\ 10^{2}$		
16		$.37871\ 10^{2}$	$.44939\ 10^{2}$		
17		$.44155\ 10^{2}$	$.49132\ 10^{2}$		
18	$\pi/2$	$.36120\ 10^{-7}$	$.31620\ 10^{-7}$	$.24153\ 10^{2}$	$.25789\ 10^{2}$
19		$.26561\ 10^{1}$	$.24462\ 10^{1}$		
20		$.10665\ 10^{0}$	$.95000\ 10^{-1}$		
21		$.41550\ 10^{-1}$	$.36600\ 10^{-1}$		
22		$.25625\ 10^{1}$	$.23331\ 10^{1}$		
23		$.36905\ 10^{1}$	$.34786\ 10^{1}$		
24	$2\pi/3$	$.11735\ 10^{-8}$	$.11735\ 10^{-8}$	$.72551\ 10^{1}$	$.11718\ 10^{2}$
25		$.19011\ 10^{-2}$	$.19957\ 10^{2}$		
26		$.30410\ 10^{1}$	$.32520\ 10^{1}$		
27		$.12927\ 10^{1}$	$.13880\ 10^{1}$		
28		$.12499\ 10^{2}$	$.13148\ 10^{2}$		
29		$.29337\ 10^{2}$	$.30237\ 10^{2}$		
30	$5\pi/6$	$.36769\ 10^{-8}$	$.36769\ 10^{-8}$	$.18011\ 10^{2}$	$.24384\ 10^{2}$
31		$.18903\ 10^{2}$	$.18530\ 10^{2}$		
32		$.10392\ 10^{2}$	$.10130\ 10^{2}$		
33		$.17172\ 10^{2}$	$.16800\ 10^{2}$		
34		$.39673\ 10^{1}$	$.39200\ 10^{1}$		
35		$.24895\ 10^{1}$	$.24795\ 10^{1}$		
36	π	$.13785\ 10^{-7}$	$.13785\ 10^{-7}$	$.31998\ 10^{1}$	$.24795\ 10^{1}$

TABLE 3

STANDARD DEVIATIONS FOR STABLE SEASONAL COEFFICIENTS
FROM ASYMPTOTIC FORMULA

	α_1^*	β_1^*	α_2^*	β_2^*	α_3^*	β_3^*	α_4^*	β_4^*	α_5^*	β_5^*	α_6^*
Coefficients	54.47	29.02	-.63	30.57	-10.65	4.49	5.27	8.54	.67	-7.63	.96
Standard Dev. from Asymptotic Formula	6.81		3.58		2.12		1.43		2.06		.53

Best Linear Unbiased Procedure

If $f_u(\lambda)$, and therefore $f_x(\lambda)$, was known then α_j and β_j may be estimated by the B.L.U. regression procedure, that is a weighted regression on $\cos\lambda_j n$ and $\sin\lambda_j n$. This procedure has been advocated by Jorgensen [39]. It is almost inconceivable that $f_u(\lambda)$ should in fact be known so that the best that could be done would be to base a B.L.U.E. on an assumed $f_u(\lambda)$. In fact the B.L.U.E. is unlikely to be used for large samples as it is known the L.S.E. is asymptotically efficient if the effect of $p(n)$ is much diminished by filtering, as must be assumed. Moreover the subsequent sections will show that the L.S.E. will prove to be quite efficient for small N provided $f_x(\lambda)$ is not markedly peaked. It should be emphasized that filtering is performed to attempt to reduce $p(n)+u(n)$ to a form approximating a stationary process with a smooth spectrum which is not markedly peaked. If this attempt is successful it will become apparent that the L.S.E. is quite an acceptable procedure, particularly as information sufficiently precise to improve on it will not normally be available.

Jorgensen [39] has suggested the case of a pure regression procedure in which the term $p(n)$ is represented as a polynomial and $p(n)$ and $s^*(n)$ are simultaneously estimated by a B.L.U. regression procedure. Normally a known $f_x(\lambda)$ is not available[16] and Jorgensen therefore recommends the use of an estimate of $f_x(\lambda)$, or parameters equivalent to it, from the residuals in an initial regression. This approach needs assumptions concerning $f_x(\lambda)$ which are equivalent to requiring it to be a polynomial in

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If the disturbances were assumed to be independent then of course $f_x(\lambda)$ and the associated lag covariances would be known and the B.L.U. procedure would be available.

$\exp(i\lambda)$ and $\exp(-i\lambda)$ so that only a finite number of parameters are to be estimated. It is difficult to assert anything concerning the merits of this approximate procedure. The term neglected in approximating $f_x(\lambda)$ as suggested may be of similar magnitude to that involved in the comparison of efficiencies. Further, for $f_x(\lambda)$ to be approximated in this way it must be comparatively smooth so that the L.S.E. then has variances which approach the optimal values fairly rapidly.

Of course, one might assume $f_x(\lambda)$ to be constant (see footnote 16) and use the L.S.E. Indeed as N increases it is known, assuming $p(n)$ to be a polynomial in n, that the efficient estimation procedure tends not to depend on $f_x(\lambda)$. However, for a high degree polynomial N needs to be very large before this is so. Although regression is not technically a filtering technique it may be thought of as one which produces a response function highly concentrated at the origin. The degree of concentration decreases as the order of the polynomial increases. In terms of $z(n)$ the task is to modify the very large concentration of spectral mass at and near the origin, and a troublesome part of this mass may not be <u>very</u> near to the origin. Thus the degree of the polynomial may have to be very large. What is required is a flexible procedure which will modify $f_u(\lambda)$ not merely right at the origin. Polynomial regression is not well adapted to do this but filtering is.[17]

Before looking in detail at examples of the actual efficiency of the L.S.E. of α_j and β_j the case where α_j and β_j are assumed to be zero for $j \neq 1$ is considered for expositional purposes only. The <u>exact</u> variances of the L.S.E. of α_1 and β_1 (see [46]) are

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17 With a high order polynomial, regression often becomes unwieldy and results in a loss of degrees of freedom. Against this must be set any loss of observations at the ends of the series due to filtering.

$$\text{var}(\hat{\alpha}_1) = \frac{4\pi}{N}\int_{-\pi}^{\pi}\Big[|S_N(\lambda-\lambda_1)|^2+|S_N(\lambda+\lambda_1)|^2+\mathcal{R}\{S_N(\lambda-\lambda_1)S_N(\lambda+\lambda_1)\}\Big]|B(\lambda)|^2 f_u(\lambda)d\lambda \tag{3.3.6}$$

$$\text{var}(\hat{\beta}_1) = \frac{4\pi}{N}\int_{-\pi}^{\pi}\Big[|S_N(\lambda-\lambda_1)|^2+|S_N(\lambda+\lambda_1)|^2-\mathcal{R}\{S_N(\lambda-\lambda_1)S_N(\lambda+\lambda_1)\}\Big]|B(\lambda)|^2 f_u(\lambda)d\lambda$$

where $\mathcal{R}\{.\}$ means the real part of the indicated function, and the function $S_N(\lambda)$ and the square of its modulus are given by

$$S_N(\lambda) = (1/\sqrt{2\pi N})\,\Sigma_1^N e^{i\lambda n}, \qquad |S_N(\lambda)|^2 = \frac{\sin^2\frac{1}{2}\lambda N}{2\pi N\sin^2\frac{1}{2}\lambda}\,. \tag{3.3.7}$$

The last expression in (3.3.7) integrates to unity and is very concentrated at $\lambda = 0$.

The B.L.U. procedure results in a variance which cannot be represented exactly in this form (see 3.4.2) but approaches $(4\pi/N)f_x(\lambda_j)$, as do the variances of $\hat{\alpha}_1$, $\hat{\beta}_1$ shown in (3.3.6). Indeed the approach to the limiting values is quite fast unless $f_x(\lambda)$ is very markedly peaked for the maxima of $|S_N(\lambda-\lambda_1)|$ are at $\lambda = \lambda_1$ and near points $\frac{3\pi}{N}$, $\frac{5\pi}{N}$, etc... away from λ_1, but are much smaller than that at λ_1. Thus $f_x(\lambda)|B(\lambda)|^2$ has to be very different from its value at $\lambda = \lambda_1$ for a large contribution to arise from anywhere other than λ_1.

3.4 Comparison of Efficiency

To compare the regression procedures a number of situations with known $f_u(\lambda)$ are considered and the exact values of the variances and covariances of the B.L.U.E. and L.S.E. of α_j and β_j, $j = 1,\ldots,6$, are computed. The B.L.U. procedure is not put forward as one usually available in practice but rather as a benchmark for the L.S. procedure. For each case considered the number of observations after filtering is N = 36, 48, 72. It does not seem necessary to consider N greater than 72 because of (3.3.2) and because the assumption of a stable seasonal pattern becomes less tenable as N is increased. The main purpose of the comparisons is to consider how the behaviour of u(n) and the method of filtering may influence the efficiency of the L.S.E. of the parameters in a stable seasonal pattern.

The possible generating models for the disturbances are limited to the three following types:

(a) an independently and identically distributed random variable with mean zero and unit variance (i.i.d.(0,1)),

(b) a stationary first order autoregressive process

$$u(n)-\alpha u(n-1) = \epsilon(n),$$

(c) a stationary second order autoregressive process

$$u(n)+\alpha_1 u(n-1)+\alpha_2 u(n-2) = \epsilon(n).$$

The random variable $\epsilon(n)$ is i.i.d.$(0,\sigma_\epsilon^2)$, where σ_ϵ^2 is selected for each choice of parameter(s) so that the variance of u(n) is always unity. The values considered for α were α = .75, .85, .90, .95, .99 and .995. The correlograms for α = .75, .95 and .995 are shown in Fig. II. For the second order process the following six sets of parameters for α_1, α_2 were investigated,

2nd Order Model No.	1	2	3	4	5	6
α_1	1.0	-.5	-1.0	-1.1	-1.25	-.75
α_2	.75	.5	.75	.3	.3	-.20

The first three sets produce complex roots; the first and third have the same amplitude but differ in that the first set produces oscillations in the correlogram with a much higher frequency than the third set. The most noticeable characteristic of the second set is the much lower amplitude of its roots and consequently the correlogram dumps out much more rapidly. The remaining sets have both roots positive and real and the only difference between them is the rate at which the lag correlations decay to zero. The rate of decay is greatest for the fourth set and declines successively for the remaining two sets. In Fig. III the correlogram is shown for the second order models Nos. (1), (3) and (5).

FIG. II

CORRELOGRAM OF 1st ORDER AUTOREGRESSIONS, MODELS $\rho = .75, .95, .995$

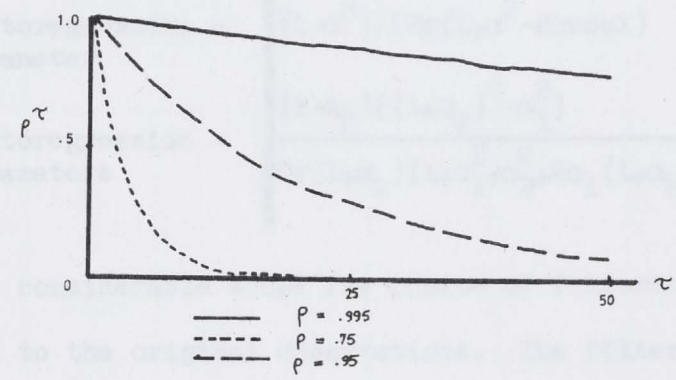
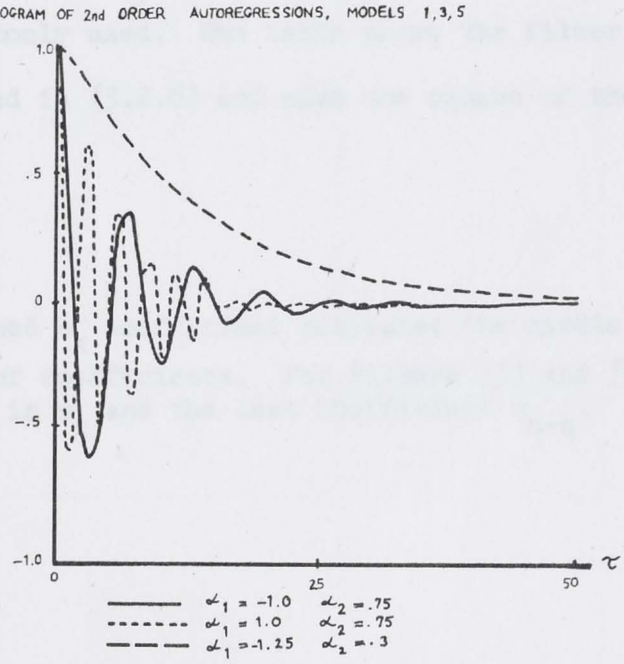


FIG. III

CORRELOGRAM OF 2nd ORDER AUTOREGRESSIONS, MODELS 1, 3, 5



The three model types and their different parameter values produce thirteen different covariance structures for the disturbances prior to filtering. The spectrum associated with each model type is set out in Table 4 below.

TABLE 4

Model Type	Spectrum $f_u(\lambda) \quad \lambda \,\varepsilon\, (-\pi,\pi)$
Independent Residuals	$\frac{1}{2\pi}$
1st Order Autoregression - Parameter	$(1-\alpha^2)/\{2\pi(1+\alpha^2-2\alpha\cos\lambda)$
2nd Order Autoregression - Parameters	$\frac{(1-\alpha_2)\{(1+\alpha_2)^2-\alpha_1^2\}}{2\pi(1+\alpha_2)(1+\alpha_1^2+\alpha_2^2+2\alpha_1(1+\alpha_2)\cos\lambda+2\alpha_2\cos2\lambda)}$.

There is considerable scope for choice of filtering routines to be applied to the original observations. The filters investigated are presented in Table 5 and have been limited to those most commonly used. The table shows the filter coefficients, b_k, as described in (3.2.6) and also the square of the gain of each filter.[18]

18
An underlined b_k coefficient indicates the middle term in a symmetric set of coefficients. For Filters (3) and (4) the first b_k coefficient is b_n and the last coefficient b_{n-q}.

TABLE 5

'TREND' REMOVING FILTERS

Description and Filter Number	b_k Coefficients	Gain Squared $\|B(\lambda)\|^2$
(1) Subtraction of 12 month moving average	$\{-\frac{1}{24}, -\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, \underline{\frac{11}{12}} \ldots\}$	$1 - \frac{\sin\lambda\sin 6\lambda}{24\sin^2\frac{1}{2}\lambda}$
(2) Subtraction of Spencer's 21 pt. moving average	$\frac{1}{350}\{1,3,5,5,2,-6,-18,-33,-47,-57,\underline{290}, \ldots\ldots\}$	$1-(\frac{1}{350})(2+2\cos\lambda-2\cos 3\lambda) \cdot \left(\frac{\sin^2\frac{5\lambda}{2}\sin\frac{7\lambda}{2}}{\sin^3\frac{\lambda}{2}}\right)$
(3) 1st Quasi-differences	$\{1,-.9\}$ $\{1,-.7\}$	$1.81-1.80\cos\lambda$ $1.5625-1.50\cos\lambda$
(4) 2nd Quasi-differences	$\{1,-2(.9),(.9)^2\}$ $\{1,-2(.7),(.7)^2\}$	$4.8961-6.516\cos\lambda+1.62\cos 2\lambda$ $3.5664-4.6875\cos\lambda+1.125\cos 2\lambda$

A notationally economic comparison of the two estimation procedures is most easily given if the model is presented in matrix terms. y is an N dimensional vector of the filtered observations. The $N \times 11$ matrix of regressors, S, is the matrix

$$\begin{bmatrix} \cos\lambda_1, \sin\lambda_1, \cos\lambda_2, \sin\lambda_2, \ldots\ldots \cos\lambda_5, \sin\lambda_5, -1 \\ \cos2\lambda_1, \sin2\lambda_1, \cos2\lambda_2, \sin2\lambda_2, \ldots\ldots \cos2\lambda_5, \sin2\lambda_5, 1 \\ \vdots \\ \cos N\lambda_1, \sin N\lambda_1, \cos N\lambda_2, \sin N\lambda_2, \ldots\ldots \cos N\lambda_5, \sin N\lambda_5, (-1)^N \end{bmatrix}$$

$\hat{\beta}$ and $\tilde{\beta}$ are the L.S.E. and the B.L.U.E. of the eleven seasonal constants in vector form. The focus of interest is not so much the estimates themselves but rather their respective variance-covariance matrices, denoted $\Gamma_{\hat{\beta}}$ and $\Gamma_{\tilde{\beta}}$ and given by [18,p 234]

$$\Gamma_{\hat{\beta}} = (S'S)^{-1}(S'\Gamma_N S)(S'S)^{-1} \tag{3.4.1}$$

and

$$\Gamma_{\tilde{\beta}} = (S'\Gamma_N^{-1}S)^{-1} \tag{3.4.2}$$

where Γ_N is the variance-covariance matrix of $x(n)$, the filtered residual. As the matrix $S'S$ is diagonal with the first ten diagonal terms $N/2$ and the last diagonal term N, the inverse $(S'S)^{-1}$ is simply obtained.

To create the elements of Γ_N (denote them by $\gamma_x(\tau)$, $\tau = 1,2,\ldots,N$) one must first generate the covariances $\gamma_u(\tau)$, $\tau = 1,2,\ldots,M$, where M is of course greater than N by the number of terms lost in filtering. Table 6 below indicates the method of generation of $\gamma_u(\tau)$ for the parameter(s) chosen.

TABLE 6

Generation of $\gamma_u(\tau)$	
Model type	$\gamma_u(\tau) \quad \tau = 0,1,\ldots,M$
Independent	$\{1,0,0,\ldots,0,0\}$
1st order autoregression	$\{1,\alpha,\alpha^2,\ldots,\alpha^M\}$
2nd order autoregression	$\{1,-\{\alpha_1/(1+\alpha_2)\},-\alpha_1\gamma_u(o)-\alpha_2\gamma_u(1), \ldots,-\alpha_1\gamma_u(M-1)-\alpha_2\gamma_u(M-2)$

Once the $\gamma_u(\tau)$ have been generated the lag covariances of $x(n)$ of which Γ_N is composed are simply obtained from the relation

$$\gamma_x(\tau) = \Sigma_{k=-p}^{q} \Sigma_{j=-p}^{q} b_j b_k \gamma_u(\tau-j+k), \quad \tau = 0,1,\ldots,N \tag{3.4.3}$$

and from the property $\gamma_u(\tau) = \gamma_u(-\tau)$, $\tau = 0,1,\ldots,M$.

A compact way of exhibiting the comparative efficiency of the B.L.U.E. and the L.S.E. is to follow the procedure suggested by Watson [55] and to present the ratio of the determinants of the variance-covariance matrices given in (3.4.1) and (3.4.2)

$$E(\Gamma_{\hat{\beta}}:\Gamma_{\tilde{\beta}}) = |S'S|^2 / \left\{|S'\Gamma S||S'\Gamma^{-1}S|\right\} \tag{3.4.4}$$

which provides a measure of the efficiency of the L.S.E. relative to the B.L.U.E. For computational purposes it proved preferable to slightly redefine the S matrix, and to rename it S_* so that the ratio of determinants becomes

$$E(\Gamma_{\hat{\beta}}:\Gamma_{\tilde{\beta}}) = 1/\left\{|S_*'\Gamma_N S_*||S_*'\Gamma_N^{-1} S_*|\right\} \tag{3.4.5}$$

where

$$S_* = \begin{bmatrix} \sqrt{2/N}\cos\lambda_1, & \sqrt{2/N}\sin\lambda_1, & \ldots\ldots, & \sqrt{2/N}\sin\lambda_5, & -\sqrt{1/N} \\ \sqrt{2/N}\cos 2\lambda_1, & \sqrt{2/N}\sin 2\lambda_1, & \ldots\ldots, & \sqrt{2/N}\sin 2\lambda_5, & +\sqrt{1/N} \\ . & . & \ldots\ldots & . & . \\ . & . & \ldots\ldots & . & . \\ . & . & \ldots\ldots & . & . \\ \sqrt{2/N}\cos N\lambda_1, & \sqrt{2/N}\sin N\lambda_1, & \ldots\ldots, & \sqrt{2/N}\sin N\lambda_5, & (-1)^N\sqrt{1/N} \end{bmatrix}$$

It has been suggested [48] that the ratio of determinants exaggerates differences and that a good case may be made for using a measure such as

$$E^*(\Gamma_{\hat{\beta}}:\Gamma_{\tilde{\beta}}) = 1/\left\{11\sqrt{|S_*'\Gamma_N S_*|\,|S_*'\Gamma_N^{-1}S_*|}\right\}. \qquad (3.4.6)$$

Both E and E* (see Table 8) are measures of efficiency which depend on the efficiency obtained for every element of the variance-covariance matrix of the parameters and because of this it is possible that an isolated very poor result may be obscured. What may be of interest to the investigator is an indication of how inefficient the L.S. procedure could be in estimating any particular α_j or β_j. In summarizing the rather detailed results this latter question of efficiency for particular α_j or β_j has been placed in a secondary position so that a more compact presentation of the results is possible. It is however not difficult to obtain a measure of this 'individual coefficient' efficiency for any situation considered by forming the ratio of the value for any α_j or β_j in the B.L.U. column to the value for the same α_j or β_j in the L.S. column of Table 7. In this Table a further characteristic of both procedures which is studied is the relation of the actual variances, that is the diagonal elements $\Gamma_{\hat{\beta}}(i,i)$, $\Gamma_{\tilde{\beta}}(i,i)$, $i = 1,2,\ldots,11$, to the appropriate asymptotic variances $\frac{4\pi}{N} f(\lambda_j)$ (for $i = 1,2$ the appropriate λ_j is of course λ_1). The only purpose in presenting this final comparison (see Table 7) is to give some indication of how effective an approximation the asymptotic value will be for various values of N.

3.5 Efficiency Results and Conclusions

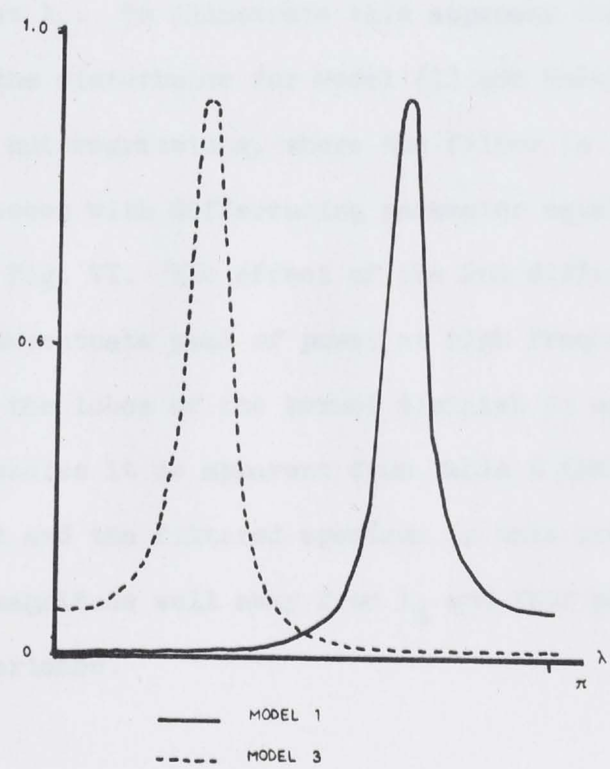
The cases presented below are necessarily selective and have been chosen to include typical as well as unusual results. To further reduce the volume of results Filter No.3, the 1st quasi-difference, is only tabled for the one differencing parameter, .7, and Filter No.4, the 2nd quasi-difference, is also limited to one differencing parameter, in this case .9.

The results for the first order autoregressive processes do not differ very much at all for the range of the parameter considered and the overall efficiencies as measured by either E or E^* are high for $\alpha = .75$ and become higher as α increases to .995. The behaviour of the ratio of actual to asymptotic variance exhibits only a minor improvement as α increases, but there is evidence that the choice of filter will influence the way in which the ratio approaches unity. The independent residuals model gives results for both the overall efficiency indicators and for the individual variance ratios which are quite as would be expected for all filters except the second quasi-difference where for the L.S. procedure it is apparent that even for N = 72 the use of the asymptotic formula would be misleading, particularly for α_1.

The three second order autoregressive models presented provide a contrast between themselves and with respect to other methods of generation. Model (5) which has a spectrum and a correlogram of a more similar nature to the first order autoregression (c.f. Model (1), see Fig. IV) performs very well on overall efficiency grounds and also when the ratio of actual to asymptotic variances is studied. Model (1) is a contrast to all other models for the disturbances (see Fig. IV). The overall efficiencies are quite disappointing and the ratios in Table 7 indicate the complete unsuitability of the asymptotic variance, particularly when the second quasi-difference filter is employed. To provide a contrast to the poor performance of Model (1) it is interesting to look at Model (3) and to note that the efficiencies are improved, although still less satisfactory than the other generating models; but most noticeable is the removal of the very large ratios of actual to asymptotic ratios associated with Model (1).

FIG. IV

SPECTRA OF 2nd ORDER AUTOREGRESSIONS, MODELS 1 AND 3



An explanation of the poor performance of Model (1) of the 2nd order autoregressions proves also to be a suitable vehicle for obtaining some understanding of the way the two estimation methods work. Returning to the discussion of L.S. in §3.3 and in particular to the expression in square brackets given in (3.3.6),[19] one may regard the L.S. variances for α_1 and β_1 as resulting (approximately) from the multiplication of the kernel (Fig. V) by the spectrum of the filtered disturbance, $|B(\lambda)|^2 f_u(\lambda)$, followed by integration over the specified range. This interpretation of the L.S.E. is valuable in suggesting how it may differ radically from the asymptotic value, which depends only on the value of the filtered disturbance at λ_1. To illustrate this approach the filtered spectrum of the disturbance for Model (1) and Model (3) of the second order autoregressions, where the filter is second quasi-differences with differencing parameter equal to .9, is presented in Fig. VI. The effect of the 2nd differencing filter has been to accentuate peak of power at high frequency in Model (1) and although the lobes of the kernel diminish in magnitude at higher frequencies it is apparent from Table 9 that the product of the kernel and the filtered spectrum in this model still has significant magnitude well away from λ_1 and this produces the large L.S. variance.

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19 The expression in square brackets is depicted in Fig. V for N = 36, 72. The difference in shape of the function shown for α_1 and β_1 is obviously due only to the different sign for the third term in the bracket.

FIG. VI

SPECTRUM OF FILTERED $U(n)$, $|B(\lambda)|^2 f_u(\lambda)$, FOR MODELS 1 AND 3 AND FILTER 4

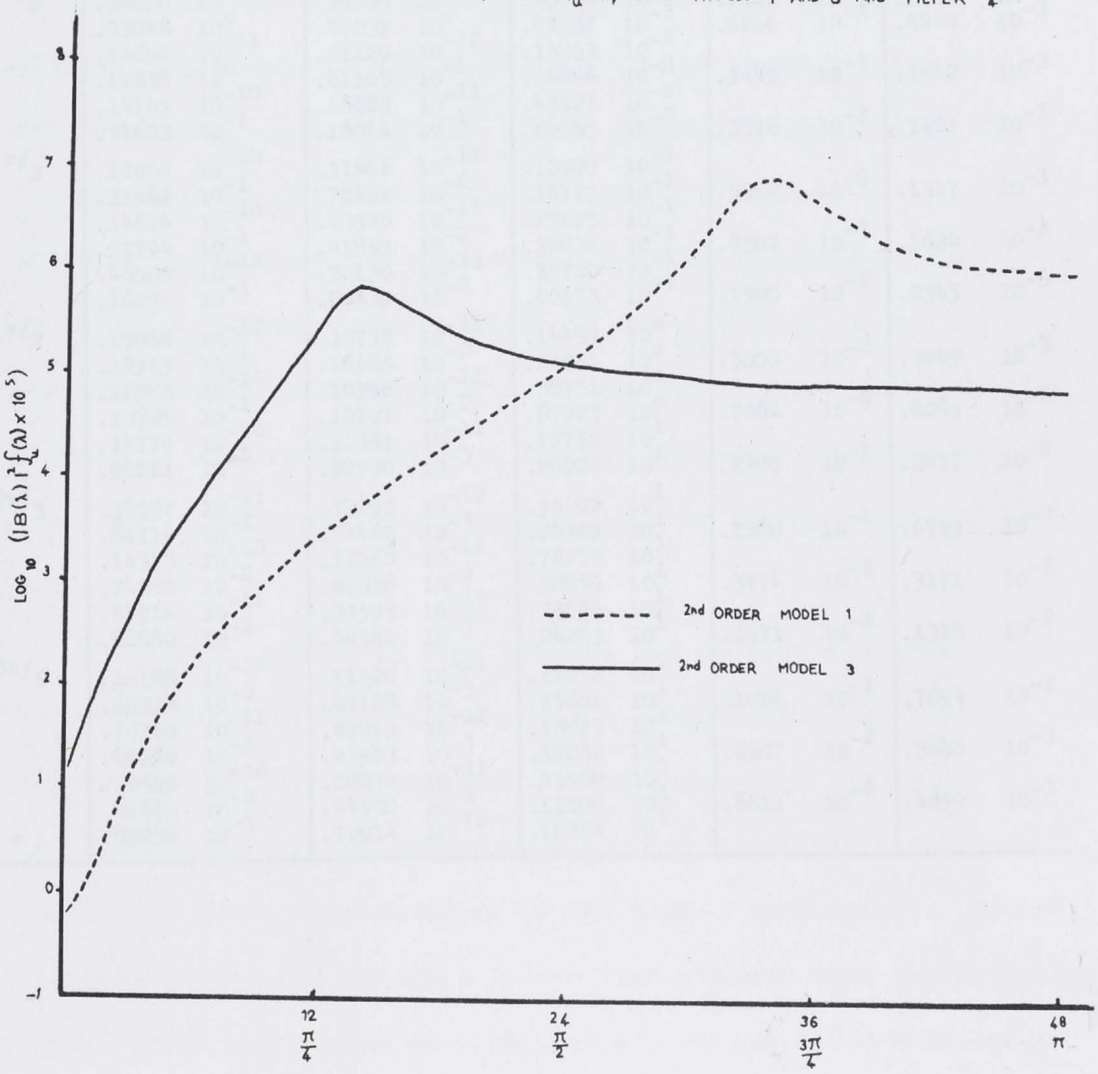

TABLE 9

KERNEL FUNCTIONS DEFINED IN (3.3.6) AND $|B(\lambda)|^2 f_u(\lambda)$ FOR MODEL 1 - FILTER 4; N = 36

Frequency j	λ	(1) Kernel for α_1 (see (3.3.6))	(2) Kernel for β_1 (see (3.3.6))	$\lvert B(\lambda)\rvert^2 f_u(\lambda)$ Model 1 Filter 4	(1) × (3)	(2) × (3)
0	0	.31673 10^{-12}	.22740 10^{-13}	.61501 10^{-6}		
1		.89401 10^{-2}	.91077 10^{-1}	.17648 10^{-5}	.1578 10^{-7}	.1607 10^{-6}
2		.30987 10^{-12}	.22266 10^{-13}	.88070 10^{-5}		
3		.47606 10^{-1}	.15463 10^{0}	.32938 10^{-4}	.1568 10^{-5}	.5093 10^{-5}
4		.32622 10^{-12}	.23422 10^{-13}	.94138 10^{-4}		
5		.68561 10^{0}	.95103 10^{0}	.22341 10^{-3}	.1532 10^{-3}	.2125 10^{-3}
6	$\pi/6$	.20000 10^{1}	.20000 10^{1}	.46630 10^{-3}	.9326 10^{-3}	.9326 10^{-3}
7		.93048 10^{0}	.70239 10^{0}	.88837 10^{-3}	.8266 10^{-3}	.6240 10^{-3}
8		.14890 10^{-11}	.51129 10^{-12}	.15835 10^{-2}		
9		.12838 10^{0}	.61105 10^{-1}	.26866 10^{-2}	.3449 10^{-3}	.1642 10^{-3}
10		.14749 10^{-10}	.18888 10^{-11}	.43927 10^{-2}		
11		.54603 10^{-1}	.18044 10^{-1}	.69885 10^{-2}	.3816 10^{-3}	.1261 10^{-3}
12	$\pi/3$	.17965 10^{-10}	.31406 10^{-11}	.10900 10^{-1}		
13		.31962 10^{-1}	.78491 10^{-2}	.16777 10^{-1}	.5362 10^{-3}	.1317 10^{-3}
14		.14624 10^{-10}	.13528 10^{-11}	.25625 10^{-1}		
15		.21794 10^{-1}	.41855 10^{-2}	.39034 10^{-1}	.8507 10^{-3}	.1634 10^{-3}
16		.48505 10^{-12}	.50130 10^{-13}	.59720 10^{-1}		
17		.16271 10^{-1}	.25439 10^{-2}	.92173 10^{-1}	.1500 10^{-2}	.2345 10^{-3}
18	$\pi/2$	.13960 10^{-10}	.10738 10^{-11}	.14459 10^{0}		
19		.12915 10^{-1}	.16985 10^{-2}	.23251 10^{0}	.3003 10^{-2}	.3949 10^{-3}
20		.12073 10^{-10}	.10196 10^{-11}	.38751 10^{0}		
21		.10725 10^{-1}	.12197 10^{-2}	.67923 10^{0}	.7284 10^{-2}	.8285 10^{-3}
22		.14779 10^{-10}	.11881 10^{-11}	.12752 10^{1}		
23		.92261 10^{-2}	.92930 10^{-3}	.26008 10^{1}	.2400 10^{-1}	.2417 10^{-2}
24	$2\pi/3$	.38597 10^{-11}	.81290 10^{-12}	.55102 10^{1}		
25		.81714 10^{-2}	.74462 10^{-3}	.90309 10^{1}	.7380 10^{-1}	.6725 10^{-2}
26		.14358 10^{-10}	.12968 10^{-11}	.78975 10^{1}		
27		.74200 10^{-2}	.62358 10^{-3}	.50859 10^{1}	.3774 10^{-1}	.3171 10^{-2}
28		.54014 10^{-11}	.39593 10^{-12}	.33625 10^{1}		
29		.68880 10^{-2}	.54341 10^{-3}	.24263 10^{1}	.1671 10^{-1}	.1318 10^{-2}
30	$5\pi/6$	.16188 10^{-10}	.11626 10^{-11}	.18912 10^{1}		
31		.65238 10^{-2}	.49128 10^{-3}	.15661 10^{1}	.1022 10^{-1}	.7694 10^{-3}
32		.70340 10^{-11}	.69019 10^{-12}	.13603 10^{1}		
33		.69260 10^{-2}	.45983 10^{-3}	.12282 10^{1}	.8507 10^{-2}	.5648 10^{-3}
34		.13680 10^{-10}	.10270 10^{-11}	.11460 10^{1}		
35		.61864 10^{-2}	.44500 10^{-3}	.11009 10^{1}	.6811 10^{-2}	.4899 10^{-3}
36	π	.89258 10^{-11}	.72914 10^{-12}	.10864 10^{1}		

Two further comments seem pertinent. The variation in the performance of the variance of α_j and β_j in Table 7 is explained by the shape of the appropriate $f_x(\lambda)$ and in particular by the difference in the shape of the kernel for α_j and β_j (see footnote (19)). For instance in Model (1) it is to be expected on the above reasoning that the variance ratios in Table 7 would be worst for α_1 and β_1, the more so for smaller N. The relation between the variance ratios for α_2 and β_2 still shows α_2 with the higher ratio but the difference is much less marked and reflects the change in the kernel due to its central location now being at λ_2. At λ_3 the α_j and β_j kernel is identical. For λ_j such that $j > 3$, there are only minor differences in the ratios of variances (the values have not been tabled) because for these λ_j although the relation of the α_j and β_j kernel is just the reverse of that for $j < 3$ there is no great concentration of power in the low frequencies for the filtered disturbances to accentuate the greater relative magnitude of the β_j kernel in this region.

The final comment is more general and is that the case of the second differencing filter in this context could be most inappropriate unless the analyst is confident that the true disturbances are not generated by a process which has a spectrum with strong power concentration in the higher frequencies. However, it should be pointed out that if the disturbances were generated by a model which had a peak at high frequencies and a 2nd difference filter was employed then the filtered data would be dominated by high frequency oscillations (obviously recognisable) which would provide an obvious warning as to the inappropriateness of the filter.

3.6 B.L.U. Procedure as a Benchmark[20]

In the examples discussed in the previous sections the place of the B.L.U.E. has been as a benchmark against which to judge the L.S. procedure. Some doubts as to the ability of the B.L.U. procedure to fulfil this role must have been raised by Lovell's proposal [42] of several axioms that seasonal adjustment procedures should satisfy and more particularly the further assertion [42,p 800] that the B.L.U. procedure does not satisfy all of these axioms in generally accepted models of a seasonal economic time series.

The series of axioms that Lovell proposed and which he requires a seasonal adjustment procedure to satisfy are given in full in the cited reference [42,pp 994-5]. Amongst these axioms are:

'Property I: An Adjustment Procedure is said to PRESERVE SUMS if and only if

$$x_t^a + y_t^a = (x_t + y_t)^a \qquad \text{for all } t,$$

where x_t and y_t are the original observations on any pair of time series and x_t^a and y_t^a are the adjusted observations',

and

'Property III: An Adjustment Procedure is ORTHOGONAL if for any time series

$$(x_t - x_t^a)x_t = 0\text{'}.$$

Lovell also shows (see Theorem 3.1 [42, p 996]) that any procedure which satisfies I and III and also satisfies

'Property IV: The Adjustment Procedure is IDEMPOTENT if

$$(x_t^a)^a = x_t^a \qquad \text{for all } t\text{'},$$

20

The notation used by Lovell in describing his axioms has been maintained in this presentation and therefore in this section only x is a vector of observations.

reduces to a least squares regression of the observed vector x, consisting of N observations x_t, on S, an appropriate matrix of k columns (and assumed to be of full rank) with N observations in each. At this point a different approach is used to prove the result previously presented by Lovell [42]. It is of interest to show how this result may be obtained without the use of Property IV. Property I implies that the vector x may be written $x_t = q_t(x)$ i.e. as an additive functional of the vector x. Now if the vector $x \in V$, where V is a finite dimensional vector space, then it follows that

$$q_t(x+y) = q_t(x)+q_t(y) \tag{3.6.1}$$

where both x and $y \in V$. Further, if the vector x and the scalar α are rational then the property

$$q_t(\alpha x) = \alpha q_t(x) \tag{3.6.2}$$

may also be established and so linearity follows. In fact unless the functional q_t is quite pathological (3.6.2) may be established and in any case this property always holds if $q_t(x)$ is bounded (i.e. $q_t(x) \leq A < \infty$) when $|x| < a$ where a and A are finite constants. Another way of establishing this result is to assume that $q_t(x)$ is a measurable and additive function of x and this implies that $q_t(x)$ is a continuous function and so the functional is both linear and bounded. Thus the expression is a linear functional if it is measurable or bounded [34,p 24] and therefore $x_t^a = q_t'x$ where q_t is a vector of N components. The vector of adjusted values, x^a, then satisfies $x^a = Qx$ where Q has q_t' as the t^{th} row. Property III implies that

$$x'Q'(I-Q)x = 0 \tag{3.6.3}$$

and that this is so for all x. Thus $Q' = Q'Q$ and Q' is symmetric. Therefore $Q = Q^2$ and Q is idempotent and a perpendicular projector. If the columns of S span the space on which Q projects then x^a is obtained by regression on these columns.

Before going further let it be said that these properties or axioms seem unacceptable on general grounds. First, it would be extraordinary if the vast array of techniques for signal detection and measurement, which have been developed over the last half century, could be reduced to the simpler problem of regression analysis. More particularly, the axioms exclude non-linear procedures and this of itself seems unacceptable. It is easy to construct examples where maximum likelihood procedures are non-linear and it would be hard to envisage realistic formulations where this was not the case. Most importantly, perhaps, Axiom or Property III must be unequivocably rejected. The statement by Lovell following this axiom [42,p 995], 'How can a nonorthogonal seasonal adjustment procedure be regarded as satisfactory? After all, if such a procedure correctly defines the seasonal movement the fact that the seasonal correction terms are correlated with the adjusted series implies that some seasonality remains in the data', elevates a particular inner product, $(x,y) = x'y$, into a premier place in relation to the vector space in which x lies. If this inner product is elevated to a premier place it can only be on the basis of prior assumptions. Appropriate prior grounds might be a model for the original observations of the form

$$x-D\delta-S\sigma = \epsilon \qquad (3.6.4)$$

where σ is a vector of k seasonal constants, $D'\delta$ is null and ϵ is composed of identically and independently distributed variables with mean zero and unit variance (i.i.d.(0,1)). The matrix D is of full rank with columns representing the non-seasonal deterministic variables and δ is the associated set of constant coefficients. If the condition on ϵ or that on S and D is not satisfied then Lovell axioms will prove to be hardly acceptable. Suppose that it is assumed that $\mathcal{E}(\epsilon\epsilon') = \Gamma$ then it will be shown that the B.L.U. procedure is the same as the L.S. procedure of regression on S alone when and only when S and D are such that $S'\Gamma^{-1}D$ is null and also the k columns of S are linear combinations of k eigenvectors of Γ. The argument proceeds as follows. A sufficient condition for $(S'S)^{-1}S'x$ to be a B.L.U.E. of σ is that $D'S = 0$ and also that the space spanned by the columns of S (call it $\mathfrak{M}(S)$) is spanned by eigenvectors of Γ. (See Watson [56]). If the L.S. regression of x on S is to be the B.L.U.E. it is certainly necessary that $D'S = 0$ since the expected value of the L.S.E. is given by

$$\mathcal{E}\left((S'S)^{-1}S'x\right) = (S'S)^{-1}S'D\delta+\sigma \tag{3.6.5}$$

and so S'D must equal zero if the expectation is to equal σ for all δ. Now consider a row vector λ',

$$\lambda' = (0:\lambda_2') \tag{3.6.6}$$

with columns corresponding to the partitioned matrix $\begin{pmatrix} D' \\ \cdots \\ S' \end{pmatrix}$.

Form the unbiassed estimator $\lambda_2'(S'S)^{-1}S'x$ of $\lambda_2'\sigma$ and let $\lambda'Ax$ be another unbiassed estimate of $\lambda_2'\sigma$. The latter estimator may be denoted $L'x$, where $L' = L_1'+L_2'$ and $L_2 \varepsilon \mathfrak{M}(S)$, $L_1 \perp \mathfrak{M}(S)$. Now the estimator L'x may be written as

$$L'x = L_1'D\delta+L_1'S\sigma+L_1'\epsilon+L_2'x$$

$$= L_1'D\delta+L_2'x+L_1'\epsilon \qquad (L_1 \perp \mathfrak{M}(S))$$

and so the expectation becomes

$$\mathcal{E}(L'x) = L_1'D\delta+\mathcal{E}(L_2'x). \tag{3.6.7}$$

Since L'x is unbiassed and the expectation does not involve δ this implies $L_1'D = 0$ and further as the expectation, $\mathcal{E}(L_2'x)$, is given by

$$\mathcal{E}(L_2'x) = L_2'S\sigma = \lambda_2'\sigma \qquad (\text{since } L_2 \varepsilon \mathfrak{M}(S) \text{ and } S'D = 0)$$

and so

$$L_2'S\sigma = \lambda_2'(S'S)^{-1}S'S\sigma$$

$$\text{i.e.} \quad (\lambda_2'(S'S)^{-1}S'-L_2')S\sigma = 0$$

$$\text{i.e.} \quad \lambda_2'(S'S)^{-1}S' = L_2'. \tag{3.6.8}$$

The variance-covariance estimator of the estimator $L'x$ is

$$L'\Gamma L = L_2'\Gamma L_2 + L_2'\Gamma L_1 + L_1'\Gamma L_1 + L_1'\Gamma L_2$$

and using (3.6.8) this expression becomes

$$L'\Gamma L = \lambda_2'(S'S)^{-1}S'\Gamma S(S'S)^{-1}\lambda_2 + \lambda_2'(S'S)^{-1}S'\Gamma L_1 + L_1'\Gamma L_1 + L_1'\Gamma S(S'S)^{-1}\lambda_2 \quad (3.6.9)$$

and therefore if $L_2'x$ is the B.L.U.E. then this implies that $\lambda_2'(S'S)^{-1}S'\Gamma L_1 = 0$. Further as $L_1 \perp \mathfrak{M}(S)$ then $\lambda_2'(S'S)^{-1}S\Gamma$ belongs to $\mathfrak{M}(S')$ and consequently $\Gamma S(S'S)^{-1}\lambda_2$ belongs to $\mathfrak{M}(S)$. As this is true for all $(S'S)^{-1}\lambda_2$ then ΓS belongs to $\mathfrak{M}(S)$ and so Γ transforms a vector in a k dimensional subspace into a vector also in that subspace (invariant transformation) and S will be spanned by k eigenvectors of (see [56]).

The only circumstances when this condition is at all likely to hold is where Γ is a scalar matrix (a numerical multiple of the identity matrix) and D is composed of a single column of units while S has columns composed of (for monthly data) $\cos \frac{2\pi j}{12} n$, $\sin \frac{2\pi j}{12} n$, for suitable values of j. Experience in seasonally adjusting economic data would lead one to believe that either D must be expanded to include other explanatory vectors as well as the unit vector or the generating process for the disturbance must be more broadly specified. For example we might have $\epsilon(n) = \rho\epsilon(n-1)+\eta(n)$ where $\eta(n)$ is i.i.d.(0,1) and ρ is close to unity but $|\rho| < 1$. Indeed it may be necessary to accept both these respecifications.

It seems therefore that Lovell's axiomatic basis for seasonal adjustment must be rejected and that a theory of seasonal adjustment must instead rest on much more elaborate techniques of signal detection and measurement.

3.7 Working Day Variation

Economic time series which register a flow of some kind over either monthly or weekly intervals may exhibit a source of variation which is not seasonal and therefore ranks for separate consideration. The original model (3.2.1) is augmented in this section to include a further signal, which on a priori grounds one could readily expect to occur. The interesting point which arises in this section is that when the signal generated by working days is incorporated in the above model its nature requires one to use a spectral regression procedure to obtain more efficient estimates than L.S.

This additional source of variation may be included if the model given in (3.2.1) is extended as follows

$$w(n) = p(n)+s^*(n)+\Sigma_{\ell 1}^{7}\alpha_{\ell}u_{\ell}(n)+u(n) \qquad (3.7.1)$$

with $u_{\ell}(n)$ the number of days of the ℓ^{th} type in the n^{th} month. In (3.7.1) it is assumed that each working day makes a specific additive contribution to the series. It may be, however, that the variation depends on the composition of the "extra days" in any month, that is those in excess of twenty eight. For example, it may not be that the Monday effect will be the same in a month ending Saturday, Sunday, Monday as it would be in a month ending Monday, Tuesday, Wednesday. An alternative model is therefore proposed which can account for certain interactions between the days in each group of "extra days". The more general model is

$$w(n) = p(n)+s^*(n)+\Sigma_{\ell 1}^{2}\Sigma_{m 1}^{7}\alpha_{\ell m}u_{\ell m}(n)+u(n) \qquad (3.7.2)$$

where the variables $u_{\ell m}(n)$ cover the fourteen possible two and three "extra days" effects.[21] At each time point n, one of the variables $u_{\ell m}(n)$ will be unity and the remainder zero. Both models (3.7.1)

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[21] No attempt has been made to estimate the "extra days" effect associated with leap year February, as this event occurs too infrequently either to give reliable estimates or to be important. All February observations are included with a zero regressor vector so that the lag correlations (and therefore the spectra) of the regressor variables are interpretable in actual time.

and (3.7.2) are suitable for the use of regression methods to estimate the proposed effects. Before dealing with appropriate regression procedures it is instructive to look at the spectra of the possible regressors, $u_{\ell}(n)$ and $u_{\ell m}(n)$, examples of which are given in Fig. VII. Certain implications are apparent. The peaks of power in the spectra of these regressors are found quite close to several seasonal frequencies.

The method of estimating the independent daily effects in (3.7.1) which has been proposed previously (see [52]) is to regress the original series on the $u_{\ell}(n)$ after trend and seasonal has been subtracted. This method will be more appropriate if the model for $s^*(n)$ is a stable one as proposed in (3.2.3). However, if as will be suggested in the next chapter, it is believed that the seasonal pattern is slowly evolving and the method of seasonal estimation is changed accordingly then at certain frequencies it is apparent that the seasonal estimates will be influenced by the working days power. Thus in a situation where the seasonal pattern is evolving it is necessary to either estimate the working days effect after <u>only</u> trend removal and before seasonal estimation or to waive explicit consideration of the working days effect.[22]

Least squares regression of $w(n)$ on $u_{\ell}'(n)$ or $u_{\ell m}'(n)$ (where the change in symbols to $w(n)$, $u_{\ell}'(n)$ and $u_{\ell m}'(n)$ indicate that trend removal filtering has been carried out) will provide estimates of the required parameters. The simple least square procedure is not however the most efficient. The filtered regression variables are

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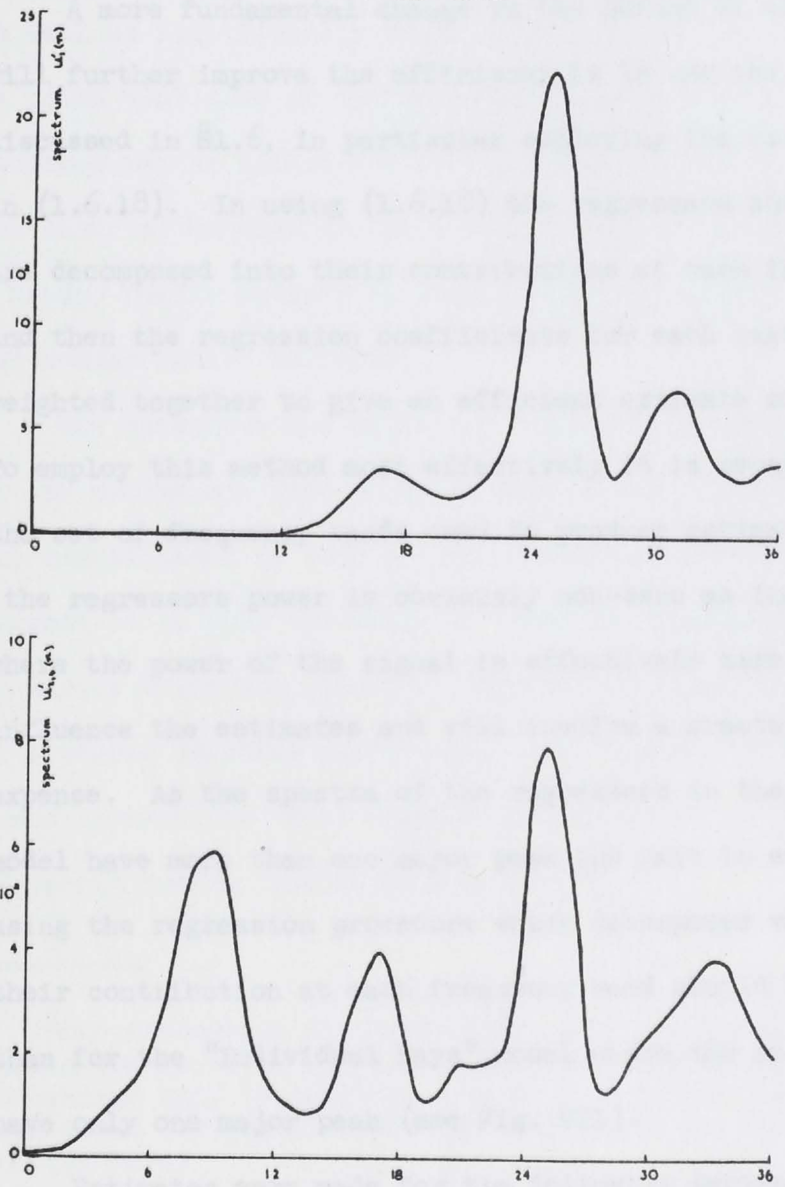
22 The reason for second possibility is easily seen when in the next chapter the response function for the evolving seasonal extraction is given and it is apparent that this will incorporate some working day power.

FIG. VII

SPECTRUM OF A REGRESSOR IN EACH WORKING DAY MODEL



close to linear dependence and so the matrix of filtered regressors is close to singularity. The appropriate one of the following restrictions,

$$\Sigma_{\ell 1}^{7}\alpha_{\ell}(n) = 0$$

$$w\,\Sigma_{1}^{7}\alpha_{1m} + (1-w)\,\Sigma_{1}^{7}\alpha_{2m} = 0 \qquad w = \frac{4}{11} \tag{3.7.3}$$

is employed to reduce by one the parameters to be estimated.

A more fundamental change in the method of estimation which will further improve the efficiency is to use the methods discussed in §1.6, in particular employing the estimator given in (1.6.18). In using (1.6.18) the regressand and the regressors are decomposed into their contributions at each frequency band and then the regression coefficients for each band are optimally weighted together to give an efficient estimate of each coefficient. To employ this method most effectively it is necessary to restrict the set of frequency bands used to produce estimates to those where the regressors power is obviously non-zero as inclusion of bands where the power of the signal is effectively zero will not influence the estimates and will involve a greater computational expense. As the spectra of the regressors in the "Excess Days" model have more than one major peak the gain in efficiency from using the regression procedure which decomposes variables into their contribution at each frequency band should be more marked than for the "Individual Days" model where the regressor spectra have only one major peak (see Fig. VII).

Estimates were made for the following series:

Australian Total Exports of Merchandise

Australian Total Imports of Merchandise Feb. 49-May 67.[23]

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[23] The source of both series of data is Commonwealth Bureau of Census and Statistics (Aust.), Monthly Review of Business Statistics.

A tabular summary of estimates employing both models on each of the series is given in Table 10. In the "Individual Days" model for both series there is only one significant coefficient. It is apparent that in this model the only significant effect discernible is the obvious negative effect associated with Sundays. The pattern of working day activity is much more apparent in the interaction "Excess Days" model and these estimates have significantly large mid week three day excesses as well as significantly small near weekend excesses. It should be noted that some of the negative effects found in the two day excesses may be influenced by the number of public holidays which can occur on Mondays in the 30 day months. Australia always has a Monday public holiday in June and April can have as many as two Monday public holidays.

The concentration of Monday holidays in two of the four 30 day months may have produced some distortion of the effects.[24] The significance of some of the excess-day coefficients does suggest it is most necessary to carry out working day corrections to series which are likely to exhibit this variation particularly if a stable seasonal pattern has been fitted. Failure to make these corrections could mislead policy makers in their assessment of recent trends in exports and imports.

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[24] Some distortion will obviously arise if public holidays occur during the month on any day because the excess days will not be determined only by those in excess of twenty-eight. More detailed work could obviously be done on this point but it has not been pursued here.

A somewhat disquieting feature of the "Excess Days" Model for Exports (although one recognizes the distorting effects of Monday holidays in the 30 day months) is that it produces a significantly negative response for Mon-Tues whereas Sun-Mon is not significantly negative. A similar situation exists for Imports in that Sun-Mon is significantly negative whereas Sat-Sun is not.

FIG. V

FUNCTIONS DEFINED IN (3.3.6), N = 36, 48, 60, 72

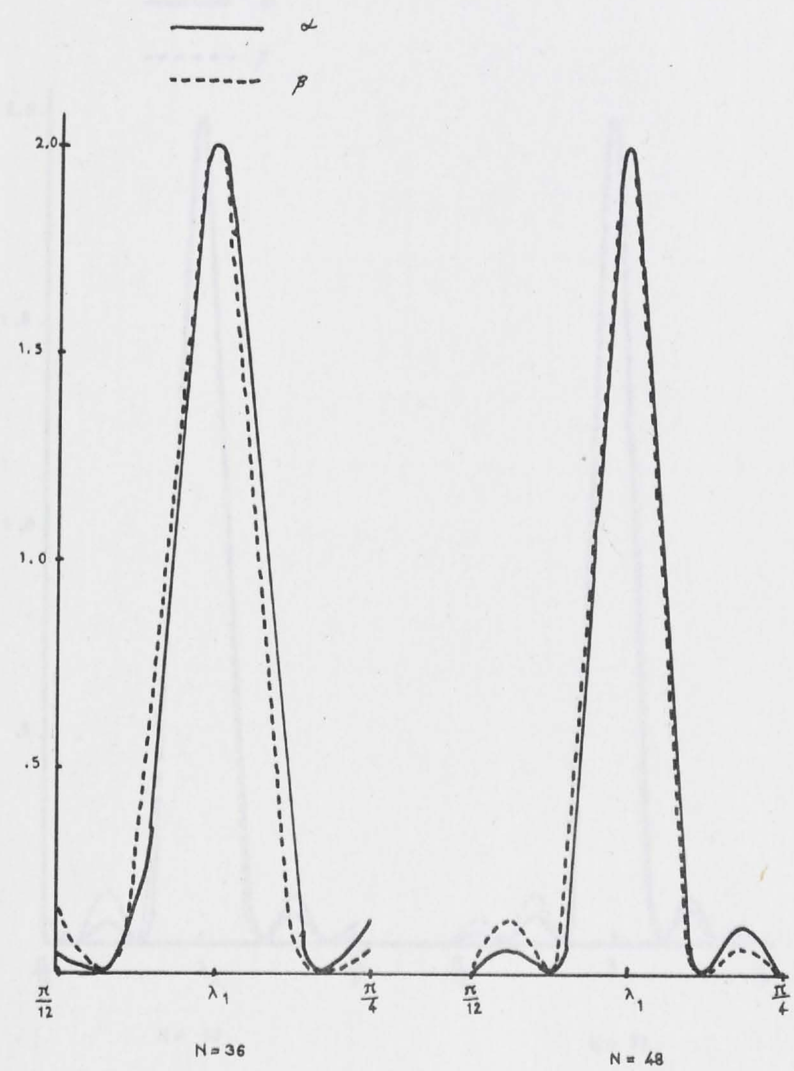


FIG. V

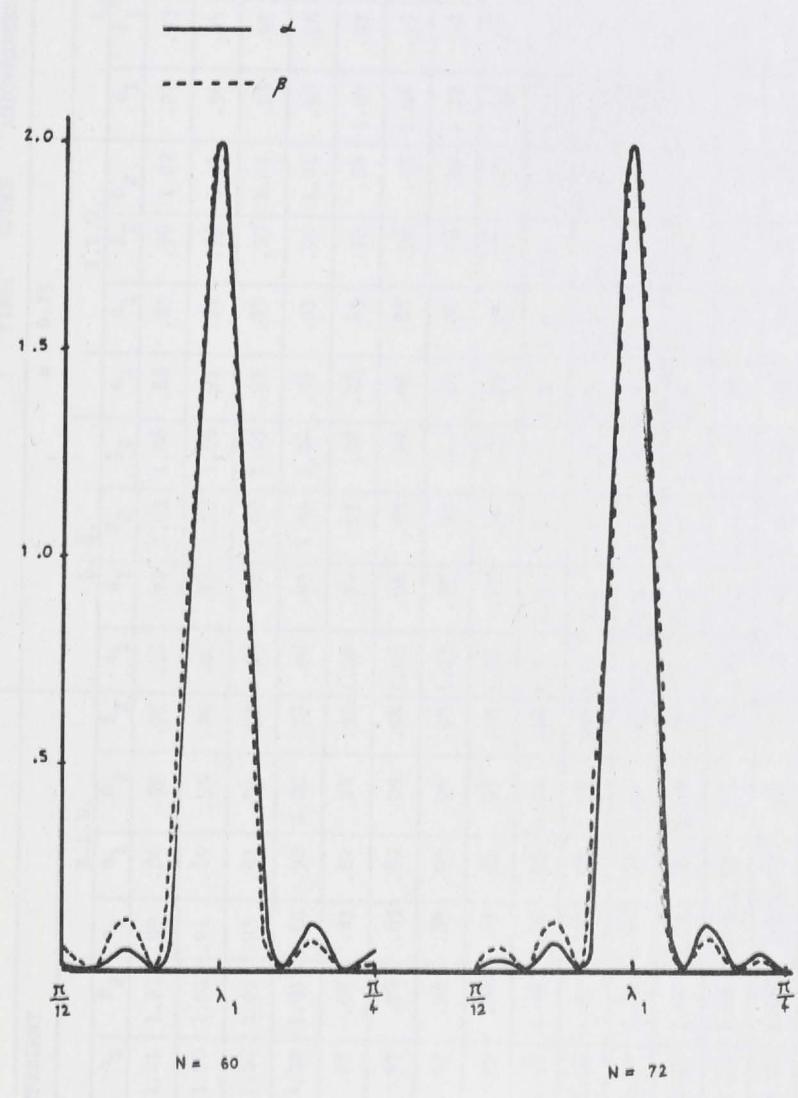


TABLE 7

RATIO OF CALCULATED VARIANCE TO ASYMPTOTIC APPROXIMATION

FILTER	N	INDEPENDENT								FIRST ORDER AUTOREGRESSIONS															
										$\alpha = 0.75$								$\alpha = 0.995$							
		L. S.				B.L.U.				L. S.				B.L.U.				L. S.				B.L.U.			
		α_1	β_1	α_2	β_2	α_1	α_1	β_2	β_2	α_1	α_1	β_2	β_2	α_1	α_1	β_2	β_2	α_1	α_1	β_2	β_2	α_1	α_1	β_2	β_2
1	36	1.01	.92	1.01	1.01	.89	.86	.99	.99	.92	.91	1.03	1.08	.88	.85	.99	1.02	.92	.93	1.04	1.10	.89	.89	1.00	1.03
	48	1.01	.94	1.00	1.01	.91	.89	.99	.99	.94	.93	1.02	1.06	.91	.87	.99	1.01	.94	.94	1.03	1.08	.91	.91	1.00	1.02
	60	1.01	.95	1.00	1.01	.93	.91	.99	.99	.95	.94	1.02	1.05	.92	.89	.99	1.01	.95	.95	1.02	1.06	.93	.93	1.00	1.02
	72	1.00	.96	1.00	1.01	1.01	.92	1.00	.99	.96	.95	1.01	1.04	.94	.91	.99	1.01	.96	.96	1.02	1.05	.94	.94	1.00	1.01
2	36	1.42	1.06	.98	.97	.81	.89	.94	.92	1.08	.94	.97	.98	.78	.85	.95	.94	1.05	.93	.97	.99	.85	.85	.95	.95
	48	1.32	1.04	.99	.98	.85	.89	.95	.94	1.06	.96	.98	.99	.82	.88	.96	.95	1.04	.95	.98	.99	.89	.88	.96	.96
	60	1.25	1.03	.99	.98	.87	.90	.96	.95	1.05	.97	.98	.99	.86	.90	.96	.95	1.03	.96	.98	.99	.90	.90	.97	.96
	72	1.21	1.03	.99	.98	.89	.91	.97	.95	1.04	.97	.99	.99	.88	.91	.97	.96	1.02	.97	.99	.99	.91	.91	.97	.97
3	36	1.27	1.00	1.05	1.00	1.07	.98	1.01	.98	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.03	1.40	1.04	1.11	1.01	1.10	1.01	1.03
	48	1.21	1.00	1.04	1.00	1.05	.98	1.00	.98	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.03	1.39	1.03	1.10	1.00	1.07	1.01	1.02
	69	1.16	1.00	1.03	1.00	1.04	.99	1.00	.99	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.03	1.38	1.03	1.10	1.00	1.06	1.00	1.01
	72	1.14	1.00	1.03	1.00	1.04	.99	1.00	.99	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.03	1.37	1.03	1.10	1.00	1.05	1.00	1.01
4	36	4.19	1.36	1.25	1.08	.95	1.05	.96	.97	1.43	1.01	1.06	1.00	.98	.96	.98	.97	1.33	1.00	1.05	1.00	1.07	0.97	1.00	.98
	48	3.39	1.27	1.18	1.06	.96	1.03	.97	.98	1.32	1.01	1.04	1.00	.99	.97	.99	.97	1.25	1.00	1.04	1.00	1.05	.98	1.00	.98
	60	2.91	1.21	1.15	1.05	.97	1.02	.97	.98	1.26	1.01	1.04	1.00	.99	.98	.99	.98	1.20	1.00	1.03	1.00	1.04	.98	1.00	.98
	72	2.59	1.18	1.12	1.04	.97	1.02	.98	.99	1.22	1.00	1.03	1.00	.99	.98	.99	.98	1.17	1.00	1.03	1.00	1.03	.99	1.00	.99

TABLE 7

RATIO OF CALCULATED VARIANCE TO ASYMPTOTIC APPROXIMATION

		SECOND ORDER AUTOREGRESSIONS																							
		MODEL 1								MODEL 3								MODEL 5							
		$\alpha_1 = 1.00$				$\alpha_2 = 0.75$				$\alpha_1 = -1.00$				$\alpha_2 = 0.75$				$\alpha_1 = -1.25$				$\alpha_2 = 0.3$			
FILTER	N	L.S.				B.L.U.				L.S.				B.L.U.				L.S.				B.L.U.			
		α_1	β_1	α_2	β_2	α_1	β_1	α_2	β_2	α_1	β_1	α_2	β_2	α_1	β_1	α_2	β_2	α_1	β_1	α_2	β_2	α_1	β_1	α_2	β_2
1	36	1.80	1.01	1.51	1.21	.89	.92	1.04	1.03	1.27	1.04	.96	1.00	.88	.88	.80	.83	.91	.93	1.07	1.15	.86	.88	.99	1.03
	48	1.60	1.01	1.38	1.16	.91	.94	1.03	1.02	1.20	1.03	.97	1.00	.90	.91	.84	.86	.93	.94	1.05	1.11	.89	.91	.99	1.02
	60	1.48	1.00	1.31	1.13	.92	.95	1.02	1.01	1.16	1.03	.98	1.00	.92	.92	.87	.88	.95	.96	1.04	1.09	.91	.92	.99	1.01
	72	1.40	1.00	1.26	1.11	.93	.96	1.02	1.01	1.14	1.02	.98	1.00	.93	.93	.89	.90	.95	.96	1.03	1.08	.92	.93	.99	1.01
2	36	5.41	1.48	1.50	1.19	.88	.89	.99	.99	2.22	1.44	.92	.94	.83	.88	.77	.77	1.01	.91	.98	1.00	.81	.82	.94	.95
	48	4.30	1.36	1.37	1.14	.89	.89	.99	.99	1.91	1.33	.94	.96	.86	.88	.82	.81	1.01	.94	.98	1.00	.85	.86	.95	.96
	60	3.63	1.29	1.30	1.11	.92	.91	.99	.99	1.73	1.26	.95	.96	.88	.88	.85	.84	1.01	.95	.99	1.00	.88	.88	.96	.96
	72	3.20	1.24	1.25	1.09	.94	.94	1.00	1.00	1.61	1.22	.96	.97	.90	.88	.87	.86	1.00	.96	.99	1.00	.90	.90	.96	.97
3	36	8.31	1.66	2.58	1.64	1.11	1.06	1.04	1.05	1.55	1.15	.90	.91	1.08	1.02	.81	.88	.99	1.17	1.01	1.07	.97	1.07	1.00	1.03
	48	6.46	1.49	2.18	1.48	1.08	1.04	1.03	1.04	1.41	1.12	.92	.93	1.06	1.01	.85	.86	.99	1.13	1.01	1.06	.98	1.04	1.00	1.02
	60	5.36	1.40	1.94	1.38	1.06	1.03	1.02	1.03	1.33	1.09	.94	.95	1.05	1.01	.88	.88	.99	1.06	1.01	1.05	.98	1.04	1.00	1.01
	72	4.64	1.33	1.79	1.32	1.05	1.02	1.02	1.02	1.27	1.08	.95	.96	1.04	1.00	.89	.90	.99	1.03	1.01	1.04	.99	1.03	.98	1.01
4	36	103.	10.3	6.19	3.13	1.01	1.11	1.00	1.02	3.16	1.66	.88	.89	.97	1.07	.81	.80	1.14	.97	1.01	.98	1.03	.95	.99	.97
	48	77.0	7.93	4.87	2.59	1.01	1.05	1.00	1.01	2.61	1.50	.91	.92	.97	1.05	.85	.83	1.11	.98	1.01	.99	1.02	.96	.99	.98
	60	61.8	6.54	4.10	2.27	1.01	1.05	1.00	1.01	2.29	1.40	.93	.93	.98	1.03	.87	.86	1.08	.98	1.01	.99	1.02	.96	.99	.98
	72	51.6	5.62	3.58	2.06	1.01	1.05	1.00	1.01	2.07	1.33	.94	.94	.98	1.03	.89	.88	1.07	.99	1.00	.99	1.01	.97	1.00	.98

TABLE 8

OVERALL EFFICIENCY INDICATORS, E and E*, FOR SOME MODELS

				1ST ORDER AUTOREGRESSION				2ND ORDER AUTOREGRESSION					
		INDEPENDENT		$\alpha = 0.75$		$\alpha = 0.995$		$\alpha_1 = -1.25,\ \alpha_2 = 0.3$		$\alpha_1 = 1.00,\ \alpha_2 = 0.75$		$\alpha_1 = -1.00,\ \alpha_2 = 0.75$	
FILTER	N	E	E*	E	E*	E	E*	E	E*	E	E*	E	E*
1	36	.761	.976	.721	.971	.726	.971	.522	.943	.199	.863	.171	.852
	48	.799	.980	.757	.975	.762	.976	.571	.950	.257	.884	.221	.872
	60	.827	.983	.787	.978	.793	.979	.614	.957	.309	.899	.267	.887
	72	.848	.985	.811	.981	.817	.982	.651	.962	.352	.910	.310	.899
2	36	.392	.918	.544	.946	.627	.959	.532	.944	.059	.774	.078	.793
	48	.466	.933	.607	.956	.686	.966	.595	.954	.034	.736	.114	.821
	60	.527	.943	.660	.963	.722	.971	.648	.961	.121	.825	.150	.841
	72	.580	.952	.701	.968	.750	.974	.690	.967	.155	.844	.184	.857
3	36	.738	.973	1.00	1.00	.676	.965	.772	.977	.058	.772	.385	.917
	48	.782	.978	1.00	1.00	.654	.962	.798	.980	.075	.794	.451	.930
	60	.814	.982	1.00	1.00	.645	.961	.822	.982	.099	.811	.506	.940
	72	.838	.984	1.00	1.00	.641	.960	.843	.985	.120	.825	.552	.947
4	36	.144	.839	.551	.947	.696	.968	.823	.983	.002	.566	.151	.842
	48	.193	.861	.623	.958	.742	.973	.857	.986	.003	.521	.206	.866
	60	.238	.878	.676	.965	.778	.978	.880	.988	.005	.617	.257	.884
	72	.279	.890	.715	.970	.806	.981	.897	.990	.006	.629	.304	.897

TABLE 10a

Working Day Effects

Individual Day Effects

Exports

Day	Sun	Mon	Tues	Wed	Thu	Fri	Sat
Regression Coefficient	-3.681	4.574	-2.333	3.308	4.134	.792	-6.794*
Standard Error	2.8	2.7	2.6	2.8	2.7	2.7	2.6

Two and Three Day Month Ending Effects

Ending	Sun Mon	Mon Tues	Tues Wed	Wed Thu	Thu Fri	Fri Sat	Sat Sun	Sun Mon Tues	Mon Tues Wed	Tues Wed Thu	Wed Thu Fri	Thu Fri Sat	Fri Sat Sun	Sat Sun Mon
Coefficient	5.85	-12.41*	-7.71	5.16	3.63	-6.45	-13.61*	-.19	12.60*	12.63*	4.79	.36	-6.78	-8.81*
Standard Error	5.4	5.6	5.4	5.3	5.4	5.4	5.6	3.8	4.0	3.8	4.0	4.0	3.9	4.0

Note: The * indicates the significant coefficients.

TABLE 10b

Working Day Effects

Individual Day Effects

Imports

Day	Sun	Mon	Tues	Wed	Thu	Fri	Sat
Regression Coefficient	-5.508*	1.013	.312	4.499	3.392	-1.389	-2.319
Standard Error	2.328	2.313	2.309	2.360	2.263	2.324	2.274

Two and Three "Excess Days" Effects

Ending	Sun Mon	Mon Tues	Tues Wed	Wed Thu	Thu Fri	Fri Sat	Sat Sun	Sun Mon Tues	Mon Tues Wed	Tues Wed Thu	Wed Thu Fri	Thu Fri Sat	Fri Sat Sun	Sat Sun Mon
Coefficient	-10.729*	-6.602	1.393	5.463	1.508	-9.667	-5.770	-2.879	11.990*	9.733*	7.542	.232	-7.318	-5.355
Standard Error	5.301	5.431	5.158	5.167	5.167	5.158	5.273	4.494	4.690	4.550	4.690	4.648	4.593	4.680

Note: The * indicates the significant coefficients.

IV SIGNAL EXTRACTION PROBLEMS

4.1 Seasonal Models and the Adjustment Problem

In this chapter the problem area is again that of seasonal adjustment, but there is no longer an exactly specified signal. Rather we begin with *a priori* ideas about the changing nature of the signal (amplitude and possibly phase modulation) and summarize these ideas on change in the spectral properties of the signal. This method appears particularly appropriate to a model of the seasonal component which must surely be represented as a sum of six narrow frequency band signals. The only extension that is proposed is to regard each signal as being amplitude modulated. Of course the seasonal signal will be superimposed on noise, however the narrow band nature of the signals means that only the average noise level over these narrow bands is of great concern. Consequently a spectral treatment of the noise will require the introduction of relatively few parameters and more detailed models will probably add very little in efficiency, while increasing the risk of invalid analysis.

The main difficulty in effecting adequate seasonal adjustment arises from the fact that the seasonal pattern may be changing. The problem of estimating such a changing seasonal pattern is an aspect of one of the most important of all scientific problems. The difficulty is simple to perceive but must be understood. If an estimation procedure is developed which is sensitive to changes in the seasonal component then the procedure will also be sensitive to chance fluctuations or noise effects. An optimal solution may be derived on the basis of an initial model. This optimal solution may be of value both for its own sake and as a standard of comparison for ad hoc procedures, but uncritical acceptance of the solution as best would be unwise as no model on which optimization procedures are based is likely to represent the truth. In any case the optimum criteria may be deficient because it fails to reflect subjective elements which are difficult to quantify, such as the reluctance of an official institution to employ methods which may entail substantive later revisions of first estimates.

There is a further point which deserves discussion in this introduction. The treatment presented is based on a model of the data, possibly after logarithmic transformation, which consists of seasonal plus 'noise', where 'noise' is all the remaining variation. It has been pointed out (see Whittle [58]) that it would be preferable to use a model in which the seasonal component is properly integrated as a part of the whole mechanism generating the series and is not merely 'stuck on' as an additional but separate component. It is as well to point out however that the use of certain seemingly more complex models leads to the additive model we have used. For example, a model of the form

$$\Sigma_{j}^{q}{}_{0}\gamma_{j}w(n-j) = e(n) \tag{4.1.1}$$

where $e(n)$ contains a component, $g(n)$, with seasonally oscillating properties is no generalization. For if one writes $e(n) = g(n)+h(n)$, where $g(n)$ produces seasonal oscillations then the general nature of the solution is

$$w(n) = p(n)+s^*(n)+u(n)$$

where $p(n)$ is the solution of the homogenous equation obtained from (4.1.1). Similarly $s^*(n)$ and $u(n)$ are the respective solutions of (4.1.1) when first $g(n)$ and then $h(n)$ replace $e(n)$. Thus the additive nature of the seasonal is maintained.

The model adopted for the seasonal alone in this chapter and Chapter V could be thought of in the following terms.

$s^*(n)$ is a solution of an expression of the form (4.1.1) with $q = 1$,

$$y_j(n) = \rho_j e^{i\lambda_j} y_j(n-1)+g^*(n) \qquad j = \pm 1, \pm 2, \ldots, \pm 5, 6 \tag{4.1.2}$$

in which $g^*(n)$ produces oscillations with frequency λ_j and with amplitude depending on the variances of the random terms $\epsilon_j(n)$ and $\eta_j(n)$ (see (4.2.3)). The advantage of the procedure adopted here is that the components p(n) and u(n) are not tied to generation by the same mechanism. Further, one obtains no generality if the polynomial, $\Sigma_{j=0}^{q}\gamma_j z^j$, is required merely to have certain roots on or very near to the unit circle and with 'argument' corresponding to the seasonal frequencies. Once again the solution of (4.1.1) is no more than a sum involving p(n) and s*(n) obtained in exactly the same manner as described above and again the model would be more restrictive rather than more general. One can propose essentially different models, such as one in which the γ_j oscillate periodically but apparently no work has yet been done on models of this nature.

The difficulty is not that of building such models, but rather of building adequate ones. Economic inter-relations are sufficiently complex so that the policy maker may be unwilling to commit himself entirely, for example, to one generating model for all components and so he would prefer to view key series with perhaps a model in mind but not restricted to it. The policy maker will want to survey series with as little done to them as possible, except for seasonal adjustment and he will probably not be prepared to use uncritically a projection of a series or a set of series, the projection having been made purely on the basis of the past of the series. This leaves an important role for seasonal corrections based on an additive model of the type presented in (3.2.1).

4.2 An Evolving Seasonal Pattern

The main task is the formulation of a suitable model and consequent statistical treatment for the case of an evolving seasonal pattern. The case considered here will be where the change in the seasonal pattern is gradual and continuous. Separate consideration should be given to the situation where sudden changes occur at randomly distributed points in time. This form of analysis is unlikely to proceed purely on the basis of the history of the data but will depend on additional related information which will be available and should be incorporated in a more complex formulation. This approach is not pursued.

As indicated in the introduction to this chapter the model of the data which is used is given in (3.2.1), but now u(n)+p(n) will often be referred to as the 'noise' component (see §3.2 - in particular the discussion associated with (3.2.4)). The traditional model for the seasonal pattern is that of a strictly periodic or stable sequence and this model has been discussed in §3.2 and in particular is characterized by (3.2.3). A simple and obvious modification is to make α^*_j and β^*_j depend on n so that

$$s^*(n) = \Sigma_{j1}^{6} s^*_j(n) = \Sigma_{j1}^{6}\left(\alpha^*_j(n)\cos n\lambda_j + \beta^*_j(n)\sin n\lambda_j\right). \tag{4.2.1}$$

Of course $\alpha^*_j(n)$ and $\beta^*_j(n)$ will need to change slowly with n otherwise the notion of a seasonal pattern fades. Deterministic variation of the $\alpha^*_j(n)$ and $\beta^*_j(n)$ sequences is not considered here (see [23], [33]), but it is preferred to treat them as determined by chance. The autocorrelation of each sequence must however be high if it is to show the smooth variation required of a reasonable model. Perhaps the simplest model is one of the form

$$\begin{aligned}\alpha^*_j(n) &= \rho_j\alpha^*_j(n-1)+\epsilon_j(n)\\ \beta^*_j(n) &= \rho_j\beta^*_j(n-1)+\eta_j(n)\end{aligned} \tag{4.2.2}$$

where $\epsilon_j(n)$ and $\eta_j(n)$ have variance σ^2_j and zero mean and all correlations between ϵ and η, for any two time points and for differing values of j vanish.

Before considering in detail the stochastic properties of the model used in further work an attempt is made to give some perspective for this choice. For this purpose we define the seasonal at each frequency λ_j as

$$s^*_j = \xi_j(n)+\overline{\xi}_j(n) \tag{4.2.3}$$

where

$$\begin{aligned}\xi_j(n) &= \overline{\xi}_{-j}(n) = \tfrac{1}{2}\left\{\alpha^*_j(n)-i\beta^*_j(n)\right\} e^{i\lambda_j n}\\ \xi_{-j}(n) &= \overline{\xi}_j(n) = \tfrac{1}{2}\left\{\alpha^*_j(n)+i\beta^*_j(n)\right\} e^{-i\lambda_j n}\end{aligned} \tag{4.2.4}$$

$$j = 1,2,\ldots,5$$

and

$$\dot{\xi}_6(n) = \alpha_6^*(n)e^{i\lambda_6 n}$$

and the complex variable $\xi_j(n)$ is written in summary form as

$$\xi_j(n) = \zeta_j(n)e^{i\lambda_j n} \qquad j = \pm1,\pm2,\ldots,\pm5,6 \tag{4.2.5}$$

where the nature of the complex random variable $\zeta_j(n)$ is obvious from the definition given in (4.2.4). Using (4.2.2) it is straightforward to derive the autoregression in the complex variable

$$\zeta_j(n) = \rho_j\zeta_j(n-1)+\psi_j(n) \tag{4.2.6}$$

where $\psi_j(n)$ is a complex random variable defined by

$$\psi_j(n) = \tfrac{1}{2}\left\{\epsilon_j(n)-i\eta_j(n)\right\}. \tag{4.2.7}$$

If one re-represents the complex random variable, $\zeta(n)$, as

$$\zeta_j(n) = |\zeta_j(n)|e^{i\theta(n)} \tag{4.2.8}$$

with $|\zeta_j(n)|$ and $\theta(n)$ the modulus and argument of $\zeta_j(n)$ then by considering in more detail the nature of $\theta(n)$ one can see the range of possibilities this formulation offers. A mixture of frequency and amplitude modulation occurs when $\theta(n)$ is of the form $n\phi(n)$ since then the signal becomes $|\zeta_j(n)|e^{in(\lambda_j+\phi(n))}$. A slowly changing $\phi(n)$ provides what has been referred to as frequency modulation provided that $\phi(n)$ does not decay to zero with n. The inclusion of frequency modulation means that the wave form changes not only because of the changing amplitude $|\zeta_j(n)|$ but also because the underlying band of frequencies in the signal is slowly changing.

However, the argument $\theta(n)$ in (4.2.8) may contain no part which may be written as $n\phi(n)$ - where $\phi(n)$ is changing slowly - but instead the signal may be $|\zeta_j(n)|e^{i\lambda_j n}e^{i\Theta(n)}$, where <u>now</u> $\Theta(n)$, the phase angle, is the slowly changing part. When the variation in the wave form arises not only from amplitude modulation due to $|\zeta_j(n)|$ but <u>also</u> from the <u>slowly</u> changing factor $\Theta(n)$ the model (4.2.8) provides a mixture of phase and amplitude modulation.

Of course it could be that $\theta(n)$ was of the form that could be partitioned into two parts; one that is of the form $n\phi(n)$ and another of the form $\Theta(n)$ so that the model would include amplitude, phase and frequency modulation.

In special circumstances (4.2.8) will produce 'pure' amplitude modulation. For if (4.2.8) becomes

$$\zeta_j(n) = |\zeta_j(n)|e^{i\theta} \qquad (4.2.9)$$

and (4.2.9) is used for $\zeta_j(n)$ and $\zeta_j(n-1)$ in (4.2.6) one derives the relation

$$|\zeta_j(n)|-\rho_j|\zeta_j(n-1)| = \psi_j(n)e^{-i\theta} . \qquad (4.2.10)$$

Assume as well a particular form for $\psi_j(n)$, namely,

$$\psi_j(n) = a(n)\cos\theta + ia(n)\sin\theta \qquad (4.2.11)$$

where θ is uniformly distributed on $(-\pi,\pi)$ and is independent of $a(n)$, a sequence of independent positive random variables. If $\mathcal{E}(a^2(n)) = \sigma^2$ then it immediately follows that

$$\mathcal{E}\left(a(n)\cos\theta\right) = \mathcal{E}\left(a(n)\sin\theta\right) = 0$$

$$\mathcal{E}\left(a^2(n)\cos^2\theta\right) = \mathcal{E}\left(a^2(n)\sin^2\theta\right) = \sigma^2/_2 \qquad (4.2.12)$$

$$\mathcal{E}\left(a(n)\cos\theta \ . \ a(n)\sin\theta\right) = 0.$$

Now the model as specified originally with $\psi_j(n)$ as defined in (4.2.7) will then correspond to pure amplitude modulation if one puts $\frac{1}{2}\epsilon_j(n) = a(n)\cos\theta$ and $\frac{1}{2}\eta_j(n) = a(n)\sin\theta$, otherwise there will be phase modulation as well. When $\epsilon_j(n)$ and $\eta_j(n)$ are in fact as prescribed for pure amplitude modulation they cannot be Gaussian random variables since they are uncorrelated but not in general independent. Thus in this model there may be more information obtainable from higher than second order moments.

When postulating in (4.2.2) the model which generated $\alpha_j^*(n)$ and $\beta_j^*(n)$ correlation between the random variables $\epsilon_j(n)$ and $\eta_j(n)$ was specifically excluded. The reason for this restriction is that if we allow correlation between $\epsilon_j(n)$ and $\eta_j(n)$ (call it $r_{\epsilon\eta}$) then the lag covariance for $s_j^*(n)$ is

$$\mathcal{E}\left(s_j^*(m)s_j^*(m+n)\right) = \frac{\sigma_j^2}{1-\rho_j^2}\,\rho_j^n\left\{\cos n\lambda_j + r_{\epsilon\eta}\sin(2m+n)\lambda_j\right\} \qquad (4.2.13)$$

so that the seasonal component would not be stationary. If however as is assumed in (4.2.2) $r_{\epsilon\eta} = 0$ then $s^*(n)$ becomes a stationary process with a covariance function,

$$\gamma_{s^*}(n) = \mathcal{E}\left(s^*(m)s^*(m+n)\right) = \Sigma_{j=1}^{6}\frac{\sigma_j^2}{1-\rho_j^2}\,\rho_j^n\cos n\lambda_j. \qquad (4.2.14)$$

It is apparent from (4.2.3) that ρ_j will have to be large for the autocorrelation sequence of $s_j^*(n)$ is now $\rho_j^n\cos n\lambda_j$ and even for $\rho_j = .98$ and $n = 60$ the autocorrelation is approximately .3 so that seasonal patterns five years could differ quite radically. It is more illuminating to express the second order properties of $s^*(n)$ and the $s_j^*(n)$ in terms of the spectrum. In this case the spectrum is related to the autocovariances by

$$\gamma_{s^*}(n) = \Sigma_j\,\gamma_j(n) = \Sigma_j\int_{-\pi}^{\pi} e^{in\lambda} f_j(\lambda)d\lambda \qquad (4.2.15)$$

where the spectrum, $f_j(\lambda)$, is the Fourier Transform of $\gamma_j(n)$, the autocovariance sequence for $s_j^*(n)$, and is given by

$$f_j(\lambda) = \frac{\sigma_j^2}{4\pi}\left\{\frac{1}{1+\rho_j^2-2\rho_j\cos(\lambda-\lambda_j)} + \frac{1}{1+\rho_j^2-2\rho_j\cos(\lambda+\lambda_j)}\right\}. \qquad (4.2.16)$$

The relation (4.2.4) may be rewritten as

$$\gamma_{s^*}(n) = \Sigma_j\int_{0}^{\pi}\cos n\lambda\,\frac{\sigma_j^2}{2\pi(1+\rho_j^2-2\rho_j\cos(\lambda-\lambda_j))}\,d\lambda \qquad (4.2.17)$$

but it is found that the complex form of (4.2.4) is easier to work with. If ρ_j is near to 1 then $f_j(\lambda)$ is very concentrated at $\pm\lambda_j$ which corresponds to the fact that $s_j^*(n)$ is, over short periods, much like a sinusoidal oscillation with frequency λ_j.

The model initially adopted is of the form given in (4.2.2) but with all $\rho_j \equiv 1$. The reason for this has already been raised and is that the ρ_j must be very near to unity in any case. Since it is most difficult to determine ρ_j accurately from the data and the model is unlikely to be correctly specified this simplification is adopted initially. One could, of course, go further and adopt a more elaborate scheme in place of that proposed in (4.2.2), for example one involving second or higher differences for $\alpha_j^*(n)$, $\beta_j^*(n)$, or generalizing in the fashion suggested in §4.1 one might consider $s^*(n)$ to be generated by a relation of order q, q > 1, such as

$$\Sigma_{j}{}_{1}^{q}\gamma_j s^*(n-j) = u(n) \qquad (4.2\ 18)$$

where the characteristic equation of (4.2.18) is $\Sigma_{j}{}_{1}^{q}\gamma_j z^j$ and has all of its roots on or outside of the unit circle and $u(n)$ is a stationary time series with known spectra (see [24]).

In principle the technique proposed (see [24] and [57]) can deal with such extensions but in practice the computational and algebraic complications become large and the additional work does not seem justified, although in connection with trend removal a second order difference scheme is dealt with. It should be noted that when $\rho_j \equiv 1$ the seasonal component $s^*(n)$ ceases to be stationary.

4.3 <u>Filtering Prior to Seasonal Extraction</u>

It should be remembered that in the introductory discussion of the evolving seasonal model that the noise can include what would usually be called trend. Thus a high proportion of its variance will be explained by very low frequency components and so it is necessary to filter $w(n)$ to eliminate the trend. After trend elimination it is assumed that the new noise term $x(n)$ is stationary with spectrum $f_x(\lambda)$. Filtering replaces $w(n)$ by $y(n)$ (see (3.2.6)),

$$y(n) = \Sigma_{-p}^{q} b_j w(n-j) \qquad n = 1,\ldots,N \tag{4.3.1}$$

and also therefore replaces $s^*(n)$ and $u(n)$ by $s(n)$ and $x(n)$,

$$s(n) = \Sigma_{j=-p}^{q} b_j s^*(n-j)$$

$$x(n) = \Sigma_{j=-p}^{q} b_j u(n-j) \qquad n = 1,\ldots,N \tag{4.3.2}$$

where N is the number of observations remaining after filtering. The coefficients $\alpha_j^*(n)$ and $\beta_j^*(n)$ become $\alpha_j(n)$ and $\beta_j(n)$ after filtering. For any of the trend-removing filters mentioned in §4.6 the small difference between the properties of the starred and unstarred coefficients may be disregarded. One of the traditional methods of forming $y(n)$, discussed in Chapter III, is the subtraction of a centred 12 months moving average from $w(n)$. A thorough consideration of an appropriate trend-removing filter is even more important in the present case than in Chapter III where $s^*(n)$ was assumed stable for the method adopted will now have to allow frequencies well below λ_1, say, to influence the estimate of $s_1(n)$ and correspondingly this estimate would be badly affected by a trend if this was inadequately removed.

4.4 Suitability of Seasonal Estimation for Optimal Procedures

The technique used to obtain an estimate of the seasonal component is founded on the use of optimal methods for the extraction of a signal, the seasonal, which were briefly sketched in §1.5. These methods have been extended to allow for a non-stationary signal (see [57] and [24]). The method is quite general and has the following virtues. It allows the data up to the latest moment to be used to estimate the seasonal component, but as well this estimate may be revised as more information comes to hand. This is important for if the seasonal is allowed to change it must be recognized that at time n a large part of the information available for the estimation of the seasonal at that time point has yet to eventuate. Second, insofar as there is a stable seasonal component, or indeed if a more elaborate model is used, a seasonal component changing according to a sufficiently simple deterministic law, this component will be exactly represented in the estimate. Thirdly, only one unknown parameter is involved at each of the seasonal frequencies, this being of the nature of a signal to noise ratio. The level at which this parameter is set reflects the compromise to be effected between a quick response, resulting in quite a variable estimate of seasonal, and the damping out of noisy fluctuations. In principle this parameter should be determined from the data.

The actual methods used involve some compromises. The first issue arises in connection with the pre-seasonal extraction filtering. In all three techniques have been used but discussion of two of these is delayed temporarily. One of the two does in fact substantially eliminate the problem now discussed while the other (see section 4) is used because it enables estimates to be made using all the data up to the current time point and does not lose us the last six observations, as does the subtraction of a centred 12 months moving average. The remaining method is the simple device of removal of a centred 12 months moving average.

As is apparent from Fig. VIII, removing a centred 12 months moving average does not affect a stable seasonal but it will do so for a changing one. The effect may be judged by considering the model arising from (4.2.1) and (4.2.2) when $|\rho_j| < 1$. It was indicated in §1.3 that the effect of filtering is to multiply the spectral densities $f_{s*}(\lambda)$, $f_u(\lambda)$ by the factor $|B(\lambda)|^2$. $B(\lambda)$ is the frequency response function, and for the subtraction of a centred 12 months moving average is given by

$$B(\lambda) = \left\{1 - \frac{1}{24}\,\frac{\sin\lambda\sin6\lambda}{\sin^2\frac{1}{2}\lambda}\right\}. \qquad (4.4.1)$$

The effect is considered at the seasonal frequency where it will usually be greatest, namely λ_1. It is $B(\lambda)$ which is more relevant than $|B(\lambda)|^2$ for $B(\lambda)$ is the factor multiplying the component at frequency λ, whose variance is $f_{s*}d\lambda$. Investigating $B(\lambda)$ in the range of frequencies in which the bulk of the spectral mass of the signal lies allows an assessment of the degree of distortion of the signal caused by filtering to remove the trend.

A good approximation to the value of a, such that $[\lambda_j-a,\lambda_j+a]$ contains a proportion p, of the total mass under the curve given in (4.2.16) is, for ρ_j near to unity and $p < 1$,

$$a = -\log\rho_j\tan\frac{\pi p}{2}. \qquad (4.4.2)$$

Indeed the proportion is very near to

$$\frac{1}{2\pi}\int_{-a}^{a}\frac{1-\rho_j^2}{1+\rho_j^2-2\rho_j\cos\lambda}\,d\lambda$$

$$= \frac{1}{2\pi}\int_{-a}^{a}\Sigma_{-\infty}^{\infty}\,\rho_j^{|k|}e^{ik\lambda}d\lambda$$

$$= \frac{1}{\pi}\Sigma_{-\infty}^{\infty}\,\rho_j^{|k|}\,\frac{\sin ak}{k} \qquad (4.4.3)$$

$$\cong \frac{2}{\pi}\int_{0}^{\infty}e^{-\alpha x}\,\frac{\sin x}{x}\,dx$$

$$= \frac{2}{\pi}\,\text{arc tan}\left(\frac{-a}{\log\rho_j}\right)$$

where $\alpha = \frac{1}{a}\log\rho_j$ and the approximation is adequate if a is small and ρ_j near to unity. For $\rho_j = .98$, $p = .90$ the value of a is 0.127 and for $p = .50$, $\rho_j = .98$ the value is .0203. The value of the response function, (4.4.1), is given at the relevant points

frequency (λ)	λ_1-0.127	λ_1-0.020	λ_1+0.020	λ_1+0.127
$B(\lambda)$	.714	.960	1.036	1.170.

This indicates that the effects of the filtering will be slight. Over the range in which 50% of the spectral mass of $s_1^*(n)$ lies the effect on the signal will be negligible. Over the remainder of the range considered the signal will be slightly diminished below λ_1 and slightly augmented above λ_1. The resulting relocation of the signal affects only 5% of the total mass. As it seems appropriate to ignore the effects of this filtering $s(n)$ and $s^*(n)$ are no longer notationally distinguished. In any case if the next suggestion for simplification is adopted these effects are reduced even further.

The second simplification is to adopt a technique which treats each $s_j(n)$ separately. The methods proposed are filtering processes and the justification for this simplification is the narrow band nature of the signal, i.e. the seasonal, for this assumes, for a given noise level, that there will be little interference between the six signals. This point is discussed in detail in Hannan [24]. To illustrate this point the responses of the seasonal extraction filters are calculated and presented in Chapter V. It will be seen that the filter used to elicit $s_j(n)$ will hardly be affected by the $s_k(n)$, $k \neq j$, because its response will be very substantially concentrated at λ_j. If, however, there was concern about possible interference another procedure could be used which eliminates both trend and $s_k(n)$, $k \neq j$, to a substantial degree. This is the procedure mentioned at the end of the preceding paragraph. In this approach one forms

$$y_j(n) = (2-\delta_j^6)\,\Sigma_{k-6}^{6} a_k w(n-k)\cos\lambda_j k \qquad (4.4.4)$$

where δ_j^6 is unity if $j = 6$ and is otherwise zero and a_k are the coefficients in a centred 12 months moving average. This produces a filter with response

$$\frac{\sin 6(\lambda-\lambda_j)\sin(\lambda-\lambda_j)}{24\sin^2\frac{1}{2}(\lambda-\lambda_j)} \quad \frac{\sin 6(\lambda+\lambda_j)\sin(\lambda+\lambda_j)}{24\sin^2\frac{1}{2}(\lambda+\lambda_j)}$$

which by elementary manipulations is reduced to

$$\frac{\sin\lambda \sin 6\lambda}{6(\cos\lambda-\cos\lambda_j)} . \tag{4.4.5}$$

The expression (4.4.5), which is illustrated in Fig. IX for $j = 1$, 2 and 3, has a zero at $\lambda = 0$ of the same order as is obtained when a centred 12 months moving average is subtracted. Whereas the response of this latter filter is like $6\lambda^2$ at $\lambda = 0$ that of (4.4.5) is like $\lambda^2(1-\cos\lambda_j)^{-1}$, which since $(1-\cos\lambda_j)^{-1}$ is larger than 6 only for $j = 1$ shows that the filter with response (4.4.5) tends to remove the trend better than the moving average subtraction when $j > 1$, and not much worse for $j = 1$. Of course at λ_k, $k \neq j$, (4.4.5) has zero response so that $s_k(n)$, $k \neq j$, is substantially removed. At λ_j, (4.4.5) is unity and tends to have a flatter, and therefore better, shape than the moving average subtraction filter has for j small, though the reverse is true for j near 6. The small j, $j = 1,2$ in particular, are most likely to be the important seasonal frequencies. Experience with practical applications has suggested that the refinement involved in the use of the filter with response (4.4.5) is not needed, (see Appendix D), particularly as it loses six observations at the end of the series.

In principle it is not necessary to take each λ_j separately any more than it is necessary to adopt a first difference scheme in representing $\epsilon_j(n)$ and $\eta_j(n)$ in terms of $\alpha_j(n)$ and $\beta_j(n)$. In practice, however, the problem of computing the optimal coefficients becomes very great unless these things are done, as high order polynomials have to be factored. Computing each $\hat{s}_j(n)$ separately has one result which must be mentioned, namely that, unless the prefiltering by means of filter (4.4.4) is carried out, it is not quite true that an additional stable seasonal component will be perfectly represented. This is because the part of the stable

FIG. VIII

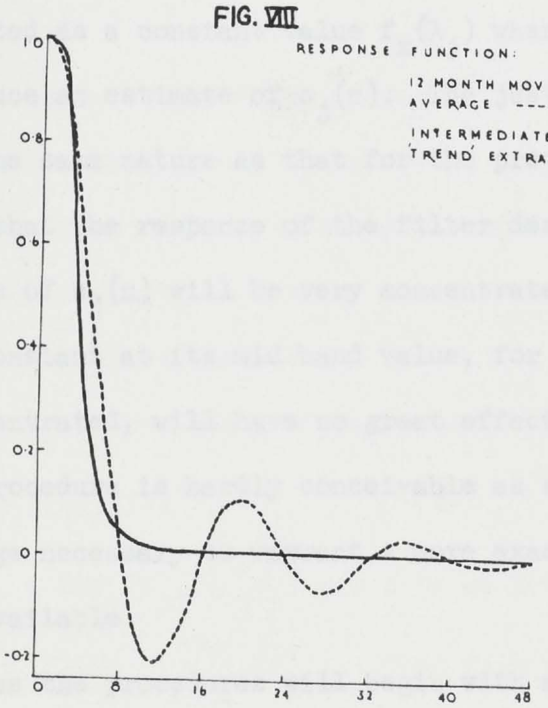
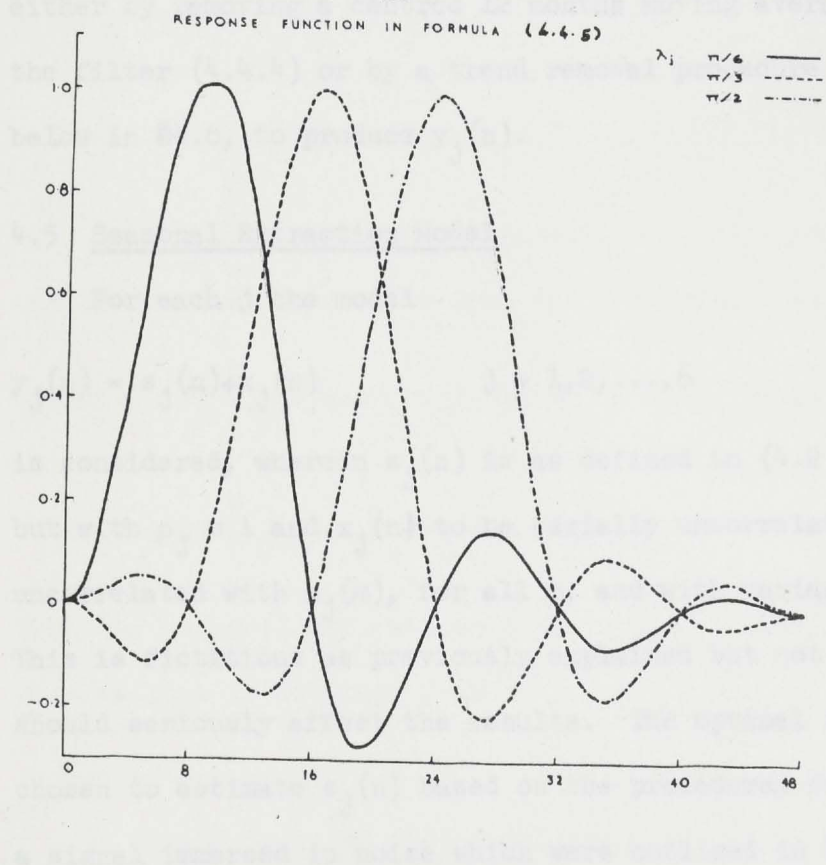


FIG. IX



seasonal due to the term $\alpha_j \cos\lambda_j n$ and $\beta_j \sin\lambda_j n$ will contribute not only to $\hat{s}_j(n)$ but also to $\hat{s}_k(n)$ for $k \neq j$. The effect will be slight however as can be judged from the response function of the filter used to compute $\hat{s}_j(n)$, which is shown in Chapter V (see Fig. XII).

The final simplification is that the noise level, i.e. $f_x(\lambda)$, is treated as a constant value $f_x(\lambda_j)$ when designing the filter to produce an estimate of $s_j(n)$. The justification for this step is of the same nature as that for the previous simplification, namely that the response of the filter designed to produce the estimate of $s_j(n)$ will be very concentrated at λ_j so that assuming $f_x(\lambda)$ constant at its mid band value, for the band where the filter is concentrated, will have no great effect. In this regard, any other procedure is hardly conceivable as sufficiently precise knowledge necessary to warrant a more exact treatment is unlikely to be available.

Thus the procedures will begin with a prefiltering of $w(n)$, either by removing a centred 12 months moving average, by use of the filter (4.4.4) or by a trend removal procedure outlined below in §4.6, to produce $y_j(n)$.

4.5 Seasonal Extraction Model

For each j the model

$$y_j(n) = s_j(n)+x_j(n) \qquad j = 1,2,\ldots,6 \qquad (4.5.1)$$

is considered, wherein $s_j(n)$ is as defined in (4.2.1) and (4.2.2) but with $\rho_j \equiv 1$ and $x_j(n)$ to be serially uncorrelated and uncorrelated with $s_j(m)$, for all m, and with variance $2\pi f_x(\lambda_j)$. This is fictitious as previously explained but not in a way which should seriously affect the results. The optimal filter is now chosen to estimate $s_j(n)$ based on the procedures for extracting a signal immersed in noise which were outlined in §1.5. The derived filter depends only upon the ratio $\{\sigma_j^2/2\pi f_x(\lambda_j)\}$, which is a form of signal to noise ratio. Some discussion of estimation of this ratio will be given below, in §6.3. The estimate $\hat{s}_j^{(\nu)}(n)$, of $s_j(n)$, using observations up to time $(n+\nu)$, $\nu \geq 0$, is then calculated, where $(n+\nu)$ is the latest time point available. The estimates for each seasonal frequency are then combined to obtain

$$\hat{s}^{(\nu)}(n) = \Sigma_{j1}^{6}\, \hat{s}_j^{(\nu)}(n). \tag{4.5.2}$$

The derivation of the formulae for $\hat{s}_j^{(\nu)}(n)$ using the model (4.5.1) is not given here (see [57] and [24], [25]) but a sketch of the main results is given only as a basis for later discussion (see Chapter V) of the implementation of these procedures and development of the estimates' characteristics.

Let β_j be the root, of less than unit modulus, of

$$1+\theta_j(1-z)(1-z^{-1}) \tag{4.5.3}$$

where $\theta_j = 2\pi f_x(\lambda_j)/(\sigma_j^2/2)$ and z is as defined in §1.5. This root is $(2\theta_j)^{-1}\{1+2\theta_j - \sqrt{(1+4\theta_j)}\}$. Then form (see [25])

$$u_j(n) = (1-\beta_j)\, \Sigma_{m0}^{\infty}\beta_j^m y(n-m)\cos m\lambda_j$$
$$\tag{4.5.4}$$
$$v_j(n) = (1-\beta_j)\, \Sigma_{m0}^{\infty}\beta_j^m y(n-m)\sin m\lambda_j.$$

If u_j and v_j are computed for some initial time point they are simply obtained thereafter from the following recursive relations

$$u_j(n+1) = \beta_j\left\{u_j(n)\cos\lambda_j - v_j(n)\sin\lambda_j\right\} + (1-\beta_j)y(n+1)$$
$$\tag{4.5.5}$$
$$v_j(n+1) = \beta_j\left\{u_j(n)\sin\lambda_j + v_j(n)\cos\lambda_j\right\}.$$

Then the seasonal estimate at each point n, based only on past observations is obtained from[25]

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25 It should be noted that calculation of the v_j defined in (4.5.4) is only necessary where the recursive formulae in (4.5.5) are used to obtain $u_j(n)$ and $v_j(n)$ for each time point, because (4.5.6) shows $\hat{s}^{(o)}(n)$ is based only on the $u_j(n)$'s.

$$\hat{s}^{(o)}(n) = \Sigma_1^6 (2-\delta_j^6)u_j(n) = \Sigma_1^6 \hat{s}_j^{(o)}(n). \tag{4.5.6}$$

To obtain $\hat{s}^{(\nu)}(n)$, for ν values > 0, one can proceed iteratively by means of

$$s^{(\nu+1)}(n) = s^{(\nu)}(n)+\Sigma_{j1}^6 \left\{(1-\beta_j)\beta_j^{\nu} \Big(y(n+\nu+1)\cos(\nu+1)\lambda_j - u_j(n+\nu+1)\cos(\nu+1)\lambda_j - v_j(n+\nu+1)\sin(\nu+1)\lambda_j \Big) (2-\delta_j^6)\right\}. \tag{4.5.7}$$

Thus as new data comes to hand $\hat{s}^{(\nu)}(n)$ is updated. The j^{th} summand in the second term is the adjustment to $\hat{s}_j^{(\nu)}(n)$ required to obtain $\hat{s}_j^{(\nu+1)}(n)$. As mentioned above, this successive modification of an estimate is essential for any efficient method for a changing seasonal as future observations contain relevant information for a correct estimate of the seasonal.

4.6 Trend Extraction Procedure

Two ways of eliminating $p(n)$, the low frequency component of $w(n)$, have already been discussed, one by subtracting a centred 12 months moving average and one by the use of the filter (4.4.4). Both methods have the disadvantage of producing a trend reduced series which stops 6 terms short of the end of the original series. In this section a technique is derived which is designed to remove trend which can be carried up to the last observation. Again the approach adopted is to propose a model and, thence, to obtain a trend representing filter which can be iteratively calculated. The model employed is not regarded as representing the truth but rather it is believed that the filter resulting will do its task reasonably well. Support of this belief is provided below.

The model is of the form

$$w(n) = p(n)+v(n) \tag{4.6.1}$$

where

$$p(n)-2p(n-1)+p(n) = \epsilon(n) \tag{4.6.2}$$

and $\epsilon(n)$ is serially independent with zero mean and constant variance σ_ϵ^2. This model produces a $p(n)$ which is the sum of a linear trend and a random component which is the result of two successive partial summations of the $\epsilon(n)$ series, i.e. of the form $\sum_{n=o}^{N} \sum_{u=o}^{n} \epsilon(u)$. As already indicated this model is only a convenient basis on which to work. This must be borne in mind when considering the rationale for an assumption that $v(n)$ has a constant spectrum. Clearly for estimation of $p(n)$ it is $s^*(n)+u(n)$ which is the noise and this sum certainly has not a uniform spectrum. The problem of solving the equations necessary to obtain an optimal filter for representing $p(n)$ when $v(n)$ is more general is a difficult one and has been avoided. The main cost will be the production of a filter with a non-zero response at the points λ_j so that the subtraction of their estimate of $p(n)$ from $w(n)$ to produce $y(n)$ will affect $s^*(n)$. However a parameter occurs in this problem, of the same nature as the θ_j occurring in the seasonal model of §4.5, which reflects the ratio of the variance $v(n)$ to that of $\epsilon(n)$. It is referred to as θ^2. By making θ^2 larger the response of the filter concentrates near to $\lambda = 0$ and thus reduces the effect of the removal of the estimate of $p(n)$ on $s^*(n)$. The reason why (4.6.2) involves second differencing, rather than just first differencing, is that only then does one obtain a filter for removing $p(n)$ which at the origin has a root of the same order as the other methods proposed above and experience suggests a root of this order is needed.

Canonical Factorization

The optimal filter may be obtained from a generating function of the form given in (1.5.37) and is written

$$c^{(\nu)}(z) = \frac{(1-z)^2}{D(z)} \left\{ \frac{g_{\epsilon\epsilon}(z)}{z^{\nu}(1-z)^2 D(z^{-1})} \right\}_+ \qquad (4.6.3)$$

where $g_{\epsilon\epsilon}(z) = \sigma_\epsilon^2/2\pi$ and the function D(z) is defined, using (1.5.33) and (1.5.34), as

$$D(z)D(z^{-1}) = g_{\epsilon\epsilon}(z)+|1-z|^4 f_v(o) \tag{4.6.4}$$

with $f_v(o)$ being the spectrum at zero of the noise v(n). The σ^2 term included in (1.5.34) has now been included in D(z) and $D(z^{-1})$. By using the definition of θ^2 previously stated $C^{(\nu)}(z)$ may be redefined using the functions F(z) and $F(z^{-1})$ arising from

$$F(z)F(z^{-1}) = 1+\theta^2|1-z|^4 \tag{4.6.5}$$

so that the generating function for the optimal filter now is

$$C^{(\nu)}(z) = \frac{(1-z)^2}{F(z)}\left[\frac{1}{z^\nu(1-z)^2F(z^{-1})}\right]_+ \tag{4.6.6}$$

remembering that for signal extraction $\nu \leq 0$ (see C in §1.5).
To progress towards an optimal prediction the canonical factors F(z) and $F(z^{-1})$ must be found so the expression (4.6.5) must be set equal to zero and solved. To facilitate this solution (4.6.5) is simply rewritten as

$$1+\theta^2(1-z)^2(1-z^{-1})^2. \tag{4.6.7}$$

If $\lambda = \rho e^{i\phi}$ is a solution of (4.6.7) then so also is $\bar{\lambda}$, λ^{-1} and $\bar{\lambda}^{-1}$. In finding the roots of (4.6.7) it is most convenient to work with

$$\lambda^* = \lambda+\lambda^{-1} \tag{4.6.8}$$

as defining this new variable allows (4.6.7) to be represented by the following quadratic

$$\lambda^{*2}+4+\frac{1}{\theta^2}-4\lambda^* = 0 \tag{4.6.9}$$

i.e. $(\lambda^*-2)^2+\frac{1}{\theta^2} = 0$.

The solutions for λ^* from (4.6.9) are

$$\lambda^* = 2 \pm i\,\frac{1}{\theta}\,. \tag{4.6.10}$$

Now the decision as to whether

$$\lambda^* = 2 + i\,\frac{1}{\theta} \qquad\qquad (4.6.11)$$

$$\bar{\lambda}^* = 2 - i\,\frac{1}{\theta}$$

or

$$\lambda^* = 2 - i\,\frac{1}{\theta} \qquad\qquad (4.6.12)$$

$$\bar{\lambda}^* = 2 + i\,\frac{1}{\theta}$$

is chosen as the appropriate pair of roots turns on whether the modulus of λ, i.e. ρ, is made greater or less than 1. As the case of interest must be where $\rho < 1$, (4.6.12) is the operative solution. Using $\lambda = \rho e^{i\phi}$ and (4.6.8), the definition of λ^*, the solutions chosen may be written

$$\rho e^{i\phi} + \rho^{-1} e^{-i\phi} = 2 - i\,\frac{1}{\theta} \qquad\qquad (4.6.13)$$

$$\rho e^{-i\phi} + \rho^{-1} e^{i\phi} = 2 + i\,\frac{1}{\theta}\,.$$

By first adding the second expression in (4.6.13) to the first and then subtracting the second from the first one obtains

$$\rho\cos\phi + \rho^{-1}\cos\phi = 2 \qquad\qquad (4.6.14)$$

$$\rho\sin\phi - \rho^{-1}\sin\phi = -\,\frac{1}{\theta}\,.$$

Multiplying the first equation in (4.6.14) by ρ a quadratic is obtained and the solutions for ρ are in fact solutions for ρ and ρ^{-1}, since multiplying (4.6.14) by ρ^{-1} gives the same quadratic expression in powers of ρ^{-1}. The solution <u>used</u> is

$$\rho = \frac{1-\sin\phi}{\cos\phi}\,, \qquad \rho^{-1} = \frac{1+\sin\phi}{\cos\phi} \qquad\qquad (4.6.15)$$

although in fact $\rho = (1 \pm \sin\phi)/\cos\phi$, and therefore solutions for ρ^{-1} of exactly this form may of course also be found. The choice exhibited in (4.6.15) is based on a desire for a pair of solutions for ρ and ρ^{-1} with the property that ρ and ρ^{-1} must be positive and ρ smaller than ρ^{-1}. This choice means that ϕ will be a small positive angle if ρ is close to 1. The product $\rho\rho^{-1}$ will equal unity.

From the second equation in (4.6.14) and the definition of ρ and ρ^{-1} in (4.6.15) the following relation between ϕ and θ may be established,

$$\sin\phi\,(\rho-\rho^{-1}) = -\frac{1}{\theta} \tag{4.6.16}$$

$$\text{i.e. } (-2\sin^2\phi)/(\cos\phi) = -\frac{1}{\theta}\,.$$

Once a value of θ^2 is specified then ϕ may be simply deduced from (4.6.16) and ρ from (4.6.15).

If a particular value of θ^2 is chosen, say θ_o^2, then using the steps suggested in the previous paragraph a root $z_o = \rho_o e^{i\phi_o}$ may be simply obtained. The remaining roots are of course $\bar{z}_o$, $\bar{z}_o^{-1}$ and z_o^{-1}. It is useful for the latter development of the response function to represent (4.6.7) explicitly in terms of its roots and thus to derive an equivalence which is required for the canonical factorization into $F(z)$ and $F(z^{-1})$ as follows

$$\begin{aligned}
\theta^2\left(\frac{1}{\theta^2}+(1-z)^2(1-z^{-1})^2\right) &= \theta^2(1-z_oz)(1-\bar{z}_oz)(1-z_o^{-1}z)(1-\bar{z}_o^{-1}z)z^{-2} \\
&= \theta^2(1-z_oz)(1-\bar{z}_oz)(z^{-1}-z_o^{-1})(z^{-1}-\bar{z}_o^{-1}) \\
&= \theta^2(1-z_oz)(1-\bar{z}_oz)\,\frac{(z^{-1}z_o-1)}{z_o}\,\frac{(z^{-1}\bar{z}_o-1)}{\bar{z}_o} \\
&= \frac{\theta^2}{z_o\bar{z}_o}(1-z_oz)(1-\bar{z}_oz)(1-z_oz^{-1})(1-\bar{z}_oz^{-1}) \\
&= \frac{\theta^2}{\rho^2}(1-z_oz)(1-\bar{z}_oz)(1-z_oz^{-1})(1-\bar{z}_oz^{-1}).
\end{aligned} \tag{4.6.17}$$

Since the roots of $(\frac{1}{\theta^2}+(1-z)^2(1-z^{-1})^2)$ are of course the same as the roots of the expression given on the left hand side of (4.6.17), the factor θ^2 has been extracted and finally $z_o\bar{z}_o = \rho^2$ has been used to redefine a new multiplicative factor, θ'^2, where $\theta' = {}^{\theta}/_{\rho}$, so that the power series in F(z) has leading coefficient of unity. The canonical factor is easily deduced from (4.6.17) and is given by

$$F(z) = {}^{\theta}/\rho\ (1-z_o z)(1-\bar{z}_o z) \tag{4.6.18}$$

and this implies directly that

$$\frac{\theta}{\rho}\ (1-z_o)(1-\bar{z}_o) = 1. \tag{4.6.19}$$

4.7 Establishing Trend Estimates from the Optimal Response Function

Using the canonical factorization from the previous section the response function may now be written

$$C^{(\nu)}(z) = \frac{(1-z)^2}{\theta'(1-z_o z)(1-\bar{z}_o z)} \left[\frac{1}{z^{\nu}(1-z)^2\theta'(1-z_o z)(1-\bar{z}_o z)} \right]_+ \tag{4.7.1}$$

where the term in the square bracket is the positive terms in a Laurent expansion in a specified annulus. It is necessary therefore to first obtain an expression for the square bracket of the form

$$\Sigma_o^{\infty}\ z^j a_j \tag{4.7.2}$$

where the individual coefficients a_j can be obtained by contour integration within the specified annulus from

$$a_j = \frac{1}{2\pi i} \oint_{\Gamma} \frac{1}{\theta' z^{\nu}(1-z)^2(1-z_o z^{-1})(1-\bar{z}_o z^{-1})z^{(j+1)}}\, dz. \tag{4.7.3}$$

The circle around which integration occurs is such that $|z_o| < |z| < \Gamma < 1$. The generating function requires a general evaluation of the expression (4.7.2) and this is obtained from substituting (4.7.3) in (4.7.2) to give

$$\Sigma_o^{\infty}\ z^j \frac{1}{\theta' 2\pi i} \oint_{\Gamma} \frac{1}{\zeta^{\nu}(1-\zeta)^2(1-z_o\zeta^{-1})(1-\bar{z}_o\zeta^{-1})\zeta^{(j+1)}}\, d\zeta$$

$$= \frac{1}{\theta' 2\pi i} \oint_{\Gamma} \frac{1}{\zeta^{(\nu+1)}(1-\zeta)^2(1-z_o\zeta^{-1})(1-\bar{z}_o\zeta^{-1})(1-\zeta^{-1}z)}\, d\zeta \tag{4.7.4}$$

$$= \frac{1}{\theta' 2\pi i} \oint_{\Gamma} \frac{1}{\zeta^{(\nu-2)}(1-\zeta)^2(\zeta-z_o)(\zeta-\bar{z}_o)(\zeta-z)}\, d\zeta\ .$$

Since (4.7.4) has poles z, z_o and $\bar{z}_o$ within Γ it therefore may be evaluated from

$$K = \frac{z_o^{(-\nu+2)}}{(1-z_o)^2\theta'(z_o-\bar{z}_o)(z_o-z)} - \frac{\bar{z}_o^{(-\nu+2)}}{(1-\bar{z}_o)^2\theta'(z_o-\bar{z}_o)(\bar{z}_o-z)} + \frac{z^{(-\nu+2)}}{(1-z)^2(z-z_o)(z-\bar{z}_o)} . \quad (4.7.5)$$

Using the expression (4.7.5) for the term in square brackets in (4.7.1) the generating function required is

$$C^{(\nu)}(z) = \frac{(1-z)^2 \quad K}{\theta'(1-z_oz)(1-\bar{z}_oz)} . \quad (4.7.6)$$

Now if $\nu = 0$ the first and second term in K may be shown to be

$$\frac{\theta'\rho^2(1-\rho^2)+z(1-\theta'(1-\rho^2))}{(z_o-z)(\bar{z}_o-z)} \quad (4.7.7)$$

by using elementary algebraic manipulations and the property that $(1-z_o)(1-\bar{z}_o) = \frac{1}{\theta'}$. The addition of the third term in K to (4.7.7) produces the expression

$$\frac{\theta'\{\theta'\rho^2(1-\rho^2)+z(1-\theta'(1-\rho^2))\}(1-z)^2+z^2}{(1-z)^2\theta'(z_o-z)(\bar{z}_o-z)} \quad (4.7.8)$$

with a numerator which is a cubic in z having complex roots z_o and $\bar{z}_o$ and is therefore divisible by $\theta'(z_o-z)(\bar{z}_o-z)$. Carrying out the suggested division and then incorporating the expression outside the square bracket one obtains

$$C^{(o)}(z) = \frac{(1-\rho^2)+z(\frac{1}{\theta'}-(1-\rho^2))}{(1-z_oz)(1-\bar{z}_oz)} . \quad (4.7.9)$$

A similar procedure is employed when $\nu = 1$ to establish the expression

$$C^{(1)}(z) = \frac{\{(1-\rho^2) - \frac{1}{\theta'}\}+z\{\frac{2}{\theta'}-(1-\rho^2)\}}{(1-z_oz)(1-\bar{z}_oz)} . \quad (4.7.10)$$

The generating function for $\hat{p}^{(o)}(n)$ is $z^{-n}C^{(o)}(z)$ and one identifies z^{-n+j} with the observation $w(n-j)$. As the two bracketed terms in the denominator may be simply expressed in terms of ρ and ϕ the expression $z^{-n}C^{(o)}(z)$ becomes

$$\begin{aligned}\hat{p}^{(o)}(n)-2\rho\cos\phi\,\hat{p}^{(o)}(n-1)+\rho^2\hat{p}^{(o)}(n-2) &= \left\{(1-\rho^2)+z\left(\tfrac{1}{\theta'}-(1-\rho^2)\right)\right\} z^{-n} \\ &= (1-\rho^2)\left\{w(n)-w(n-1)\right\} + \tfrac{1}{\theta'}w(n-1).\end{aligned} \tag{4.7.11}$$

In general, the generating function for $\hat{p}^{(\nu)}(n)$ is $z^{-n-\nu}C^{(\nu)}(z)$ (if $\nu = -1, -\infty$, $\hat{p}^{(\nu)}(n)$ is written $\hat{p}^{(-\nu)}(n)$ to simplify presentation); for $\nu = -1$ it is

$$\hat{p}^{(1)}(n) = 2\rho\cos\phi\,\hat{p}^{(1)}(n-1)-\rho^2\hat{p}^{(1)}(n-2)+\left(1-\rho^2-\tfrac{1}{\theta'}\right)\Big(w(n+1)-w(n)\Big)+\tfrac{1}{\theta'}w(n). \tag{4.7.12}$$

The recursive relations in n, (4.7.11) and (4.7.12), must clearly have two starting values. Rather than guessing two values, which could be done reasonably effectively, it is possible to find exact starting values at two starting points, say (n-1) and (n-2), by noting that the denominator of (4.7.9) may be written as follows

$$\begin{aligned}\frac{1}{(1-z_o z)(1-\bar{z}_o z)} &= \frac{1}{(z_o-\bar{z}_o)}\left\{\frac{z_o}{(1-z_o z)} - \frac{\bar{z}_o}{(1-\bar{z}_o z)}\right\} \\ &= \frac{1}{2i\rho\sin\phi}\sum_{j=o}^{\infty}\left(z_o^{j+1}-\bar{z}_o^{j+1}\right)z^j \\ &= \sum_{j=o}^{\infty}\rho^j\,\frac{\sin(j+1)\phi}{\sin\phi}\,z^j .\end{aligned} \tag{4.7.13}$$

Thus the summations,

$$\begin{aligned}\hat{p}^{(o)}(n) &= (1-\rho^2)\sum_{j=o}^{\infty}\rho^j\frac{\sin(j+1)\phi}{\sin\phi}w(n-j)+\left\{\tfrac{1}{\theta'}-(1-\rho^2)\right\}\sum_{j=o}^{\infty}\rho^j\frac{\sin(j+1)\phi}{\sin\phi}w(n-j-1) \\ \hat{p}^{(1)}(n) &= \left\{(1-\rho^2)-\tfrac{1}{\theta'}\right\}\sum_{j=o}^{\infty}\rho^j\frac{\sin(j+1)\phi}{\sin\phi}w(n+1-j)+\left\{\tfrac{2}{\theta'}-(1-\rho^2)\right\} \\ &\qquad \sum_{j=o}^{\infty}\rho^j\frac{\sin(j+1)\phi}{\sin\phi}w(n-j)\end{aligned} \tag{4.7.14}$$

provide an explicit evaluation of $\hat{p}^{(1)}(n)$ and $\hat{p}^{(o)}(n)$.

The expression $z^{-n-\nu}C^{(\nu)}(z)$ which generates the coefficients for $\hat{p}^{(\nu)}(n)$ is obtained from (4.7.6) and is

$$z^{-n-\nu}C^{(\nu)}(z) = \frac{z^{-n}}{\theta'(1-z_o z)(1-\bar{z}_o z)\theta'(1-z_o z^{-1})(1-\bar{z}_o z^{-1})} + \frac{z^{-n-\nu}(1-z)^2}{(z_o-\bar{z}_o)\theta'^2(1-z_o z)(1-\bar{z}_o z)} . \quad (4.7.15)$$

$$\left\{\frac{z_o^{(\nu+2)}}{(1-z_o)^2(z_o-z)} - \frac{\bar{z}_o^{(\nu+2)}}{(1-\bar{z}_o)^2(\bar{z}_o-z)}\right\}.$$

If (4.7.15) is used to form $z^{-n-\nu}C^{(\nu)}(z)-2z^{-n-\nu+1}C^{(\nu-1)}(z)+z^{-n-\nu+2}C^{(\nu-2)}(z)$ which produces $\hat{p}^{(\nu)}(n)-2\hat{p}^{(\nu-1)}(n)+\hat{p}^{(\nu-2)}(n)$, then after some simplification one obtains

$$z^{-n-\nu}C^{(\nu)}(z)-2z^{-n-\nu+1}C^{(\nu-1)}(z)+z^{-n-\nu+2}C^{(\nu-2)}(z) = \frac{z^{-n-\nu}(1-z)^2}{(1-z_o z)(1-\bar{z}_o z)} .$$

$$\left\{\frac{z_o^{(\nu+1)}(1-\bar{z}_o)^2-\bar{z}_o^{\nu+1}(1-z_o)^2}{(z_o-\bar{z}_o)} - \frac{z(z_o^{\nu}(1-\bar{z}_o)^2-\bar{z}_o^{\nu}(1-z_o)^2)}{(z_o-\bar{z}_o)}\right\} \quad (4.7.16)$$

$$= z^{-n-\nu}\left[\frac{(1-z)^2}{(1-\bar{z}_o z)(1-z_o z)}\{a+bz\}\right] ,$$

where,

$$a = \rho^{\nu}(1-\rho^2)\cos\nu\phi, \quad b = \rho^{\nu-1}(1-\rho^2)\cos(\nu-1)\phi. \quad (4.7.17)$$

From the formula for $C^{(o)}(z)$ given in (4.7.9) it may be deduced that the difference between the observation at time point $(n+\nu)$ and the estimated trend based on observations prior to $(n+\nu)$ at the same time point is generated by

$$w(n+\nu)-\hat{p}^{(o)}(n+\nu) = z^{-(n+\nu)}\left\{1 - \frac{(1-\rho^2)+(\frac{1}{\theta'}-(1-\rho^2))z}{(1-z_o z)(1-\bar{z}_o z)}\right\} \quad (4.7.18)$$

which may be further simplified by using $\rho^2 = z_o\bar{z}_o$ and $\frac{1}{\theta'} = (1-z_o)(1-\bar{z}_o)$ to become

$$w(n+\nu)-\hat{p}^{(o)}(n+\nu) = \frac{z^{-(n+\nu)}\rho^2(1-z)^2}{(1-z_o z)(1-\bar{z}_o z)} . \tag{4.7.19}$$

Since the generating function when lagged one time period is given by

$$w(n+\nu-1)-\hat{p}^{(o)}(n+\nu-1) = \frac{z^{-(n+\nu)}\rho^2(1-z)^2 z}{(1-z_o z)(1-\bar{z}_o z)} \tag{4.7.20}$$

it is obvious that the equivalence

$$\frac{a}{\rho^2}\left(w(n+\nu)-\hat{p}^{(o)}(n+\nu)\right) + \frac{b}{\rho^2}\left(w(n+\nu-1)-\hat{p}^{(o)}(n+\nu-1)\right) = \frac{z^{-(n+\nu)}(1-z)^2(a+bz)}{(1-z_o z)(1-\bar{z}_o z)} \tag{4.7.21}$$

holds and that by using the left hand side of (4.7.16) and the right hand side of (4.7.21) the following iteration on ν is derived,

$$\begin{aligned}\hat{p}^{(\nu)}(n) = {} & 2\hat{p}^{(\nu-1)}(n)-\hat{p}^{(\nu-2)}(n)+\rho^{\nu-2}(1-\rho^2)\cos\nu\phi\{w(n+\nu)-\hat{p}^{(o)}(n+\nu)\} \\ & -\rho^{\nu-3}(1-\rho^2)\cos(\nu-1)\phi\{w(n+\nu-1)-\hat{p}^{(o)}(n+\nu-1)\} .\end{aligned} \tag{4.7.22}$$

4.8 Additional Value from the Response Function

The response functions established in §4.7 are the source of estimating relations for the trend based on the model (4.6.1) and (4.6.2). Because of the generative properties of these response functions they provide valuable insight into the effects of the optimal filter they represent on a complex harmonic. The gain, the square of the modulus of the response function, provides information on how the spectrum of the observations has been affected by the optimal filters. To develop convenient formulae for the responses for varying values of ν it is necessary to return to (4.7.15). For simplicity only, the response function is obtained for the time point n = 0, although generalization to any time point n merely requires multiplication by z^{-n}. The general response function for $\hat{p}^{(\nu)}(o)$ is therefore

$$h_o^{(\nu)}(z) = z^{-\nu}C^{(\nu)}(z) = \frac{1}{\theta'(1-z_oz)(1-\bar{z}_oz)\theta'(1-z_oz^{-1})(1-\bar{z}_oz^{-1})}$$

$$+ \frac{z^{-\nu}(1-z)^2}{\theta'^2(1-z_oz)(1-\bar{z}_oz)(z_o-\bar{z}_o)} . \qquad (4.8.1)$$

$$\left\{ \frac{z_o^{(\nu+2)}}{(1-z_o)^2(z_o-z)} - \frac{\bar{z}_o^{(\nu+2)}}{(1-\bar{z}_o)^2(\bar{z}_o-z)} \right\} .$$

It is convenient to simplify (4.8.1) considerably by elementary algebraic manipulations to obtain

$$\frac{1+\theta'^2z^{-(\nu+2)}(1-z)^2\{\rho^{\nu+1}(1-\rho^2)(\rho\cos\nu\phi - z\cos(\nu+1)\phi)\}}{\theta'^2(1-z_oz)(1-\bar{z}_oz)(1-z_oz^{-1})(1-\bar{z}_oz^{-1})} . \qquad (4.8.2)$$

As ν becomes large it is apparent that the second term in (4.8.2) will become very small as ρ is less than one and the response will then be closely approximated by

$$h_o^{(\infty)}(z) = \frac{1}{\theta'^2(1-z_oz)(1-\bar{z}_oz)(1-z_oz^{-1})(1-\bar{z}_oz^{-1})} \qquad (4.8.3)$$

which on employing (4.6.17) becomes

$$h_o^{(\infty)}(z) = \frac{1}{1+\theta^2(1-z)^2(1-z^{-1})^2} = \frac{1}{1+\theta^2(2(1-\cos\lambda))^2} = \frac{1}{1+16\theta^2\sin^4\frac{1}{2}\lambda} . \qquad (4.8.4)$$

To evaluate the response at any ν value which is not large (4.8.2) is best rewritten as

$$h_o^{(\nu)}(z) = \frac{1+\theta'^2z^{-(\nu+2)}(1-z)^2(c+dz)}{1+16\theta^2\sin^4\frac{1}{2}\lambda} \qquad (4.8.5)$$

where the constants c and d are defined by

$$c = \rho^{\nu+2}(1-\rho^2)\cos\nu\phi, \qquad d = \rho^{\nu+1}(1-\rho^2)\cos(\nu+1)\phi . \qquad (4.8.6)$$

As the expression (4.8.5) is complex it is best to consider the gain of the optimal filter which is

$$|h_o^{(\nu)}(z)|^2 = \frac{|1+\theta'^2 z^{-(\nu+2)}(1-z)^2(c+dz)|^2}{1+16\theta^2\sin^4\frac{1}{2}\lambda} . \tag{4.8.7}$$

An example of this function, when $\theta = 15$, is given in Table 11 in Chapter V.

4.9 Trend Estimation Formulae for Large ν

It is convenient to leave the derivation of formulae for estimating a trend value at a time point which is both preceded by and followed by a reasonably large number of observations until this juncture because these formulae are most easily deduced from the expression (4.8.3). From the symmetry of (4.8.3) it is apparent that its expression in partial fractions must be of the form

$$\frac{1}{\theta'^2}\left\{\frac{c+dz}{(1-z_o z)(1-\bar{z}_o z)} + \frac{c+dz^{-1}}{(1-z_o z^{-1})(1-\bar{z}_o z^{-1})} + e\right\} . \tag{4.9.1}$$

On simplifying and equating like terms it is found that $c = -e$. By setting the term in z or z^{-1} equal to zero the equivalence

$$c = -d(1+z_o\bar{z}_o)/z_o\bar{z}_o(z_o+\bar{z}_o) \tag{4.9.2}$$

may be established. Further, the equating of constant terms provides the following expression in c alone,

$$1 = 2c+2c\,\frac{(z_o\bar{z}_o)(z_o+\bar{z}_o)^2}{1+z_o\bar{z}_o} - c\left(1+(z_o\bar{z}_o)^2+(z_o+\bar{z}_o)^2\right) . \tag{4.9.3}$$

Elementary manipulations of (4.9.2) and (4.9.3) result in the following expressions for c and d, which depend only on ρ and ϕ,

$$c = \frac{1+\rho^2}{(1-\rho^2)(1+\rho^2+2\rho\cos\phi)(1+\rho^2-2\rho\cos\phi)}$$

$$d = \frac{-\rho^2(2\rho\cos\phi)}{(1-\rho^2)(1+\rho^2+2\rho\cos\phi)(1+\rho^2-2\rho\cos\phi)} . \tag{4.9.4}$$

Thus if (4.9.1) is rewritten to include the values of c and d given in (4.9.4) it becomes

$$\frac{1}{\theta'^2(1-\rho^2)(1+\rho^2+2\rho\cos\phi)(1+\rho^2-2\rho\cos\phi)}$$

$$\left\{\frac{C-Dz^{-1}}{(1-z_oz^{-1})(1-\bar{z}_oz^{-1})}+\frac{C-Dz}{(1-z_oz)(1-\bar{z}_oz)}-(1-\rho^2)\right\} \quad (4.9.5)$$

where new constants $C = 1+\rho^2$ and $D = \rho^2(2\rho\cos\phi)$ have been employed. The expression (4.9.5) allows decomposition of the generating function so that the trend estimate $\hat{p}(o)$ may be considered as having three parts. One component involves observations prior to the time point of estimate and is denoted $\hat{p}''(o)$. Another component involves observations after the time point of the trend estimate and is denoted $\hat{p}'(o)$. The last component merely weights the observations at the same time point as the required trend estimate. The term which produces $\hat{p}'(o)$ is the one on which attention is first focussed. It is simply expressed in partial fractions as follows,

$$\frac{C-Dz^{-1}}{(1-z_oz^{-1})(1-\bar{z}_oz^{-1})}=\frac{\gamma}{(1-z_oz^{-1})}+\frac{\delta}{(1-\bar{z}_oz^{-1})} \quad (4.9.6)$$

where $\gamma = \dfrac{Cz_o-D}{z_o-\bar{z}_o}$ and $\delta = \dfrac{D-Cz_o}{z_o-\bar{z}_o}$.

Expanding each term on the right hand side of (4.9.6) in a geometric series, collecting like terms and rewriting z_o and $\bar{z}_o$ in terms of ρ and ϕ, one obtains the following simplification of (4.9.6)

$$\frac{C-Dz^{-1}}{(1-z_oz^{-1})(1-\bar{z}_oz^{-1})}=C\sum_{m}^{\infty}{}_{o}\,\rho^m\frac{\sin(m+1)\phi}{\sin\phi}z^{-m}-D\sum_{m}^{\infty}{}_{o}\,\rho^{m-1}\frac{\sin m\phi}{\sin\phi}z^{-m}.$$

(4.9.7)

In an exactly analogous manner it is found that the expression generating $\hat{p}''(o)$ may be simplified to become

$$\frac{C-Dz}{(1-z_o z)(1-\bar{z}_o z)} = C \sum_m {}_o^\infty \rho^m \frac{\sin(m+1)\phi}{\sin\phi} z^m - D \sum_m {}_o^\infty \rho^{m-1} \frac{\sin m\phi}{\sin\phi} z^m . \qquad (4.9.8)$$

The expressions in (4.9.7) and (4.9.8) may be generalized simply to relate to estimates at any time point, say n, and when applied to the relevant observation give the following component estimates,

$$\hat{p}'(n) = C \sum_m {}_o^\infty \rho^m \frac{\sin(m+1)\phi}{\sin\phi} w(n+m) - D \sum_m {}_o^\infty \rho^{m-1} \frac{\sin m\phi}{\sin\phi} w(n+m)$$

$$\hat{p}''(n) = C \sum {}_o^\infty \rho^m \frac{\sin(m+1)\phi}{\sin\phi} w(n-m) - D \sum {}_o^\infty \rho^{m-1} \frac{\sin m\phi}{\sin\phi} w(n-m).$$

The overall estimate $\hat{p}^{(\infty)}(n)$ is obtained by recombining the component parts according to (4.9.5) to give

$$\hat{p}^{(\infty)}(n) = \frac{\{\hat{p}'(n)+\hat{p}''(n)-Cw(n)\}}{\theta'^2(1-\rho^2)(1+\rho^2+2\rho\cos\phi)(1+\rho^2-2\rho\cos\phi)} = \frac{\{\hat{p}'(n)+\hat{p}''(n)-Cw(n)\}}{\theta'(1-\rho^2)(1+\rho^2+2\rho\cos\phi)} . \qquad (4.9.10)$$

V COMPUTATION OF SEASONAL AND TREND ESTIMATES AND THEIR PROPERTIES

5.1 Introduction

The calculations described are those necessary for the deseasonalizing of an economic time series, when it is important for the whole series to be adequately adjusted. If it is only necessary to have adequate adjustment for the latter part of the data available some savings in computer storage and processing time may easily be had.

The series $y(n)$ is assumed to be available for $n = 1,\ldots,N$. To aid in the description of the computations to be carried out the available sequence of filtered observations is divided into three parts, from 1 to M+1, M+1 to N-M and N-M to N. These divisions will henceforth be referred to as the lower, intermediate and upper segments of the series. To proceed with the explanation of M it is necessary to return to the definition of the seasonal $s_j(n)$ in terms of $\xi_j(n)$ and $\xi_{-j}(n)$, first introduced in §4.2 (see (4.2.3) and §4.2). In the next section a formula is proposed for estimating $\xi_j(n)$ and the time point M will be chosen so that the estimate $\hat{\xi}_j(M)$ can be regarded as being only negligibly influenced by observations at the beginning of the series. Similarly the estimate, $\hat{\xi}_j(N-M)$, is chosen so that it is only influenced in a minor way by observations at the end of the series.

5.2 The Intermediate Segment Seasonal Estimates

So far in the discussion of signal extraction formulae the point of signal extraction has been located by the parameter $\nu \geq 0$. Thus by using a procedure discussed by Hannan [25] and Whittle [57] the estimation of $\xi_j^{(\nu)}(N-\nu)$, the signal at a time point ν observations from the end of the filtered sequence, is given by

$$\hat{\xi}_j^{(\nu)}(N-\nu) = H \sum_{m=o}^{\infty} \left\{\beta_j^{|m-\nu|} + \beta_j^{m+\nu+1}\right\} e^{i(m-\nu)\lambda_j} y(N-m) \qquad (5.2.1)$$

where $H = (1-\beta_j)/(1+\beta_j)$. This expression is easier to develop if a change of variable $m = k+\nu$ is made so that (5.2.1) may be rewritten as

$$\hat{\xi}_j^{(\nu)}(N-\nu) = H \sum_{k=-\nu}^{\infty} \left\{\beta_j^{|k|}+\beta_j^{k+2\nu+1}\right\} e^{ik\lambda_j} y(N-k-\nu). \qquad (5.2.2)$$

As ν becomes large the second term in braces becomes negligible and the estimator is then denoted

$$\hat{\xi}_j^{(\infty)}(N-\nu) = H \sum_{k=-\infty}^{\infty} \beta_j^{|k|} e^{ik\lambda_j} y(N-k-\nu). \qquad (5.2.3)$$

Estimates of $\xi_j^{(\infty)}(n)$ at time points in the intermediate segment are obtained using the formula (5.2.3) for all λ_j, $j = \pm1,\pm2,\ldots,\pm5,6$. Of course, the limits of the summation in (5.2.3) are in practice 1 and N, but if M is appropriately chosen the difference between the actual and theoretical limits on the summation will result in only an insignificant mis-specification. For ease of exposition of the computational procedure (5.2.3) may be rewritten as[26]

$$\hat{\xi}_j(n) = \hat{\xi}'_j(n)+\hat{\xi}''_j(n)-Hy(n) \qquad (5.2.4)$$

where

$$\hat{\xi}'_j(n) = H\, \Sigma_0^{\infty} \beta_j^m e^{im\lambda_j} y(n-m)$$
$$\hat{\xi}''_j(n) = H\, \Sigma_0^{\infty} \beta_j^m e^{-im\lambda_j} y(n+m). \qquad (5.2.5)$$

---

26 The upper parenthesis (∞) attached to $\hat{\xi}_j(n)$ in (5.2.3) emphasizes the <u>theoretical</u> number of observations <u>assumed</u> to be available between n and N. Henceforth an upper parenthesis is attached only if (∞) is not the appropriate one.

The real and complex parts of $\hat{\xi}'_j(n)$ and $\hat{\xi}''_j(n)$ are easily obtained from (5.2.5) and are

$$u'_j(n) = H\,\Sigma_0^\infty \beta_j^m y(n-m)\cos\lambda_j m$$

$$v'_j(n) = H\,\Sigma_0^\infty \beta_j^m y(n-m)\sin\lambda_j m$$

$$u''_j(n) = H\,\Sigma_0^\infty \beta_j^m y(n+m)\cos\lambda_j m \qquad (5.2.6)$$

$$v''_j(n) = H\,\Sigma_0^\infty \beta_j^m y(n+m)\sin\lambda_j m.$$

The quantities detailed in (5.2.6) may be evaluated at $n = M+1$ *and* $N-M$ to provide a basis for the evaluation of the same quantities for all time points in the intermediate segment. As each quantity, for each time point in the intermediate segment, would involve a separate summation from 1 to N computing time is minimized by developing an iterative relation starting from the value obtained at M+1. It should be added that if the iterative approach is not adopted there would be no need to calculate $v'_j(n)$ and $v''_j(n)$ because the estimate of the seasonal at each frequency does not involve these quantities.[27]

To establish the recursive relations needed for iterative evaluation of the quantities in (5.2.6) it is easiest to first obtain a recursion for $\hat{\xi}'_j(n)$ and $\hat{\xi}''_j(n)$ of the form[28]

$$\hat{\xi}'_j(n+1) = H\beta_j e^{i\lambda_j}\hat{\xi}'_j(n) + Hy(n+1)$$

$$\hat{\xi}''_j(n-1) = H\beta_j e^{-i\lambda_j}\hat{\xi}''_j(n) + Hy(n-1). \qquad (5.2.7)$$

---

27 The reason for this comment will become evident as the seasonal estimate $\hat{s}_j(n)$, given in (5.2.10), is expressed in terms of $u'_j(n)$, $u''_j(n)$ and $y(n)$.

28 A recursion for $\hat{\xi}''_j(n+1)$ in terms of $\hat{\xi}''_j(n)$ and $y(n)$ may be obtained but is difficult to use for computer tabulation because the factor multiplying $\hat{\xi}''_j(n)$ involves β_j^{-1} and any errors are greatly magnified.

On substituting for the complex quantities $\hat{\xi}'_j(n)$ and $\hat{\xi}''_j(n)$ using the expressions

$$\begin{aligned}\hat{\xi}'_j(n) &= u'_j(n)+iv'_j(n)\\ \hat{\xi}''_j(n) &= u''_j(n)-iv''_j(n)\end{aligned} \tag{5.2.8}$$

one can equate the real and complex parts in both equations in (5.2.7) to give the following iterative relations

$$\begin{aligned}u'_j(n+1) &= \beta_j\left\{u'_j(n)\cos\lambda_j-v'_j(n)\sin\lambda_j\right\} + Hy(n+1)\\ v'_j(n+1) &= \beta_j\left\{u'_j(n)\sin\lambda_j+v'_j(n)\cos\lambda_j\right\}\\ u''_j(n-1) &= \beta_j\left\{u''_j(n)\cos\lambda_j-v''_j(n)\sin\lambda_j\right\} + Hy(n-1)\\ v''_j(n-1) &= \beta_j\left\{u''_j(n)\sin\lambda_j+v''_j(n)\cos\lambda_j\right\} .\end{aligned} \tag{5.2.9}$$

The relations in (5.2.9) and the value calculated at $n = M+1$ and $N-M$ may be used to produce values for $n = M+2$ up to $N-M$. The associated estimated seasonal for this segment is then obtained from inserting the estimated quantities in (4.2.3) to give

$$\begin{aligned}\hat{s}_j(n) &= \hat{\xi}_j(n)+\hat{\xi}_{-j}(n)\\ &= (2-\delta^6_j)\left(u'_j(n)+u''_j(n)-Hy(n)\right) .\end{aligned} \tag{5.2.10}$$

5.3 The Upper and Lower Segment Seasonal Estimates

Attention is now focussed on the upper or most recent segment of data, where the estimate of the seasonal extracted will depend more on past and less on future observations as the point of estimation approaches N.[29] The estimate $\hat{\xi}_j(N)$ has no future observations on which it could be based. To begin the calculations

---

29 Past and future is defined in relation to the time point at which the estimate is being made.

for this segment an estimate is made for each n therein which is based only on information from time points which precede n. These estimates are obtained from

$$\hat{\xi}_j^{(o)}(n) = \Sigma_o^{\infty}(1-\beta_j)\beta_j^m e^{im\lambda_j} y(n-m). \qquad (5.3.1)$$

The real and imaginary parts of these estimates are the $u_j(n)$ and $v_j(n)$ defined in (4.5.4) and are obtained from the recursive relations (4.5.5). To begin these iterative procedures one can use the values $u'(N-M)$ and $v'(N-M)$ after multiplication by $(1+\beta_j)$. The estimate of the seasonal for each frequency λ_j based only on the past is

$$\hat{s}_j^{(o)}(n) = \hat{\xi}_j^{(o)}(n)+\hat{\xi}_{-j}^{(o)}(n). \qquad (5.3.2)$$

Although $\hat{s}_j^{(o)}(n)$ is the best estimate one can obtain at N it is apparent that for other points of time one should make use of future observations as well. In particular at $N-\nu$ there are ν future observations available. It is therefore possible to obtain $\hat{s}_j^{(\nu)}(N-\nu)$ for $\nu = 0,1,\ldots,M$ and this estimate can be obtained by employing the iterative relation which is

$$\hat{s}_j^{(\nu)}(n) = \hat{s}_j^{(\nu-1)}(n)+(2-\delta_j^6)(1-\beta_j)\beta_j^{(\nu-1)}\Big\{y(n+\nu)\cos\lambda_j\nu$$

$$-u_j(n+\nu)\cos\lambda_j\nu - v_j(n+\nu)\sin\lambda_j\nu\Big\} \qquad (5.3.3)$$

$$\nu = 1,\ldots,M; \quad n = N-M,\ldots,N; \quad j = 1,2,\ldots,6.$$

This recursion in ν, when summed over j, is just the recursion presented in (4.5.7). To start the iteration based on (5.3.3) the quantities $\hat{s}_j^{(o)}(n)$, $u_j(n)$ and $v_j(n)$ are needed. Figure X illustrates that (M+1) values of $\hat{s}_j^{(o)}(n)$, $u(n)$ and $v(n)$ produce, through (5.3.3), M values of $\hat{s}_j^{(1)}(n)$. Each step of the iterative procedure provides estimates at one less time point. The estimate at any time point $N-\nu$ containing most information is the value $\hat{s}^{(\nu)}(N-\nu)$, marked in the figure by a cross. This method has the virtue of not only reducing computer processing time but also of allowing a view of the way in which the estimates of the seasonal at each point of time stabilize.

FIG. X

ITERATIVE UPDATING OF ESTIMATES IN THE UPPER AND LOWER SEGMENTS

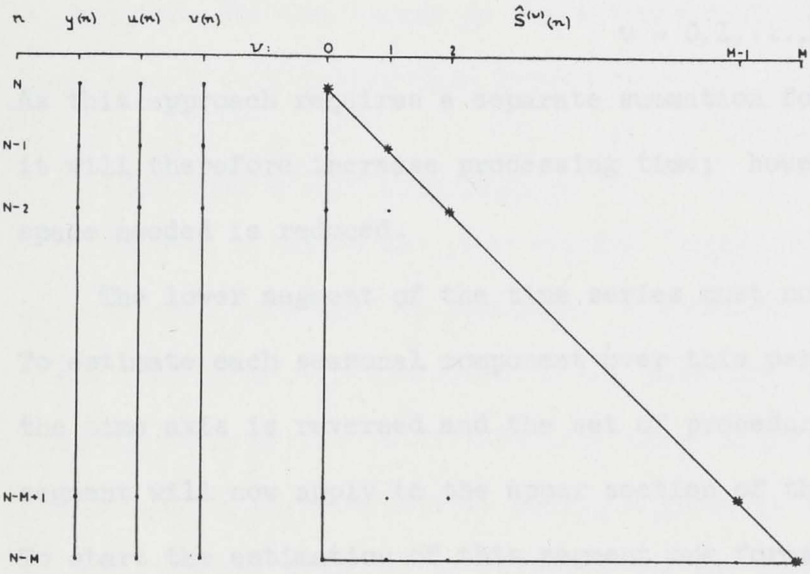
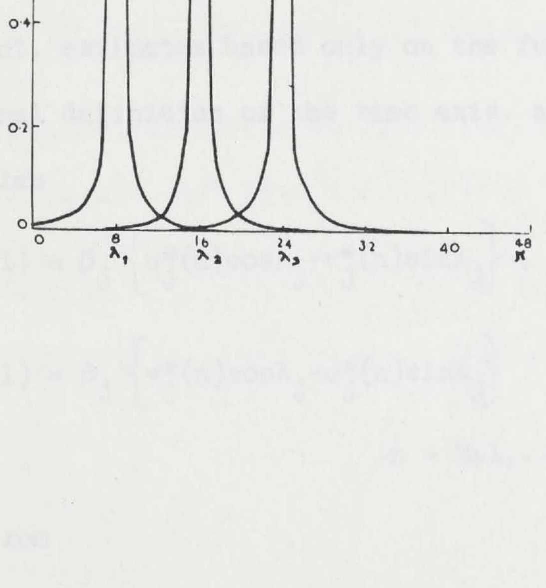


FIG. XI

INTERMEDIATE SEGMENT SEASONAL EXTRACTOR RESPONSE FUNCTION



If computer storage is a constraining factor and one is willing to forgo the advantage of following the effect of additional information on the estimate then a more direct estimate may be obtained from

$$\hat{s}_j^{(\nu)}(N-\nu) = \hat{\xi}_j^{(\nu)}(N-\nu)+\hat{\xi}_{-j}^{(\nu)}(N-\nu) \tag{5.3.4}$$

where

$$\hat{\xi}_j^{(\nu)}(N-\nu) = H\,\Sigma_{m}{}_{0}^{\infty}\left\{\beta_j^{|m-\nu|}+\beta_j^{m+\nu+1}\right\} e^{i(m-\nu)\lambda_j} y(N-m) \tag{5.3.5}$$

$$\nu = 0,1,\ldots,M.$$

As this approach requires a separate summation for each estimate it will therefore increase processing time; however the storage space needed is reduced.

The lower segment of the time series must now be considered. To estimate each seasonal component over this period we may imagine the time axis is reversed and the set of procedures for the upper segment will now apply to the upper section of the reversed series. To start the estimation of this segment one forms

$$u_j^*(M+1) = (1-\beta_j)\,\Sigma_0^\infty \beta_j^m y(M+1+m)\cos m\lambda_j$$

$$v_j^*(M+1) = (1-\beta_j)\,\Sigma_0^\infty \beta_j^m y(M+1+m)\sin m\lambda_j . \tag{5.3.6}$$

These values need not however be obtained <u>ab initio</u> as $u_j^*(M+1) = (1+\beta_j)u_j''(M+1)$ and $v_j^*(M+1) = (1+\beta_j)v_j''(M+1)$. In a completely analogous manner to that described for the upper segment, estimates based only on the future, with respect to the original definition of the time axis, are obtained from the iterative formulae

$$u_j^*(n-1) = \beta_j\left\{u_j^*(n)\cos\lambda_j - v_j^*(n)\sin\lambda_j\right\} + (1-\beta_j)y(n-1)$$

$$v_j^*(n-1) = \beta_j\left\{v_j^*(n)\cos\lambda_j - u_j^*(n)\sin\lambda_j\right\} \tag{5.3.7}$$

$$n = M+1,\ldots,2$$

and from

$$\hat{s}_j^{(o)}(n) = \hat{\xi}_j^{(o)}(n)+\hat{\xi}_{-j}^{(o)}(n) = (2-\delta_j^6)u_j^*(n) \tag{5.3.8}$$

$$j = 1,2,\ldots,6; \quad n = 1,\ldots,M+1.$$

Exactly the same updating procedure in ν (see 5.3.3) is carried out to obtain estimates for each time point using successively more past information until each estimate is based on as many past observations as possible. The recurrence relation (5.3.3) allows construction of a table similar to that shown in Fig. X except that the time scale on the extreme left of the figure now runs from 1 to M rather than from N to N-M.

5.4 Trend Extraction Computations

In §4.6 a method was proposed for "trend extraction". The calculations necessary to implement this method closely resemble those that have been discussed in the last two sections of this chapter. The series is divided into three segments as before and evaluation is begun either by making a guess at the trend value at M, M-1, N-M and N-M+1 or by carrying out the summations detailed in (4.9.7) and (4.9.8) i.e. evaluating $\hat{p}'(n)$ at $n = N-M$, $N-M+1$ and $\hat{p}''(n)$ at $n = M, M-1$. To economically compute the values in the intermediate segment, i.e. $n = M+1,\ldots,N-M-1$ two simple recursive formulae may be used. In section 4.9 the generating function of $\hat{p}'(o)$ was presented in (4.9.6). If the trend estimate considered is now generalized to refer to the time point n then the left hand side of (4.9.6) implies the following recursive relation

$$\hat{p}'(n)-2\rho\cos\phi\hat{p}'(n+1)+\rho^2\hat{p}'(n+2) = Cy(n)-Dy(n+1). \tag{5.4.1}$$

Similarly, the left hand side of (4.9.8) may be used to obtain recursion for $\hat{p}''(n)$,

$$\hat{p}''(n)-2\rho\cos\phi\hat{p}''(n-1)+\rho^2\hat{p}''(n-2) = Cy(n)-Dy(n-1). \tag{5.4.2}$$

Once the values of $\hat{p}'(n)$ and $\hat{p}''(n)$ have been established for all n in the intermediate segment then the trend estimate for all points is derived from (4.9.10).

Trend estimation of the upper segment of the series requires an evaluation of both the summation in (4.7.14) at n = N-M, N-M+1 to start the recurrence relation in time for $\nu = 0$ given in (4.7.11) and for $\nu = 1$ given in (4.7.12). The values of $\hat{p}^{(o)}(n)$ and $\hat{p}^{(1)}(n)$ when they have been calculated for the complete upper segment (it should be noticed that $\hat{p}^{(1)}(n)$ can only be obtained up to N-1) provide the basis for a further iteration in ν using (4.7.22). This iteration is used to build a triangle of estimates similar in form to that depicted in Fig. X and discussed in §5.3. The body of this figure allows investigation of how rapidly the trend estimate stabilizes as ν increases.

The filtered observations y(n) are obtained from $w(n)-\hat{p}^{(\nu)}(n)$, using the largest ν possible, and naturally these observations are input to the seasonal extraction procedure. As new data comes to hand there must be a revision of past y(n) and therefore of already computed values such as $\hat{s}^{(\nu)}(n)$. Of course the trend should soon stabilize so that the updating may only have to be carried back for a few steps, perhaps 12 to 15. Such recalculations must be made if adequate estimates of the seasonal are to be made for the most recent data points because any trend correction at these points must be incorporated as further very relevant information comes to hand.

To obtain the estimated "trend" for the lower segment the methods just described for the upper segment are available if once again it is imagined that the time series is temporarily reversed. The formulae, (4.7.11), (4.7.12) and (4.7.22) are then applied to the variables $\hat{q}^{(\nu)}(n)$ and r(n) where these newly defined variables are $r(n) = y(N+1-n)$ and $q^{(\nu)}(n) = p^{(\nu)}(N+1-n)$ for $n = N-M,\ldots,N$.

5.5 Evaluating Response Functions of Optimal Filters

To evaluate how effectively the 'optimal' procedure is extracting the seasonal signal it is necessary to obtain an expression for the response function of the seasonal extraction procedure and to evaluate this function for various values of ν. To accomplish this one can obtain the following expression for $\hat{s}_j^{(\nu)}(N-\nu)$ by substituting in (5.3.4) using the definition given in (5.3.5),

$$\hat{s}_j^{(\nu)}(N-\nu) = H \sum_{m=0}^{\infty} \left\{\beta_j^{|m-\nu|} + \beta_j^{m+\nu+1}\right\} \left(e^{i(m-\nu)\lambda_j} + e^{-i(m-\nu)\lambda_j}\right) y(N-m). \tag{5.5.1}$$

As the expression for $\hat{s}_j^{(\nu)}(N-\nu)$ in (5.5.1) is a filtering of $y(N-m)$ to produce $\hat{s}_j^{(\nu)}(N-\nu)$ and is of the form $\Sigma_0^{\infty} a_j^{(\nu)}(m)y(N-m)$ the response function is given by

$$\Sigma_0^{\infty} a_j^{(\nu)}(m)e^{i\lambda m} = h_j^{(\nu)}(\lambda-\lambda_j)e^{i\nu\lambda_j} + h_j^{(\nu)}(\lambda+\lambda_j)e^{-i\nu\lambda_j} \tag{5.5.2}$$

where

$$h_j^{(\nu)}(\lambda) = \frac{1-\beta_j}{(1-\beta_j e^{i\lambda})} \left\{e^{i\nu\lambda} + \beta_j \frac{(e^{i\nu\lambda}-\beta_j^{\nu})(1-e^{i\lambda})}{e^{i\lambda}-\beta_j}\right\}, \quad \nu \geqq 0. \tag{5.5.3}$$

In Fig. XI the response function (5.5.2) is depicted for the first three seasonal frequencies for $\beta_j = .96$ and for $\nu = \infty$. The figure reveals in the shape of the filter just how well the signal (and the power) at each individual frequency is reproduced by the estimation procedure in the intermediate segment. It should be noted however that only when in Fig. XII the modulus of the sum over all seasonal frequencies is graphed for $\nu = \infty$ is it apparent from the values of this gain function how adequately the overall seasonal will be reproduced. In particular, the effectiveness is very poor when $\nu = 0$, as is shown by the gain of the sum over all frequencies in Fig.XII; however the performance is quite adequate when $\nu = 20$ as can be seen from Fig. XII. A detailed tabulation

FIG. XII

GAIN OF SUM OVER ALL FREQUENCIES OF (5.5.3)

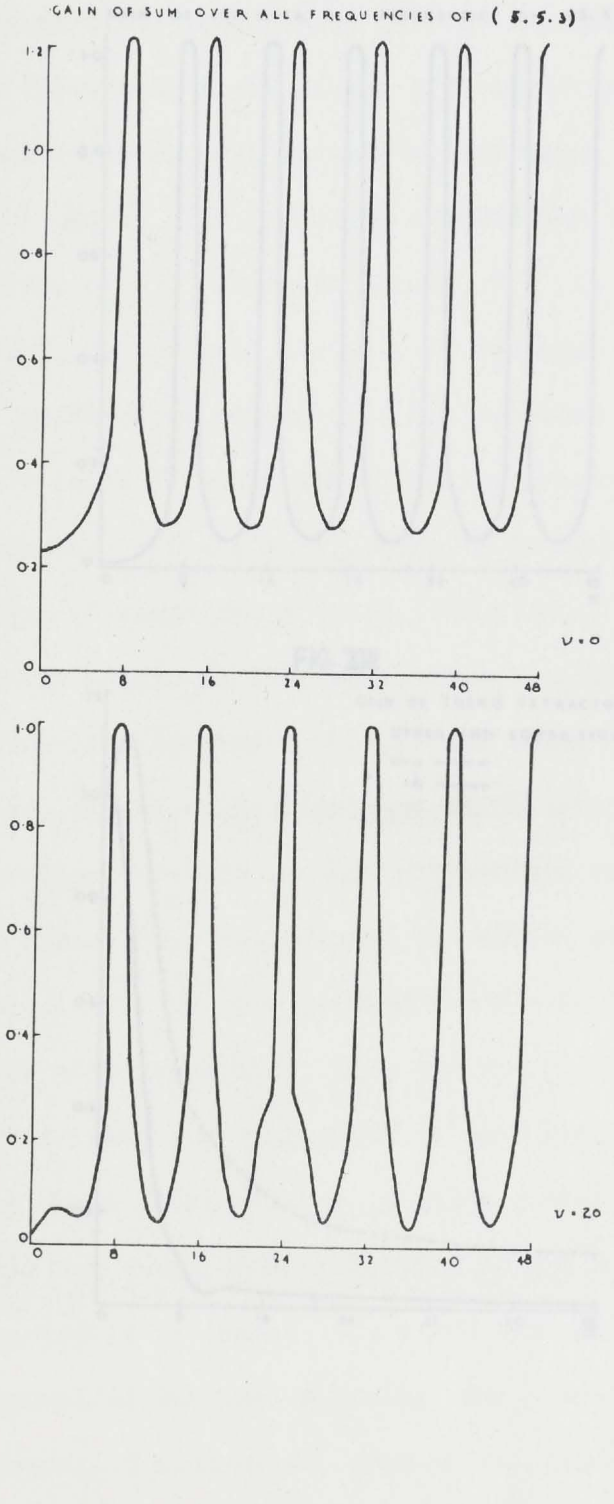


FIG. XII

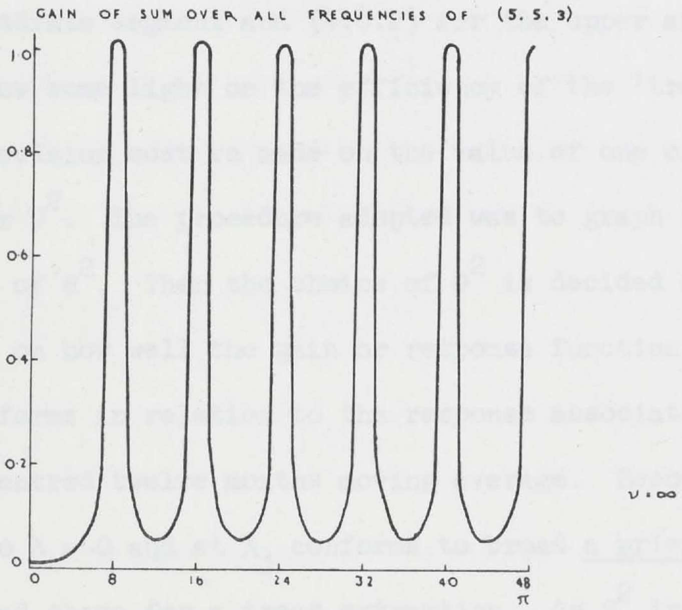
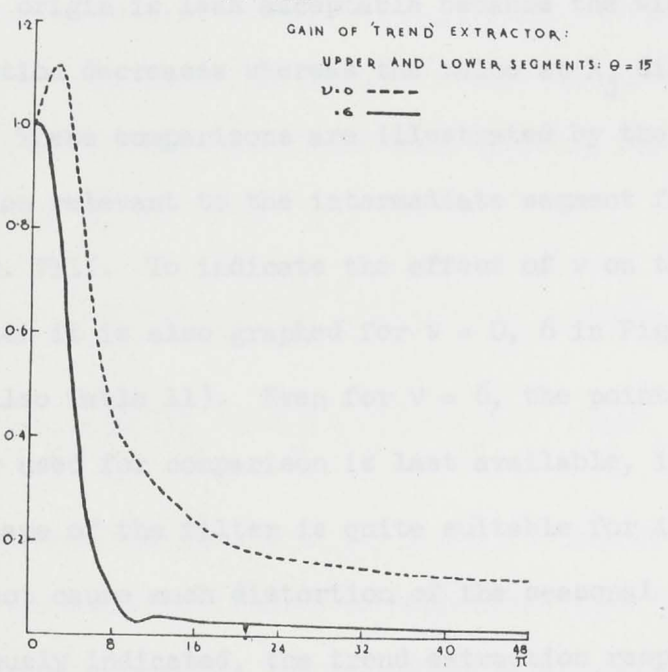


FIG. XIII



of the gain in question shows that after $\nu = 20$ the reproduction of the overall signal is probably adequate (see Table 12). It is also rather obvious that the shape of the gain function will depend on β_j. For the present $\beta_j = .96$ is used for all j and the matter of discussing an appropriate choice of β_j will be postponed until Chapter VI.

Before one can use the response functions (4.8.3) for the intermediate segment and (4.8.2) for the upper and lower segments, to throw some light on the efficiency of the 'trend' extractor some decision must be made on the value of one of the parameters ρ, ϕ or θ^2. The procedure adopted was to graph several possible values of θ^2. Then the choice of θ^2 is decided on two counts. First, on how well the gain or response function for a particular θ^2 performs in relation to the response associated with the removal of a centred twelve months moving average. Second, on how the shape near to $\lambda = 0$ and at λ_j conforms to broad <u>a priori</u> ideas on the required shape for a trend extraction. As θ^2 increases the shape at the origin is less acceptable because the width of effective extraction decreases whereas the value at λ_j differs by less from zero. These comparisons are illustrated by the graphing of the function relevant to the intermediate segment for $\theta = 15$, $\nu = \infty$ in Fig. VIII. To indicate the effect of ν on the shape of the function it is also graphed for $\nu = 0, 6$ in Fig. XIII when $\theta = 15$ (see also Table 11). Even for $\nu = 6$, the point at which the filter used for comparison is last available, it is clear that the shape of the filter is quite suitable for its purpose and will not cause much distortion of the seasonal signal. As is previously indicated, the trend extraction response, unlike removal of the twelve months moving average, is non zero at the seasonal frequencies. However in comparing the relative merits of each response it must be borne in mind that the seasonal component is now represented by a band of power about each j and thus each filter will cause minor distortion.

5.6 Characteristics of the Seasonal Estimates

The estimate proposed at each λ_j is $\hat{s}_j^{(\infty)}(n)$ in the intermediate segment and $\hat{s}_j^{(\nu)}(n)$ for the largest ν available in the other segments. This seasonal estimate is free to evolve and the nature of this evolutionary pattern is depicted by presenting an estimate of the amplitude (and phase) for each point of time. As is apparent from the signal at λ_j as given in (4.2.4) an estimate of the amplitude of $s_j(n)$ depends only on $\hat{\alpha}_j(n)$ and $\hat{\beta}_j(n)$, the estimates of the real and complex parts of the amplitude of $\xi_j(n)$. If $\xi_j(n)$ is replaced by its estimate for the highest ν available in (5.1.2) the logical estimate for $\alpha_j(n)$ and $\beta_j(n)$ is

$$\hat{\alpha}_j(n) = \mathcal{R}\left(\hat{\xi}_j^{(\nu)}(n)e^{-i\lambda_j n}\right)(2-\delta_j^6)$$
$$\hat{\beta}_j(n) = \mathcal{J}\left(\hat{\xi}_j^{(\nu)}(n)e^{-i\lambda_j n}\right)(2-\delta_j^6). \tag{5.6.1}$$

In the intermediate segment these estimates may be obtained from quantities already calculated for the seasonal estimation procedure. By using in (5.6.1) either the expression for $\hat{\xi}_j(n)$ given in (5.2.3), or rather more conveniently the further decompositions presented in (5.2.4) and (5.2.6) one finds the following expression for $\hat{\alpha}_j(n)$ and $\hat{\beta}_j(n)$,

$$\hat{\alpha}_j(n) = \hat{s}_j(n)\cos n\lambda_j + (2-\delta_j^6)\left(v_j'(n)-v_j''(n)\right)\sin n\lambda_j$$
$$\hat{\beta}_j(n) = \hat{s}_j(n)\sin n\lambda_j - (2-\delta_j^6)\left(v_j'(n)-v_j''(n)\right)\cos n\lambda_j. \tag{5.6.2}$$

Before estimates of $\hat{\alpha}_j(n)$ and $\hat{\beta}_j(n)$ may be obtained for the other two segments it must be emphasized that the $\hat{s}_j(n)$ for these periods are obtained from an iterative formula (5.3.3) involving only the real part of $\xi_j^{(\nu)}(n)$, since $\hat{s}_j^{(\nu)}(n) = (2-\delta_j^6)\mathcal{R}(\hat{\xi}_j^{(\nu)}(n))$. As is apparent from (5.6.1) both $\hat{\alpha}_j(n)$ and $\hat{\beta}_j(n)$ will depend on $\mathcal{J}(\hat{\xi}_j^{(\nu)}(n))$ it is therefore necessary to evaluate $\mathcal{J}(\hat{\xi}_j^{(\nu)}(n))$

for every time point in both segments. The optimal seasonal estimate at a given time point with $(\nu+1)$ additional observations available after the time point of estimate is related to the estimate at the same time point when only ν extra observations are available by the formula given in Hannan [25, p 1075]. By equating imaginary parts in this relation an iteration in ν of the following form is obtained

$$\mathcal{I}\left(\xi_j^{(\nu)}(n)\right) = \mathcal{I}\left(\xi_j^{(\nu-1)}(n)\right) - (1-\beta_j)\beta_j^{(\nu-1)}\left\{\left(y(n+\nu)-u_j(n+\nu)\right)\sin\nu\lambda_j + v_j(n+\nu)\cos\nu\lambda_j\right\}, \quad v = 1,2,\ldots,M+1. \tag{5.6.3}$$

Now defining the quantity $\overline{\hat{s}_j^{(\nu)}(n)} = (2-\delta_j^6)\,\mathcal{I}(\hat{\xi}_j^{(\nu)}(n))$ then $\hat{\alpha}_j(n)$ and $\hat{\beta}_j(n)$ are obtained from

$$\begin{aligned} \hat{\alpha}_j(n) &= \hat{s}_j^{(\nu)}(n)\cos n\lambda_j + \overline{\hat{s}_j^{(\nu)}(n)}\sin n\lambda_j \\ \hat{\beta}_j(n) &= \hat{s}_j^{(\nu)}(n)\sin n\lambda_j - \overline{\hat{s}_j^{(\nu)}(n)}\cos n\lambda_j \end{aligned} \tag{5.6.4}$$

$$n = 1,2,\ldots,M+1 \text{ and } N-M,\ldots,N$$

for the upper and lower segments.

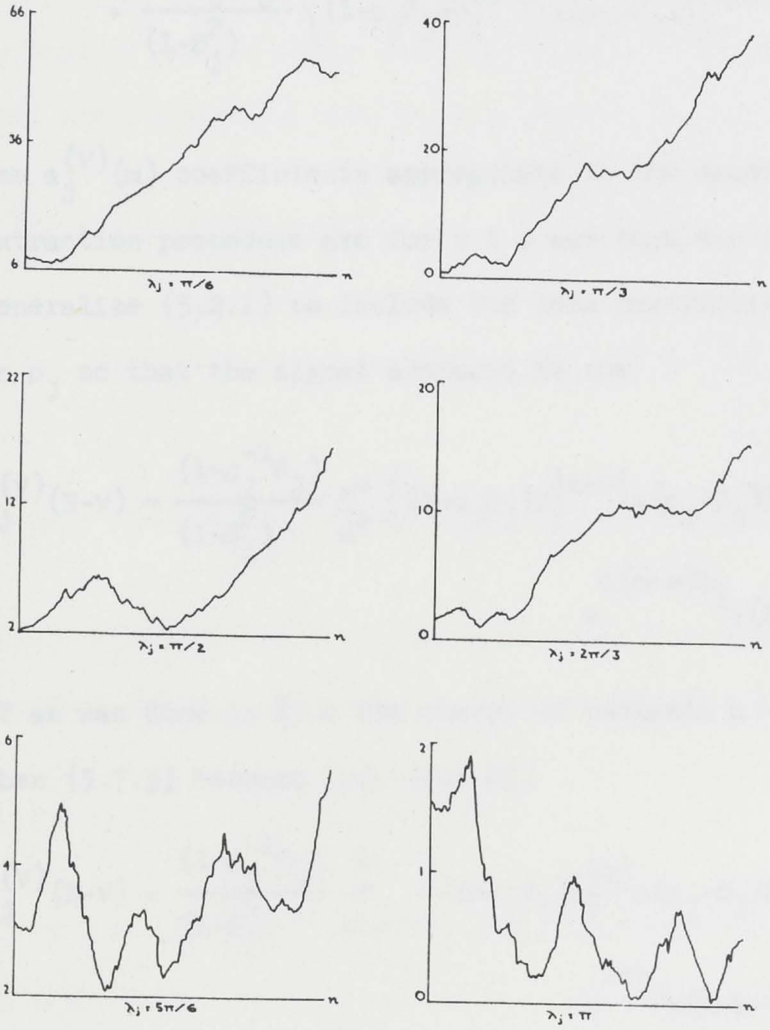
The reason for constructing the values of $\hat{\alpha}_j(n)$ and $\hat{\beta}_j(n)$ over the whole history of the series (see Fig. XIV) is to depict the evolutionary nature of the signal at each seasonal frequency. The characteristic used to focus attention on this evolution is $\hat{R}_j(n)$, an estimate of the amplitude of the j^{th} seasonal frequency at a point of time, and $\hat{\theta}_j(n)$ a measure of the changing phase at the j^{th} seasonal frequency at a point of time. The estimates of $R_j(n)$ and $\theta_j(n)$ are simply

$$\hat{R}_j(n) = \sqrt{\hat{\alpha}_j^2(n)+\hat{\beta}_j^2(n)}\,, \qquad n = 1,2,\ldots,N$$

$$\hat{\theta}_j(n) = \arctan\frac{\hat{\beta}_j(n)}{\hat{\alpha}_j(n)}\,, \qquad n = 1,2,\ldots,N.$$

FIG. XIV

EVOLUTION OF $\hat{R}_j(n)$; $n = 1,\ldots,N$



5.7 Computing Procedure when ρ_j is not assumed equal to Unity

To generalize our estimation procedure to allow ρ_j to vary (i.e. $-1 \leqq \rho_j \leqq 1$) we write following Whittle [57] and Hannan [25]

$$\hat{\xi}_j^{(\nu)}(N-\nu) = \Sigma_{m}{}_{o}^{\infty} y(N-m) a_j^{(\nu)}(m) \qquad (5.7.1)$$

where

$$a_j^{(\nu)}(m) = (1-\rho_j^{-1}\beta_j)\rho_j^{-\nu}\beta_j e^{i(m-\nu)\lambda_j} \qquad \nu \leqq 0$$

$$= \frac{(1-\rho_j^{-1}\beta_j)}{(1-\beta_j^2)} \left\{ (1-\rho_j\beta_j)\beta_j^{|m-\nu|} + (\rho_j-\beta_j)\beta_j^{m+\nu+1} \right\} e^{i(m-\nu)\lambda_j} \qquad \nu \geqq 0. \qquad (5.7.2)$$

The $a_j^{(\nu)}(m)$ coefficients appropriate to the seasonal (signal) extraction procedure are for $\nu \geqq 0$ and thus one may easily generalize (5.2.1) to include the less restrictive assumptions on ρ_j so that the signal estimate is now

$$\hat{\xi}_j^{(\nu)}(N-\nu) = \frac{(1-\rho_j^{-1}\beta_j)}{(1-\beta_j^2)} \Sigma_{m}{}_{o}^{\infty} \left\{ (1-\rho_j\beta_j)\beta_j^{|m-\nu|} + (\rho_j-\beta_j)\beta_j^{(m+\nu+1)} \right\} . \; e^{i(m-\nu)\lambda_j} y(N-m). \qquad (5.7.3)$$

If as was done in §5.2 the change of variable $m = k+\nu$ is made then (5.7.3) becomes (cf. (5.2.2))

$$\hat{\xi}_j^{(\nu)}(N-\nu) = \frac{(1-\rho_j^{-1}\beta_j)}{(1-\beta_j^2)} \sum_{k=-\nu}^{\infty} \left\{ (1-\rho_j\beta_j)\beta_j^{|k|} + (\rho_j-\beta_j)\beta_j^{(k+2\nu+1)} \right\} . \; e^{ik\lambda_j} y(N-k-\nu) \qquad (5.7.4)$$

which, as ν becomes large, may be written (cf. (5.2.3))

$$\hat{\xi}_j^{(\infty)}(N-\nu) \cong \frac{(1-\rho_j^{-1}\beta_j)(1-\rho_j\beta_j)}{(1-\beta_j^2)} \Sigma_{k-\infty}^{\infty} \beta_j^{|k|} e^{ik\lambda_j} y(N-\nu-k)$$

$$\cong \frac{(1-\rho_j^{-1}\beta_j)(1-\rho_j\beta_j)}{(1-\beta_j^2)} H^{-1} \left[H \Sigma_{k-\infty}^{\infty} \beta_j^{|k|} e^{ik\lambda_j} y(N-\nu-k) \right] \quad (5.7.5)$$

$$\cong \frac{(1-\rho_j^{-1}\beta_j)(1-\rho_j\beta_j)}{(1-\beta_j)^2} \left[H \Sigma_{k-\infty}^{\infty} \beta_j^{|k|} e^{ik\lambda_j} y(N-\nu-k) \right] .$$

Now, it is convenient for computations in the intermediate segment to express the estimates with variable ρ_j in terms of the quantities used for computation when $\rho_j \equiv 1$, thus (5.7.5) is re-expressed as

$$\hat{\xi}_j^{(\infty)}(n) \cong K\left\{\hat{\xi}_j' + \hat{\xi}_j'' - Hy(n)\right\}$$

$$\cong K\left\{u_j'(n) + u_j''(n) - Hy(n)\right\} + iK\left\{v_j'(n) - v_j''(n)\right\} \quad (5.7.6)$$

where $K = \frac{(1-\rho_j^{-1}\beta_j)(1-\rho_j\beta_j)}{(1-\beta_j^2)}$ and the quantities $\hat{\xi}_j'(n)$, $\hat{\xi}_j''(n)$, $u_j'(n)$, $u_j''(n)$, $v_j'(n)$ and $v_j''(n)$ are defined in (5.2.5) and (5.2.6). The seasonal estimate for the intermediate sector will then be (cf. (5.2.10))

$$\hat{s}_j(n) = (2-\delta_j^6)\ K\left\{u_j'(n) + u_j''(n) - y(n)\right\} . \quad (5.7.7)$$

If now one turns to the general expression (5.7.3) and begins as was done in §5.2 with the estimator based only on the past observations with respect to each time point one obtains

$$\hat{\xi}_j^{(o)}(n) = \frac{(1-\rho_j^{-1}\beta_j)}{(1-\beta_j^2)} \Sigma_{m\,o}^{\infty} \left\{(1-\rho_j\beta_j)\beta_j^{|m|}+(\rho_j-\beta_j)\beta_j^{m+1}\right\} e^{im\lambda_j} y(n-m)$$

$$= \frac{(1-\rho_j^{-1}\beta_j)(1-\beta_j^2)}{(1-\beta_j^2)} \Sigma_{m\,o}^{\infty}\beta_j^m e^{im\lambda_j} y(n-m)$$

$$= \frac{(1-\rho_j^{-1}\beta_j)}{(1-\beta_j)} \left[(1-\beta_j)\,\Sigma_{m\,o}^{\infty}\beta_j^m e^{im\lambda_j} y(n-m)\right] \qquad (5.7.8)$$

$$= L\left[(1-\beta_j)\,\Sigma_{m\,o}^{\infty}\beta_j^m e^{im\lambda_j} y(n-m)\right]$$

$$= L\left\{u_j(n)+iv_j(n)\right\}$$

where $L = (1-\rho_j^{-1}\beta_j)/(1-\beta_j)$, and where $u_j(n)$ and $v_j(n)$ were defined in (4.5.4) and may be obtained for all time points from the recursive formulae, (4.5.5). The details of starting off the recursion are exactly as given in §5.3. The seasonal estimate based only on the past is exactly as given in (5.3.2) but we would now use (5.7.8) to compute $\hat{\xi}_j^{(o)}(n)$ and $\hat{\xi}_{-j}^{(o)}(n)$.

To complete the seasonal calculations, with ρ_j not restricted to the value unity, a recurrence relation developed by Hannan [25] is used in the same way as is suggested in §5.3 (cf. (5.3.3)). The iterative relation employed is

$$\hat{\xi}_j^{(\nu)}(n) = \hat{\xi}_j^{(\nu-1)}(n)+(\beta_j-\rho_j)\beta_j^{(\nu-1)}e^{-i\nu\lambda_j}\left(\hat{\xi}_j(n+\nu)-y(n+\nu)\right) \qquad (5.7.9)$$

and taking real and imaginary parts and simplifying we obtain the following two recursions for $\hat{s}_j^{(\nu)}(n)$ and $\overline{\hat{s}_j^{(\nu)}}(n)$ (see §5.6 for their definition) which involves only quantities already computed and constants, (cf. (5.3.3) and (5.6.3))

$$\hat{s}_j^{(\nu)}(n) = \hat{s}_j^{(\nu-1)}(n)+(2-\delta_j^6)(\rho_j-\beta_j)\beta_j^{(\nu-1)}\left[y(n+\nu)\cos\nu\lambda_j - L(\cos\nu\lambda_j u_j(n+\nu) + \sin\nu\lambda_j v_j(n+\nu))\right]$$

$$\overline{\hat{s}_j^{(\nu)}}(n) = \overline{\hat{s}_j^{(\nu-1)}}(n)+(2-\delta_j^6)(\rho_j-\beta_j)\beta_j^{(\nu-1)}\left[y(n+\nu)\sin\nu\lambda_j - L(\sin\nu\lambda_j u_j(n+\nu) - v_j(n+\nu)\cos\nu\lambda_j)\right]. \qquad (5.7.10)$$

The second recursion in (5.7.10) is not required for computation of the seasonal estimates but is required for the calculation of the seasonal characteristics, $\hat{\alpha}_j(n)$, $\hat{\beta}_j(n)$, for the non-intermediate segment. The procedures given in formulae (5.6.4) are still appropriate although now the quantities $\hat{s}_j^{(\nu)}(n)$ and $\overline{\hat{s}_j^{(\nu)}(n)}$ must be obtained from (5.7.10), not from (5.3.3) and (5.6.3).

The intermediate segment $\hat{\alpha}_j(n)$ and $\hat{\beta}_j(n)$ are very simply computed for (5.7.6) shows that the only difference in the seasonal estimate for this segment is a constant multiple K. This means that all that is necessary is for the quantities obtained using (5.6.2) to be multiplied by the factor K.

As a final comment it is worth presenting formulae for the variance of estimate of the signal extraction procedure which is a function of ν and which is given by Hannan [25, p 1075-6]

$$\operatorname{var} \hat{\xi}_j^{(\nu)}(n) = \frac{\sigma_j^2}{2(1-\beta_j^2)} \left\{ \beta_j \rho_j^{-1} + \frac{(1-\rho_j^{-1}\beta_j)}{(1-\rho_j\beta_j)} \beta_j^{2\nu+2} \right\} \qquad (5.7.11)$$

which must be doubled at each frequency to become the appropriate variance for $\hat{s}_j^{(\nu)}(n)$. It is also argued (see [25, p 1075-6]) that because the filter is highly concentrated about each λ_j, that $\mathcal{E}(s_j^{(\nu)}(n)s_k^{(\nu)}(n)) \cong 0$, $j \neq k$, and so one could compute an approximate prediction variance for $\hat{s}^{(\nu)}(n)$ from the expression,

$$\operatorname{var}\left(\hat{s}^{(\nu)}(n) - s(n)\right) = \Sigma_{j1}^{6} \frac{\sigma_j^2}{(1-\beta_j^2)} \left\{ \beta_j \rho_j^{-1} + \frac{(1-\rho_j^{-1}\beta_j)}{(1-\rho_j\beta_j)} \beta_j^{2\nu+2} \right\} . \qquad (5.7.12)$$

Now it is worth setting out two special cases of (5.7.11) for $\nu = 0$ and $\nu = \infty$. Thus we have

$$\text{var } \hat{\xi}_j^{(o)}(n) = \frac{\sigma_j^2}{2(1-\beta_j^2)} \left\{ \beta_j \rho_j^{-1} + \frac{(1-\rho_j^{-1}\beta_j)}{(1-\rho_j\beta_j)} \beta_j^2 \right\}$$

$$= \frac{\sigma_j^2}{2} \frac{1}{(1-\beta_j^2)} \left\{ \beta_j \rho_j^{-1} - \beta_j^2 + \beta_j^2 - \rho_j^{-1} \beta_j^2 \right\} \qquad (5.7.13)$$

$$= \frac{\sigma_j^2}{2} \frac{\beta_j \rho_j^{-1}}{(1-\rho_j\beta_j)}$$

and

$$\text{var } \hat{\xi}_j^{(\infty)}(n) = \frac{\sigma_j^2}{2} \frac{\beta_j \rho_j^{-1}}{(1-\beta_j^2)} \qquad (5.7.14)$$

which in the special case of $\rho_j \equiv 1$ become

$$\text{var } \hat{\xi}_j^{(o)} = \frac{\sigma_j^2}{2} \frac{\beta_j}{(1-\beta_j)}$$

$$(5.7.15)$$

$$\text{var } \hat{\xi}_j^{(\infty)} = \frac{\sigma_j^2}{2} \frac{\beta_j}{(1-\beta_j^2)} .$$

These variances are not put forward as anything but a rough guide for after all the model was only really proposed as a basis for a filtering routine.

TABLE 11

TREND EXTRACTION : ONE SIDED GAIN VALUES.. θ =15.0, ρ = .833, ϕ = .1818

I*PI/96:	I	0	1	2	3	4	6	12	24	36	48	INF
0	0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1	1.013	1.008	1.005	1.002	1.000	0.997	0.998	1.000	1.000	1.000	1.000
	2	1.050	1.031	1.017	1.005	0.997	0.987	0.988	0.996	0.996	0.996	0.996
	3	1.102	1.063	1.031	1.006	0.987	0.966	0.968	0.979	0.979	0.980	0.980
	4	1.157	1.092	1.039	0.996	0.964	0.928	0.930	0.936	0.938	0.938	0.938
	5	1.198	1.108	1.031	0.969	0.922	0.867	0.868	0.861	0.862	0.862	0.862
	6	1.212	1.100	1.002	0.921	0.858	0.785	0.781	0.752	0.751	0.751	0.751
	7	1.194	1.065	0.951	0.854	0.777	0.687	0.678	0.624	0.620	0.620	0.620
	8	1.148	1.009	0.884	0.776	0.689	0.585	0.567	0.493	0.489	0.489	0.489
	9	1.085	0.942	0.811	0.697	0.602	0.487	0.461	0.375	0.374	0.375	0.375
	10	1.015	0.872	0.739	0.622	0.523	0.400	0.364	0.280	0.283	0.283	0.283
	11	0.945	0.804	0.673	0.554	0.454	0.326	0.281	0.209	0.214	0.213	0.213
	12	0.878	0.742	0.614	0.496	0.395	0.264	0.211	0.158	0.162	0.161	0.161
	13	0.817	0.686	0.561	0.447	0.346	0.213	0.153	0.123	0.123	0.123	0.123
	14	0.762	0.636	0.516	0.405	0.306	0.172	0.106	0.097	0.094	0.095	0.095
	15	0.713	0.592	0.477	0.369	0.272	0.139	0.068	0.077	0.073	0.074	0.074
	16	0.669	0.554	0.443	0.338	0.245	0.112	0.038	0.060	0.059	0.058	0.058
	17	0.630	0.520	0.413	0.312	0.222	0.091	0.015	0.046	0.047	0.047	0.047
	18	0.595	0.489	0.387	0.290	0.203	0.075	0.011	0.036	0.038	0.038	0.038
	19	0.563	0.462	0.364	0.271	0.187	0.063	0.023	0.029	0.030	0.031	0.031
	20	0.535	0.438	0.344	0.254	0.173	0.054	0.032	0.025	0.025	0.025	0.025
	21	0.510	0.417	0.326	0.239	0.162	0.049	0.039	0.022	0.021	0.021	0.021
	22	0.487	0.397	0.309	0.226	0.152	0.046	0.043	0.019	0.018	0.018	0.018
	23	0.466	0.379	0.295	0.215	0.144	0.044	0.045	0.016	0.015	0.015	0.015
	24	0.446	0.363	0.282	0.205	0.137	0.044	0.045	0.013	0.013	0.013	0.013
	25	0.429	0.348	0.270	0.196	0.131	0.044	0.044	0.010	0.010	0.011	0.011
	26	0.413	0.335	0.259	0.188	0.126	0.044	0.041	0.008	0.009	0.010	0.009
	27	0.398	0.323	0.249	0.180	0.121	0.045	0.038	0.007	0.009	0.008	0.008
	28	0.384	0.311	0.240	0.173	0.117	0.045	0.035	0.007	0.008	0.007	0.007
	29	0.372	0.301	0.231	0.167	0.113	0.045	0.031	0.007	0.006	0.006	0.006
	30	0.360	0.291	0.224	0.162	0.110	0.045	0.028	0.007	0.005	0.006	0.006
	31	0.349	0.282	0.217	0.157	0.107	0.045	0.025	0.006	0.005	0.005	0.005
	32	0.339	0.273	0.210	0.152	0.105	0.044	0.023	0.004	0.005	0.004	0.004
	33	0.329	0.266	0.204	0.148	0.102	0.043	0.021	0.003	0.004	0.004	0.004
	34	0.320	0.258	0.198	0.144	0.100	0.043	0.021	0.003	0.004	0.004	0.004
	35	0.312	0.251	0.193	0.140	0.098	0.042	0.021	0.003	0.003	0.003	0.003
	36	0.304	0.245	0.188	0.137	0.096	0.041	0.021	0.003	0.003	0.003	0.003
	37	0.296	0.239	0.183	0.134	0.094	0.040	0.022	0.004	0.003	0.003	0.003
	38	0.289	0.233	0.179	0.131	0.092	0.039	0.022	0.003	0.003	0.002	0.002
	39	0.283	0.228	0.175	0.128	0.091	0.038	0.022	0.003	0.002	0.002	0.002
	40	0.277	0.223	0.171	0.125	0.089	0.036	0.022	0.002	0.002	0.002	0.002
	41	0.271	0.218	0.167	0.123	0.087	0.035	0.022	0.001	0.002	0.002	0.002
	42	0.265	0.213	0.164	0.120	0.086	0.034	0.021	0.001	0.002	0.002	0.002
	43	0.260	0.209	0.160	0.118	0.085	0.033	0.021	0.001	0.002	0.002	0.002
	44	0.255	0.205	0.157	0.116	0.083	0.032	0.020	0.002	0.002	0.001	0.001
	45	0.250	0.201	0.154	0.114	0.082	0.031	0.019	0.002	0.001	0.001	0.001
	46	0.246	0.198	0.152	0.112	0.081	0.030	0.018	0.002	0.001	0.001	0.001
	47	0.242	0.194	0.149	0.110	0.079	0.030	0.017	0.002	0.001	0.001	0.001
	48	0.238	0.191	0.146	0.109	0.078	0.029	0.017	0.001	0.001	0.001	0.001
	49	0.234	0.188	0.144	0.107	0.077	0.028	0.016	0.000	0.001	0.001	0.001
	50	0.230	0.185	0.142	0.106	0.076	0.028	0.016	0.001	0.001	0.001	0.001
	51	0.226	0.182	0.140	0.104	0.075	0.027	0.016	0.001	0.001	0.001	0.001
	52	0.223	0.179	0.138	0.103	0.074	0.027	0.016	0.001	0.001	0.001	0.001
	53	0.220	0.177	0.136	0.101	0.073	0.027	0.016	0.002	0.001	0.001	0.001
	54	0.217	0.174	0.134	0.100	0.072	0.026	0.016	0.001	0.001	0.001	0.001
	55	0.214	0.172	0.132	0.099	0.071	0.026	0.016	0.001	0.001	0.001	0.001
	56	0.211	0.170	0.131	0.098	0.070	0.026	0.016	0.001	0.001	0.001	0.001
	57	0.209	0.167	0.129	0.097	0.069	0.026	0.016	0.000	0.000	0.001	0.001
	58	0.206	0.165	0.127	0.096	0.068	0.026	0.016	0.000	0.001	0.001	0.001
	59	0.204	0.163	0.126	0.094	0.067	0.025	0.015	0.001	0.001	0.001	0.001
	60	0.201	0.162	0.125	0.093	0.066	0.025	0.015	0.001	0.001	0.001	0.001
	61	0.199	0.160	0.123	0.093	0.065	0.025	0.015	0.001	0.000	0.001	0.001
	62	0.197	0.158	0.122	0.092	0.065	0.025	0.014	0.001	0.000	0.001	0.001
	63	0.195	0.157	0.121	0.091	0.064	0.025	0.014	0.001	0.001	0.001	0.001
	64	0.193	0.155	0.120	0.090	0.063	0.025	0.014	0.000	0.001	0.000	0.000
	65	0.192	0.154	0.119	0.089	0.062	0.024	0.014	0.000	0.001	0.000	0.000
	66	0.190	0.152	0.118	0.088	0.062	0.024	0.013	0.000	0.000	0.000	0.000
	67	0.188	0.151	0.117	0.088	0.061	0.024	0.013	0.001	0.000	0.000	0.000
	68	0.187	0.150	0.116	0.087	0.061	0.024	0.014	0.001	0.000	0.000	0.000
	69	0.185	0.148	0.115	0.086	0.060	0.023	0.014	0.001	0.001	0.000	0.000
	70	0.184	0.147	0.114	0.085	0.060	0.023	0.014	0.001	0.001	0.000	0.000
	71	0.182	0.146	0.113	0.085	0.059	0.023	0.014	0.001	0.000	0.000	0.000
	72	0.181	0.145	0.113	0.084	0.059	0.023	0.014	0.000	0.000	0.000	0.000
	73	0.180	0.144	0.112	0.084	0.058	0.023	0.014	0.000	0.000	0.000	0.000
	74	0.179	0.143	0.111	0.083	0.058	0.022	0.013	0.001	0.001	0.000	0.000
	75	0.178	0.142	0.111	0.083	0.058	0.022	0.013	0.001	0.001	0.000	0.000
	76	0.177	0.142	0.110	0.082	0.057	0.022	0.013	0.001	0.000	0.000	0.000
	77	0.176	0.141	0.109	0.082	0.057	0.022	0.013	0.001	0.000	0.000	0.000
	78	0.175	0.140	0.109	0.081	0.057	0.022	0.013	0.001	0.000	0.000	0.000
	79	0.174	0.139	0.108	0.081	0.057	0.021	0.012	0.000	0.000	0.000	0.000
	80	0.173	0.139	0.108	0.080	0.056	0.021	0.012	0.000	0.001	0.000	0.000
	81	0.172	0.138	0.107	0.080	0.056	0.021	0.012	0.000	0.000	0.000	0.000
	82	0.172	0.138	0.107	0.080	0.056	0.021	0.012	0.001	0.000	0.000	0.000
	83	0.171	0.137	0.107	0.079	0.056	0.021	0.012	0.001	0.000	0.000	0.000
	84	0.170	0.137	0.106	0.079	0.056	0.021	0.012	0.001	0.000	0.000	0.000
	85	0.170	0.136	0.106	0.079	0.056	0.021	0.012	0.001	0.000	0.000	0.000
	86	0.169	0.136	0.106	0.078	0.055	0.021	0.013	0.001	0.000	0.000	0.000
	87	0.169	0.135	0.105	0.078	0.055	0.021	0.013	0.000	0.000	0.000	0.000
	88	0.169	0.135	0.105	0.078	0.055	0.021	0.013	0.000	0.000	0.000	0.000
	89	0.168	0.135	0.105	0.078	0.055	0.021	0.013	0.000	0.000	0.000	0.000
	90	0.168	0.135	0.105	0.078	0.055	0.021	0.012	0.001	0.000	0.000	0.000
	91	0.168	0.134	0.105	0.078	0.055	0.021	0.012	0.001	0.000	0.000	0.000
	92	0.168	0.134	0.105	0.077	0.055	0.021	0.012	0.001	0.000	0.000	0.000
	93	0.167	0.134	0.105	0.077	0.055	0.021	0.012	0.001	0.000	0.000	0.000
	94	0.167	0.134	0.104	0.077	0.055	0.021	0.012	0.001	0.000	0.000	0.000
	95	0.167	0.134	0.104	0.077	0.055	0.021	0.012	0.000	0.000	0.000	0.000
π	96	0.167	0.134	0.104	0.077	0.055	0.021	0.012	0.000	0.000	0.000	0.000

TABLE 12

ONE SIDED GAIN OF THE SUM OVER ALL FREQUENCIES OF (5.5.3)

$\nu = 0,1,2,6,12,30,48,60,72,84,96,\ \beta = .96$

$\tau \backslash \nu$	0	1	2	6	12	30	48	60	72	84	96
0	0.2394	0.1918	0.1479	0.0043	0.1543	0.0140	0.0507	0.0387	0.0314	0.0269	0.0241
1	0.2420	0.1944	0.1504	0.0205	0.1490	0.0272	0.0114	0.0177	0.0249	0.0265	0.0248
2	0.2504	0.2027	0.1585	0.0421	0.1356	0.0225	0.0542	0.0254	0.0114	0.0255	0.0268
3	0.2664	0.2186	0.1738	0.0679	0.1236	0.0198	0.0150	0.0468	0.0242	0.0251	0.0311
4	0.2943	0.2462	0.2004	0.1024	0.1347	0.0601	0.0691	0.0205	0.0472	0.0282	0.0394
5	0.3446	0.2954	0.2478	0.1549	0.1932	0.0573	0.0371	0.0593	0.0650	0.0429	0.0573
6	0.4467	0.3942	0.3429	0.2518	0.3214	0.0662	0.1383	0.1288	0.0904	0.0905	0.1049
7	0.7075	0.6439	0.5837	0.4908	0.6114	0.3942	0.2714	0.2499	0.2667	0.2866	0.2970
8	1.2156	1.1347	1.0735	1.0153	1.1369	1.0139	1.0418	1.0309	1.0242	1.0201	1.0176
9	0.6144	0.5773	0.5667	0.5937	0.5399	0.4400	0.2698	0.2617	0.2777	0.2930	0.3000
10	0.3602	0.3307	0.3309	0.3765	0.2423	0.0777	0.1396	0.1220	0.0960	0.1024	0.1116
11	0.2864	0.2520	0.2500	0.2995	0.1127	0.0918	0.0345	0.0774	0.0680	0.0591	0.0692
12	0.2790	0.2379	0.2292	0.2765	0.0835	0.1387	0.0871	0.0373	0.0676	0.0489	0.0603
13	0.3157	0.2686	0.2518	0.2908	0.1570	0.0821	0.0399	0.0740	0.0739	0.0577	0.0702
14	0.4142	0.3612	0.3363	0.3578	0.2966	0.0875	0.1443	0.1324	0.0985	0.1011	0.1135
15	0.6795	0.6165	0.5838	0.5619	0.5932	0.4381	0.2758	0.2588	0.2753	0.2938	0.3031
16	1.2182	1.1518	1.1283	0.9874	1.1406	1.0065	1.0463	1.0354	1.0287	1.0246	1.0221
17	0.6364	0.6198	0.6330	0.5144	0.5594	0.4189	0.2741	0.2632	0.2794	0.2957	0.3034
18	0.3758	0.3679	0.3907	0.3043	0.2595	0.0842	0.1429	0.1271	0.0988	0.1041	0.1142
19	0.2929	0.2803	0.3055	0.2357	0.1228	0.0685	0.0373	0.0783	0.0714	0.0606	0.0714
20	0.2778	0.2563	0.2801	0.2202	0.0780	0.1186	0.0886	0.0389	0.0693	0.0506	0.0620
21	0.3089	0.2777	0.2973	0.2420	0.1473	0.0790	0.0399	0.0762	0.0743	0.0595	0.0716
22	0.4045	0.3637	0.3767	0.3177	0.2880	0.0729	0.1448	0.1320	0.0995	0.1028	0.1147
23	0.6699	0.6183	0.6222	0.5383	0.5862	0.4193	0.2762	0.2607	0.2772	0.2951	0.3040
24	1.2187	1.1719	1.1657	1.0192	1.1413	1.0190	1.0471	1.0362	1.0295	1.0254	1.0229
25	0.6450	0.6460	0.6478	0.5651	0.5666	0.4308	0.2752	0.2631	0.2794	0.2961	0.3041
26	0.3825	0.3928	0.4029	0.3482	0.2666	0.0750	0.1439	0.1288	0.0995	0.1043	0.1149
27	0.2962	0.3037	0.3201	0.2738	0.1279	0.0859	0.0383	0.0783	0.0725	0.0609	0.0719
28	0.2774	0.2772	0.2980	0.2530	0.0765	0.1312	0.0890	0.0393	0.0697	0.0511	0.0625
29	0.3056	0.2952	0.3183	0.2693	0.1424	0.0765	0.0398	0.0770	0.0742	0.0602	0.0720
30	0.3993	0.3778	0.3994	0.3381	0.2832	0.0863	0.1449	0.1316	0.0998	0.1035	0.1150
31	0.6644	0.6313	0.6435	0.5456	0.5821	0.4323	0.2763	0.2615	0.2779	0.2956	0.3043
32	1.2189	1.1914	1.1658	0.9881	1.1415	1.0075	1.0474	1.0365	1.0298	1.0257	1.0232
33	0.6501	0.6616	0.6279	0.5298	0.5708	0.4259	0.2756	0.2629	0.2792	0.2961	0.3044
34	0.3867	0.4080	0.3826	0.3203	0.2708	0.0852	0.1443	0.1297	0.0998	0.1043	0.1151
35	0.2984	0.3200	0.3023	0.2511	0.1313	0.0721	0.0388	0.0782	0.0731	0.0609	0.0721
36	0.2773	0.2943	0.2834	0.2347	0.0759	0.1245	0.0892	0.0394	0.0699	0.0512	0.0627
37	0.3034	0.3125	0.3068	0.2555	0.1392	0.0819	0.0395	0.0775	0.0740	0.0605	0.0721
38	0.3957	0.3945	0.3902	0.3301	0.2798	0.0737	0.1448	0.1311	0.0999	0.1038	0.1152
39	0.6605	0.6476	0.6332	0.5488	0.5791	0.4240	0.2762	0.2620	0.2784	0.2959	0.3045
40	1.2190	1.2055	1.1427	1.0195	1.1416	1.0194	1.0475	1.0366	1.0299	1.0258	1.0233
41	0.6539	0.6664	0.6063	0.5560	0.5738	0.4271	0.2759	0.2626	0.2790	0.2961	0.3045
42	0.3899	0.4128	0.3617	0.3383	0.2740	0.0743	0.1445	0.1303	0.0999	0.1042	0.1152
43	0.3001	0.3270	0.2805	0.2639	0.1340	0.0838	0.0391	0.0780	0.0735	0.0608	0.0722
44	0.2772	0.3038	0.2608	0.2433	0.0757	0.1278	0.0892	0.0395	0.0700	0.0513	0.0627
45	0.3017	0.3241	0.2837	0.2598	0.1365	0.0742	0.0393	0.0778	0.0737	0.0607	0.0722
46	0.3927	0.4075	0.3672	0.3290	0.2769	0.0857	0.1447	0.1307	0.0999	0.1041	0.1152
47	0.6572	0.6608	0.6104	0.5377	0.5764	0.4292	0.2761	0.2623	0.2787	0.2960	0.3045
48	1.2190	1.2106	1.1291	0.9882	1.1417	1.0077	1.0475	1.0366	1.0299	1.0258	1.0233

VI A RE-APPRAISAL OF THE METHODS ADOPTED

6.1 Introduction

In §4.1 a brief excursion was made to extend the seasonal model used for estimation purposes to incorporate phase modulation. In this chapter there are two aims. The first is to reconsider the model generating the α_j, β_j and further to reconsider the estimating procedure for these constants with a view to improving these estimates. The source of this improvement will be additional information on the parameters of the generating model for the seasonal, ρ_j and σ_j^2, and also an estimate of the spectral power of the non-seasonal frequencies. The second task is much dependent on the results of the first and is an attempt to give some indication of the accuracy of the seasonal estimates. The use of cross-spectral techniques for summarizing the efficiency of the seasonal extraction techniques is also considered.

6.2 Possible Generalization of the Seasonal Model

In §4.2 the seasonal generation model was extended sufficiently to consider models which included both amplitude and phase modulation. It is necessary to consider and then dismiss a further generalization of the model which was introduced in §4.2. In §4.2 a Markov relation is given for the complex variable $\zeta_j(n)$ and the parameter ρ_j in this relation is a real constant. An obvious extension is to consider the relation

$$\zeta_j(n) = \mu_j\zeta_j(n-1)+\psi_j(n) \tag{6.2.1}$$

where $\mu_j = \rho_j - i\tau_j$, ρ_j and τ_j are both real parameters, and $|\mu_j| < 1$. The seasonal, using the definition given in (5.1.1) is therefore given by

$$s_j(n) = \zeta_j(n)e^{in\lambda_j}+\bar{\zeta}_j(n)e^{-in\lambda_j} \tag{6.2.2}$$

and for $s_j(n)$ to be stationary it is required that

$$\mathcal{R}\left(\mathcal{E}(\zeta_j(n)\zeta_j(n-m)e^{i(2n-m)\lambda_j}\right) = \phi(m), \tag{6.2.3}$$

that is the expression described in (6.2.3) only depends on m. Using (6.2.1) it is straight forward to establish that the lag covariance or order m of $\zeta(.)$ is given by

$$\mathcal{E}\left(\zeta_j(n)\zeta_j(n-m)\right) = \frac{\mathcal{E}(\psi_j^2(m))\mu_j^m}{1-\mu_j^2} \tag{6.2.4}$$

so that the requirement of (6.2.3) implies that $\mathcal{E}(\psi_j^2(m)) = 0$. Two direct consequencies of this latter equality are that the variance of the residuals $\epsilon_j(n)$ and $\eta_j(n)$ must be equal and have zero covariance, i.e.

$$\left.\begin{aligned} \mathcal{E}\left(\epsilon_j^2(n)\right) = \mathcal{E}\left(\eta_j^2(n)\right) = \sigma^2 \\ \mathcal{E}\left(\epsilon_j(n)\eta_j(n)\right) = 0 \end{aligned}\right\} \tag{6.2.5}$$

and also that $\mathcal{E}(\zeta_j(n)\zeta_j(n-m))$ is equal to zero for all m. A further implication is derived from the special case m = 0 of the latter equality; since $\mathcal{E}(\zeta_j^2(n)) = 0$ equating of real and complex parts provide the following restrictions,

$$\begin{aligned} \mathcal{E}\left(\alpha_j^2(n)\right) = \mathcal{E}\left(\beta_j^2(n)\right) \\ \mathcal{E}\left(\alpha_j(n)\beta_j(n)\right) = 0. \end{aligned} \tag{6.2.6}$$

The model presented in (6.2.1) may be more informatively presented. By equating real and complex parts the model is shown to be a bivariate autoregressive process in $\alpha_j(n)$ and $\beta_j(n)$ of the following form,

$$\begin{pmatrix} \alpha_j(n) \\ \beta_j(n) \end{pmatrix} = \begin{pmatrix} \rho_j & \tau_j \\ -\tau_j & \rho_j \end{pmatrix} \begin{pmatrix} \alpha_j(n-1) \\ \beta_j(n-1) \end{pmatrix} + \begin{pmatrix} \epsilon_j(n) \\ \eta_j(n) \end{pmatrix}. \tag{6.2.7}$$

The complex Markov process, (6.2.1), may be solved in terms of current and lagged $\psi_j(n)$ to give the following expression for $\zeta_j(n)$,

$$\zeta_j(n) = \sum_{k=0}^{\infty} \mu_j^k \psi_j(n-k). \tag{6.2.8}$$

Formula (6.2.8) is conjugated and the expression

$$\mathcal{E}\left(\zeta_j(n)\bar{\zeta}_j(n)\right) = \mathcal{E}\left(\alpha_j^2(n)\right) + \mathcal{E}\left(\beta_j^2(n)\right) = \frac{2\sigma^2}{1-|\mu_j|^2} \qquad (6.2.9)$$

is simply derived. An analogous procedure allows the derivation of the recursion

$$\mathcal{E}\left(\zeta_j(n)\bar{\zeta}_j(n-1)\right) = \mu_j \mathcal{E}\left(\zeta_j(n)\bar{\zeta}_j(n)\right) = \frac{(\rho_j - i\tau_j)2\sigma_j^2}{1-(\rho_j^2+\tau_j^2)} = \gamma_j e^{-i\theta} \qquad (6.2.10)$$

and this suggests the definition of a new variable $\Lambda_j(n)$, which is related to $\zeta_j(n)$ by the expression

$$\Lambda_j(n) = \zeta_j(n)e^{in\theta} \qquad (6.2.11)$$

and which has a lag covariance given by

$$\begin{aligned} \mathcal{E}\left(\Lambda_j(n)\bar{\Lambda}_j(n-1)\right) &= \mathcal{E}\left(\zeta_j(n)e^{in\theta}\bar{\zeta}_j(n-1)e^{-in\theta}\right) \\ &= \gamma_j . \end{aligned} \qquad (6.2.12)$$

The expression for $s_j(n)$, (6.2.2), when rewritten in terms of $\Lambda_j(n)$ thus becomes

$$\begin{aligned} s_j(n) &= \Lambda_j(n)e^{-in\theta}e^{in\lambda_j} + \bar{\Lambda}_j(n)e^{in\theta}e^{-in\lambda_j} \\ &= \Lambda_j(n)e^{+in(\lambda_j-\theta)} + \bar{\Lambda}_j(n)e^{-in(\lambda_j-\theta)} \end{aligned} \qquad (6.2.13)$$

and it is therefore apparent that a model of greater generality (see (6.2.1) or (6.2.7)) will result in frequency modulation. As it seems unwise on <u>a priori</u> grounds to entertain a scheme which allows the local peak to be at other than the seasonal frequency this more general model will not be pursued.

6.3 Information on the Model Parameters from Seasonal Estimates

This discussion is based on the estimates of seasonal introduced in Chapter V and in particular the expression for $s_j(n)$ in terms of $\xi_j(n)$ and $\bar{\xi}_j(n)$ given in (4.2.3) and (4.2.4). An estimate of $\xi_j(n)$ is presented in (5.2.2) and if the estimate is based only on past observations it was defined in (5.3.1) as

$$\begin{aligned}\hat{\xi}_j^{(o)}(n) &= \Sigma_{m\,o}^{\infty}(1-\beta_j)\beta_j^m e^{im\lambda_j} y(n-m)\\ &= u_j(n)+iv_j(n).\end{aligned} \tag{6.3.1}$$

Now consider the expression

$$\Delta\left\{\hat{\xi}_j^{(o)}(n)e^{-in\lambda_j}\right\} \tag{6.3.2}$$

where Δ is the first differencing operator.

The estimate in (6.3.1) is not much affected by $s_k(n)$, $k \neq j$, particularly if the filter proposed in (4.4.4) is employed to obtain the filtered series. In any case since the response of the filter with output $\hat{\xi}_j^{(o)}(n)$ is $\dfrac{1-\beta_j}{1-\beta_j e^{i(\lambda+\lambda_j)}}$ and is confined to a fairly narrow band around $-\lambda_j$, if β_j is near to unity, then whether or not the filter suggested in (4.4.4) is used $\hat{\xi}_j^{(o)}(n)$ will be mainly affected by $\xi_j(n)$ and not by $\xi_k(n)$. Thus $\hat{\xi}_j^{(o)}(n)$ may be regarded as having been obtained from an input $\xi_j(n)+x_j(n)$ where $x_j(n)$ has a constant spectrum $f_x(\lambda_j)$. The quantity put forward for consideration in (6.3.2) then has a spectrum, relocated at the origin of frequencies, given by

$$\frac{2(1-\beta_j)^2}{2\pi(1+\beta_j^2-2\beta_j\cos\lambda)}\left\{\frac{\sigma_j^2}{4}+2\pi f_x(\lambda_j)(1-\cos\lambda)\right\}. \qquad (6.3.3)$$[30]

If the statistic,

$$s_j^2 = \frac{1}{N}\Sigma|\Delta\left\{\hat{\xi}_j^{(o)}(n)e^{-in\lambda_j}\right\}|^2, \qquad (6.3.4)$$

is formed this estimates the variance obtained by integration from (6.3.3), which is

$$\frac{2(1-\beta_j)^2}{(1-\beta_j^2)}\left\{\frac{\sigma_j^2}{4}+2\pi f_x(\lambda_j)(1-\beta_j)\right\}. \qquad (6.3.5)$$

The computations defined in (6.3.4) are carried out most conveniently by using the quantities $u_j(n)$ and $v_j(n)$ defined in (4.5.4) to obtain expressions for $\hat{\alpha}_j^{(o)}(n)$ and $\hat{\beta}_j^{(o)}(n)$. From the definition (4.2.4) and the estimator (5.3.1) the following equivalences

$$\hat{\xi}_j^{(o)}(n)e^{-in\lambda_j} = \tfrac{1}{2}\left(\hat{\alpha}_j^{(o)}(n)-i\hat{\beta}_j^{(o)}(n)\right)$$

and

$$\tfrac{1}{2}\left(\hat{\alpha}_j^{(o)}(n)-i\hat{\beta}_j^{(o)}(n)\right) = \left(u_j(n)+iv_j(n)\right)e^{-in\lambda_j}$$

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30 It should be noted that the spectrum of $\hat{\xi}_j^{(o)}(n)$, relocated at the origin, is

$$\frac{\sigma_j^2}{2}\frac{1}{2\pi}\frac{1}{(1+\rho_j^2-2\rho_j\cos\lambda)}\frac{(1-\beta_j)^2}{|1-\beta_je^{i\lambda}|^2}\frac{f_x(\lambda_j)(1-\beta_j)^2}{|1-\beta_je^{i\lambda}|^2},$$

and that $f_x(\lambda_j)$ is the spectrum of the noise at all λ_j, $j = \pm1,\pm2,\ldots,\pm5,6$. This requires a slight modification of the definition of α_j used for calculation of the optimal β_j (see (6.3.15)) to produce compatibility with the computations in Chapter V.

are derived.[31] By equating real and complex parts in the latter equivalence the expressions for $\hat{\alpha}_j^{(o)}(n)$ and $\hat{\beta}_j^{(o)}(n)$ as simple functions of already computed quantities are

$$\hat{\alpha}_j^{(o)}(n) = (2-\delta_j^6)\left(u_j(n)\cos n\lambda_j + v_j(n)\sin n\lambda_j\right)$$

$$\hat{\beta}_j^{(o)}(n) = (2-\delta_j^6)\left(u_j(n)\sin n\lambda_j - v_j(n)\cos n\lambda_j\right). \qquad (6.3.6)$$

For convenience the actual statistic computed is

$$\hat{s}_j^2 = \frac{1}{2}\left(\frac{1+\beta_j}{1-\beta_j}\right) s_j^2 \qquad (6.3.7)$$

which estimates $\sigma_j^2/4 + 2\pi f_x(\lambda_j)(1-\beta_j)$. Of course one must use values of β_j close to unity and indeed as β_j approaches unity one obtains an estimate of $\sigma_j^2/4$ alone, while the slope of a graph of $\hat{s}_j^2$ against different β_j values will estimate $-2\pi f_x(\lambda_j)$. Also by varying β_j one may obtain estimates of both σ_j^2 and $2\pi f_x(\lambda_j)$; however the results of adopting this procedure were unsatisfactory as for most estimates made with these β_j couplets varying between .95 and .99 the estimate of $\sigma_j^2/4$ was negative. It was therefore decided to make $\hat{\xi}_j^{(\nu)}(n)$, for larger ν, the basis of further efforts because there was a better chance of obtaining some meaningful estimates of $\sigma_j^2/4$ and $2\pi f_x(\lambda_j)$. The virtue of taking larger ν is that the response function which replaces the factor outside the bracket in (6.3.3) is more concentrated and will therefore better justify the assumptions on the nature of the input. The value of ν used is infinite, and although this is not computable it is clear from the tabulated examples (see Table 12) that there is little change in the estimates as ν increases above a moderate value, say 12.

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31 The differencing operation in (6.3.4) could more simply have been carried out on expressions which are simply functions of $\cos n\lambda_j$, $\sin n\lambda_j$, $u_j(n)$ and $v_j(n)$; however as the $\hat{\alpha}_j^{(o)}(n)$ and $\hat{\beta}_j^{(o)}(n)$ are needed for later investigations an expression for these quantities is developed.

To be ultra-cautious the calculations were based only on the estimates obtained from the intermediate segment (see §5.2). Naturally as β_j varies the number of observations included in the intermediate segment also varies, i.e. the value of M, introduced in §5.1 depends on β_j. If (5.2.3) is rewritten as

$$\hat{\xi}_j^{(\infty)}(n) = \left(\frac{1-\beta_j}{1+\beta_j}\right)\Sigma_{-\infty}^{\infty} y(n-m)\beta_j^{|m|} e^{im\lambda_j} \qquad n = M, M+1, \ldots, N-M \tag{6.3.8}$$

then one is led to form the quantity,

$$\Delta\left\{\hat{\xi}_j^{(\infty)}(n)e^{-in\lambda_j}\right\} \tag{6.3.9}$$

which has the spectrum (approximately),

$$2\left(\frac{1-\beta_j}{1+\beta_j}\right)\frac{(1-\beta_j^2)}{(1+\beta_j^2-2\beta_j\cos\lambda)^2}\,\frac{1}{2\pi}\left\{\frac{\sigma_j^2}{4} + 2\pi f_x(\lambda_j)(1-\cos\lambda)\right\} \tag{6.3.10}$$

integrating to

$$2\left(\frac{1-\beta_j}{1+\beta_j}\right)^2\left(\frac{1+\beta_j^2}{1-\beta_j^2}\right)\left\{\frac{\sigma_j^2}{4} + 2\pi f_x(\lambda_j)\frac{(1-\beta_j)^2}{(1+\beta_j^2)}\right\}. \tag{6.3.11}$$

Using the N-2M values of $\Delta\{\hat{\xi}_j^{(\infty)}(n)e^{-in\lambda_j}\}$ the statistic,

$$\tilde{s}_j^2 = \tfrac{1}{2}\left(\frac{1+\beta_j}{1-\beta_j}\right)^2\frac{(1-\beta_j^2)}{(1+\beta_j^2)}\,\frac{1}{N-2M}\,\Sigma\left|\Delta\left\{\hat{\xi}_j^{(\infty)}(n)e^{-in\lambda_j}\right\}\right|^2 \tag{6.3.12}$$

is constructed and estimates $\sigma_j^2/4 + 2\pi f_x(\lambda_j)\dfrac{(1-\beta_j)^2}{(1+\beta_j^2)}$. Again the simplest way to compute the expression to be differenced is to write it as

$$\hat{\xi}_j^{(\infty)}(n)e^{-in\lambda_j} = \hat{\alpha}_j^{(\infty)}(n) - i\hat{\beta}_j^{(\infty)}(n)$$
$$= \left[u_j'(n)+u_j''(n) - \frac{1-\beta_j}{1+\beta_j}y(n) + i(v_j'(n)-v_j''(n))\right]\left[\cos n\lambda_j - i\sin n\lambda_j\right] \tag{6.3.13}$$

(where this construction and the quantities involved are developed in (5.2.3) - (5.2.6)) and to use the final representation in (6.3.13). However as the $\hat{\alpha}_j^{(\infty)}(n)$, $\hat{\beta}_j^{(\infty)}(n)$ are needed for further investigation they are derived using (5.6.1) and the first representation in (6.3.13) is actually the basis of the computations of $\tilde{s}_j^2$. The values of $\tilde{s}_j^2$ are plotted against a number of large β_j (e.g. $\beta_j = .95, .96, \ldots, .99$). Given a marked change in the downward slope as β_j approaches unity this could suggest that the extraction procedure was omitting some signal and therefore implying that a β_j has been reached for which the model is no longer appropriate.

The graph of $\tilde{s}_j^2$ against β_j, shown in Fig. XV, does not provide a completely satisfactory clue as to an appropriate value of β_j although it does seem that $\beta_j = .99$ is too severe except possibly for j = 5 and 6. In Table 13 all possible couplets of β_j, for $\beta_j = .95, .96, .97, .98, .99$, are presented together with the implied estimate of $\sigma_j^2/2$ and $2\pi f_x(\lambda_j)$ resulting from the estimated $\tilde{s}_j^2$ for the series Bank Advances. The table also included for each couplet an estimate of the optimal β_j assuming $\rho_j \equiv 1$ (see (6.3.14) below). This estimate is a direct application of the procedure suggested by Whittle and Hannan (see [57], [25]) which shows that the optimal value of the coefficient β_j in a signal extraction problem similar to that proposed for the seasonal is given by

$$\beta_j = \frac{1+\theta_j(1+\rho_j^2)-\Delta_j}{2\theta_j\rho_j} \leqq \rho_j \leqq 1 \qquad (6.3.14)$$

where

$$\Delta_j = \sqrt{1+2\theta_j(1+\rho_j^2)+\theta_j^2(1-\rho_j^2)^2}$$

$$\theta_j = \left(2\pi f_x(\lambda_j)\right)/(\sigma_j^2/2) \qquad j = \pm1,\pm2,\ldots,\pm5 \qquad (6.3.15)$$

$$= \pi f_x(\lambda_6)/\sigma_6^2 \qquad j = 6.$$

FIG. XV

GRAPH OF $\hat{S}_j^2$ AGAINST β_j, $j = 1, 2, \ldots, 6$

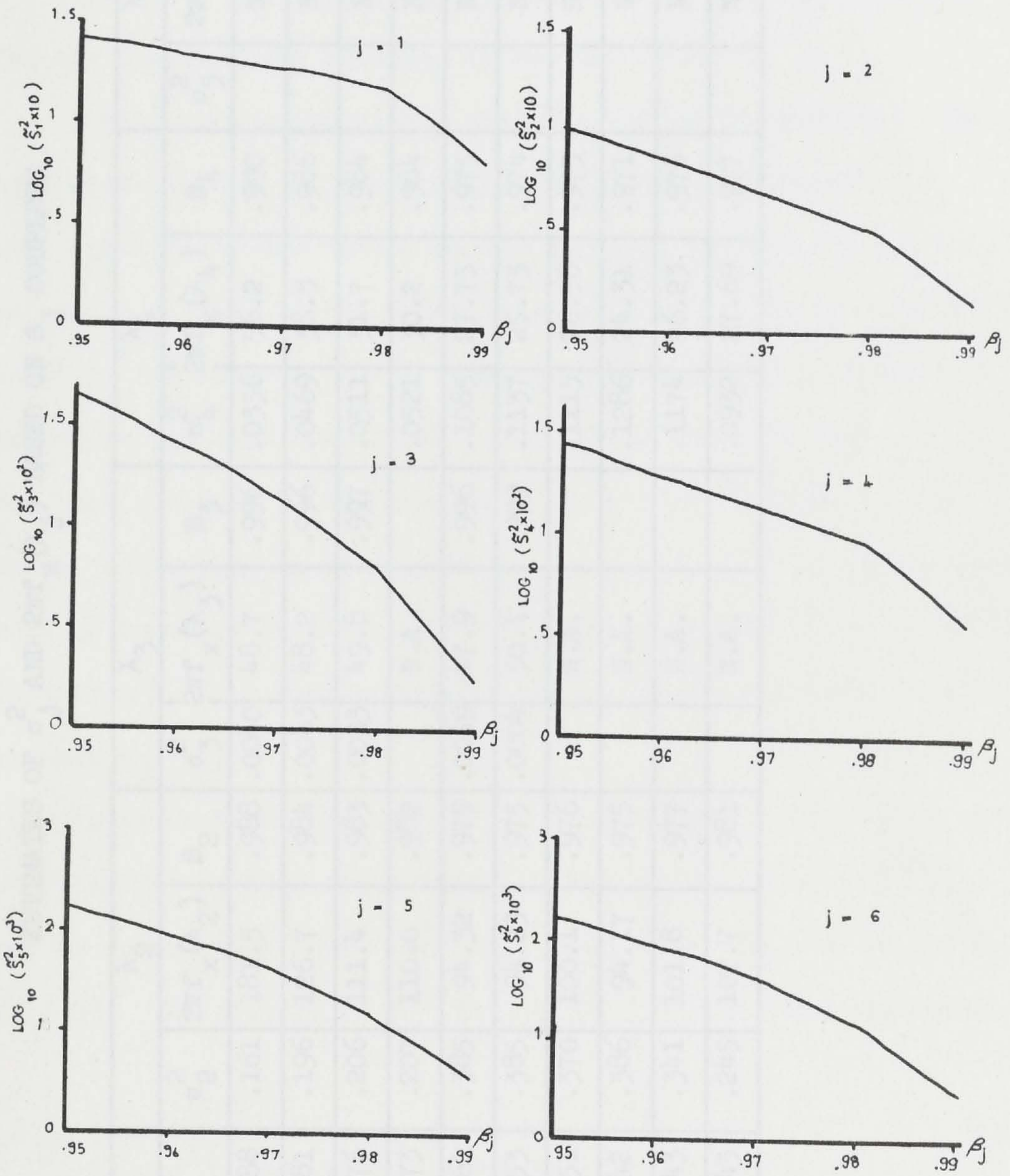


TABLE 13

ESTIMATES OF σ_j^2 AND $2\pi f_x(\lambda_j)$ BASED ON β_j COUPLETS

		λ_1			λ_2			λ_3			λ_4			λ_5			λ_6		
β_j	β_k	σ_1^2	$2\pi f_x(\lambda_1)$	β_1	σ_2^2	$2\pi f_x(\lambda_2)$	β_2	σ_3^2	$2\pi f_x(\lambda_3)$	β_3	σ_4^2	$2\pi f_x(\lambda_4)$	β_4	σ_5^2	$2\pi f_x(\lambda_5)$	β_5	σ_6^2	$2\pi f_x(\lambda_6)$	β_6
.99	.98	.798	859.9	.988	.161	181.5	.988	.0040	48.7	.996	.0356	56.2	.990		N.A.			N.A.	
.99	.97	1.05	470.7	.981	.196	126.7	.984	.0043	48.2	.996	.0469	38.3	.986		N.A.			N.A.	
.99	.96	1.15	312.1	.976	.206	111.4	.983	.0033	49.8	.997	.0511	31.7	.984		N.A.			N.A.	
.99	.95	1.19	245.9	.973	.207	110.0	.982		N.A.		.0521	30.2	.984		N.A.			N.A.	
.98	.97	2.39	240.5	.961	.385	94.32	.975	.0059	47.9	.996	.1085	27.75	.975		N.A.			N.A.	
.98	.96	2.55	178.2	.953	.385	94.23	.975	.0004	50.1	.999	.1137	25.73	.974		N.A.			N.A.	
.98	.95	2.59	161.0	.951	.370	100.1	.976		N.A.		.1115	26.58	.975		N.A.		.2536	3.412	.897
.97	.96	3.01	134.4	.942	.386	94.17	.975		N.A.		.1286	24.31	.971		N.A.			N.A.	
.97	.95	2.99	136.7	.943	.341	101.8	.977		N.A.		.1174	26.23	.974		N.A.			N.A.	
.96	.95	2.96	138.4	.943	.245	107.7	.981		N.A.		.0932	27.69	.977		N.A.			N.A.	

Although several couplets are rejected as inadmissible, because they produce negative variances, some guidance is obtained on the possible magnitude of $\sigma_j^2/2$ and $2\pi f_x(\lambda_j)$. It seems likely that one should place most confidence in those couplets involving high values of β_j for then the assumptions as to the nature of the input to the filter will be closer to correct. A choice along these lines is made (see Table 14) and the values of $\sigma_j^2/2$, $j = 1,2,\ldots,6$, are used to approximate the prediction variance for the intermediate seasonal estimates.

Another point suggests a slightly different approach. The assumption that $\rho_j \equiv 1$ is unlikely to be exactly correct. A more appropriate assumption may be that $\alpha_j(n)$ and $\beta_j(n)$ are still generated by (4.2.2) but not with ρ_j equal to unity as stipulated in §4.5, and as used in the computations discussed in Chapter V. Indeed as was mentioned in §4.2 the reason for choosing $\rho_j \equiv 1$ was that although it was unknown it must nevertheless, in the model considered, be very close to unity. If $\rho_j < 1$, the approach must be somewhat amended and this relaxation results, as ρ_j differs more from 1, in a smaller concentration of spectral mass in $s(n)$ at each λ_j.

Consider first the case of $\nu = 0$. On the same basis as was used earlier in this section the spectral properties of $\hat{\xi}_j^{(o)}(n)$ may be found, with $\xi_j(n)$ now being given by a relation of the form (4.2.2). The spectra of $\hat{\alpha}_j^{(o)}(n)$ and $\hat{\beta}_j^{(o)}(n)$ are then approximately

$$\frac{\sigma_j^2}{2}\,\frac{2}{2\pi}\,\frac{(1-\beta_j)^2}{|1-\rho_j e^{i\lambda}|^2\,|1-\beta_j e^{i\lambda}|^2} + \frac{f_x(\lambda_j)(1-\beta_j)^2}{|1-\beta_j e^{i\lambda}|^2} \qquad (6.3.16)$$

and these spectra are approximately incoherent, that is all lag correlations vanish. The spectrum in (6.3.16) is that of a mixed autoregressive-moving average process, second order autoregression, first order moving average (A.R.-M.A. 2:1), which may be more revealingly represented as

TABLE 14

PREDICTION VARIANCES USING σ_j^2 ESTIMATES FROM TABLE 13 AND $\beta_j = .96$

Freq.	Max. σ_j^2 From Table 13		Min. σ_j^2 From Table 13		Best Guess σ_j^2 From Table 13	
	$var(\hat{s}_j^{(o)}(n))$	$var(\hat{s}_j^{(\infty)}(n))$	$var(\hat{s}_j^{(o)}(n))$	$var(\hat{s}_j^{(\infty)}(n))$	$var(\hat{s}_j^{(o)}(n))$	$var(\hat{s}_j^{(\infty)}(n))$
λ_1	72.24	36.86	19.15	9.77	57.36	29.27
λ_2	9.26	4.73	3.86	1.97	9.24	4.71
λ_3	.142	.072	.010	.005	.142	.072
λ_4	3.09	1.58	.854	.436	2.60	1.33
λ_5	N.A.*	N.A.	N.A.	N.A.	N.A.	N.A.
λ_6	-	-	-	-	3.04*	1.55

* One σ_6^2 estimate only is given in Table 13 and N.A. indicates there are no estimates of σ_5^2 in Table 13.

$$\frac{1}{2\pi}\frac{\kappa_j^2|1-\tau_j e^{i\lambda}|^2}{|1-\rho_j e^{i\lambda}|^2|1-\beta_j e^{i\lambda}|^2} \qquad (6.3.17)$$

where the relation between the parameters in (6.3.16) and (6.3.17) is given by

$$\kappa_j^2\tau_j = 2\pi f_x(\lambda_j)\rho_j(1-\beta_j)^2$$

$$\text{i.e.}\quad 2\pi f_x(\lambda_j) = \kappa_j^2\frac{\tau_j}{\rho_j}\frac{1}{(1-\beta_j)^2}$$

(6.3.18)

and

$$\kappa_j^2(1+\tau_j^2) = 2(\frac{\sigma_j^2}{2})(1-\beta_j)^2+2\pi f_x(\lambda_j)(1-\beta_j)^2(1+\rho_j^2)$$

$$\text{i.e.}\quad 2(\frac{\sigma_j^2}{2})(1-\beta_j)^2 = \kappa_j^2(1+\tau_j^2)-\kappa_j^2\frac{\tau_j}{\rho_j}(1+\rho_j^2)$$

$$= \frac{\kappa_j^2}{\rho_j}(\rho_j-\tau_j)(1-\rho_j\tau_j).$$

The unknown parameters in (6.3.17), i.e. ρ_j, κ_j^2 and τ_j may be estimated by the methods proposed by Box and Jenkins [6] and [7] or alternatively by methods suggested by Durbin [12] or Hannan [27]. Preliminary estimates of ρ_j, τ_j and κ_j^2 based on $\hat{\alpha}_j^{(o)}(n)$ and $\hat{\beta}_j^{(o)}(n)$ were attempted using the latter methods. As can be seen from (6.3.18) if one accepts on a priori grounds that ρ_j must be positive then for acceptable estimates of σ_j^2 and $f_x(\lambda_j)$ τ_j must also be positive and the inequality $\rho_j > \tau_j$ holds. The failure of all estimates but those for λ_1 to meet these restrictions stimulated the following approach. Since the estimates of $\hat{\alpha}_j^{(o)}(n)$ and $\hat{\beta}_j^{(o)}(n)$ have been formed using a particular chosen value of β_j the parameter estimation may be simplified somewhat if weighted partial sums of the $\hat{\alpha}_j^{(o)}(n)$ and $\hat{\beta}_j^{(o)}(n)$ are formed as follows

$$S\hat{\alpha}_j^{(o)}(n) = \Sigma_{k_o}^{n-M}\beta_j^k\hat{\alpha}_j^{(o)}(n-k)$$

$$S\hat{\beta}_j^{(o)}(n) = \Sigma_{k_o}^{n-M}\beta_j^k\hat{\beta}_j^{(o)}(n-k) \qquad n = M,\ldots,N.$$

Of course if the $\hat{\alpha}_j^{(o)}(n)$ and $\hat{\beta}_j^{(o)}(n)$ were available for the whole period $n = 1,\ldots,N$ then the upper summation limit is $(n-1)$. The spectra of the weighted partial sum sequences of $\hat{\alpha}_j^{(o)}(n)$ and $\hat{\beta}_j^{(o)}(n)$ are of the form of a mixed first order autoregression - first order moving average process (A.R.-M.A. (1:1)) and the estimates of ρ_j, τ_j and κ_j thus obtained may be the basis of the estimates of $\sigma_j^2/2$ and $2\pi f_x(\lambda_j)$.

The chosen value of β_j could also be used in the recursive representation of $\epsilon(n)$,

$$\epsilon(n) = \hat{\alpha}_j^{(o)}(n)-(\beta_j+\rho_j)\hat{\alpha}_j^{(o)}(n-1)+\beta_j\rho_j\hat{\alpha}_j^{(o)}(n-2)+\tau_j\epsilon(n-1)$$

to compute a sequence of $\epsilon(n)$ for each admissible value of ρ_j and τ_j. The sum of squares for each grid point in the ρ_j, τ_j plane is then scanned for the minimal sum of squares given the restrictions on ρ_j and τ_j and the associated estimate of ρ_j, τ_j. It was quite clear from investigating the grid in the ρ_j, τ_j plane that the model was unsatisfactory for frequencies other than λ_1. The data does not support the restrictions and values for ρ_j and τ_j are insignificant. In fact even for λ_1 the estimates of ρ_1, τ_1 are not a satisfactory support for a priori ideas, on the value of ρ_1 in particular, as can be seen from Table 15.

There will of course be two estimates of each of these constants - one from $\hat{\alpha}_1^{(o)}(n)$ and one from $\hat{\beta}_1^{(o)}(n)$ - and these estimates may be averaged to produce better estimates.

TABLE 15

ESTIMATES OF ρ_1, τ_1, κ_1 BASED ON $\hat{\alpha}_1^{(o)}(n)$ $\hat{\beta}_1^{(o)}(n)$

	ρ_1	τ_1	κ_1^2
From $\hat{\alpha}_1^{(o)}(n)$	.56	.001	.92
From $\hat{\beta}_1^{(o)}(n)$	.60	.001	.66

Rather than proceed with the task of attempting to improve the estimates based on $\hat{\alpha}_1^{(o)}(n)$ and $\hat{\beta}_1^{(o)}(n)$ it was thought more effective to direct attention to the possibility of obtaining better estimates from $\hat{\alpha}_j^{(\infty)}(n)$, $\hat{\beta}_j^{(\infty)}(n)$. The narrower response function used in producing $\hat{\xi}_j^{(\infty)}(n)$ will make the specified signal plus noise model more appropriate and thus the estimates arising from these quantities more suitable. It is quite simple to consider estimates of $\alpha_j(n)$, $\beta_j(n)$ from the intermediate segment. The spectra of $\hat{\alpha}_j^{(\infty)}(n)$ and $\hat{\beta}_j^{(\infty)}(n)$ are approximately

$$\frac{\sigma_j^2}{2\pi}\frac{1}{|1-\rho_j e^{i\lambda}|^2}\frac{(1-\beta_j)^4}{\{|1-\beta_j e^{i\lambda}|^2\}^2}+\frac{2\pi f_x(\lambda_j)}{2\pi}\frac{(1-\beta_j)^4}{\{|1-\beta_j e^{i\lambda}|^2\}^2} \qquad (6.3.19)$$

which may be rewritten as

$$\frac{\sigma_j^2(1-\beta_j)^4+2\pi f_x(\lambda_j)(1-\beta_j)^4(1+\rho_j^2-2\rho_j\cos\lambda)}{2\pi\{|1-\beta_j e^{i\lambda}|^2\}^2\;|1-\rho_j e^{i\lambda}|^2}\;. \qquad (6.3.20)$$

The expression in (6.3.20) represents a mixed moving average autoregressive process, but it is (A.R.-M.A. (3:1)). However, using again the known value of β_j one forms weighted partial double summations as follows

$$DS\hat{\alpha}_j^{(\infty)}(n) = \Sigma_{k^1}^{n-T}(k+1)\beta_j^k\hat{\alpha}_j^{(\infty)}(n-k)$$

$$DS\hat{\beta}_j^{(\infty)}(n) = \Sigma_{k^1}^{n-T}(k+1)\beta_j^k\hat{\beta}_j^{(\infty)}(n-k)$$

where T may be set at some value < M (e.g. 20). It has already been noted that at say $\nu = 20$ the response of the extraction procedure is not markedly different from the intermediate response i.e. $\nu = \infty$, and so the (∞) notation here includes the estimates $\hat{\alpha}_j^{(\nu)}(n)$, $\hat{\beta}_j^{(\nu)}(n)$ using the maximum ν available for time points

$n = T,\ldots,M$ and $n = N-M,\ldots,N-T$. As before the upper summation limit would be $(n-1)$ if the $\hat{\alpha}_j^{(\infty)}(n)$, $\hat{\beta}_j^{(\infty)}(n)$ were available for $n = 1,\ldots,N$.

The most convenient procedure is to form a sequence of $\epsilon(n)$ for each ρ_j, τ_j couplet in the region $-1 < \rho_j, \tau_j < 1$ and to select the estimate of ρ_j, τ_j as the values which produce the minimum sum of squares, $\Sigma\epsilon^2(n)$. The minimum is however chosen subject to the restrictions on the ρ_j, τ_j plane which are derived from (6.3.21) below. If the optimal ρ_j, τ_j and κ_j^2 are to produce positive estimates of σ_j^2 and $2\pi f_x(\lambda_j)$ then as for the previous search when ρ_j is assumed positive a priori τ_j must also be positive and as well ρ_j must be greater than τ_j. As β_j is known the $\epsilon(n)$ are obtained recursively for each ρ_j, τ_j using

$$\epsilon(n) = \tau_j\epsilon(n-1)+\hat{\alpha}_j^{(\infty)}(n)-(\rho_j+2\beta_j)\hat{\alpha}_j^{(\infty)}(n-1)+(2\beta_j\rho_j+\beta_j^2)$$
$$\hat{\alpha}_j^{(\infty)}(n-2)-\beta_j^2\rho_j\hat{\alpha}_j^{(\infty)}(n-3).$$

Otherwise the estimation problem may again be reduced to that of estimating an (A.R.-M.A. (1:1)) process by the above partial sum procedures. The equivalences between the parameter estimates obtained from the above (A.R.-M.A. (3:1)) process and $\sigma_j^2/2$, $2\pi f_x(\lambda_j)$ are

$$2\pi f_x(\lambda_j) = \frac{\tau_j}{\rho_j}\frac{\kappa_j^2}{(1-\beta_j)^4}$$

and

$$\sigma_j^2 = \left\{\kappa_j^2(1+\tau_j^2)\right\} / (1-\beta_j)^4 - (1+\rho_j^2)2\pi f_x(\lambda_j) \qquad (6.3.21)$$

$$= \frac{\kappa_j^2}{\rho_j(1-\beta_j)^4}(\rho_j-\tau_j)(1-\rho_j\tau_j)$$

and these equivalences are used to turn the parameter estimates for ρ_1, τ_1 and κ_1^2 in Table 16 below into the estimates of σ_1^2 and $2\pi f_x(\lambda_1)$ in Table 17.

TABLE 16

ESTIMATES OF ρ_1, τ_1 AND κ_1^2 BASED ON $\hat{\alpha}_1^{(\infty)}(n)$, $\hat{\beta}_1^{(\infty)}(n)$

	ρ_1	τ_1	κ_1^2
From $\hat{\alpha}_1^{(\infty)}(n)$	.80	.01	.048
From $\hat{\beta}_1^{(\infty)}(n)$	.66	.01	.006

Basing the estimates on $\hat{\alpha}_j^{(\infty)}(n)$ and $\hat{\beta}_j^{(\infty)}(n)$ has somewhat improved the estimates of ρ_j for frequency λ_1, but has also confirmed that the data does not support the restrictions for the other frequencies. It is apparent that the model is useful only for the first seasonal frequency, which from Fig. XX can be seen to provide the major portion of seasonal power. In fact with the benefit of hindsight it is clear that it would be very difficult to obtain the parameters for the signal plus noise model in the latter seasonal frequencies for Bank Advances as these frequencies have so little power.

The estimates of σ_1^2 and $2\pi f_x(\lambda_1)$ based on ρ_1, τ_1 and κ_1^2 from Table 16 are presented below and could be used (see however §6.7) to guess at a possible choice of the optimal β_1.

TABLE 17

ESTIMATES OF σ_1^2 AND $2\pi f_x(\lambda_1)$ USING $\hat{\rho}_1$, $\hat{\tau}_1$ AND $\hat{\kappa}_1^2$ FROM TABLE 16

σ_1^2	$2\pi f_x(\lambda_1)$
18000 1500	284 35

To obtain the prediction variance $\mathcal{E}\{s_j(n)-\hat{s}_j^{(\nu)}(n)\}^2$ when $s_j(n)$ or rather the components $\xi_j(n)$ and $\bar{\xi}_j(n)$ are generated by (4.2.3) but the estimation procedure is based on $\rho_j \equiv 1$, the prediction variance $\mathcal{E}\{\xi_j(n)-\hat{\xi}_j^{(\nu)}(n)\}^2$ is found from the expression,

$$\int_{-\pi}^{\pi}\left\{|1-h_j^{(\nu)}(\lambda-\lambda_j)|^2\left\{\frac{\sigma_j^2}{2}\frac{1}{2\pi}\frac{1}{|1-\rho_j e^{i\lambda}|^2}\right\}+|h^{(\nu)}(\lambda-\lambda_j)|^2 f_x(\lambda_j)\right\}d\lambda \tag{6.3.22}$$

which becomes when relocated at the origin

$$\int_{-\pi}^{\pi}\left\{|1-h_j^{(\nu)}(\lambda)|^2\left\{\frac{\sigma_j^2}{2}\frac{1}{2\pi}\frac{1}{|1-\rho_j e^{i\lambda}|^2}\right\}+|h^{(\nu)}(\lambda)|^2 f_x(\lambda_j)\right\}d\lambda. \tag{6.3.23}$$

If $\nu = 0$ then $h_j^{(o)}(\lambda) = \dfrac{(1-\beta_j)}{1-\beta_j e^{i\lambda}}$ and by straight forward contour integration one obtains the expression,

$$\mathcal{E}\left\{\xi_j(n)-\hat{\xi}_j^{(o)}(n)\right\}^2 = \frac{\sigma_j^2}{2}\left\{\frac{2\beta_j^2}{(1-\rho_j\beta_j)(1+\beta_j)(1+\rho_j)}+\frac{2\pi f_x(\lambda_j)(1-\beta_j)^2}{(\sigma_j^2/2)(1-\beta_j^2)}\right\}. \tag{6.3.24}$$

The prediction variance in (6.3.24) may be related to that given in §5.7 for $\mathcal{E}(s_j(n)-\hat{s}_j^{(o)})^2$, where of course $\rho_j \equiv 1$, by recalling (see (4.2.3)) that $s_j(n) = \xi_j(n)+\bar{\xi}_j(n)$ and by setting ρ_j to unity in (6.3.24). Then, by noting that $\{2\pi f_x(\lambda_j)/(\sigma_j^2/2)\}$ is equal to θ_j and that θ_j may be rewritten in terms of β_j as (see (4.5.3))

$$\theta_j = -\frac{1}{(1-\beta_j)(1-\beta_j^{-1})} = \frac{\beta_j}{(1-\beta_j)^2} \tag{6.3.25}$$

if the optimal β_j is used, (6.3.24) is equivalent to (5.7.15). If $\nu = \infty$ then $h_j^{(\infty)}(\lambda) = \dfrac{(1-\beta_j)^2}{|1-\beta_j e^{i\lambda}|^2}$ and again contour integration followed by simple but lengthy algebraic manipulations provide the following prediction variance,

$$\mathcal{E}\left(\xi_j(n)-\hat{\xi}_j^{(\infty)}(n)\right)^2 = \frac{2\pi f_x(\lambda_j)(1-\beta_j)(1+\beta_j^2)}{(1+\beta_j)^3}$$

$$+\frac{\sigma_j^2}{2}\frac{1}{(1-\rho_j^2)}\left\{1+\frac{2\rho_j\beta_j(1-\beta_j)^2}{(1-\rho_j\beta_j)^2(1+\beta_j)^2}+\frac{(1-\beta_j)(1+\beta_j^2)(1+\rho_j\beta_j)}{(1+\beta_j)^3(1-\rho_j\beta_j)}\right. \qquad (6.3.26)$$

$$\left.-\frac{2(1-\beta_j)(1+\rho_j\beta_j)}{(1+\beta_j)(1-\rho_j\beta_j)}\right\}.$$

Again the cumbersome expression in (6.3.26) may be reduced to the very simple expression (5.7.15) by using the same expression for θ_j in terms of β_j (6.3.25) and by letting $\rho_j \to 1$ and applying L'Hospital's rule to the second term.

6.4 Demodulation and Modelling

There must of course be a good deal of uncertainty as to whether the model proposed in (4.2.3) is a suitable one. Before going into detail on the points to be examined the demodulation procedure will be briefly sketched. The technique of investigating a particular frequency, or more correctly a narrow band about a particular frequency has a long history. Two recent discussions of this approach are Granger [15] and Bingham [5]. The demodulated series is composed of a real and a complex part and one of the purposes of this section is to investigate what sort of stochastic processes might generate the real and complex parts.

One begins with the original observations $w(n)$ and as will be the general approach in this section the removal of long term, low frequency, power will be carried out by subtracting a twelve month moving average from $w(n)$ giving the series $y(n)$. The demodulated series results from a further filtering of $y(n)$ and the success of this operation depends on the choice of coefficients in the filter. The filtering of $y(n)$ proceeds as follows to give

$$A_j(n) = \Sigma_{k-T}^{T} b_k e^{-i(n-k)\lambda_j} y(n-k) \tag{6.4.1}$$

$$n = 6+T+1,\ldots,N-6-T; \quad j = 1,2,\ldots,6$$

which is a sequence of complex numbers. How narrow a band about λ_j is represented by the sequence depends of course on the chosen b_k coefficients. Perhaps the simplest method is to choose the b_k as a centred moving average, that is to first form

$$M_j(n) = \Sigma_{-T}^{T} \left(1/(2T+1)\right) e^{-i(n-k)\lambda_j} y(n-k) \tag{6.4.2}$$

and then to centre the $M_j(n)$ values at integral time points if this is not so by then forming

$$A_j(n) = \tfrac{1}{2}\left\{M_j(n-\tfrac{1}{2})+M_j(n+\tfrac{1}{2})\right\}. \tag{6.4.3}$$

The frequency response of the b_k coefficients associated with $A_j(n)$ as defined in (6.4.3) is, when $(2T+1) = 48$,

$$B(\lambda) = (\tfrac{1}{96}) \frac{\sin 24\lambda \sin\lambda}{\sin^2 \tfrac{1}{2}\lambda}. \tag{6.4.4}$$

The success of the demodulation depends on how effectively the response function of the chosen b_k represents only the desired narrow band about λ_j. To attempt to improve the result another set of filter coefficients was used. These coefficients represent the repetition of a moving average and may be represented as

$$A_j(n) = \Sigma_{k-M}^{M} \Sigma_{\ell-M}^{M} \left\{\frac{1}{2M+1}\right\}^2 e^{-i(n-k-\ell)\lambda_j} y(n-k-\ell)$$

$$= \left\{\frac{1}{2M+1}\right\}^2 \Sigma_{k-2M}^{2M} e^{-i(n-k)\lambda_j} (2M+1-|k|) y(n-k) \tag{6.4.5}$$

where if one selects $2M = 24$ then the number of terms lost in demodulating will be approximately the same as in (6.4.3) and the response function is

$$B(\lambda) = \frac{1}{(25)^2}\left\{\frac{\sin {}^{25}/_2\,\lambda}{\sin\frac{1}{2}\lambda}\right\}^2 . \tag{6.4.6}$$

A comparison of the response functions of the two proposed demodulating averages is given in Fig. XVI. Now the quantities which will be the subject of further investigation are $\mathcal{R}(A_j(n))$, the real part of $A_j(n)$, and $\mathcal{I}(A_j(n))$, the imaginary part, and which are given by

$$\begin{aligned}\mathcal{R}(A_j(n)) &= \mathcal{R}\left\{e^{-in\lambda_j}\left[\Sigma_{k=-T}^{T} b_k e^{ik\lambda_j} y(n-k)\right]\right\} \\ &= \cos n\lambda_j \Sigma_{k=-T}^{T} b_k \cos k\lambda_j y(n-k) + \sin n\lambda_j \Sigma_{k=-T}^{T} b_k \sin k\lambda_j y(n-k)\end{aligned} \tag{6.4.7}$$

and by

$$\begin{aligned}\mathcal{I}(A_j(n)) &= \mathcal{I}\left\{e^{-in\lambda_j}\left[\Sigma_{k=-T}^{T} b_k e^{ik\lambda_j} y(n-k)\right]\right\} \\ &= \sin n\lambda_j \Sigma_{k=-T}^{T} b_k \cos k\lambda_j y(n-k) - \cos n\lambda_j \Sigma_{k=-T}^{T} b_k \sin k\lambda_j y(n-k).\end{aligned} \tag{6.4.8}$$

The quantities calculated using (6.4.7) and (6.4.8) are slightly modified to produce

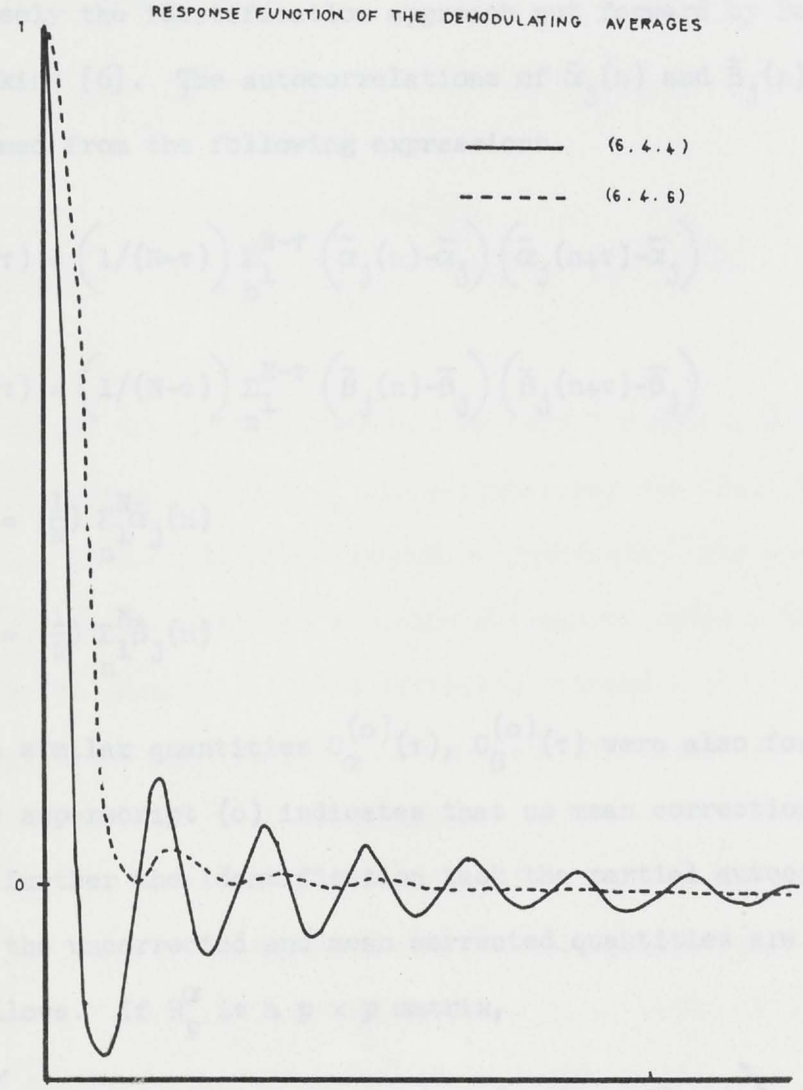
$$\begin{aligned}\tilde{\alpha}_j(n) &= \mathcal{R}(A_j(n))(2-\delta_j^6) \\ \tilde{\beta}_j(n) &= \mathcal{I}(A_j(n))(2-\delta_j^6)\end{aligned} \tag{6.4.9}$$

and one may then easily contrast what is being done in this section with the approach based on a model which specifies the probabilistic development of the $\alpha_j(n)$ and $\beta_j(n)$. To do this the expressions in the first line of (6.4.7) and (6.4.8) should be compared with those given in (5.6.1), assuming ν is set equal to infinity. Now $\hat{\xi}_j^{(\nu)}(n)$ and $\hat{\xi}_{-j}^{(\nu)}(n)$ are given by (5.2.3) (except that $N-\nu$ is set equal to n) and so

$$\hat{\xi}_j^{(\infty)}(n) = H\,\Sigma_{-\infty}^{\infty} \beta_j^{|k|} e^{ik\lambda_j} y(n-k). \tag{6.4.10}$$

FIG. XVI

RESPONSE FUNCTION OF THE DEMODULATING AVERAGES



The reason for repeating the expression for $\hat{\xi}_j^{(\infty)}(n)$ in (6.4.10) is to emphasize that the quantities to be studied, $\tilde{\alpha}_j(n)$, $\tilde{\beta}_j(n)$, differ from $\hat{\alpha}_j(n)$, $\hat{\beta}_j(n)$ only in that the b_k term in the square bracket in (6.4.7) and (6.4.8) do not spring from a <u>specific a priori</u> assumption as to the nature of the stochastic process generating $\alpha_j(n)$, $\beta_j(n)$.

To obtain guidance as to the kind of process by which $\alpha_j(n)$ and $\beta_j(n)$ might be generated the quantities were tested in the following manner. The procedure adopted here follows closely the identification approach put forward by Box and Jenkins [6]. The autocorrelations of $\tilde{\alpha}_j(n)$ and $\tilde{\beta}_j(n)$ were formed from the following expressions

$$
\begin{aligned}
C_\alpha(\tau) &= \Big(1/(N-\tau)\Big) \sum_{n}{}_{1}^{N-\tau} \Big(\tilde{\alpha}_j(n)-\bar{\alpha}_j\Big)\Big(\tilde{\alpha}_j(n+\tau)-\bar{\alpha}_j\Big) \\
C_\beta(\tau) &= \Big(1/(N-\tau)\Big) \sum_{n}{}_{1}^{N-\tau} \Big(\tilde{\beta}_j(n)-\bar{\beta}_j\Big)\Big(\tilde{\beta}_j(n+\tau)-\bar{\beta}_j\Big) \\
\bar{\alpha}_j &= \left(\tfrac{1}{N}\right) \sum_{n}{}_{1}^{N} \tilde{\alpha}_j(n) \\
\bar{\beta}_j &= \left(\tfrac{1}{N}\right) \sum_{n}{}_{1}^{N} \tilde{\beta}_j(n)
\end{aligned}
\tag{6.4.11}
$$

and similar quantities $C_\alpha^{(o)}(\tau)$, $C_\beta^{(o)}(\tau)$ were also formed where the superscript (o) indicates that no mean correction was made. To further the identification task the partial autocorrelations of the uncorrected and mean corrected quantities are obtained as follows. If R_p^α is a $p \times p$ matrix,

$$
\begin{pmatrix}
1 & R_\alpha(1) & \cdots & R_\alpha(p-2) & R_\alpha(p-1) \\
R_\alpha(1) & 1 & \cdots & R_\alpha(p-3) & R_\alpha(p-2) \\
\vdots & \vdots & & \vdots & \vdots \\
R_\alpha(p-2) & \cdot & \cdots & 1 & R_\alpha(1) \\
R_\alpha(p-1) & R_\alpha(p-2) & \cdots & R_\alpha(1) & 1
\end{pmatrix}
\tag{6.4.12}
$$

where $R_\alpha(\tau) = \{C_\alpha(\tau)/C_\alpha(o)\}$ and if r is the vector $\{R_\alpha(1), R_\alpha(2), \ldots, R_\alpha(p)\}$ then a vector of estimates $\phi^\alpha = \{\phi_{p1}, \phi_{p2}, \ldots, \phi_{pp}\}$ is obtained from

$$\phi^\alpha = (R_p^\alpha)^{-1}\, r. \qquad (6.4.13)$$

To interpret ϕ^α it should be understood that ϕ_{pj} is the j^{th} autoregressive parameter in a process of order p, and in particular the partial autocorrelation of order p is ϕ_{pp}. The simultaneous equations (6.4.13) may be solved by using the recursive relations given by Durbin [12]

$$\phi_{\tau+1,j} = \phi_{\tau,j} - \phi_{\tau+1,\tau+1}\phi_{\tau,\tau-j+1} \qquad j = 1, \ldots, \tau$$

$$\phi_{\tau+1,\tau+1} = \frac{R(\tau+1) + \Sigma_{j1}^{\tau}\phi_{\tau,j}R(\tau+1-j)}{1+\Sigma_{j1}^{\tau}\phi_{\tau,j}R(j)}, \quad \phi_{11} = -R(1) \qquad (6.4.14)$$

to obtain the partial autocorrelations for $\tilde{\alpha}_j(n)$, $\tilde{\beta}_j(n)$.[32] The statistics $R(\tau)$ and $\phi_{\tau,\tau}$ are suitable for the identification of moving average and autoregressive processes. For example if the $\tilde{\alpha}_j(n)$ are generated by a moving average of order k it is well known that the theoretical autocorrelation beyond k will be zero. To aid identification then one would compare R(k) with its standard error on the assumption that the process is a moving average of order (k-1). In this case for τ values less than k one would expect $R(\tau)$ to be large in relation to its standard error. The variance of R(k) when it is assumed that the process is of order (k-1) is given by the following formula proposed by Bartlett [4]

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32 The equations in (6.4.14) have dropped the superscript, α, to give a less cluttered formulation. Of course it is also the case that (6.4.12), (6.4.13) and (6.4.14) are used also for $\beta_j(n)$.

$$\text{var}\left\{R(k)\right\} = \left(\frac{1}{N}\right)\left\{1+2\left(\rho^2(1)+\rho^2(2)+\ldots\ldots+\rho^2(k-1)\right)\right\} \qquad (6.4.15)$$

where the $\rho(\tau)$ are the population lag correlation coefficients. It is usually necessary in practice to use instead of $\rho(\tau)$ the $R(\tau)$ values from the sample of size N. The statistic tabled in Table 18 is

$$N_k = \left\{R(k)/\sqrt{\text{var } R(k)}\right\} \qquad k = 1,2,\ldots,24 \qquad (6.4.16)$$

which is approximately a standard normal variate.

Identifying a purely autoregressive process on the basis of the autocorrelations may be based on a judgement that the autocorrelation function "tails off" in contrast to the situation for the moving average process of order k where as has been pointed out the autocorrelations after the k^{th} "cut off". It is very much simpler however to instead concentrate on the partial autocorrelations for it may be shown that for an autoregression of order k that the <u>partial</u> autocorrelations "cut off", and that the point of "cut off" will be the order of the autoregression. The test statistic which is computed on the basis of the partial autocorrelations to aid in identifying the "cut off" point is

$$M_k = \left\{\phi_{k,k}/\sqrt{N-k}\right\}; \qquad k = 1,2,\ldots,24 \qquad (6.4.17)$$

where M_k is a standard normal variate, since the variance of $\phi_{k,k}$, if the process is autoregressive of the order (k-1) is

$$\text{var } \phi_{k,k} = \left\{1/(N-k)\right\}. \qquad (6.4.18)$$

Tables of demodulated values of $\tilde{\alpha}_j(n)$, $\tilde{\beta}_j(n)$ are presented in Table 18 for the two filtering methods and mean corrected $\tilde{\alpha}_j(n)$, $\tilde{\beta}_j(n)$ are based on only one filter, that given in (6.4.5). The tabulations are most space consuming so the latter seasonal frequencies, λ_4, λ_5 and λ_6 are omitted. It is however already apparent in λ_3 that a simple first order autoregression model is not the appropriate one for that frequency and similar results obtain for the latter frequencies. In fact as the frequencies after λ_2 are for many series quite low in spectral power[33] and therefore their contribution to the total seasonal variation is quite minor it seems possible to justify the following approach. Suppose that for these latter frequencies there is a stable seasonal coefficient which would thus produce a spike in the spectrum for these λ_j. This spike will then be approximated by using the model proposed in (4.2.3), and further assuming that $\rho_j \equiv 1$ for $j = 3,4,5,6$, as was done for all λ_j previously.

A rather general point is inserted before discussing in more detail the problems of identification. The series which is used for the modelling in identification procedures is "Bank Advances" and this series has characteristics which do indicate it is evolving (see Fig. XIV). For any series under consideration it is only sensible to do some preliminary investigation to establish whether the series is in fact evolving before attempting to establish the nature of the evolution. A fairly obvious indication can be obtained from the study of a periodogram of residuals after regressing off a stable seasonal pattern as will be suggested in the next section.

Returning to the constructed series $\tilde{\alpha}_j(n)$, $\tilde{\beta}_j(n)$, which are to form the basis of an enquiry into three simple generating models

$$\begin{aligned} &(a) \quad \alpha_j(n) = m_j+\epsilon_j(n) \\ &(b) \quad \alpha_j(n) = \rho_j\alpha_j(n-1)+\epsilon_j(n) \\ &(c) \quad (\alpha_j(n)-m_j) = \rho_j(\alpha_j(n-1)-m_j)+\epsilon_j(n) \end{aligned} \tag{6.4.19}$$

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33 The normalized spectra for wool presented in §6.7 does seem to have an unusual make-up of spectral power. (See Fig. XX).

where $\epsilon_j(n)$ is a random disturbance which is I.I.D. (0,1) and m_j is the population mean of $\alpha_j(n)$.[34] It proves to be convenient in certain contexts to consider (c) in the form

$$\alpha_j(n) = m_j(1-\rho_j)+\rho_j\alpha_j(n-1)+\epsilon_j(n). \qquad (6.4.20)$$

To reiterate the variables which form the basis of tables are $\tilde{\alpha}_j(n)$, $\tilde{\beta}_j(n)$, $\tilde{\alpha}_j(n)-\overline{\alpha}_j$ and $\tilde{\beta}_j(n)-\overline{\beta}_j$ and the estimate of ρ_j, denoted $\hat{\rho}_j$, is the first autocorrelation for both the mean corrected and non-mean corrected data in Table 18. If it was the case that (a) was the correct model but that the estimate of $\hat{\rho}_j$ was based on model (b) then it is easily shown that the probability limit of $\hat{\rho}_j$ is

$$\text{plim } \hat{\rho}_j = \frac{1}{\{\frac{\sigma_\epsilon^2}{m_j^2} + 1\}} = \frac{1}{\{1 + \frac{1}{m_j^2}\}} \qquad (6.4.21)$$

and so unless m_j^2 was very small one would expect $\hat{\rho}_j$ to be close to unity. On the other hand if model (a) is true and a procedure to estimate ρ_j appropriate to model (c) is employed then the probability limit of $\hat{\rho}_j$ is

$$\text{plim } \hat{\rho}_j = 0. \qquad (6.4.22)$$

As the samples are reasonably large (always greater than 100) and the ρ_j estimates do not differ markedly depending on whether or not mean corrections are made <u>and</u> the estimates of ρ_j are quite large it appears unlikely that (a) is the appropriate model. In any case if a process of (a) type was generating $\alpha_j(n)$ and $\beta_j(n)$ it is *a priori* much more likely that the disturbance will not be

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[34] If in model (a) the disturbance $\epsilon_j(n)$ was replaced by a disturbance $\eta_j(n)$ such that $\eta_j(n) = \rho_j\eta_j(n-1)+\epsilon_j(n)$, $|\rho_j| < 1$, then model (a) and model (c) are identical.

independent (see footnote 34). So in this rather restricted set of models attention is now centred on (b) and (c) as formulated in (6.4.20). If model (c) is the true model and the estimation procedure is based on model (b) then there is a bias in the estimate of $\hat{\rho}_j$, which is for large N equal to

$$(1-\rho_j)/\left\{\frac{\sigma^2_{\alpha_j}}{m_j^2+1}\right\} \qquad (6.4.23)$$

where $\sigma^2_{\alpha_j}$ is the population variance of $\alpha_j(n)$. Thus with ρ_j values close to 1 the asymptotic expression for this bias will be small. Another possibility is if model (b) is true but the estimate of ρ_j is appropriate to model (c). In those circumstances there is no bias in the estimate of ρ_j, however there is a loss of efficiency in the estimate of ρ_j. The variance for $\hat{\rho}_j$ in the true model is

$$\sigma^2_{\epsilon_j}\left\{\sum_{n=1}^{N}\alpha_j^2(n)\right\}^{-1} \qquad (6.4.24)$$

and for the assumed model is

$$\sigma^2_{\epsilon_j}\left\{\sum_{n=1}^{N}\alpha_j^2(n) - \frac{(\Sigma\alpha_j(n))^2}{N}\right\}^{-1} \qquad (6.4.25)$$

and so the loss of efficiency for estimating ρ_j using the incorrect model will be given by

$$\left\{1 - \left\{\left(\sum_{n=1}^{N}\alpha_j(n)\right)^2 \Big/ \left(N\sum_{n=1}^{N}\alpha_j^2(n)\right)\right\}\right\}. \qquad (6.4.26)$$

So far only the influence of the choice of an incorrect model on the parameter ρ_j has been considered but as is apparent from the discussion of an optimal β_j (see (6.3.14)) the effect of incorrect model specification on $\sigma^2_{\epsilon_j}$ is also important. We consider again the case when (c) is true and (b) is employed. The estimate of $\sigma^2_{\epsilon_j}$ is based on the vector of calculated residuals, $\tilde{\epsilon}$, where

$$\tilde{\epsilon} = \tilde{\alpha}_j(n) - \hat{\rho}_j\tilde{\alpha}(n-1) \tag{6.4.27}$$

and is given by,

$$\hat{\sigma}^2_{\epsilon_j} = \left\{\tilde{\epsilon}'\tilde{\epsilon}/(N-1)\right\} \tag{6.4.28}$$

which as is shown in App. A is not an unbiassed estimate of $\sigma^2_{\epsilon_j}$. The bias is presented for the general situation in App. A and in the above situation is given by,

$$m_j^2(1-\rho_j)^2 \left\{ \Big(N/(N-1)\Big) - \Big(\Sigma\alpha_j(n-1)\Big)^2 / \Big((N-1)\Sigma\alpha_j^2(n-1)\Big) \right\}$$

$$= m_j^2(1-\rho_j^2) \left\{ \Big(N\Sigma\alpha_j^2(n-1) - \Big(\Sigma\alpha_j(n-1)\Big)^2\Big) / (N-1)\Sigma\alpha_j^2(n-1) \right\} \tag{6.4.29}$$

and the asymptotic expression for this bias is,

$$m_j^2(1-\rho_j^2)\left\{1 - \frac{m_j^2}{\sigma^2_{\alpha_j} + m_j^2}\right\} \tag{6.4.30}$$

which will be small if m_j is small or if ρ_j is close to 1. If model (b) is true and (c) is used again one finds there is no bias in the estimate of $\sigma^2_{\epsilon_j}$, but the variance of the estimate is larger than would be the case if the true model was employed. (See App. A).

The estimates of $\sigma^2_{\epsilon_j}$ actually obtained for each situation is the first figure given in the variance of residuals now in Table 18 and it is quite clear that the estimates do depend substantially on whether a mean correction is appropriate or not. As the question of whether a mean correction should be made must therefore be faced a test of whether the parameter m_j in model (c) is significantly different from zero must be considered. As has been noted this is exactly the same as testing whether m_j is significantly different from zero in model (a) if $\epsilon_j(n)$ is replaced by $\eta_j(n)$ defined in footnote 34. The testing procedure can be expressed as an extremely simple regression problem, where the only regressor is the unit vector but where the variance-covariance matrix of residuals is the familiar one associated with a first order autoregressive disturbance with parameter ρ_j. Thus if ρ_j is as expected close to 1 it will be difficult to find a significant constant term since its variance is given by[35]

$$\operatorname{var} \bar{\alpha}_j = \frac{\hat{\sigma}^2_{\epsilon_j}}{N} \frac{1}{(1-\hat{\rho}_j)^2} = \frac{\hat{\sigma}^2_{\alpha_j}}{N} \frac{(1-\hat{\rho}^2_j)}{(1-\hat{\rho}_j)^2} \tag{6.4.31}$$

where $\hat{\sigma}^2_{\alpha_j}$ is the variance of $\tilde{\alpha}_j(n)$, where the generalized least squares estimate of m_j is just $\bar{\alpha}_j$, and where the second expression merely employs the population relation between the variance of $\alpha_j(n)$ and the variance of $\epsilon_j(n)$.

Only rather tentative conclusions on the nature of the parameters needed to develop the "optimal" model can be drawn. It appears that the estimates of $\hat{\rho}_j$ and $\sigma^2_{\epsilon_j}$ obtained from $\tilde{\alpha}_j(n)$ and $\tilde{\beta}_j(n)$ are useful only for j 1,2, - which are usually the main sources of power. As mentioned before the values of ρ_j are not very much different whether mean corrections are made or not but the estimates of $\sigma^2_{\epsilon_j}$ are quite different so that unless one is able to decide with some certainty whether $m_j = 0$ a range of $\sigma^2_{\epsilon_j}$ values would have to be considered. The latter frequencies, $j \geq 2$, do not appear to be easily handled by the models proposed

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35 It may be more convenient on occasions to use the spectral representation of the variance of $\bar{\alpha}_j$ which is given by

$\operatorname{var} \bar{\alpha}_j = \frac{2\pi f(o)}{N}$ where $f(\lambda)$ is the spectrum of $\eta_j(n)$.

and short of extending the models under consideration a rather familiar approach has to be adopted - partly justified by the usually minor role of the latter seasonal frequencies. It is assumed that the ρ_j for these frequencies are identically equal to unity and thus an estimate of σ_j^2 will be made from the first differences of the $\tilde{\alpha}_j(n)$ and $\tilde{\beta}_j(n)$ for $j = 2,3,4,5,6$.

6.5 Periodogram Estimates

To make further progress with the task of establishing the quantities on which the "optimal" method of signal (seasonal) extraction depends it is necessary to know something of the nature of the process generating the "non-seasonal". At least one must have some idea of the "non-seasonal" power or magnitude, if as is suggested in Chapter IV the simplifying assumption is made that the level of non-seasonal power or 'noise' is a constant over a band at each seasonal frequency. A crude estimate of this power will be derived from an analysis of the periodogram ordinates of $y(n)$ - the trend removed series. This section also includes a discussion of the periodogram ordinates of the original observations, $w(n)$ (see (3.2.1)), and of the residual series after a stable seasonal pattern has been removed, $r(n)$.

The value in inspecting the periodogram ordinates of $r(n)$ is easily demonstrated. Before beginning the detailed work of developing an evolving seasonal estimation procedure as suggested in Chapters IV and V it is as well to attempt some assessment of whether a stable pattern is adequate. A quick guide to the adequacy of the stable pattern is to graph the periodogram ordinates of $r(n)$ and to note whether significant power remains in frequencies in a band about any λ_j. This has been done for two series:

All (Australian) Cheque Paying Banks: Loans, Advances and Bills Discounted \$m (Bank Advances) September 1945 to May 1967 inclusive[36]

Number of bales of greasy wool sold in Australia 000 (Wool) July 1948 to June 1967 inclusive[37]

and in Figs. XVII a, b, the periodogram ordinates, these ordinates after smoothing by a three term average, and after smoothing by a five term average are shown. For those averages which include an ordinate which is $\{\pi j/6\}$, $j = 1,2,3,4,5,6$, that term is omitted from the average and the denominator reduced by unity. It is quite clear that a considerable amount of power remains close to the seasonal frequencies. For each of the series this is most marked for λ_1; however there are also other seasonal frequencies where it does appear that the extraction of the power in a band about λ_j would remove some of the peaks close to these seasonal frequencies in the spectrum of $r(n)$.

A further useful enterprise is the consideration of the periodogram of four original series, the two previously introduced above and Registrations of New Motor Vehicles in Australia (Motor Vehicles) and Production of Electricity in Australia (Electricity)[36] (the logarithm of the published series is $w(n)$). It was conjectured that careful scrutiny of the periodogram for each series might indicate appreciable differences in band width at seasonal frequencies for different series. To make this comparison the series were all restricted to the most recent 204 observations and thus to the same number of periodogram ordinates. It is possible to

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36 The source of these series is the Monthly Review of Business Statistics (Commonwealth Bureau of Census and Statistics, Canberra, Australia).

37 The source of this series is the National Council of Wool Selling Brokers.

FIG. XVII a

5 TERM AVERAGE OF PERIODOGRAM ORDINATES FOR RESIDUALS FROM A STABLE SEASONAL PATTERN

a) BANK ADVANCES

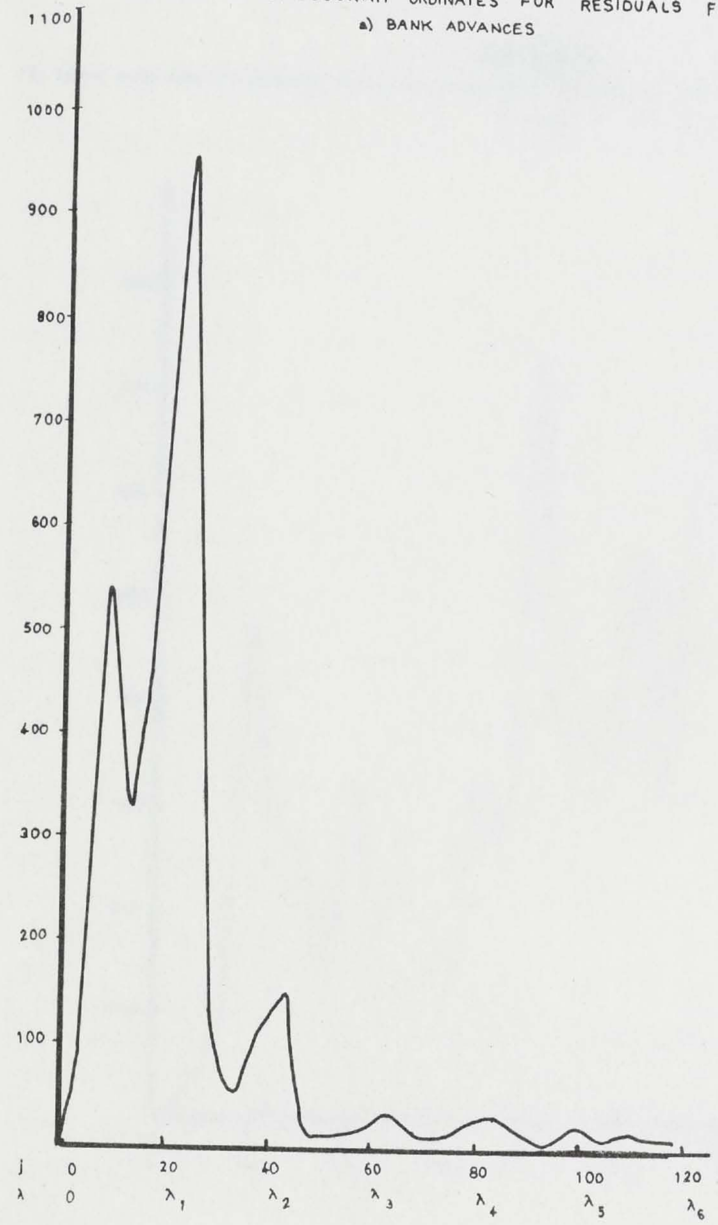
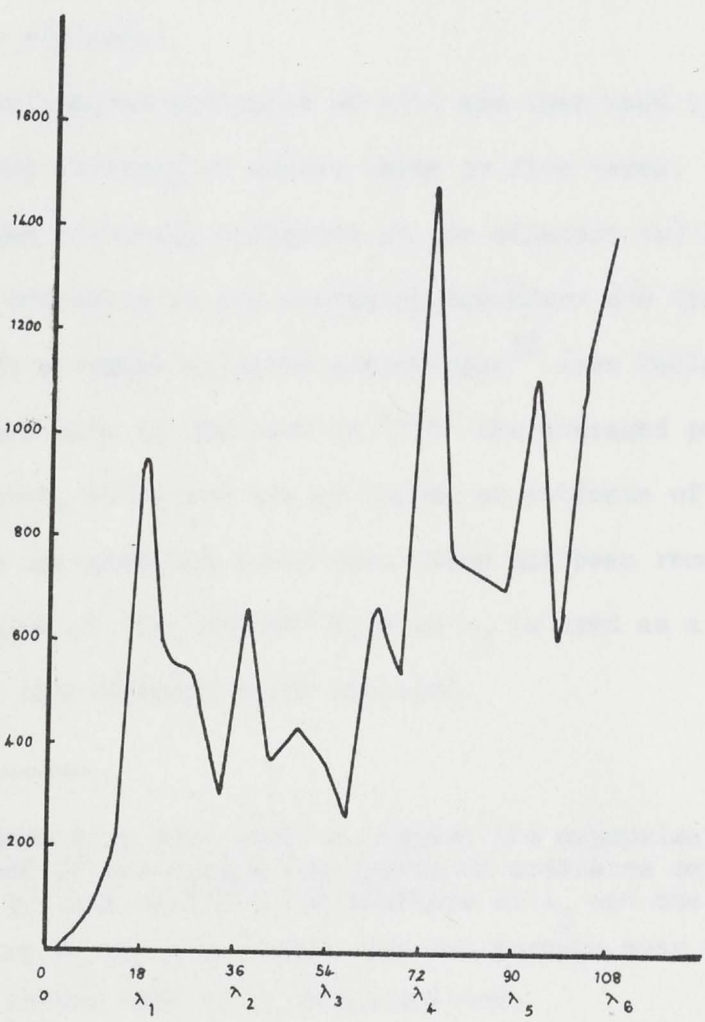


FIG. XVII b

5 TERM AVERAGE OF PERIODOGRAM ORDINATES FOR RESIDUALS FROM A STABLE SEASONAL PATTERN

b) WOOL



recognize the difference between the series Bank Advances and Wool on the basis of the bandwidth of the seasonal signals (see Fig. XVIII). It was however particularly difficult to find any of the series which over a period as long as 17 years produced a periodogram with seasonal spikes.

To complete the search for parameters necessary to help in the development of an 'optimal' seasonal extraction procedure one must at least estimate the power of $x(n)$, i.e. the non-seasonal noise after trend has been removed, in a band about each λ_j. A guiding estimate of these quantities may be obtained from the periodogram of $y(n)$, which is defined by (see §4.3)

$$y(n) = s(n)+x(n). \qquad (6.5.1)$$

The periodogram ordinates of $y(n)$ are then used to form equally weighted averages of either three or five terms. Once again averages involving ordinates at (or adjacent to) λ_j should omit those ordinates in the averaging procedure and the denominator of such averages adjusted accordingly[38] (see Table 19). A freehand line is now used to "fit" the averaged periodogram ordinates, which are now of course an estimate of the spectrum of the non-seasonal noise when trend has been removed (see Fig. XIX). The value of the "fitted" line at λ_j is used as a rough estimate of the non-seasonal power required.

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[38] There is a need here to compare the approximate spectral shape obtained if one varies the number of ordinates omitted, that is begin by just omitting the ordinate at λ_j and one either side and then try λ_j and two either side and perhaps even the next step of three either side of λ_j being omitted.

FIG. XVIII

3 TERM AVERAGE OF PERIODOGRAM ORDINATES OF ORIGINAL DATA

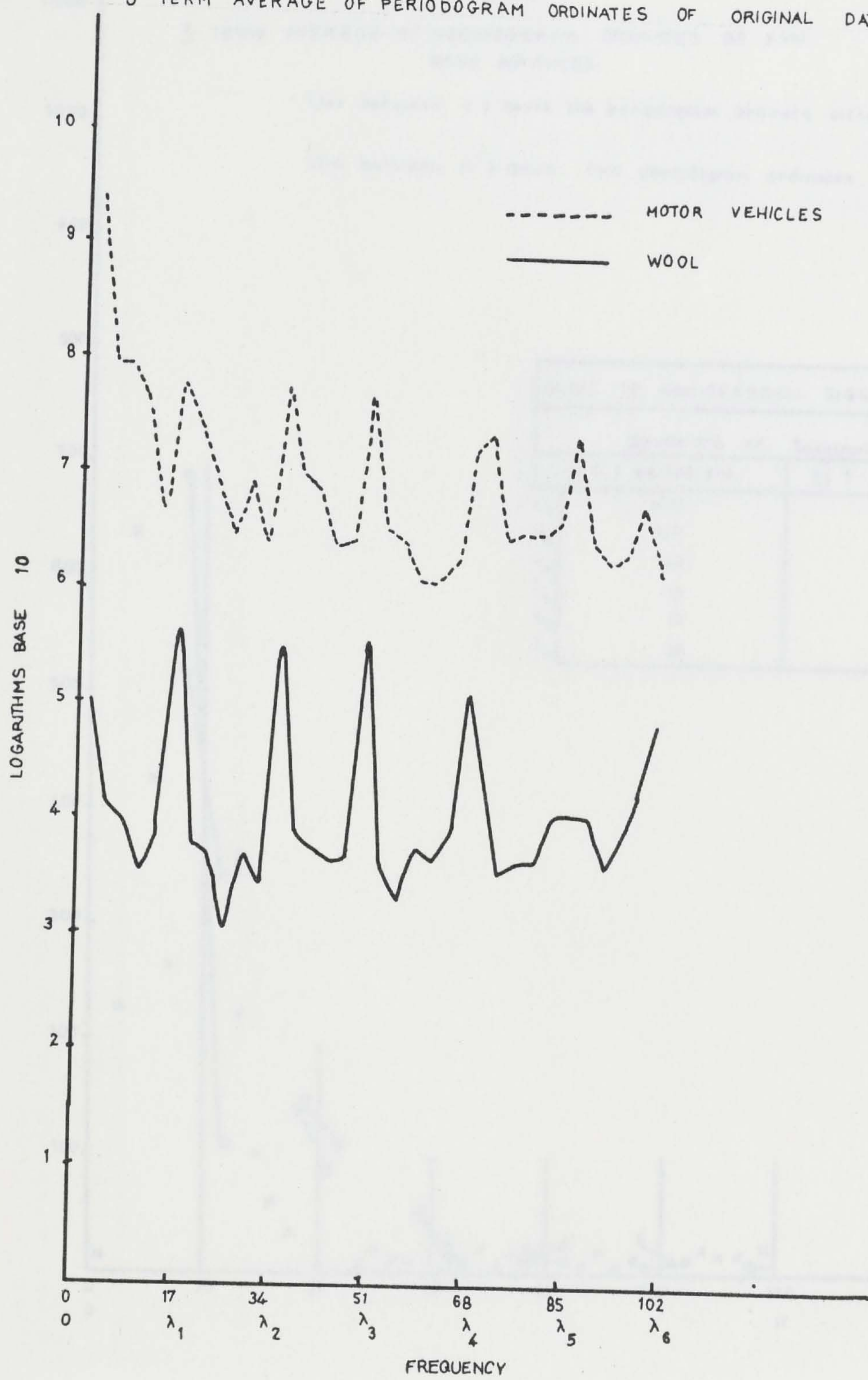
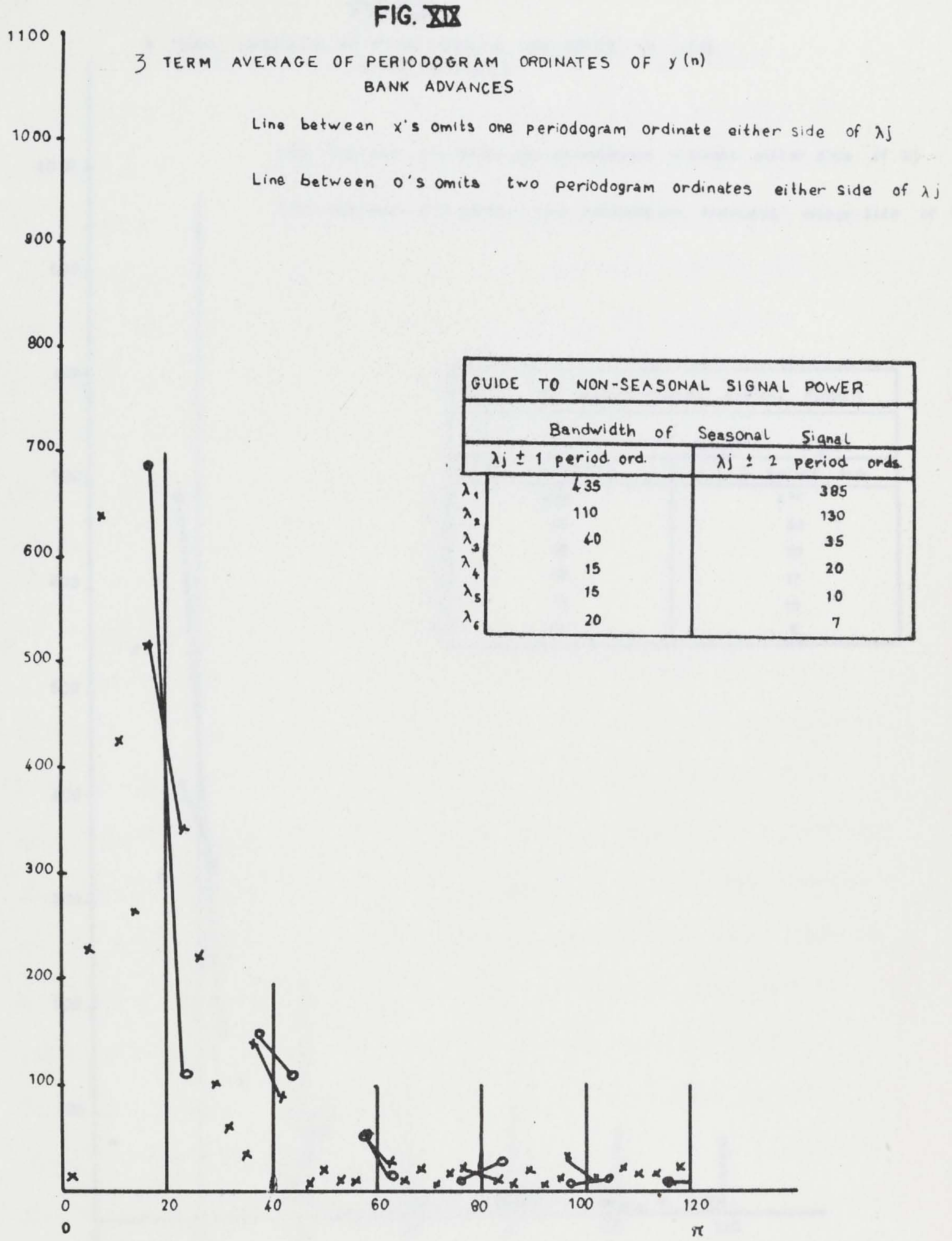


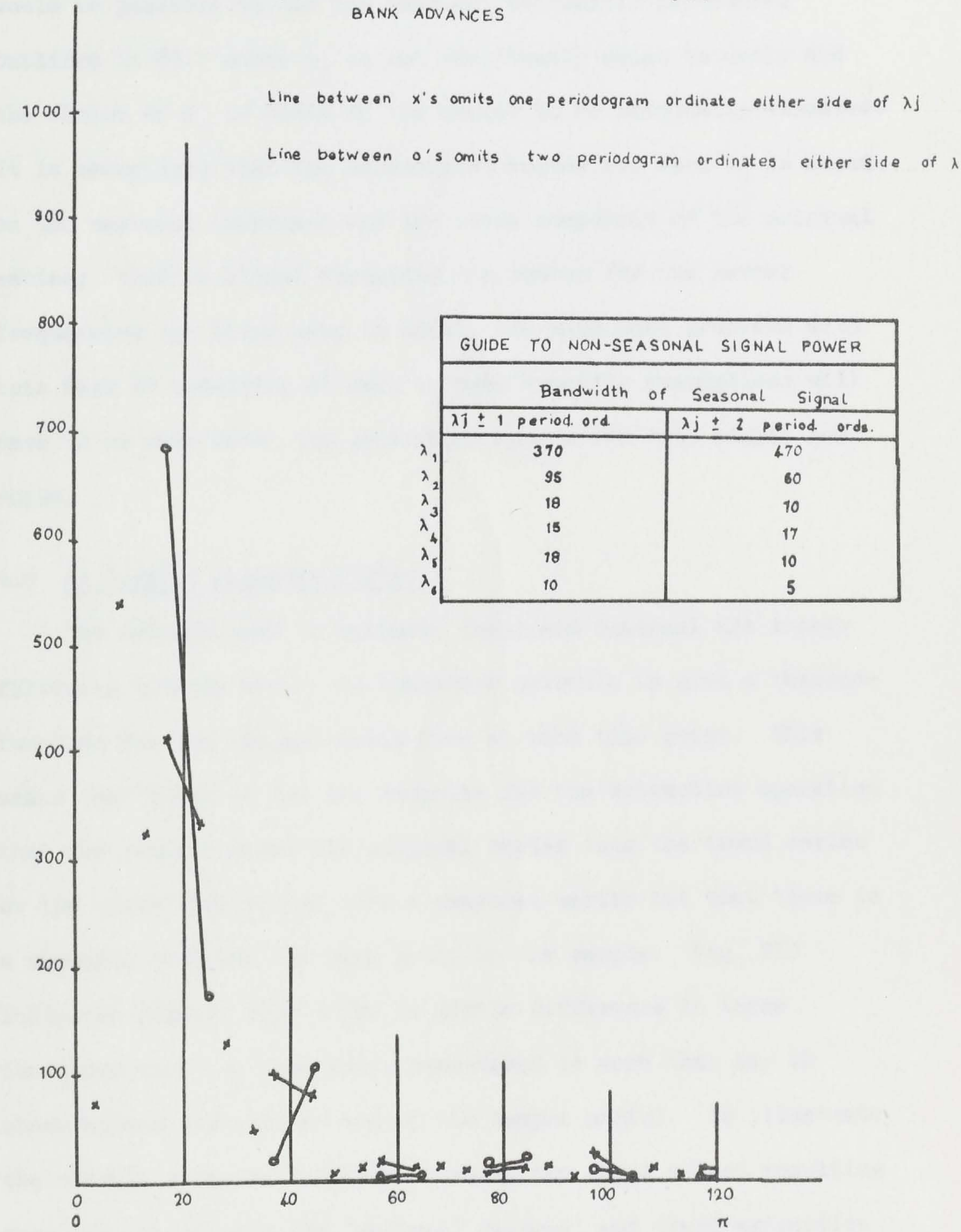
FIG. XIX



GUIDE TO NON-SEASONAL SIGNAL POWER		
	Bandwidth of Seasonal Signal	
	λj ± 1 period ord.	λj ± 2 period ords.
λ_1	435	385
λ_2	110	130
λ_3	40	35
λ_4	15	20
λ_5	15	10
λ_6	20	7

FIG. XIX

5 TERM AVERAGE OF PERIODOGRAM ORDINATES OF y(n)
BANK ADVANCES



	GUIDE TO NON-SEASONAL SIGNAL POWER	
	Bandwidth of Seasonal Signal	
	$\lambda j \pm 1$ period. ord.	$\lambda j \pm 2$ period. ords.
λ_1	370	470
λ_2	95	60
λ_3	18	10
λ_4	15	17
λ_5	18	10
λ_6	10	5

Now that estimates of ρ_j, $\sigma^2_{\epsilon_j}$ and $f_x(\lambda_j)$ are available it would be possible to use the seasonal estimation procedures outlined in §5.7 where ρ_j is not identically equal to unity and the choice of β_j is based on the series to be seasonally adjusted. It is recognized that the demodulated series for each λ_j is based on the seasonal component and the noise component of the original series; this is almost certainly the reason for the latter frequencies not being easy to model. To make much progress with this task of modelling at each λ_j more specific assumptions will have to be made about the generating models for both signal and noise.

6.6 An Overall Implicit Filter

The methods used to estimate trend and seasonal are linear filtering methods and it is therefore possible to give a response function for the optimal extraction at each time point. This means that there is not one response for the extraction operation that for example takes the original series into the trend series or the trend free series into a seasonal series but that there is a separate response for each point in the sample. Fig. XII indicates however that there is little difference in these functions once the time point considered is more than say 12 observations from either end of the sample period. To illustrate the overall response function, meaning the total effect resulting from the use of both the "optimal" seasonal and trend extraction appropriate to each time point, the approach employed by Nerlove [45] is introduced. The situation considered by Nerlove was of course quite different in that the B.L.S. procedure was a non-linear one and therefore not easily describable by a theoretical response function. An implicit response function derived from the cross-spectrum was the only practical guide to the overall effect of the B.L.S. procedure. The cross-spectrum under consideration is between the original series and that series after subtraction of a seasonal estimate, i.e. an adjusted series.

To briefly outline what was done the symbols T and S denote the operations of trend and seasonal extraction, where extraction is synonomous with estimation and not with removal. The trend is therefore estimated from the original series by Tw(n). Seasonal extraction is then performed on the trend removed series, (1-T)w(n); an additional operation produces a seasonal estimate, S(1-T)w(n), and the adjusted series is therefore

$$\begin{aligned} a(n) &= w(n)-S(1-T)w(n) \\ &= (1-S)w(n)+STw(n). \end{aligned} \qquad (6.6.1)$$

The extent to which the estimate of the gain from the cross-spectrum between a(n) and w(n) approximates the gain of the filtering operation (1-S), which should we know from Fig. XII have a shape very close to that sought for by Nerlove [45] and Fishman [14], depends on two rather straightforward points. The first is the extent to which one has been able to design a trend extraction filter with a response function which does not have a significant overlap with the response function of the seasonal extraction filter. Thus an effectively designed seasonal filter will fail to produce an implicit response function with acceptable gain and phase characteristics if the trend filter is poorly designed.

It would be very difficult indeed to produce a trend filter which had no overlap problems and if overlap does exist then the extent to which the implicit function fails to represent the gain of the operation (1-S) depends on the power of w(n) around points of significant overlap. Quite apart from the consideration of what the implicit response is actually measuring there are practical difficulties in adequately estimating a frequency response function, especially when there are peaks in the spectrum as there will be at or close to zero frequency and at the seasonal frequencies (see [37]). As the presence of peaks in the spectrum will result in smudged cross spectral estimates the estimation was repeated for an increasing number of spectral points. This is described as "window closing" [37] and if as the number of points increase there is little change in the gain and phase, an indication is obtained that little "smudging" of the cross-spectral estimate is taking place.[39]

Two series, Bank Advances and Wool, and their respective adjusted series, are subjected to cross-spectral analysis; the number of lag covariances, m, used in the estimates are allowed to vary as follows: m = 24,36,48,60,72. A measure of the relative distribution of power in each original series is obtained by normalizing the spectral quantity at each frequency by dividing by the sum of the spectral power over all frequencies. It is plotted in Fig. XX for each m. Both series have been restricted to the most recent 216 observations and it could be argued that one should preclude from serious consideration those spectra and cross-spectra based on larger m than 48. The wool graphs certainly seem to support this argument and so the gain and phase for m = 48 is focusssed on (see Fig. XXl). There is no significant phase change introduced by the filtering procedures as the maximum phase change is approximately $\frac{\pi}{12}$ at λ_1 and as

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[39] It might be argued in this section that it may also be sensible to use a different covariance averaging kernel such as the Tukey-Hanning kernel since the presence of negative spectra and then removal as m increases may be an effective way of recognizing the disappearance of excessive blurring.

coherence at this frequency is .044 the variance of the phase estimate would be large at this frequency.[40] No attempt will be made to explain in detail the failure of the gain to reproduce the complement of the gain given in Fig. XII. The only comment is that if as is suggested most of the smudging effects have been minimized then there remains a minor distortion which can be ascribed to the design of the filtering routine piece by piece, which in practical terms seems to be the only way to proceed.

In the Bank Advances series one might consider the spectra based on 60 lag covariances as there is no indication of oscillations in the spectral and cross-spectral quantities until $m = 72$. Even with this large number of spectral bands the gain of the implicit response does not clearly represent the complement of the gain of the seasonal extraction operation (see Fig. XII). The largest phase change indicated is a change of approximately $\pi/6$ at $\frac{39\pi}{60}$ - a time lag of about one quarter of a month. Again however the coherence at $39\pi/60$ is .038 and little significance should be attached to this change. It would be surprising if this coherence was not small as $39\pi/60 \cong \frac{2\pi}{3}$ i.e. a harmonic of the frequency $\pi/6$.

It should also be noted that even with the same number of observations in each series the implicit response produced is far from identical. Some of the differences are due to chance variation but some may also be due to the different spectral structure of the series (compare Figs. on normalized spectra with gain for each series).

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40 Since the estimate of the phase can be shown to be asymptotically normal with mean $\theta_{ij}(\lambda)$ and variance

$$\frac{M}{2N}\left[\int_{-\infty}^{\infty} k^2(x)dx\right]\left\{\frac{1}{W_{ij}^2(\lambda)} - \frac{\{c_{ij}^2(\lambda)-q_{ij}^2(\lambda)\}^2}{\{c_{ij}^2(\lambda)+q_{ij}^2(\lambda)\}^2}\right\}.$$

6.7 Concluding Example

Chapters IV and V dealt with a special model of the seasonal signal and presented methods for a detailed analysis of a set of data using this special model. Chapter VI has suggested ways in which one might use the same set of data with a slightly more general and efficient model. The modifications to the model derive from further analysis of the data and are used to re-estimate the seasonal component in a fashion suggested in §5.7.

As an illustrative example the series Bank Advances is first seasonally adjusted on the basis of Model I in which $\rho_j \equiv 1$ and $\beta_j = .96$ for all λ_j and subsequently on the basis of Model II where

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
ρ_j	.99	1.0	1.0	1.0	1.0	1.0
β_j	.93	.99	.98	.99	.99	.98

The results of the deseasonalizing procedures based on Models I and II are printed out in full detail in Appendix C. Model II involves parameter estimates derived from the demodulation approach suggested in §6.4 and from the periodogram ordinates of $y(n)$. It is relatively easy to support the decision to use these estimates notwithstanding the difficulties associated with this approach which are discussed in §6.4. However as the estimates in Tables 16 and 17 are unsuitable for further use some explanation for the obvious inability to recover satisfactory estimates of ρ_j, τ_j and κ_j^2 from the $\hat{\alpha}_j(n)$, $\hat{\beta}_j(n)$ is necessary. The approximate spectrum of the $\hat{\alpha}_j^{(\infty)}(n)$, $\hat{\beta}_j^{(\infty)}(n)$ presented in (6.3.20) is equated to the general formula for the appropriate mixed moving average autoregression, i.e.

$$\frac{\kappa_j^2|1-\tau_j e^{i\lambda}|^2}{\{|1-\beta_j e^{i\lambda}|^2\}^2\{|1-\rho_j e^{i\lambda}|^2\}} \tag{6.7.1}$$

to derive the relations given in (6.3.21).

As a satisfactory method for estimating the parameters of (6.7.1), subject to the necessary inequality constraints, is not available the search procedure on ρ_j and τ_j outlined in §6.3 was employed. The values in Tables 16 and 17 for $\hat{\tau}_1$, $\hat{\sigma}_1^2$ and $2\pi\hat{f}_x(\lambda_1)$ are a reflection of the fact that the plotted likelihood changes very little in magnitude as τ_1 is varied - indicating of course a very high variance for $\hat{\tau}_1$. As a consequence of the wide range in which τ_1 may lie the associated estimates of σ_1^2 and $2\pi f_x(\lambda_1)$ will also have large variances.

A possible reason for the likelihood discriminating so poorly with respect to τ_1 is the invalidity of the assumption that the noise level is constant over the band considered. It is almost certain however that the real cause is that one is trying to estimate the shape of a sharply changing spectrum over a very narrow frequency band. Very many sets of parameter values will do almost equally as well with the information available so that the likelihood function is extremely flat in the neighbourhood of its maximum.

TABLE 18a

CENTRED 48 TERM M.A. - MEAN CORRECTED

α_1

-7.11	-6.79	-6.78	-6.26	-7.09	-6.00	-6.44	-6.13	-6.86	-5.41	-5.41	-4.47
-3.75	-3.52	-3.76	-2.48	-1.48	0.40	0.69	0.22	0.29	0.93	2.02	3.06
4.12	4.77	4.49	3.62	1.04	0.59	-0.25	-0.30	-1.50	-1.39	-3.32	-4.66
-6.14	-6.82	-6.77	-6.21	-6.32	-4.67	-4.00	-4.13	-3.82	-3.71	-4.90	-6.12
-6.49	-5.73	-4.56	-3.25	-3.21	-3.31	-5.45	-7.43	-7.43	-6.63	-7.70	-8.54
-10.61	-11.73	-12.14	-13.13	-16.50	-18.45	-20.71	-22.10	-21.56	-21.05	-20.65	-23.52
-26.92	-28.87	-30.17	-30.20	-30.40	-28.64	-26.89	-26.61	-26.26	-26.14	-26.87	-27.07
-27.91	-26.32	-25.38	-25.21	-27.37	-27.52	-28.30	-29.00	-28.51	-29.55	-30.49	-33.43
-33.78	-32.99	-33.05	-33.90	-37.52	-39.64	-40.28	-40.46	-39.14	-38.86	-39.36	-40.21
-41.79	-41.10	-41.61	-41.23	-41.89	-40.64	-39.81	-39.50	-38.13	-37.68	-38.88	-42.13
-43.64	-42.81	-43.04	-42.24	-44.05	-45.09	-47.12	-49.75	-49.52	-48.36	-46.91	-46.92
-47.20	-45.76	-46.35	-46.10	-47.56	-46.64	-46.22	-47.43	-46.97	-46.84	-47.64	-48.60
-48.92	-46.78	-47.82	-47.36	-48.69	-46.25	-44.55	-44.60	-44.73	-45.53	-46.58	-47.87
-47.21	-44.21	-44.20	-44.63	-47.15	-47.34	-47.03	-47.86	-47.00	-46.55	-48.08	-49.75
-49.48	-47.09	-47.42	-47.12	-49.15	-47.32	-45.92	-45.17	-43.36	-43.54	-47.24	-53.13
-55.82	-53.66	-53.54	-54.51	-57.77	-59.15	-59.32	-60.88	-60.24	-59.84	-60.92	-62.50
-61.52	-59.15	-59.57	-59.85	-62.02	-59.69	-57.74	-58.24	-57.36			

AUTOCORRELATIONS

0.990	0.978	0.965	0.953	0.940	0.928	0.914	0.900	0.884	0.868	0.853	0.838
0.821	0.805	0.790	0.776	0.761	0.747	0.733	0.718	0.702	0.687	0.675	0.665

VARIANCE OF RESIDUALS

0.8164E 01	0.8066E 01	0.8066E 01	0.8066E 01	0.8060E 01	0.8059E 01	0.8055E 01	0.8024E 01
0.7991E 01	0.7988E 01	0.7970E 01	0.7970E 01	0.7901E 01	0.7897E 01	0.7856E 01	0.7855E 01
0.7856E 01	0.7852E 01	0.7850E 01	0.7841E 01	0.7824E 01	0.7790E 01	0.7733E 01	0.7718E 01

VARIANCES OF AUTOCORRELATIONS

0.005	0.015	0.024	0.034	0.043	0.051	0.060	0.068	0.076	0.084	0.092	0.099
0.106	0.112	0.119	0.125	0.131	0.137	0.142	0.148	0.153	0.158	0.163	0.167

SIG. STATISTIC

14.034	8.057	6.201	5.207	4.559	4.093	3.735	3.445	3.201	2.993	2.818	2.665
2.525	2.399	2.290	2.193	2.103	2.019	1.942	1.867	1.795	1.730	1.675	1.626

AUTOREGR.COEFFS.

-1.083	0.083	0.003	-0.014	0.010	-0.011	-0.027	0.001	0.025	0.066	-0.032	-0.095
0.099	0.050	-0.060	-0.007	-0.018	0.007	-0.018	-0.021	0.109	0.025	-0.038	-0.044

PARTIAL AUTOCORRELS.

0.990	-0.110	0.002	-0.007	-0.029	-0.013	-0.022	-0.062	-0.065	-0.018	0.048	-0.005
-0.093	0.021	0.073	0.011	-0.001	-0.020	-0.016	-0.036	-0.047	0.065	0.086	0.044

VARIANCES OF PARTIAL AUTOCORRELS

0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005
0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.006

SIG. STATISTIC

14.034	-1.546	0.035	-0.101	-0.402	-0.185	-0.307	-0.858	-0.896	-0.248	0.661	-0.075
-1.280	0.290	0.991	0.144	-0.019	-0.271	-0.213	-0.481	-0.624	0.873	1.144	0.581

TABLE 18a

CENTRED 48 TERM M.A. - MEAN CORRECTED

β_1

-6.87	-7.16	-8.01	-8.44	-7.33	-7.52	-8.25	-7.03	-8.82	-8.53	-8.63	-8.70
-8.54	-8.63	-11.08	-12.82	-13.34	-13.54	-13.06	-13.58	-15.34	-15.96	-16.22	-15.77
-14.97	-13.32	-12.49	-10.71	-9.83	-10.60	-10.90	-11.57	-13.59	-12.80	-11.48	-12.27
-13.03	-14.25	-15.89	-16.78	-17.65	-18.59	-18.49	-19.10	-20.47	-20.43	-20.73	-21.45
-22.71	-23.56	-24.59	-24.27	-23.64	-23.20	-23.09	-25.17	-27.06	-27.79	-26.21	-25.77
-25.81	-25.84	-24.02	-21.55	-19.12	-19.03	-19.16	-20.86	-21.24	-20.29	-20.01	-19.61
-20.47	-21.51	-22.00	-23.26	-24.32	-25.69	-25.08	-24.94	-24.18	-24.32	-24.27	-25.22
-26.60	-26.40	-24.16	-21.97	-19.25	-18.68	-17.47	-17.61	-16.55	-14.51	-13.12	-12.98
-14.90	-14.42	-11.95	-9.72	-6.62	-6.84	-6.85	-7.68	-6.80	-5.32	-3.99	-4.03
-5.57	-5.85	-6.46	-8.29	-8.51	-9.64	-9.12	-9.30	-8.57	-7.01	-5.45	-4.46
-6.42	-6.90	-7.11	-8.33	-6.57	-6.76	-7.00	-9.70	-12.33	-14.29	-15.06	-14.99
-16.27	-15.84	-16.61	-17.70	-17.23	-18.37	-17.67	-18.54	-18.55	-17.96	-17.46	-17.86
-19.28	-17.77	-17.48	-19.74	-19.69	-21.35	-20.12	-19.27	-17.94	-17.16	-16.93	-17.48
-18.71	-15.98	-14.37	-12.78	-11.62	-12.63	-11.86	-12.78	-12.58	-12.13	-10.72	-11.13
-12.49	-10.81	-9.52	-8.98	-7.35	-8.96	-8.39	-8.64	-6.94	-3.01	0.99	1.70
-0.76	1.00	3.81	7.93	10.35	9.67	10.04	9.08	9.50	10.12	11.02	9.93
9.12	11.53	12.68	12.94	14.07	12.81	14.28	14.01	14.05			

AUTOCORRELATIONS

0.973	0.938	0.902	0.869	0.835	0.798	0.758	0.714	0.674	0.633	0.592	0.552
0.508	0.466	0.424	0.384	0.344	0.304	0.268	0.232	0.199	0.164	0.132	0.108

VARIANCE OF RESIDUALS

0.5185E 01	0.5088E 01	0.5077E C1	0.5063E 01	0.5051E 01	0.5029E 01	0.5000E 01	0.4974E 01
0.4961E 01	0.4947E 01	0.4940E 01	0.4937E 01	0.4866E 01	0.4862E 01	0.4860E 01	0.4858E 01
0.4854E C1	0.4850E 01	0.4843E 01	0.4842E 01	0.4841E 01	0.4819E 01	0.4808E 01	0.4753E 01

VARIANCES OF AUTOCORRELATIONS

0.005	0.014	0.023	0.031	0.039	0.046	0.052	0.058	0.063	0.067	0.071	0.075
0.078	0.080	0.083	0.084	0.086	0.087	0.088	0.089	0.089	0.090	0.090	0.090

SIG. STATISTIC

13.788	7.823	5.928	4.916	4.241	3.735	3.323	2.972	2.689	2.440	2.216	2.019
1.820	1.642	1.476	1.324	1.175	1.032	0.903	0.779	0.666	0.547	0.440	0.359

AUTOREGR.COEFFS.

-1.100	0.084	0.116	-0.109	0.020	-0.048	0.007	0.133	-0.112	0.023	0.056	-0.141
0.139	-0.038	0.017	-0.036	-0.002	0.070	-0.045	0.018	-0.081	0.110	0.071	-0.108

PARTIAL AUTOCORRELS.

0.973	-0.137	-0.046	0.053	-0.049	-0.065	-0.077	-0.072	0.052	-0.054	-0.038	0.024
-0.120	0.030	-0.019	-0.017	-0.029	-0.028	0.039	-0.017	-0.010	-0.068	0.048	0.108

VARIANCES OF PARTIAL AUTOCORRELS

0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005
0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.006

SIG. STATISTIC

13.788	-1.931	-0.649	0.746	-0.690	-0.914	-1.071	-0.995	0.715	-0.739	-0.522	0.330
-1.644	0.410	-0.259	-0.228	-0.400	-0.385	0.532	-0.222	-0.132	-0.905	0.635	1.430

TABLE 18a

CENTRED 48 TERM M.A. - MEAN CORRECTED

α_2

-1.60	-0.80	0.01	-0.20	-0.46	-1.20	-1.99	-1.05	-2.44	-1.83	-1.87	-1.47
-0.86	-0.24	1.86	2.45	1.74	0.11	0.04	-0.16	-2.00	-1.89	-0.85	0.10
0.62	-0.35	-1.43	-2.88	-0.89	0.12	0.65	-0.22	-1.91	-0.88	-1.49	-3.47
-4.58	-3.51	-0.95	-0.53	0.11	-0.58	-0.48	-1.21	-3.27	-3.10	-4.50	-5.94
-5.65	-3.92	-2.53	-3.85	-4.25	-4.36	-2.72	-3.28	-4.86	-4.32	-3.85	-5.26
-7.36	-8.11	-10.03	-12.66	-11.91	-9.75	-7.09	-7.70	-8.91	-6.91	-5.94	-9.35
-12.07	-12.86	-13.18	-12.33	-10.68	-10.73	-11.67	-12.38	-12.10	-11.11	-11.92	-13.70
-13.25	-11.21	-13.47	-16.29	-16.71	-16.67	-15.40	-15.07	-14.36	-12.52	-12.12	-15.85
-15.30	-14.49	-17.51	-19.71	-18.81	-16.48	-16.16	-17.28	-17.31	-15.08	-14.21	-16.08
-16.38	-14.73	-15.24	-14.34	-13.05	-13.74	-14.64	-15.43	-15.52	-13.16	-13.16	-16.73
-16.09	-14.05	-15.41	-15.53	-14.69	-13.40	-12.47	-13.28	-15.91	-16.33	-15.56	-16.73
-16.14	-14.25	-14.31	-14.12	-12.77	-12.74	-12.86	-12.63	-12.80	-11.41	-12.27	-14.81
-13.67	-12.19	-13.90	-13.05	-10.92	-11.11	-12.24	-12.07	-11.05	-9.65	-10.42	-13.50
-12.50	-11.60	-13.64	-15.80	-14.13	-13.21	-13.47	-13.71	-13.91	-12.44	-13.24	-16.29
-14.29	-12.74	-15.12	-16.60	-15.69	-16.22	-18.11	-19.84	-19.20	-14.85	-14.82	-20.94
-21.17	-20.36	-24.08	-27.69	-26.06	-24.48	-24.65	-24.22	-24.14	-22.37	-23.19	-26.49
-23.89	-23.38	-25.78	-26.46	-24.73	-26.15	-27.43	-27.54	-27.63			

AUTOCORRELATIONS

0.963	0.922	0.900	0.878	0.849	0.821	0.809	0.805	0.790	0.763	0.753	0.743
0.710	0.684	0.670	0.644	0.602	0.562	0.538	0.521	0.496	0.468	0.464	0.465

VARIANCE OF RESIDUALS

0.4031E 01	0.4001E 01	0.3728E 01	0.3716E 01	0.3707E 01	0.3705E 01	0.3605E 01	0.3573E 01
0.3555E 01	0.3486E 01	0.3345E 01	0.3296E 01	0.3176E 01	0.3091E 01	0.3089E 01	0.2919E 01
0.2864E 01	0.2863E 01	0.2854E 01	0.2854E 01	0.2850E 01	0.2850E 01	0.2785E 01	0.2782E 01

VARIANCES OF AUTOCORRELATIONS

0.005	0.014	0.023	0.031	0.038	0.046	0.052	0.059	0.065	0.071	0.077	0.083
0.088	0.093	0.098	0.103	0.107	0.110	0.113	0.116	0.119	0.121	0.124	0.126

SIG. STATISTIC

13.656	7.732	5.979	5.012	4.331	3.849	3.540	3.320	3.095	2.856	2.710	2.581
2.389	2.238	2.139	2.010	1.842	1.693	1.597	1.528	1.439	1.343	1.321	1.310

AUTOREGR.COEFFS.

-1.084	0.245	-0.106	-0.003	-0.032	0.112	-0.052	-0.088	-0.162	0.310	-0.190	-0.106
0.275	-0.110	-0.176	0.092	0.127	0.043	-0.054	0.051	-0.087	0.170	-0.113	-0.034

PARTIAL AUTOCORRELS.

0.963	-0.086	0.261	-0.057	-0.049	0.025	0.164	0.094	-0.070	-0.139	0.201	-0.121
-0.192	0.163	-0.028	-0.234	-0.138	0.020	0.056	0.009	0.037	-0.016	0.150	0.034

VARIANCES OF PARTIAL AUTOCORRELS

0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005
0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.006

SIG. STATISTIC

13.656	-1.209	3.679	-0.796	-0.680	0.343	2.289	1.308	-0.970	-1.923	2.773	-1.664
-2.626	2.226	-0.379	-3.188	-1.867	0.265	0.749	0.115	0.497	-0.214	2.003	0.454

TABLE 18a

CENTRED 48 TERM M.A. - MEAN CORRECTED

α_3

-5.79	-4.99	-5.37	-5.69	-4.62	-4.32	-2.77	-3.71	-2.70	-5.44	-4.74	-4.81
-4.22	-3.40	-3.79	-5.84	-5.30	-3.97	-4.00	-3.15	-3.51	-4.43	-3.36	-2.45
-2.42	-3.63	-3.97	-1.81	-1.87	-3.47	-3.37	-1.56	1.03	0.75	1.80	-0.47
-0.68	2.76	3.09	0.62	1.02	1.34	0.50	3.09	2.77	1.51	1.48	0.22
0.98	2.32	1.20	0.48	0.72	0.22	-0.21	1.94	0.77	0.25	2.44	0.16
-0.78	0.06	-1.31	0.15	2.90	1.25	-1.31	1.72	0.78	0.36	2.72	-0.97
-2.28	-1.51	-0.43	-0.18	-1.29	-3.03	-2.15	-0.28	-2.11	-1.54	-0.37	-3.75
-1.39	-0.80	-4.23	-2.83	-0.96	-1.28	-2.39	-1.71	-3.86	-2.54	0.06	-4.48
-2.28	-2.60	-4.33	-1.46	0.01	-2.75	-2.20	-0.49	-3.38	-3.12	-0.86	-4.10
-2.18	-1.19	-2.49	-2.08	-3.30	-3.68	-2.58	-1.65	-4.05	-3.13	-1.33	-5.61
-1.89	-1.64	-3.92	-2.65	-2.82	-5.41	-4.16	-0.98	-2.67	-4.52	-3.28	-5.93
-3.51	-2.34	-3.93	-3.52	-3.69	-4.78	-4.67	-3.77	-4.72	-3.34	-2.68	-7.57
-3.78	-4.00	-5.16	-3.70	-4.69	-7.55	-5.62	-4.27	-6.43	-5.47	-3.26	-7.68
-5.53	-5.96	-8.16	-5.77	-4.96	-6.84	-6.34	-5.51	-7.15	-6.09	-5.11	-10.33
-5.72	-6.54	-8.72	-6.87	-6.37	-7.90	-5.03	-3.52	-7.74	-6.07	-2.51	-9.08
-6.19	-7.97	-10.09	-6.39	-6.36	-8.91	-7.95	-7.33	-8.60	-6.68	-5.65	-11.47
-6.36	-8.74	-10.24	-7.90	-8.68	-9.02	-7.83	-6.78	-9.04			

AUTOCORRELATIONS

0.783	0.755	0.851	0.728	0.745	0.831	0.690	0.649	0.770	0.652	0.63C	0.736
0.574	0.606	0.668	0.506	0.522	0.594	0.478	0.476	0.549	0.399	0.402	0.523

VARIANCE OF RESIDUALS

0.3655E 01	0.3169E 01	0.2155E 01	0.2127E 01	0.2044E 01	0.1897E 01	0.1825E 01	0.1693E 01
0.1557E 01	0.1546E 01	0.1540E 01	0.1503E 01	0.1438E 01	0.1348E 01	0.1279E 01	0.1168E 01
0.1148E 01	0.1146E 01	0.1146E 01	0.1146E 01	0.1145E 01	0.1143E 01	0.1114E 01	C.1087E 01

VARIANCES OF AUTOCORRELATIONS

0.005	0.011	0.017	0.024	0.029	0.035	0.042	0.046	0.051	0.056	0.061	0.065
C.070	0.073	0.077	0.081	0.084	0.087	0.090	0.092	0.095	0.098	0.099	0.101

SIG. STATISTIC

11.107	7.168	6.573	4.703	4.356	4.457	3.383	3.015	3.427	2.743	2.558	2.897
2.170	2.240	2.408	1.775	1.803	2.018	1.591	1.565	1.785	1.277	1.276	1.647

AUTOREGR.COEFFS.

-0.569	-0.105	-0.148	-0.003	-0.102	-0.302	0.170	0.387	-0.334	-0.023	0.082	-0.155
0.188	-0.385	0.198	0.324	-0.176	-0.005	0.033	-0.011	0.006	-0.112	0.244	-0.154

PARTIAL AUTOCORRELS.

0.783	0.365	0.566	-0.114	0.197	0.269	-0.194	-0.269	0.283	-0.083	-0.063	0.155
-0.209	0.249	-0.227	-0.294	0.132	0.038	-0.007	0.004	0.022	0.045	-0.160	0.154

VARIANCES OF PARTIAL AUTOCORRELS

0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005
0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.006

SIG. STATISTIC

11.107	5.144	7.959	-1.604	2.758	3.753	-2.701	-3.740	3.928	-1.148	-0.863	2.127
-2.863	3.409	-3.097	-4.001	1.796	0.520	-0.096	0.060	0.292	0.607	-2.134	2.044

TABLE 18b

REPEATED 24 TERM M.A. - NO MEAN CORRECTION

α_1

AUTOCORRELATIONS	0.991	0.982	0.972	0.963	0.956	0.948	0.939	0.930	0.920	0.910	0.901	0.893
	0.885	0.877	0.868	0.861	0.854	0.847	0.839	0.831	0.823	0.816	0.810	0.804

VARIANCE OF RESIDUALS	0.3207E 02	0.3207E 02	0.3198E 02	0.3186E 02	0.3175E 02	0.3172E 02	0.3170E 02	0.3157E 02
	0.3154E 02	0.3153E 02	0.3150E 02	0.3150E 02	0.3150E 02	0.3147E 02	0.3147E 02	0.3138E 02
	0.3138E 02	0.3135E 02	0.3135E 02	0.3130E 02	0.3128E 02	0.3125E 02	0.3125E 02	0.3124E 02

VARIANCES OF AUTOCORRELATIONS	0.005	0.015	0.024	0.034	0.043	0.052	0.061	0.070	0.078	0.087	0.095	0.103
	0.111	0.119	0.127	0.134	0.141	0.149	0.156	0.163	0.170	0.176	0.183	0.190
SIG. STATISTIC	14.049	8.086	6.231	5.244	4.610	4.154	3.804	3.520	3.286	3.089	2.924	2.781
	2.654	2.543	2.441	2.352	2.271	2.196	2.126	2.060	1.999	1.944	1.893	1.846
AUTOREGR.COEFFS.	-0.989	-0.061	0.111	-0.001	-0.074	-0.000	-0.041	0.032	0.049	0.016	-0.015	-0.018
	0.023	-0.037	0.065	-0.047	-0.029	0.017	-0.032	0.061	0.011	-0.023	0.008	-0.017
PARTIAL AUTOCORRELS.	0.991	-0.002	-0.054	0.059	0.061	-0.028	-0.027	-0.065	-0.028	0.017	0.030	0.013
	-0.008	0.028	-0.011	0.054	0.007	-0.030	-0.008	-0.039	0.021	0.033	0.009	0.017
VARIANCES OF PARTIAL AUTOCORRELS	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005
	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.006
SIG. STATISTIC	14.049	-0.025	-0.762	0.833	0.849	-0.387	-0.374	-0.899	-0.385	0.236	0.408	0.184
	-0.109	0.381	-0.146	0.738	0.094	-0.403	-0.114	-0.531	0.283	0.447	0.120	0.227

TABLE 18b

REPEATED 24 TERM M.A. - NO MEAN CORRECTION

β_1

AUTOCORRELATIONS

0.992	0.982	0.969	0.959	0.951	0.939	0.924	0.905	0.887	0.871	0.855	0.839
0.818	0.797	0.776	0.756	0.737	0.715	0.690	0.664	0.641	0.621	0.601	0.581

VARIANCE OF RESIDUALS

0.6174E 01	0.6058E 01	0.5930E 01	0.5729E 01	0.5654E 01	0.5046E 01	0.4835E 01	0.4802E 01
0.4628E 01	0.4626E 01	0.4575E 01	0.4572E 01	0.4492E 01	0.4465E 01	0.4452E 01	0.4405E 01
0.4387E 01	0.4326E 01	0.4313E 01	0.4264E 01	0.4226E 01	0.4219E 01	0.4202E 01	0.4200E 01

VARIANCES OF AUTOCORRELATIONS

0.005	0.015	0.024	0.034	0.043	0.052	0.061	0.069	0.077	0.085	0.093	0.100
0.107	0.114	0.120	0.126	0.132	0.137	0.142	0.147	0.151	0.155	0.159	0.163

SIG. STATISTIC

14.063	8.079	6.210	5.223	4.592	4.124	3.753	3.442	3.192	2.985	2.811	2.655
2.502	2.365	2.240	2.131	2.032	1.932	1.831	1.734	1.649	1.575	1.507	1.440

AUTOREGR.COEFFS.

-1.067	-0.237	0.406	0.118	-0.364	-0.049	0.078	0.263	-0.147	0.017	0.001	-0.153
0.192	-0.041	-0.060	0.056	-0.119	0.015	0.187	0.029	-0.073	-0.106	0.086	-0.019

PARTIAL AUTOCORRELS.

0.992	-0.138	-0.145	0.184	0.114	-0.328	-0.204	-0.084	0.190	-0.021	-0.106	-0.022
-0.133	0.077	0.054	-0.103	-0.064	-0.119	-0.056	0.106	0.094	0.041	-0.065	0.019

VARIANCES OF PARTIAL AUTOCORRELS

0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005
0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.006

SIG. STATISTIC

14.063	-1.942	-2.043	2.586	1.600	-4.581	-2.846	-1.162	2.637	-0.291	-1.463	-0.307
-1.818	1.059	0.742	-1.406	-0.867	-1.608	-0.750	1.432	1.261	0.552	-0.869	0.255

TABLE 18b

REPEATED 24 TERM M.A. - NO MEAN CORRECTION

α_2

AUTOCORRELATIONS

0.980	0.956	0.930	0.908	0.890	0.880	0.873	0.867	0.862	0.853	0.841	0.830
0.816	0.800	0.785	0.766	0.749	0.736	0.726	0.719	0.717	0.714	0.710	0.705

VARIANCE OF RESIDUALS

0.1024E 02	0.1013E 02	0.1010E 02	0.1003E 02	0.9909E 01	0.9724E 01	0.9716E 01	0.9695E 01
0.9686E 01	0.9668E 01	0.9653E 01	0.9625E 01	0.9573E 01	0.9538E 01	0.9537E 01	0.9398E 01
0.9382E 01	0.9366E 01	0.9355E 01	0.9323E 01	0.9249E 01	0.9248E 01	0.9238E 01	0.9230E 01

VARIANCES OF AUTOCORRELATIONS

0.005	0.015	0.024	0.032	0.040	0.048	0.056	0.064	0.071	0.078	0.086	0.093
0.100	0.106	0.113	0.119	0.125	0.130	0.136	0.141	0.146	0.151	0.156	0.161

SIG. STATISTIC

13.890	7.930	6.051	5.056	4.429	4.005	3.687	3.438	3.232	3.045	2.874	2.726
2.586	2.455	2.338	2.222	2.123	2.040	1.972	1.917	1.877	1.837	1.797	1.757

AUTOREGR.COEFFS.

-1.044	0.025	0.132	-0.009	0.028	-0.089	-0.006	-0.011	-0.066	-0.012	0.120	-0.117
0.023	0.063	-0.123	0.153	-0.001	-0.018	0.026	0.032	-0.105	0.044	-0.003	-0.029

PARTIAL AUTOCORRELS.

0.980	-0.103	-0.053	0.084	0.110	0.137	0.029	0.047	0.030	-0.044	-0.039	0.054
-0.074	-0.061	-0.001	-0.121	0.041	0.042	0.034	0.058	0.090	-0.011	0.033	0.029

VARIANCES OF PARTIAL AUTOCORRELS

0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005
0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.006

SIG. STATISTIC

13.890	-1.446	-0.751	1.174	1.534	1.911	0.403	0.646	0.422	-0.604	-0.541	0.740
-1.011	-0.835	-0.013	-1.647	0.554	0.568	0.459	0.786	1.202	-0.143	0.440	0.381

TABLE 18b

REPEATED 24 TERM M.A. - NO MEAN CORRECTION

β_2

AUTOCORRELATIONS

0.971	0.959	0.925	0.904	0.887	0.886	0.879	0.886	0.887	0.891	0.881	0.885
0.865	0.859	0.835	0.816	0.790	0.779	0.766	0.768	0.768	0.774	0.768	0.771

VARIANCE OF RESIDUALS

0.6534E 01	0.6061E 01	0.5458E 01	0.5439E 01	0.5057E 01	0.4584E 01	0.4531E 01	0.4486E 01
0.4364E 01	0.4344E 01	0.4035E 01	0.3679E 01	0.3589E 01	0.3526E 01	0.3389E 01	0.3367E 01
0.3350E 01	0.3313E 01	0.3298E 01	0.3293E 01	0.3269E 01	0.3243E 01	0.3222E 01	0.3216E 01

VARIANCES OF AUTOCORRELATIONS

0.005	0.014	0.024	0.032	0.040	0.048	0.056	0.063	0.071	0.079	0.087	0.095
0.103	0.110	0.117	0.124	0.131	0.137	0.143	0.149	0.155	0.161	0.167	0.173

SIG. STATISTIC

13.769	7.999	6.035	5.053	4.427	4.044	3.722	3.518	3.321	3.168	2.986	2.876
2.700	2.591	2.438	2.314	2.183	2.105	2.024	1.989	1.953	1.931	1.882	1.856

AUTOREGR.COEFFS.

-0.818	-0.533	0.468	0.364	-0.411	-0.248	0.362	-0.032	-0.326	-0.002	0.353	-0.287
0.085	-0.124	0.005	0.150	0.172	-0.163	-0.045	0.176	-0.077	-0.130	0.116	-0.043

PARTIAL AUTOCORRELS.

0.971	0.269	-0.316	0.059	0.265	0.306	-0.108	0.099	0.165	0.067	-0.267	0.297
-0.157	-0.133	-0.197	-0.081	-0.070	0.105	0.068	-0.040	0.085	0.088	-0.081	0.043

VARIANCES OF PARTIAL AUTOCORRELS

0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005
0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.006

SIG. STATISTIC

13.769	3.795	-4.440	0.824	3.708	4.271	-1.508	1.382	2.287	0.933	-3.674	4.082
-2.150	-1.814	-2.681	-1.103	-0.956	1.423	0.922	-0.534	1.144	1.182	-1.080	0.577

TABLE 18b

REPEATED 24 TERM M.A. - NO MEAN CORRECTION

α_3

AUTOCORRELATIONS

0.844	0.818	0.818	0.762	0.795	0.841	0.763	0.694	0.714	0.677	0.674	0.751
0.641	0.611	0.597	0.536	0.564	0.591	0.531	0.480	0.481	0.439	0.446	0.511

VARIANCE OF RESIDUALS

0.9408E 01	0.8131E 01	0.7480E 01	0.7476E 01	0.6954E 01	0.6023E 01	0.5844E 01	0.4764E 01
0.4697E 01	0.4486E 01	0.4435E 01	0.4288E 01	0.4075E 01	0.4075E 01	0.4005E 01	0.3850E 01
0.3765E 01	0.3711E 01	0.3700E 01	0.3673E 01	0.3661E 01	0.3609E 01	0.3598E 01	0.3598E 01

VARIANCES OF AUTOCORRELATIONS

0.005	0.012	0.019	0.025	0.031	0.037	0.044	0.050	0.055	0.060	0.065	0.069
0.075	0.079	0.083	0.086	0.089	0.092	0.096	0.098	0.101	0.103	0.105	0.107

SIG. STATISTIC

11.967	7.451	5.974	4.786	4.501	4.344	3.620	3.093	3.041	2.760	2.650	2.856
2.345	2.176	2.077	1.825	1.890	1.948	1.716	1.529	1.516	1.367	1.375	1.562

AUTOREGR.COEFFS.

-0.438	-0.463	-0.054	0.266	-0.135	-0.520	0.127	0.572	-0.140	-0.380	0.151	-0.029
0.074	-0.274	0.154	0.386	-0.150	-0.208	0.066	0.158	-0.030	-0.139	0.058	-0.009

PARTIAL AUTOCORRELS.

0.844	0.368	0.283	-0.022	0.264	0.366	-0.173	-0.430	0.118	0.212	-0.106	0.182
-0.223	0.011	-0.131	-0.197	0.148	0.119	-0.055	-0.065	0.057	0.120	-0.054	0.009

VARIANCES OF PARTIAL AUTOCORRELS

0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005
0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.006

SIG. STATISTIC

11.967	5.197	3.982	-0.312	3.701	5.110	-2.403	-5.973	1.634	2.934	-1.463	2.501
-3.055	0.145	-1.784	-2.680	2.014	1.611	-0.748	-1.145	0.766	1.600	-0.727	0.116

TABLE 18b

REPEATED 24 TERM M.A. - NO MEAN CORRECTION

β_3

AUTOCORRELATIONS

0.825	0.812	0.803	0.733	0.790	0.809	0.770	0.710	0.750	0.740	0.732	0.837
0.699	0.683	0.661	0.587	0.630	0.634	0.598	0.542	0.565	0.542	0.539	0.629

VARIANCE OF RESIDUALS

0.1203E 02	0.9995E 01	0.9315E 01	0.9268E 01	0.8301E 01	0.7577E 01	0.7577E 01	0.6602E 01
0.6046E 01	0.5390E 01	0.5296E 01	0.5021E 01	0.4466E 01	0.4444E 01	0.4094E 01	0.3942E 01
0.3941E 01	0.3907E 01	0.3889E 01	0.3774E 01	0.3720E 01	0.3643E 01	0.3643E 01	0.3620E 01

VARIANCES OF AUTOCORRELATIONS

0.005	0.012	0.018	0.025	0.030	0.036	0.043	0.049	0.054	0.059	0.065	0.070
0.077	0.082	0.087	0.091	0.094	0.098	0.102	0.106	0.109	0.112	0.115	0.118

SIG. STATISTIC

11.702	7.494	5.935	4.660	4.556	4.246	3.720	3.218	3.237	3.039	2.878	3.159
2.518	2.402	2.245	1.945	2.049	2.022	1.868	1.666	1.713	1.619	1.590	1.833

AUTOREGR.COEFFS.

-0.381	-0.356	-0.099	0.333	-0.234	-0.417	0.067	0.338	-0.138	-0.405	0.124	-0.135
0.138	-0.241	0.215	0.306	0.004	-0.170	0.070	0.233	-0.051	-0.111	0.022	-0.080

PARTIAL AUTOCORRELS.

0.825	0.412	0.261	-0.071	0.323	-0.295	0.001	-0.359	0.290	0.329	-0.132	0.228
-0.332	-0.069	-0.281	-0.192	-0.019	0.093	-0.067	-0.172	0.120	0.143	0.008	0.080

VARIANCES OF PARTIAL AUTOCORRELS

0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005
0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.006

SIG. STATISTIC

11.702	5.806	3.668	-1.002	4.522	4.125	0.017	-4.983	4.020	4.552	-1.826	3.132
-4.559	-0.943	-3.831	-2.617	-0.260	1.253	-0.905	-2.313	1.612	1.918	0.113	1.069

TABLE 18c

REPEATED 24 TERM M.A. - MEAN CORRECTED

α_1

-4.78	-6.05	-5.83	-7.50	-7.66	-8.97	-5.74	-7.71	-4.06	-7.38	-7.66	-9.51
-8.66	-9.20	-7.65	-8.31	-7.45	-8.00	-7.10	-6.10	-5.55	-5.46	-6.00	-5.84
-4.84	-2.74	0.18	0.55	4.08	2.57	4.60	4.66	5.65	5.36	8.0[illegible]	6.74
5.71	6.40	6.64	8.00	10.33	8.13	6.49	4.08	3.07	4.17	5.36	4.58
2.56	-0.76	-6.29	-10.90	-14.40	-16.55	-16.62	-15.48	-15.69	-17.45	-20.37	-24.26
-25.07	-23.60	-20.26	-18.66	-17.22	-20.01	-22.47	-22.81	-24.17	-25.59	-26.85	-30.17
-30.76	-30.14	-26.91	-27.56	-27.40	-30.23	-31.59	-30.37	-29.54	-30.41	-30.89	-34.62
-33.72	-34.00	-32.77	-35.72	-36.21	-39.56	-38.02	-36.24	-35.07	-32.10	-38.40	-36.67
-40.90	-38.89	-36.62	-38.08	-38.36	-40.06	-40.35	-38.74	-39.78	-41.11	-44.31	-47.33
-46.91	-46.85	-44.46	-45.42	-44.82	-45.58	-45.18	-44.21	-46.53	-50.22	-54.33	-57.91
-55.76	-55.26	-52.77	-56.49	-58.70	-60.08	-53.90	-54.86	-53.82	-53.37	-56.65	-58.98
-58.28	-57.66	-52.72	-53.47	-52.28	-53.42	-53.62	-51.45	-52.76	-54.62	-58.16	-61.46
-59.29	-57.65	-52.33	-55.55	-55.56	-60.32	-58.78	-54.58	-50.22	-47.32	-48.83	-51.78
-53.10	-52.81	-49.79	-51.55	-49.68	-51.85	-50.76	-48.87	-50.55	-53.35	-57.72	-59.96
-57.99	-55.78	-51.24	-55.27	-54.56	-56.71	-54.44	-51.54	-51.64	-53.70	-57.38	-59.71
-58.47	-58.23	-56.05	-59.40	-61.90	-61.56	-62.94	-59.81	-61.84	-67.32	-70.89	-74.25
-73.02	-69.81	-67.70	-70.25	-71.07	-72.91	-73.05	-70.54	-73.43			

AUTOCORRELATIONS

0.985	0.969	0.951	0.938	0.927	0.916	0.902	0.884	0.866	0.850	0.835	0.822
0.806	0.791	0.776	0.764	0.752	0.739	0.724	0.707	0.692	0.679	0.667	0.655

VARIANCE OF RESIDUALS

0.1705E 02	0.1704E 02	0.1694E 02	0.1665E 02	0.1650E 02	0.1645E 02	0.1628E 02	0.1621E 02
0.1617E 02	0.1612E 02	0.1611E 02	0.1611E 02	0.1605E 02	0.1599E 02	0.1599E 02	0.1592E 02
0.1591E 02	0.1582E 02	0.1580E 02	0.1576E 02	0.1574E 02	0.1571E 02	0.1571E 02	0.1571E 02

VARIANCES OF AUTOCORRELATIONS

0.005	0.015	0.024	0.033	0.042	0.050	0.059	0.067	0.074	0.082	0.089	0.096
0.103	0.109	0.115	0.121	0.127	0.133	0.138	0.144	0.149	0.153	0.158	0.162

SIG. STATISTIC

13.961	8.014	6.145	5.164	4.539	4.084	3.723	3.424	3.172	2.968	2.797	2.650
2.514	2.394	2.283	2.191	2.108	2.026	1.947	1.867	1.795	1.734	1.676	1.620

AUTOREGR.COEFFS.

-0.991	-0.095	0.195	-0.010	-0.124	-0.040	0.007	0.009	0.117	-0.024	-0.012	-0.061
0.089	-0.053	0.075	-0.073	-0.056	0.030	-0.021	0.094	0.000	-0.054	0.010	0.005

PARTIAL AUTOCORRELS.

0.985	-0.019	-0.078	0.130	0.096	-0.057	-0.102	-0.065	-0.047	0.059	0.011	-0.002
-0.062	0.060	0.004	0.070	-0.010	-0.078	-0.034	-0.050	0.033	0.039	-0.015	-0.005

VARIANCES OF PARTIAL AUTOCORRELS

0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005
0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.006

SIG. STATISTIC

13.961	-0.268	-1.096	1.826	1.340	-0.798	-1.417	-0.902	-0.647	0.816	0.146	-0.023
-0.846	0.817	0.055	0.953	-0.135	-1.049	-0.460	-0.679	0.509	0.517	-0.200	-0.067

TABLE 18c

REPEATED 24 TERM M.A. - MEAN CORRECTED

β_1

-6.60	-6.37	-6.58	-5.02	-6.03	-5.62	-2.16	-8.84	-3.71	-8.79	-8.35	-8.45
-8.36	-10.27	-9.38	-9.91	-10.23	-10.45	-12.34	-13.45	-14.30	-14.52	-15.14	-16.31
-17.49	-20.58	-19.32	-19.67	-18.61	-17.36	-19.18	-20.18	-22.47	-20.88	-21.47	-18.50
-20.74	-23.69	-25.31	-26.39	-23.87	-21.03	-20.86	-23.07	-22.60	-24.41	-20.67	-18.07
-16.19	-14.13	-12.28	-13.68	-14.92	-18.51	-22.52	-23.96	-24.30	-23.33	-23.89	-26.67
-31.43	-34.74	-36.25	-33.29	-31.38	-28.71	-31.33	-32.52	-34.16	-32.50	-31.90	-31.35
-32.66	-35.31	-35.66	-33.46	-33.52	-31.97	-34.54	-35.28	-35.22	-31.07	-28.78	-26.29
-25.03	-23.43	-20.83	-17.14	-18.26	-18.92	-22.72	-22.87	-22.65	-20.76	-17.23	-18.46
-16.36	-15.24	-13.65	-9.13	-9.31	-7.90	-9.99	-8.78	-8.41	-5.89	-5.21	-6.61
-6.59	-6.59	-4.64	-1.87	-2.09	-1.29	-3.15	-2.74	-1.33	-0.39	-0.52	-2.83
-2.80	-2.27	0.56	3.54	0.07	-2.82	-8.71	-11.31	-11.86	-12.08	-11.76	-15.37
-16.59	-19.69	-20.09	-17.91	-19.89	-19.07	-22.90	-23.76	-24.16	-23.59	-24.15	-27.85
-29.72	-32.09	-31.15	-26.68	-29.49	-29.90	-36.63	-39.82	-40.79	-36.56	-32.52	-30.37
-28.79	-30.38	-25.98	-22.71	-19.82	-16.27	-17.95	-15.27	-13.76	-9.96	-9.44	-10.34
-8.97	-9.21	-4.62	0.21	-0.75	1.39	-0.93	0.06	2.11	4.74	5.68	4.86
6.67	6.13	10.86	12.02	11.18	10.48	7.12	7.23	8.06	11.06	10.06	10.04
8.49	8.85	11.74	14.77	13.93	13.53	8.92	9.76	12.09			

AUTOCORRELATIONS

0.975	0.946	0.912	0.880	0.854	0.820	0.778	0.731	0.685	0.643	0.604	0.562
0.512	0.461	0.411	0.363	0.316	0.261	0.203	0.144	0.092	0.046	0.002	-0.043

VARIANCE OF RESIDUALS

0.8924E 01	0.8855E 01	0.8749E 01	0.8728E 01	0.8669E 01	0.8318E 01	0.8066E 01	0.8034E 01
0.8021E 01	0.8018E 01	0.8015E 01	0.7984E 01	0.7781E 01	0.7775E 01	0.7772E 01	0.7742E 01
0.7717E 01	0.7573E 01	0.7504E 01	0.7502E 01	0.7479E 01	0.7470E 01	0.7453E 01	0.7448E 01

VARIANCES OF AUTOCORRELATIONS

0.005	0.014	0.023	0.032	0.039	0.047	0.053	0.059	0.065	0.069	0.073	0.077
0.080	0.083	0.085	0.087	0.088	0.089	0.090	0.090	0.090	0.090	0.090	0.090

SIG. STATISTIC

13.820	7.873	5.970	4.953	4.307	3.798	3.370	3.001	2.695	2.445	2.230	2.026
1.807	1.603	1.410	1.233	1.065	0.877	0.677	0.481	0.305	0.154	0.008	-0.144

AUTOREGR.COEFFS.

-0.992	-0.116	0.170	0.094	-0.237	0.001	0.070	0.063	0.019	0.009	-0.058	-0.104
0.154	-0.059	-0.009	0.024	-0.088	0.017	0.109	0.055	-0.024	-0.084	0.024	0.025

PARTIAL AUTOCORRELS.

0.975	-0.088	-0.109	0.049	0.082	-0.201	-0.174	-0.063	0.041	0.019	-0.019	-0.063
-0.160	0.026	-0.022	-0.062	-0.057	-0.136	-0.096	0.015	0.056	0.034	-0.048	-0.025

VARIANCES OF PARTIAL AUTOCORRELS

0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005
0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.006

SIG. STATISTIC

13.820	-1.246	-1.536	0.691	1.154	-2.808	-2.429	-0.874	0.565	0.256	-0.256	-0.862
-2.188	0.361	-0.299	-0.838	-0.774	-1.846	-1.291	0.206	0.754	0.455	-0.639	-0.327

TABLE 18c

REPEATED 24 TERM M.A. - MEAN CORRECTED

α_2

1.19	-0.02	0.94	1.44	2.12	2.02	4.08	-1.27	1.58	-2.62	-2.31	-2.03
-1.58	-0.92	-0.33	1.18	0.05	0.13	-0.88	-1.86	-2.20	-3.26	-4.29	-3.44
-3.32	-0.38	-0.79	0.89	-3.04	-1.70	-3.71	-3.71	-3.60	-2.89	-1.40	-0.40
0.96	4.48	2.48	2.42	-1.81	-3.27	-5.26	-5.90	-3.50	-4.34	0.49	0.96
0.07	-3.07	-5.87	-2.76	-1.53	0.67	-0.75	-3.36	-5.28	-7.25	-8.86	-9.87
-7.52	-5.05	-2.86	-4.60	-8.65	-10.28	-12.70	-14.03	-13.56	-15.45	-15.73	-14.77
-14.80	-12.64	-10.80	-11.01	-11.08	-12.79	-16.09	-18.51	-19.51	-21.21	-20.28	-18.96
-18.80	-19.61	-16.97	-16.02	-15.11	-12.48	-15.62	-17.14	-18.37	-17.25	-21.14	-17.07
-19.97	-18.92	-15.76	-16.16	-15.63	-14.90	-15.95	-17.33	-17.57	-18.48	-20.72	-20.59
-20.09	-20.24	-16.65	-16.06	-17.18	-16.17	-16.93	-18.51	-16.88	-20.17	-22.98	-23.31
-23.72	-24.37	-19.39	-17.09	-13.64	-11.10	-11.91	-16.67	-17.60	-19.03	-20.57	-19.52
-18.54	-17.26	-12.74	-12.29	-12.49	-12.98	-13.74	-15.24	-14.18	-16.82	-19.28	-19.18
-17.85	-16.00	-10.95	-9.00	-7.24	-6.78	-11.76	-17.20	-19.62	-18.99	-18.22	-14.39
-13.54	-11.48	-10.58	-7.58	-11.55	-11.08	-12.67	-13.52	-13.54	-16.22	-20.85	-20.47
-21.72	-19.05	-15.55	-13.47	-13.27	-12.60	-14.06	-17.25	-17.74	-19.81	-22.85	-23.46
-26.28	-25.89	-24.96	-24.01	-22.98	-22.90	-23.07	-27.56	-27.23	-31.92	-35.44	-34.78
-36.12	-33.59	-30.64	-29.95	-30.14	-29.01	-32.26	-35.00	-34.24			

AUTOCORRELATIONS!

0.955	0.901	0.838	0.783	0.744	C.726	0.713	0.713	0.713	0.706	0.690	0.672
0.643	0.607	0.569	0.520	0.479	0.448	0.428	0.419	0.424	0.425	0.425	0.418

VARIANCE OF RESIDUALS

0.7575E 01	0.7445E 01	0.7344E 01	0.7305E 01	0.7143E 01	0.6941E 01	0.6941E 01	0.6847E 01
0.6840E 01	0.6829E 01	0.6817E 01	0.6789E 01	0.6709E 01	0.6657E 01	0.6649E 01	0.6420E 01
0.6412E 01	0.6406E 01	0.6395E 01	0.6389E 01	0.6315E 01	0.6311E 01	0.6308E 01	0.6301E 01

VARIANCES OF AUTOCORRELATIONS

0.005	0.014	0.022	0.029	0.035	0.041	0.046	0.051	0.056	0.061	0.066	0.071
0.075	0.079	0.083	0.086	0.089	0.091	0.093	0.095	0.097	0.099	0.100	0.102

SIG. STATISTIC

13.544	7.601	5.635	4.590	3.965	3.594	3.325	3.155	3.009	2.854	2.682	2.527
2.341	2.153	1.975	1.770	1.605	1.482	1.401	1.359	1.363	1.354	1.340	1.307

AUTOREGR.COEFFS.

-1.026	-0.021	0.171	0.043	-0.015	-0.127	0.095	-0.112	-0.037	-0.024	0.131	-0.145
0.054	0.031	-0.141	0.210	-0.012	-0.000	-0.004	0.074	-0.136	0.047	0.013	-0.035

PARTIAL AUTOCORRELS.

0.955	-0.131	-0.117	0.073	0.149	0.168	-0.001	0.116	0.034	-0.039	-0.042	0.064
-0.108	-0.088	-0.035	-0.186	0.035	0.031	0.042	0.031	0.108	-0.023	0.022	0.035

VARIANCES OF PARTIAL AUTOCORRELS

0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005
0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.006

SIG. STATISTIC

13.544	-1.845	-1.644	1.026	2.086	2.345	-0.019	1.615	0.469	-0.543	-0.583	0.881
-1.486	-1.208	-0.474	-2.526	0.469	0.423	0.561	0.412	1.442	-0.309	0.298	0.463

TABLE 18c

REPEATED 24 TERM M.A. - MEAN CORRECTED

β_2

-8.45	-9.03	-8.83	-10.24	-8.89	-9.75	-13.19	-8.40	-14.32	-9.08	-10.43	-10.75
-9.54	-10.71	-9.14	-9.28	-7.60	-7.81	-5.85	-5.77	-5.19	-4.68	-5.52	-6.00
-5.47	-5.74	-3.68	-4.01	-2.32	-4.04	-1.57	-2.38	-3.09	-1.21	-3.01	0.68
-1.63	0.96	2.42	3.16	4.35	2.49	1.41	-0.66	-0.84	-2.62	-2.40	-0.09
1.10	1.63	-3.22	-5.42	-5.37	-3.14	0.89	1.41	1.66	1.18	-0.65	-3.48
-6.38	-6.15	-2.96	-0.24	1.60	-1.08	0.99	-1.72	-0.62	-0.08	-3.26	-3.18
-3.49	-5.02	-1.55	-2.20	0.51	0.13	2.09	-0.39	-1.25	-2.11	-6.47	-6.25
-6.01	-8.93	-8.56	-10.40	-9.41	-9.30	-4.97	-6.52	-7.24	-11.69	-11.35	-16.05
-12.90	-15.20	-13.91	-14.51	-13.55	-15.21	-11.69	-13.18	-12.22	-12.42	-14.15	-17.49
-16.89	-19.92	-18.58	-17.74	-16.66	-18.43	-14.97	-16.17	-13.90	-11.52	-13.80	-16.76
-16.00	-21.18	-20.31	-21.64	-19.83	-19.06	-12.26	-11.87	-11.69	-13.09	-14.46	-18.70
-16.53	-19.39	-15.05	-15.00	-12.11	-14.66	-11.05	-12.59	-10.08	-9.18	-11.64	-15.90
-15.54	-19.16	-14.95	-17.00	-11.83	-11.52	-5.13	-7.04	-10.57	-14.28	-17.59	-19.00
-16.04	-17.33	-15.26	-15.82	-12.68	-16.00	-11.29	-12.57	-9.88	-8.14	-10.70	-13.68
-12.90	-17.46	-13.59	-16.51	-12.74	-15.06	-9.70	-9.66	-8.75	-7.37	-8.28	-9.59
-7.54	-11.95	-9.76	-11.59	-11.33	-11.97	-6.86	-8.09	-6.07	-2.21	-7.05	-6.99
-9.17	-12.37	-10.82	-11.05	-9.66	-9.85	-3.73	-6.36	-2.03			

AUTOCORRELATIONS

0.912	0.881	0.781	0.730	0.688	0.698	0.695	0.732	0.751	0.773	0.752	0.770
0.713	0.709	0.642	0.596	0.528	0.508	0.474	0.491	0.497	0.522	0.509	0.523

VARIANCE OF RESIDUALS

0.6201E 01	0.5687E 01	0.4996E 01	0.4976E 01	0.4553E 01	0.4087E 01	0.4072E 01	0.4022E 01
0.3792E 01	0.3763E 01	0.3481E 01	0.3044E 01	0.3023E 01	0.2914E 01	0.2711E 01	0.2676E 01
0.2642E 01	0.2641E 01	0.2626E 01	0.2623E 01	0.2620E 01	0.2611E 01	0.2553E 01	0.2519E 01

VARIANCES OF AUTOCORRELATIONS

0.005	0.013	0.021	0.027	0.032	0.037	0.042	0.047	0.052	0.058	0.064	0.069
0.075	0.080	0.085	0.089	0.093	0.096	0.098	0.100	0.103	0.105	0.108	0.111

SIG. STATISTIC

12.936	7.649	5.395	4.437	3.826	3.627	3.396	3.388	3.291	3.220	2.979	2.926
2.602	2.503	2.200	1.996	1.733	1.643	1.512	1.549	1.551	1.609	1.549	1.574

AUTOREGR.COEFFS.

-0.704	-0.622	0.540	0.438	-0.488	-0.335	0.415	0.098	-0.573	-0.070	0.489	-0.177
-0.171	-0.165	0.191	0.169	-0.011	-0.139	0.162	0.119	-0.146	-0.088	0.228	-0.116

PARTIAL AUTOCORRELS.

0.912	0.288	-0.349	0.064	0.292	0.320	-0.062	0.110	0.239	0.080	-0.276	0.354
-0.084	-0.190	-0.264	-0.115	-0.113	-0.006	-0.077	-0.036	0.030	0.061	-0.149	0.116

VARIANCES OF PARTIAL AUTOCORRELS

0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005
0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.006

SIG. STATISTIC

12.936	4.061	-4.905	0.896	4.084	4.464	-0.866	1.533	3.314	1.100	-3.803	4.869
-1.155	-2.595	-3.595	-1.562	-1.533	-0.081	-1.033	-0.479	0.401	0.814	-1.987	1.539

TABLE 18c

REPEATED 24 TERM M.A. - MEAN CORRECTED

α_3

-5.34	-6.44	-6.32	-6.97	-8.67	-7.31	-13.28	-7.43	-13.11	-7.35	-9.47	-6.85
-6.94	-6.12	-5.28	-5.81	-6.25	-5.24	-4.47	-4.60	-3.67	-4.18	-6.45	-3.44
-3.99	-2.79	-2.08	-1.77	-0.71	-1.01	1.71	-0.18	-1.60	0.57	-0.97	0.85
3.94	1.41	-1.03	1.61	0.31	2.10	6.97	2.70	2.13	2.95	4.03	3.04
4.23	3.20	5.38	2.88	3.46	2.44	1.15	2.32	5.27	2.03	-1.47	1.60
3.19	1.08	2.61	0.65	-1.78	0.22	4.27	-0.32	0.23	-0.94	-6.44	-1.64
-2.93	-2.13	-2.55	-2.30	-6.62	-2.57	1.22	-2.36	-0.17	-2.62	-8.33	-2.89
-5.00	-2.67	0.03	-2.65	-2.47	-2.35	-0.58	-2.36	0.28	-2.33	-8.49	-2.41
-5.30	-2.25	0.42	-1.96	-2.58	-1.97	-1.50	-1.91	-0.01	-2.08	-6.17	-2.16
-4.01	-1.97	1.82	-2.16	-1.50	-2.55	-2.71	-2.67	-1.08	-2.95	-7.82	-3.29
-8.68	-3.11	2.27	-3.30	-4.14	-3.89	-6.09	-3.77	-1.02	-4.19	-9.50	-4.58
-6.12	-4.48	-2.67	-4.72	-7.11	-4.86	-4.05	-4.72	-1.76	-4.67	-10.21	-5.03
-8.04	-4.88	-1.86	-5.09	-10.51	-5.41	-2.55	-5.19	-3.09	-5.48	-13.71	-5.69
-5.19	-5.72	-2.92	-5.97	-7.27	-6.30	-6.09	-6.56	-3.93	-6.98	-13.21	-7.71
-13.97	-7.81	-5.50	-8.06	-10.05	-8.27	-10.03	-8.04	-6.03	-8.02	-11.89	-8.23
-14.49	-8.48	-6.34	-8.32	-8.03	-8.42	-10.15	-8.31	-5.99	-8.09	-15.12	-8.23
-14.25	-7.89	-3.92	-7.79	-7.52	-7.84	-5.85	-7.76	-1.54			

AUTOCORRELATIONS

0.719	0.683	0.690	0.602	0.673	0.769	0.643	0.528	0.589	0.534	0.552	0.705
0.514	0.470	0.456	0.356	0.418	0.479	0.375	0.292	0.311	0.242	0.270	0.397

VARIANCE OF RESIDUALS

0.8669E 01	0.7643E 01	0.7026E 01	0.7025E 01	0.6460E 01	0.5276E 01	0.5231E 01	0.4206E 01
0.4134E 01	0.3870E 01	0.3841E 01	0.3726E 01	0.3564E 01	0.3562E 01	0.3454E 01	0.3132E 01
0.3108E 01	0.3047E 01	0.3035E 01	0.2996E 01	0.2992E 01	0.2908E 01	0.2894E 01	0.2890E 01

VARIANCES OF AUTOCORRELATIONS

0.005	0.010	0.015	0.020	0.023	0.028	0.034	0.038	0.040	0.044	0.047	0.050
0.055	0.057	0.059	0.062	0.063	0.065	0.067	0.068	0.069	0.070	0.071	0.071

SIG. STATISTIC

10.191	6.791	5.683	4.308	4.430	4.630	3.511	2.720	2.932	2.552	2.553	3.161
2.197	1.963	1.871	1.434	1.668	1.884	1.452	1.119	1.182	0.914	1.015	1.485

AUTOREGR.COEFFS.

-0.317	-0.518	-0.055	0.314	-0.136	-0.614	0.088	0.696	-0.152	-0.511	0.104	0.084
0.028	-0.384	0.160	0.533	-0.105	-0.287	0.071	0.214	-0.022	-0.209	0.058	0.039

PARTIAL AUTOCORRELS.

0.719	0.344	0.284	-0.011	0.284	0.428	-0.092	-0.443	0.130	0.253	-0.088	0.173
-0.209	-0.021	-0.174	-0.305	0.088	0.140	-0.063	-0.113	0.036	0.168	-0.071	-0.039

VARIANCES OF PARTIAL AUTOCORRELS

0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005
0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.006

SIG. STATISTIC

10.191	4.855	3.997	-0.153	3.970	5.979	-1.281	-6.150	1.807	3.493	-1.207	2.372
-2.862	-0.283	-2.375	-4.153	1.189	1.891	-0.852	-1.521	0.486	2.241	-0.943	-0.516

TABLE 18c

REPEATED 24 TERM M.A. - MEAN CORRECTED

β_3

0.68	-0.14	1.12	2.97	1.53	2.39	1.94	3.15	2.44	5.08	2.79	1.18
3.24	2.47	4.15	6.84	4.69	5.63	5.19	5.97	5.97	8.21	6.48	5.49
7.25	7.87	7.79	11.42	8.46	10.14	8.75	11.13	9.12	10.72	9.47	10.56
9.49	15.41	9.63	11.08	9.24	6.54	8.99	13.72	9.00	10.38	8.85	9.20
8.44	9.12	8.21	10.33	7.73	9.28	7.08	5.23	6.81	10.51	6.63	3.83
6.27	5.38	5.90	8.90	5.81	5.09	5.79	9.54	5.63	10.32	5.06	2.69
4.36	2.44	3.55	6.06	3.13	0.36	2.48	4.14	2.27	8.31	1.71	-0.52
0.66	-4.89	0.03	2.03	-0.67	0.56	-1.13	-1.48	-1.67	-1.16	-1.90	-8.22
-2.16	-7.10	-2.14	-0.15	-1.77	0.61	-1.60	-0.98	-1.31	1.66	-1.26	-4.75
-1.36	-7.11	-0.88	1.76	-0.38	3.20	-0.23	1.58	0.15	5.23	0.37	-1.63
0.11	-9.86	0.20	3.81	0.63	4.60	0.30	-0.11	0.55	4.43	0.51	-5.09
0.23	-5.01	0.73	4.22	1.17	3.32	1.40	2.91	1.71	7.23	2.18	-1.63
2.15	-4.58	2.79	9.43	3.43	3.11	3.47	4.91	4.05	9.46	4.42	-2.21
4.69	3.16	5.22	12.05	5.40	9.16	5.26	5.54	5.24	10.08	5.49	3.38
5.33	-1.77	5.72	10.96	5.93	10.24	5.73	5.36	6.01	9.64	6.29	5.29
6.34	-0.17	6.20	8.02	6.48	10.79	6.61	8.04	6.81	14.74	7.05	7.78
6.60	-0.22	6.53	9.29	6.56	9.35	6.39	7.93	6.51			

AUTOCORRELATIONS

0.672	0.651	0.637	0.512	0.625	0.668	0.598	0.484	0.566	0.556	0.547	0.756
0.502	0.487	0.443	0.316	0.405	0.423	0.362	0.258	0.310	0.276	0.280	0.463

VARIANCE OF RESIDUALS

0.1092E 02	0.9470E 01	0.8923E 01	0.8866E 01	0.7978E 01	0.7182E 01	0.7178E 01	0.6158E 01
0.5686E 01	0.4870E 01	0.4814E 01	0.4509E 01	0.4051E 01	0.4042E 01	0.3688E 01	0.3481E 01
0.3471E 01	0.3449E 01	0.3419E 01	0.3259E 01	0.3236E 01	0.3174E 01	0.3174E 01	0.3168E 01

VARIANCES OF AUTOCORRELATIONS

0.005	0.009	0.014	0.018	0.020	0.024	0.029	0.032	0.035	0.038	0.041	0.044
0.049	0.052	0.054	0.056	0.057	0.059	0.061	0.062	0.063	0.064	0.064	0.065

SIG. STATISTIC

9.526	6.692	5.447	3.849	4.385	4.294	3.534	2.697	3.042	2.859	2.705	3.615
2.255	2.135	1.900	1.332	1.693	1.744	1.469	1.037	1.240	1.093	1.104	1.812

AUTOREGR.COEFFS.

-0.318	-0.357	-0.104	0.357	-0.206	-0.467	0.037	0.387	-0.120	-0.473	0.105	-0.106
0.122	-0.307	0.200	0.350	0.023	-0.191	0.070	0.272	-0.032	-0.121	0.011	-0.045

PARTIAL AUTOCORRELS.

0.672	0.364	0.240	-0.080	0.316	0.316	0.022	-0.377	0.277	0.379	-0.107	0.252
-0.319	-0.047	-0.296	-0.237	-0.055	0.078	-0.093	-0.216	0.084	0.138	0.004	0.045

VARIANCES OF PARTIAL AUTOCORRELS

0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005
0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.006

SIG. STATISTIC

9.526	5.135	3.384	-1.119	4.430	4.412	0.306	-5.238	3.834	5.236	-1.475	3.460
-4.371	-0.637	-4.038	-3.222	-0.744	1.058	-1.259	-2.908	1.130	1.851	0.050	0.604

TABLE 19 a

PERIODOGRAM AND SMOOTHED PERIODOGRAMS OF y(n)

PERIODOGRAM ORDINATES FOR 192 OBSERVATIONSOF MOTOR VEHICLE REGISTRATIONS SERIES

K	$(\pi+(K-1)5\pi)/96$	$(2\pi+(K-1)5\pi)/96$	$(3\pi+(K-1)5\pi)/96$	$(4\pi+(K-1)5\pi)/96$	$(5\pi+(K-1)5\pi)/96$
1	0.89385E 00	0.40658E 01	0.10957E 02	0.21046E 02	0.37652E 01
2	0.52899E 02	0.77827E 02	0.14614E 02	0.43956E 02	0.31067E 02
3	0.57782E 02	0.85233E 02	0.79743E 02	0.20359E 02	0.14347E 02
4	0.17855E 04	0.41987E 03	0.37555E 03	0.12893E 03	0.63864E 02
5	0.27334E 03	0.35708E 02	0.61663E 02	0.45001E 02	0.48912E 02
6	0.64495E 02	0.85831E 02	0.81603E 02	0.11875E 02	0.31571E 01
7	0.23019E 02	0.11801E 04	0.72340E 02	0.23795E 02	0.35253E 01
8	0.95463E 00	0.64053E 02	0.60929E 02	0.89247E 01	0.88225E 01
9	0.36434E 02	0.41647E 01	0.15552E 02	0.13430E 02	0.29972E 02
10	0.65569E 02	0.12463E 03	0.10750E 04	0.38042E 01	0.35382E 02
11	0.11950E 03	0.59895E 01	0.23702E 01	0.60981E 01	0.15057E 02
12	0.12991E 02	0.91306E 01	0.30411E 02	0.95028E 01	0.48988E 01
13	0.11128E 02	0.20211E 02	0.17433E 02	0.31432E 03	0.58107E 01
14	0.26219E 01	0.53364E 03	0.35084E 02	0.23854E 02	0.31002E 02
15	0.40873E 02	0.89979E 01	0.66345E 02	0.11637E 02	0.46100E 01
16	0.34140E 02	0.12334E 03	0.83183E 00	0.10523E 02	0.49187E 03
17	0.14084E 02	0.44626E 02	0.56954E 02	0.31537E 01	0.19539E 02
18	0.45075E 02	0.89155E 01	0.17556E 02	0.17305E 02	0.36670E 02
19	0.32138E 02	0.11217E 02	0.48789E 02	0.22020E 02	0.72214E 01
20	0.79935E 01				

THREE-TERM AVERAGES

K	$(2\pi+(K-1)5.3\pi)/96$	$(5\pi+(K-1)5.3\pi)/96$	$(8\pi+(K-1)5.3\pi)/96$	$(11\pi+(K-1)5.3\pi)/96$	$(14\pi+(K-1)5.3\pi)/96$
1	0.53073E 01	0.25904E 02	0.45466E 02	0.58027E 02	0.38318E 02
2	0.39771E 03	0.15555E 03	0.47457E 02	0.66429E 02	0.32212E 02
3	0.47680E 02	0.94249E 01	0.44636E 02	0.16474E 02	0.19651E 02
4	0.95099E 02	0.52894E 02	0.48193E 01	0.12393E 02	0.14938E 02
5	0.16257E 02	0.42163E 01	0.19752E 03	0.26958E 02	0.27531E 02
6	0.52770E 02	0.12304E 02	0.34911E 02	0.24510E 02	0.23844E 02
7	0.30715E 02	0.14621E 02			

FIVE-TERM AVERAGES

K	$(3\pi+(K-1)5.5\pi)/96$	$(8\pi+(K-1)5.5\pi)/96$	$(13\pi+(K-1)5.5\pi)/96$	$(18\pi+(K-1)5.5\pi)/96$	$(23\pi+(K-1)5.5\pi)/96$
1	0.81467E 01	0.44073E 02	0.51594E 02	0.24705E 03	0.93024E 02
2	0.49402E 02	0.30670E 02	0.28737E 02	0.19910E 02	0.57346E 02
3	0.29803E 02	0.13387E 02	0.13646E 02	0.12524E 03	0.26493E 02
4	0.16383E 03	0.27671E 02	0.25105E 02	0.24277E 02	0.0

TABLE 19b

PERIODOGRAM AND SMOOTHED PERIODOGRAMS OF y(n)

PERIODOGRAM ORDINATES FOR 240 OBSERVATIONSOF BANK ADVANCES SERIES

K	(π+(K-1)5π)/120	(2π+(K-1)5π)/120	(3π+(K-1)5π)/120	(4π+(K-1)5π)/120	(5π+(K-1)5π)/120
1	0.33653E 02	0.22041E 01	0.12081E 02	0.56793E 02	0.28049E 03
2	0.35044E 03	0.53396E 03	0.26151E 03	0.11267E 04	0.43086E 03
3	0.31227E 02	0.81235E 03	0.81820E 02	0.48914E 03	0.22074E 03
4	0.66373E 03	0.70568E 03	0.16815E 03	0.68294E 03	0.97890E 04
5	0.34481E 04	0.80647E 03	0.15532E 03	0.67754E 02	0.30353E 03
6	0.33685E 03	0.14299E 02	0.25043E 03	0.53139E 02	0.10204E 01
7	0.49796E 02	0.11354E 03	0.16305E 02	0.58942E 02	0.13607E 02
8	0.27314E 02	0.84937E 01	0.27003E 03	0.14651E 03	0.17029E 04
9	0.31764E 03	0.89313E 02	0.14534E 03	0.12049E 03	0.69928E 02
10	0.57341E 00	0.17246E 02	0.14428E-01	0.44881E 01	0.25105E 02
11	0.30745E 02	0.50826E 01	0.10999E 02	0.18003E 02	0.11278E 02
12	0.17379E 01	0.11404E 02	0.48836E 02	0.10968E 02	0.27931E 03
13	0.98778E 02	0.26552E 02	0.19936E 02	0.16025E 02	0.95932E 01
14	0.68374E 01	0.29530E 02	0.27912E 02	0.56667E 01	0.30512E 01
15	0.12181E 02	0.42317E 01	0.38967E 02	0.16833E 01	0.65752E 01
16	0.27258E 02	0.55578E 00	0.11939E 02	0.62261E 02	0.30804E 03
17	0.97762E 02	0.17317E 01	0.66675E 00	0.55163E 02	0.74807E 01
18	0.68950E 01	0.80413E 01	0.24694E 02	0.30224E 02	0.11079E 02
19	0.17042E 01	0.89498E 01	0.73274E 00	0.18441E 00	0.50361E 01
20	0.22699E 02	0.46044E 01	0.46723E 02	0.14905E 02	0.46417E 02
21	0.56467E 01	0.11162E 02	0.17501E 02	0.35655E 01	0.25775E 01
22	0.38216E 02	0.89141E 01	0.40623E 01	0.13287E 02	0.20871E 02
23	0.95175E 01	0.14379E 02	0.37263E 01	0.23059E 02	0.39157E 01
24	0.78549E 01	0.59023E 01	0.22091E 02	0.66869E 01	0.11240E 02

THREE-TERM AVERAGES

K	(2π+(K-1)5.3π)/120	(5π+(K-1)5.3π)/120	(8π+(K-1)5.3π)/120	(11π+(K-1)5.3π)/120	(14π+(K-1)5.3π)/120
1	0.15980E 02	0.22924E 03	0.64073E 03	0.42481E 03	0.26390E 03
2	0.51252E 03	0.20655E 04	0.34318E 03	0.21823E 03	0.10153E 03
3	0.59880E 02	0.33287E 02	0.14168E 03	0.20348E 03	0.11192E 03
4	0.59446E 01	0.20112E 02	0.11362E 02	0.81399E 01	0.29902E 02
5	0.48422E 02	0.10819E 02	0.21036E 02	0.64880E 01	0.15742E 02
6	0.13251E 02	0.80011E 02	0.19187E 02	0.74723E 01	0.21999E 02
7	0.37956E 01	0.93066E 01	0.22078E 02	0.84043E 01	0.78813E 01
8	0.17064E 02	0.14559E 02	0.13888E 02	0.58910E 01	0.14389E 02

FIVE-TERM AVERAGES

K	(3π+(K-1)5.5π)/120	(8π+(K-1)5.5π)/120	(13π+(K-1)5.5π)/120	(18π+(K-1)5.5π)/120	(23π+(K-1)5.5π)/120
1	0.77044E 02	0.54070E 03	0.32706E 03	0.22205E 04	0.95623E 03
2	0.13115E 03	0.50438E 02	0.45235E 03	0.14854E 03	0.94853E 01
3	0.15222E 02	0.72945E 02	0.34177E 02	0.14599E 02	0.12728E 02
4	0.10201E 03	0.32561E 02	0.16187E 02	0.33215E 01	0.88932E 02
5	0.80905E 01	0.17070E 02	0.11020E 02	0.42535E 02	

TABLE 19c

PERIODOGRAM AND SMOOTHED PERIODOGRAMS OF y(n)

PERIODOGRAM ORDINATES FOR 216 OBSERVATIONS OF WOOL SERIES

K	$(\pi+(K-1)5\pi)/108$	$(2\pi+(K-1)5\pi)/108$	$(3\pi+(K-1)5\pi)/108$	$(4\pi+(K-1)5\pi)/108$	$(5\pi+(K-1)5\pi)/108$
1	0.46854E 00	0.16170E 01	0.60756E 00	0.90204E 00	0.30461E 02
2	0.25709E 02	0.50749E 02	0.52456E 02	0.18658E 03	0.14093E 02
3	0.68529E 02	0.12312E 02	0.33626E 02	0.46367E 03	0.45654E 03
4	0.44811E 03	0.38091E 03	0.12349E 06	0.18337E 04	0.11317E 04
5	0.10941E 03	0.13784E 04	0.11743E 03	0.67775E 03	0.51123E 03
6	0.28750E 03	0.51620E 03	0.72106E 03	0.47615E 03	0.69261E 03
7	0.44396E 03	0.10574E 03	0.11299E 03	0.99321E 02	0.73861E 03
8	0.97546E 05	0.47146E 03	0.11306E 03	0.18191E 04	0.23351E 03
9	0.27197E 03	0.27230E 03	0.60992E 03	0.66797E 02	0.62439E 03
10	0.86564E 02	0.11580E 04	0.17281E 02	0.72167E 03	0.18287E 03
11	0.14990E 03	0.36891E 03	0.63355E 03	0.10764E 06	0.30598E 03
12	0.79348E 03	0.93782E 02	0.79150E 02	0.59848E 02	0.27933E 03
13	0.51450E 02	0.15891E 03	0.51273E 03	0.41306E 03	0.21934E 04
14	0.48690E 03	0.12634E 03	0.11257E 04	0.58616E 03	0.38692E 03
15	0.82047E 03	0.38016E 05	0.14141E 04	0.25279E 03	0.34272E 04
16	0.21329E 03	0.64993E 03	0.94082E 03	0.13182E 04	0.69477E 03
17	0.50157E 03	0.71869E 02	0.20913E 04	0.63645E 03	0.31613E 03
18	0.18758E 03	0.72639E 03	0.16759E 04	0.23581E 03	0.17586E 04
19	0.14989E 04	0.43868E 03	0.94915E 03	0.18155E 04	0.85967E 03
20	0.78651E 03	0.54542E 03	0.50540E 03	0.40718E 03	0.78117E 03
21	0.79124E 03	0.22125E 03	0.11546E 03	0.22358E 04	0.24368E 04
22	0.11559E 04	0.16371E 04	0.20737E 05		

THREE-TERM AVERAGES

K	$(2\pi+(K-1)5.3\pi)/108$	$(5\pi+(K-1)5.3\pi)/108$	$(8\pi+(K-1)5.3\pi)/108$	$(11\pi+(K-1)5.3\pi)/108$	$(14\pi+(K-1)5.3\pi)/108$
1	0.89769E 00	0.19024E 02	0.96595E 02	0.31645E 02	0.31795E 03
2	0.41451E 03	0.10249E 04	0.72453E 03	0.43831E 03	0.62994E 03
3	0.22090E 03	0.41896E 03	0.80122E 03	0.25943E 03	0.43370E 03
4	0.42061E 03	0.35148E 03	0.50123E 03	0.39775E 03	0.13944E 03
5	0.24103E 03	0.10311E 04	0.61273E 03	0.60370E 03	0.16981E 04
6	0.60135E 03	0.83485E 03	0.94986E 03	0.41020E 03	0.95588E 03
7	0.96224E 03	0.11539E 04	0.48600E 03	0.59789E 03	0.15960E 04
8	0.13965E 04				

FIVE-TERM AVERAGES

K	$(3\pi+(K-1)5.5\pi)/108$	$(8\pi+(K-1)5.5\pi)/108$	$(13\pi+(K-1)5.5\pi)/108$	$(18\pi+(K-1)5.5\pi)/108$	$(23\pi+(K-1)5.5\pi)/108$
1	0.68112E 01	0.65917E 02	0.20694E 03	0.94861E 03	0.55884E 03
2	0.53870E 03	0.30013E 03	0.65929E 03	0.36917E 03	0.43327E 03
3	0.36459E 03	0.26112E 03	0.66592E 03	0.54240E 03	0.14787E 04
4	0.76140E 03	0.73345E 03	0.28262E 04	0.11124E 04	0.60514E 03
5	0.11601E 04	0.27930E 04			

FIG. XX
NORMALIZED SPECTRAL DENSITY FOR ORIGINAL SERIES
BANK ADVANCES

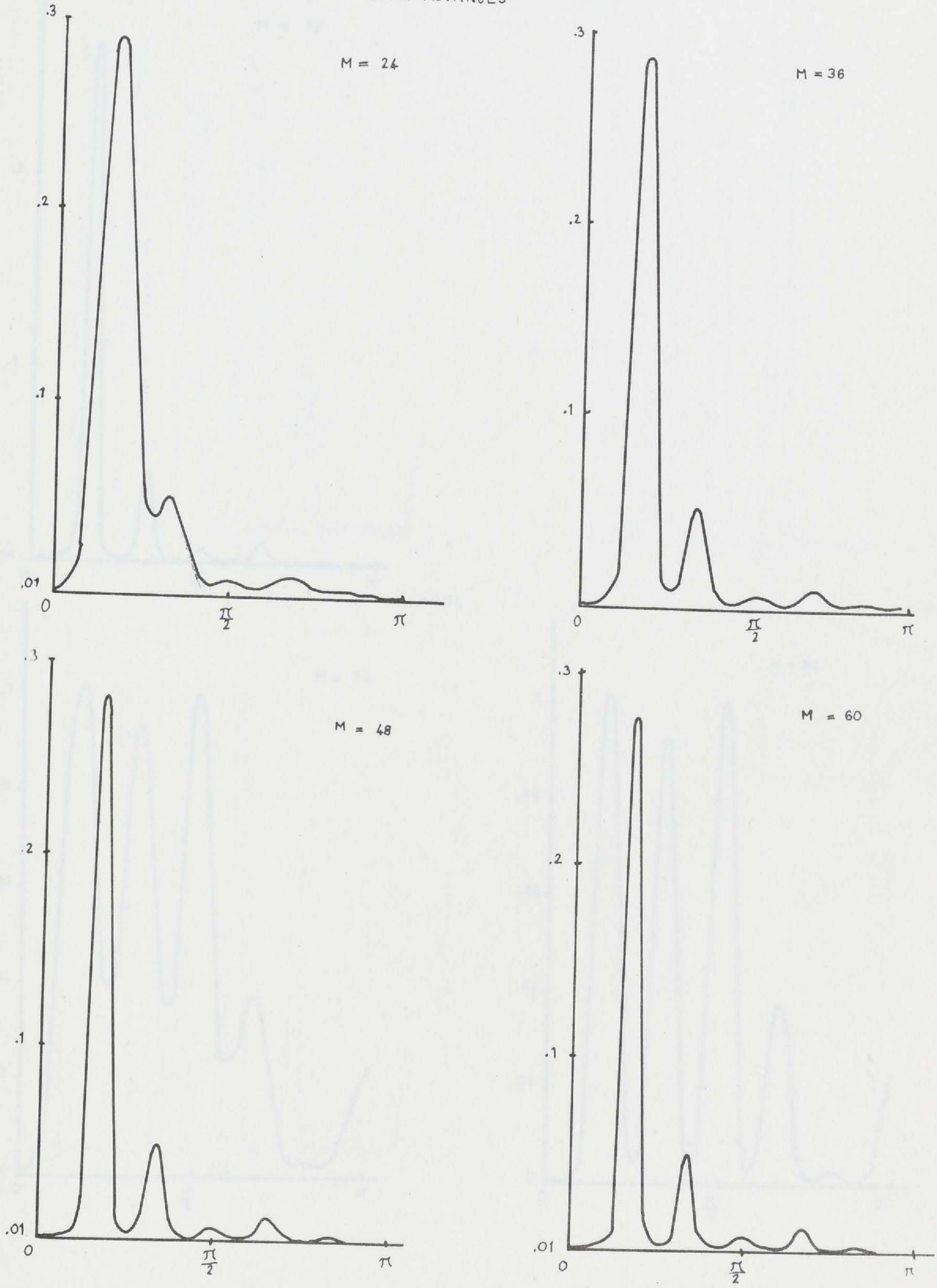


FIG. XX

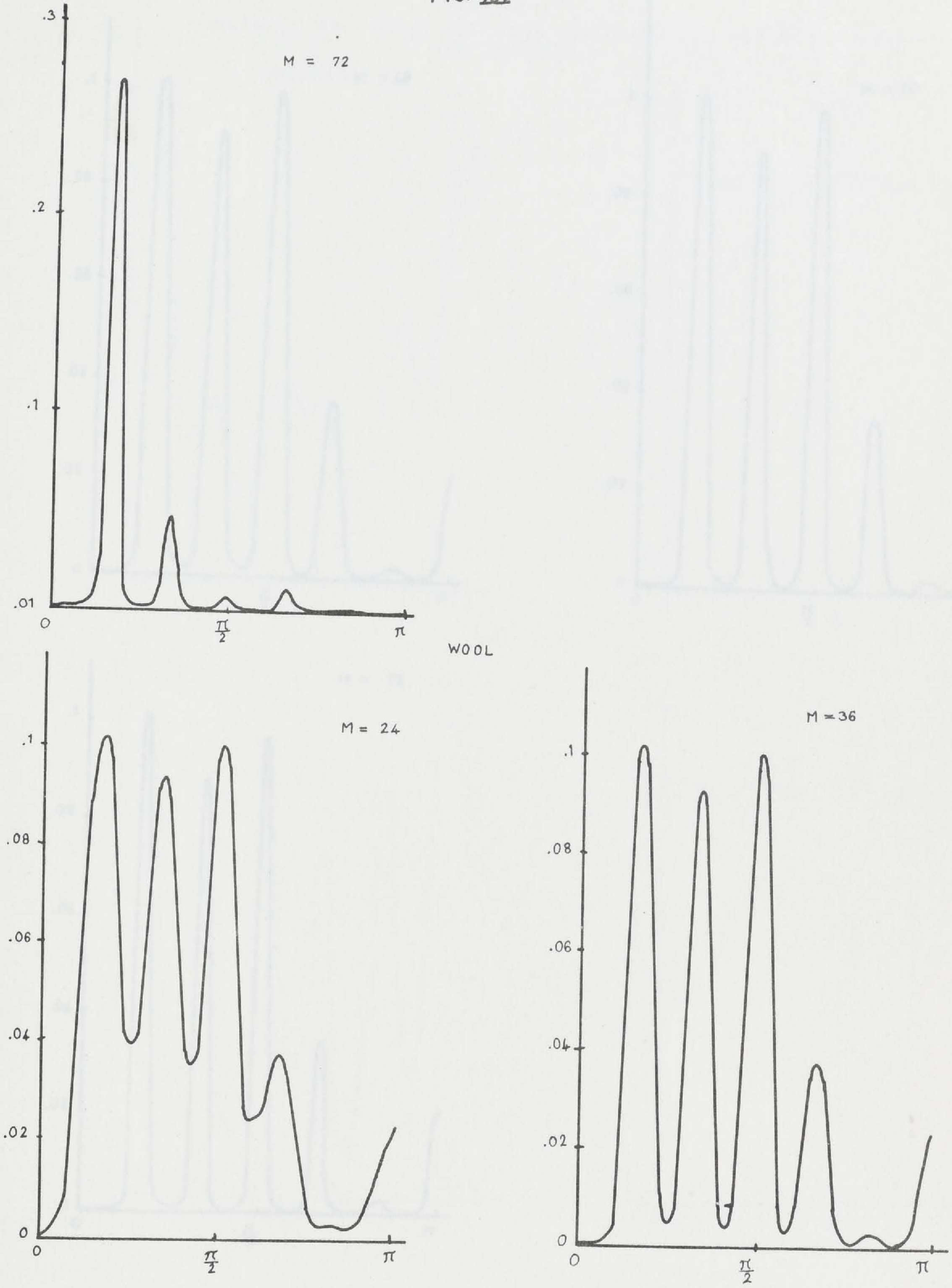
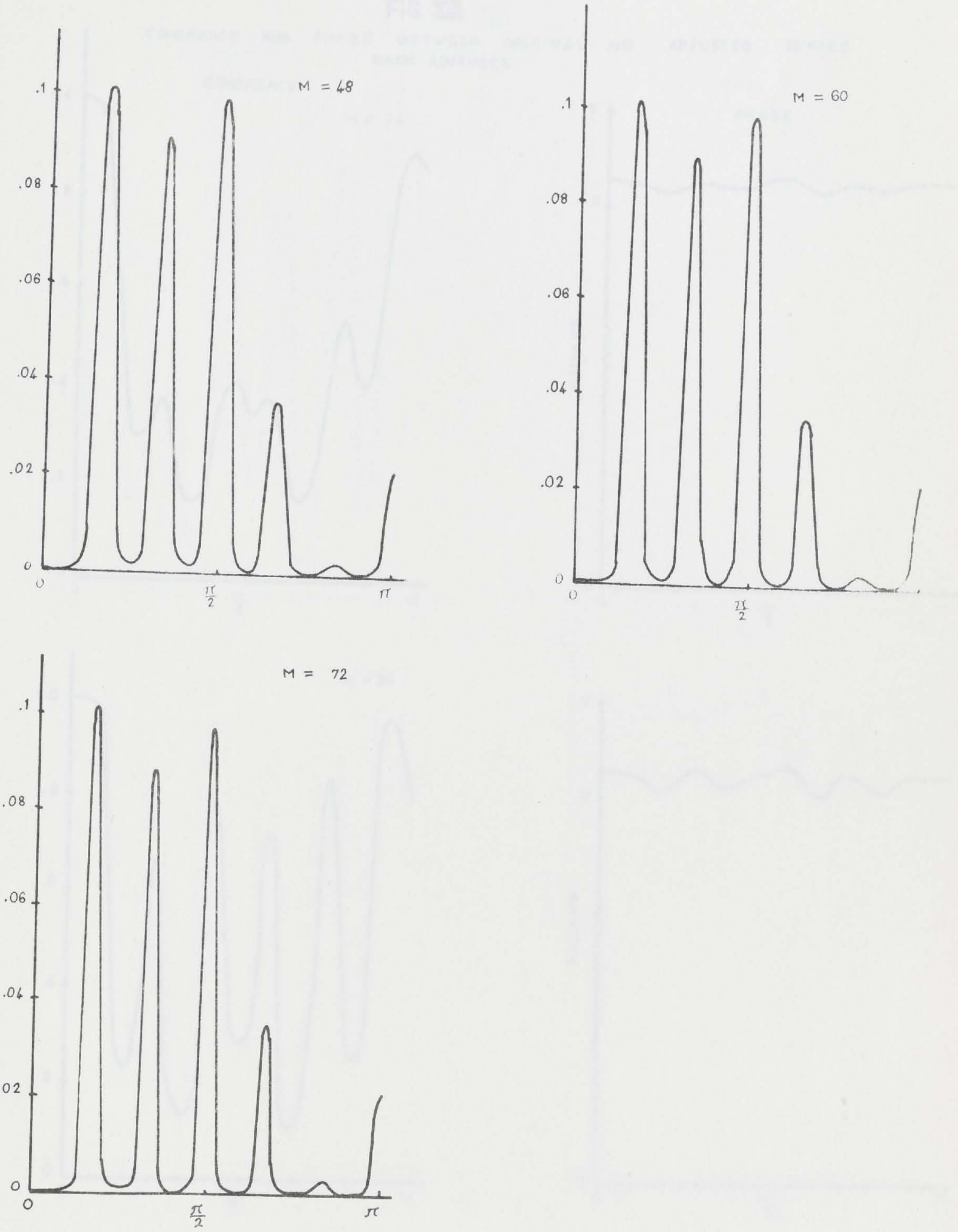

M = 72
.3
.2
.1
.01
0
π/2
π
WOOL
M = 24
.1
.08
.06
.04
.02
0
0
π/2
π
M = 36
.1
.08
.06
.04
.02
0
0
π/2
π

FIG. XX



M = 48
.1
.08
.06
.04
.02
0
0
π/2
π
M = 60
.1
.08
.06
.04
.02
0
0
π/2
M = 72
.1
.08
.06
.04
.02
0
0
π/2
π

FIG. XXI

COHERENCE AND PHASE BETWEEN ORIGINAL AND ADJUSTED SERIES
BANK ADVANCES

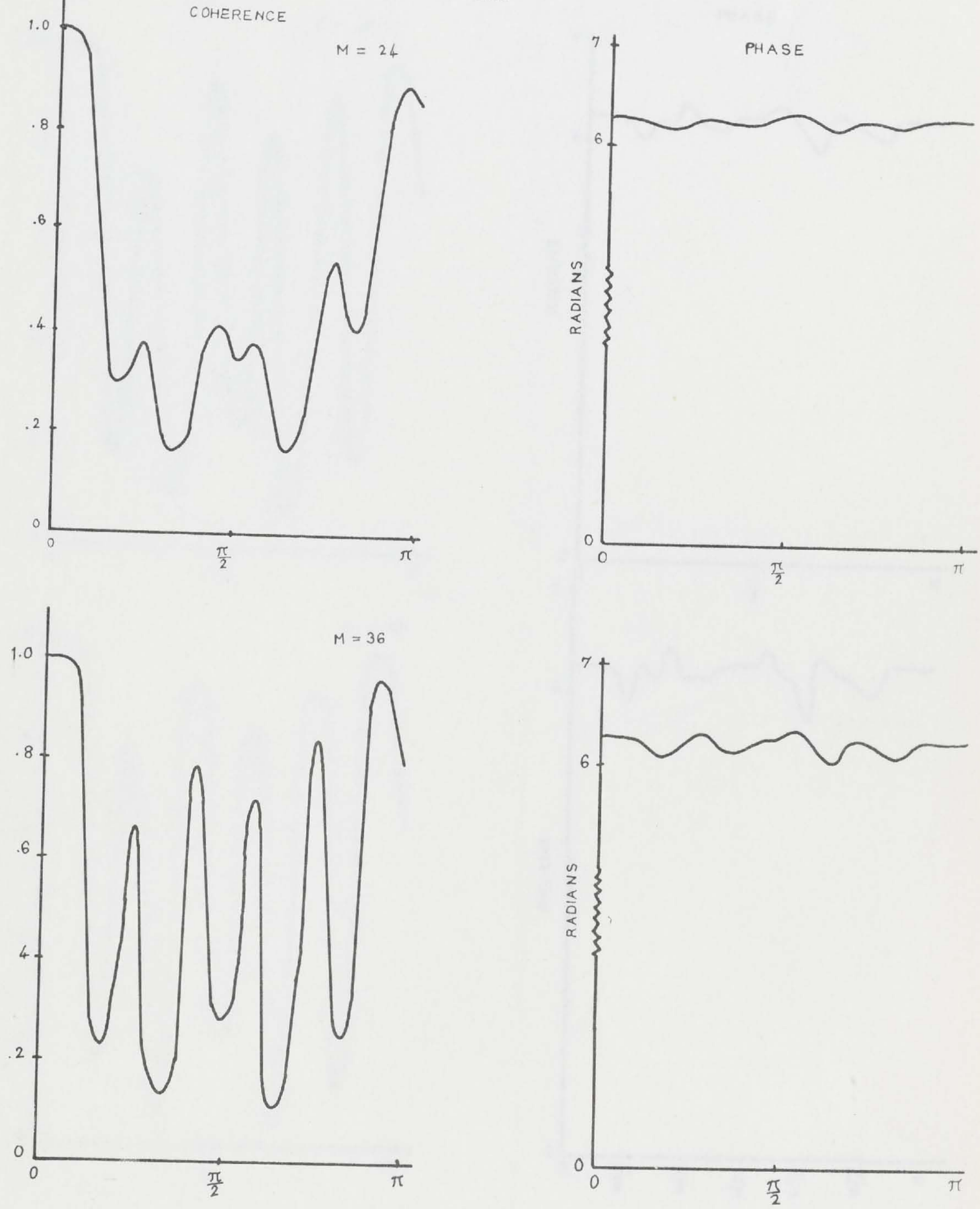


FIG. XXI

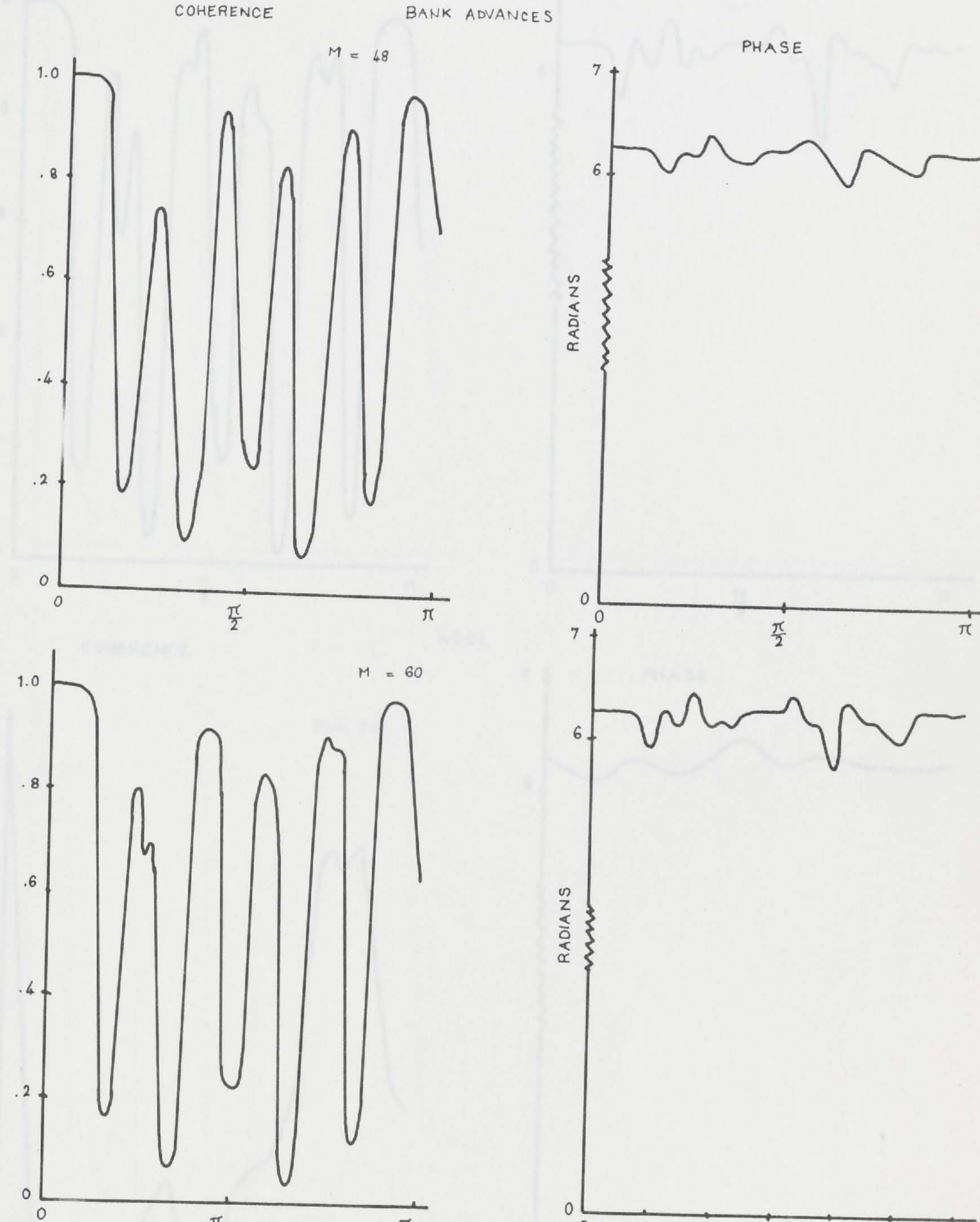

COHERENCE
BANK ADVANCES
PHASE
M = 48
M = 60
RADIANS
1.0
.8
.6
.4
.2
0
7
6
π/2
π
π/6
π/3
2π/3
5π/6

FIG. XXI

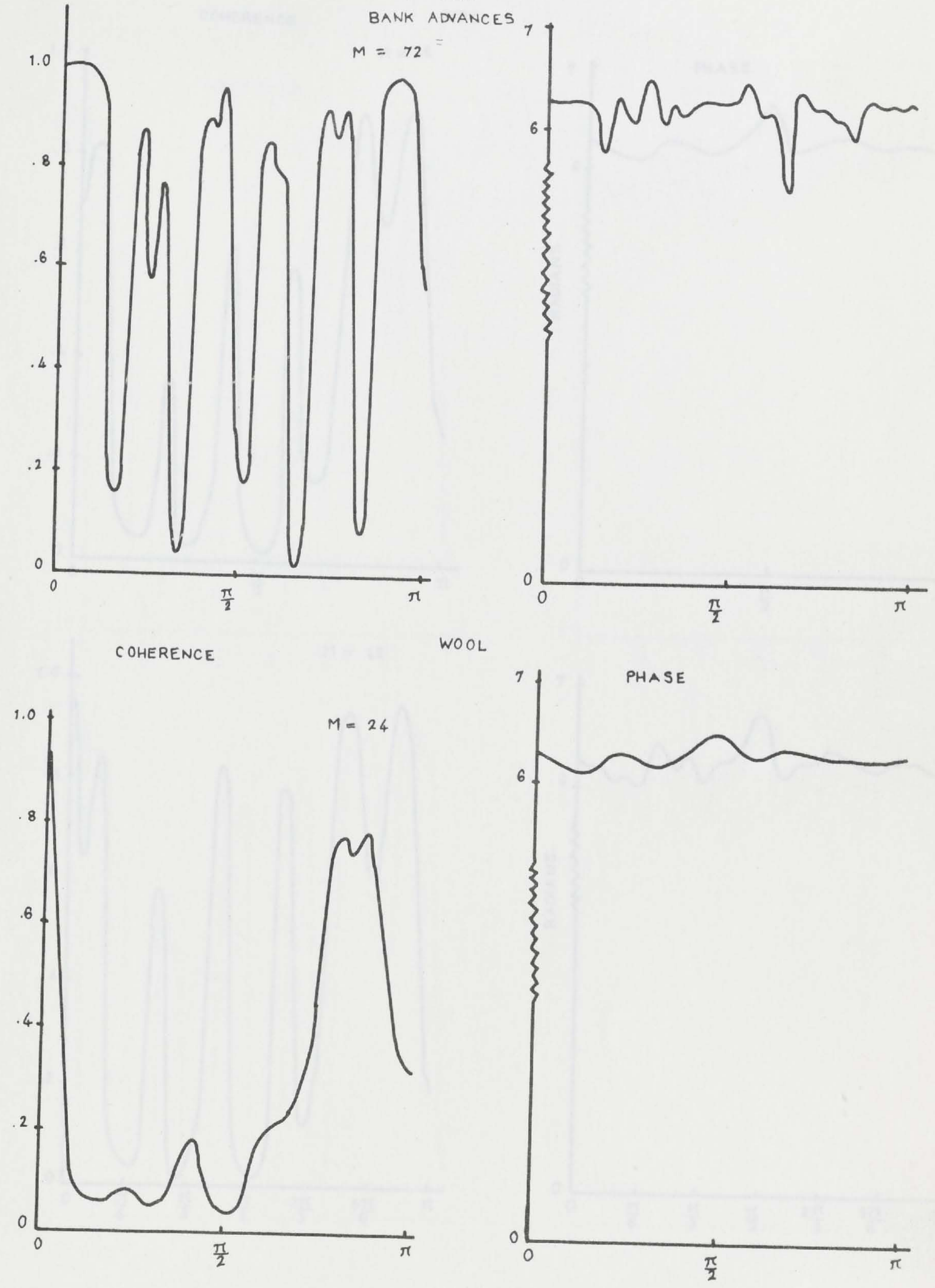
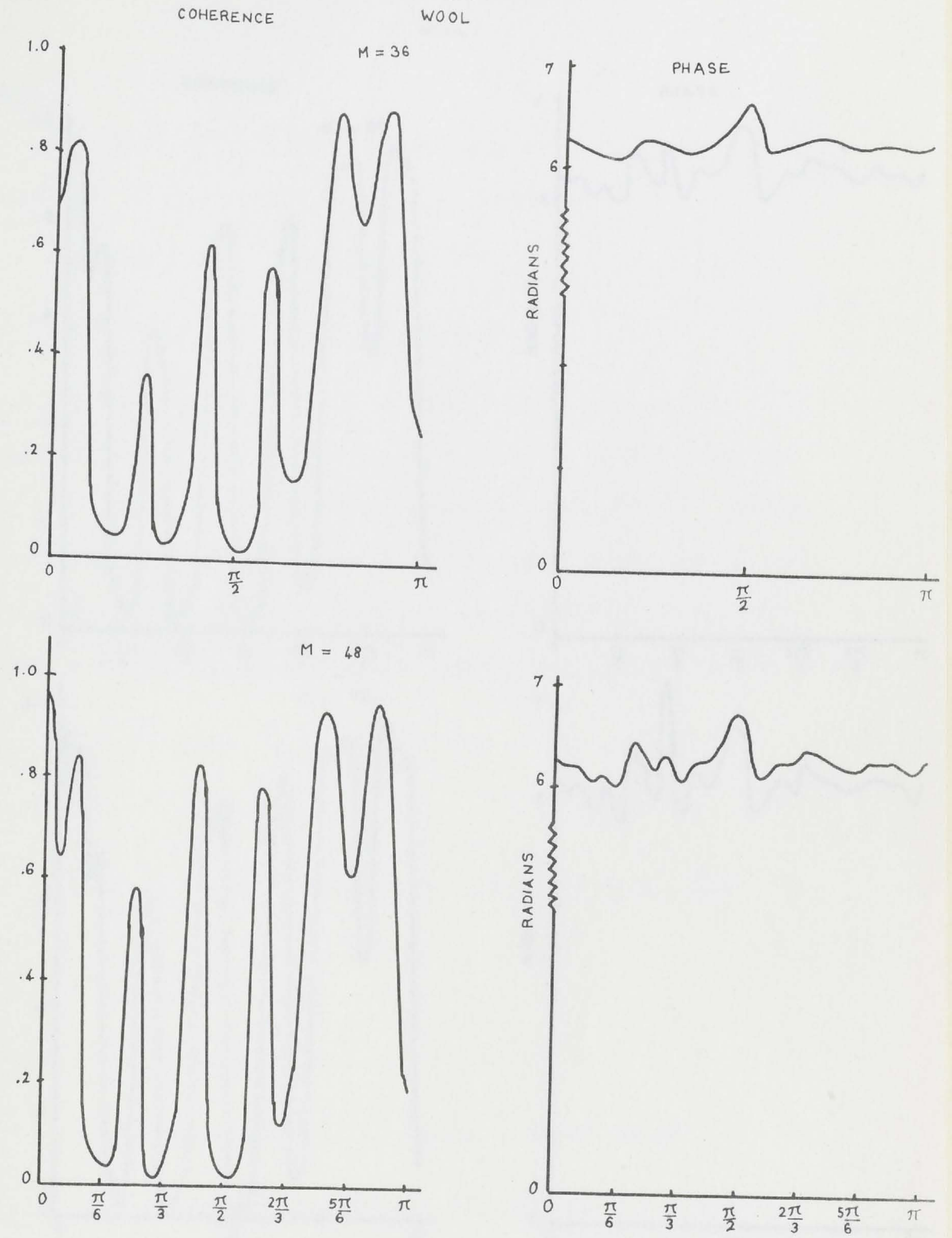

BANK ADVANCES
M = 72
1.0
.8
.6
.4
.2
0
0
π/2
π
7
6
0
0
π/2
π
COHERENCE
WOOL
PHASE
M = 24
1.0
.8
.6
.4
.2
0
0
π/2
π
7
6
0
0
π/2
π

FIG. XXI


COHERENCE
WOOL
M = 36
PHASE
RADIANS
M = 48
RADIANS
1.0
.8
.6
.4
.2
0
7
6
0
π/2
π
π/6
π/3
2π/3
5π/6

APPENDIX A EFFECTS OF MIS-SPECIFICATION OF MODEL ON ESTIMATE OF σ_ϵ^2

Case I

The mis-specification involves only erroneous inclusion or exclusion of explanatory variables. Initially

$$\begin{aligned} &\text{Assumed Model:} \quad y = X_1\beta_1+\eta \\ &\text{True Model:} \quad y = X_1\beta_1+X_2\beta_2+\epsilon \end{aligned} \tag{A.1}$$

and the data matrix in the true model is $N \times K$ and may be partitioned as follows, $X = (X_1, X_2)$, where X_1 is $N \times L$ and X_2 is $N \times (K-L)$. The disturbance term ϵ is N.I.D. $(0, \sigma_\epsilon^2)$. If the residuals from the true model were computed, denote them $\hat{\epsilon}$, then the mean of the statistic $\hat{\epsilon}'\hat{\epsilon}/(N-K)$ is $\mathcal{E}(\hat{\epsilon}'\hat{\epsilon}/(N-K)) = \sigma_\epsilon^2$. The residuals from the assumed model, denote them $\hat{\eta}$, are obtained from

$$\hat{\eta} = y - X_1\tilde{\beta}_1, \quad \text{where} \quad \tilde{\beta}_1 = (X_1'X_1)^{-1}X_1'y \tag{A.2}$$

and may be written as

$$\begin{aligned} \hat{\eta} &= X_1\beta_1+X_2\beta_2+\epsilon-X_1(X_1'X_1)^{-1}X_1'X_1\beta_1-X_1(X_1'X_1)^{-1}X_1'X_2\beta_2-X_1(X_1'X_1)^{-1}X_1'\epsilon \\ &= (I-P_1)(X_2\beta_2+\epsilon) \end{aligned} \tag{A.3}$$

where $P_1 = (X_1(X_1'X_1)^{-1}X_1')$.

The statistic $(\hat{\eta}'\hat{\eta})/(N-L)$, easily obtained using (A.3), is

$$(N-L)^{-1}\hat{\eta}'\hat{\eta} = (X_2\beta_2+\epsilon)'(I-P_1)(X_2\beta_2+\epsilon)(N-L)^{-1} \tag{A.4}$$

and the mean of (A.4) is

$$\mathcal{E}\left((N-L)^{-1}\hat{\eta}'\hat{\eta}\right) = \sigma_\epsilon^2+\beta_2' \frac{\{X_2'X_2-X_2'X_1(X_1'X_1)^{-1}X_1'X_2\}}{(N-L)} \beta_2. \tag{A.5}$$

The second term in (A.5) is a positive definite quadratic form in β_2 and so the estimate $\hat{\eta}'\hat{\eta}(N-L)^{-1}$ has a mean which over estimates σ_ϵ^2 by an amount which depends on the degree of correlation between the sets of variables in X_1 and X_2.

Case II

Here one considers the case of erroneous inclusion of variables, i.e.

Assumed Model: $y = X_1\beta_1+X_2\beta_2+\eta$

True Model: $y = X_1\beta_1+\epsilon$.

The residuals from the assumed model are again $\hat{\eta}$ and are

$$\hat{\eta} = y-X\tilde{\beta}, \quad \text{where} \quad \tilde{\beta} = (X'X)^{-1}X'y. \tag{A.6}$$

On using the estimate of $\tilde{\beta}$ given in (A.6) $\hat{\eta}$ becomes,

$$\hat{\eta} = y-X(X'X)^{-1}X'y$$

$$= X_1\beta_1+\epsilon-X(X'X)^{-1}X'X\begin{pmatrix}\beta_1\\ \cdots\\ 0\end{pmatrix}-X(X'X)^{-1}X'\epsilon \tag{A.7}$$

$$= (I-P)\epsilon$$

where $P = I-X(X'X)^{-1}X'$.

The statistic $\hat{\eta}'\hat{\eta}(N-K)^{-1}$ has a mean which is,

$$\mathcal{E}\left(\hat{\eta}'\hat{\eta}(N-K)^{-1}\right) = \sigma_\epsilon^2 \tag{A.8}$$

and so will give an unbiassed estimator of σ_ϵ^2.

Turning to the residuals from the <u>true</u> regression, and denoting them $\hat{\epsilon}$ one can simply show that

$$\hat{\epsilon}'\hat{\epsilon} = \epsilon'(I-P_1)\epsilon \tag{A.9}$$

and so deduce that $\hat{\epsilon}'\hat{\epsilon}/\sigma_\epsilon^2 \sim \chi^2_{N-L}$. Thus the statistic $\hat{\epsilon}'\hat{\epsilon}(N-L)^{-1}$ has a mean and variance given by

$$\mathcal{E}\left(\hat{\epsilon}'\hat{\epsilon}(N-L)^{-1}\right) = \sigma_\epsilon^2$$

$$\text{var}\left(\hat{\epsilon}'\hat{\epsilon}(N-L)^{-1}\right) = 2(\sigma_\epsilon^2)^2(N-L)^{-1}. \quad \text{(A.10)}$$

However the statistic based on the residuals from the assumed relation, $\hat{\eta}'\hat{\eta}/\sigma_\epsilon^2$, is distributed as χ^2_{N-K} and so while $\mathcal{E}(\hat{\eta}'\hat{\eta}(N-K)^{-1}) = \sigma_\epsilon^2$ the variance is,

$$\text{var}\left(\hat{\eta}'\hat{\eta}(N-K)^{-1}\right) = 2(\sigma_\epsilon^2)^2(N-K)^{-1}.$$

The second case gives an unbiassed estimate of σ_ϵ^2 unlike the first but the efficiency will be given by the expression $(\frac{N-K}{N-L})$.

APPENDIX B

PERIODOGRAM AND SMOOTHED PERIODOGRAM OF ORIGINAL OBSERVATIONS

PERIODOGRAM ORDINATES FOR 204 OBSERVATIONS OF BANK ADVANCES SERIES

K	(π +(K-1)5π)/102	(2π+(K-1)5π)/102	(3π +(K-1)5π)/102	(4π +(K-1)5π)/102	(5π+(K-1)5π)/102
1	0.16866E 07	0.87967E 06	0.47038E 06	0.24863E 06	0.27602E 06
2	0.22807E 06	0.90478E 05	0.32947E 05	0.56029E 05	0.31877E 05
3	0.44355E 05	0.41722E 05	0.30084E 05	0.23685E 05	0.27662E 05
4	0.19837E 05	0.52573E 05	0.77462E 04	0.78370E 04	0.10576E 05
5	0.81909E 04	0.62435E 04	0.80379E 04	0.85837E 04	0.75790E 04
6	0.67043E 04	0.70317E 04	0.52492E 04	0.50122E 04	0.44076E 04
7	0.37277E 04	0.26849E 04	0.37855E 04	0.63749E 04	0.60836E 04
8	0.51348E 04	0.40660E 04	0.36438E 04	0.28642E 04	0.25887E 04
9	0.29211E 04	0.32327E 04	0.23417E 04	0.28117E 04	0.26505E 04
10	0.26672E 04	0.23670E 04	0.20713E 04	0.22198E 04	0.21101E 04
11	0.24309E 04	0.29959E 04	0.19820E 04	0.18601E 04	0.17499E 04
12	0.16051E 04	0.19906E 04	0.20331E 04	0.16966E 04	0.18495E 04
13	0.17004E 04	0.13246E 04	0.18446E 04	0.17976E 04	0.14613E 04
14	0.17021E 04	0.14716E 04	0.23978E 04	0.78197E 03	0.11852E 04
15	0.10587E 04	0.11369E 04	0.10459E 04	0.12355E 04	0.12657E 04
16	0.12657E 04	0.12701E 04	0.12782E 04	0.10823E 04	0.11969E 04
17	0.11070E 04	0.10387E 04	0.13651E 04	0.11062E 04	0.73676E 03
18	0.12391E 04	0.98311E 03	0.99795E 03	0.97256E 03	0.92721E 03
19	0.11892E 04	0.87974E 03	0.10677E 04	0.13236E 04	0.93843E 03
20	0.10330E 04	0.10345E 04	0.88653E 03	0.12144E 04	0.11353E 04
21	0.97219E 03	0.10260E 04			

THREE-TERM AVERAGES

K	(2π +(K-1)5.3π)/102	(5π+(K-1)5.3π)/102	(8π+(K-1)5.3π)/102	(11π+(K-1)5.3π)/102	(14π+(K-1)5.3π)/102
1	0.10122E 07	0.25090E 06	0.59818E 05	0.39318E 05	0.27143E 05
2	0.26719E 05	0.68631E 04	0.76217E 04	0.71047E 04	0.48896E 04
3	0.33994E 04	0.58644E 04	0.35246E 04	0.29142E 04	0.26013E 04
4	0.23685E 04	0.22536E 04	0.22793E 04	0.17919E 04	0.18764E 04
5	0.16232E 04	0.16537E 04	0.15505E 04	0.11269E 04	0.11823E 04
6	0.12713E 04	0.11287E 04	0.11700E 04	0.98631E 03	0.96591E 03
7	0.10455E 04	0.10984E 04	0.10451E 04	0.10445E 04	

FIVE-TERM AVERAGES

K	(3π+(K-1)5.5π)/102	(8π+(K-1)5.5π)/102	(13π+(K-1)5.5π)/102	(18π+(K-1)5.5π)/102	(23π+(K-1)5.5π)/102
1	0.71226E 06	0.87880E 05	0.33501E 05	0.19714E 05	0.77268E 04
2	0.56810E 04	0.45313E 04	0.36595E 04	0.27915E 04	0.22871E 04
3	0.22038E 04	0.13450E 04	0.16257E 04	0.15077E 04	0.11485E 04
4	0.12186E 04	0.10707E 04	0.10240E 04	0.10797E 04	0.10607E 04
5	0.99912E 03				

APPENDIX B

PERIODOGRAM AND SMOOTHED PERIODOGRAM OF ORIGINAL OBSERVATIONS

PERIODOGRAM ORDINATES FOR 204 OBSERVATIONS OF WOOL SERIES

K	$(\pi+(K-1)5\pi)/102$	$(2\pi+(K-1)5\pi)/102$	$(3\pi+(K-1)5\pi)/102$	$(4\pi+(K-1)5\pi)/102$	$(5\pi+(K-1)5\pi)/102$
1	0.26901E 05	0.46047E 04	0.12048E 04	0.20651E 04	0.12377E 04
2	0.77043E 03	0.62067E 03	0.14850E 04	0.74608E 03	0.61612E 01
3	0.50063E 03	0.54804E 03	0.11742E 04	0.61839E 03	0.31122E 03
4	0.70457E 03	0.11950E 06	0.14756E 04	0.74093E 03	0.24902E 03
5	0.79935E 03	0.34533E 03	0.88225E 03	0.29670E 03	0.20565E 03
6	0.69447E 02	0.43582E 02	0.10860E 04	0.10564E 03	0.26969E 03
7	0.36519E 02	0.39237E 02	0.79180E 03	0.93754E 05	0.54643E 03
8	0.60861E 02	0.18665E 04	0.93915E 02	0.17562E 03	0.83197E 03
9	0.25411E 03	0.62473E 03	0.26475E 03	0.80070E 03	0.27675E 03
10	0.26227E 03	0.85527E 03	0.33070E 03	0.15471E 03	0.98411E 03
11	0.10108E 06	0.62563E 03	0.55674E 03	0.19993E 02	0.12267E 03
12	0.24815E 03	0.23150E 03	0.13071E 03	0.11572E 04	0.35780E 03
13	0.10085E 04	0.74246E 02	0.33015E 03	0.17673E 04	0.41952E 03
14	0.24423E 03	0.97498E 03	0.38194E 05	0.11260E 04	0.24916E 03
15	0.54779E 04	0.52533E 03	0.40219E 02	0.59282E 02	0.99003E 03
16	0.23372E 03	0.75127E 02	0.10086E 04	0.49276E 03	0.13067E 03
17	0.74645E 03	0.13614E 04	0.14417E 04	0.48807E 03	0.19785E 04
18	0.13164E 04	0.20890E 03	0.15391E 04	0.15297E 04	0.23741E 03
19	0.11205E 04	0.11973E 02	0.10857E 03	0.10366E 04	0.76027E 03
20	0.29413E 03	0.40797E 02	0.18020E 04	0.32361E 04	0.88012E 03
21	0.17551E 04	0.20921E 05			

THREE-TERM AVERAGES

K	$(2\pi+(K-1)5.3\pi)/102$	$(5\pi+(K-1)5.3\pi)/102$	$(8\pi+(K-1)5.3\pi)/102$	$(11\pi+(K-1)5.3\pi)/102$	$(14\pi+(K-1)5.3\pi)/102$
1	0.10904E 05	0.13577E 04	0.95058E 03	0.35161E 03	0.70128E 03
2	0.40559E 05	0.59643E 03	0.50809E 03	0.10623E 03	0.48712E 03
3	0.28918E 03	0.31454E 05	0.71202E 03	0.57027E 03	0.44740E 03
4	0.48275E 03	0.34072E 05	0.40079E 03	0.20077E 03	0.54857E 03
5	0.47098E 03	0.81036E 03	0.13432E 05	0.20841E 04	0.36318E 03
6	0.43913E 03	0.45663E 03	0.10970E 04	0.11679E 04	0.11021E 04
7	0.41369E 03	0.69699E 03	0.17096E 04	0.78520E 04	

FIVE-TERM AVERAGES

K	$(3\pi+(K-1)5.5\pi)/102$	$(8\pi+(K-1)5.5\pi)/102$	$(13\pi+(K-1)5.5\pi)/102$	$(18\pi+(K-1)5.5\pi)/102$	$(23\pi+(K-1)5.5\pi)/102$
1	0.72027E 04	0.72566E 03	0.63050E 03	0.24534E 05	0.50586E 03
2	0.31488E 03	0.19034E 05	0.60578E 03	0.44421E 03	0.51741E 03
3	0.20480E 05	0.42507E 03	0.71996E 03	0.81576E 04	0.14185E 04
4	0.38817E 03	0.12032E 04	0.96631E 03	0.60758E 03	0.12606E 04
5	0.11338E 05				

APPENDIX B

PERIODOGRAM AND SMOOTHED PERIODOGRAMS OF r(n)

PERIODOGRAM ORDINATES FOR 240 OBSERVATIONSOF BANK ADVANCES SERIES

K	(π+(K-1)5π)/120	(2π+(K-1)5π)/120	(3π+(K-1)5π)/120	(4π+(K-1)5π)/120	(5π+(K-1)5π)/120
1	0.33653E 02	0.22041E 01	0.12082E 02	0.56793E 02	0.28049E 03
2	0.35043E 03	0.53396E 03	0.26151E 03	0.11267E 04	0.43086E 03
3	0.31228E 02	0.81235E 03	0.81823E 02	0.48913E 03	0.22074E 03
4	0.66372E 03	0.70567E 03	0.16815E 03	0.68290E 03	0.39515E-05
5	0.34480E 04	0.80645E 03	0.15532E 03	0.67751E 02	0.30353E 03
6	0.33685E 03	0.14299E 02	0.25042E 03	0.53137E 02	0.10204E 01
7	0.49795E 02	0.11354E 03	0.16305E 02	0.56942E 02	0.13607E 02
8	0.27314E 02	0.84931E 01	0.27003E 03	0.14651E 03	0.15354E-05
9	0.31761E 03	0.69306E 02	0.14532E 03	0.12047E 03	0.69929E 02
10	0.57318E 00	0.17242E 02	0.14451E-01	0.44872E 01	0.25109E 02
11	0.30737E 02	0.50801E 01	0.11000E 02	0.18010E 02	0.11278E 02
12	0.17338E 01	0.11402E 02	0.48838E 02	0.10967E 02	0.79874E-06
13	0.98302E 02	0.26553E 02	0.19929E 02	0.16022E 02	0.95932E 01
14	0.68403E 01	0.29530E 02	0.27918E 02	0.56693E 01	0.30509E 01
15	0.12185E 02	0.42325E 01	0.38964E 02	0.16339E 01	0.65776E 01
16	0.27259E 02	0.55531E 00	0.11937E 02	0.62269E 02	0.95301E-06
17	0.97766E 02	0.17299E 01	0.66603E 00	0.55186E 02	0.74838E 01
18	0.68970E 01	0.80362E 01	0.24691E 02	0.30221E 02	0.11082E 02
19	0.17050E 01	0.83435E 01	0.73255E 00	0.18518E 00	0.50339E 01
20	0.22698E 02	0.46057E 01	0.46721E 02	0.14908E 02	0.30327E-06
21	0.56471E 01	0.11159E 02	0.17498E 02	0.35645E 01	0.25761E 01
22	0.38213E 02	0.89153E 01	0.40625E 01	0.13286E 02	0.20862E 02
23	0.95164E 01	0.14878E 02	0.37277E 01	0.23068E 02	0.39168E 01
24	0.78533E 01	0.59009E 01	0.22087E 02	0.66849E 01	0.24610E-06

THREE-TERM AVERAGES

K	(2π+(K-1)5.3π)/120	(5π+(K-1)5.3π)/120	(8π+(K-1)5.3π)/120	(11π+(K-1)5.3π)/120	(14π+(K-1)5.3π)/120
1	0.15980E 02	0.22924E 03	0.64073E 03	0.42481E 03	0.26390E 03
2	0.51251E 03	0.20655E 04	0.34317E 03	0.21822E 03	0.10153E 03
3	0.59380E 02	0.33237E 02	0.14168E 03	0.20346E 03	0.11191E 03
4	0.59434E 01	0.20111E 02	0.11363E 02	0.81396E 01	0.29902E 02
5	0.48428E 02	0.10819E 02	0.21039E 02	0.64895E 01	0.15742E 02
6	0.13251E 02	0.80017E 02	0.19174E 02	0.74723E 01	0.21998E 02
7	0.37954E 01	0.93058E 01	0.22078E 02	0.84033E 01	0.78795E 01
8	0.17064E 02	0.14555E 02	0.13891E 02	0.58903E 01	0.14386E 02

FIVE-TERM AVERAGES

K	(3π+(K-1)5.5π)/120	(8π+(K-1)5.5π)/120	(13π+(K-1)5.5π)/120	(18π+(K-1)5.5π)/120	(23π+(K-1)5.5π)/120
1	0.77044E 02	0.54070E 03	0.32705E 03	0.22204E 04	0.95621E 03
2	0.13114E 03	0.50433E 02	0.45234E 03	0.14853E 03	0.94852E 01
3	0.15221E 02	0.72945E 02	0.34180E 02	0.14602E 02	0.12729E 02
4	0.10202E 03	0.32566E 02	0.16185E 02	0.33210E 01	0.88933E 02
5	0.80390E 01	0.17068E 02	0.11021E 02	0.42526E 02	

APPENDIX B

PERIODOGRAM AND SMOOTHED PERIODOGRAMS OF r(n)

PERIODOGRAM ORDINATES FOR 216 OBSERVATIONSOF WOOL SERIES

K	(π+(K-1)5π)/108	(2π+(K-1)5π)/108	(3π+(K-1)5π)/108	(4π+(K-1)5π)/108	(5π+(K-1)5π)/108
1	0.46851E 00	0.16170E 01	0.60718E 00	0.90200E 00	0.30461E 02
2	0.25712E 02	0.50749E 02	0.52459E 02	0.18658E 03	0.14096E 02
3	0.68525E 02	0.12313E 02	0.33630E 02	0.46366E 03	0.45658E 03
4	0.44312E 03	0.38090E 03	0.34741E-04	0.18335E 04	0.11317E 04
5	0.10937E 03	0.13735E 04	0.11745E 03	0.67773E 03	0.51122E 03
6	0.28749E 03	0.51615E 03	0.72102E 03	0.47614E 03	0.69260E 03
7	0.44393E 03	0.10577E 03	0.11296E 03	0.99320E 02	0.73880E 03
8	0.61785E-04	0.47100E 03	0.11312E 03	0.18190E 04	0.23356E 03
9	0.27205E 03	0.27279E 03	0.60997E 03	0.66816E 02	0.62437E 03
10	0.86575E 02	0.11579E 04	0.17295E 02	0.72162E 03	0.18278E 03
11	0.14980E 03	0.36844E 03	0.63460E 03	0.44680E-03	0.30577E 03
12	0.79307E 03	0.93894E 02	0.79098E 02	0.59768E 02	0.27934E 03
13	0.51424E 02	0.15831E 03	0.51281E 03	0.41325E 03	0.21937E 04
14	0.48700E 03	0.12636E 03	0.11251E 04	0.58594E 03	0.38720E 03
15	0.81955E 03	0.27398E-03	0.14130E 04	0.25285E 03	0.34275E 04
16	0.21323E 03	0.64976E 03	0.94061E 03	0.13179E 04	0.68486E 03
17	0.50147E 03	0.71799E 02	0.20915E 04	0.68640E 03	0.31630E 03
18	0.18756E 03	0.72679E 03	0.16758E 04	0.23579E 03	0.57205E-04
19	0.14986E 04	0.43883E 03	0.94933E 03	0.18155E 04	0.35956E 03
20	0.78555E 03	0.54538E 03	0.50532E 03	0.40731E 03	0.78101E 03
21	0.79133E 03	0.22135E 03	0.11554E 03	0.22354E 04	0.24365E 04
22	0.11563E 04	0.16363E 04	0.89892E-04		

THREE-TERM AVERAGES

K	(2π+(K-1)5.3π)/108	(5π+(K-1)5.3π)/108	(8π+(K-1)5.3π)/108	(11π+(K-1)5.3π)/108	(14π+(K-1)5.3π)/108
1	0.89755E 00	0.19025E 02	0.96594E 02	0.31644E 02	0.31796E 03
2	0.41451E 03	0.10243E 04	0.72455E 03	0.43829E 03	0.62992E 03
3	0.22089E 03	0.41906E 03	0.80105E 03	0.25947E 03	0.43372E 03
4	0.42059E 03	0.35140E 03	0.50152E 03	0.39758E 03	0.13940E 03
5	0.24102E 03	0.10313E 04	0.61247E 03	0.60337E 03	0.16978E 04
6	0.60120E 03	0.83474E 03	0.94991E 03	0.41022E 03	0.95579E 03
7	0.96225E 03	0.11539E 04	0.48600E 03	0.59789E 03	0.15958E 04
8	0.13963E 04				

FIVE-TERM AVERAGES

K	(3π+(K-1)5.5π)/108	(8π+(K-1)5.5π)/108	(13π+(K-1)5.5π)/108	(18π+(K-1)5.5π)/108	(23π+(K-1)5.5π)/108
1	0.68111E 01	0.65918E 02	0.20694E 03	0.94855E 03	0.55885E 03
2	0.53868E 03	0.30016E 03	0.65918E 03	0.36920E 03	0.43323E 03
3	0.36465E 03	0.26103E 03	0.66600E 03	0.54232E 03	0.14782E 04
4	0.76127E 03	0.73350E 03	0.28259E 04	0.11124E 04	0.60511E 03
5	0.11600E 04	0.27926E 04			

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