Errata

Page 23 line 8 "completely" should read "incompletely"
Page 27 line 23 "on" should read "in"
Page 26 line 17 " f' = \Phi " should read " f' = \Phi "
Page 64 line 11 delete "on"
Page 85 line 1 "control" should read "control variable"
Page 145 line 18 \frac{\partial L}{\partial K}\text{ should be equated to zero}
Page 156 line 8 " r is less than r^* " should read " r^* is less than r "
Page 157 line 12 " p(F' - 5 - r^* K_P) " should read " p(F' - 5 - r^* K_P) "
Page 162 line 6 "capital" should read "labour"
Page 163 line 26 " I^* " should read " I^d "
Page 163 line 27 " I^* " should read " I^d "
Page 168 line 2 " \delta r^* " should read " r^* "
Page 178 line 20 "\delta" should read "6"
Except where otherwise indicated this thesis is my own work.

Rahim A. Bade
I am grateful for the supervision of Professor J.D. Pitchford.
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INTRODUCTION

In the world today the gap between the rich developed countries and the poor under-developed countries appears to be widening. This has created much discussion in recent years of the part which international capital movements can play in reducing these international differences by stimulating growth of the poorer economies.

Australia cannot be classified as a "poor" country although its growth has always been restricted by a lack of capital and a small population. Thus the greater part of Australia has remained undeveloped through its history. However with the rapid inflow of foreign capital in the past decade, coupled with an active immigration policy, Australia's north and west are at present in the midst of great change. This sudden inflow of foreign-owned capital into Australia has led 'economists here to ask such questions as "what is the effect of foreign investment on the national economy?", "should the level of foreign investment be controlled?", and if so, "what policy should the government adopt in controlling the inflow of new capital and the outflow of dividends or profits?". It is questions such as these which have prompted the following theoretical analysis of international capital movements.

Traditional pure theory of international trade is based on the assumption of perfect mobility of factors of production within each country and complete immobility of factors of production between countries. The earliest theory of international capital movements was in terms of the transfer payments - discussing specifically the problems of raising the necessary capital for the transfer.
payment and the effects of this transfer on the barter terms of trade. However, recent literature\(^1\) in the field of international trade has relaxed part of the traditional assumptions to include the movement of capital between countries. The term "international capital movements" has been defined to include only the movements of the factor of production, capital, in either the sense of foreign investment or imports of capital. This definition of international capital movements will also be assumed in the following analysis.

Almost all of the recent analyses\(^2\) of international capital movements have assumed a static framework. However, since capital itself is a dynamic concept, it would seem reasonable that any analysis of international capital movements would be more correctly carried out within a dynamic model. Thus the following analysis is concerned with international capital movements within the context of optimal economic growth. The over-riding question of this analysis is, "what policy should a country adopt regarding international capital movements if its aim is to achieve optimal economic growth?"

A survey of the static analyses of international capital movements is presented in Chapter I whilst Chapters II and III survey the dynamic analyses. Chapter II considers purely descriptive models but Chapter III investigates the question in the context of optimal economic growth.

The nature of international capital movements, when the term "international capital movements" includes only the movements of the factor of production capital, is considered in Chapter IV. Most of this

\(^1\) For example Jones [13], Kemp [16 to 20] Bardhan [7]

\(^2\) For example Jones [13], Kemp [16 to 20]
discussion is centred around the question of mobility of capital internationally and the nature of the return payable on foreign borrowed capital.

Chapters V, VI and VII consider the question of optimal economic growth when the international capital movement takes the form of an international loan. Chapter V examines the effect of the assumption of restricted mobility of foreign capital on the borrowing country's optimal growth paths. This analysis is carried out under the condition of a linear objective function. Then in Chapter VI the objective function is assumed to be non-linear by introducing the concept of diminishing marginal utility of consumption. Both Chapters V and VI include the assumption made by Bardhan [7] that the rate of interest payable on a foreign loan is a function of the amount borrowed per head of the population in the borrowing country. From the discussion in Chapter IV it will be seen that this assumption is not realistic. Thus Chapter VII considers the question of optimal foreign borrowing when the more reasonable assumption is made that the rate of interest payable on an international loan is a function of the absolute size of the loan.

Chapters VIII and IX discuss optimal economic growth when the international capital movements take the form of direct investment. Chapter VIII abstracts from explicit international trade by assuming that only one commodity is produced in the world economy. International trade is then introduced explicitly, by assuming the existence of two goods in the world economy, in Chapter IX.

The concluding chapter gives a comparison between international capital movements in the form of an international loan and those in the form of foreign direct investment. It also discusses the value of this work to economic theory in general.
Much of the discussion of international capital movements has been carried out within the static framework. The earliest discussions began with the "transfer problem". As was noted in the Introduction the "transfer problem" was not concerned with the international movement of the factor of production capital but, as the effect of a transfer payment on the terms of trade does carry over into the analysis of real international capital movements, section I of this chapter contains a brief discussion of the "transfer problem".

The earliest of the modern analyses of the movement of the factor of production capital are examined in section II. All later theories on international capital movements can be explained in terms of two concepts which were developed in these early analyses. The first is the effect of international capital movements on the rate of return to capital invested in the country. This concept has been discussed by MacDougall [22] and Kemp [19] in their examination of the benefits and costs of private foreign investment. The second is the effect of international capital movements on the pattern of production and the terms of trade. This question was first considered by Johnson [12] and later it was extended by Amano [2]. Both these concepts will be discussed in section II.

The recent general theories on foreign investment within the static trade model which have been put forward by Kemp [16, 17, 20], Jones [13], Nadel [26], Corden [9] and Rowan and Pearce [31] will...
be considered in section III. Jones and Kemp examine the case of full optimization in which the home country is free to adjust its commercial and fiscal policies so as to maximise its welfare. This "first best" case will be considered in part (a) of section III. However, often the home country is not free to control both its terms of trade and its level of foreign investment. Thus the "second best" case in which the home country can only control its level of foreign investment has been discussed by Rowan and Pearce, Corden and Jones. This "second best" case will be considered in part (b) of section III.

As the analysis here and in all subsequent chapters is in terms of barter models it is assumed that markets including the "foreign transactions markets" are in equilibrium. If they were not, short-run capital flows would be necessary to correct the disequilibrium and this is impossible in a barter model.

I

Economists have long discussed the effects of a transfer payment from one country to another on the terms of trade. Its long history is a clear indication of the difficulty they have had in reaching agreement on the proper analysis. The problem simply stated is to find what happens to the barter terms of trade when country A makes a transfer payment to country B. However despite this extensive history most economic theorists subscribe to what is called the "orthodox" doctrine: any increase in unilateral payments will probably shift the terms of trade against the paying country; any reduction in its unilateral payments will probably shift the terms of trade in its favour.

The debate between Keynes and Ohlin on the effects of a transfer
payment is well-known. Keynes argued that if country A made to country B a unilateral payment then the terms of trade would turn against country A. In addition to the "primary burden" on country A of the direct payment, there would be a "secondary burden" as her export prices deteriorated relative to her import prices. Keynes' analysis was in terms of price-elasticity.

Ohlin adopted what is called the "modern view". This took into account income effects and the purchasing power passing between the two countries and concluded that no change in the terms of trade was necessarily implied by the transfer. Samuelson [36] interprets Ohlin's Theory 3 as follows. "The Marshallian international-trade-offer curves of both countries are shifted in opposite directions, with this important result: the implied qualitative effect on the terms of trade does not depend upon the price-elasticity of one offer curve alone, or on the price-elasticities of the two curves; rather the more crucially important parameters are the income-elasticities or propensities in the two countries".

By the 1930's the holders of the orthodox viewpoint had refined their analyses to take account of the mutual shifts of the offer curves. They then concluded that a deterioration in the terms of trade of the paying country is not inevitable, but there is a strong presumption that the income-elasticities of the different goods in the different countries will be such as to create some deterioration in the terms of trade. At the same time holders of the modern view admitted that a change in the terms of trade might take place but argued that it could be in either direction and

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that there was no presumption as to its direction.

Samuelson [35, 36] has carried out an exhaustive analysis of this effect on the terms of trade. In his first paper [36] he considers the problem with the assumption that transport costs are zero and that no impediments to trade or imperfections of competition exist. He concludes, in this case, that the orthodox presumption that the terms of trade of the paying country will tend to deteriorate need not hold. In his analysis he has established for every case put forward in favour of the orthodox view, that somewhere in the chain of reasoning impediments to trade have been directly or indirectly considered. The crucial point of this analysis is that under the assumption of zero transport costs there is absolutely no localisation of demand and thus there can be no a priori correlation between national patterns of production and tastes. Then in this case of zero transport costs it appears that Ohlin's view is correct. Namely, there is no presumption that the terms of trade will deteriorate rather than favour the paying country.

However, in Samuelson's second paper [35], he finds that the existence of transport costs and impediments to trade may create a presumption that the terms of trade will move against the paying country but this presumption is still weak.

In the traditional theory of international capital movements transport costs are considered to be zero but the imposition of a tariff policy is permissible. From Samuelson's analysis above there is no clear cut solution regarding the effect of an international capital transfer on the terms of trade of the countries involved in the transfer.
Johnson [12] and Mundell [25] have also considered the "transfer problem" with the main aim of translating Samuelson's results into modern trade terms. Their discussions of the "transfer problem" have been carried out within the standard two-country, two-good trade model. Under both the classical and the Keynesian assumptions, Johnson concludes that the movement in the terms of trade depends on whether the sum of the marginal propensities to import of the two countries exceeds or falls short of unity. On the other hand Mundell, assuming the classical assumptions, derives the additional result that the transfer payment causes a rise in the real income in the recipient country and a fall in the real income of the transferring country. This theory of transfer payment as presented by Johnson and Mundell is conventionally accepted by present day international trade theorists. However it should be noted that regarding the effect of an international capital transfer on the terms of trade, the solution depends on the comparison of the marginal propensities to import of the two countries and is thus not clear cut.

II

(a) The effects of international capital movements on the rate of return on capital

MacDougall [22] and Kemp [16, 19] have considered the benefits and costs of private foreign investment. MacDougall's analysis is concerned with the effects of foreign borrowing on the country's national revenue. Kemp, on the other hand, considers the benefits and costs to both the borrowing country and the lending country of private foreign investment. The costs and benefits of Kemp's analysis refer to the effects of foreign investment on the country's real income.
The analyses are carried out under the strict assumptions of perfect competition, constant returns to scale, the absence of taxation and external economies and the independence of the terms of trade of international capital movements. It is also assumed here as well as in all the analyses to be discussed in this chapter that there exists two countries, two factors of production (labour and capital) and that international capital movements take place instantaneously without cost to either country.

Any increase in the level of foreign investment reduces the marginal product of capital and thus the rate of return on capital in the borrowing country. Because the analysis is within the static framework, the level of domestically-owned capital is constant. Therefore the total return to the borrowing country's capitalists falls but this loss is merely a redistribution of output to labour. The total return on the increased level of foreign investment rises but the loss, through the reduced rate of return, on the original level of foreign investment goes now to the factor of production labour in the borrowing country. Therefore total real income in the borrowing country increases as a result of an increase in the level of foreign investment.

MacDougall makes two further points in his analysis. The first is that the foreign capitalists will continue to invest abroad until the rates of return on capital in the two countries are equated. Secondly, he concludes that the level of foreign investment should be restricted by the lending country because the real income created by increased foreign investment falls short of that which would be created if the investment was made in the lending country by the proportion of the increased output which goes to labour in the borrowing country. It is these two points which Kemp discusses in
more detail in his analyses.

The first of two conclusions made by MacDougall is discussed by Kemp in [19]. Because foreign capital flows into the borrowing country until the rates of return in the two countries are equal, it appears that the borrowing country should encourage further capital inflows until the rate of return on capital becomes zero. That is, it appears that the borrowing country should subsidize foreign investment so that the inflow of foreign capital will continue until the net benefit to the borrowing country is reduced to zero.

Kemp discusses this possibility by maximizing the real income of the borrowing country subject to the condition that the net rates of return in both countries are equal. He derives that it is always optimal for the borrowing country to tax capital earnings.

This result can be explained as follows. Assume initially that the capital stock owned by the two countries is distributed freely between them so as to make the rates of return on capital equal. If the borrowing country imposes a small tax on capital earnings, then the gross rates of return in the two countries are forced apart by the amount $t f_2'$, where $t$ is the tax rate and $f_2'$ is the marginal product of capital in the borrowing country. This results in an outflow of foreign capital until the net rates of return are once again equated. Because the total capital stock invested in the borrowing country has decreased, output has also decreased. Then the borrowing country's real income is affected by two forces - an increase in real income resulting from the taxation policy and a decrease in real income resulting from the reduced output. The optimal taxation policy will then be such as to balance these two opposing forces.
In Kemp's second paper [16] he considers the question of foreign investment from the point of view of the lending country. He analyses the case where the lending country is free to choose its terms of trade and the level of its private foreign lending so as to maximise its national welfare. These chosen levels of private foreign lending and terms of trade are then maintained by appropriate fiscal and commercial policies.

Foreign investment has a depressing influence both on the price of the good to which the investment is directed and on the average physical productivity of the capital invested. The private investor is ignorant of both these influences and invests abroad until the value of the marginal product of capital is equated in both countries. But, from the national point of view, the optimal level of foreign investment is reached when the value of the marginal product at home is equated to the magnitude which is marginal to the curve of total earnings abroad. Then from Kemp's analysis it follows that foreign investment is always excessive and should be curbed via a tax on repatriated foreign earnings.

In Kemp's analysis he assumes that earnings on foreign investment are subject both to the tariff, because foreign earnings are repatriated in the form of goods, and then to a tax because they are earnings on foreign investment. Then the tax imposed is the optimal derived tax minus the optimal tariff imposed on goods. The optimal tariff is always positive. Therefore Kemp discovers that the optimal tax imposed on foreign earnings can be either positive, negative or zero. But it is unnecessary to consider the problem in this way, particularly as it does not carry over into the real world. The optimal derived tax is always positive which is in agreement with the conclusions drawn by MacDougall [22] and later by Jones [13].
Johnson's pioneering work [12] on the effects of international capital movements on the pattern of production and the terms of trade is well known. His analysis is carried out within the two-country, two-good, two-factor of production model with both the countries (1 and 2) incompletely specialized in the production of the two goods (A and B). Country 1's natural export is good A which is the capital-intensive good while country 2's natural export is good B which is labour-intensive.

Johnson considers the effect of an act of foreign investment by country 1 in country 2 on the world pattern of production for given terms of trade in order to infer what change in the terms of trade is necessary to restore international equilibrium. This movement of capital in the form of foreign investment gives rise to both a short-run and a long-run transfer problem. In the short-run the movement of capital from country 1 to country 2 would require the expenditure of part of country 1's current income in a different way than it otherwise would have been spent. In the long-run the international flow of interest or dividends from country 2 to country 1 would result in a different pattern of expenditure than if the income (dividends) were retained in country 2. In both cases the transfer payment will turn the terms of trade in favour of the country for whose export good demand is increased as the result of the transfer. From the discussion of this short-run transfer problem in section I, there is no presumption that the terms of trade will turn in favour of one country or the other. But as far as the long-run transfer is concerned, if it is assumed that capital is invested abroad by the country whose income per head is higher, the terms of trade will tend to move in favour of the investing
country. That is, under Johnson's assumption above, the transfer of interest or dividends from country 2 to country 1 would tend to turn the terms of trade in favour of country 1.

Then ignoring the short-run transfer problem, the act of foreign investment has the same effect as a decumulation of capital in country 1, capital accumulation in country 2, accompanied by a transfer of income from country 2 to country 1. The decumulation of capital in country 1 will reduce country 1's real income and thus reduce its demand for imports whilst its output of the labour-intensive good is increased. This change in the pattern of production is determined by the Rybczynski Effect [32] which shows that, for given terms of trade, if one of the factors of production is increased then the output of the good which is intensive in the use of that factor of production will increase also. That is, country 1's own production of its imported good rises and thus its demand for imports falls. Thus, other things being equal, country 1's terms of trade would tend to improve.

Similarly, country 2's demand for imports will fall because its production of its imported good will increase by more than its increase in total output (via the Rybczynski Effect again). Thus, other things being equal, country 2's terms of trade would tend to improve.

Apart from the effect of the long-run transfer of dividend payments on the terms of trade, foreign investment by country 1 reduces the level of foreign trade below what it otherwise would be. But, because it reduces demand for imports of both countries, its effect on the terms of trade might be either favourable or unfavourable to country 1, the lending country.
Johnson examines this effect on the terms of trade by considering the effect of foreign investment on the world demand and supply of the capital-intensive good A at constant prices.

In the case of foreign direct investment under competitive conditions, the return payable on foreign investment is equal to the marginal product of capital in country 2. Therefore all the increase in output in country 2 due to the foreign investment will accrue as income to country 1. The total change in the demand for good A due to the act of foreign investment will therefore be determined completely by the above increase in country 1's real income. As part of this increase in country 1's real income increases its demand for good A, the consumption effect of foreign investment is favourable to the lending country, country 1.

On the production side, the effect of foreign investment is favourable or unfavourable to country 1 according as the decrease in country 1's output of good A is greater or less than the increase in country 2's production of good A. Although it is more likely that total production will decrease and thus turn the terms of trade in favour of country 1, no general statement can be made unless more information on the production conditions is given.

Amano [2] takes Johnson's analysis further by considering the effect of factor-intensity reversal on the analysis. If good A above is assumed to be capital-intensive in country 1 but labour-intensive in country 2 a movement of capital from country 1 to country 2 will reduce the output of good A in country 1 and in country 2 (via the Rybczynski Effect [32]). That is, the total production of good A falls whilst demand rises. Therefore the terms of trade will turn in favour of country 1. Amano concludes that, in general, if there is a factor-intensity reversal between the lending and the borrowing
country, international capital movements will turn the terms of trade in favour of or against the lending country depending on whether its export good is capital-intensive or labour-intensive.

III

In section II the effects of international capital movements on (a) the rate of return to capital and (b) the pattern of production and the terms of trade have been discussed. These two concepts can be used to explain the recent theories of international capital movements.

In part (a) the "first best" case or the full optimisation case will be discussed. In this case the home country can adjust both the level of international indebtedness and the terms of trade with the aim of maximising its national welfare. In part (b) the "second best" case will be considered. In this case it is assumed that the home country is not free to adjust the terms of trade and can only attempt to maximise its national welfare by adjusting the level of international indebtedness.

(a) The "first best" case

Kemp [20] and Jones [13] consider international capital movements within the standard two-country, two-commodity trade model in which the factors of production are capital and labour. The factor of production, labour, is internationally immobile while capital is assumed to be completely mobile between countries. Both Kemp and Jones are concerned with static analysis; therefore there is no growth in the supplies of factors of production. That is, in each country the labour supply is constant and net savings are zero. The factor markets are considered perfectly competitive with full employment of factors and constant returns to scale. It is also assumed
that there is no retaliation by a country against a restrictive trade policy introduced by the other country.

The aim of the home country is to adopt whatever commercial and fiscal policies it chooses so as to maximise its national welfare. That is, it maximises its welfare with respect to the commodity terms of trade $p^*$ and the level of international indebtedness $K$. Movements in both $p^*$ and $K$ can change the home country's real income in two ways - directly and indirectly. A favourable movement in the relative price of any commodity or factor exported by the home country increases the home country's real income directly. If the home country is a lender of capital, a change in the level of international indebtedness will have a direct effect on the home country's real income which will be favourable or unfavourable depending on whether the rate of return on capital in the foreign country is greater than or less than that on capital in the home country. Jones has also shown explicitly that $p^*$ and $K$ affect real income indirectly by changing the rate of return on capital in the foreign country $r^*$ and the foreign country's excess demand for the home country's natural export good (good 2) $E^*_2$.

That is, $r^* = r^*(p^*, K)$ and $E^*_2 = E^*_2(p^*, K)$.

Jones derives the following expression for the change in the home country's real income $y$

$$dy = \left(E^*_2dp^* + Kdr^*\right) + \left((p^*-p)dE^*_2 + (r^*-r)dk\right).$$

It should be noted that if there is no interference in the goods and factor markets, $p^* = p$ and $r^* = r$, so that the second bracket becomes zero. However, with positive taxes and tariffs the home country's real income can be increased by the movement of goods and factors abroad for which the world price and rate of return exceeds
The nature of the functional relationships \( r^* = r^*(p^*, K) \) and \( E^*_2 = E^*_2(p^*, K) \) is important in determining the total effect of a change in the commodity terms of trade or a change in the level of international indebtedness on the home country's real income. These functions are dependent on the degree of specialization in the foreign country's production. Consider firstly \( r^* = r^*(p^*, K) \).

If the foreign country's production is incompletely specialized, \( r^* \) is determined uniquely by \( p^* \) and not by \( K \). Then the effect of a change in the commodity terms of trade on the return on capital in the foreign country depends on the factor-intensity ranking of the industries - the Stolper-Samuelson Effect [34]. If the foreign country's production is completely specialized, the return on capital is its marginal physical product and hence depends only on \( K \). Therefore, in general, \( r^* = r^*(p^*, K) \) so that

\[
\frac{dr^*}{r^*} = \gamma^* \frac{dp^*}{p^*} + \delta^* \frac{dK}{K},
\]

where \( \gamma^* = \frac{p^*}{r^*} \frac{2r^*}{p^*} \) and \( \delta^* = \frac{K}{r^*} \frac{2r^*}{p^*} \)

and either \( \gamma^* \) or \( \delta^* \) equals zero.

Consider the relationship \( E^*_2 = E^*_2(p^*, K) \). For given amounts of foreign indebtedness and fixed resources, the relationship between \( E^*_2 \) and \( p^* \) is given by the foreign offer curve for imports with the elasticity of demand along this curve \( \eta^*_2 = \frac{2E^*_2}{\partial p^*} \frac{p^*}{E^*_2} \). For given terms of trade the relationship between \( E^*_2 \) and \( K \) depends on the pattern of specialization in the foreign country. If production in the foreign country is completely specialized in the first good, a change in \( K \) does not affect foreign production of the second good but it does affect the demand because an increase
in the level of foreign investment reduces the rate of return on invested capital \( r^k \), thus reducing the burden of indebtedness in the foreign country. Therefore \( \frac{\delta E^h}{\delta K} = -\frac{m_h}{p^h} r^k \delta h \), where \( m_h \) is the marginal propensity to consume the imported good in the foreign country.

If the production is incompletely specialized, changes in foreign investment at constant terms of trade do not change the return on capital and thus cannot affect the foreign country's real income or demand for the second good. But production is affected because, from the Rybczynski Effect [32], an inflow of capital at constant prices raises production of the capital-intensive good. Using Samuelson's reciprocity theorem [33], \( \frac{\delta \chi^h_2}{\delta K} = \frac{E^h_2}{K} \mu^h \gamma^h_2 \), where \( \chi^h_2 \) is the foreign country's output of good 2 and \( \mu^h = \frac{E^h_2}{E^h_2 K} \).

Then the total change in \( E^h_2 \) due to changes in \( p^h \) and \( K \) is given by

\[
dE^h_2 = E^h_2 \left( \frac{\delta h}{p^h} \right) \frac{dp^h}{p^h} - \mu^h \left( \frac{m^h \delta h + \gamma^h_2}{K} \right) \frac{dK}{K},
\]

with one of the coefficients of \( dK \) equal to zero.

Substituting for \( dE^h_2 \) and \( dr^h \) in the equation for \( dy \), the total change in home welfare produced by a change in \( p^h \) and \( K \) is given by

\[
dy = \frac{3v}{3p^h} dp^h + \frac{3v}{3K} dK,
\]

where \( \frac{3v}{3p^h} = E^h_2 \left( 1 + \mu^h \gamma^h_2 + \frac{p^h - p^2}{p^h} \eta^h_2 \right) \)

and \( \frac{3v}{3K} = \left[ 1 + (m^h + \frac{E^h_2}{p^h} m^h_2) \delta h + \frac{E^h_2 - p^2 \gamma^h_2 h^2}{E^h_2 \gamma^h_2} \right] r^h - r \).

It should be noted that Jones has preferred to discuss the case of
incomplete specialization in the foreign country in terms of the effect of the factor-intensities in preference to the Rybczynski Effect. But, by Samuelson's reciprocity theorem \( \frac{3X}{2} = \frac{3r}{2pK} \), so these two effects are duals.

The home country is free to choose its tariff and tax policies so as to maximise its welfare. That is, by equating \( \frac{3y}{3pH} \) and \( \frac{3y}{3K} \) to zero the home country determines its optimal tariff and optimal tax policies respectively. The resulting optimal policies depend on the pattern of specialization of production in the countries. Therefore the various patterns of production will be considered one at a time.

The foreign country's production completely specialized

In this case, the rate of return on capital in the foreign country is a function of \( K \) only and not of \( p^* \). Then the model reduces to the early Kemp model discussed in section II above. In this case it is optimal for the home country to levy a positive tariff on goods and a positive tax on capital flows whether it is a net lender or borrower of capital.

The foreign country's production incompletely specialized

The rate of return payable on capital in the foreign country is now a function of \( p^* \) only - assuming that any change in the level of international indebtedness is small so that it will not bring about complete specialization of production in the foreign country, in which case \( r^* \) then becomes a function of \( K \).

Assume initially that there is free trade in goods and unrestricted movement of capital. Then \( \frac{3y}{3p^*} \) reduces to \( E^*(1 + \mu^*r^*) \). That is, a movement in the terms of trade changes the real income via
its effect on $r^*$. This effect of $p^*$ on $r^*$ depends on the technology of the foreign country's production. If the sign of $\frac{3y}{3p^*}$ is positive then the home country's income will increase as a result of an increase in $p^*$. That is, it is optimal for the home country to impose a tariff so as to increase the terms of trade and thus increase its real income.

Recalling that $\mu^*$ is the ratio of foreign capital earnings to foreign export earnings $\left(\frac{\lambda r^*}{p^*}\right)$, $\mu^*$ is positive if the home country is a lender of capital and negative if the home country is a borrowing country. Also $\mu^*$ will be larger the more important foreign capital earnings are to the home country as a source of income. The sign of $\gamma^*$ depends on the factor-intensity of the foreign country's production of good 2 (the home country's natural export good). If good 2 is labour-intensive $\gamma^* < 0$ but if good 2 is capital-intensive $1 < \gamma^*$ - from the Stolper-Samuelson theorem [34].

The home country's optimal policy is to impose a tariff in all cases except

(i) when the home country is primarily a lender of capital and the foreign country's production of good 2 is labour-intensive, and the symmetrical case

(ii) when the home country is primarily a borrower of capital and good 2 is capital-intensive abroad.

In these two cases $\frac{3y}{3p^*}$ is negative and it is optimal for the home country to subsidize exports of good 2.

The optimal tax policy of the home country is derived by equating

$$\frac{3y}{3p^*} = [1 + \frac{p^*}{p^*} (\gamma^*)]r^* - r$$

to zero. Because the foreign country is incompletely specialized, the return on capital abroad $r^*$ varies only with a change in the commodity terms of trade via the foreign
technology. Therefore the optimal tax policy of the home country which links $r$ and $r^\#$ via $r = r^\#(1 + t)$, depends on its tariff policy and the technology of the foreign country.

Taking $\frac{3y}{3x} = 0$ and $r = r^\#(1 + t)$ together, the optimal tax $t$ is $\frac{p^H - p}{p^H} y^\#$. Then the optimal tax is positive if and only if the tariff is positive and the foreign production of the home country's export good is capital-intensive or the home country subsidizes its commodity exports and the foreign production of this good is labour-intensive.

Then the possible combinations of optimal tax and optimal tariff when the home country is a lender of capital are:

(i) Both the optimal tariff and optimal tax are positive. This occurs if and only if the favourable movement in the commodity terms of trade produces a favourable movement in the return on capital abroad. That is, if the foreign production of the home country's natural export is capital-intensive.

(ii) The optimal tariff will be positive but the capital exports should be subsidized if foreign production of the home country's export good is labour-intensive and the home country is primarily a commodity exporting country. That is, capital exports are encouraged so as to increase the foreign production of the good with a relatively higher value at home.

(iii) The optimal tax will be positive and the home country will subsidize its commodity exports if foreign production of this export good is labour-intensive while the home country is primarily a lender of capital. By subsidizing commodity exports the return on capital in the foreign country will rise. But capital exports are discouraged because the good whose production is increased abroad as a result of the capital flows has a relatively lower value at home.
The case in which the home country is a borrower of capital is symmetrical. Therefore the optimal tax and optimal tariff can be both positive and negative but it is never optimal to subsidize both capital exports and commodity exports. These results then differ from those of Kemp [20]. In his analysis, Kemp fails to take account of the Stolper-Samuelson Theorem [34] which shows that if $\gamma^2$ is positive then it is also greater than unity. This error led to Kemp's paradoxical result that both the optimal tax and the optimal tariff can be negative.

Jones has considered only cases in which at least one of the two countries is completely specialized as the result of capital flows induced by the home country's tariff and tax policies. He realized that if both countries were incompletely specialized it would only be by chance that the optimal tariff and optimal tax policies would satisfy the Stolper-Samuelson Theorem [34]. He therefore concludes that production in at least one country must become completely specialized. Thus the Stolper-Samuelson Theorem no longer holds and the system is not constrained by the magnification effect ($\gamma^2 > 1$, if production of good 2 is capital-intensive). Nadel [26] disagrees.

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4. Both countries are incompletely specialized in production. Therefore in each country the rate of return on capital and the country's price ratio are linked by the country's technology. When the home country levies the optimal tariff, the price ratios in the two countries are known and separated by the tariff. The rate of return on capital in each country is then determined via its technology. It will only be by chance that these two rates of return on capital will be separated by the optimal tax. That is, it is only by chance that the Stolper-Samuelson Theorem is satisfied.
with Jones' conclusion on the grounds that market forces will not take the system to complete specialization in at least one country and that incomplete specialization of production remains in both countries but that the pattern of trade changes.

Nadel considers the effects of interferences such as taxes and tariffs on Mundell's theory of perfect substitutability of goods flow with factor movements when production in both countries is completely specialized. Mundell [24] showed that in the absence of trade impediments and provided the production functions were the same in each country, factor movements can generate exactly the same results as free trade in goods with factor immobility - the only differences being that capital is distributed differently between countries and that there is a continuous repatriation of foreign earnings to the lending country in place of the continuous two way flow of goods. Also, with mobility of the factor of production, capital, no barter trade takes place between the two countries as imports are paid for out of foreign earnings and not by an equivalent amount of exports of final goods. Then apart from the repatriation of foreign earnings the two countries are self-sufficient.

Using these generalized results Nadel [26] shows that the solution to the incompatibility between incomplete specialization and tariff effectiveness, as recognised by Jones [13], is that incomplete specialization is retained but that the difference in relative prices between the two countries is less than the tariff protection. That is, it is not always certain that a tariff which is levied is automatically effective. Then firstly, the good which was imported before the tariff was levied may cease to be imported afterwards - that is, the pattern of trade may change under the influence of tariffs, taxes and capital mobility, and secondly, even if the good continues to be imported subject to the tariff, this necessitates
absolute prices of the imported good differing in the two countries by the amount of the tariff but does not necessitate relative goods prices differing by that magnitude.

The introduction of factor mobility into the case where both countries are incompletely specialized does not increase the optimization potential a country has when it seeks to interfere with barter goods trade under conditions of capital mobility, but merely complicates attempted interference in the barter goods trade. Nadel then makes no attempt to find the optimal tariff and optimal tax applicable for this case where both countries are incompletely specialized.

(b) The "Second Best" case

Jones [13], Rowan and Pearce [31] and Corden [9] have considered the "second best" case in which the home country aims to maximise its real income or national welfare by controlling only the movement of capital internationally. Often in the real world, a country is not free to adjust its tariff policy (for example, as the result of an international tariff agreement) so the "second best" cases are important and probably much more realistic than the "first best" cases.

The tariff policy is assumed to be given and is non-optimal. For simplicity it is assumed that the tariff rate is held at zero level. That is, free trade in commodities is guaranteed.

In the full optimisation case of part (a) above the expression Jones derives for the change in real income is

\[ dy = \frac{\partial y}{\partial p} dp + \frac{\partial y}{\partial K} dK, \]

where

\[ \frac{\partial y}{\partial p} = E\frac{\hat{a}}{2}(1 + \mu \eta \hat{\eta} + \frac{\hat{a} \hat{P}}{p \hat{\eta} \eta}). \]
and \( \frac{\partial y}{\partial K} = [1 + (m_2^2 + \frac{D}{p^m} m_2^2) \delta^2 - \frac{p_2 m_2}{p^m} \gamma^2] \eta - r \).

In this "second best" case, the change in real income as the result of a change in the level of international indebtedness, assuming initially that free trade in goods and unimpeded capital flows exist, is given by

\[
\frac{dy}{dK} = \frac{\partial y}{\partial p^m} \frac{dp^m}{dK} + \frac{\partial y}{\partial K} ,
\]

where \( \frac{\partial y}{\partial p^m} = \frac{E_2^2 (1 + \mu \gamma \delta^2)}{2} \) and

\( \frac{\partial y}{\partial K} = \delta \phi \gamma \eta^2 \).

That is, a change in the level of foreign investment, \( K \), in general alters real income directly, as of given commodity terms of trade and indirectly, through requiring a change in these terms of trade. \( p^\alpha \) is the price ratio which clears the world markets, and in general there exists a different equilibrium \( p^\alpha \) for each value of \( K \).

Then \( \frac{dp^\alpha}{dK} \) is determined by the change in \( p^\alpha \) necessary as the result of the change in \( K \) to clear the world market for good 2. That is, \( \frac{dp^\alpha}{dK} \) is determined from \( d(E_2 + E_2^2) = 0 \).

The factors changing \( E_2 \) are given in the following equation:

\[
dE_2 = E_2^2 \left( \frac{\partial p^m}{p} + \frac{m_2^2}{p^m} \frac{dp^m}{dK} \right) dy + \mu \gamma_2 \frac{dK}{K} .
\]

An increase in \( p \) must decrease \( E_2 \) via the substitution effects in consumption and production. The second term is the increase in \( E_2 \) as a result of any increase in real income through \( m_2 \), the home country's marginal propensity to consume good 2. Finally, any change in the supply of good 2 as the result of capital flows produces a change in \( E_2 \). By Samuelson's reciprocity theorem [33] the change in production at constant home prices is given by \( \gamma \).

A similar expression can be obtained for the foreign country. Since
initially prices and rates of return are equal in the two countries - \( dy^2 = dy \) and \( dp = dp^2 \) because the tariff is assumed to be fixed at zero level, then \( d(E_2 + E_2^2) \) is given by

\[
d(E_2 + E_2^2) = E_2^2 \left[ -\Delta \frac{dp^2}{dp^2} + \mu^2 \left( (y - y^2) + (m_2 - m_2^2)\delta^2 \right) \right] \frac{dK}{K},
\]

where \( \Delta = -(\hat{n}_2 + \hat{n}_2^2 + (m_2 - m_2^2)(1 + \mu^2 y^2)) \).

Assuming that commodity markets are stable \( \frac{\partial(E_2 + E_2^2)}{\partial p^2} \), for given \( K \), is negative. That is, the coefficient of \( dp^2 \) is negative, or \( E_2^2 \) is positive.

Then equating \( d(E_2 + E_2^2) \) to zero

\[
\frac{dp^2}{dK} = \frac{E_2^2}{E_2^2} \left( (y - y^2) + (m_2 - m_2^2)\delta^2 \right).
\]

The total effect of a change in the level of foreign investment on the home country's real income is given by the following equation which is derived by substituting the above expression for \( \frac{dp^2}{dK} \) into the equation for \( \frac{dy}{dK} \):

\[
\frac{dy}{dK} = \frac{E_2^2}{E_2^2} \left( (1 + \mu^2 y^2)(y - y^2) - (\hat{n}_2 + \hat{n}_2^2)\delta^2 \right).
\]

This general equation can be used to determine the optimal taxation policy to be adopted in the free trade world. If, from an initial position of unimpeded capital flows, an increase in \( K \) increases \( y \), foreign investment should be encouraged by a subsidy if the home country is a net lender and discouraged by a tax if the home country is a net borrower of capital.

If the foreign country is incompletely specialized and the home country completely specialized in production the sign of the optimal tax on capital flows is identical to that of the full-optimisation case, although the process through which interference in the inter-
national flow of capital affects the home country's real income differs in these two cases.

Since it is assumed that free trade in goods is maintained, the direct effect of capital flows on the home country's income $\frac{\partial y}{\partial K}$ is zero. Capital flows can affect real income only via the indirect effect they have on the terms of trade. An increase in $K$ will raise or lower $p^*$ depending on the factor-intensity of the production of good 2 in the foreign country. In this case $\frac{dp^*}{dK} = \frac{\gamma^*}{E^R} (1 - \gamma^*)$.

Then this change in $p^*$ will raise or lower the home country's real income. The direction of this change depends uniquely on the sign of $E^R(1 + \mu^*\gamma^*)$. If the home country is a lender of capital, real income will be raised by an increase in $p^*$ unless the foreign country's production of good 2 is labour-intensive and the home country is primarily a lender of capital. Similar results are obtained for the case in which the home country is a borrower of capital.

If the foreign country is completely specialized in production capital flows have both a direct and an indirect influence on the home country's real income. In the full optimisation case capital flows had only a direct effect on real income; therefore the additional indirect effect on real income (via its effect on $p^*$) may produce different results.

On this case $\delta^R$ is non-zero. Therefore the determination of the effect of capital flows on the commodity terms of trade is more complicated than in the case of incomplete specialization abroad. Not only are pure supply effects involved, $(\gamma - \gamma^*)$, which in this case reduces to $\gamma$, but also the change in the rate of return on capital necessarily involves a transfer of real income from one
country to the other. This transfer of real income affects the world
demand for goods if the marginal propensities to consume good 2
differ between countries. From the discussion of the "transfer
problem" above in section I, it is not clear in which direction the
terms of trade will move.

With the foreign country completely specialized in production, the
total effect of capital flows on the home country's real income
reduces to

$$\frac{dy}{dk} = \frac{r^*}{E_{\Delta K}} \frac{E_y}{2} \left( \gamma - n_2 \right) \frac{\Delta}{2} \delta \alpha \right).$$

At unchanged terms of trade $\frac{dy}{dk} = r^* \delta \alpha$. This real income transfer
affects $p^*$ which further affects $y$. The net effect on real
income, ignoring the supply effects $\gamma$, is

$$\frac{\Delta}{\alpha} (r^* \delta \alpha).$$

This effect must have the same sign as $r^* \delta \alpha$ and will be larger or
smaller than it depending on the direction in which the terms of trade
move. However an increase in $K$ also alters the world production
of good 2 by an amount $\gamma$ as capital moves out of the home country.
It is this effect on the world supply of good 2 which introduces
the possibility that the optimal taxation policy in this case may be
different from that of the full optimisation case.

If the home country is a lender, $\delta \alpha$ is negative and ignoring supply
effects $\frac{dy}{dk}$ would be negative so that it would be optimal to tax
capital outflows. But the imposition of a tax would bring capital
into the home country and, at constant $p^*$, increase the production
of the capital-intensive good. If this good is good 2, $\gamma$ is positive
and $p^*$ tends to fall which would reduce the gains brought about
by increasing $r^*$. If the terms of trade effect of the supply changes
brought about by the change in $K$ outweighs the beneficial effects
of raising $r^*$ via the imposition of a tax, it would pay the home
country to subsidize foreign investment.

If the home country is also completely specialized \( \gamma \) is unity. The above "paradoxical" result can be obtained if the home country is a lender but it is not possible when the home country is a borrower. A subsidy is optimal only if \( \frac{dy}{dk} \) is negative which would require \( (\hat{n}_2 + \hat{n}_2)\hat{\alpha}_2 \) to exceed unity. However when the home country is a borrower this expression is negative so that the "paradoxical" result cannot occur. When the home country is a borrower it is optimal to impose a tax on foreign investment to lower the rate of return payable on the borrowed capital and to restrict world production of its export good.

As discussed above in part (a) of this section Jones does not consider the case in which both countries are incompletely specialized. Corden [9] and Rowan and Pearce [31], however, do discuss this "second best" case. Corden considers the case in which the tariff is fixed above its optimal level for some industry and the resultant efficiency of a tax policy in controlling investment in this over-protected industry. Rowan and Pearce, on the other hand examine what appears to be a much more realistic problem than that considered by Jones and Kemp. Instead of attempting to reallocate world capital optimally, Rowan and Pearce assume some new capital formation in the lending country which is then lent abroad. Initially this act of foreign investment is considered within the standard two-country, two-good, two-factor of production trade model. Then Rowan and Pearce introduce a third good into each country of the model. These two goods are assumed to be non-traded goods.

In the standard two-good trade model Rowan and Pearce derive the following equation for the total change in the home country's
national welfare \((W)\) resulting from this movement of capital abroad:

\[
dW = \frac{dx_2}{dK} \frac{dp^2}{dK} + r^2 + k \frac{dr^2}{dK},
\]

where \(x_2\) is the difference between the production \(X_2\) of and the demand \(D_2\) for, the home country's export good (No. 2) and \(p^2\) is the terms of trade in terms of the price of good 1.

Rowan and Pearce have chosen to examine the effect of foreign investment on the home country's welfare in terms of (a) the change in the terms of trade and (b) the change in the supply of goods abroad as a result of the foreign investment. In the two-good case this is the Rybczynski Effect but as Rowan and Pearce later extend the model to include non-traded goods, they prefer not to discuss the results in terms of factor-intensities. However, the analysis of the two-good case can perhaps be more clearly explained in terms of Jones' analysis which shows explicitly the Stolper-Samuelson Effect. (The Stolper-Samuelson Effect and the Rybczynski Effect are duals.)

From the Balance of Payments equation and the home country's budget constraint:

Total Demand by home country = Total Production by home country - exports + imports + return on foreign investment

\(i.e. \ D_1 + p^2D_2 = X_1 + p^2X_2 - p^2X_2 - p^2E_2^2 + M_1 + r^2K,\)

where \(M_1\) represents imports of good 1. Re-arranging the terms

\((X_1 - D_1) + p^2(X_2 - D_2) + r^2K = p^2E_2^2 - M_1,\)

In Rowan and Pearce's notation

\(x_1 + p^2x_2 + r^2K = p^2E_2^2 - M_1.\)
Differentiating with respect to $K$

$$\frac{d}{dK} (x_1 + p^\theta x_2 + r^\theta K) = x_2 \frac{dp^\theta}{dK} + p^\theta = E^\theta \frac{dp^\theta}{dK}$$

Therefore the Rowan and Pearce equation reduces to

$$\frac{dW}{dK} = E^\theta \frac{dp^\theta}{dK} + \lambda \frac{dp^\theta}{dK}.$$ 

Now

$$\frac{dW}{dK} = \frac{\partial p^\theta}{\partial \lambda} + \frac{\partial p^\theta}{\partial p^\theta} \frac{dp^\theta}{dK}.$$ 

Because the foreign country’s production is incompletely specialized the level of foreign investment does not affect the rate of return on capital at constant prices, $\frac{\partial p^\theta}{\partial \lambda}$ is zero. It should be noted that $\frac{\partial p^\theta}{\partial p^\theta}$ is the concept discussed in Kemp’s early paper [19]. Therefore the effect of foreign investment in this case can be explained in terms of the Stolper-Samuelson Effect and the terms of trade effect.

From the discussion of Jones’ derivation of $\frac{dy}{dK}$ above,

$$\frac{\partial p^\theta}{\partial y} = \frac{E^\theta}{2} \frac{\lambda^\theta}{\lambda^\theta}.$$ 

Therefore, in more explicit terms, the Rowan and Pearce equation becomes

$$\frac{dW}{dK} = \frac{E^\theta}{2} (1 + \lambda^\theta \lambda^\theta) \frac{dp^\theta}{dK}.$$ 

From Jones’ discussion of the effect of capital flows on the terms of trade, the sign of $\frac{dp^\theta}{dK}$ depends on the sign of $(\gamma - \gamma^\theta)$. However, in this model the capital invested abroad is newly created capital so that its withdrawal from the home country has no effect on the home country’s supply of goods. Thus $\gamma$ becomes zero and the sign of $-\gamma^\theta$ determines the direction of the movement in the terms of trade resulting from the investment of capital abroad. If the foreign country’s production of good 2 is capital-intensive the terms of trade deteriorate but if good 2 is labour-intensive abroad the terms of trade improve. Because the home country is the lender
Rowan and Pearce conclude that
(a) if the home country's export good is capital-intensive abroad the terms of trade deteriorate and the home country's welfare falls as the result of investing the newly formed capital abroad. It is optimal in this case to levy a tax on capital flows out of the home country,
(b) if the home country's natural export good is labour-intensive abroad the terms of trade improve and it is not clear whether the home country's welfare falls or rises. However, from the discussion above of the Rowan and Pearce problem in terms of Jones' analysis, it is clear that it is optimal to subsidize capital flows unless the home country is a more important investor of capital abroad than an exporter of good 2 (i.e. $\mu^*$ large) in which case it is optimal to tax the capital flow from the home country.

From the analysis of the more general model Rowan and Pearce are unable to make any conclusions regarding the change in the home country's welfare because it is now not known how the production of the three goods in the foreign country will change as a result of the capital inflow. However as the analysis is entirely in terms of price-elasticities of demand, elasticities of supply and elasticities of supply with respect to factor endowment changes, conclusions can be made for any given quantitative system.

The above analysis of Jones and Kemp (part (a) section III) have been carried out within the static trade model. However in the dynamic framework of optimal economic growth to be developed later, it will be seen that their conclusions drawn from the "first best" cases still hold.
Chapter II

THE ROLE OF INTERNATIONAL CAPITAL MOVEMENTS IN GROWTH THEORY

In this chapter only the effects of international capital movements on the economic growth of a country will be considered. The question of optimal economic growth will not be discussed until the following chapter.

Johnson [12], Ball [5, 6], Massell [23] and Amano [1] consider the effects of international capital movements on the equilibrium growth rate of a single economy. Their analyses will be discussed in section I. In section II Johnson's extension of this analysis to the two country case will be considered. Discussions of the paths of capital accumulation and international debt which take the economy to the position of equilibrium growth are carried out by Hamada [10], Oniki and Uzawa [28], and Kemp [18]. These discussions will be considered in section III.

I

Johnson [12] and Ball [5] extended the Harrod-Domar growth model from the closed economy case to a two-country international economy so that conclusions could be drawn regarding equilibrium growth in an open economy.

The Harrod-Domar growth model consists of two basic relations:
(a) The investment required to employ existing capital fully is equal to a constant fraction of income saved multiplied by the capacity output of the existing equipment. That is, $I_t = sY_t$. 

33.
(b) The increment to capacity output resulting from this investment is

$$\frac{dY}{dt} = \frac{I_t}{k} = aL_t,$$

where \( k \), the capital coefficient, is the constant ratio of capital to output and \( a \), the output coefficient, is its reciprocal.

From these two relations it follows that the rate of growth of investment, capacity and income which maintains full employment of the productive capacity is

$$r = \frac{1}{Y_t} \frac{dY}{dt} = \frac{s}{k} = as.$$  

Then for consistency the model must assume that either population growth rate is at least \( r \) or that technical progress of a type that leaves the capital coefficient \( k \) unchanged is going on rapidly or steadily enough to prevent the capital stock from outgrowing the labour required to work it. It is also assumed that the price level is constant, that output is measured in units such that the price of a unit of output is one unit of currency and that units of currency are so defined that exchange rates are initially equal to unity.

In the closed economy the only outlet for saving is investment. However, in the open economy imports as well as savings constitute leakages from effective demand, while exports as well as investment may fill the gap between domestic demand for and supply of home production at capacity output. Then the first equation of the Harrod-Domar model becomes

$$(s + m)Y_t = I_t + X_t,$$

where \( m \) is the constant fraction of income spent on imports and
the current rate of exports. Then

$$r_t = a(s - b_t),$$

where

$$b_t = \frac{X_t}{Y_t} - m$$

is the country's current export surplus expressed as a fraction of its production.

Then from these equations Johnson [12] draws the following conclusions:

(a) If the proportional export surplus is falling over time, the equilibrium rate of growth rises over time.

(b) By differentiating the above equation for $r_t$

$$\frac{dr}{dt} = -a \frac{X_t}{Y_t}(x - r_t),$$

where $x$ is the rate of growth of exports. Then if the rate of growth of exports exceeds the equilibrium rate of growth, the latter is falling over time.

(c) The equilibrium growth rate will be higher the higher the output coefficient and the savings ratio and the lower the export surplus (or the higher the import surplus) as a fraction of total output.

These results are dependent on the three simplifying assumptions:

(a) that investment utilizes only domestic output,

(b) that domestic production has no import content - imports are required only for consumption,

(c) that trade surpluses or deficits are financed in a way that does not give rise to international interest payments - such as unilateral payments or movements of international reserves.

Now, if the creation of new productive capacity requires a certain proportion of imported goods, the equation for the equilibrium growth rate becomes

$$r_t = \frac{a}{1-m} (s + m - \frac{X_t}{Y_t}),$$
where \( m' \) is the fraction of investment expenditure spent on imported goods, assumed to be constant. If \( a \) is redefined as the ratio of output to the domestically produced fraction of capital, all the preceding results apply, with the extension that the output coefficient varies with both output per unit of total capital and the proportion of investment goods imported.

Relaxing the second assumption so that production now has an import content income is no longer identical with production, but is less to the extent of the import content. Then to maintain full-capacity production, investment must be large enough to offset the gap between exports and the sum of (i) expenditure on the import content of domestic production and (ii) saving and expenditure on imports from domestically earned income. Then the equilibrium rate of growth is

\[
rt = a(s + m + q - \frac{X}{Y})
\]

where \( q \) is the fractional import content of domestic output which is assumed to be constant and \( c(= 1 - s - m) \) is the fraction of income spent in home produced goods \((c > 0)\). All the conclusions previously derived still stand with the difference now that the initial equilibrium growth rate \( r_0 \) is given by

\[
r_0 = a(s + m + q - \frac{X_0}{Y_0}) = a(s - sq - b_0)
\]

and the further conclusion follows that the equilibrium level of income and equilibrium rate of growth will be higher the higher the fractional import content of domestic production.

If the third of Johnson's simplifying assumptions is now relaxed so that trade deficits and surpluses are financed by capital movements, the problem becomes much more complicated since income is no longer
determined exclusively by current output. Income arising from current production will be increased or decreased by interest payments from or to the outside world. Leakages from domestic earned income may be offset by consumption of domestic goods out of interest earned on foreign investment, as well as by exports and investments in new capacity, or such leakages may be augmented by the payment of interest to foreigners.

Johnson [12] assumes that differences between receipts and payments on current account, including both the trade balance and interest payments on past debts, are financed by international capital movements bearing a fixed interest rate $i$. Then ignoring the two qualifications introduced above, the equilibrium rate of growth becomes

$$ r_t = a(s + m - \frac{cZ_t}{y_t} - \frac{X_t}{y_t}) , $$

where $Z_t$ is the interest income or the negative of the interest payments determined by

$$ \frac{dZ}{dt} = i(X_t + Z_t - mY_t - mZ_t) . $$

That is, $r_t = a(s + s \frac{Z_t}{y_t} - b)$ and $\frac{dZ}{dt} = ibY_t$, where $b$ is now the current account surplus expressed as a fraction of domestic production. Then the equilibrium rate of growth will be higher the lower the ratio of foreign investment income to domestic production as well as the higher the output coefficient and the savings and import ratios, and the lower the ratio of exports to output.

The behaviour of the equilibrium growth rate now depends on its relation to the rate of growth of interest payments as well as its relation to the rate of growth of exports. Therefore if the equilibrium growth rate is initially below the rate of growth of exports it is no longer possible to conclude that it will remain there. To
consider this behaviour Johnson derived expressions for $\dot{Y}_t$ and $Z_t$ from the equations immediately above. However these equations are too complex to allow the formulation of simple relationships between the constants of the system. He does, however, conclude that it is only under certain conditions that there exists a rate of growth of exports which would allow the economy to maintain equilibrium between effective demand and productive capacity by growing at a steady rate.

The simplest case is when initially trade is balanced and interest payments (or receipts) are zero. Then $x = r = 0$. Or more generally, a rate of growth of exports permitting equilibrium growth at a constant rate exists only when initially the equilibrium growth of output is equal to the initial rate of growth of interest income (or interest payments). A special case is when initially trade is balanced. Then equilibrium growth at a constant rate $i$ is possible if exports are expanded at the rate $x = i$. However, if initially the current account is balanced via the existence of interest payments, then no rate of growth of exports exists which would permit equilibrium growth at a constant rate. These are particular cases however and no simple but general statement can be made about the behaviour of the equilibrium growth rate.

Ball [5] and Massell [23] also consider the effect on the rate of growth of a country of financing its interest payments on international debt by further borrowing. Ball considers the rate of growth of the economy as the rate of growth of its national income. But in a model with international borrowing or lending, as Massell explains, the distinction between income growth and output growth is important. If a country borrows from abroad, or is the recipient of direct foreign investment, then its output will rise provided the
productivity of the new investment is greater than zero. But some part of the increased output, possibly all of it, will be paid outside the country either as interest payments or dividends. As Massell points out Ball's failure to make this distinction is the basis of his paradoxical results that capital imports result in an accelerated rate of growth in the recipient country and that the burden of debt thus incurred can be shifted entirely to the capital exporting country.

The basic structure of Ball's model is somewhat different to that of Johnson's. Ball assumes that international debt adjusts to increases in national income via \[ \Delta D = \lambda \Delta Y \], where the parameter \( \lambda \) is positive, and that the balance of trade adjusts to changes in the economy's external capital position

\[ X - M = rD - \Delta D \]

whereas Johnson assumes that differences between receipts and payments on current account, including both trade balance and interest payments on past debts, are financed by further borrowing.

Massell then concludes that the rate of growth of output will be increased by foreign borrowing if the net increase in debt exceeds that part of the interest payments that would be allocated to investment in the absence of foreign debt. In Ball's formulation of the problem with no distinction between income and output, the rate of growth of both income and output in the presence of foreign borrowing appears higher regardless of the magnitude of the interest payments. This is because a balance of payments surplus is automatically created to enable the economy to meet its interest payments and this is done at no cost to the economy. But in Massell's analysis, the fraction \( s \) of interest payments must be subtracted
from current foreign borrowing and this difference determines whether or not the rate of growth of output is increased and by how much. Thus an economy servicing a debt incurred in the past, but no longer receiving new foreign loans, will have a growth rate of output less than that of the Harrod-Domar model with no international capital movements; whereas, according to Ball, the growth rate of output will always be at least as great as that of the closed Harrod-Domar rate, regardless of the size of the debt or the interest rate payable on this debt.

Considering the rate of growth of income, Massell finds that capital imports will increase this growth rate if the marginal output-capital ratio exceeds the interest rate. Amano's criticism [1] of Massell's model is that it fails to show the explicit link between the domestic rate of profit on capital and its marginal output-capital ratio—and hence the actual direction of capital movements. Amano attempts to find this link.

Amano assumes that the country is relatively small and characterised by perfect competition and full utilisation of economic resources. He also assumes that commodity prices and the rate of profit on capital in the rest of the world are given. Then, without capital movements, this country grows with zero trade balance at constant terms of trade. Its long-run equilibrium growth rate equals the sum of the rate of growth of working population and the rate of productivity increase. Moreover, along the equilibrium growth path the domestic rate of profit is constant and is determined by the relative share of capital, the equilibrium growth rate and the economy's average propensity to save. These equilibrium growth rates differ from those of the Harrod-Domar growth model because Amano has assumed substitutability of factors of production. In the Harrod-
Domar growth model the rate of growth of both factors of production must be equal along the equilibrium growth path so as to maintain full employment of both factors.

Amano assumes that the country's aggregative production function is the Cobb-Douglas type, 

\[ P(t) = K(t)^a L(t)^{1-a} \]

where \( a \) is the elasticity of output with respect to capital input. Under the assumed condition of perfect competition \( a \) also represents the relative share of capital and \( \frac{aP(t)}{K(t)} \) the domestic rate of profit on capital. The country's saving is a constant fraction (s) of its national income while the labour force is assumed to be growing at a constant rate \( n \). Then with no international capital movements, the output-capital ratio along the equilibrium growth path is \( \frac{n}{s} \), so that the rate of profit is \( \frac{an}{s} \).

When free international capital movements are introduced into the model, a capital inflow or outflow will start, depending on whether the domestic rate of profit is higher or lower than the profit rate given in the rest of the world. If capital is perfectly mobile internationally, capital will flow immediately so as to equate the domestic rate of profit to the fixed international level.

In the case of the capital-importing country, if the foreign rate of profit is \( r \) then \( r < \frac{an}{s} \). Amano concludes that if free capital inflow is allowed to the country with a potentially high rate of profit, the continuous capital inflow raises the rate of growth of national income of the capital-importing country above its long-run equilibrium growth rate in the absence of capital movements. But the rate of growth of national income declines as capital inflow continues and after a sufficiently long period it will revert to the original level. Since labour grows at the constant rate \( n \), the rate
of growth of income per head increases initially but as time goes to infinity the rate of growth of income per head approaches zero.

Also the ratio of foreign borrowed capital to the total capital stock is a decreasing function of time, approaching a finite value as time goes to infinity. Thus it is not possible for the growth of foreign capital to continuously outstrip the growth of domestic capital forever.

In the case of the capital-exporting country, Amano derives similar results. That is, whenever the foreign rate of profit exceeds the average output-capital ratio in the absence of capital exports, the rate of growth of national income in the capital-exporting country is greater than its potential growth rate for a given time period.

Thus the main conclusion of Amano's analysis is that free international capital movements based on profit motives are beneficial to both the capital-importing and capital-exporting countries, in the sense that they raise the rates of economic growth above their potential rates of growth in the absence of international capital movements, at least for a sufficiently long period.

II

The analysis so far has been concerned with the requirements for equilibrium growth in a single country, the rate of growth of whose exports is given. But except in the case of a relatively small country this analysis is unsatisfactory because it does not consider the requirements of equilibrium growth in the rest of the world nor does it allow for repercussions of variations in a country's imports on incomes abroad and through them on its own exports.
Abstracting from foreign content of investment expenditure, the import content of domestic production and the payment of interest on international loans, the simple extension of Johnson's previous model gives the result that if the equilibrium growth rate of one country is greater than that of the other, then its equilibrium growth rate must rise and the other must fall continually over time.

That is

$$\frac{dr_i}{dt} = \frac{a_i m_j Y_i}{Y_i} (r_i - r_j),$$

where $r_i$ is the rate of growth of investment, capacity and income which keeps productive capacity continuously fully employed, $a_i$ the constant ratio of output to capital, $Y_i$ the income, $m_i$ the fraction of income spent on imports and $i$ and $j$ represent the two countries.

Assuming that domestic production has an import content, the behaviour of the two countries' equilibrium growth rates over time depends, as above, on the relative magnitude of their initial equilibrium growth rates. But if part of investment expenditure is spent on imported goods then:

$$\frac{dr_i}{dt} = \frac{a_i (m_i + m_i') Y_i}{1 - m_i' - m_i'} \frac{Y_i}{Y_i} (r_i - r_j)$$

(where $m_i'$ is the fraction of investment expenditure spent on imports) from which it is seen that the behaviour of the equilibrium growth rate over time depends on the relative magnitudes of the two growth rates and the sum of the two foreign content fractions of investment expenditure. If this sum is less than unity then unless the equilibrium growth rates are the same in the two countries, they will diverge over time. If the sum of the two fractional foreign contents of investment is greater than unity then provided the equilibrium growth rates are not the same, they will converge.
over time. But in the limiting case when the above sum is unity, the equilibrium growth rates of the two countries are completely indeterminate.

Johnson [12] finds that the introduction of interest payments on international lending makes it impossible to deduce the future course of the equilibrium growth rates from their initial magnitudes. However, he does draw some conclusions about the influences of interest payments and receipts on the equilibrium growth rates at any given time. The receipt of interest payments by one country from the other may tend either to raise or lower its equilibrium growth rate depending on whether the fraction of interest receipts it spends on its own good is less than or greater than the fraction of interest payments by which the paying country reduces its imports from the receiving country—and conversely for the payment of interest by one country to the other. In the limiting case where these two fractions are equal, interest payments and receipts would have no influence on the equilibrium growth rates. If no part of interest receipts were saved or if no part of payments dissaved, the effect of interest transfers on the equilibrium growth rates would depend on whether the sum of the fractions of interest receipts spent on imports were less than or greater than unity. Then the effects of interest payments are similar to the effects of foreign contents of investment expenditure.

III

Sections I and II have been concerned with the effect of international capital movements on the equilibrium rate of growth of output of a country. In this section the time paths of capital accumulation and international debt which take the economy to the equilibrium position will be examined.
Hamada's analysis [10] is in terms of the two-country model of economic growth with international capital movements. Capital is assumed to be perfectly mobile between countries while labour is completely immobile internationally. It is assumed that the two countries produce only one good which is the same in both countries. Thus Hamada's model abstracts from international trade and the terms of trade effect. On the other hand, Oniki and Uzawa's [28] and Kemp's [18] analyses are within the standard two-country, two-good, two-factor of production model of international trade but the process of capital accumulation is determined with the aid of Uzawa's two sectoral growth theory. Hamada's analysis will be considered first and then the analyses of Oniki and Uzawa, and Kemp will be discussed.

In Hamada's model the two factors of production, labour and capital, are used to produce one commodity. The factor of production labour is growing at a constant rate in each country, but not necessarily at the same rate in both countries. Capital is assumed to move instantaneously so as to equate the marginal productivities of capital in the two countries. Then the rate of interest payable on borrowed capital is assumed to equal this common marginal product of capital. It is assumed that saving is a constant fraction of income and that the production functions which are linear, homogeneous, concave and twice-differentiable functions of capital and labour are identical in both countries.

The optimal paths of capital accumulation in each country and of international debt depend on the relationship existing between the two labour growth rates and the saving ratios.

(a) The case when each country has the same labour growth rates and the same saving ratios, namely \( \lambda_1 = \lambda_2 = \lambda \); \( s_1 = s_2 = s \).
If the initial capital endowments are proportional to the initial available labour, then there is no need for international capital movements since capital growth, as well as labour growth, is at the same rate in both countries. Therefore the marginal product of capital will continue to be the same in both countries. But if the initial capital-labour ratios are not identical, capital will move instantaneously from one country to the other so as to equalize the rates of return to capital. Thus initially the level of international debt is $B_0$.

At any time, the total amount of saving or the increase in the total stock of capital is

$$K = S = s(P_1 - rB) + s(P_2 + rB) = s(P_1 + P_2) = sP.$$  

Production ($P_i$) in each country is proportional to its labour. Also, capital has been adjusted to be proportional to labour so as to equalize its marginal productivity in the two countries. To maintain these conditions investment in each country must equal its saving. That is $K_i = sP_i$. But interest payments must also be paid on the foreign debt by country 1. Therefore saving in country 1 is smaller than that required to maintain these above conditions for balanced growth by an amount $srB$. Then this amount must be borrowed anew from country 2. Thus the level of international debt is increasing continuously:

$$B = srB.$$  

Hamada finds that on this balanced growth path the amount of inter-

---

5. Assume $\frac{K}{L}$ in country 1 < $\frac{K}{L}$ in country 2.
national debt increases exponentially at the rate $sr$ where $r$ is the marginal productivity of capital at the equilibrium capital-labour ratio $k = \frac{s f(k)}{\lambda}$, namely $r = f'(k)$. But the ratio of existing international debt to the total capital stock of the two countries \( \left( \frac{B}{K} \right) \) continuously decreases at a constant rate on the equilibrium balanced growth path.

(b) The case with different labour growth rates and the same saving ratios, namely $\lambda_1 > \lambda_2$; $s_1 = s_2 = s$.

If initially the capital-labour ratios are not equal, capital will move instantaneously so as to equate the capital-labour ratios and thus the marginal productivities of capital in the two countries will be identical. As in the previous case, $k = K_1 + K_2 = s(P_1 + P_2)$.

Because the labour growth rates are no longer identical $K_1$ and $K_2$ must equal $sP_1 + (\lambda_1 - \lambda_2) \frac{K_1K_2}{K} \quad \text{and} \quad sP_2 - (\lambda_1 - \lambda_2) \frac{K_1K_2}{K}$ respectively so as to maintain equal capital-labour ratios in the two countries.

Savings in each country are

\[ S_1 = s(P_1 - rB) \quad \text{and} \quad S_2 = s(P_2 + rB) \] .

Therefore to maintain equal marginal productivities in each country, international debt must grow continuously, namely

\[ \dot{B} = K_1 - S_1 = (\lambda_1 - \lambda_2) \frac{K_1K_2}{K} + srB \] .

But Hamada finds that regardless of the initial value of international debt, the proportion of international debt to total capital stock

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6. Assuming the initial debt $B_0$ is positive.
becomes negligible in the long-run.

(c) The case where profits equal savings and the labour growth rates are different.

In this case assume again that $\lambda_1 > \lambda_2$. The total increase in capital is

$$\dot{K} = \dot{K}_1 + \dot{K}_2 = S_1 + S_2 = r(K_1 - B) + r(K_2 + B) = rK.$$ 

In order to maintain equal marginal productivities of capital

$$\dot{K}_1 = rK + (\lambda_1 - \lambda_2) \frac{K_1K_2}{K}; \quad \dot{K}_2 = rK - (\lambda_1 - \lambda_2) \frac{K_1K_2}{K}$$

and the required increase in international debt is

$$B = S_1 - S_2 = (\lambda_1 - \lambda_2) \frac{K_1K_2}{K} + rB.$$ 

As in the previous case if $B_0 > 0$ then $B > 0$. But Hamada concludes in this case where saving always equals profits in both countries, that the debt-capital ratio converges to a positive value which is determined by the initial endowments of capital. This result reflects the self-reproductive nature of the savings behaviour assumption.

(d) The case of different saving ratios with the same labour growth rates, namely $s_1 < s_2$ and $\lambda_1 = \lambda_2 = \lambda$.

In this case the increase in the total stock of capital is

$$\dot{K} = S_1 + S_2 = s_1(P_1 - rB) + s_2(P_2 + rB)$$

$$= s_1P_1 + s_2P_2 + (s_2 - s_1)rB.$$ 

This additional term $(s_2 - s_1)rB$ appears because the saving ratios are constant fractions of income which is output net of interest payments.
In the previous cases $s_1 = s_2$ and $K = sP$. The $K_i$ necessary for each country to equate its marginal productives are

$$
K_1 = \frac{s_1K_1 + s_2K_2}{K} P_1 + (s_2 - s_1) \frac{K_1}{K} rB \quad \text{and}
$$

$$
K_2 = \frac{s_1K_1 + s_2K_2}{K} P_2 + (s_2 - s_1) \frac{K_2}{K} rB.
$$

The second terms enter again here because of the assumption of different saving ratios. Therefore the required increase in international debt is

$$
\dot{B} = \dot{K}_1 - (s_1P_1 - s_1rB).
$$

Then the system can be described by the two equations (in per capita terms)

$$
\dot{k} = s f(k) + (s_2 - s_1)rB - \lambda k,
$$

$$
\dot{b} = (s_2 - s_1)k f(k) + srb - \lambda b
$$

where $k = \frac{L_1}{L}$, $b = \frac{L_2}{L}$, $s = \frac{L_2}{L} s_1 + \frac{L_1}{L} s_2$, and $s = \frac{L_1}{L} s_1 + \frac{L_2}{L} s_2$.

The equilibrium solutions found by equating $\dot{k}$ and $\dot{b}$ to zero give one positive level and one negative level of $b$. For the positive equilibrium solution Hamada derives that

$$
\frac{k^*}{f(k^*)} > \frac{s}{\lambda}
$$

where $s$ is the average savings ratio weighted by the existing labour in each country and thus can be interpreted as the average savings ratio of the total world economy. Then $\frac{s}{\lambda}$ is the equilibrium capital-output ratio of the asymptotic growth path in an economy where labour grows at the rate $\lambda$ and the saving ratio is $s$. Then if the equilibrium amount of per capita indebtedness (b) is positive, capital is deepened more than in the case of no international capital movements. This result reflects the fact that interest payments from a country saving less to that saving more increases the total average propensity to save in the world.
This case with different saving ratios and the same labour growth rates is the most realistic because if labour growth rates are different then in the very long-run one country becomes dominant and the other negligible.

(e) The case where both saving ratios and labour growth rates are different.

Assuming $\lambda_1 > \lambda_2$ and $s_1 \neq s_2$, the increase in the total capital stock is

$$ K = s_1 P_1 + s_2 P_2 + (s_2 - s_1) rB. $$

Then in order to maintain equal capital-labour ratios in the two countries,

$$ K_1 = \frac{s_1 K_1 + s_2 K_2}{K} P_1 + (s_2 - s_1) \frac{K_1}{K} rB + (\lambda_1 - \lambda_2) \frac{K_1 K_2}{K}, $$

and

$$ K_2 = \frac{s_1 K_1 + s_2 K_2}{K} P_2 + (s_2 - s_1) \frac{K_2}{K} rB - (\lambda_1 - \lambda_2) \frac{K_1 K_2}{K}. $$

The first term in each is $sP$, where $s$ is the average world saving ratio. Then the second and third terms are adjustments to $K_i$ to take account of the different saving ratios and the different labour growth rates respectively. Therefore the amount of new borrowing is

$$ B = (s_2 - s_1) \frac{K_2}{K} P + (\lambda_1 - \lambda_2) \frac{K_1}{K} rB + \frac{s_1 K_1 + s_2 K_2}{K} rB. $$

Then under the condition of this analysis, namely that capital is completely mobile internationally and that the production functions of the two countries are identical, the increase in the level of international debt depends on (i) the difference of the saving ratios, (ii) the difference of the labour growth rates and (iii) a compensating term for the international interest payment transfer.
Hamada has assumed that the production functions are identical in each country. As this assumption removes much of the rationale for international capital movements, it would have been interesting to see the effect of different production functions on the optimal paths of capital accumulation and international debt.

Oniki and Uzawa [28] have assumed that one of the two goods traded between the two countries is a consumption good and the other an investment good. For given technology and consumers' tastes in each country, the volume and terms of trade and the pattern of specialization of production in each country depends on the factor endowments of the country. Each commodity is assumed to be produced by the two factors of production, labour and capital. The supply of labour in each country is given and it is assumed to be growing at a constant rate n in both countries. International capital movements in the model put forward by Oniki and Uzawa exist only in the form of trade in capital goods and not in the form of foreign investment. Thus the rate of capital accumulation is determined by the amount of domestic output and the imports of the investment good. The comparative advantages of each country thus vary over time as the process of capital accumulation continues and the terms of trade in turn affects the rate at which capital is accumulated.

Oniki and Uzawa use the reciprocal demand curve to determine the patterns of specialization of production and of trade and the effect of factor endowments on these patterns. They derive the dynamic path of capital accumulation and show that, under their assumption that consumption goods are capital-intensive, the accumulation path is always stable with the capital-labour ratio of each country converging to the long-run stationary ratio. This long-run stationary capital-labour ratio of each country is deter-
mined relative to the average propensity to save and the technological conditions of the other country, as well as of its own. If the average propensities to save of both countries are different the production of one of the countries may be specialized either in consumption goods or in investment goods at the long-term stationary state.

Kemp [18] attempts to extend the above analysis by allowing the possibility of movement of capital internationally in the form of foreign borrowing or lending. As in the Oniki and Uzawa model, Kemp assumes that there exist two countries, two goods and the two factors of production, capital and labour. He assumes that trade between the two countries is unimpeded and that capital is completely mobile internationally. Thus capital moves instantaneously so as to equalize the returns on capital in the two countries.

Kemp makes the additional assumption that technology is not only given but is the same in both countries. Then each country is incompletely specialized in production and thus from the production side this international economy behaves like a closed two-sector economy.

Considering the static case, or the case in which the labour growth rate is zero, the equilibrium distributions of capital and labour are determined by balancing the world demand for and supply of newly created capital goods. These distributions are found to be unique provided that the total capital-labour ratio is a monotonic function of the wage-rental ratio and that all equilibria are locally stable.

Introducing constant labour growth into the model, Kemp draws the
following conclusions:

(i) a steady-growth solution always exists
(ii) not all steady-growth solutions are locally stable
(iii) global stability is not assured
(iv) for local instability it suffices that the macro-elasticity of factor substitution \( \sigma \) is less than \( c \)
(v) for local stability it suffices that the consumption good industry be relatively capital-intensive and that \( \sigma > c \).

Kemp also concludes that since international trade and foreign indebtedness are not considered explicitly, their actual levels are indeterminate. However, Kemp attempts to evaluate the limits of international indebtedness along the steady growth path. But, from Mundell's discussion [24] of international trade and factor mobility, since international trade is unimpeded, there will be no trade when the system is in equilibrium. Because capital is assumed to be perfectly mobile between countries, it will move so as to equate the marginal return on capital in the two countries. Therefore, the limits of international indebtedness are obvious.

7. \( c \) is defined by

\[
\frac{s^L(1 - s^K)k_1 - s^K(1 - s^L)k_2}{s^L[s^K(k_2 + \omega) + (1 - s^K)(k_1 + \omega)]}
\]

where \( s^L \) is the labour-weighted average of the propensities to save,

\( s^K \) is the capital-weighted average of the two propen-
sities to save,

\( k_1 \) is the capital-labour ratio in the \( i \)th-industry,

\( \omega \) is the wage-rental ratio.
Kemp's model is somewhat uninteresting from the point of view of international capital movements because, as he assumes that the production functions, prices and labour growth rates are the same in the two countries, the international economy behaves in the same way as that of a single country. International capital movements would play a more important role in a model where either the production functions were not identical and international trade was restricted, or the mobility of capital internationally was restricted. Hamada [10] has considered the case in which the production functions and labour growth rates are identical in both countries. But because it is assumed that both countries are completely specialized in producing the same good, it is possible to determine the accumulation path of international indebtedness as well as that of total capital.

In section I Johnson [12] and Ball [5] extend the well-known Harrod-Domar growth model and draw conclusions regarding capital accumulation in the open model. The two factors of production, labour and capital, are assumed to be fully-employed. Then for equilibrium growth it is assumed, if labour is not growing at the same rate as capital, that technical progress of a type which maintains a constant capital-output ratio is going on at such a pace as to prevent the capital stock outgrowing the labour required to work it.

Johnson concludes that the equilibrium growth rate will be higher the higher the output-capital ratio, the saving ratio and the import surplus as a fraction of total output. However, if capital is also imported in the form of investment goods, the equilibrium rate of growth rises as the fractional import content of domestic production increases. Johnson does not consider the possibility of capital movements in the form of foreign investment although he
does consider the case in which trade deficits and surpluses are financed by capital movements which bear a fixed interest rate. The equilibrium growth rate will be higher the lower the ratio of foreign interest income to domestic production. The behaviour of the equilibrium growth rate then depends on its relation to the rate of growth of interest payments as well as its relation to the rate of growth of exports. Johnson is unable to draw any general solution as the system of equations is too complex but he does give answers to particular cases.

Ball [5] considers a slightly different problem to that discussed by Johnson. Ball assumes that capital movements take the form of foreign investment and that the level of foreign investment adjusts to increases in national income via a debt-accelerator. It is also assumed that the Balance of Trade adjusts to changes in the economy's external capital position. As Massell [23] points out, Ball assumes that output and income growth are identical but this is incorrect when capital movements take the form of foreign investment. In this case Massell concludes that if the economy is servicing a debt incurred in the past but is not receiving new loans, the rate of growth of output is less than that in the closed Harrod-Domar growth model.

Amano examines the relationship between the domestic rate of profit on capital and its marginal output-capital ratio, to determine the direction of the flow of capital in Massell's model. Amano assumes that in the small country labour is growing at a constant rate and that the country's aggregative production function is of the Cobb-Douglas type. Capital is assumed to be completely mobile and moves between countries so as to equalize the rate of profit in the two countries. He concludes that in both the lending and
the borrowing countries the rate of growth of national income exceeds its potential growth rate in the absence of capital movements, for a sufficiently long period of time.

Except in the case of a relatively small country it is unsatisfactory to assume that the level of exports is given, as in the case discussed by Johnson [12] and Ball [5]. In section II Johnson takes account of the repercussions of variations in a country's imports on incomes in the rest of the world and through them on its own exports. Then Johnson derives that if a country's equilibrium growth rate is above that of the other, its growth rate must rise and the other must fall continually over time, provided that domestic investment has no import content and that trade surpluses and deficits are financed in a way that does not give rise to international interest payments. However if these two assumptions are relaxed the relationship between the two growth rates depends on whether the sum of the two fractional foreign contents of investment is greater than or less than one and whether the fraction of interest receipts one country spends on its own good is less than or greater than the fraction of interest payments by which the paying country reduces its imports from the receiving country.

In section III the paths of capital accumulation and international debt for a country from any initial position to the position of equilibrium growth are considered. Hamada's model abstracts from trade as he assumes that only one good is produced in the two-country world. As he also assumes that the production functions are identical, the international movement of capital is determined by the factor of production endowments of the two countries, the relationship between the saving ratios and that between the labour growth rates of the two countries. International capital movements
considered by Oniki and Uzawa exist only in the form of trade in capital goods and not in the form of foreign investment. Thus capital accumulation is determined by the amount of domestic output and the imports of investment goods. Then the terms of trade influence the rate of capital accumulation. Kemp has discussed the capital accumulation path of the two-country two-good model when international capital movements take the form of foreign investment. However as Kemp assumes that the production functions and the constant labour growth rates of the two countries are identical much of the rationale for international capital movements is removed.
Chapter III

INTERNATIONAL CAPITAL MOVEMENTS AND
OPTIMAL ECONOMIC GROWTH

Whilst many economists have been interested in the optimal accumula-
tion path of a national economy, most of the analyses have been
confined to the optimal growth of a closed economy. However recently
Bardhan [7], Hamada [10], Negishi [27] and Pitchford [29] have dis-
cussed optimal growth in an open economy. Bardhan and Hamada have
considered the case of a country facing an international capital
market while Negishi and Pitchford consider the case of direct foreign
investment.

The central problem of the theory of optimal economic growth is how
national income should be allocated between investment and consumption
so as to maximise national welfare. From the theory of optimal
economic growth of a closed economy, a balanced growth path or a
golden-age growth path is that along which the two factors of pro-
duction, capital and labour, grow at the same rate. If the saving
ratio is constant there is only one balanced growth path. That is
the path along which $\frac{f(k)}{k} = \frac{\omega}{s}$, where $s$ is the constant saving
ratio, $\omega$ is the constant rate of capital depreciation, $k$ is the
capital-labour ratio and $f(k)$ is the production function. If
however $s$ is not constant there exists a number of golden-age

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8. $f(k)$ can be considered to be either a neoclassical production
function or a production function with fixed coefficients.
paths which are all logarithmically parallel. The golden-age path along which consumption per head \(c\) is maximised is called the Golden Rule path. Then the Golden Rule path is that golden-age path along which \(f'(k) = \omega\).

Introducing diminishing marginal utility of consumption, the optimal growth path is defined as that path along which the accumulated difference between the actual utility of consumption and the maximum obtainable level of utility (or Bliss) is minimised. This path is defined by the Ramsey Rule which states that the rate of saving multiplied by the marginal utility of consumption should always equal Bliss minus the actual rate of utility enjoyed.

If, on the other hand, a rate of time preference \(\rho\) is introduced so that the welfare of each generation is weighted according to \(ce^{-\rho t}\) or \(U(c)e^{-\rho t}\), the golden-age path along which welfare is maximised is given by \(f'(k) = \omega + \rho\). This balanced growth path is termed the modified Golden Rule path.

In section I the borrowing models of Bardhan [7] and Hamada [10] will be considered and then in section II the two-country models of Negishi [27] and Pitchford [29] will be discussed. In all of these papers it is assumed that there exists only one good which may be

9. Logarithmically-parallel means that the logarithm of the capital stock on one golden-age path differs by a constant from that on any other particular golden-age path. There is such logarithmic parallelism in terms of any variable. Thus the consumption path corresponding to one golden-age path never intersects the consumption path corresponding to any other golden-age path.
either consumed or accumulated. Then these models may be thought of as either incorporating or not incorporating international trade. However, it is not necessary to consider trade explicitly, so that the theory abstracts from the problems associated with international trade and thus allows a discussion of long-run international capital movements in the context of optimal growth theory.

I

The models put forward by Bardhan [7] and Hamada [10] which deal with optimal foreign borrowing, assume that capital is perfectly mobile internationally. However, as the other basic assumptions of these two models are very different, the papers will be discussed separately.

Bardhan considers an economy which can borrow from, but never lend to, an imperfect international capital market such that the rate at which it borrows is a function of the total amount of capital borrowed. Bardhan's objective is to find the optimal capital accumulation path along which the economy maximises, over an infinite time period, the accumulated difference between the actual level and the maximum sustainable level of social welfare. He defines social welfare as the difference between the social utility of per capita consumption and the social dis-utility of having foreign-owned capital in the home country. The population growth rate is assumed to be constant, savings per head is a constant fraction of per capita national income and depreciation is ignored. Then Bardhan formulates the problem such that the level of domestically-owned capital per head is the only state variable and the level of foreign-owned capital per head and the savings ratio are the two control variables. State variables are those which define the state of the economy at each instant of time and control variables are
those which are assumed to be adjusted so as to determine the growth path of the economy.

The solution of the problem depends on the relationship between the population growth rate and the rate of interest ruling in the international capital market when the amount of foreign borrowing of the home country is zero. This rate of interest is denoted by $r_0$.

(a) The case where the rate of population growth is greater than the rate of interest $r_0$.

If initially the level of per capita domestically-owned capital is less than its optimal level, then along the optimal growth path the amount per capita of domestically-owned capital is steadily increased while the amount of foreign-owned capital per head decreases asymptotically to approach the Golden Rule path. Along this path the level of foreign-owned capital per head is positive. The opposite holds when the initial level of per capita domestically-owned capital is greater than its optimal level.

(b) The case where the rate of population growth is less than or equal to the rate of interest $r_0$.

Along the optimal growth path, if the level of domestically-owned capital per head is less than its optimal level initially, the amount of per capita domestically-owned capital steadily increases and the amount of foreign-owned capital per head is decreased to ultimately become zero—or is always zero.

An important feature of Bardhan's model is the rule derived for the acquisition of foreign-owned capital:

$$f' - r = \frac{D'}{U_T} + r'k_f > 0$$

or $$f' - \phi = \frac{D'}{U_T} > 0$$,
where \( f' \) is the rate of return on all capital invested in the borrowing country, \( r \) the rate of interest in the international capital market \((r' > 0)\), \( D' \) the marginal disutility of foreign-owned capital in the borrowing country and \( U' \) the marginal utility of consumption in the borrowing country. Also \( \varphi = r + r'k_f \) is the marginal cost of foreign borrowing.

Private borrowers would borrow up to the point where \( f' = r \).

Therefore, private borrowing tends to be excessive because of their failure to take account of the fact that an increase in the amount borrowed for the country as a whole raises the total borrowing cost and also of the increased disutility for the country as a whole from the extra borrowing. That is, private borrowers tend to over-borrow and thus it would be optimal for the borrowing country to tax the earnings from borrowed capital.

To compare this Bardhan rule for acquiring foreign-owned capital with that derived in the static model, consider Jones [13] where the rule derived is \( f' = \varphi = 0 \). That is, foreign borrowing should continue until the marginal return on capital in the borrowing country just equals the marginal cost of borrowing. This rule differs from that derived by Bardhan because in the static model it is assumed that there exists no disutility from foreign-owned capital in the borrowing country. Therefore, under the static model assumptions optimal foreign borrowing is greater than that determined from the Bardhan model.

It should be noted that the endpoint determined from within the Bardhan model satisfies the Golden Rule. That is, the marginal product of capital equals the rate of growth. However, I have not been able to discover a sufficiency theorem which would establish
that the assumptions of the Bardhan model (particularly the form of the objective function) are sufficient to ensure that an optimal path exists.

Hamada assumes in his analysis that the economy faces a competitive international capital market where it can borrow or lend capital at an interest rate which is constant over time and independent of the amount it borrows or lends. That is, Hamada considers the case of lending or borrowing by a small country whose dealings with the international capital market are such as to have no effect on the interest rate ruling in the market.

Hamada's aim is to determine the optimal path of capital accumulation which is defined as the choice of the levels of consumption, real capital net of depreciation existing in the country and the total international debt of the economy (all in per capita terms), which maximises the discounted sum of future utilities of consumption per head over an infinite time period. This maximisation is carried out subject to the conditions that (a) the initial value of the national wealth of the country is given and (b) a wealth-indebtedness restriction in the case of borrowing and a wealth-creditness restriction in the case of lending exist. The wealth-indebtedness restriction limits the country's level of indebtedness \( B_t \) to a certain proportion \( \beta \) of its net wealth \( K_t \). That is, \( B_t \leq \beta K_t \) (\( \beta < 1 \)). Similarly the wealth-creditness restriction does not permit the creditor country's foreign credit or lending to exceed a certain fraction of its total wealth. Hamada assumes that labour is growing at a constant positive rate and that all variables are net of depreciation.

Because the value of national wealth enters into the model as the
difference between the total capital existing in the economy and the total international debt of the economy, it is implied that the country can increase or decrease its capital stock by increasing or decreasing its indebtedness instantaneously. Hence perfect mobility of capital is assumed. By introducing the wealth-indebtedness restriction (or wealth-creditness restriction) there is a restriction on the upper limit to borrowing (or to lending) but within this limit there remains complete mobility of capital.

The solution of this problem depends on the relationship between the sum of the rate of growth of the labour force and the discount rate and on the interest rate ruling in the international capital market.

(a) If the sum of the rate of growth of labour and the discount rate is greater than the interest rate, the optimal capital accumulation path eventually approaches a balanced path where capital and foreign indebtedness grow at the rate of growth of labour. Under these conditions the economy will benefit eventually from borrowing abroad. If the initial endowment of domestically-owned capital is relatively small, the economy will assume a borrowing position immediately. If it is relatively large, the economy will first assume a lending position and then shift from lending to borrowing, keeping the capital-labour ratio constant such that the marginal product equals the interest rate. But when the maximum possible indebtedness-wealth ratio is reached, the economy gradually approaches the balanced path where capital and foreign indebtedness grow at the rate of labour growth. If the initial endowment is intermediate the economy starts at the shifting position. That is, it increases its borrowing until the maximum possible indebtedness-wealth ratio is reached whilst the capital-labour ratio remains constant. Then when this maximum possible indebtedness-wealth ratio is reached, the
economy gradually approaches the balanced path while maintaining this maximum indebtedness-wealth ratio.

To consider this balanced growth path refer to the diagram below.

The modified golden rule growth path in the closed economy is shown by $\dot{k}$ where domestically-owned capital grows at the rate $\lambda$ while consumption per head is $\dot{c}$. But Hamada's balanced growth path is shown by $k^\ast$ where consumption per head $c^\ast$ is greater than $\dot{c}$. That is, the possibility of borrowing from an international capital market at a fixed interest rate can eventually sustain a higher level of per capita consumption than can a closed economy with the same production technology. At this endpoint, output is distributed such that the domestically-owned capital stock $(k^\ast - b^\ast)$ grows at the rate $\lambda$, $ib^\ast$ is paid as interest owing on the foreign borrowed capital $b^\ast$ and the balance goes to consumption. However, it is a stable endpoint because the economy can maintain the level of international indebtedness $b = \beta k$. That is, $b = \frac{\beta}{1-\beta} (k - b)$ so
if \((k - b) = \lambda(k - b)\) then \(b = \lambda b\). That is, to maintain \(b\) at the greatest level, it must grow at the same rate as \((k - b)\) and thus at the same rate at \(k\). Then \(b\) grows at the rate \(\lambda\) such that part of the new borrowing is used for interest payments on existing debt and the balance is allocated to consumption. Therefore, Hamada's growth path is both balanced and stable but this results directly from the assumption that the interest rate is constant, independent of the amount of international indebtedness and that it is less than \(\lambda + \rho\). If however, the interest rate was made a function of the level of foreign borrowing, as in the Bardhan case, these results would not necessarily hold.

It should be noted that because Hamada's initial condition involves only the level of national wealth and because it is assumed that the rate of interest payable on foreign borrowed capital is constant, without the wealth-Indebtedness restriction the optimal solution would be the trivial one of borrowing as much as possible and consuming as much as possible. Thus the assumption of the wealth-Indebtedness restriction is vital to Hamada's model. Firstly, it eliminates the trivial solution and secondly, it allows the endpoint, which is unstable in the optimal growth theory of the closed economy, to be stable because the capital-labour ratio can be maintained, as labour grows, by borrowing forever. This endpoint of Hamada's analysis then allows consumption per head to be increased above that given by the Golden Rule endpoint.

The results of the other cases when \(\lambda + \rho \leq i\) are similar to those above.

(b) If the sum of the rate of growth of labour and the discount rate is smaller than the rate of interest, the optimal capital accumulation path approaches asymptotically a balanced path where
both domestic capital and foreign assets grow at the rate of growth of labour. Under these conditions, the economy will benefit eventually from investing abroad. If the initial endowment of national wealth is relatively large, the economy assumes a lending position from the beginning. If the initial endowment of national wealth is relatively small, the economy assumes a borrowing position. Then the economy shifts from a borrower to a lender keeping the capital-labour ratio such that the marginal product equals the rate of interest.

As in the above case, the asymptotic level of consumption is higher than that of a closed economy with the same production function and the same labour growth rate. Also in this case the capital-labour ratio at the endpoint is greater than in the closed economy case. The country can afford to deepen its domestic capital because of the favourable international investment opportunity. The added incentive to deepen comes from the introduction of the wealth-creditness restriction; in order to take advantage of the foreign investment opportunity, the country must match any increase in foreign lending with a certain amount of domestic investment.

(c) If the sum of the rate of growth of labour and the discount rate equals the rate of interest, the optimal paths of capital accumulation and consumption approach asymptotically the stationary values of capital per head (which is determined by equating the marginal product to the interest rate) and the corresponding level of consumption. If the initial endowment of national wealth is relatively small (or relatively large) then the economy assumes a position of borrowing (or lending) and then moves to the stationary solution. In this boundary case, the asymptotic level of consumption is not improved by the existence of an international capital market.
Negishi [27] and Pitchford [29] have considered capital movements between two countries when foreign investment takes the form of direct investment and not foreign borrowing. They both assume that capital is perfectly mobile and labour is completely immobile internationally. In the last part of Pitchford's analysis he restricts the mobility of capital by assuming that it is irreversible internationally. That is, once capital is installed it can change ownership but it cannot be moved between countries. Negishi determines the optimal level of foreign investment in the long-run but Pitchford's analysis is concerned with finding the optimal growth path to this optimal long-run solution. Both Negishi and Pitchford assume that the two countries involved in the international capital movements produce the same good which can be both consumed and accumulated. That is, both countries are completely specialized in the production of the same good. As a consequence of this, international trade does not enter explicitly into the analyses of Negishi and Pitchford.

Negishi discusses international capital movements between two countries which have the same growth rates but different saving ratios. He assumes initially that both economies are in a stationary state but later relaxes this assumption to include the case where the uniform growth rate is positive. The optimal level of foreign investment is such that the home country maximises its long-run level of maintained consumption per head subject to the stationary state condition that gross savings must equal depreciation in each country. But because capital is assumed to be perfectly mobile internationally, it will instantaneously move between countries so as to equate the marginal productivities of capital in both countries. Then the home country controls the movement of capital by adopting
the appropriate fiscal policy so as to maintain the optimal level of foreign investment.

Negishii considers two different savings assumptions, firstly that only capitalists save and secondly, that gross saving is a constant proportion of gross income.

Using the first of these savings assumptions it should be noted that the rate of return on capital in the foreign country is constant because the level of foreign-owned capital adjusts to a change in foreign investment so as to keep the total capital stock per head constant. Hence the rate of return abroad is constant. In this case, Negishii finds that the optimal fiscal policy is to subsidize its foreign lending. Comparing this result with the discussion in Chapter II, this optimal policy is "paradoxical".

With the second saving assumption, the optimal policy is for both countries to impose a tax on earnings from foreign investment. If however the saving assumptions are such that in the foreign country only capitalists save and that in the home country gross saving is a proportion of gross income, then it is optimal for capital to move internationally so as to equate the marginal productivities of capital in both countries. He also shows that extending the analysis to include a positive growth rate gives similar optimal fiscal policies.

Both Kemp [15] and Jones [13] have discussed Negishii's "paradoxical" result in relation to their analyses. Because Negishii includes growth in his model, there is the possibility that changes in foreign investment may induce an increase in the level of savings and capital endowments and thus a rise in future maintainable consumption.
In considering Jones' model, the effect of increased foreign investment in stimulating further capital accumulation at home must be the dominant factor near the initial point where returns to capital in both countries are equated and thus the sole determinant of the optimal fiscal policy. Kemp continues this discussion of the effect of subsidizing capital movements on the steady-growth income.

From neoclassical growth theory, the steady-growth rate is independent of the savings assumption but the level of income per head is not. Thus any policy which turns the distribution of income in favour of capitalists will raise the level of steady-growth income. But to subsidize foreign investment is to forgo possible monopoly-monopsony gains which, from the short-run point of view, is to misallocate resources. Thus the subsidization of capital movements has two conflicting effects on steady-growth income, namely an expansionary effect and a misallocation effect. Therefore the home country, in choosing its optimal fiscal policy, has to weigh up these two effects on the steady-growth income. Then the problem confronting the home country is that of choosing the optimal rate of subsidy.

Pitchford, on the other hand, determines the optimal fiscal policy necessary to control the level of foreign investment and the optimal current domestic investment so as to move the economy along the growth path which will maximise the accumulated utility of consumption along this path to the optimal long-run endpoint. It should be noted that Pitchford treats the level of current domestic investment or saving as a control variable. Therefore his model is differs from Negishi's model in which the saving ratio is assumed to be constant.

Pitchford's results are similar to those of the static models dis-
cussed in Chapter I. At any point of time the optimal level of foreign investment and the optimal tax rate agree with those derived from the static analysis with the same assumption of complete specialization of production in both countries. However, the optimal long-run solution is not derived explicitly in the static analysis. If the rate of discount of future utilities $\delta$ is less than $r$, the lowest rate of return at which any country will lend, it is optimal for the home country to have no foreign direct investment in the long-run. If however $\delta$ is greater than $r$, the optimal level of foreign investment remains positive for all time.

When Pitchford restricts the mobility of capital internationally by assuming that capital is irreversible he does not develop the complete set of optimal growth paths for any initial levels of foreign investment and domestically-owned capital but instead considers the characteristics of various paths.

As I have independently developed a similar model for the case of direct foreign investment under the various patterns of specialization of production, any further discussion of Pitchford's analysis will be considered when this model is presented in Chapter VIII.

In the following discussion of optimal foreign borrowing, there are two concepts which Bardhan and Hamada have used which will not be utilised. The first is the concept of disutility of foreign capital in the home country, which was considered by Bardhan. The second is Hamada's wealth-indebtededness restriction (or wealth-creditness restriction). This assumption seems to have been included in his model to eliminate the possibility of obtaining a trivial solution. As both of these concepts appear artificial in nature they will not be used in the following theory.
Chapter IV

THE NATURE OF INTERNATIONAL CAPITAL MOVEMENTS

There are many factors, both economic and non-economic, the behaviour of which it is necessary to examine to give a clear understanding of the nature of international capital movements. These factors include the degree of mobility of capital internationally, the characteristics of the international capital market, the existing level of international debt and the stability of the country both politically and economically.

The standard model of international trade theory assumes perfect mobility of factors of production within each country but complete immobility of factors between countries. On the other hand, in recent literature dealing with international capital movements the assumptions which have been made are not far short of implying perfect mobility of capital between countries. The true situation however is a complex mixture of partial, total and zero mobility which is difficult, if not impossible, to represent generally.

There are two aspects to international capital mobility in a growth model. The first is the question of reversibility of foreign investment. In the literature referred to above it is assumed that

1. For example, recent papers by R.W. Jones [13], P.K. Bardhan [7], K. Hamada [10].
reversibility is achieved by the physical withdrawal of foreign-owned capital if foreign borrowing has been excessive, but reversibility could also be achieved by buying out foreign investments as an alternative to domestic investment. In the models to be presented below, I have assumed that neither form of reversibility is possible. The first because in some cases it is obviously physically impossible and in others the expense would be prohibitive. Initially I have omitted the possibility of buying out foreign-owned capital because it often appears difficult in practice to persuade foreign owners to sell their invested capital—perhaps this may be partly for non-economic reasons. Then foreign ownership of capital can only be reduced by the non-replacement of foreign-owned capital and thus the stock of such capital is allowed to depreciate. However, this assumption is later relaxed so that the implications of permitting buying-out of foreign-owned capital can be investigated.

The second aspect of capital mobility is concerned with the supply of and the demand for foreign-owned capital in the international capital market. When the home country is a borrower there is a limit to the rate of expansion of foreign-owned capital in the home country, determined by the supply of capital in the international capital market. The supply of such capital is a function of the rate of return on capital in the international capital market. Similarly when the home country is a lender of capital there is a limit to the amount of foreign lending, set by the demand for capital by the borrowing countries. This demand is also determined by the rate of return on capital in the international capital market. However there is a natural upper limit to the supply of capital available for lending which is determined by the limited capacity of the lending country's capital goods industries.
In the models to be presented below it is assumed that capital is neither completely mobile nor completely immobile between countries. Both Bardhan [7] and Hamada [10] have assumed prefect mobility of international capital. Therefore the introduction of this more realistic assumption should modify their results.

The behaviour of the rate of return on capital borrowed or lent plays an important role in the theory of international capital movements. If the form of the international capital movement is that of direct investment in the borrowing country then the gross rate of return on the foreign-owned capital is the same as that on the domestically-owned capital. That is, the gross rate of return equals the marginal product of total capital employed in the economy. Then the gross rate of return on the foreign investment is a decreasing function of the total capital in the borrowing country.

When the international capital movement is of the form of an international loan then the rate of return depends on the relative size of the country, the size of the existing international debt and risk factors such as political and economic stability. The political stability of the borrowing country is a difficult factor to define precisely and a difficult factor for the lending country to measure. However economic stability and the ability to pay the interest on the international loan are directly related. Thus economic and political stability are included in the risk factors associated with international capital movements. In the models to follow it is assumed that the world economy is riskless. This assumption has been made in most growth models although in the real world risk is one of the more important factors entering into any investment decision.
When the country is relatively small, the rate of return is equal to that ruling in the international capital market. That is, the rate of return is independent of both the size of the current international loan and the level of the existing international debt. However when the country is relatively large, its borrowing or lending influences the rate of return ruling in the international capital market. The rate of return is a function of the size of the loan and of the existing international indebtedness of the country. In the models below this function is approximated by assuming that the rate of return is a function of the total level of international indebtedness.

It has been noted in Chapter III above that Bardhan has assumed that the rate of return is a function of the level of international indebtedness per head. That is, when the country is a borrower, the rate of return on the foreign borrowing is a function of the amount of the foreign borrowing per head in the borrowing country. This function is extremely unrealistic because the population of the borrowing country is not considered by the lender in determining the rate of interest at which it is willing to make an international loan to the borrowing country. A similar argument exists for the case when the home country is an international lender of capital. The rate of interest which the borrowing country is willing to pay on foreign loans is determined independently of the population of the lending country.

If \( i(t) \) is the interest rate payable on a loan made at time \( t \) and \( I_f(t) \) the borrowing at time \( t \), the total interest payable on loans over the time period \( T \) is given by

\[
\int_{n-T}^{n} i(t) I_f(t) \, dt.
\]

This equals \( i(t) K_f \) only if \( i(t) \) is constant for all \( t \).
That is,

\[ \int_{n-t}^{n} I_F(t) \, dt = i K_f(n), \]

where \( K_f \) is the total amount of foreign borrowing over the time period \( t \). Then the Bardhan model is only an inexact approximation unless \( i \) is constant or such fixed interest loans are not made.

In addition to the previous chapter, capital in the real world is

The more realistic assumptions relating to the nature of international capital movements, namely that the mobility of capital is restricted and that the rate of return on foreign loans, when the home country is relatively large, is a function of the total amount of international debt, are introduced into the models below, one at a time, so that full effect of each new assumption can be determined.

He assumes that the social objective of the country is to maximize over an infinite horizon, the integral of the shortfall of the actual rate of social welfare from the maximum sustainable level of each welfare by controlling amongst other things the level of foreign borrowing from the international capital market. That is, the country can affect the level of foreign borrowing instantaneously by assigning the foreign-owned capital to or from the home country. His model may be made more realistic by postulating the movement of foreign-owned capital. I have introduced such a restriction into the model by assuming irreversibility of capital. To do this it is necessary to re-formulate the problem so that the country controls not the level of total foreign borrowing but the level of per capita current foreign borrowing.

In section I the foreign problem will be re-visited and then in section II the case where the mobility of foreign capital is rest...
Chapter V

OPTIMAL GROWTH WITH RESTRICTED MOBILITY
OF FOREIGN CAPITAL

As discussed in the previous chapter, capital in the real world is neither completely mobile nor completely immobile internationally but is rather one of restricted mobility.

Bardhan [7] discusses the optimal capital accumulation of a relatively large country which borrows from an international capital market, when capital is assumed to be completely mobile internationally. He assumes that the social objective of the country is to minimise, over an infinite horizon, the integral of the shortfall of the actual rate of social welfare from the maximum sustainable level of such welfare by controlling amongst other things the level of foreign borrowing from the international capital market. That is, the country can adjust the level of foreign borrowing instantaneously by shipping the foreign-owned capital to or from the home country. His model may be made more realistic by restricting the movement of foreign-owned capital. I have introduced such a restriction into the model by assuming irreversibility of capital. To do this it is necessary to reformulate the problem so that the country controls not the level of total foreign borrowing but the level of per capita current foreign borrowing.

In section I the Bardhan problem will be re-worked and then in section II the case where the mobility of foreign capital is res-
tricted will be considered. By comparing these results the full implications of the above restriction will be seen explicitly. In section III the effects of permitting the home country to repay foreign loans will be considered. In all three sections of this chapter the objective function is assumed to be linear. In the following chapter the effects of a non-linear objective function on the optimal growth paths will be examined.

The country faces an imperfect international capital market from which it borrows but to which it can never lend. It is assumed that all capital whether borrowed or domestically-owned is fully employed. If $k_d$ is the domestically-owned capital/head ($k_d \geq 0$) and $k_f$ the foreign-owned capital/head in the home country ($k_f \geq 0$), then the total capital/head employed is

$$ k = k_d + k_f. $$

The domestically-owned capital accumulation equation is

$$ \dot{k}_d = i_d - \mu k_d - \delta k_d $$

or

$$ \dot{k}_d = i_d - \omega k_d, $$

where $i_d$ is the current domestic investment/capita ($i_d \geq 0$), $\mu$ the constant rate of population growth and $\delta$ the constant rate of capital depreciation ($\omega = \mu + \delta$).
It is assumed that the country has a neo-classical production function in per capita terms under conditions of constant returns to scale with

\[ f'(k) > 0, \quad f''(k) < 0 \]
\[ f(0) = 0, \quad f'(0) = \infty \]
\[ f(\infty) = \infty, \quad f'(\infty) = 0. \]

The rate of interest payable on borrowed capital is a function of the amount borrowed per head of population in the home country.\(^2\)

That is,

\[ r = r(k_f) > 0 \]
and \[ r'(k_f) > 0. \]

There is however an upper limit \( \gamma \) to \( k_f \) with \( r(\gamma) = \infty \). Then the marginal cost of borrowing is \( \ddot{\phi}(k_f) = r(k_f) + r'(k_f)k_f \) such that \( \ddot{\phi}'(k_f) > 0 \).

If \( c \) is the current consumption/capita \((c \geq 0)\) and \( \bar{c} \) the maximum sustainable level of consumption per capita, then the home country's budget constraint is

\[ f(k_d + k_f) \geq r(k_f)k_f + i_d + c. \]

Now the home country's aim is to minimise the difference between the actual level and the maximum sustainable level of consumption along the accumulation path from the initial point to a fixed end-point by choosing whatever amounts of current per capita domestic investment, current per capita consumption and the level of foreign

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\(2. \) As discussed in Chapter IV, this is the assumption which Burdhan makes regarding the rate of interest payable on a foreign loan.
borrowing which satisfy the budget constraint. That is, the objective is to minimise
\[ \int_0^T (\dot{c} - c) \, dt, \] where \( T \) is unspecified, \( (1) \)
along the path from any initial point to the endpoint subject to
\[ f(k) - r(k_f)k_f - i_d - c \geq 0, \] \( (2) \)
\[ \dot{k}_d = i_d - \omega d k, \] \( (3) \)
\[ c \geq 0 \text{ and } \] \( (4) \)
\[ i_d \geq 0. \] \( (5) \)

The fixed endpoint \((k_d, k_f)\) is such that sustainable consumption or net output per head is maximised. Net output is
\[ y = f(k_d + k_f) - r(k_f)k_f - \omega d k. \] To maximise net output with respect to \(k_d\) and \(k_f\)
\[ \frac{\partial y}{\partial k_d} = f' - \omega = 0 \]
\[ \frac{\partial y}{\partial k_f} = f' - \phi = 0. \]

That is, the endpoint is defined by \( f' = \omega = \phi \). This is illustrated in the diagram below. The shapes of the curves have been derived in Appendix No.1.
This endpoint is dependent on the initial assumption that
\[ \omega > r(0) > \delta \]. That is, that the rate of depreciation on domestically-owned capital plus the rate of population growth is greater than the rate of interest in the international capital market before the home country borrows. \( r(0) \) is the gross rate of interest payable on a foreign loan. Therefore \( r(0) \) must be greater than \( \delta \), the rate of capital depreciation, for the lending country to be attracted to the home country's foreign loan raising activities. Then \( f' = \phi \) becomes \( f' = r(0) \) at \( k_f = 0 \) and the curve \( f' = \phi \) cuts the \( k_d \) axis at a point beyond the point where \( f' = \omega \). The curves \( f' = \omega \) and \( f' = \phi \) will intersect on the \( k_f = 0 \) axis if \( \omega = r(0) > \delta \) initially but there is no point of intersection if initially \( \omega < r(0) > \delta \). But when the rate of depreciation on domestically-owned capital plus the rate of population growth is less than the rate of interest in the international capital market, at the endpoint the home country would only be interested in lending domestically-owned capital to the market although initially it may be optimal to borrow foreign-owned capital. Therefore I have chosen to omit the case where \( \omega < r(0) > \delta \) from this section on foreign borrowing and to consider only the case where \( \omega > r(0) > \delta \) initially.

In the optimization problem which must now be solved, the control variables \( c \), \( i_d \), \( k_f \) can be chosen subject to the given constraints to determine the movement of the state variable \( k_d \). This problem is solved by Arrow's Method of Solution\(^3\) which is based on Pontryagin's Maximum Principle.\(^4\)

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3. Arrow [3].

4. Pontryagin [30], page 19, Theorem No. 1.
Define the Hamiltonian

\[ H = -(\dot{c} - c) + \lambda_1 [i_d - ak_d] \quad (6) \]

and the Lagrangian

\[ L = H + \pi_d + tc + \rho[f(k_d + k_f) - r(k_f)k_f - c - i_d], \quad (7) \]

where \( \lambda_1 \), the auxiliary variable, has the interpretation of the social price of net domestic investment and \( \rho, t \) and \( \rho \) are the Lagrange multipliers.

Necessary conditions for an optimal solution are:

(a) \( \lambda_1 \) is a continuous function of time given by

\[ \lambda_1 = -\frac{\partial L}{\partial k_d} = -[\rho f' - \lambda_1 w]. \quad (8) \]

(b) The constraint satisfies the Constraint Qualification \( ^5 \) at any point of time and the Hamiltonian is maximised subject to this constraint. That is

\[ \frac{\partial L}{\partial c} = 1 + t - \rho = 0 \quad (9) \]
\[ \frac{\partial L}{\partial k_d} = \lambda_1 + \pi - \rho = 0 \quad (10) \]
\[ \frac{\partial L}{\partial k_f} = \rho (f' - 4) = 0 \quad (11) \]
\[ \pi \geq 0; \ \pi_d = 0 \quad (12) \]
\[ t \geq 0; \ \pi_c = 0 \quad (13) \]
\[ \rho \geq 0; \ \rho [f(k) - r(k_f)k_f - c - i_d] = 0 \quad (14) \]

These necessary conditions are also sufficient for an optimal solution.

5. Arrow, Hurwicz, Uzawa [4].
solution because the Hamiltonian $H$ is concave in the state variable $k_d$, given $\lambda_1$ and $t$. This sufficiency condition is given by Arrow [3], lecture 1, proposition 4.

From (9) and (13), $\rho \geq 1$ (15)

Therefore using (14), all net output is used for either $i_d$ or $c$. Then all possible policy combinations satisfying the given constraint are

<table>
<thead>
<tr>
<th>Policy</th>
<th>$i_d$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>F</td>
</tr>
<tr>
<td>B</td>
<td>F</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>&lt;F(&gt;0)</td>
<td>&lt;F(&gt;0)</td>
</tr>
</tbody>
</table>

where $F = f - rk_f$.

Properties of these policies

Policy A

$i_d = 0$; $c = F$

From (12) to (15)

$p \geq 0$; $t = 0$; $\rho = 1$

Substituting into (10) and (8)

$\lambda_1 \leq 1$; $\lambda_1 = [-f' - \lambda_1 \omega]$.

Policy B

$i_d = F$; $c = 0$

From (12) to (15)
\[ p = 0; \quad t \geq 0; \quad \rho \geq 1. \]

Substituting into (10) and (8)

\[ \dot{\lambda}_1 \geq 1; \quad \ddot{\lambda}_1 = -[\lambda_1(f' - \omega)]. \]

**Policy C**

\[ \dot{i}_d > 0; \quad c > 0 \]

From (12) to (15)

\[ p = 0; \quad t = 0; \quad \rho = 1. \]

Substituting into (10) and (8)

\[ \lambda_1 = 1; \quad \ddot{\lambda}_1 = -[f' - \omega]. \]

But \( \lambda_1 = 1 \), therefore \( \ddot{\lambda}_1 = 0 \)

i.e. \( f' = \omega \)

That is, policy C can only be optimal at the endpoint.\(^6\)

---

6. As is explained in the next paragraph, \( k_f \) is freely adjusted so that \( f' = \phi \) always. But along policy C, \( f' = \omega \). Therefore policy C can only be optimal at the endpoint where \( f' = \phi \) and \( f' = \omega \).
85.

The Phase Diagram

Because \( k_f \) is a control, the amount of foreign borrowing can be adjusted immediately so that \( f' = \phi \) and then by applying the appropriate policy, A or B, the system moves along \( f' = \phi \) to the endpoint \((\hat{k}_d, \hat{k}_f)\).

Along \( f' = \omega \), \( k_d + k_f = \hat{k} \). Therefore in the region \( f' > \omega \), \( k < \hat{k} \) and along \( f' = \phi \) in this region \( k_d < \hat{k}_d \). Then the optimal policy is B along which all of net output is allocated to domestic investment and nothing to consumption. Also in the region \( f' < \omega \), \( k_d > \hat{k}_d \) along \( f' = \phi \). Therefore the optimal policy is A, along which all of net output is consumed and domestically-owned capital is allowed to depreciate at the maximum possible rate.

II

The assumptions of this model are the same as those of the previous model but now the movement of \( k_f \) is restricted by introducing \( i_f \), the current foreign borrowing/capita, as a control variable. Then from the assumption of restricted mobility of foreign capital,
0 \leq i_f \leq V \text{ where } V \text{ is the upper limit to the amount of current foreign borrowing imposed by the physical limitations on supply. All other variables are as defined for the previous model. Then the foreign-owned capital accumulation equation is}

\dot{k}_f = i_f - \omega k_f.

Then the objective is to minimise

\int_0^T (\hat{c} - c) dt,

where \( T \) is unspecified, along the path from any initial point to the fixed endpoint, subject to

\begin{align*}
\alpha(k_d + k_f) - r(k_f)k_f - c - i_d &\geq 0 \quad (16) \\
\dot{k}_d &= i_d - \omega k_d \quad (17) \\
\dot{k}_f &= i_f - \omega k_f \quad (18) \\
c &\geq 0 \quad (19) \\
i_d &\geq 0 \quad (20) \\
0 &\leq i_f \leq V. \quad (21)
\end{align*}

In the optimization problem which must now be solved, the control variables \( c, i_d, i_f \) can be chosen, subject to the given budget constraint, to determine the movement of the state variables \( k_d \) and \( k_f \). Using Arrow's method of solution again, the Hamiltonian is

\[ H = - (\hat{c} - c) + \lambda_1 [i_d - \omega k_d] + \lambda_2 [i_f - \omega k_f] \quad (22) \]

and the Lagrangian is

7. Arrow [3].
\[ L = H + p_i_d + q_i_f + v[V - i_f] + tc \]
\[ + p[f(k) - r(k_f).k_f - c - i_d] , \]  

where \( \lambda_1 \) and \( \lambda_2 \) are the auxiliary variables and \( p_i, q_i, v, t, p \) are the Lagrange multipliers. These auxiliary variables have the economic interpretation of social prices of net domestic and net foreign borrowing respectively.

Necessary conditions for an optimal solution are

(a) \( \lambda_1 \) and \( \lambda_2 \) are continuous functions of time given by

\[
\dot{\lambda}_1 = -\frac{\partial L}{\partial t} = -[pf' - \lambda_1 \omega] \tag{24}
\]

\[
\dot{\lambda}_2 = -\frac{\partial L}{\partial t} = -[p(f' - \phi) - \lambda_2 \omega] . \tag{25}
\]

(b) The constraint satisfies the Constraint Qualification at any point of time and the Hamiltonian is maximised subject to this constraint. That is

\[
\frac{\partial L}{\partial c} = 1 + t - p = 0 \tag{26}
\]

\[
\frac{\partial L}{\partial d} = \lambda_1 + p - p = 0 \tag{27}
\]

\[
\frac{\partial L}{\partial f_d} = \lambda_2 + q - v = 0 \tag{28}
\]

\[
p \geq 0 ; \quad p_i_d = 0 \tag{29}
\]

\[
q \geq 0 ; \quad q_i_f = 0 \tag{30}
\]

\[
v \geq 0 ; \quad v[V - i_f] = 0 \tag{31}
\]

\[
t \geq 0 ; \quad tc = 0 \tag{32}
\]

\[
\rho \geq 0 ; \quad p[f(k) - r(k_f).k_f - c - i_d] = 0 . \tag{33}
\]

8. Arrow, Hurwicz, Uzawa [4].
A further necessary condition from the Pontryagin Theorem\(^9\) is that this optimal value of \( H \) be zero so that for optimality, \( c \), \( i_d \) and \( i_f \) from (22) satisfy

\[
(\dot{c} - c) = \lambda_1 [i_d - \omega k_d] + \lambda_2 [i_f - \omega k_f].
\]

These necessary conditions are also sufficient for optimality from Arrow [3], lecture 1, proposition 4.

From (26) to (32), \( \rho \geq 1 \).

Therefore, from (33) all net output is used for either \( i_d \) or \( c \).

Then all possible policy combinations satisfying the given constraint are

<table>
<thead>
<tr>
<th>Policy</th>
<th>( i_d )</th>
<th>( c )</th>
<th>( i_f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>F</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>F</td>
<td>( &gt;0(&lt;V) )</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>F</td>
<td>( V )</td>
</tr>
<tr>
<td>D</td>
<td>( F )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>E</td>
<td>( F )</td>
<td>0</td>
<td>( &gt;0(&lt;V) )</td>
</tr>
<tr>
<td>F</td>
<td>( F )</td>
<td>0</td>
<td>( V )</td>
</tr>
<tr>
<td>G</td>
<td>( &lt;F(&gt;0) )</td>
<td>( &lt;F(&gt;0) )</td>
<td>0</td>
</tr>
<tr>
<td>H</td>
<td>( &lt;F(&gt;0) )</td>
<td>( &lt;F(&gt;0) )</td>
<td>( &gt;0(&lt;V) )</td>
</tr>
<tr>
<td>I</td>
<td>( &lt;F(&gt;0) )</td>
<td>( &lt;F(&gt;0) )</td>
<td>( V )</td>
</tr>
</tbody>
</table>

where \( F = f(k) - r(k_f)k_f \).

Properties of these policies

By substituting the values of \( i_d \), \( i_f \) and \( c \) along each policy into the equations (24) to (33) and (35) it is possible to determine the values of \( \lambda_1 \), \( \lambda_2 \), \( \lambda_1' \) and \( \lambda_2' \). This detailed working is

\[9. \text{ Pontryagin [30], page 19, Theorem No. 1.}\]
provided in Appendix No.1 but the following table gives a sufficient summary of this information.

<table>
<thead>
<tr>
<th>Policy</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>Region in which policy is optimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$\leq 1$</td>
<td>$\leq 0$</td>
<td>$f' \leq \omega, f' \leq \phi$</td>
</tr>
<tr>
<td>B</td>
<td>$\leq 1$</td>
<td>$= 0$</td>
<td>$f' \leq \omega, f' = \phi$</td>
</tr>
<tr>
<td>C</td>
<td>$\leq 1$</td>
<td>$\geq 0$</td>
<td>$f' \leq \omega, f' \geq \phi$</td>
</tr>
<tr>
<td>D</td>
<td>$\geq 1$</td>
<td>$\leq 0$</td>
<td>$f' \geq \omega, f' \leq \phi$</td>
</tr>
<tr>
<td>E</td>
<td>$\geq 1$</td>
<td>$= 0$</td>
<td>$f' \geq \omega, f' = \phi$</td>
</tr>
<tr>
<td>F</td>
<td>$\geq 1$</td>
<td>$\geq 0$</td>
<td>$f' \geq \omega, f' \geq \phi$</td>
</tr>
<tr>
<td>G</td>
<td>$= 1$</td>
<td>$\leq 0$</td>
<td>$f' = \omega, f' \leq \phi$</td>
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<tr>
<td>H</td>
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<td>$f' = \omega, f' = \phi$</td>
</tr>
<tr>
<td>I</td>
<td>$= 1$</td>
<td>$\geq 0$</td>
<td>$f' = \omega, f' \geq \phi$</td>
</tr>
</tbody>
</table>

Possible policy switches

The shadow prices, $\lambda_1$ and $\lambda_2$, which are continuous functions of time and of the state variables $(k_d, k_f)$ also possess continuous derivatives with respect to time. It is the combination of these two properties which allows the policy switches to be determined. The possible switches are set out in the table below which is based on Appendix No.1. Policies B, E, G, I cannot switch into any other policy. Therefore they have been omitted from that part of the table. A zero indicates that no switch is possible while the other information given relates to the conditions necessary for the switch to take place. Condition (i) must exist immediately before the switch and condition (ii) must exist at the "switching surface".

1. All possible policy switches are derived in Appendix No. 1.
Using all the information summarized in this table it is possible to construct the phase diagram showing the paths of the optimal policies.
I. Region $f' > \omega$

(A) $f' < \phi$

In this region $k < \hat{k}$, $k_d < \hat{k}_d$, $k_f > \hat{k}_f$. Therefore the optimal policy is $D$, along which all net output is allocated to domestic investment, nothing to consumption and foreign capital is allowed to depreciate at the maximum possible rate. If at the initial position $k_f$ is such that $F < \omega k_d$, then although all of net out-

2. The shapes of the curves $F = 0$, $F = \omega k_d$ and $f''(F - \omega k) + \omega k_f \phi' = 0$ are derived in Appendix No. 1. In drawing this phase diagram it is assumed that $f''(F - \omega k_d) + \omega k_f \phi' = 0$ cuts the $\dot{k}_d = 0$ axis above $f' = \phi$ so that policy $E$ lies along the whole of $f' = \phi$ in the region $f' \geq \omega$. 
put is allocated to domestic investment it is not greater than the amount of depreciation so \( k_d \) as well as \( k_f \) falls until \( F > \omega k_d \). Then by increasing total capital per head, \( f' \) is reduced until \( f' \) equals either \( \phi \) or \( \omega \). (This depends on the initial position.)

(i) If the system reaches \( f' = \omega \), the optimal path switches from policy D to G along which \( k_f \) depreciates at the maximum possible rate, thus reducing \( \phi \) so that at the endpoint \( f' = \omega = \phi \). But the path of G is along \( f' = \omega \) where \( \dot{k} = 0 \). This is achieved by allocating the required amount from net output to domestic investment and the balance to consumption.

(ii) If the system reaches \( f' = \phi \), then the optimal path switches from policy D to E, which takes the system along \( f' = \phi \) to the endpoint. Along policy E all net output is invested \( (k_d > 0) \), nothing is allocated to consumption and \( \dot{k}_f < 0 \) but \( > -\omega k_f \). That is, \( f' \) is reduced to \( \omega \) by increasing \( k \) by allocating net output to \( k_d \) and allowing \( k_f \) to depreciate so as to keep \( f' = \phi \).

That is, the optimal policy D switches to either policy E or G. The difference between these policies arises from the fact that along G, \( k = \hat{k} \) but along E, \( k < \hat{k} \).

\( f' > \phi \)

Because \( k < \hat{k} \), the optimal policy is F which allocates all of total net output to domestic investment and current foreign investment is at the maximum amount possible (i.e. \( V \)) until either \( f' \) equals \( \phi \) or \( \omega \).

(i) If \( f' \) becomes equal to \( \omega \) first, the optimal policy
switches from policy F to policy I along which $k_f$ continues to increase at the maximum possible rate until $f'$ is decreased such that $f' = \omega = \phi$. But the path of policy I is along $f' = \omega$ where $k = \dot{k}$ so that to reach the endpoint it is necessary to change only the composition of ownership of capital and not the total amount of capital. This is achieved by allocating the required amount from total net output to domestic investment and the balance to consumption.

(ii) If along policy F $f'$ becomes equal to $\phi$ then the optimal policy switches from F to E. This has been discussed in (A) above.

II. Region $f' < \omega$

(A) $f' > \phi$

The optimal policy in this region where $k > \dot{k}$, $k_d > \dot{k}_d$, $k_f < \dot{k}_f$ is policy C along which all net output is consumed, nothing is allocated to domestic investment and foreign-owned capital is increased at the maximum possible rate. Thus, by decreasing total capital per head, $f'$ is increased until it equals either $\omega$ or $\phi$. (This depends on the initial position.)

(i) If along policy C the system reaches $f' = \omega$ first, the optimal policy switches from policy C to I. This has been discussed in I (B) above.

(ii) If the system reaches $f' = \phi$ first, the optimal policy switches from policy C to B along which all net output is allocated to consumption, nothing to investment ($k_d < 0$) and $k_f > 0$. Thus $f'$ is increased to $\omega$ by reducing $k$ by allowing $k_d$ to depreciate at the maximum possible rate and by increasing $k_f$ so as to keep $f' = \phi$. 
(B) $f' < \phi$

Because in this region $k > \hat{k}$, the optimal policy is A which allocates all of total net output to consumption and allows both $k_d$ and $k_f$ to depreciate until either $f'$ equals $\omega$ or $\phi$. (Neglect for the moment all points of this region which do not lie on a path of policy A which cuts $f' = \omega$ in the $F > w k_d$ region, that is, the shaded area on the phase diagram.)

(i) If the system reaches $f' = \omega$ first, the optimal policy switches from A to G. This has been discussed in I (A) above.

(ii) If policy A takes the system to $f' = \phi$ first, the optimal policy switches from A to B. This then has been discussed in II (A) above.

(C)

The remaining region is the shaded area on the phase diagram. In this region $k > \hat{k}$ and $k_f > \hat{k}_f$. Therefore the optimal policy is either A or D. Both policies A and D can exist in this region but if the initial policy is A then it must switch to policy D somewhere inside the region. These policies differ mainly in the rate of depreciation of $k_d$. Policy A allocates nothing to $k_d$ thus allowing maximum depreciation of $k_d$, whilst policy D allocates all of total net output to $k_d$ but this is not sufficient to cover depreciation so $k_d$ does not increase. The actual level of $k_d$ at which the policy switch occurs is indeterminate.

Qualification

If the curve $f''(F - w k) + w k f' \phi' = 0$ cuts the line $f' = \phi$ in the region $f' > \omega$ then policy E is no longer optimal along the
whole of $f' = \phi$ in this region. Thus the phase diagram is as given below.

The optimal policies remain the same in most regions of the phase diagram but it should be noted that because policy E does not lie along $f' = \phi$ in the region $f''(F - \omega k) + \omega k_f \phi' < 0$, policy F will still take the economy to the $f' = \phi$ curve but there it switches into policy D which as before switches to E at $f' = \phi$ in the region $f''(F - \omega k) + \omega k_f \phi' > 0$. Policy E then takes the system to the fixed endpoint.

Comparison of the two models
In the first model, as $k_f$ is a control variable, the optimal policy is to adjust the level of $k_f$ instantaneously so that the marginal product of foreign borrowing is equated to its marginal cost. Then subsequent adjustments to the endpoint depend on whether $k_d > k_d'$. If $k_d < k_d'$, say, domestic capital is increased at the maximum possible rate with instantaneous adjustments of $k_f$ such as to maintain the relationship $f' = \phi$.

The introduction of the above restrictions on the movement of
foreign-owned capital in the second model does not allow the instantaneous adjustment of \( k_f \) as \( k_f \) is restricted to
\[ -wk_f \leq k_f \leq V - wk_f. \]
In this model there are two important relationships which determine the optimal policy—namely the relationship between \( f' \) and \( \phi \) and that between \( f' \) and \( \omega \), that is, (a) the marginal product and the marginal cost of foreign borrowing and (b) the size of the total capital stock relative to its optimum size.

To illustrate the conclusions for the second model take the case, say, of a country for which \( k < \hat{k} \).

(i) If the country has less than optimal foreign borrowing then the optimal policy is \( F \) along which \( k_d \) and \( k_f \) are increased at the maximum possible rate until either \( f' = \phi \) or \( k = \hat{k} \).

(a) If the marginal product of the foreign-owned capital becomes equal to its marginal cost then the optimal policy switches from \( F \) to \( E \) or, as discussed in the qualification, from \( F \) to \( D \) and then into \( E \). Along policy \( E \), \( k_d \) continues to increase at the maximum possible rate and \( k_f \) is controlled so as to maintain the equality between \( f' \) and \( \phi \) until the endpoint is reached where \( k_d = \hat{k}_d, k_f = \hat{k}_f \).

(b) If the total capital stock reaches the optimal capital stock size, then the optimal policy switches from \( F \) to \( I \). Along \( I \), only the composition of the capital stock is changed by continuing to increase \( k_f \) at the maximum possible rate and adjusting \( k_d \) so as to keep \( k = \hat{k} \) until the endpoint is reached.

(ii) If the country has excessive foreign borrowing and hence a deficiency in domestically-owned capital, the optimal policy is \( D \) along which domestic investment is increased at the
maximum possible rate and the foreign-owned capital is reduced by allowing it to depreciate at the fastest possible rate until either \( k = \hat{k} \) or \( f' = \phi \).

(a) If along policy D the total capital stock is increased to the optimal level \( k \), then the optimal policy switches from policy D to G along which the composition of \( k \) is adjusted by continuing to allow the foreign-owned capital to depreciate at the maximum possible rate and increasing \( k_d \) so that \( k \) remains at the optimal level until the amount of foreign borrowing is no longer excessive; that is, until the endpoint where \( f' = \phi \) and \( k = \hat{k} \).

(b) If the excess foreign borrowing is reduced so that along D the marginal product of the foreign-owned capital becomes equal to its marginal cost then the optimal policy switches from D to E. Along E foreign-owned capital is adjusted so that \( f' = \phi \) and \( k_d \) is increased at the maximum possible rate until the endpoint is reached where \( k = \hat{k} \).

Comparing these two sets of results it is clearly seen that the assumption of restricted mobility of international capital does modify the answer. When the movement of foreign capital is unrestricted the optimal policy is always to adjust the level of foreign borrowing instantaneously so that the marginal product of the foreign-owned capital equals its marginal cost. But when the mobility of foreign capital is restricted, instantaneous adjustment of foreign capital is no longer possible and both domestically-owned and foreign-owned capital are adjusted simultaneously until either the marginal product of the foreign-owned capital equals its marginal cost or the total capital stock equals the optimal amount of capital stock. Then by a policy switch the system is taken to the endpoint where both criteria are satisfied.
In this section the home country is permitted to reduce its international debt by repaying part of its international loans. Then the home country's current domestic investment per head \((i_d)\) has two possible components. They are \(i_{d1}\) the current domestic investment per head out of the country's net output and \(i_{d2}\) the current domestic investment per head resulting from the repayment of part of the foreign loans. Then \(i_d = i_{d1} + i_{d2}\) with both \(i_{d1}\) and \(i_{d2}\) restricted to non-negative values.

The capital accumulation equations are now

\[
\begin{align*}
\dot{k}_d &= i_{d1} + i_{d2} - wk_d \\
\dot{k}_f &= i_f - i_{d2} - wk_f.
\end{align*}
\]

Then the home country's objective is to minimise

\[
\int_0^T (\hat{c} - c) dt,
\]

where \(T\) is unspecified along the path from any initial point to the endpoint subject to

\[
\begin{align*}
f(k) &= r(k_f)k_f - c - i_{d1} - i_{d2} \\
\dot{k}_d &= i_{d1} + i_{d2} - wk_d \\
\dot{k}_f &= i_f - i_{d2} - wk_f \\
c &\geq 0 \\
i_{d1} &\geq 0 \\
i_{d2} &\geq 0 \\
0 &\leq i_f \leq V.
\end{align*}
\]

This solution is presented in Appendix No. 2 and is similar to that given in section II except that now there are four control variables, namely \(i_{d1}, i_{d2}, i_f\) and \(c\). As before all of net output is allocated to either \(i_{d1}, i_{d2}\) or \(c\). Therefore all the possible
policy combinations can be listed. When $i_{d_2} = 0$ the problem is the same as that in section II above. Thus policies A to I are identical to those given there, with $i_{d_2}$ held at zero level.

The additional possible policies are

<table>
<thead>
<tr>
<th>Policy</th>
<th>$i_{d_1}$</th>
<th>$i_{d_2}$</th>
<th>$c$</th>
<th>$i_f$</th>
</tr>
</thead>
<tbody>
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<td>0</td>
</tr>
<tr>
<td>K</td>
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<td>F</td>
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<td>0</td>
</tr>
<tr>
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<td></td>
<td>$0 &lt; i_f &lt; V$</td>
</tr>
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<td>L</td>
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<td>V</td>
</tr>
<tr>
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<td>$0 &lt; i_{d_1} &lt; F$</td>
<td>$0 &lt; c &lt; F$</td>
<td>0</td>
</tr>
<tr>
<td>Q</td>
<td>0</td>
<td>$0 &lt; i_{d_1} &lt; F$</td>
<td>$0 &lt; c &lt; F$</td>
<td>$0 &lt; i_f &lt; V$</td>
</tr>
<tr>
<td>R</td>
<td>0</td>
<td>$0 &lt; i_{d_1} &lt; F$</td>
<td>$0 &lt; c &lt; F$</td>
<td>V</td>
</tr>
<tr>
<td>S</td>
<td>$0 &lt; i_{d_1} &lt; F$</td>
<td>$0 &lt; i_{d_2} &lt; F$</td>
<td>$0 &lt; c &lt; F$</td>
<td>0</td>
</tr>
<tr>
<td>T</td>
<td>$0 &lt; i_{d_1} &lt; F$</td>
<td>$0 &lt; i_{d_2} &lt; F$</td>
<td>$0 &lt; c &lt; F$</td>
<td>$0 &lt; i_f &lt; V$</td>
</tr>
<tr>
<td>U</td>
<td>$0 &lt; i_{d_1} &lt; F$</td>
<td>$0 &lt; i_{d_2} &lt; F$</td>
<td>$0 &lt; c &lt; F$</td>
<td>V</td>
</tr>
</tbody>
</table>

The possible policy switches are derived in Appendix No. 2 and are summarised in the following table.

### Table of Switches

<table>
<thead>
<tr>
<th>Switches out of</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>F</th>
<th>I</th>
<th>J</th>
<th>$D, E, K$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$M, N, O$</td>
</tr>
<tr>
<td>A</td>
<td>-</td>
<td>(i) $f' &lt; \phi$</td>
<td>(ii) $f' &gt; \omega K_d$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>(i) $f' = \phi$</td>
<td>-</td>
<td>(i) $f' &lt; \omega$</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>- (i) $f' &gt; \omega$</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>J</td>
<td>(i) $f' &lt; \omega$</td>
<td>(ii) $f' = \phi$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>- (i) $f' &gt; \omega$</td>
<td>(ii) $f' = \lambda_d \omega$</td>
</tr>
</tbody>
</table>
Policies $G, P, S$ have been omitted from the above table because they are non-optimal policies. Similarly, policies $H, L, Q, R, T$ and $U$ have been omitted as they are only internally consistent at the endpoint. Also the policies $B, D, E, I, K, M, N$ and $O$ cannot switch into another policy. Therefore they have been omitted from that part of the table.

A zero indicates that no switch is possible while the other information given relates to the conditions necessary for the switch to take place. Condition (i) must exist immediately before the switch and condition (ii) must exist at the "switching surface".

Policies $D, E, K, M, N$ and $O$ are all optimal along $f' = \phi$ in the region $f' > \omega$. Policies $D, E, M$ have been examined in Appendix No. 2, but so far the policies $K, N, O$ have not been discussed. Along the policies $K, N, O$ the home country increases its foreign borrowing ($i_f > 0$) and at the same time repays part of its international loan ($i_d > 0$), but for the policies to be consistent $i_f$ and $i_d$ must be such that $k_f$ decreases. Thus policies $K, N, O$ have the same effect on $k_d$ and $k_f$ as does policy $E$.

As it is optimal for the home country to reduce its foreign borrowing, it does not seem rational to both increase and decrease the amount of foreign borrowing to achieve this. Thus policies $K, N, O$ appear to be irrational policies for the home country to adopt. The economy can be moved along that same part of the $f' = \phi$ curve by adopting policy $E$. In the real world policies $K, N, O$ would be rational policies only in the case in which part of the international loan matures and by agreement must be repaid. It would be optimal to repay this matured loan and then to increase the level of foreign borrowing so as to move the economy along $f' = \phi$. 
Comparing this phase diagram with the one presented in section II of this chapter, it is seen that the optimal growth paths in the region $f' \geq \phi$ are identical.

In the region $f' < \phi$, $k_f > \hat{k}_f$. It is now optimal to reduce the level of foreign borrowing by repaying part of the international debt. The optimal policy is $J$ along which all of net output is allocated to the repayment of foreign loans. However, when the economy approaches $f' = \phi$ the region $f' < \omega$, $k_d > \hat{k}_d$ and $k_f$ approaches its optimal level, the optimal policy switches from $J$ to $A$. As derived in the Appendix No. 2, this switch occurs when $\lambda_2 = \frac{f' - \omega}{u}$. Then along policy $A$ all of net output is allocated to consumption and both the level of domestically-owned capital and foreign borrowing are allowed to depreciate. When the level of foreign borrowing is such that $f' = \phi$ the policy switches from
A to B which, as before, takes the economy along \( f' = \phi \) to the endpoint.

**Conclusion**

The introduction of the possibility of repaying foreign loans has modified the solution somewhat. When foreign borrowing is excessive and the level of domestically-owned stock is either below its optimal level or not as excessive as the foreign borrowing, it is optimal to allocate all of net output to the repayment of foreign loans until the level of foreign borrowing is either reduced to its optimal level corresponding to the stock of domestically-owned capital or reduced so that it is in no greater excess than is the stock of domestically-owned capital. Then by the policy switches discussed above the economy is taken to the endpoint.
Chapter VI

OPTIMAL GROWTH WITH A NON-LINEAR OBJECTIVE FUNCTION

In the models presented in the previous chapter, the home country's aim is to minimise the difference between the actual level and maximum sustainable level of consumption along its capital accumulation path. That is, the objective function is assumed to be linear. Bardhan [7], on the other hand, assumes a non-linear objective function. Therefore, the introduction of diminishing marginal utility of consumption into the model will allow a strict comparison to be made with the Bardhan model. Initially, capital will be assumed to be perfectly mobile internationally as this agrees with Bardhan's assumption. Then in section II, the case in which the mobility of foreign capital is restricted and the repayment of foreign loans is not permitted will be discussed. Then comparing the results of this model with those of the model with a linear objective function (discussed in section II of Chapter V), an analysis of the effects of diminishing marginal utility on the determination of the optimal growth paths can be made.

Using the variables as defined in the preceding chapter, the home country's objective is now to minimise

\[ \int_0^T [u(c) - u(x)] dt, \]

where \( T \) is unspecified, along the path from any initial point to the fixed endpoint subject to
\[ f(k_d + k_f) - r(k_f)k_f - i_d - c \geq 0 , \]
\[ k_d' = i_d' - \omega k_d , \]
\[ c \geq 0 \text{ and } i_d \geq 0 . \]

It is also assumed that \( U'(c) > 0 , U''(c) < 0 \) and \( U'(0) = \omega . \)

The fixed endpoint is such that sustainable consumption per head is
maximised. As derived in Chapter V the endpoint \((k_d^*, k_f^*)\) is
defined by \( f' = \phi = \omega . \)

In the optimisation problem which must now be solved, the control
variables \( i_d \), \( c \), \( k_f \) can be chosen, subject to the given budget con­
straint, to determine the movement of the state variable \( k_d . \)

Applying Arrow's Method of Solution\(^3\) again, the Hamiltonian is
\[ H = -[U'(c) - U'(c)] + \lambda_1 [i_d - \omega k_d] \]
and the Lagrangian is
\[ L = H + p_i d + tc + \rho [f(k_d + k_f) - r(k_f)k_f - c - i_d] . \]

Necessary conditions for an optimal solution are:

(a) The shadow price \( \lambda_1 \) is a continuous function of time given by
\[ \lambda_1' = -\frac{3L}{\partial k_d} = -[\rho f' - \lambda_1 \omega] . \]

(b) The constraint satisfies the Constraint Qualification\(^4\) at any
point of time and the Hamiltonian is maximised subject to this con­
straint. That is,

\[ \text{3. Arrow [3]} \]
\[ \text{4. Arrow, Hurwicz, Uzawa [4]} \]
\[
\frac{\partial L}{\partial c} = U'(c) + t - \rho = 0
\]
\[
\frac{\partial L}{\partial \lambda_1} = \lambda_1 + p - \rho = 0
\]
\[
\frac{\partial L}{\partial \lambda_1} = \rho[f' - \phi] = 0
\]
\[
p \geq 0; \quad p_i = 0
\]
\[
t \geq 0; \quad t_c = 0
\]
\[
\rho \geq 0; \quad \rho[f(k) - r(k_f)k_f - c - i_d] = 0.
\]

(c) The optimal value of the Hamiltonian is zero. That is,

\[
U(c) - U(c) = \lambda_1[i_d - \omega k_d].
\]

These necessary conditions are also sufficient for an optimal solution because the Hamiltonian is a concave function of the state variable \(k_d\) for given \(\lambda_1\) and \(t\). This sufficiency condition is given by Arrow [3], lecture 1, proposition 4.

From \(\frac{\partial L}{\partial c} = 0\) it follows that \(\rho = U'(c) + t\). But, as \(t\) is positive, \(\rho \geq U'(c)\). Then all of net output is distributed between consumption and current domestic investment. Thus all the possible policies can be determined but before listing these it should be noted that since \(U'(c) = 0\) when consumption is zero, it can never be optimal to allocate all of net output to current domestic investment. Then all the possible policies with the allocation to consumption at the non-zero level are

<table>
<thead>
<tr>
<th>Policy</th>
<th>(i_d)</th>
<th>(c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>F</td>
</tr>
<tr>
<td>B</td>
<td>(0 &lt; i_d &lt; F)</td>
<td>(0 &lt; c &lt; F)</td>
</tr>
</tbody>
</table>

where \(F = f(k) - r(k_f)k_f\).
Properties of these policies

Policy A

\[ i_d = 0 ; \quad c = F \]
then \[ p \geq 0 , \quad t = 0 \quad \text{and} \quad \rho = U'(F) . \]
Thus \[ \lambda_1 \leq U'(F) \quad \text{and} \quad \lambda_1 = -[pf' - \lambda_1 \omega] . \]

Policy B

\[ i_d > 0 ; \quad c > 0 \]
then \[ p = 0 , \quad t = 0 \quad \text{and} \quad \rho = U'(c) . \]
Thus \[ \lambda_1 = U'(c) \quad \text{and} \quad \lambda_1 = -U'(c)[f' - \omega] . \]

Since \( k_f \) is a control variable, the amount of foreign borrowing can be adjusted instantaneously so that \( f' = \phi \). Then policies A and B lie along \( f' = \phi \). Along policy A \( k_d = -\omega k_d \). Therefore policy A can only be optimal in the region \( f' < \omega \). But as \( k_d \) can be either positive or negative along policy B it is possible for B to be optimal in both the \( f' < \omega \) region and the \( f' > \omega \) region. However, from the fact that the auxiliary variable is a continuous function of time and of the state variable and the necessary condition for optimality is \( \lambda_1 = -[pf' - \lambda_1 \omega] \), policy A switches into policy B at \( \lambda_1 = U'(F) \).

The exact position and shape of this "switching surface" on the phase diagram cannot be determined because of the presence of the shadow price \( \lambda_1 \) in the equation of the surface. However some information regarding its position on the phase diagram can be determined.

Consider \( \lambda_1 = U'(c) \). For a constant \( c \), \( \lambda_1 \) is constant and also the larger \( c \) the smaller \( \lambda_1 \). In the region \( f' \geq \omega \), \( c < F \) and therefore \( U'(c) > U'(F) \) and \( \lambda_1 > U'(F) \). Thus the
"switching surface" \( \lambda_1 = U'(c) \), where \( c = F' \), must lie entirely to the right of the curve \( f' = \omega \). That is, the lower boundary to the surface \( \lambda_1 = U'(c) \) must lie entirely in the region \( f' < \omega \).

The Phase Diagram

Capital is assumed to be completely mobile internationally. Therefore the level of foreign borrowing adjusts instantaneously so that \( f' = \phi \).

When the stock of domestically-owned capital is below its optimal level \( k_d \), the optimal policy is B, along which all of net output is distributed between consumption and current domestic investment. This distribution is determined by the relationship \( \lambda_1 = U'(c) \) which can be interpreted as follows:
$\lambda_1$ is the social demand price of a unit of domestic investment (at time $t$), measured in terms of utility units. That is, $\lambda_1$ is the amount of domestic investment which society would exchange for a unit of utility or, $\frac{1}{\lambda_1}$ is the amount of utility society would exchange for a unit of domestic investment.

$U'(c) = \frac{AU}{Ac}$ is the amount of utility which society can exchange for a unit of consumption. However, from the budget constraint, $Ac = -\Delta d$ (at time $t$). Therefore at time $t$, $\frac{AU}{Ac} = -\frac{AU}{\Delta d}$ which is the amount of utility society will forgo by investing one more unit of net output or the amount of utility society will gain by investing one unit less. That is, $U'(c)$ is the social supply price of a unit of domestic investment measured in terms of utility.

Therefore the social optimal utility at time $t$ is determined by equating the social demand and social supply prices of a unit of domestic investment. That is, the social optimal utility is given by $\lambda_1 = U'(c)$.

If initially the stock of domestically-owned capital exceeds its optimal level and the level of net output $F$ is such that the social demand price $\lambda_1$ of a unit of domestic investment is less than the social supply price $U'(c)$ of a unit of domestic investment for

5. The Hamiltonian $H$ measures the sum of national income in terms of consumption over the time interval $0$ to $T$. If future national income is discounted, the Hamiltonian is the current value of the future flow of national income in terms of utility. However, because future national income flows are not discounted in this model, the Hamiltonian has the same value at all points of time.
all possible values of $c$, the optimal policy is $A$ along which all of net output is consumed and the stock of domestically-owned capital depreciates as quickly as possible until the social supply and demand prices of a unit of domestic investment, measured in terms of utility, are equal. At this point the optimal policy switches from $A$ to $B$ which takes the economy to the endpoint. Along policy $B$, all of net output is allocated between current domestic investment and consumption, in such a way as to maintain social optimal utility along its path. The allocation to current domestic investment is less than the depreciation of the domestically-owned capital, so that the stock of this capital continues to fall until the endpoint is reached.

Along policy $B$, $\lambda_1 = U'(c)$
so that $\lambda_1 = U''(c)c$.

But $\lambda_1 = -U'(c)[f' - \omega]$.

Therefore $c = \frac{U'(c)}{U''(c)} [f' - \omega]$.

Then along policy $B$ consumption per head is rising when the total capital stock is below its optimal level and falling when the total capital exceeds its optimal level. (The optimal total capital stock is given by $f'(k) = \omega$.)

Comparison with Bardhan's Model
The above model differs from Bardhan's model (Bardhan [7]) in two important ways. Firstly, Bardhan assumes that the borrowing country derives disutility from its dependence on foreign borrowed capital. As stated in Chapter III, I have chosen not to include this concept in the models presented here. Secondly, Bardhan formulates his problem so that the endpoint is free. That is, the endpoint is determined from within the model. In the above model the endpoint
is fixed or specified in the formulation of the optimisation problem. The implications of these two differences will be seen in the following comparison.

In Bardhan's model the level of foreign borrowing is adjusted instantaneously so that

$$f'(k) - \phi(k_f) = \frac{D'(k_f)}{U'(c)} > 0.$$  

If, however, Bardhan had not included the disutility of foreign borrowed capital the level of foreign borrowing would have been adjusted instantaneously so that

$$f'(k) - \phi(k_f) = 0.$$  

This rule is identical to that derived in the above model for the adjustment of the level of foreign borrowing. Then, for all levels of domestically-owned capital per head, the level of foreign borrowing per head is greater in the above model than it is in Bardhan's model.

Bardhan's endpoint satisfies the Golden Rule and is therefore equivalent to the above fixed endpoint, after allowance has been made for Bardhan's assumptions regarding the disutility of foreign borrowed capital and capital depreciation. Also it can be shown, after the same allowances have been made, that the capital accumulation paths of Bardhan's model are similar to those of the above model. Because capital is assumed to be perfectly mobile internationally the level of foreign borrowing adjusts instantaneously

$$D'(k_f)$$

so that

$$f'(k) - \phi(k_f) = \frac{D'(k_f)}{U'(c)}.$$  

Therefore the optimal capital accumulation paths\(^6\) are along this curve to the endpoint. It is

---

6. Bardhan \[7\], page 123.
then obvious that Bardhan's capital accumulation paths along which part of national income is saved are equivalent to the paths of policy B above and that the part of the capital accumulation path along which all of national income is consumed corresponds to the path of policy A above.

From this comparison, it appears that Bardhan's capital accumulation paths are similar to those of the above model. However, the important difference lies in the fact that the capital accumulation paths of the above model are optimal paths whereas those of the Bardhan model may or may not be optimal. 7

II

In section I capital is assumed to be perfectly mobile between countries. The more realistic assumption of restricted mobility of foreign capital will now be introduced into the model of section I. It will also be assumed that the repayment of foreign loans is not permitted. Then using the variables as defined in the preceding chapter, the objective is now to minimise

\[ \int_0^T [U(c) - U(c)] dt , \]

where \( T \) is unspecified, along the path from any initial point to the fixed endpoint, subject to

\[
\begin{align*}
 f(k_d + k_f) - r(k_f)k_f - i_d - c &\geq 0 \\
 k_d &= i_d - \omega k_d \\
 k_f &= i_f - \omega k_f \\
 c &\geq 0
\end{align*}
\]

7. This point has been noted in Chapter III page 62
\[ i_d \geq 0 \]
\[ 0 \leq i_f \leq V. \]

It is also assumed that \( U'(c) > 0 \), \( U''(c) < 0 \) and \( U'(0) = \infty \).

The fixed endpoint is such that sustainable consumption per head is maximised. As derived in the above chapter the endpoint \((k_d^*, k_f^*)\) is defined by \( f' = \phi = \omega \).

This optimisation problem is solved by applying Arrow's Method of Solution.\(^8\) The detailed working of this solution is given in Appendix No. 3. As in the previous model the budget constraint is satisfied with equality. Therefore all of net output is allocated to either current domestic investment or to consumption. However, because \( U'(c) = \infty \) when consumption is zero, policies along which all of net output is allocated to current domestic investment are ruled out as they can never be optimal. Then the possible policy combinations satisfying the budget constraint and having the allocation to consumption at some positive level are:

<table>
<thead>
<tr>
<th>Policy</th>
<th>( i_d )</th>
<th>( c )</th>
<th>( i_f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>F</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>F</td>
<td>&gt;0</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>F</td>
<td>V</td>
</tr>
<tr>
<td>D</td>
<td>( 0 &lt; i_d &lt; F )</td>
<td>( 0 &lt; c &lt; F )</td>
<td>0</td>
</tr>
<tr>
<td>E</td>
<td>( 0 &lt; i_d &lt; F )</td>
<td>( 0 &lt; c &lt; F )</td>
<td>&gt;0</td>
</tr>
<tr>
<td>F</td>
<td>( 0 &lt; i_d &lt; F )</td>
<td>( 0 &lt; c &lt; F )</td>
<td>V</td>
</tr>
<tr>
<td>G</td>
<td>( F - c )</td>
<td>( \hat{c} )</td>
<td>0</td>
</tr>
<tr>
<td>H</td>
<td>( F - c )</td>
<td>( \hat{c} )</td>
<td>&gt;0</td>
</tr>
<tr>
<td>I</td>
<td>( F - c )</td>
<td>( \hat{c} )</td>
<td>V</td>
</tr>
</tbody>
</table>

where \( F = f(k_d + k_f) - r(k_f)k_f \).

\(^8\) Arrow [3]
The properties of these policies and the possible policy switches are discussed in Appendix No. 3. However, the following table gives the possible policy switches and the conditions necessary for the switch to take place. Condition (i) must be satisfied immediately before the switch and condition (ii) must exist at the "switching surface". A zero in the table indicates that no switch is possible. It should be noted that policies G and I which are non-optimal and H which is consistent only at the endpoint, are excluded from the table. Also policy E cannot switch into any other policy. Therefore it has been excluded from the relevant part of the table.

Table of Switches

<table>
<thead>
<tr>
<th>Switches out of</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>-</td>
<td>(i) ( f' &lt; \phi )</td>
<td>(ii) ( \lambda_1 \leq U'(F) )</td>
<td>0</td>
<td>(ii) ( \lambda_1 \leq U'(F) )</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>-</td>
<td>0</td>
<td></td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>(i) ( f' &gt; \phi )</td>
<td>(ii) ( \lambda_1 \leq U'(F) )</td>
<td>-</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td>(i) ( f' &lt; \phi )</td>
</tr>
<tr>
<td>F</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td>(i) ( f' &gt; \phi )</td>
</tr>
</tbody>
</table>

The "switching surface" \( \lambda_1 = U'(F) \) has been discussed in section I. Its shape is indeterminate but it has been shown (in section I) that this "switching surface" lies completely within the region \( f' < \omega \).
I. Region $f' < \phi$

In this region $f' < \phi$, the level of foreign borrowing exceeds its optimal level. If, at the same time, the social supply price $U'(c)$ of a unit of domestic investment, with all of net output $F$ being consumed, exceeds the social demand price $\lambda_1$ of a unit of domestic investment, the optimal policy is $A$. Along $A$, all of net output is allocated to consumption and both domestically-owned capital and foreign borrowed capital are allowed to depreciate at the fastest possible rate until either the social demand and supply prices of a unit of domestic investment become equal ($\lambda_1 = U'(F)$) or the level of foreign borrowing is at its optimal level corresponding to the stock of domestically-owned capital ($f' = \phi$).

(i) If $f'$ becomes equal to $\phi$, $A$ switches into policy $B$. Along
B all of net output is consumed and the stock of domestically-owned capital continues to depreciate at the maximum possible rate whilst the level of current foreign borrowing is such as to maintain the equality between \( f' \) and \( \phi' \). Because the social demand price \( \lambda_1 \) is less than the social supply price \( U'(c) \) of a unit of domestic investment, the stock of domestically-owned capital is reduced at the maximum possible rate until these two prices become equal \( (\lambda_1 = U'(F)) \). Then, at \( \lambda_1 = U'(F) \), policy B switches into policy E. Along policy E part of net output is allocated to consumption and part to current domestic investment so as to maintain the social optimal utility \( (\lambda_1 = U'(c)) \). The level of current foreign borrowing is such as to maintain the equality between \( f' \) and \( \phi' \). In this region the stock of domestically-owned capital is greater than its optimal level. Therefore along policy E, the allocation to current domestic investment is not great enough to cover depreciation so that the stock of domestically-owned capital decreases. Policy E takes the economy to the endpoint \( (k_d, k_f) \).

(ii) If along policy A the social demand and supply price of a unit of domestic investment become equal, the optimal policy switches from A to policy D at the "switching surface" \( \lambda_1 = U'(F) \). Along policy D all of net output is distributed between consumption and current domestic investment so as to maintain the level of utility at its social optimum. Because \( f' \) is less than \( \phi' \), the stock of foreign borrowed capital is allowed to depreciate at its maximum possible rate. Policy D remains the optimal policy until the level of foreign borrowing falls to its optimal level corresponding to the stock of domestically-owned capital. Then at \( f' = \phi' \) the optimal policy switches from D to E which takes the economy to the endpoint. Along E the allocation of net output between consumption and current domestic investment is such as to maintain social optimal utility
\( (\lambda_1 = U'(c)) \). But along policy E the level of current foreign borrowing is such as to maintain the equality between \( f' \) and \( \phi \).

II. Region \( f' > \phi \)

When the stock of domestically-owned capital is so large that the social demand price is less than the social supply price of a unit of domestic investment (for all possible levels of consumption) the optimal policy is \( C \), along which all of net output is allocated to consumption and the stock of domestically-owned capital is allowed to depreciate at the maximum possible rate. Since \( f' \) is greater than \( \phi \) the level of foreign borrowing is below its optimal level corresponding to the stock of domestically-owned capital. Therefore along policy \( C \) the stock of foreign borrowed capital is increased at the fastest possible rate. Policy \( C \) is the optimal policy until either \( f' = \phi \) or the social demand and supply prices of a unit of domestic investment become equal.

(i) If the economy reaches the point where \( f' = \phi \) the optimal policy switches from \( C \) to \( B \) which takes the economy along \( f' = \phi \) to the "switching surface" \( \lambda_1 = U'(F) \) where the optimal policy switches to \( E \). This movement along \( f' = \phi \) to the endpoint, using policies \( B \) and \( E \), has been discussed in I (i) above.

(ii) If policy \( C \) takes the economy to the "switching surface" \( \lambda_1 = U'(F) \), the optimal policy switches at this surface from \( C \) to \( F \). Along policy \( F \) part of net output is allocated to consumption and part to current domestic investment so as to maintain the equality between the social demand and supply prices of a unit of domestic investment. Since \( f' > \phi \), the stock of foreign borrowed capital is below its optimal level corresponding to the level of domestically-owned capital. Therefore along policy \( F \) the stock of foreign borrowed capital is increased at the fastest possible
rate until $f'$ and $\phi$ become equal. At $f' = \phi$, the optimal policy switches from F to E which takes the economy to the endpoint. The switch to policy E has been discussed in I (ii) above.

Conclusions

The introduction of diminishing marginal utility into the model with restricted mobility of capital internationally, has produced some marked changes in the pattern of the optimal growth paths. Firstly, it is no longer optimal for the allocation of net output to consumption to be zero and secondly, the relationship between the total capital stock and the optimal total capital stock is no longer of any importance in determining the optimal growth paths.

It is assumed in this model that $U'(0) = \infty$. Therefore the zero level of consumption can never be optimal and thus along all the optimal growth paths there is greater emphasis on the level of consumption. The reason for this is that now the utility derived from a small increase in the consumption level when consumption is relatively low, is higher. Therefore it pays the country to consume more in the earlier stages of, and also throughout, the growth process, whereas in the linear model a unit increase in the consumption level has the same value at all levels of consumption or at any stage of the growth process. Then it pays to move the economy as quickly as possible to the endpoint where consumption is held at the maximum sustainable level.

Along the optimal growth paths where not all of net output is allocated to consumption, consumption per head adjusts over time according to

$$c = - \frac{U'(c)}{U''(c)} [f' - \omega].$$
That is, when \( f' < \omega \) the allocation to consumption is falling and when \( f' > \omega \) consumption per head is rising. In the linear model, all of net output is allocated to consumption when \( f' < \omega \) but when \( f' > \omega \) consumption is held at the zero level until the endpoint is reached.

Consider the second difference in the pattern of the optimal growth paths. In the linear model the "switching surfaces" are \( f' = \omega \) and \( f' = \phi \). The optimal growth path is the one which moves the economy as fast as possible to either one of these curves and then switches to a policy which takes the economy along the curve to the endpoint. That is, the system is moved as quickly as possible to a position where the marginal product of capital equals the marginal cost of borrowing, or where the total capital stock is at the optimal level. In this model the switching surfaces are \( \lambda_1 = U'(F) \) and \( f' = \phi \). \( \lambda_1 = U'(F) \) is the lower bound to the surface \( \lambda = U'(c) \) over which social utility is optimised. If initially the social demand is less than the supply price of a unit of domestic investment the optimal policy is such that all of net output is allocated to consumption so that the economy moves as quickly as possible to the switching surface where the social demand and supply prices of a unit of investment become equal. There the optimal policy switches to a policy which, while maintaining the equality between social demand and social supply price of a unit of domestic investment, moves the economy as quickly as possible to \( f' = \phi \). Then at \( f' = \phi \) the optimal policy switches to one which while maintaining social optimal utility, moves the system along \( f' = \phi \) to the endpoint.

The relationship between \( f' \) and \( \omega \) no longer plays an important role in the pattern of the optimal growth paths. This results from
the emphasis placed on social optimal utility of consumption in this model. In the linear model the allocation to consumption along \( f' = \omega \) is the residual of the net output after the allocation to current domestic investment has been made. The amount going to current domestic investment is such as to move the economy along \( f' = \omega \).

In this model, \( \lambda' = 0 \) along the curve \( f' = \omega \) and thus the allocation to consumption must be held constant at the maximum sustainable level. But there is no policy along which \( c = \hat{c} \) and the balance of net output which is allocated to current domestic investment maintains social optimal utility and is just sufficient to move the economy along \( f' = \omega \). Thus the optimal policy is always one of maintaining social optimal utility while moving the economy as quickly as possible to the \( f' = \phi \) curve and, after a policy switch, along the \( f' = \phi \) curve to the endpoint.
Chapter VII

OPTIMAL FOREIGN BORROWING

BY A LARGE COUNTRY

Hamada [10] has considered the case of a small country borrowing from an international capital market. In this case the rate of interest payable on an international loan is constant, independent of the size of the loan. On the other hand, Bardhan [7] has considered the case of a large country which also borrows from an international capital market. The rate of interest payable on foreign loans is no longer constant but Bardhan assumes that it is a function of the level of foreign borrowing per head of the population in the borrowing country. However, it seems that a more realistic assumption would be that the rate of interest is a function of the absolute level of foreign borrowing. The interest rate at which a market would lend capital would seem to be independent of the population of the borrowing country. 9 The inclusion of this assumption in place of Bardhan's leads to very different conclusions.

Initially, the problem as posed by Bardhan will be re-worked. It will be assumed in this model that the objective function is linear. Then, in the second section, the assumption of restricted mobility of foreign capital will be introduced into the model. In these two

9. The rate of interest has been discussed in Chapter IV, page 74
sections an examination of the assumption that the interest rate payable on foreign borrowed capital is a function of the absolute level of foreign borrowing will be made.

As discussed in Chapter IV there is an upper limit to the amount of current foreign borrowing and this is determined by the current supply of capital \( V \) in the international capital market. It is reasonable to assume that the current supply of such capital is an increasing function of the rate of interest payable on capital in the international capital market. The introduction of this assumption into the model in section II will allow the effects of such a limit to the current supply of foreign capital to be examined.

In section III diminishing marginal utility will be introduced into the model. This section gives no new results and is included only for completeness.

I

The basic assumptions and variables of the Bardhan problem have been discussed and defined in Chapter V. All these assumptions and variables remain the same here except for those relating to the nature of the rate of interest payable on the capital borrowed from the international capital market. The rate of interest is now assumed to be a function of the absolute level of foreign borrowing. That is \( r = r(K_f) > 0 \).

Or in per capita terms \( r = r(L_kf) > 0 \) with \( r'(L_kf) > 0 \).

It is assumed that there exists an upper limit \( \gamma \) to the amount of foreign borrowing which is defined by \( r(\gamma) = 0 \).

The marginal cost of borrowing is now

\[
\Phi(L_kf) = r(L_kf) + r'(L_kf)L_kf,
\]
where \( \Phi'(L_k) \) is positive. It is worth repeating here that the population \( L \) grows at a constant rate \( \mu \). That is, \( \frac{dL}{dt} = \mu \).

Then the optimisation problem is to minimise

\[
\int_0^\infty (\hat{c} - c) \, dt
\]

along the path from any initial point to the fixed endpoint\(^1\) subject to

\[
f(k_d + k_f) - r(L_k)k_f - l_d - c \geq 0,
\]

\[
k_d = l_d - \omega k_d,
\]

\[
c \geq 0 \quad \text{and} \quad l_d \geq 0.
\]

That is, the home country’s aim is to minimise the difference between the actual level and the maximum sustainable level of consumption along the accumulation path from the initial position to the fixed endpoint, by choosing whatever amounts of current domestic investment per head, current consumption per head and the level of foreign borrowing, which satisfy the budget constraint.

As in the previous models the fixed endpoint is such that sustainable consumption per head is maximised.

Net output per head is

\[
y = f(k_d + k_f) - r(L_k)k_f - \omega k_d.
\]

Then maximising \( y \) with respect to \( k_d, k_f \)

\footnote{1. It can be shown by integration by parts that a sufficient condition for convergence of this integral is that \( \lim_{t \to \infty} \frac{\partial Lc}{\partial t} = 0 \) and that \( \frac{d\left[\hat{c} - c\right]}{dt} \) is bounded on \((0, \infty)\). These conditions do hold along the optimal growth paths and thus the integral converges.}
Then maximum sustainable consumption per head is achieved when

$$f'(k) = \Phi(Lk_f) = \omega.$$ 

That is, the fixed endpoint is defined by $k = \hat{k}$ and $K_f = \hat{K_f}$. 

In the above models the endpoint is defined in terms of $k_d$ and $k_f$. By describing the endpoint in terms of $k_d$ and $k_f$, it can be compared directly with that of the previous models.

By definition $Lk_f = K_f$. Therefore since $L = L_0 e^{ult}$ where $L_0$ is the population at the initial time $t_0$, $k_f = k_f L_0 e^{-ult}$. The optimal level of total borrowing $\dot{k}_f$ is constant. Therefore the level of per capita foreign borrowing decreases at the rate $\omega$ over time and thus approaches zero as time goes to infinity.

Also at the endpoint, $k = \hat{k}$, so that $\dot{k}_d = -k_f$. That is, $k_d$ grows at the rate $\omega$ so as to maintain the constant total capital-labour ratio $k$. That is, as time approaches infinity, $k_d \to \hat{k}$. Then the fixed endpoint as time goes to infinity is $(k_d, 0)$, where $\dot{k}_d = \hat{k}$.

This fixed endpoint can also be derived in another way by considering how the curve $f'(k) = \Phi(Lk_f)$ moves over time. Differentiating with respect to time

$$\frac{d[f'(k) - \Phi(Lk_f)]}{dt} = f''(k_d + k_f) - \Phi'[k_f \dot{L} + k_L k_f].$$

But $f'(k) = \Phi(Lk_f)$ and therefore

$$k_f \left[ f'(k) = \Phi(Lk_f) \right] = \frac{\Phi' k_f \dot{L} - k_f \ddot{L}}{\ddot{f} - \Phi' L}.$$
Therefore assuming \(k_d\) to be fixed over time, \(k_f\) is negative. Also for a fixed \(k_f\), \(k_d\) is negative. Therefore the curve \(f'(k) = \phi(Lk_f)\) shifts downwards to the left. From the slope \(\frac{dk_f}{dk_d} = \frac{-f''}{f'' - L{k_f}'}\) of the curve, it can be seen that the curve becomes flatter as it moves downwards. The following diagram illustrates this point.

This diagram is dependent on the initial assumption that \(\omega > r(0) > \delta\). This has been discussed in Chapter V above.

The optimisation problem is solved again by applying Arrow's Method of Solution.2 The detailed working of this solution is

---

2. Arrow [3], lecture 2, page 12.
presented in Appendix No. 4. As in the previous models the budget constraint is satisfied with equality. Therefore all of net output is allocated either to current domestic investment or to consumption.

The possible policy combinations satisfying the given constraints are:

<table>
<thead>
<tr>
<th>Policy</th>
<th>i_d</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>F</td>
</tr>
<tr>
<td>B</td>
<td>F</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>0&lt;i_d&lt;F</td>
<td>0&lt;c&lt;F</td>
</tr>
</tbody>
</table>

where \( F = f(k) - r(Lk_f)k_f \).

A summary of the properties of these policies which are derived in Appendix No. 4 is:

<table>
<thead>
<tr>
<th>Policy</th>
<th>( \lambda_1 )</th>
<th>Region in which policy is optimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>( \leq 1 )</td>
<td>( f' = \phi ), ( f' \leq \omega )</td>
</tr>
<tr>
<td>B</td>
<td>( \geq 1 )</td>
<td>( f' = \phi ), ( f' \geq \omega )</td>
</tr>
<tr>
<td>C</td>
<td>( = 1 )</td>
<td>( f' = \phi ), ( f' = \omega )</td>
</tr>
</tbody>
</table>
When the total capital-labour ratio in the country is less than the optimal level, the problem is one of optimal capital accumulation. Because $k_f$ is a control variable the level of foreign borrowing can be adjusted instantaneously so that $f'(k) = \phi(Lk_f)$ and then subsequently adjusted over time to maintain this equality. Thus the optimal policy is B along which all of net output is allocated to current domestic investment and nothing to consumption. That is, the economy moves such that $f'(k) = \phi(Lk_f)$ until the total capital-labour ratio reaches the optimal level where the optimal policy switches from B to C. Along policy C the absolute level of foreign borrowing $(Lk_f)$ is constant because $k_f$ is adjusted so as to maintain the equality $f'(k) = \phi(Lk_f)$. But because labour is growing at the rate $\mu$, $k_f$ is decreasing at the same rate so as
to maintain this constant $Lk_f$. Therefore along policy C current domestic investment is adjusted so as to maintain the fixed optimal level of the total capital-labour ratio $\hat{k}$. From the above discussion of the fixed endpoint $\dot{k}_d = -k_f$. But $\dot{k}_f = -\omega k_f$ and therefore $\dot{k}_d$ must equal $-\omega k_f$. By definition $\dot{k}_d = i_d - \omega k_d$. Therefore the allocation from net output to current domestic investment is $\omega k_d - \omega k_f$.

If initially the total capital-labour ratio in the country is greater than the optimal level, the optimal capital decumulation path is along policy A where the level of foreign borrowing is controlled so that $f'(k) = \Phi(Lk_f)$ until the total capital-labour ratio is reduced to the optimum $\hat{k}$. Then the optimal policy switches from A to C which, as above, takes the economy to the fixed endpoint $(\hat{k}_d, 0)$.

Conclusions
The effect of the introduction of the assumption $r = r(k_f)$ into the model can be determined by comparing these results with those of the model where $r = r(k_f)$ in Chapter V section I.

In the model where $r = r(k_f)$ the fixed endpoint is $(\hat{k}_d, \hat{k}_f)$ with, in the case $\omega > r(0) > \delta$, both $\hat{k}_d$ and $\hat{k}_f$ positive quantities. Now with $r = r(k_f)$ the fixed endpoint is defined by $\hat{k}$ and $\hat{k}_f$. That is, it is defined by the optimal total capital-labour ratio and the optimal level of absolute foreign borrowing. In both models, because capital is assumed to be perfectly mobile internationally, the level of foreign borrowing adjusts instantaneously so as to equate the marginal product and the marginal cost of borrowing. Then the optimal policy is to move the economy as quickly as possible, while maintaining this equality, to the optimal total capital-labour ratio. In the earlier model with $r = r(k_f)$, this endpoint
is reached in finite time. However in this model with \( r = r(K_f) \),
the optimal total capital-labour ratio may be reached within finite
time but the composition \( (k_d + k_f) \) of the optimal total capital-
labour ratio changes over time because \( k_f = k_f e^{-ut} \). The fixed
endpoint is approached as time goes to infinity when \( k_d \) goes to
\( \dot{k} \) and \( k_f \) goes to zero.

II

The assumption of complete mobility of international capital has
already been discussed in Chapter IV as being very unrealistic.
The assumption of restricted mobility of international capital will
be introduced again into the model as it was in Chapter V. As
before, the level of current foreign borrowing will be introduced
as the control variable thus eliminating Bardhan's assumption of
reversibility of foreign borrowed capital.

Bardhan [7] also considers the supply of foreign capital as unlimited.
However he prevents the level of foreign borrowing per head from
going to infinity by assuming firstly that social disutility is
derived from the presence of foreign-owned capital in the home
country and secondly that at some large value \( \gamma \) of foreign
borrowing per head, the interest rate goes to infinity. But in
the real world the level of current foreign borrowing is limited by
the current supply of capital in the international capital market.
This limit to the current supply of capital in the international
capital market has been discussed in Chapter IV. The limit is an
increasing function of the rate of interest payable on loans from
the market and is thus assumed to be represented by \( V(r) \), where
\( V'(r) \) is positive. Then in this model, the assumption of restricted
mobility of international capital is \( 0 \leq i_f \leq V(r) \). However
there is a natural limit to \( V(r) \) which is determined by the capacity
of lending country's capital-goods industries.

Then the home country's objective is to minimise

$$\int_0^T (c - c) \, dt,$$

along the path from any initial point to the fixed endpoint subject to

$$f(k_d + k_f) - r(Lk_f)k_f - c - l_d \geq 0,$$

$$k_d = i_d - \omega k_d,$$

$$k_f = i_f - \omega k_f,$$

$$c \geq 0,$$

$$l_d \geq 0,$$

$$0 \leq i_f \leq V(r),$$

$$L = L_0 e^{ut}.$$

In the optimisation problem which must now be solved, the control variables $c, i_d, i_f$ can be chosen, subject to the budget constraint, to determine the movement of the state variables $k_d$ and $k_f$. The fixed endpoint is again $(k_d, 0)$. Using Arrow's Method of Solution 3 the optimisation problem has been solved and the solution presented in Appendix No. 4. The following table gives all the possible policies found in solving the optimisation problem. From the characteristics of these policies the phase diagram is constructed.

---

3. Arrow [3]
The Phase Diagram

\[
\text{where } F = f(k) - r(Lk_f)k_f
\]

Policies E and B lie along the curve \( f'(k) = \Phi(Lk_f) \). However as this curve shifts downwards over time the paths of the policies E
and B are as shown above. At any point in time the pattern of optimal
growth paths is similar to that derived in Chapter V where restricted
mobility of foreign capital is introduced into the model with
\( r = r(k_f) \). However as the curve \( f'(k) = \phi(Lk_f) \) shifts down to the
left over time the optimal growth paths are considerably different.
Initially the optimal policy for any given level of \( k_d \) and \( k_f \) is
the same as in the earlier model of Chapter V with \( r = r(k_f) \). Then,
with the appropriate policy switches at \( f'(k) = \omega \) and \( f'(k) = \phi(Lk_f) \),
the economy approaches the fixed endpoint \((k_d,0)\). If at some finite
time the total capital-labour ratio and the absolute level of foreign
borrowing become equal to \( \hat{k} \) and \( \hat{k}_f \) respectively, the optimal
policy switches to policy \( H \) which moves the economy along \( f'(k) = \omega \)
so that the fixed endpoint is approached as time goes to infinity.

It should be noted that if initially \( k_d \) and \( k_f \) are such as to
lie on the curve \( f'(k) = \omega \) below the intersection of \( f'(k) = \omega \)
and \( f'(k) = \phi(Lk_f) \), then \( L_0k_f \) is less than the optimal level
of total foreign borrowing \( \hat{k}_f \). The optimal policy is I which
moves the economy along \( f'(k) = \omega \) until \( L_t k_f = \hat{k}_f \) (assuming
that \( \hat{k}_f \) is reached at time \( T \)). On \( f'(k) = \omega \) the total capital-
labour ratio is at its optimal level. Then, as with all the other
optimal paths, once the economy has reached this position where
\( k = \hat{k} \) and \( Lk_f = \hat{k}_f \), the optimal policy switches to \( H \) which takes
the economy along \( f'(k) = \omega \) to approach the fixed endpoint
\((k_d,0)\) as time goes in infinity. Along policy \( H \), \( i_f \) is such
as to maintain \( Lk_f = \hat{k}_f \) and the allocation of net output
to current domestic investment is such as to maintain \( k = \hat{k} \),

4. See page 91
whilst the balance goes to consumption.

Bardhan [7] shows that if borrowing from abroad is left to private borrowers, it tends to be excessive. This results from the fact that private borrowers do not take into account the effect of their borrowing on the rate of interest payable on foreign loans. That is, they borrow up to the point where

\[ f'(k) = \Phi(Lk_f) = r(Lk_f) + r'(Lk_f)k_f. \]

Therefore it is optimal to tax foreign borrowing so that along \( f'(k) = \Phi(Lk_f) \)

\[ (1 - t)f'(k) = r(Lk_f), \]

where \( t \) is the tax rate.

Since taxation enters into the budget constraint only as a transfer payment, the optimal tax rate is given by

\[ \frac{r'(Lk_f)k_f}{\Phi(Lk_f)}. \]

Now that the supply of capital in the international capital market is considered to be a function of the rate of interest payable on foreign loans and not unlimited as in the Bardhan case, there may exist values of \( r(K_f) \) for which it is optimal to subsidize foreign borrowing. To consider this possible implication of assuming \( V \) to be a function of \( r \), the model will be simplified by making the Bardhan assumption that \( r = r(k_f) \). If \( s \) is the rate of subsidisation, the optimisation problem is now to minimise

---

5. Capitalists borrow at the rate of interest \( r \) and earn the rate of return \( f' \). To restrict the level of foreign borrowing, the government taxes the capitalists earnings. These tax payments are merely transfer payments between the capitalists and the government and do not affect the rate of interest payable on foreign loans.
where $T$ is unspecified, along the accumulation path from any initial point to the fixed endpoint subject to

\[
\begin{align*}
    f(k_d + k_f) - (1 + s)r(k_f)k_f - c - i_d & \geq 0 \\
    k_d = i_d - \omega k_d \\
    \dot{k}_f = i_f - \omega k_f \\
    i_d & \geq 0 \\
    c & \geq 0 \\
    0 & \leq i_f \leq V(1 + s) r(k_f) \\
    s & \geq 0
\end{align*}
\]

The fixed endpoint is defined, as above, to be such that sustainable consumption per head is maximised. If $y$ is net output per head then

\[
y = f(k_d + k_f) - (1 + s)r(k_f)k_f - \omega k_d .
\]

Maximising $y$ with respect to $k_d$, $k_f$, $s$

\[
\begin{align*}
    \frac{\partial y}{\partial k_d} = f' - \omega = 0 \\
    \frac{\partial y}{\partial k_f} = f' - (1 + s)\phi = 0 \\
    \frac{\partial y}{\partial s} = -r k_f & \leq 0
\end{align*}
\]

That is, at the endpoint the subsidy $s$ is zero (its lower bound) and then the endpoint is the same as for the Bardhan model. That is, the endpoint is defined by $f'(k) = \omega$ and $f'(k) = \phi(k_f)$.

In the optimisation problem to be solved, the control variables $c$, $i_d$, $i_f$, $s$ can be chosen, subject to the budget constraint, to determine the movement of the state variables $k_d$ and $k_f$.

Using Arrow's Method of Solution, the Hamiltonian is

---

6. Arrow [3]
\[ H = -(\dot{c} - c) + \lambda_1 [i_d - \omega k_d] + \lambda_2 [i_f - \omega k_f] \]

and the Lagrangian

\[
L = H + p \dot{i}_d + q i_f + v[V(1 + s r(k_f)) - i_f] + \rho (f(k_d + k_f) - (1 + s)r(k_f)k_f - c - i_d] + ws + tc ,
\]

where \( \lambda_1 \) and \( \lambda_2 \) are the auxiliary variables and \( p, q, \rho, v, w, t \) are the Lagrange multipliers.

Necessary conditions for an optimal solution are

(a) \( \lambda_1 \) and \( \lambda_2 \) are continuous functions of time given by

\[
\begin{align*}
\dot{\lambda}_1 &= -\frac{\partial L}{\partial i_d} = -(pf' - \lambda_1 \omega) \\
\dot{\lambda}_2 &= -\frac{\partial L}{\partial i_f} = -[\rho(f' - \frac{1}{1 + s} \phi) - \lambda_2 \omega + v(1 + s) V' \phi] .
\end{align*}
\]

(b) The constraint satisfies the Constraint Qualification\(^7\) at any point of time and the Hamiltonian is maximised subject to this constraint. That is,

\[
\begin{align*}
\frac{\partial L}{\partial c} &= 1 + t - \rho = 0 \\
\frac{\partial L}{\partial i_d} &= \lambda_1 - \rho + p = 0 \\
\frac{\partial L}{\partial i_f} &= \lambda_2 + q - v = 0 \\
\frac{\partial L}{\partial s} &= -\rho k_f + v V' r + w = 0 \\
p &\geq 0 ; \quad p i_d = 0 \\
q &\geq 0 ; \quad q i_f = 0 \\
v &\geq 0 ; \quad v[V - i_f] = 0 \\
w &\geq 0 ; \quad ws = 0 \\
t &\geq 0 ; \quad tc = 0 \\
\rho &\geq 0 ; \quad \rho [f(k) - (1 + s)r(k_f)k_f - c - i_d] = 0 .
\end{align*}
\]

---

\(^7\) Arrow, Hurwicz, Uzawa \([4]\)
These necessary conditions are also sufficient for optimality since the Hamiltonian is concave in the state variables, given the auxiliary variables and time. This sufficiency condition is given by Arrow [3], pages 10 and 12.

The pattern of optimal growth paths is obviously the same as that derived in the similar model in Chapter V where $V$ is a given constant. Therefore it is only necessary to consider the control variable $s$ along these paths.

From $\frac{3L}{3S} = -\rho k_f + W' v + w$, the subsidy $s$ is zero if $\frac{3L}{3S} < 0$ and positive if $\frac{3L}{3S} = 0$. On paths along which $v = 0$,

$\frac{3L}{3S} = -\rho k_f + w$. Since $sw = 0$ and $k_f \neq 0$ along such paths $\frac{3L}{3S}$ is negative so that it is not optimal to subsidise foreign borrowing along these paths. From Appendix No. 1 the paths along which $v = 0$ are the optimal paths when the level of foreign borrowing is either greater than or equal to its optimal level.

For paths along which $v > 0$ the optimal subsidy is such that $W' = \frac{\rho k_f}{v}$. From the necessary conditions for optimality above it can be shown that if $v > 0$, $\lambda_2 = v$ and $\rho \geq 1$. Then the optimal subsidy along these paths is non-zero if $W'$ is greater than or equal to $\frac{k_f}{\lambda_2}$. Depending on the exact nature of the supply function it may be optimal to subsidise foreign borrowing along part of the optimal growth paths when foreign borrowing is below its optimal level.

In the static analyses discussed in Chapter I subsidisation of

---

8. See page 91
international capital movements appears only in relation to the terms of trade effect. It now appears that it may be optimal to subsidize foreign borrowing even in the absence of such an effect. If this investigation of the supply function had been carried out in the non-autonomous model, where \( r \) is assumed to be a function of \( Lk_f \), the results regarding the optimal subsidy would have been identical.

III

To complete this analysis of optimal foreign borrowing by a large country consider the inclusion of diminishing marginal utility in this model. The effects of diminishing marginal utility on the Bardhan model have been discussed in Chapter VI, section II. The effects on this model are similar and thus this section is included only for completeness. Therefore the model will not be worked in detail and only a discussion of the effects of diminishing marginal utility on the model will be given.

The introduction of diminishing marginal utility into the Bardhan model in Chapter VI produced a change in the "switching surfaces". The curve \( f'(k) = \omega \) is no longer of any importance in the determination of the pattern of optimal growth paths and \( \lambda_1 = U'(c) \) when \( c = F \) is the new "switching surface". However \( F \) is an explicit function of time. Therefore the "switching surface" \( \lambda_1 = U'(F) \), as well as \( f'(k) = \Phi(Lk_f) \), will move over time.

At the "switching surface" \( \lambda_1 = U'(F) \), \( \lambda_1 \) is positive. Then
\[
\frac{d}{dt} [\lambda_1 - U'(F)] = \lambda_1 - U''(f' k_d + \dot{k}_d) - \dot{k}_f r - k_f r' (L_{k_f} + \dot{k}_L).
\]
Therefore along \( \lambda_1 = U'(F) \)
\[
\dot{\lambda}_1 = U''(f' k_d + \dot{k}_d) - \dot{k}_f r - k_f r' (L_{k_f} + \dot{k}_L).
\]
If \( \dot{k}_f = 0 \), \( \dot{k}_d \) is positive. Therefore over time the "switching surface" moves out to the right.
Then the pattern of optimal growth paths at any point of time is similar to that of the Bardhan model with restricted mobility of capital (Chapter VI). However, over time, both the "switching surfaces" move. Thus this pattern of optimal growth paths is modified in the same way as was that of the model with the linear objective function, in section II of this chapter.

Conclusions

The aim of this chapter is to investigate the effect of making the rate of interest payable on foreign loans a function of the absolute level of foreign borrowing and to consider the implications of assuming that the limit to current foreign borrowing is the supply of capital in the international capital market which is itself assumed to be a function of the rate of interest.

As shown in section I the assumption that the rate of interest payable on a foreign loan is a function of the absolute level of foreign borrowing does lead to very different results from those derived under the assumption that the rate of interest is a function of the level of foreign borrowing per head of population in the borrowing country. Perhaps the most important aspect of these different results is that the optimal endpoint is now defined in terms of the total capital-labour ratio and the absolute level of foreign borrowing. Therefore, as time goes to infinity the optimal level of foreign borrowing per head goes to zero. That is, in the long-run the optimal foreign-owned capital-labour ratio is zero. Thus, this differs greatly from the Bardhan results in which the optimal foreign-owned capital-labour ratio remains positive over time.

The assumption that the limit to current foreign borrowing is a function of the supply of capital in the international capital
market which is itself a function of the rate of interest, leads to very interesting results. It now appears that it is optimal to subsidise foreign borrowing for some levels of foreign borrowing depending on the exact nature of the supply function of capital in the international capital market. The subsidisation of international capital movements has only appeared before in relation to the terms of trade effect. Therefore these results that it may be optimal to subsidise foreign borrowing even in the absence of the terms of trade effect are new and very interesting.

This chapter concludes the analysis of the topic of optimal foreign borrowing. The following chapters will be concerned with the optimal level of direct foreign investment.
Chapter VIII

OPTIMAL FOREIGN INVESTMENT IN THE
ONE COMMODITY WORLD

In this chapter the question of optimal international capital movements when the borrowed capital takes the form of direct investment is considered. It is assumed here, as in the above chapters, that only one good is produced in the world economy. The case involving the production of more than one good will be discussed in the following chapter, where both international trade and optimal international capital movements will be examined.

Direct foreign investment differs from foreign borrowing in many ways. The gross rate of return on direct foreign investment is not determined by the amount of foreign investment as in the case of foreign borrowing but it is equal to the gross rate of return on domestically-owned capital invested in the borrowing country. That is, the gross rate of return to foreign-owned capital invested in the borrowing country is equal to the marginal product of the total capital employed in the borrowing country.

In the static analyses of direct foreign investment by MacDougall [22] and Kemp [16, 19, 20] it is assumed that capital is perfectly mobile between countries so that the upper limit to the supply of foreign-owned capital is the foreign country's total capital stock. However, as discussed in Chapter IV, capital is not completely mobile between countries. Capital is irreversible and once invested cannot
be moved internationally. Therefore any additions to foreign investment can only be made out of newly-created capital goods and any reductions in foreign investment can only be made by either allowing the invested capital to depreciate or by selling the equipment to the borrowing country. Therefore the concept of irreversibility of capital internationally cannot be introduced into the analysis unless capital growth is assumed to exist at least in the lending country. If the borrowing country is to be permitted to buy out foreign invested capital then the borrowing country's capital stock must also be growing.

The rate of growth of any economy depends on the rates of growth of its capital stock and of its population. Labour is usually assumed to be immobile internationally. If the rate of population growth is assumed to be greater in one country and constant over time then, in the long-run, this country will dominate the other in the size of its labour force. If the rates of population growth are assumed to be constant and equal in both countries the introduction of population growth into the model will add little to the conclusions to be drawn from a model in which the population is assumed to be constant. Perhaps the only meaningful assumption that can be made about the factor of production, labour, is that it has restricted mobility and is controlled by the country's immigration policy. However, the standard assumption of international trade theory that labour is immobile internationally is maintained in the following analysis.

Capital, on the other hand, is assumed to have some degree of mobility. Then, if the capital stock owned by each country is growing, the question of one country becoming dominant does not arise. The total capital stock of one country may become greater
but capital is invested internationally so that the returns to capital are equated everywhere. Then the factor of production, capital, will be distributed between countries and not accumulated in the one country. That is, one country may dominate as far as the ownership of capital is concerned but it is unlikely that, in the long-run, the stock of capital employed in any one country will dominate that employed in any other.

In this chapter capital growth will be introduced into Kemp's static model of foreign investment (Kemp [16]) and then in section II the optimal growth model will be developed where capital is considered to be irreversible.

I

Kemp [16] assumes that the world consists of two countries which produce the same consumption good. That is, he assumes that the capital stock owned by each country is constant but, as a result of the assumption of complete mobility of capital internationally, this total world stock of capital is distributed between the two countries so that the return to capital in each country is the same. To allow for capital growth it is now assumed that the one commodity produced can be either consumed or invested. Initially it is assumed that the foreign country is in a static state but this assumption will be relaxed later so that its effect on the analysis can also be determined.

Assume that the foreign country is free to invest its capital stock in either country but that the home country is permitted to borrow foreign-owned capital (in the form of direct investment) but never to lend its domestically-owned capital stock. It is further assumed that all capital in both the home and the foreign country is fully-employed.
As in the previous models the total capital stock employed in the home country \((K)\) is equal to the sum of its domestically-owned capital \((K_d)\) and the foreign-owned capital \((K_f)\) which has been invested in the home country. The starred variables such as \(K^*_d\), \(K^*_d\) refer to quantities in the foreign country. Then \(K = K_d + K_f\) and \(K^* = K^*_d - K^*_f\).

The accumulation equation for domestically-owned capital is

\[
\dot{K}_d = I_d - \delta K_d,
\]

where \(I_d\) is the current domestic investment \((I_d \geq 0)\) and \(\delta\) the rate of capital depreciation in the home country. It has been assumed that the foreign country's capital stock is not growing. Therefore \(K^*\) is a fixed quantity over all time.

The gross rate of return on capital in the home country is \(F'(K)\).

Then the net rate of return on invested capital is defined by

\[
r = (1 - \tau)[F'(K) - \delta],
\]

where \(\tau\) is the rate of tax imposed by the home country on capital earnings and \(F(K)\) is the neo-classical production function such that \(F' > 0\), \(F'' < 0\). It has been assumed that the home country can never invest its domestically-owned capital abroad. Therefore the rate of return on capital abroad \(r^*_f\) must be less than or equal to the rate of return on capital in the home country \(r\) for the foreign country to be willing to invest in the home country.

It is assumed that the production function of the foreign country \(F^*(KK^*)\) is also neo-classical with \(\frac{dF^*}{dK^*} > 0\), \(\frac{d^2F^*}{dK^*_2} < 0\). The rate of return on capital invested in the foreign country is

\[
r^*_f(K^*) = \frac{dF^*}{dK^*}.\]

However from \(K^* = K^*_d - K^*_f\), since it is assumed that \(K^*_d\) is given, \(\Delta K^*_f = -\Delta K^*_f\). Therefore \(\frac{3r^*_f}{3K^*_f} = -\frac{d^2F^*}{dK^*_2}\) which
is positive. That is, given the total stock of capital owned by the foreign country, the rate of return on capital invested in the foreign country rises as the level of foreign investment in the home country rises.

If $C$ is current consumption then the home country's budget constraint is

$$F(k_d + k_r) - F'(k_d + k_r)K_r + \tau F'(k_d + k_r) - \delta K_r \geq I_d + C.$$ 

That is, the total domestic output minus earnings on foreign investment plus the tax revenue on these foreign earnings must be at least as great as the sum of current domestic investment and current consumption. It should be noticed that if all capital earnings are taxed then the taxation income derived from taxing the earnings on domestically-owned capital is merely a transfer payment. Also in the case of foreign borrowing from an international capital market, any restrictive policy such as taxing the earnings on the foreign borrowed capital does not enter into the analysis explicitly but is purely a transfer payment within the home country's economy. This has been discussed in Chapter VII (footnote 5) and is thus another difference between foreign borrowing and foreign direct investment.

Then the home country's aim is to minimise the difference between the maximum level of sustainable social welfare and its actual level along the capital accumulation path from any initial point to a fixed endpoint subject to its budget constraint and the boundary conditions. That is, the objective is to minimise

$$\int_0^T [U(C) - U(C)]dt ,$$

where $T$ is unspecified, along the accumulation path from any initial point to the endpoint subject to
\[ F(K_d + K_f) - F'(K_d + K_f)K_f + \tau(F'(K_d + K_f) - \delta)K_f - C = I_d \geq 0 \]

\[ \dot{K}_d = I_d - \delta K_d \]

\[ C > 0 \]

\[ I_d \geq 0 \]

\[ K_f \geq 0 \]

\[ r = (1 - \tau)(F' - \delta) \]

\[ r \geq r^* \]

where \( U'(C) > 0 \), \( U''(C) < 0 \) and \( U'(0) = \infty \).

The fixed endpoint \((\hat{K}_d, \hat{K}_f)\) is such that sustainable consumption is maximised. Net output is

\[ Y = F(K_d + K_f) - F'(K_d + K_f)K_f + \tau(F' - \delta)K_f - \delta K_d \]

so that

\[ \frac{\partial Y}{\partial K_d} = F' - \tau F'K_f + \tau F''K_f - \delta = 0 \]

and

\[ \frac{\partial Y}{\partial K_f} = -F'K_f + \tau[F' - \delta + F''K_f] = 0 \]

That is,

\[ \hat{K}_f = \frac{\tau(F' - \delta)}{(1 - \tau)F''} \]

Substituting into \( \frac{\partial Y}{\partial K_d} = 0 \) for \( \hat{K}_f \) gives

\[ (1 - \tau)(F' - \delta) = 0 \]

Since \( r \) goes to zero and thus \( K_f = 0 \) if \( \tau = 1 \), then sustainable consumption is maximised at \( F' = \delta \). That is, \( \hat{K}_d \) is defined by \( F'(K) = \delta \) and the endpoint is \((\hat{K}_d, 0)\).

In the optimisation problem to be solved, the control variables \( C \), \( I_d \), \( \tau \) and \( K_f \) can be chosen subject to the given constraint to determine the movement of the state variable \( K_d \). This problem is

---

9. Because it is assumed that \( U'(C) = \infty \) when consumption is zero, it can never be optimal to allocate nothing to consumption.
solved by Arrow's Method of Solution (Arrow [3]) which is based on Pontryagin's Maximum Principle (Pontryagin [30]).

The Hamiltonian is defined by

$$ H = -[U'(C) - U'(C)] + \lambda_1[I_d - \delta K_f], $$

where $\lambda_1$ is the social price of net domestic investment and the Lagrangian is

$$ L = H + \rho[F(K_d + K_f) - F'(K_d + K_f)K_f + \tau[F'(K) - \delta]K_f - C - I_d] $$
$$ + pI_d + nC + z[r - r^\delta]. $$

Necessary conditions for an optimal solution are:

(a) $\lambda_1$ is a continuous function of time given by

$$ \frac{\partial L}{\partial K_d} = -\rho(F' - F''K_f + \tau F''K_f) - \lambda_1 \delta. $$

(b) The constraint satisfies the Constraint Qualification at any point of time and the Hamiltonian is maximised subject to the constraint. That is

$$ \frac{\partial L}{\partial t} = U'(C) + n - \rho = 0 $$
$$ \frac{\partial L}{\partial t} = \lambda_1 + p - \rho = 0 $$
$$ \frac{\partial L}{\partial t} = \rho(F' - \delta)K_f + z(F' - \delta) = 0 $$
$$ \frac{\partial L}{\partial t} = \rho[-F''K_f + \tau(F' - \delta) + \tau F''K_f] + z[r' - r^\delta]. $$

The Lagrange Multipliers are such that

1. Arrow, Hurwicz, Uzawa [4].

2. $r' = \frac{\partial r}{\partial K_f}$, $r^\delta = \frac{\partial r^\delta}{\partial K_f}$. 
These necessary conditions are also sufficient for optimality since the Hamiltonian is concave in the state variable, given the auxiliary variables and time. This sufficiency condition is given by Arrow [3], page 10.

Since $\frac{\partial L}{\partial c} = 0$, $\rho$ is positive. Therefore the budget constraint is satisfied with equality. That is, all net output is allocated between domestic investment and consumption.

From $\frac{\partial L}{\partial t} = 0$, either $F' = 0$ or $z = \rho K_F$. If $F' = 0$, then $r$ is zero. But foreign investors would not lend their capital if the net return on the invested capital were zero. Therefore $z = \rho K_F$ and is positive for all $K_f > 0$. That is, the optimal taxation policy is such as to equate the returns to capital in the two countries. Then from $\frac{\partial L}{\partial K_f} = 0$, the optimal tax is

$$\tau_{opt} = \frac{K_f \rho' F'}{F' - \delta}.$$

Because $\rho' F'$ is positive and because foreign capitalists would only lend their capital if $F' > 0$, the optimal tax is positive for $K_f > 0$ and zero for $K_f = 0$.

But $r^\ast = r = [1 - \tau_{opt}][F' - \delta]$ or $F' - \delta = r^\ast + K_f \rho^\ast'$. That is, for any $K_d$, $K_f$ is controlled so that $F' - \delta = r^\ast + K_f \rho^\ast'$. 
Then the corresponding phase diagram is:

The feasible phase space lies below the \( F' = \delta \) curve where \( F' \) is greater than \( \delta \) and along the \( K_d \) axis. Foreign investors will lend their capital until the return on their foreign investment equals the return on capital invested in their own country. Therefore, at all points in the phase space \( r = r^* \).

Assume initially that the level of foreign direct investment and the level of domestically-owned capital are such that the economy is at point A.

---

3. The phase diagram is constructed on the assumption that \( r^* \neq 0 \). If \( r^* = 0 \) the two curves \( f' - \delta = r^* \) and \( f' - \delta = r^* + K_f r^* \) would intersect \( F' = \delta \) at \( K_f = 0 \).
Then \( F' - \delta > r^* + K_f p^* \nu \)
\[ \text{or } F' - \delta > (1 - \tau)(F' - \delta) + K_f p^* \nu \]

which implies that \( \tau > \tau_{\text{opt}} \). Because capital is assumed to be completely mobile internationally, the optimal policy is to reduce the tax rate to the optimal rate \( \frac{K_f p^* \nu}{F' - \delta} \) so that the inflow of foreign capital is such as to take the system to the point \( A' \).

Then the optimal growth path for the economy is along the curve \( F' - \delta = r^* + K_f p^* \nu \) to the endpoint \( (\hat{k}_d, 0) \). Along this growth path all of the home country’s net output is allocated between current consumption and current domestic investment.

Similarly, if the initial levels of foreign investment and domestically-owned capital are such that the economy is at point \( B \), the existing tax rate is less than the optimal tax rate. By increasing the tax rate to its optimal level foreign capital will flow out of the home country instantaneously and the economy is moved to \( B' \). Then the optimal growth path is along the curve \( F' - \delta = r^* + K_f p^* \nu \) to the endpoint \( (\hat{k}_d, 0) \).

If the level of domestically-owned capital is greater than \( \hat{k}_d \) then the optimal policy is to allow the capital to depreciate at the fastest possible rate until \( K_d \) becomes equal to \( \hat{k}_d \). That is, all of net output of the home country is allocated to current consumption. The rate of growth along \( F' - \delta = r^* + K_f p^* \nu \) depends on the distribution of the home country’s net output between current consumption and current direct investment. The allocation to current domestic investment must be greater than the amount of depreciated capital and the economy’s rate of growth is an increasing function of the amount allocated to current domestic investment.

In the analysis so far it has been assumed that the capital stock
owned by the foreign country is constant and not subject to growth. A more realistic assumption would be that the foreign country's economy is moving along a positive growth path. To simplify the discussion of the effect of growth in the foreign country on the above analysis, assume that the total capital stock owned by the foreign country is growing exponentially at the rate $\alpha$. That is, $K^f = e^{\alpha t}$.

Assume that initially the stock of capital owned by the foreign country is $K^f_0$. Then $K^f_0$ is allocated between the two countries so as to equalize the rates of return on capital in the two countries. Then at the end of the first period the national income of the foreign country is composed of the total output produced in its own country plus its earnings from capital invested abroad in the home country. The national income is distributed between current investment and current consumption so as to satisfy the foreign country's budget constraint. The allocation to current investment must be

$I^f = (\alpha + \delta)K^f$ so as to maintain the chosen growth rate $\alpha$ of the foreign country's capital stock. The balance of the national income then goes to current consumption. That is, at the beginning of the next period the total capital stock owned by the foreign country is increased to $K^f_1$, where $K^f_1 = K^f_0 e^{(\alpha + \delta)(t_1 - t_0)}$. Then, as in the first period, $K^f_1$ is allocated between $K^f_d$ and $K^f_f$ so as to once again equalize the rates of return to capital in the two countries.

The main effect of growth in the foreign country on the above analysis is its effect on the rate of return on capital. The rate
of return on capital in the foreign country is

\[ r^* = \frac{d \hat{K}^*}{d \hat{K}} = \frac{\hat{K}^*}{K^*_f - K^*_0} \]. When there is no economic growth in the foreign country \((K^*_f = K^*_0)\), \(r^*\) is an increasing function of \(K^*_f\), as shown below.

If \(K^*_0\) is distributed between \(K^*_0\) and \(K^*_f\) so that \(K^*_f = K^*_0\), then \(r^* = r^*_0\) \((= r)\). At the end of the first period \(K^*_0\) increases to \(K^*_1\). Then for a given \(K^*_f\) \((say\ K^*_0)\) \(r^*\) falls. This is true for all values of \(K^*_f\). Therefore the curve \(r^*(K^*_f)\), when the total capital stock owned by the foreign country increases to \(K^*_1\), lies below that when \(K^*_f = K^*_0\). Then before allocation of the foreign capital by the foreign country.

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4. The rate of return on capital in the foreign country also depends on the rate of growth of labour in the foreign country and on the rate of technical progress. The rate of growth of labour has been discussed earlier in this chapter and is assumed to be zero. It is also assumed throughout this work that there is no technical change.
country's capital stock between $K_d^*$ and $K_f$, the rate of return on capital in the foreign country falls to $r_f^*$. If $r > r_f^*$, $K_f$ increases and $r_f^*$ adjusts along the curve $r_f^* | K_f = K_d^*$ until the rates of return in the two countries are again equal. Thus there are now two effects on the rates of return in the foreign country. The growth of the capital stock owned by the foreign country forces $r_f^*$ down and the allocation of this capital stock between $K_d^*$ and $K_f$ changes $r_f^*$ along the appropriate $r_f^*$ curve. The effect of $K_f$ on $r_f^*$ remains unchanged (that is, $\frac{\partial r_f^*}{\partial K_f}$ is positive) but the total effect of growth and the distribution of this capital stock between $K_d^*$ and $K_f$ can be positive, negative, or zero. Then the previous analysis holds for both the case of no growth and the case of positive growth in the foreign country.

II

The above analysis is based on the assumption of complete mobility of capital internationally. This analysis can now be made more realistic by assuming capital to be irreversible. That is, an increase in the stock of capital can only be made out of the current output of capital goods, regardless of whether this stock of capital is the domestic investment in either country of the foreign investment in the home country. Also the stock of foreign-owned capital in the home country can only be reduced either by allowing this stock of capital to depreciate or by the buying-out of foreign invested capital by the home country.

To introduce this assumption of irreversibility of capital, it is necessary to assume that the foreign country's capital stock is
growing. Let this rate of growth\(^5\) be \(a\) so that \(\dot{K}_f = aK_f\) or \(I_f = (a + \delta)K_f\), where \(I_f\) is the current addition to the capital stock owned by the foreign country and \(\delta\) is the rate of depreciation of capital. Then \(I_f\), the current direct foreign investment in the home country, must be part of \(I_f\). That is, \(I_f\) is such that \(0 \leq I_f \leq (a + \delta)K_f\). The effect of growth in the foreign country on the optimal growth paths of the home country has been discussed immediately above and will not be considered again here.

If the level of foreign investment in the home country is above its optimal level at any point of time, the foreign investment can be reduced either by allowing this capital stock to depreciate or by the buying-out of foreign invested capital by the home country. To be able to reduce the level of foreign investment by buying it out, the home country must allocate part of its national income for this purpose. Then the net output of the home country

\[
[F - F'K_f + \tau(F' - \delta)K_f] \] can be allocated between current consumption \(C\), current domestic investment in new capital equipment \(I_{d1}\) and current investment in previously owned foreign investment \(I_{d2}\).

Then the home country's budget constraint becomes

\[
F(K_d + K_f) - F'(K)K_f + \tau[F'(K) - \delta]K_f \geq C + I_{d1} + \pi I_{d2}
\]

where \(C > 0\), \(I_{d1} \geq 0\) and \(\pi\) is the price paid per unit of capital when the home country buys out foreign invested capital in the home country \((I_{d2} > 0)\) or the home country sells some of its domestic capital stock to the foreign country. In the case in which the home country's capital stock is to remain constant, \(\pi > 0\) must be greater than unity.

\(^5\) The rate of growth of the foreign country's capital stock is the decision of the foreign country. It is assumed here that this rate of growth is given exogenously and is \(a\).
investment to foreign owners \((I_{d2} < 0)\). The price \(\pi\) is a function of \(I_{d2}\) with \(\pi'(I_{d2}) < 0\). This price \(\pi\) depends on the supply and demand conditions present in the capital market. The price of a newly-created good is assumed to be unity. Therefore as \(I_{d2}\) goes to zero \(\pi\) goes to unity. That is, \(\pi(0) = 1\).

If the home country's policy were to reduce the level of foreign investment, the price it would have to offer or pay for the foreign-owned capital would depend on the supply conditions of capital in the foreign country. These supply conditions are reflected in the willingness of the foreign country to invest more capital in the home country. If the foreign country is not willing to invest any of its newly-created capital in the home country, this is because \(r^* > r\). Then the foreign country would be willing to sell part of its investment in the home country at a price which is less than unity so that it can increase its capital investment in its own country. That is, it can invest more of its capital in its own country where the return to capital is greater. Then if \(I_f = 0\) and \(I_{d2} > 0\), \(\pi\) must be less than unity.

If the home country's policy is to reduce the level of foreign investment but the foreign country is willing to increase its investment in the home country because \(r^*\) is less than \(r\), the home country would have to offer a price for the foreign-owned capital which exceeds unity. That is, if \(0 < I_f \leq (\alpha + \delta)k^*\) and \(I_{d2} > 0\), \(\pi\) must be greater than unity.

In the case in which the home country's policy is to reduce its stock of domestically-owned capital by offering part of it for sale to the foreign country, the price \(\pi\) at which it is offered depends on the willingness of the foreign country to have more of its capital
stock invested in the home country. If the foreign country is willing to increase its investment in the home country in order to take advantage of the better investment opportunity in the home country \((r > r^\alpha)\), then the foreign country would be willing to pay more than unity for domestically-owned capital. That is, if \(0 < I_f \leq (a + \delta)K^\alpha\) and \(I_{d2} < 0\), \(\pi\) would be greater than unity. However, if in this case the foreign country was not willing to increase its investment in the home country, the home country would have to offer part of its domestically-owned capital at a price less than unity. That is, if \(I_f = 0\) and \(I_{d2} < 0\), \(\pi\) must be less than unity.

Then assuming that the home country cannot lend its capital and that the rate of growth of the foreign country's capital \(a\) is given exogenously, the aim of the home country is to minimise

\[
\int_0^T \left[ U(\dot{C}) - U(C) \right] dt ,
\]

where \(T\) is unspecified, along the accumulation path from any initial point to the endpoint subject to the budget constraint

\[
P(K_d + K_f) - F'(K)K_f + \frac{rF'(K) - \delta}{C - I_{d1} - \pi I_{d2}} \geq 0 ,
\]

the capital accumulation equations

\[
\begin{align*}
\dot{K}_d &= I_{d1} + I_{d2} - \delta K_d \\
\dot{K}_f &= I_f - I_{d2} - \delta K_f ,
\end{align*}
\]

and the boundary conditions

\[
\begin{align*}
C &\geq 0 \\
I_{d1} &\geq 0 \\
I_{d2} &\leq 0 \\
0 &\leq I_f \leq (a + \delta)K^\alpha \\
r &\geq r^\alpha
\end{align*}
\]
where \( n = \pi(I_{d_2}) \) and \( r = (1-\tau)[F'(K) - \delta] \).

The fixed endpoint \((\hat{K}_d, 0)\) is defined and derived in part I of this chapter. This optimisation problem is solved in Appendix No. 5 giving the following phase diagram:

The Phase Diagram

where the optimal policies are defined by

<table>
<thead>
<tr>
<th>Policy</th>
<th>( I_{d_1} )</th>
<th>( I_{d_2} )</th>
<th>( C )</th>
<th>( I_f' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>( 0 &lt; I_{d_1} &lt; f )</td>
<td>( 0 &lt; I_{d_2} &lt; f )</td>
<td>( 0 &lt; C &lt; f )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>E</td>
<td>( 0 &lt; I_{d_1} &lt; f )</td>
<td>( 0 &lt; I_{d_2} &lt; f )</td>
<td>( 0 &lt; C &lt; f )</td>
<td>( 0 &lt; I_{d_2} &lt; (a+\delta)K_f )</td>
</tr>
<tr>
<td>F</td>
<td>( 0 &lt; I_{d_1} &lt; f )</td>
<td>( 0 &lt; I_{d_2} &lt; f )</td>
<td>( 0 &lt; C &lt; f )</td>
<td>( (a+\delta)K_f )</td>
</tr>
<tr>
<td>J</td>
<td>( 0 &lt; I_{d_1} &lt; f )</td>
<td>( 0 &lt; I_{d_2} &lt; f )</td>
<td>( 0 &lt; C &lt; f )</td>
<td>( 0 )</td>
</tr>
</tbody>
</table>

where \( f = F - F'K_f + \tau(F' - \delta)K_f \).
If initially the levels of $K_d$ and $K_f$ are such that the economy is at a point above the curve $F' - \delta = r^* + K_f r^{d*}$, then the stock of foreign-owned capital invested in the home country is greater than the optimal level corresponding to the given domestically-owned capital stock. The foreign country would not have invested this amount of capital in the home country if $r^*$ were to exceed $r$. Therefore this distribution of foreign-owned capital must be such that $r$ is less than $r^*$. Then the optimal policy for the home country to adopt is policy J. Along J a tax is imposed so that $r = r^*$ and the level of foreign investment is reduced by buying out foreign-owned capital invested in the home country at the price $\pi^*$. The balance of net national output is distributed between current domestic investment and consumption so that the social demand $(\lambda_1)$ and supply $(U'(C))$ prices of a unit of domestic investment are equal and social utility is maximised. That is, foreign investment is reduced at a greater rate than the normal capital depreciation rate until the economy lies on the curve $F' - \delta = r^* + K_f r^{d*}$. Then the stock of foreign-owned capital invested in the home country is at its optimal level corresponding to the stock of domestically-owned capital in the home country. The optimal policy then switches from policy J to either D or E.

Along policies D and E the home country allocates all of net output between current domestic investment and current consumption. Then the level of current foreign investment is such as to maintain the stock of foreign-owned capital in the home country at its optimal level. If the level of current foreign investment is zero the optimal

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6. The equality between $\lambda_1$ and $U'(C)$ has been discussed in detail in Chapter VI page 108.
policy is D but if it is positive (and less than \((a + \delta)K^*\)) the optimal policy is E. Policy J may initially switch into D and then from D into E at some point along the curve \(F' - \delta = r^*_d + r^*_f K^*_f\). Then policy E takes the economy to the endpoint \((K^*_d, 0)\), where the stock of foreign-owned capital invested in the home country is zero.

If initially the levels of \(K_d\) and \(K_f\) are such that the economy lies somewhere below the curve \(F' - \delta = r^*_d + K^*_f r^*_f\), then the level of \(K_f\) is below its optimal value for the given level of \(K_d\). The optimal policy is F along which current foreign investment is at its maximum \((a + \delta)K^*\) and \(r^*_d\) and \(r^*_f\) are controlled by taxation so that \(r = r^*_d\). It can be seen from the properties of policy F in Appendix No. 5 and from

\[
\lambda_2 = -[\rho(F' - \delta - r^*_f K^*_f) + \nu(a + \delta) - \lambda_2 \delta]
\]

which is negative along policy F, that is may be optimal for the home country to subsidise foreign investment depending on the nature of the function \(r^*_f = r^*_f(K^*_f)\), on the relationship between \(r^*_d\) and \(r\) and on the growth rate \(\alpha\). Net national output is distributed between current domestic investment and current consumption in such a way as to maximise social utility. Policy F takes the economy to the position where the level of foreign investment is at its optimal level corresponding to the level of domestically-owned capital in the home country. Then the optimal policy switches from F to either D or E which then takes the economy along the curve \(F' - \delta = r^*_d + K^*_f r^*_f\) to the endpoint \((K^*_d, 0)\).

**Conclusions**

From static analysis the optimal level of direct foreign investment for the given domestically-owned capital stock of the home country and the optimal taxation policy for the home country to adopt so as to maintain this level of direct foreign investment can be found. The dynamic analysis is concerned with finding the optimal capital
accumulation paths over time.

The first model above is similar to Kemp's static model (Kemp [16]) except that growth of domestically-owned capital in the home country is introduced into the model. This model gives the static model results for any given level of domestically-owned capital but it also gives the optimal growth path for both the stock of domestically-owned capital and foreign investment over time. That is, because of the assumption of complete mobility of capital internationally, the stock of foreign-owned capital invested in the home country is adjusted instantaneously to its optimal level corresponding to the stock of domestically-owned capital. This is identical with the results of the static model. As the domestically-owned capital stock grows, the level of foreign investment is adjusted according to the static results until the optimal domestically-owned capital stock is reached. At this endpoint where sustainable consumption per head is maximised, the amount of foreign investment is zero.

In the second model above, the assumption of irreversibility of capital is introduced. The inclusion of this assumption produces two changes. Firstly, if the level of foreign investment is above its optimal level corresponding to the given domestically-owned capital stock, it cannot be reduced instantaneously, as in the first model, but can only be reduced by allowing the foreign-owned capital to depreciate or by the buying-out of part of the foreign investment by the home country at a price determined by the supply and demand conditions ruling in the market. Secondly, any additions to direct foreign investment can only be made out of newly-created capital goods. Therefore, there is a natural limit to the rate of expansion of direct foreign investment in the home country.
Thus, the inclusion of the assumption of irreversibility of capital does not allow the economy to jump instantaneously to a position where the amount of foreign direct investment is at its optimal level corresponding to the given level of domestic investment, as in the first model. Now there is an optimal growth path along which the amount of direct foreign investment is adjusted, as the stock of domestically-owned capital grows, until it is at its optimal level corresponding to the level of domestic investment. When this is achieved the optimal capital accumulation policy switches to that which maintains the optimal level of direct foreign investment as the domestically-owned capital stock continues to grow until the optimal endpoint \((K_d, \theta)\) is reached. That is, the amount of direct foreign investment is adjusted first but this adjustment process is no longer an instantaneous one. However the level of direct foreign investment is always adjusted as quickly as possible.

These conclusions are perhaps the most significant ones but it is interesting to consider what is happening to the level of consumption along these optimal growth paths.

Along all these optimal growth paths social utility is optimised. That is \(\lambda_1 = U'(C)\). From the necessary and sufficient conditions for optimality, \(\dot{\lambda}_1 = -[\rho F' - \lambda_1 \delta]\). But \(\dot{\lambda}_1 = -U''(C)\dot{C}\) and therefore \(\dot{C} = -\frac{U'}{U''} [F' - \delta]\) which is always positive in the region \(F' > \delta\). That is, along these optimal growth paths the level of consumption is increasing until the maximum sustainable consumption level is reached at the endpoint.

Comparing this second model above with that of Pitchford [29], it should be noticed that he includes the assumption of irreversibility of capital internationally but has not introduced growth into the
lending country. As discussed early in section II any increase in the level of direct foreign investment when capital is assumed to be irreversible can only be made from newly-created capital stock. In the model of section II it is assumed that the foreign country has chosen a constant growth rate $\alpha$. It would have been more interesting if the foreign country's growth rate was not fixed, but somehow determined from within the model. However the system would be far more complicated and it is doubtful if indeed it could be analysed. The assumption of a constant growth rate $\alpha$ is not as limited as it first appears. If $\alpha$ varies from one time period to the next but is constant for each period, the above results still hold, provided the time periods are not infinitesimal.

As noted earlier it may be optimal to subsidise foreign investment as a result of the assumption of irreversibility of capital. This assumption of restricted mobility of capital will allow the rates of return on capital in the two countries to differ. Also the natural limit to the amount of current foreign borrowing is the supply of newly-created capital in the foreign country. However the amount of capital which the foreign country is willing to invest in the home country depends on the relationship between the rates of return in the two countries. Therefore the supply of capital which the foreign country is willing to invest in the home country is an implicit function of the rate of return on capital in the home country. Then, as noted in Chapter VII for the case of optimal foreign borrowing, it may be optimal for the home country to subsidise foreign investment because the supply of capital available for current foreign investment is limited by the rate of return on capital in the home country.
In Chapter VIII it was assumed that only one commodity is produced in the international economy, so that the model abstracted from the problems associated with international trade. In this chapter, international trade is introduced explicitly by discussing optimal direct investment within the two-country, two-factor of production, two-commodity model of international trade, in which the factor of production, labour, is assumed to be immobile between countries. The home country is then free to choose any fiscal and commercial policies it pleases, with the aim of minimising the accumulated difference between actual social welfare and the maximum level of sustainable social welfare along its growth path. It is assumed that the foreign country does not interfere in either the commodity or the factor markets. Also the factors of production are assumed to be fully-employed in both countries.

As in the previous chapters it is assumed that the home country can accept foreign investment from abroad but that it is not permitted to invest outside of its own country. As in the previous chapter labour is assumed to be constant in both countries. It is also assumed that labour is immobile between industries. This assumption differs from that of the standard trade model in which labour is assumed to be mobile within each country. This assumption has been made for two reasons. Firstly, it appears to be a difficult task to introduce foreign investment into the standard two-sector growth model.
model, as no general theory of economic growth with international capital movements taking the form of foreign direct investment, has yet been put forward. Secondly, it is not a completely unrealistic assumption because labour, particularly skilled-labour, is to some degree mobile only within the one industry. It would be of great interest to relax this assumption of immobility of capital between industries. However, no attempt to do this will be made in this work.

Initially it is assumed that economic growth is taking place only in the home country. Because it is unrealistic to assume that the home country's growth is completely dependent on imports of capital goods, it is assumed that the home country always produces good 2, which can be either consumed or invested. Good 2 is then also the home country's natural export good. The natural export good of the foreign country is good 1 which is the consumption good. Because it is assumed initially that the foreign country is in a static state, its demand for good 2 is for consumption purposes and for the replacement of capital stock which has depreciated. However, in section IV, the difficulties of relaxing this assumption and introducing growth in the foreign country will be discussed.

Several patterns of specialization of production in the two countries have to be considered. The general model will be developed in section I and the different cases of specialization of production will be considered in the following two sections. Section II assumes that the home country's production is completely specialized whereas in section III production in the home country is assumed to be incompletely specialized.

The foreign country is assumed to be in a static state. Then the
stock of capital owned by the foreign country $K_f$ is constant. Assuming immobility of labour between industries, the outputs of good 1 and 2 in the foreign country are $F_1(K_1^f)$ and $F_2(K_2^f)$ respectively. If, however, the foreign country is completely specialized in production the output of good 2, $F_2(K_2^f)$, is zero. Assuming $K_f$ to be the stock of capital invested in the home country, it follows from the assumption of full-employment of factors of production that $K^h - K_f$ equals $K_1^d + K_2^d$.

The output of good 1 in the home country is $F_1(K_1^h)$, where the total capital stock employed in the home country, $K_1 + K_2$, is the sum of the domestically-owned capital in the home country $K_1^d$ and the level of foreign investment $K_f$. If the home country is completely specialized in its production, $F_1(K_1^h)$ is zero.

All these functions are assumed to be neo-classical production functions under constant returns to scale. That is, for example,

$$F_1'(K_1^h) > 0, \quad F_1''(K_1^h) < 0$$

$$F_1(0) = 0, \quad F_1'(0) = \infty$$

$$F_1(\infty) = \infty, \quad F_1'(\infty) = 0,$$

The accumulation of domestically-owned capital in the home country is given by

$$\dot{K}_d = I_d - \delta K_d,$$

where $I_d$, the current domestic investment in the home country, is non-negative and $\delta$ is the rate of capital depreciation in both countries. Since the foreign country is in a static state, $\dot{K}_f$ is zero. Therefore, from the equation for the static state ($\dot{K}_f = I_f - \delta K_f = 0$), the level of current domestic investment in the foreign country is $I_f = \delta K_f$. 
Capital is assumed to be completely mobile between industries in the one country. Therefore the gross rate of return in the foreign country is

\[
R^* = \frac{3p^t_2(k^*_2)}{\partial k^*_2} - \frac{3p^t_1(k^*_1)}{\partial k^*_1},
\]

where \( p^t_1 = \frac{P_1}{P_2} \) is the world terms of trade. Or alternatively, the price-weighted sum of changes in production, at constant prices, resulting from an inflow of an extra unit of capital must equal the gross rate of return on capital. That is,

\[
R^* = -\left[\frac{3p^t_2}{\partial k_f} + \frac{3p^t_1}{\partial k_f}\right].
\]

Because it is assumed that the foreign country does not interfere in the commodity and factor markets, the gross rate of return \( R^* \) equals the net rate of return \( r^* \) on capital employed in the foreign country.

Similarly, the gross rate of return on capital invested in the home country is

\[
R = \frac{3F_2}{\partial k_f} + p \frac{3F_1}{\partial k_f}.
\]

The net rate of return on capital is \( r = (1 - \tau)(R - \delta) \), where \( \tau \) is the rate of tax imposed by the home country on earnings from foreign investment after depreciation allowances have been made.

It is assumed initially that capital is also completely mobile between countries. That is, capital moves instantaneously between countries so as to equalize the net rates of return on capital in both countries. That is, capital moves so as to maintain the equality \( r = r^* \) at all points of time. It has been assumed that the home country can never lend capital. Therefore, for international capital movements, \( r^* \) must not exceed \( r \). That is, \( r \geq r^* \).
The more realistic assumption of restricted mobility of capital internationally will be introduced into the model in section III.

The utility of consumption in the home country is given by

$$ U = U(D_1, D_2) = U_1(D_1) + U_2(D_2), $$

where $U_i$ is the utility derived from the consumption $D_i$ of good $i$. As it is also assumed that $D_i$ is non-negative and $U_i'$ is positive with $U_i'(0) = \infty$, it is never optimal for the home country to consume nothing of either good. That is, $D_i$ is always positive. Similarly, foreign consumption of either good must also be non-zero, or $D_f$ is always positive.

If $p$ is the price of good 1 in terms of good 2 in the home country, then the home country's demand for the two goods will be such that

$$ \frac{U_1'}{p} = U_2'. $$

The four constraints of the optimisation problem are the home country's budget constraint, the two market clearing equations and the Balance of Payments constraint.

The home country's budget constraint is such that output allocated to consumption of the two goods and to current domestic investment, must not exceed the home country's output net of earnings paid to foreign capitalists. That is,

$$ F_2(K_2) - RK_f - \tau(R - \delta)K_f \geq D_2 + pD_1 + I_d + I_d. $$

The market-clearing equations are

$$ F_2(K_2) + F_2(K_2) = D_2 + D_f + I_d + I_d $$

and

$$ F_1(K_1) + F_1(K_1) = D_1 + D_f. $$
The home country's Balance of Payments inequality is such that its exports of good 2 do not fall short of the sum of its imports of good 1 and foreign earnings paid abroad. That is,

\[(D^*_2 - F^*_2) \geq p^a(D_1 - F_1) + (RK_f - \tau[R - \delta]K_f).\]

However, since these four constraints are not independent it is necessary to include only the first three of these constraints in the model.

If \(U(D_1, D_2)\) is the maximum sustainable level of social welfare, the optimisation problem is to minimise

\[\int_0^T [U(D_1, D_2) - U_1(D_1) - U_2(D_2)]dt,\]

where \(T\) is unspecified, along the capital accumulation path from any initial point to the fixed endpoint, subject to

\[F_2(K_2) - R K_f + \tau(R - \delta)K_f - p^a(D_1 - F_1) - D_2 - I_d \geq 0;\]
\[F_1(K_1) + F^a(K^*_1) = D_1 + \delta K^*_1;\]
\[F_2(K_2) + F^a(K^*_2) = D_2 + \delta K^*_2 + I_d + \delta K^*_2;\]
\[K_d = I_d - \delta K^*_d;\]
\[I_d \geq 0;\]
\[K_f \geq 0;\]
\[D_i \Delta K^*_i > 0 \ (i = 1, 2);\]
\[r = (1 - \tau)(R - \delta) \geq \varphi;\]
\[p = p^a(1 + \tau_1),\]

where \(\tau_1\) is the tariff imposed by the home country on its exports of good 2,

\[R = \frac{3F_2(K_2)}{3K_2} + p \frac{3F_1(K_1)}{3K_1} \text{ and } p = \frac{U_1(D_1)}{U_2(D_2)}.\]

The fixed endpoint is such that sustainable consumption is maximised.

The net output depends on the specialization of production in the
home country. If the home country is completely specialized in the production of good 2, the net output is

\[ Y = F_2 - F'_2K_f + \tau(F'_2 - \delta)K_f - \delta K_d. \]

Then

1. \[ \frac{\partial Y}{\partial K_d} = F'_2 - (1 - \tau)F''_2K_f - \delta = 0 \quad \text{and} \]
2. \[ \frac{\partial Y}{\partial K_f} = -(1 - \tau)F''_2K_f + \tau(F'_2 - \delta) = 0. \]

Substituting (2) into (1), either \( F'_2(K_f) = \delta \) or \( \tau = 1 \). But if \( \tau = 1 \), \( \tau \) is zero and the level of foreign investment is zero.

Then, in the one sector economy, sustainable output is maximised if \( F'_2 = \delta \). That is, \((K_d', 0)\) is the fixed endpoint which is shown below.

If the home country is incompletely specialized in production, net output is

\[ Y = pF_1 + F_2 - RK_f + \tau(R - \delta)K_f - \delta K_d. \]

Then

1. \[ \frac{\partial Y}{\partial K_d} = pF'_1 + F'_2 - \delta = R - \delta = 0, \]
2. \[ \frac{\partial Y}{\partial K_f} = pF'_1 + F'_2 - R + \tau(R - \delta) = \tau(R - \delta) = 0 \quad \text{and} \]
3. \[ \frac{\partial Y}{\partial p} = F'_1 - \frac{3R}{\partial p}K_f(1 - \tau) = 0. \]
But \( r^\alpha = (1 - \tau)(R - \delta) \).

Therefore \( \frac{3Y}{3p} = F_1 - \frac{3R}{3p} K_f R - \delta = 0 \)

or \( (R - \delta)F_1 - \frac{3R}{3p} K_f r^\alpha = 0 \).

From \( \frac{3Y}{3K_d} = \frac{3Y}{3K_f} = 0 \), \( R - \delta = 0 \).

From \( \frac{3Y}{3p} = 0 \) and \( R - \delta = 0 \), \( K_f = 0 \).

That is, the optimal endpoint is again defined by \( R - \delta = 0 \) and \( K_f = 0 \).

In the optimisation problem which must now be solved the control variables \( I_d, D_1, D_2, D_2, K_f, T, t_1 \) and \( p^\alpha \) can be chosen subject to the given constraints to determine the movement of the state variable \( K_d \). This problem is solved by Arrow's Method of Solution (Arrow [3]) which is based on Pontryagin's Maximum Principle (Pontryagin [30]).

The Hamiltonian is defined by

\[
H = -[(U_1(D_1, D_2) - U_1(D_1) - U_2(D_2)] + \lambda_1[I_d - \delta K_d],
\]

where \( \lambda_1 \) is the social price of a unit of net domestic investment in terms of utility units and the Lagrangian is

\[
L = H + p[F_2(K_2) - RK_f + \tau(R - \delta)K_f - p^\alpha(D_1 - F_1) - D_2 - I_d]
+ z[F_1(K_1) + p^\alpha(K_1) - D_1 - D_1]
+ w[F_2(K_2) + p^\alpha(K_2) - D_2 - D_2 - I_d - \delta K_d]
+ h[r - r^\alpha] + v[p - (1 + r_1)p^\alpha].
\]

Necessary conditions for an optimal solution are:

(a) \( \lambda_1 \) is a continuous function of time given by

\[
\lambda_1 = -\frac{\partial L}{\partial K_d} = -(2 + p^\alpha F_1 + p + w F_2)
+ (1 - \tau)\frac{3R}{3K_d}(h - pK_f - \lambda_1 \delta).
\]
(b) The constraints satisfy the Constraint Qualification \(^7\) at any point of time and the Hamiltonian is maximised subject to this constraint. That is,

\[
\frac{\partial L}{\partial \lambda} = \lambda_1 - \rho - w + f = 0
\]

\[
\frac{\partial L}{\partial \alpha_1} = U_1^{(D_1)} - \rho \alpha_1 - z + \nu \frac{U_1'}{U_2'} = 0
\]

\[
\frac{\partial L}{\partial \alpha_2} = U_1^{(D_2)} - \rho - w - \nu \frac{U_2}{U_2^*} = 0
\]

\[
\frac{\partial L}{\partial \beta} = -w = 0
\]

\[
\frac{\partial L}{\partial \alpha} = -(\rho(D_1 - F_1) + zD_1' - wD_2' + h \frac{\partial}{\partial \alpha} + \nu(1 + \tau_1') = 0
\]

\[
\frac{\partial L}{\partial \beta} = (\rho K_f - h)(R - \delta) = 0
\]

\[
\frac{\partial L}{\partial \gamma} = -\nu \alpha = 0
\]

\[
\frac{\partial L}{\partial \beta_f} = \rho[F_2' + p \beta' F_1' - R - (1 - \tau)R'K_f + \tau(R - \delta)]
\]

\[
+ z[F_1' + F_1''] + w[F_2' + F_2''] + h[\frac{\partial}{\partial \beta_f} - \frac{\partial}{\partial K_f}] = 0.
\]

(c) The Lagrange Multipliers \(\rho\), \(h\) and \(f\) which are associated with the inequality constraints must satisfy the following conditions:

\[
\rho \geq 0 ; \quad \rho[F_2 - RK_f + \tau(R - \delta)K_f - p \alpha(D_1 - F_1) - D_2 - I_d] = 0
\]

\[
h \geq 0 ; \quad h[r - \alpha] = 0
\]

\[
f \geq 0 ; \quad \forall I_d = 0.
\]

The values of the other Lagrange Multipliers \(z\), \(\nu\) and \(w\) are not restricted as they are associated with equality constraints.

Since \(\frac{\partial L}{\partial \beta} = 0\), either \(h\) equals \(\rho K_f\) or \(R\) equals \(\delta\). Foreign

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capitalists would not be willing to invest capital in the home country if the net return \((R - \delta)\) on capital were zero. Therefore \(h - \rho K_f\) is zero. Also since \(\frac{3L}{3L_1} = 0\), \(\frac{3L}{3D_1} = 0\) and \(\frac{3L}{3D_2} = 0\), \(v\) and \(w\) are zero and \(\rho\) equals \(U_2^r\) which is always positive. Therefore the budget constraint is satisfied with equality and thus the Balance of Payments is always in equilibrium. Since \(\frac{3L}{3D_1} = 0\), \(z\) equals \(U_1^r - \rho p^\alpha\). Then substituting for \(z\), \(h\), \(\rho\), \(w\) and \(v\) in \(\frac{3L}{3p^\alpha} = 0\) and using the relationships \(U_1^r = pU_2^r\) and \(p = p^\alpha(1 + \tau_1)\), the optimal tariff rate is given by

\[
\tau_1 \text{ opt.} = \frac{-E_1}{p^\alpha \mu} [1 - \mu \gamma^\alpha],
\]

where \(E_1\), the home country's excess demand for good 1, equals \(D_1 - F_1\), \(w\), the ratio of net earnings of foreign capitalists from the home country to the value of the home country's imports of good 1, equals \(\frac{K_f^r}{p^\alpha E_1}\) and \(\gamma^\alpha\), which equals \(\frac{3p^\alpha}{3p^\alpha}\), depends on the factor-intensity of good 2 in the foreign country. If good 2 is capital-intensive abroad, \(\gamma^\alpha\) is positive but if it is labour-intensive abroad \(\gamma^\alpha\) is negative.

Similarly, since \(\frac{3L}{3K_f} = 0\), the optimal rate of tax is given by

\[
\tau_1 \text{ opt.} = \frac{D_1^\alpha \gamma^\alpha K_f + E_1(1 - \mu \gamma^\alpha)F_1^\alpha}{D_1^\alpha (R - \delta)}.
\]

The signs of the optimal tax and tariff rates depend on the pattern of specialization of production. These signs will be considered in section II.

It should be noted that since \(h\) equals \(U_2^r K_f\), \(h\) is positive for all \(K_f\) greater than zero and thus the net rates of return on capital in the two countries are always equal. That is, the foreign country's capital stock adjusts instantaneously so that \(r\) and \(r^\alpha\) are equal at all points of time. Then from the assumption of perfect
mobility of capital internationally, the optimal capital accumu-
ation path is given by
\[ r^* = (1 - \tau_{\text{opt}})(R - \delta). \]

Also, substituting for the Lagrange Multipliers, the equation for \( \dot{\lambda}_1 \) reduces to:
\[ \dot{\lambda}_1 = -[U'_2 R - \lambda_1 \delta]. \]

The nature of the optimal growth paths varies with the optimal tax-
ation and tariff policies of the home country. These, in turn depend
on the pattern of specialization of production in the two countries
and on the pattern of international trade. Therefore optimal
commercial and fiscal policies of the home country will be discussed
initially when production of the home country is completely specialized.
Then in section III the case in which the home country is incom-
pletely specialized will be considered.

II
A: Home Country Completely Specialized

When the home country is completely specialized in the production of
good 2 \( R \) equals \( F'_2 \) and the signs of the optimal tariff and tax
rates depend on the degree of specialization of production in the
foreign country.

(1) When the foreign country is completely specialized in the pro-
duction of good 1, \( r^* = \frac{\partial F_1}{\partial K_1} = \frac{\partial F_2}{\partial K_2} \) which is independent of the
terms of trade \( p^* \). \( (k^*_d \) is defined as the stock of capital
employed in the foreign country.) Then \( \gamma^* \) is zero and \( \frac{\partial r^*}{\partial K_F} \) is
positive because the foreign country is assumed to be a lender of
capital. The derived equations for the optimal tariff and tax
reduce to
respectively. As these are both positive, it is always optimal for the home country to impose a tax on foreign earnings paid abroad and a tariff on its exports.

(ii) If the foreign country is incompletely specialized in its production, the rate of return on capital invested in the foreign country is

\[ r^*_f = \frac{\frac{\partial F_f}{\partial K_f}}{\frac{\partial F_f}{\partial r_f} + p_2} \]

Then \( \frac{\partial r^*_f}{\partial K_f} \) is zero and the sign of \( \gamma^*_f \), which equals \( \frac{1}{\frac{\partial q^*_f}{\partial p_2}} \), depends on the factor-intensity of good 2 in the foreign country. The optimal tariff and tax rates now become

\[ t_1 \text{ opt.} = \frac{\frac{-E_1}{p} K_f}{E_1 F_f}(1 - \mu \gamma^*_f) \] \[ t_2 \text{ opt.} = \frac{E_1 F_f'(1 - \mu \gamma^*_f)}{1}(R - \delta) \]

respectively.

The signs of the optimal tax and tariff rates now depend on the pattern of trade which is reflected in the magnitude of \( \mu \), as well as on \( \gamma^*_f \), the factor-intensity of good 2 in the foreign country.

If the home country's export good (2) is labour-intensive abroad then \( \gamma^*_f < 0 \) and via the Rybczynski Theorem [32] \( F_f' < 0 \), so that it is always optimal for the home country to impose a tariff on its exports of good 2 and a tax on foreign earnings paid abroad. A tariff on exports increases the price of good 2 abroad so that the return on capital in the foreign country falls and capital tends to flow into the home country until the returns to capital in the two
countries are equated. The inflow of capital is controlled, however, by imposing a tax on capital earnings so that the net rates of return on capital in the two countries are the same.

If the home country's export good is capital-intensive abroad, \( \gamma_2 \) is positive and greater than 1 and \( F_1' \) is positive from the Rybczynski Theorem [32] so that it is always optimal to impose a tariff on exports unless the home country is a more important importer of capital than of good 1 (\( \mu \) large), in which case it is optimal to subsidise exports. It is also optimal for the home country to tax the earnings on foreign investment unless the home country is a greater importer of good 1 than of capital (\( \mu \) small), in which case it is optimal to subsidise foreign investment.

The above results show that the optimal tax and tariff can be either positive or negative but it is never optimal to subsidise both exports and foreign investment. These results agree with those derived by Jones [13] for the static case.

The optimal growth paths for the home country when its production is specialized, are given below. The shapes of the curves are discussed in Appendix No. 6.
The Phase Diagrams

(i) Foreign country completely specialized
(ii) Foreign country incompletely specialized

Because capital is assumed to be completely mobile internationally, the level of $K_f$ is adjusted instantaneously so that

$$r = r^* = (1 - \tau_{opt})(R - \delta) .$$

Then the optimal growth path of the home economy is along

$$r^*(1 - \tau_{opt})(R - \delta)$$
to the endpoint $(k_d, 0)$. The optimal tax rate and the optimal tariff on exports are defined above.

Conclusions

When both countries are completely specialized in the production of different goods, the optimal rate of taxation on foreign earnings and the optimal tariff on exports are given by

$$\tau_{opt.} = \frac{B^{\tau_{opt}}K_f + E_rF^{\tau_{opt}}}{B^{\tau_{opt}}(R - \delta)}$$
and
respectively. In the previous chapter, where it is assumed that only one good is produced in both countries and thus no explicit account is taken of the movement of this good, the optimal tax rate is

\[ t_{\text{opt.}} = -\frac{E_1}{p_{\text{import}}}, \]

From a comparison of these two sets of results it can be seen that when the model includes international trade in goods \((E_1 \neq 0)\) as well as the movement of capital, it is optimal for the home country to impose a tariff on its exports (so as to reduce the world terms of trade) and not only to tax foreign earnings (so as to lower the rate of return payable on foreign investment) but to impose a larger tax than is optimal in the model with no explicit trade in goods. The reason for this increased optimal tax on foreign earnings is as follows.

The home country's aim is to reduce both the world terms of trade and the rate of return payable on foreign-owned capital invested in the home country. By levying a tax on foreign earnings, the net rate of return payable on foreign investment is reduced. Because capital is assumed to be completely mobile between countries, capital will flow out of the home country until the net rates of return on capital in the two countries become equal.

By imposing a tariff on exports the world terms of trade fall. That is, the relative price of good 1 (the home country's import good) falls. But as \(p^*\) falls the demand for good 1 in the foreign country \(D^*_1\) rises. If the world supply of good 1 \((F_1)\) is constant, the relative price of good 1 will tend to rise. The home country can maintain the lower \(p^*\) be forcing some of the foreign-
owned capital to flow out of the home country and thus increasing F*. This outflow of foreign-owned capital is produced by increasing the rate of taxation on foreign earnings so that foreign capitalists will adjust their capital stocks so as to equate the net rates of return on capital in the two countries. Therefore, for any given domestically-owned capital stock, the optimal level of foreign direct investment is lower in the presence of explicit international trade than in its absence. Also, along the optimal growth path of the home country, the level of foreign investment approaches its optimal level (zero) at a point in time when the level of domestically-owned capital is lower.

When the foreign country is incompletely specialized in its production, the optimal growth path for the home country depends on foreign technology and on whether the home country is a greater importer of capital than of good 1. The foreign technology links the world terms of trade with the rate of return on capital in the foreign country.

It is always optimal for the home country to levy a tax on foreign earnings, except when the home country's export good is capital-intensive abroad. In this case a tariff on exports produces a rise in the rate of return on capital in the foreign country which results in an outflow of capital from the home country. If the home country is primarily an importer of capital then the optimal policy is one of subsidising foreign investment. The optimal level of foreign investment at any given level of domestically-owned capital stock is greater than that in the case which abstracts from explicit international trade - except when the level of foreign investment falls to its optimal level, zero.

Therefore when more than one commodity is produced in the world,
international trade in goods and the degree of specialization of production must be considered. Each of these plays a part in modifying the results obtained in the one-commodity model of Chapter VIII.

III

When the home country is incompletely specialized in production, the optimal commercial and fiscal policies depend primarily on the degree of specialization in the foreign country. If the foreign country is completely specialized in the production of good 1, the optimal tariff and tax rates are

\[ T_{1 \text{ opt.}} = \frac{-E_1}{pM_P} \quad \text{and} \quad T_{\text{opt.}} = \frac{R^1 + E_1 P'}{D^1(R - \delta)} \]

respectively. As these are both positive, it is always optimal to levy a tariff on commodity exports and a tax on foreign earnings paid abroad.

Comparing these results with those in section II, when both countries are completely specialized, it is found that the two sets of results are identical. Thus the optimal commercial and fiscal policies depend primarily on the degree of specialization of production in the foreign country.

The shapes of the various curves necessary for the phase diagram are derived in Appendix No. 8. However the phase diagram is identical with that when both countries are completely specialized (section II) and thus will not be repeated here.

When both countries are incompletely specialized the analysis becomes more complex as the pattern of trade and the optimal tariff and taxation policies are interdependent. Nadel [26] has discussed this
interdependence in his analysis of the static model. It is only by chance that the tariff and tax levied by the home country will be related to each other in the exact proportions that the technology requires. Nadel has shown that if this situation does not occur, the tariff becomes inefficient and thus a change in the pattern of trade occurs. Therefore the optimal tariff and taxation policies of the home country and the pattern of international trade are interdependent.

Assume initially that there exists free trade and unrestricted capital flows. Mundell [24] has shown that when capital is assumed to be completely mobile internationally the free trade results of the standard trade model with factor immobility can be achieved by exporting the factor of production, capital, instead of exporting the good whose production is capital-intensive. Capital moves between the countries until the rates of return on capital are the same in the two countries. Then both countries become self-sufficient and the only flow between them is the one-way repatriation of foreign earnings from the home country. This repatriation of foreign earnings can be viewed as a consequence of the assumption of labour immobility between countries. An additional assumption will be made that the factor-intensity of each good is the same in both countries and that no reversal of factor-intensities occurs.

If the foreign earnings paid by the home country are in terms of good 2, $E_1$ is zero and it is not optimal for the home country to interfere in the flow of goods or of capital.

If on the other hand, the home country exports good 1 in order to pay for the foreign earnings, $E_1 = D_1 - F_1$ is negative, $\mu$ equals -1 and the optimal tariff and tax rates become
Good 2 is capital-intensive because the foreign country has substituted capital exports for exports of the good which is capital-intensive. Therefore it is optimal for the home country to subsidise exports of good 1 and to tax foreign earnings.

However, from Nadel's analysis [26] of the static case, the imposition of the optimal subsidy and tax may stimulate a movement of capital internationally which, because of the given technology, may change the pattern of trade.

The nominal return on capital is \( r_{p_2} \) and the relative change in the nominal return is given by \( \frac{dr_{p_2}}{r_{p_2}} \) which equals \( \frac{dr}{r} \) since \( p_2 \) is constant. From the given technology of the home country \( \frac{dr}{r} \) equals \( -\frac{dp}{p} \), where \( \gamma \) depends on the factor-intensity of good 2. Combining these two results, \( \frac{dr_{p_2}}{r_{p_2}} \) equals \( -\gamma \frac{1}{1} \).

The tax on foreign earnings reduces the nominal return by 100%. Therefore

(i) if \( -\gamma \frac{1}{1} > 0 \), foreign capitalists will increase the level of their investment in the home country,

(ii) if \( -\gamma \frac{1}{1} < 0 \), foreign capitalists will withdraw their capital from the home country, assuming that under the initial assumption of unrestricted capital movements, the level of foreign investment is positive. If this initial level of foreign investment is zero then there will be no relocation of foreign-owned capital as the home country is not permitted to lend its capital.
Since good 2 is capital-intensive \((\gamma > 0)\), \(r_1 < 0\) and \(r > 0\), \(-\frac{\gamma y_1}{r}\) is positive and capital will flow into the home country. As capital flows in, the output of good 2 rises at home and falls abroad whilst the output of good 1 falls at home and rises abroad. This results from the Rybczynski Theorem [32]. The excess demand for good 1 in the foreign country falls. Therefore as capital continues to flow into the home country, exports of good 1 continue to fall until they eventually equal zero. At this point the optimal tariff and tax become zero and capital ceases to move between countries. The home country now exports good 2 to pay for the foreign earnings on investment in the home country. These exports also satisfy an excess demand for good 2 which has been created in the foreign country as a result of the outflow of capital to the home country.

That is, in the case where both countries are incompletely specialized in production, free trade in goods and unrestricted capital movements are the optimal policies for the borrowing country to adopt. However, if initially the borrowing country is an exporter of its labour-intensive good it is optimal for the home country to subsidise exports but to tax foreign earnings until the foreign country's production of the labour-intensive good satisfies its demand, when the free trade in goods and unrestricted capital movements again become the optimal policy.

These results agree with those put forward by Nadel [26] for the static case. However, Nadel considers the extra problem of the tariff imposed on all exports to the lending country. From the formula developed for the optimal tax and tariff, the idea that all exports should be subjected to the tariff does not arise. This is because as soon as exports of good 1 - the lending country's natural export under conditions of immobility of factors of production - fall to
zero, it is no longer optimal to impose a tariff.

Thus the optimal commercial and fiscal policies of the borrowing country are those of free commodity trade and unrestricted movement of capital internationally, except when under these conditions the borrowing country pays the earnings on foreign investment by exporting the good which as defined above, is the lending country's natural export. This optimal growth path eventually takes the home country to its optimal endpoint where the level of foreign investment is zero and the terms of trade are such that $R$ equals $\delta$.

IV

If the more realistic assumption of restricted mobility of capital internationally is introduced into the above model in which the home country is not permitted to lend abroad, international capital movements would cease after the first time period unless growth is also introduced in the foreign country. This has been discussed in Chapter VIII. However the introduction of growth in the foreign country complicates the international economy further and thus makes the analysis of the home country's growth a little more difficult.

When the foreign country is completely specialized the effect of growth on $r^*_{Kf}$ is composed of two forces. The increase in $\kappa^*$ forces $r^*_{Kf}$ to fall but as capital is relocated between the two countries so as to maintain the equality between $r$ and $r^*_{Kf}$, $r^*_{Kf}$ rises along the curve $r^*_{Kf} = r^*_{Kf}(K_f)$, given $\kappa^*$. The total effect of growth in the foreign country on the rate of return to capital in the foreign country can be either positive, negative or zero. This has been discussed in detail for the one-commodity model in the previous chapter.
If the foreign country's production is incompletely specialized, growth in the foreign country does not affect \( r^* \) directly but indirectly by increasing the output of the capital-intensive good which forces its commodity terms of trade to change so as to clear the goods markets. This movement in the terms of trade changes the rate of return on capital in the foreign country. If good 1 is capital-intensive abroad, growth in the foreign country increases its production of good 1 which forces the terms of trade to fall and thus the rate of return on capital in the foreign country decreases. Foreign-owned capital is then re-distributed between the two countries so that \( r^* \) and \( r \) remain equal. The total effect of growth on the rate of return on capital in the foreign country can again be positive, negative or zero. A similar effect can be shown to exist for the case in which good 1 is labour-intensive.

Defining goods 1 and 2 as above, the foreign country's capital stock can be increased either by accumulating part of its output of good 2 if its production is incompletely specialized or by accumulating part of its imports of good 2. The stock of capital employed in the home country can be increased either by current domestic investment or by accepting more foreign investment. However, because it is assumed that the mobility of capital is restricted internationally, the level of foreign investment can only be reduced either by allowing it to depreciate or by buying it out.

If it is assumed here as it was in the previous chapter that the foreign country is growing exponentially at the rate \( \alpha \), then the introduction of restricted mobility of capital between countries modifies the results obtained in sections II and III of this chapter in the same way as its introduction modified the results of the
previous chapter. That is, instead of adjusting the level of foreign investment instantaneously so that \( r^* = (1 - \tau_{opt})(R - \delta) \), the economy is moved as quickly as possible to that position and then, as before, along this curve to the optimal endpoint. If the level of foreign investment exceeds that for which \( r^* = (1 - \tau_{opt})(R - \delta) \), given the stock of domestically-owned capital, then it is optimal to reduce \( K_f \) by buying it out. If the stock of foreign-owned capital is below that for which \( r^* = (1 - \tau_{opt})(R - \delta) \) given \( K_d \), then it is optimal to accept all of the capital which the foreign country offers for investment in the home country.

If good 1 is assumed to be an investment good as well as a consumption good, the home country can increase its capital stock by current domestic investment, by accepting foreign investment or by importing capital from abroad. Then, in the above model, the home country's demand for imports of good 1 is either for consumption or for investment purposes.

As before it is optimal for the home country to move as quickly as possible to the position where \( r^* = (1 - \tau_{opt})(R - \delta) \) and then along this curve to the endpoint. However it should be noted that without specific formulation of the production functions it is impossible to determine the exact composition of the home country's current investment along the optimal growth path. Once the allocation to consumption along the growth path is known, the balance of the home country's output net of foreign earnings is allocated to current domestic investment. This investment can be composed of good 1 or good 2 or a combination of the two, depending on the specific production functions. However the total domestically-owned capital stock is independent of which good is accumulated. If for example, it is assumed that capital in the form of good 1 or good 2 can be used
just as efficiently to produce either good 1 or good 2 then from the accumulation point of view the two goods can be classified as one single good. This then reduces the problem to that of the one-good type, as far as the accumulation process is concerned.

The formulae for the optimal commercial and fiscal policies are unaffected by the introduction of restricted mobility of capital between countries and of growth in the lending country. However, the introduction of growth in the foreign country may effect the position of the curve \( r^* = (1 - t_{opt})(R - \delta) \) on the phase diagram, with the result that the optimal absolute level of foreign investment may be greater than or less than that determined in the model where the foreign country's total capital stock is constant.

It is assumed in this chapter that labour is immobile between industries, while the standard international trade model assumes perfect mobility of labour within each country. However, it should be noted that the signs of the optimal tariff and taxation policies put forward by Jones [13], Kemp [20] and Nadel [26] remain the same under this different assumption of immobility of labour between industries.

When the country's production is completely specialized these two assumptions are identical but when the country's production is incompletely specialized these two assumptions do differ. A better analysis of this case, where the production is incompletely specialized in either country, would have to be carried out under the assumption

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8. The phase diagram is that presented in sections II and III depending on the pattern of specialization and trade.
of perfect mobility of labour within each country. This however
would involve the use of a two-sector growth model with international
capital movements and international trade. Before this analysis
can be attempted it will be necessary to introduce international
capital movements alone into a two-sector growth model. To date
no such attempt has been made and this is probably because the
introduction of international capital movements complicates the two-
sector growth model in such a way that any analysis of the model
appears very difficult.
Conclusion

The foregoing dynamic analysis of international capital movements is composed of two parts. International capital movements have been examined firstly, in the form of foreign borrowing and secondly, in the form of foreign direct investment. The results of each model have been discussed and comparisons have been made between these results and (i) the results of other similar analyses which are included in the survey Chapters I, II and III and (ii) the results of models in which the international capital movements are assumed to take the same form. However, no comparison has been made between the results derived from models where international capital movements take the form of foreign borrowing and those derived from models where international capital movements take the form of foreign direct investment. Such a comparison can be made by discussing any of the models in Chapter VIII and the corresponding models in Chapters V, VI and VII.

As the nature of the objective function plays an important role in these models, compare the results of the models of Chapter VI with those of the corresponding models of Chapter VIII in which the objective functions are identical. In these models the only difference between international capital movements in the form of foreign direct investment and those in the form of foreign borrowing is the rate of return on the foreign-owned capital in the home country. As discussed in Chapter IV the rate of return on foreign direct investment is the same as that on the domestically-owned capital. That is, the rate of return equals the marginal product of total capital employed in the home country, whereas the rate of return on foreign borrowed capital depends on the level of foreign borrowing. Thus there is no difference in the structure of these corresponding models and the resulting patterns of optimal growth paths.
for the economy are similar.

In the real world, however, the nature of foreign borrowing and that of foreign direct investment are different. This difference arises principally because much foreign borrowing is at a fixed interest rate and because it can be used for the import of consumption goods as well as of capital. However as neither of these implications are contained in these borrowing models the results of the corresponding models of Chapters VI and VIII are similar.

A more realistic analysis of international capital movements in the form of foreign borrowing would have to be carried out within the standard two-country, two-good, two-factor of production trade model. However, as stated in Chapter IX, before any such analysis can be made, it will be necessary to introduce international capital movements into the standard two-sector growth model.

One important difference in the results of these models can be found in the levels of per capita foreign-owned capital which are optimal in the long-run. In the case of foreign direct investment its optimal per capita level at the endpoint is zero, whereas in the case of foreign borrowing it is always optimal to have a positive amount of foreign-owned capital per head in the borrowing country (assuming \( w > r(O) > 0 \)). However, in the models of Chapter VII, in which the rate of interest payable on foreign borrowing is assumed to be a function of the absolute level of foreign borrowing, the optimal level of per capita foreign borrowing falls to zero in the long-run. Therefore, it appears that the optimal level of foreign-owned capital per head employed in the home country goes to zero in the long-run.

Static analysis gives the level of foreign-owned capital which it is
optimal for the home country to employ for any given level of
domestically-owned capital. This result corresponds to a point of
the optimal capital accumulation path derived by dynamic analysis.
Therefore static analysis, which is simpler, is useful but from
static analysis it is not possible to determine the optimal long-
run endpoint and the pattern of optimal growth paths which take the
economy to this endpoint. This information can be determined by
the use of dynamic analysis although the analysis becomes much more
complex.

Although this work on the theory of international capital movements
may not have any direct practical application, it is hoped that
some useful information has been obtained and that the results may
provide some insight into the behaviour of international capital
movements in the real world.
Appendix No. 1.

Shapes of the curves \( f' = w \) and \( f' = \phi \)

To find the shapes of \( f' = w \) and \( f' = \phi \), differentiate each totally. Then

\[
\begin{align*}
\frac{d k_d}{d f} & |_{f' = w} = -1 \\
\frac{d k_f}{d k} & |_{f' = \phi} = -\frac{f''}{f'' - \phi'}
\end{align*}
\]

and \( 0 > \frac{d k_f}{d k} |_{f' = \phi} > -1 \).

Also \( f' = \phi \) lies below \( k_f = \gamma \) because at \( k_f = \gamma, \phi = \infty \) but \( f' = \infty \) only at \( k = 0 \).

Properties of the Policies

Policy A

\[ i_d = 0; \quad i_f = 0; \quad c = F(>0) \]

From (29) to (33) and (35)

\[ p \geq 0; \quad q \geq 0; \quad v = 0; \quad t = 0; \quad \rho = 1. \]

Substituting into (26) to (28)

\[ \lambda_1 \leq 1; \quad \lambda_2 \leq 0. \]

From (24) and (25)

\[ \dot{\lambda}_1 = -(f' - \lambda_1 w); \quad \dot{\lambda}_2 = -(f' - \phi - \lambda_2 w). \]

Slope of path of policy A:

Along A, \( \dot{k}_d = -\omega k_d \); \( \dot{k}_f = -\omega k_f \), so that

\[ \frac{d k_f}{d k} = \frac{k_f}{k_d}. \]

That is, the path of A is a ray which passes through the origin.

190.
Policy B

\[ i_d = 0 ; \quad 0 < i_f < V ; \quad c = F \]

Then from equations (26) to (33) and (35)

\[ \lambda_1 \leq 1 ; \quad \lambda_2 = 0 . \]

From (24) and (25)

\[ \dot{\lambda}_1 = -[f' - \lambda_1 \omega] ; \quad \dot{\lambda}_2 = -[f' - \phi] . \]

But \( \lambda_2 = 0 \) so that \( \dot{\lambda}_2 = 0 \).

Hence along policy B, \( f' = \phi \) which has a negatively sloped path.

Therefore, this policy will be consistent only if \( i_f > wk_f \).

Furthermore if policy B is to take the system to the endpoint then it must be optimal in the region \( f' < \omega \). But if along policy B in this region \( \lambda_1 \) becomes equal to one then \( \dot{\lambda}_1 \) is positive.

That is, if \( \lambda_1 = 1 \) at any point on the path of policy B in this region then \( \lambda_1 \) becomes greater than one and thus policy B is no longer the optimal policy. Therefore, if policy B is to take the system to the endpoint, \( \lambda_1 \) must be less than one and \( \dot{\lambda}_1 \) must be positive along its path in the region \( f' < \omega \).

Policy C

\[ i_d = 0 ; \quad i_f = V ; \quad c = F \]

From equations (26) to (33) and (35)

\[ \lambda_1 \leq 1 ; \quad \lambda_2 \geq 0 . \]

From (24) and (25)

\[ \dot{\lambda}_1 = -[f' - \lambda_1 \omega] ; \quad \dot{\lambda}_2 = -[(f' - \phi) - \lambda_2 \omega] . \]

Policy D

\[ i_d = F ; \quad i_f = 0 ; \quad c = 0 \]

From equations (26) to (33) and (35)

\[ \lambda_1 \geq 1 ; \quad \lambda_2 \leq 0 . \]

From (24) and (25)

\[ \dot{\lambda}_1 = -\lambda_1 [f' - \omega] ; \quad \dot{\lambda}_2 = -[\lambda_1 (f' - \phi) - \lambda_2 \omega] . \]
Policy E

\[ i_d = F; \quad 0 < i_f < V; \quad c = 0 \]

From equations (26) to (33) and (35)

\[ \lambda_1 \geq 1; \quad \lambda_2 = 0. \]

From (24) and (25)

\[ \dot{\lambda}_1 = -\lambda_1 [f' - w]; \quad \dot{\lambda}_2 = -\lambda_1 [f' - \phi]. \]

But \( \lambda_2 = 0 \), therefore \( \dot{\lambda}_2 = 0 \) and \( f' = \phi \) which has a negatively sloped path. Therefore, this policy will be consistent only if \( i_f < \omega k_f \). For policy E to lie along \( f' = \phi \) the slope of E must equal that of \( f' = \phi \), so that

\[ \frac{df}{dk} \bigg|_{E} = \frac{i_f - \omega k_f}{F - \omega k_d}, \quad \frac{df}{dk} \bigg|_{f' = \phi} = \frac{-f''}{f''' - \phi}. \]

Then policy E will lie along \( f' = \phi \) only if

\[ i_f = \frac{-f''(F - \omega k) + \omega k_f f' + \phi'}{f''' - \phi}. \]

But along policy E \( i_f \) is positive. Therefore policy E can only lie along that part of \( f' = \phi \) where \( f''(F - \omega k) + \omega k_f f' + \phi' > 0. \)

Then using a similar argument as for policy B above, it will be found that if policy E is the optimal policy which takes the system to the endpoint then \( \lambda_1 \) must be greater than one along its path in the region \( f' > w \).

Policy F

\[ i_d = F; \quad i_f = V; \quad c = 0 \]

From equations (26) to (33) and (35)

\[ \lambda_1 \geq 1; \quad \lambda_2 \geq 0. \]

From equations (24) and (25)

\[ \dot{\lambda}_1 = -\lambda_1 [f' - w]; \quad \dot{\lambda}_2 = -[\lambda_1 (f' - \phi) - \lambda_2 \omega]. \]
Policy G

0 < i_d < F ; i_f = 0 ; 0 < c < F
From equations (26) to (33) and (35)

\[ \lambda_1 = 1 ; \lambda_2 \leq 0 . \]
From (24) and (25)

\[ \dot{\lambda}_1 = -(f' - w) ; \dot{\lambda}_2 = -(f' - \phi) - \lambda_2 \omega . \]

But \( \lambda_1 = 1 \), therefore \( \dot{\lambda}_1 = 0 \) and \( f' = w \) which has a negatively sloped path. Therefore this policy is consistent only if \( i_d > \omega k_d \).

Then if policy G is to take the system to the endpoint, it must be optimal in the region \( f' < \phi \). If \( \lambda_2 \) becomes zero along policy G in this region, then \( \dot{\lambda}_2 \) is positive and policy G is no longer the optimal policy. Therefore, if policy G is to take the system to the endpoint, then \( \lambda_2 \) must be less than zero and \( \dot{\lambda}_2 \) must be positive along its path in the region \( f' < \phi \).

Policy H

0 < i_d < F ; 0 < i_f < V ; 0 < c < F
From equations (26) to (33) and (35)

\[ \lambda_1 = 1 ; \lambda_2 = 0 . \]
From (24) and (25)

\[ \dot{\lambda}_1 = -(f' - w) ; \dot{\lambda}_2 = -(f' - \phi) . \]

But \( \lambda_1 = 1 \), therefore \( \dot{\lambda}_1 = 0 \) and \( \lambda_2 = 0 \), therefore \( \dot{\lambda}_2 = 0 \).

That is, this policy is optimal only at the endpoint where \( f' = w = \phi \).

Policy I

0 < i_d < F ; i_f = V ; 0 < c < F
From equations (26) to (33) and (35)

\[ \lambda_1 = 1 ; \lambda_2 \geq 0 . \]
From (24) and (25)

\[ \dot{\lambda}_1 = -(f' - w) ; \dot{\lambda}_2 = -(f' - \phi) - \lambda_2 \omega . \]
But $\lambda_1 = 1$ therefore $\dot{\lambda}_1 = 0$ and $f' = \omega$ which has a negatively sloped path. Therefore this policy is consistent only if $i_d < \omega k_d$. Then using similar argument as for $G$ above it will be found that if policy $I$ is the optimal policy which takes the system to the endpoint then $\lambda_2$ must be greater than zero and $\dot{\lambda}_2$ must be negative along its path in the region $f' > \phi$.

Possible Policy Switches

From policies $G$ and $I$, it is known that $\lambda_1 = 1$ along that part of $f' = \omega$ which lies in the region $F > \omega k_d$. Also from policies $E$ and $B$, it is known that $\lambda_2 = 0$ along $f' = \phi$. Then using the fact that the auxiliary variables are continuous functions of time and of the state variables and that the necessary conditions for optimality are

\[
\dot{\lambda}_1 = -[\rho f' - \lambda_1 \omega] \quad \text{and} \quad \dot{\lambda}_2 = -[\rho (f' - \phi) - \lambda_2 \omega],
\]

all possible policy switches can be determined.

In the region $f' \geq \phi$, $\lambda_2 \leq 0$ if $\lambda_2 \leq 0$. Therefore, if policies with $\lambda_2 \leq 0$ are optimal in this region they cannot switch to any other policy in this region nor take the system from this region to the endpoint.

Therefore policies $A$ and $D$ are not optimal policies in the region $f' > \phi$. Similarly, policies $C$ and $F$ are not optimal policies in the region $f' < \phi$.

Policy $A$

\[
\lambda_1 \leq 1; \quad \lambda_2 \leq 0 \quad \Rightarrow \quad \dot{\lambda}_1 = -[f' - \lambda_1 \omega]; \quad \dot{\lambda}_2 = -[(f' - \phi) - \lambda_2 \omega].
\]
\[
\begin{align*}
\dot{\lambda}_1 &= 0 \quad \text{if } f' < w \\
\dot{\lambda}_1 &= 0 \quad \text{if } f' = w \\
\dot{\lambda}_1 &= < 0 \quad \text{if } f' > w \\
\dot{\lambda}_2 &= 0 \quad \text{if } f' = \phi \\
\dot{\lambda}_2 &= > 0 \quad \text{if } f' < \phi \\
\dot{\lambda}_2 &= < 0 \quad \text{if } f' = \phi \\
\dot{\lambda}_2 &= ? \quad \text{if } f' < \phi 
\end{align*}
\]

If \( \lambda_1 = 1 \)

\[
\begin{align*}
\dot{\lambda}_1 &= 0 \quad \text{if } f' = w \\
\dot{\lambda}_1 &= < 0 \quad \text{if } f' > w \\
\dot{\lambda}_1 &= < 0 \quad \text{if } f' = \phi \\
\dot{\lambda}_2 &= ? \quad \text{if } f' < \phi 
\end{align*}
\]

If \( \lambda_1 < 1 \)

Then from \( \dot{\lambda}_1 \) above, \( \lambda_1 \) cannot equal zero anywhere in the region \( f' \leq \phi \) other than at \( f' = \phi \), if policy A is to switch into either policy B or policy C. Therefore in the region \( f' < \omega \), along policy A \( \lambda_1 < 1 \). Then from \( \dot{\lambda}_1 \) above, policy A cannot switch into policy D or E in this region \( f' > \omega \) nor take the system to the endpoint.

Consider all the paths of policy A which cut \( f' = \omega \) in the region \( F > \omega_{k1} \). Then from \( \dot{\lambda}_1 \) above, \( \lambda_1 \) cannot equal one anywhere in the region \( f' \leq \omega \) other than at \( f' = \omega \), if policy A is to switch into either policy B or policy C. Therefore in the region \( f' < \omega \), along policy A \( \lambda_1 < 1 \). Then from \( \dot{\lambda}_1 \) above, policy A cannot switch into policy D or E in this region \( f' > \omega \) nor take the system to the endpoint.

1. Later it will be shown that if policy D exists in this region \( f' < \omega \), \( \lambda_1 > 1 \) and never \( \lambda_1 = 1 \). Therefore, then neither A can switch into D nor D into B. That is, this cannot lead to the endpoint.
policy B or G. Therefore, in the region \( f' < \phi \) policy A has \( \lambda_2 < 0 \).

Consider all the paths of policy A which cut \( f' = \omega \) in the region \( F \leq \omega k_d \). In this region it is not known where \( \lambda_1 = 1 \). From \( \lambda_1 \) above, \( \lambda_1 \) can have the values

\[
\begin{align*}
\lambda_1 &< 1 \text{ in } f' < \omega \\
\lambda_1 &= 1 \text{ at } f' = \omega \\
\lambda_1 &< 1 \text{ in } f' > \omega
\end{align*}
\]

so that policy A is possible over the entire region or \( \lambda_1 \) is less than one initially in \( f' < \omega \) region, but somewhere in \( f' < \omega \), \( \lambda_1 = 1 \) and the policy is no longer possible beyond this point because \( \lambda_1 \) becomes greater than one.

In the region \( f' < \phi \), \( \lambda_2 < 0 \) along both policy A and policy D, it therefore seems possible that policy A will switch into Policy D somewhere in the region \( F \leq \omega k_d \).

\[
\begin{align*}
\text{If } \lambda_1 &= 1 \\
\dot{\lambda}_1 &= 0 \text{ if } f' = \omega \\
\dot{\lambda}_1 &= 0 \text{ if } f' < \omega \\
\end{align*}
\]

\[
\begin{align*}
\text{If } \lambda_1 &= 0 \text{ if } f' > \omega \\
\dot{\lambda}_1 &= 0 \text{ if } f' = \omega \\
\dot{\lambda}_1 &= 0 \text{ if } f' < \omega \\
\end{align*}
\]

Note: Only those paths of policy A which cut \( f' = \omega \) in the region \( F \leq \omega k_d \).
Consider the region in which $F > w_k$. Then from $\lambda_1$ above, $\lambda_1$ cannot equal one anywhere in the region $f' \geq \omega$ other than at $f' = \omega$ if the policy $D$ is to switch into either policy $E$ or policy $G$. Therefore in the region $f' > \omega$, along policy $D$, $\lambda_1 > 1$.

Then from $\lambda_1$ above, $D$ cannot switch into $A$ and $B$ in the region $f' < \omega$ nor take the system to the endpoint.

Consider the region in which $F \leq w_k$. Because in the region $f' > \omega$, $\lambda_1 < 0$ and at $f' = \omega$, $\lambda_1 = 0$, it is necessary that $\lambda_1$ be greater than one at all points in the region defined by $F \leq w_k$ and $f' \geq \omega$. But in the region $f' < \omega$, $\lambda_1 > 0$ therefore it is possible for $\lambda_1$ to be equal to one somewhere in this region so that $\lambda_1$ will be greater than one at $f' = \omega$.

From the discussion of policy $A$, $\lambda_1$ along $A$ becomes one somewhere in the shaded region.

Therefore it is possible that policy $A$ will switch into policy $D$ somewhere in the shaded region. 3

3. In this region policy $A$ cannot be optimal beyond the $\lambda_1 = 1$ point because $\lambda_1 > 0$ in this region.
Policy B

Policy B does not switch into any other policy but takes the system to the final endpoint.

Policy C

\[ \begin{align*}
\lambda_1 &\leq 1 ; \quad \lambda_2 \geq 0 \\
\dot{\lambda}_1 &= -[f' - \lambda_1 \omega] ; \quad \dot{\lambda}_2 = -[(f' - \phi) - \lambda_2 \omega] \\
\end{align*} \]

If \( \lambda_1 = 1 \)

\[ \begin{align*}
\dot{\lambda}_1 &= 0 \quad \text{if } f' < \omega \\
\dot{\lambda}_1 &= 0 \quad \text{if } f' = \omega \\
\dot{\lambda}_1 &< 0 \quad \text{if } f' > \omega \\
\end{align*} \]

If \( \lambda_1 < 1 \)

\[ \begin{align*}
\dot{\lambda}_1 &= \varphi \quad \text{if } f' < \omega \\
\dot{\lambda}_1 &= 0 \quad \text{if } f' = \omega \\
\dot{\lambda}_1 &< 0 \quad \text{if } f' > \omega \\
\end{align*} \]

If \( \lambda_1 < 0 \), \( \dot{\lambda}_1 < 0 \) for all \( f' \).

From \( \dot{\lambda}_1 \) above, \( \lambda_1 \) cannot equal one anywhere in the region \( f' \leq \omega \) except at \( f' = \omega \) if policy C is to switch into either policy B or I. 4 Therefore in the region \( f' < \omega \), along policy C \( \lambda_1 < 1 \).

Then from \( \dot{\lambda}_1 \) above, policy C cannot switch into either F or E in the region \( f' > \omega \) nor take the system to the endpoint.

\[ \begin{align*}
\dot{\lambda}_2 &= 0 \quad \text{if } f' = \phi \\
\dot{\lambda}_2 &< 0 \quad \text{if } f' > \phi \\
\end{align*} \]

---

4. Later it will be shown that if policy F is to be optimal in this region \( f' < \omega , \lambda_1 > 1 \) and never equal to one. Therefore, neither can C switch into F nor F into B.
If \( \lambda_2 > 0 \) \[ \begin{cases} \lambda_2 > 0 & \text{if } f' = \phi \\ \lambda_2 = 0 & \text{if } f' > \phi \end{cases} \]

Then from \( \lambda_2 \) above, \( \lambda_2 \) cannot equal zero anywhere in the region \( f' \geq \phi \) except at \( f' = \phi \) if policy C is to switch into either policy I or B. Therefore in the region \( f' > \phi \), policy C has \( \lambda_2 > 0 \).

**Policy D**

\[ \lambda_1 \geq 1; \lambda_2 \leq 0 \]

\[ \lambda_1 = -\lambda_1[f' - \omega]; \lambda_2 = -[\lambda_1(f' - \phi) - \lambda_2 \omega] \]

If \( \lambda_2 = 0 \), \( \lambda_2 = -\lambda_1[f' - \phi] > 0 \) in the whole region \( f' < \phi \).

Therefore, along D, \( \lambda_2 \neq 0 \) at any point within this region \( f' \leq \phi \) other than at \( f' = \phi \) if policy D is to switch into policies E and G and in fact, if it is to be optimal. Also, \( \lambda_2 \) must be less than zero if policy A is to switch into policy D.

**Policy E**

Policy E does not switch into any other policy but takes the system to the final endpoint.

**Policy F**

\[ \lambda_1 \geq 1; \lambda_2 \geq 0 \]

\[ \lambda_1 = -\lambda_1[f' - \omega]; \lambda_2 = -[\lambda_1(f' - \phi) - \lambda_2 \omega] \]

If \( \lambda_1 = 1 \) \[ \begin{cases} \lambda_1 > 0 & \text{if } f' < \omega \\ \lambda_1 = 0 & \text{if } f' = \omega \\ \lambda_1 < 0 & \text{if } f' > \omega \end{cases} \]

If \( \lambda_1 > 1 \) \[ \begin{cases} \lambda_1 > 0 & \text{if } f' < \omega \\ \lambda_1 = 0 & \text{if } f' = \omega \\ \lambda_1 < 0 & \text{if } f' > \omega \end{cases} \]
Then from $\dot{\lambda}_1$ above, $\lambda_1$ cannot equal one anywhere in the region $f' \geq \omega$ other than $f' = \omega$ if policy $F$ is to switch into policy $E$, policy $I$, or policy $D$. Therefore in the region $f' > \omega$, along policy $F$ $\lambda_1 > 1$. Then from $\dot{\lambda}_1$ above, $F$ cannot switch into $C$ or $B$ in the region $f' < \omega$ nor take the system to the endpoint.

If $\lambda_2 = 0$, $\lambda_2 > 0$ in the region $f' > \phi$. Therefore, along policy $F$ $\lambda_2 \neq 0$ at any point within this region $f' \geq \phi$ other than at $f' = \phi$ if policy $F$ is to switch into either policy $E$ or $I$ and in fact, if it is to be optimal. Policy $F$ will switch into policy $D$ at $f' = \phi$ only if part of $f' = \phi$ lies in the region $f''(f - w) + wk_2\phi' < 0$.

Policies $G$ and $I$ - same as for policies $B$ and $E$.

Shape of the curve $F = 0$

$$F = f(k_d + k_F) - r(k_F)k_F = 0$$

By total differentiation

$$\frac{d k_F}{dk_d} = -\frac{f'}{f' - \phi},$$

i.e. $\frac{d k_F}{dk_d}$ is

- $< 0$ if $f' > \phi$
- $> 0$ if $f' < \phi$
- $= \infty$ if $f' = \phi$
- $= 0$ if $\phi = \infty$

Because the slope of $F = 0$ is less than the slope of the $f' = \phi$ curve, if $F = 0$ lies below $f' = \phi$ initially it will remain below it for all $k_d$.

Also the endpoint is defined as the point where sustainable consumption per head is maximised. That is, net output is maximised.
over $k_d, k_f$.

Net output = $f - rk_f - wk_d = F - wk_d$

But $c \geq 0$, so that $F \geq wk_d \geq 0$.

Therefore, for the endpoint to be stable it must lie in the region $F \geq 0$. That is, it must be either on $F = 0$ or in the phase space between the origin and $F = 0$. But $F$ cannot be zero at the endpoint because this means that the curve must initially lie below the $f' = \phi$ curve and from the above, the curve must then remain below $f' = \phi$.

Therefore the $F = 0$ curve must lie beyond the endpoint and in fact must lie completely beyond the $f' = \phi$ curve or intersect it at the $k_f$ axis.

Thus the curve is rising everywhere until $k_f = \gamma$ where it becomes horizontal.

Shape of the $F = wk_d$ curve

By totally differentiating $f - rk_f - wk_d = 0$,

$$\frac{dk_f}{dk_d} = \frac{f' - w}{f' - \phi}.$$

Then $\frac{dk_f}{dk_d}$ is

- $< 0$ when $f' < \phi$, $f' < w$
- $= 0$ when $f' = w$
- $> 0$ when $f' < \phi$, $f' > w$
- $= \infty$ when $f' = \phi$

At $k_d = 0$, $F = wk_d$ intersects the curve $F = 0$. 
At $k_f = 0$,

Therefore, $F = \omega k_d$ curve cuts the $k_d$ axis at $\bar{f}$ in the region $f' < \omega$.

Shape of $f''(F - \omega k) + \omega k f' = 0$

Along this curve, $F - \omega k_d$ is greater than zero at all points except at $k_f = 0$ where it equals zero. Therefore this curve lies wholly below the $F - \omega k_d = 0$ curve and intersects it only at the $k_f = 0$ axis.

The fixed endpoint is defined such that sustainable consumption per head is maximised and at this endpoint there exists an optimal policy such that $\dot{k}_d = k_f = 0$. Therefore the curve $f''(F - \omega k) + \omega k f' = 0$ must lie above the endpoint.

At the $k_d = 0$ axis, both the slope of the curve and its intercept on the axis are somewhat indeterminate. However the important question is whether the curve cuts the $k_d = 0$ axis above or below the $f' = \phi$ line. For example:
Define the Hamiltonian

\[ H = H(x, \dot{x}, t) \]

and the Lagrangian

\[ L = T - V \]

The necessary and sufficient conditions for optimality are given by the Euler-Lagrange equations and the Hamiltonian equations. The variables \( x_0 \) and \( x_1 \) are continuous functions of time.

(b) The constraint satisfies the constraint qualification (Herron, et al. [5]) at any point of time and the Hamiltonian is non-negative subject to this constraint. That is:

\[ H(x, \dot{x}, t) \geq 0 \]
Appendix No. 2.

To minimise

$$\int_0^T (\dot{q} - \omega) dt$$

subject to

$$f(k) - r(k) k_f - C - i_d - i_d' \geq 0$$

$$k_d = i_{d_1} + i_{d_2} - w_{kd}$$

$$k_f = i_{f_1} - i_{d_2} - w_{kf}$$

$$c \geq 0$$

$$i_{d_1} \geq 0$$

$$i_{d_2} \geq 0$$

$$0 \leq i_f < V$$

Define the Hamiltonian

$$H = \left[ \dot{q} - v(c) \right] + \lambda_1 \left[ i_{d_1} + i_{d_2} - w_{kd} \right]$$

$$+ \lambda_2 \left[ i_{f_1} - i_{d_2} - w_{kf} \right]$$

and the Lagrangian

$$L = H + \rho \left[ f(k) - r(k) k_f - C - i_{d_1} - i_{d_2} \right] + tc$$

$$+ p i_{d_1} + m i_{d_2} + q i_f + v[v - i_f]$$

The necessary conditions for an optimal solution are

(a) that the auxiliary variables $\lambda_1$ and $\lambda_2$ are continuous functions of time given by

$$\dot{\lambda}_1 = -\frac{3L}{d^d} = -\left[ p f' - \lambda_1 \omega \right]$$

and

$$\dot{\lambda}_2 = -\frac{3L}{d^f} = -\left[ p f' - \phi - \lambda_2 \omega \right]$$

(b) The constraint satisfies the Constraint Qualification (Arrow et al [4]) at any point of time and the Hamiltonian is maximised subject to this constraint. That is:

$$\frac{\partial L}{\partial c} = 1 + t - \rho = 0$$

204.
\[
\begin{align*}
\frac{\partial L}{\partial d_1} &= \lambda_1 + p - \rho = 0 \\
\frac{\partial L}{\partial d_2} &= \lambda_1 - \lambda_2 - \rho + m = 0 \\
\frac{\partial L}{\partial i_f} &= \lambda_2 + q - v = 0 \\
p &\geq 0; \quad p_{d_1} = 0 \\
q &\geq 0; \quad q_{i_f} = 0 \\
v &\geq 0; \quad v[V - i_f] = 0 \\
t &\geq 0; \quad t_c = 0 \\
m &\geq 0; \quad m_{d_2} = 0 \\
\rho &\geq 0; \quad \rho[f - r_k - c - i_{d_1} - i_{d_2} - i_d] = 0.
\end{align*}
\]

These necessary conditions are also sufficient for optimality from Arrow [3], lecture 1, proposition 4.

As \( t \geq 0 \) and \( \rho = 1 + t, \rho > 0 \). Therefore the budget constraint is satisfied with equality and the possible policy combinations are

<table>
<thead>
<tr>
<th>Policy</th>
<th>( d_1 )</th>
<th>( d_2 )</th>
<th>( c )</th>
<th>( i_f )</th>
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<tr>
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<td>0</td>
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<td>C</td>
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<td>(&lt;F)&gt;</td>
<td>V</td>
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</tbody>
</table>

where \( F = f(k) - r(k_f)k_f \)

**Properties of the policies**

**Policy A**

\[
\begin{align*}
  i_{d_1} &= 0; \quad i_{d_2} = 0; \quad c = F; \quad i_f = 0 \\
  \text{Then } p &\geq 0; \quad m \geq 0; \quad t = 0; \quad q \geq 0; \quad v = 0, \quad \rho = 1 \\
  \text{and } \lambda_1 &= \rho - p \leq 1 \\
  \lambda_2 &= -q \leq 0, \text{ so that } \\
  \lambda_1 - \lambda_2 &= \rho - p + q. \\
  \text{But } \lambda_1 - \lambda_2 &= p - m. \\
  \text{Therefore } p - q &= m, \\
  \text{or } p &\geq q.
\end{align*}
\]

Also \( \lambda_1 = -[f' - \lambda_1 \omega] \) and \( \lambda_2 = -[(f' - \phi) - \lambda_2 \omega] \).

**Policy B**

\[
\begin{align*}
  i_{d_1} &= 0; \quad i_{d_2} = 0; \quad c = F; \quad 0 < i_f < V \\
  \text{Then } p &\geq 0; \quad m \geq 0; \quad t = 0; \quad \rho = 1; \quad v = q = 0, \\
  \lambda_1 &= 1 - p \leq 1 \\
  \lambda_2 &= 0 \\
  \lambda_1 - \lambda_2 &= 1 - m.
\end{align*}
\]

Therefore along policy B, \( p = m \).

But since \( \lambda_2 = 0 \), \( \lambda_2 = 0 \) and therefore policy B is only consistent
if it lies along $\lambda_2 = 0$ or along $f' = \phi$.

**Policy C**

$$i_{d_1} = 0; \quad i_{d_2} = 0; \quad c = F; \quad i_f = V$$

Then $p \geq 0; \quad m \geq 0; \quad t = 0; \quad \rho = 1; \quad q = 0; \quad v \geq 0$.

and $\lambda_1 = 1 - p \leq 1$,

$\lambda_2 = v \geq 0$,

$\lambda_1 - \lambda_2 = 1 - m$.

But $\lambda_1 - \lambda_2 = 1 - (p + v)$.

Therefore along policy C, $p + v = m$.

**Policy D**

$$i_{d_1} = F; \quad i_{d_2} = 0; \quad c = 0; \quad i_f = 0$$

Then $p = 0; \quad m \geq 0; \quad t \geq 0; \quad \rho \geq 1; \quad q \geq 0; \quad v = 0$

and $\lambda_1 = \rho \geq 1$,

$\lambda_2 = -q \leq 0$,

$\lambda_1 - \lambda_2 = \rho - m$.

But $\lambda_1 - \lambda_2 = \rho + q$.

Therefore along policy D, $q = -m$.

But this is impossible unless $q = m = 0$, that is, $\lambda_2 = 0$ and thus $\lambda_2 = 0$. Then policy D is only consistent if it lies along the curve $f' = \phi$.

**Policy E**

$$i_{d_1} = F; \quad i_{d_2} = 0; \quad c = 0; \quad 0 < i_f < V$$

Then $p = 0; \quad m \geq 0; \quad t \geq 0; \quad \rho \geq 1; \quad v = q = 0$

and $\lambda_2 = \rho \geq 1$,

$\lambda_2 = 0$,

$\lambda_1 - \lambda_2 = \rho$.

But $\lambda_1 - \lambda_2 = \rho - m$. 


Therefore along policy E,  \( m = 0 \).

Since \( \lambda_2 = 0 \), \( \lambda_2 = 0 \). Then policy E is only consistent if it lies along the curve \( f' = \phi \).

**Policy F**

\[
\begin{align*}
    i_{d_1} &= F; & i_{d_2} &= 0; & c &= 0; & i_f &= V
\end{align*}
\]

Then \( p = 0; \ m \geq 0; \ t \geq 0; \ \rho \geq 1; \ q = 0; \ v \geq 0 \)

and

\[
\begin{align*}
    \lambda_1 &= \rho \geq 1,
    
    \lambda_2 &= v \geq 1,
    
    \lambda_1 - \lambda_2 &= \rho - m.
\end{align*}
\]

But \( \lambda_1 - \lambda_2 = \rho - v \).

Therefore along policy F, \( v = m \).

**Policy G**

\[
\begin{align*}
    i_{d_1} > 0; & & i_{d_2} &= 0; & c &= 0; & i_f &= 0
\end{align*}
\]

Then \( p = 0; \ m \geq 0; \ t = 0; \ \rho = 1; \ q \geq 0; \ v = 0 \)

and

\[
\begin{align*}
    \lambda_1 &= 1,
    
    \lambda_2 &= -q \leq 0,
    
    \lambda_1 - \lambda_2 &= 1 - m.
\end{align*}
\]

But \( \lambda_1 - \lambda_2 = 1 + q \).

Therefore along policy G, \( q = -m \) which is impossible unless both \( q \) and \( m \) are zero. That is, policy G is only consistent if

\[
\begin{align*}
    \lambda_1 &= 1 \quad \text{and} \quad \lambda_2 = 0,
\end{align*}
\]

which makes policy G only consistent at the endpoint. But since \( \dot{k}_f < 0 \) and \( \dot{k}_d > 0 \), this policy is non-optimal at the endpoint which is defined by \( \dot{k}_d = \dot{k}_f = 0 \).

**Policy H**

\[
\begin{align*}
    i_{d_1} > 0; & & i_{d_2} &= 0; & c &= 0; & 0 < i_f < V
\end{align*}
\]

Then \( p = 0; \ m \geq 0; \ t = 0; \ q = v = 0, \rho = 1 \)
and

\[ \lambda_1 = 1 , \]
\[ \lambda_2 = 0 , \]
\[ \lambda_1 - \lambda_2 = 1 - m . \]

Therefore \( m = 0 \) along policy \( H \). But policy \( H \) is only optimal at the endpoint where \( i_{d_1} \) and \( i_f \) are such as to maintain \( k_d = k_f = 0 \).

**Policy I**

\[ i_{d_1} > 0 ; \quad i_{d_2} = 0 ; \quad c > 0 ; \quad i_f = v \]

Then \( p = 0 ; \quad m \geq 0 ; \quad t = 0 ; \quad \rho = 1 ; \quad v \geq 0 ; \quad q = 0 \)

and

\[ \lambda_1 = 1 , \]
\[ \lambda_2 = v \geq 0 , \]
\[ \lambda_1 - \lambda_2 = 1 - m . \]

But \( \lambda_1 - \lambda_2 = 1 - v \).

Therefore along policy I, \( m = v \).

But \( \lambda_1 = 1 \) and therefore \( \lambda_1 = 0 \). That is, policy I is only consistent along \( f' = \omega \).

**Policy J**

\[ i_{d_1} = 0 ; \quad i_{d_2} = F ; \quad c = 0 ; \quad i_f = 0 \]

Then \( p \geq 0 ; \quad m = 0 ; \quad t \geq 0 ; \quad \rho \geq 1 ; \quad q \geq 0 ; \quad v = 0 \)

and

\[ \lambda_1 = \rho - p , \]
\[ \lambda_2 = -q , \]
\[ \lambda_1 - \lambda_2 = \rho . \]

Therefore \( p = q \) along policy J.

Also \( \lambda_1 = -[\rho f' - \lambda_1 \omega] \)

and \( \lambda_2 = -[\rho (f' - \phi) - \lambda_2 \omega] \).

The slope of the path of policy J is given by

\[ \frac{dk_f}{dt} = \frac{k_f}{t} = -\frac{(F + \omega_k F)}{F - \omega k} \leq -1 . \]
Therefore policy J is steeper than the curve \( f' = \omega \).

Policy K

\[
\begin{align*}
& i_{d1} = 0 \quad i_{d2} = F \quad c = 0 \quad i_f > 0 \\
\end{align*}
\]

Then \( p \geq 0 \), \( m = 0 \), \( t \geq 0 \), \( v = q = 0 \), \( \rho \geq 1 \)
and
\[
\begin{align*}
& \lambda_1 = \rho - p \\
& \lambda_2 = 0 \\
& \lambda_1 - \lambda_2 = \rho - p
\end{align*}
\]

But \( \lambda_1 - \lambda_2 = \rho \).

Therefore \( p = 0 \) and \( \lambda_1 \geq 1 \).

Since \( \lambda_2 = 0 \), \( \lambda_2 = 0 \) and thus policy K is only internally consistent along \( f' = \phi \). As \( \lambda_1 \) is greater than unity only in the region \( f' > \omega \) and \( f' = \phi \) is negatively sloped, policy K is only consistent if \( i_f - F < \omega k_f \).

Policy L

\[
\begin{align*}
& i_{d1} = 0 \quad i_{d2} = F \quad c = 0 \quad i_f = V \\
\end{align*}
\]

Then \( p \geq 0 \), \( m = 0 \), \( t \geq 0 \), \( q = 0 \), \( v \geq 0 \), \( \rho \geq 1 \)
and
\[
\begin{align*}
& \lambda_1 = \rho - p \\
& \lambda_2 = v \\
& \lambda_1 - \lambda_2 = \rho - p - v
\end{align*}
\]

But \( \lambda_1 - \lambda_2 = \rho \).

Therefore \( p = v = 0 \). That is \( \lambda_1 = 1 \) and \( \lambda_2 = 0 \). Then policy L is only consistent at the endpoint.

Policy M

\[
\begin{align*}
& i_{d1} < F \quad i_{d2} < F \quad c = 0 \quad i_f = 0 \\
\end{align*}
\]

Then \( p = 0 \), \( m = 0 \), \( t \geq 0 \), \( \rho \geq 1 \), \( q \geq 0 \), \( v = 0 \)
and
\[
\begin{align*}
& \lambda_1 = \rho \\
& \lambda_2 = -q \\
& \lambda_1 - \lambda_2 = \rho
\end{align*}
\]
Therefore \( q = 0 \) and thus \( \lambda_2 = 0 \).

Also \( \lambda_1 = -\lambda_1[f' - \omega] \)
and \( \lambda_2 = -[\rho(f' - \phi)] \). Since \( \lambda_2 = 0 \), \( \lambda_2 = 0 \) and thus policy M is only internally consistent along \( f' = \phi \). But \( f' = \phi \) has a negative slope and therefore this policy is only consistent if \( F > \omega k_d \).

**Policy N**

\[
\begin{align*}
i_1 &< F; \quad i_{d_1} < F; \quad c = 0; \quad i_f > 0 \\
\text{Then} \quad p = 0; \quad m = 0; \quad t \geq 0; \quad \rho \geq 1; \quad q = v = 0
\end{align*}
\]

and

\[
\begin{align*}
\lambda_1 &= \rho, \\
\lambda_2 &= 0, \\
\lambda_1 - \lambda_2 &= \rho.
\end{align*}
\]

Since \( \lambda_2 = 0 \), \( \lambda_2 = 0 \) and policy N is only internally consistent along \( f' = \phi \).

**Policy 0**

\[
\begin{align*}
i_{d_1} &< F; \quad i_{d_2} < F; \quad c = 0; \quad i_f = V \\
\text{Then} \quad p = 0; \quad m = 0; \quad t \geq 0; \quad \rho \geq 1; \quad q = 0; \quad v \geq 0
\end{align*}
\]

and

\[
\begin{align*}
\lambda_1 &= \rho, \\
\lambda_2 &= v, \\
\lambda_1 - \lambda_2 &= \rho.
\end{align*}
\]

Therefore \( v = 0 \) and \( \lambda_2 = 0 \). That is \( \lambda_2 = 0 \) and policy 0 is only consistent along \( f' = \phi \).

**Policy P**

\[
\begin{align*}
i_{d_1} &= 0; \quad i_{d_2} < F; \quad c < F; \quad i_f = 0 \\
\text{Then} \quad p \geq 0; \quad m = 0; \quad t = 0; \quad \rho = 1; \quad q \geq 0; \quad v = 0
\end{align*}
\]

and

\[
\begin{align*}
\lambda_1 &= 1 - p, \\
\lambda_2 &= -q, \\
\lambda_1 - \lambda_2 &= 1.
\end{align*}
\]
But $\lambda_1 - \lambda_2 = 1 - p + q$.

Therefore $p = q$ along policy $P$.

Also $\lambda_1 - \lambda_2 = 1$ and therefore $\lambda_1 - \lambda_2 = 0$. Then policy $P$ is only consistent along $\phi = \omega$. But $\phi$ equals $\omega$ only at $k_f = k_d$.

Policy $P$ is therefore non-optimal because at the endpoint $k_f = k_d = 0$ and here $k_f < 0$.

Policy $Q$

\[
\begin{align*}
 i_d^1 &= 0; i_d^2 = F; c < F; i_f > 0 \\
 \text{Then } p &\geq 0; m = 0; t = 0; \rho = 1; v = q = 0 \\
 \text{and} & \\
 \lambda_1 &= 1 - p, \\
 \lambda_2 &= 0, \\
 \lambda_1 - \lambda_2 &= 1.
\end{align*}
\]

But $\lambda_1 - \lambda_2 = 1 - p$.

Therefore $p = 0$ and $\lambda_1 = 1$ along policy $Q$. That is, policy $Q$ is only internally consistent at the endpoint.

Policy $R$

\[
\begin{align*}
 i_d^1 &= 0; i_d^2 = F; c < F; i_f = V \\
 \text{Then } p &\geq 0; m = 0; t = 0; \rho = 1; q = 0; v \geq 0 \\
 \text{and} & \\
 \lambda_1 &= 1 - p, \\
 \lambda_2 &= v, \\
 \lambda_1 - \lambda_2 &= 1.
\end{align*}
\]

But $\lambda_1 - \lambda_2 = 1 - p - v$.

Therefore policy $R$ is only consistent if $p = v = 0$, that is if $\lambda_2 = 0$ and $\lambda_1 = 1$. Then policy $R$ is only consistent at the endpoint.

Policy $S$

\[
\begin{align*}
 i_d^1 < F; i_d^2 < F; c < F; i_f = 0
\end{align*}
\]
Then \( p = 0 \); \( m = 0 \); \( t = 0 \); \( \rho = 1 \); \( q \geq 0 \); \( v = 0 \) and
\[
\lambda_1 = 1, \\
\lambda_2 = -q, \\
\lambda_1 - \lambda_2 = 1.
\]
But \( \lambda_1 - \lambda_2 = 1 + q \).

Therefore \( q = 0 \) and \( \lambda_2 = 0 \).

Then this policy satisfies the necessary conditions of the theorem at the endpoint. But as the endpoint is defined by \( k_d = k_f = 0 \), this policy is non-optimal because \( k_f < 0 \).

**Policy T**

\[
i_{d_1} < F; i_{d_2} < F; C < F; i_f > 0
\]
Then \( p = 0 \); \( m = 0 \); \( t = 0 \); \( \rho = 1 \); \( v = q = 0 \) and
\[
\lambda_1 = 1, \\
\lambda_2 = 0.
\]

Then this policy is only consistent at the endpoint.

**Policy U**

\[
i_{d_1} < F; i_{d_2} < F; C < F; i_f = V
\]
Then \( p = 0 \); \( m = 0 \); \( t = 0 \); \( \rho = 1 \); \( q = 0 \); \( v \geq 0 \) and
\[
\lambda_1 = 1, \\
\lambda_2 = v, \\
\lambda_1 - \lambda_2 = 1.
\]
But \( \lambda_1 - \lambda_2 = 1 - v \).

Therefore this policy is only consistent if \( v = 0 \) and thus the policy is only consistent at the endpoint.

**Possible Policy Switches**

The possible policy switches are determined by the necessary condition that the auxiliary variables must be continuous functions.
of time given by

\[ \lambda_1 = -[\rho f' - \lambda_1 \omega] ; \quad \lambda_2 = -[\rho (f' - \phi) - \lambda_2 \omega]. \]

From the properties of policies B, D, E and M, it is known that \( \lambda_2 = 0 \) along \( f' = \phi \). Also from policy I, \( \lambda_1 = 1 \) along the \( f' = \omega \) curve in the region \( f' \geq \phi \).

In Appendix No. 1 it was shown that policies A and C switch into B at \( f' = \phi \), policies C and F switch into I at \( f' = \omega \) and policy F switches into E at \( f' = \phi \). Since D, M, N and O can also lie along \( f' = \phi \) in the region \( f' > \phi \), policy F can also switch into D, K, M, N and O at \( f' = \phi \).

Policy J

\[ \lambda_1 = \rho - p, \quad \lambda_2 = -q \quad \text{and} \quad p = q. \]

Because \( p = q \) along policy J, as \( \lambda_2 \) goes to zero so does \( p \) and \( \lambda_1 \) is greater than or equal to one. Therefore policy J cannot switch into policy B which is characterised by \( \lambda_1 < 1 \) and \( \lambda_2 = 0 \) (except at the endpoint where \( \lambda_1 = 1 \)).

Also policy A cannot switch into policy E (and thus into D, K, M, N or O) as discussed in Appendix No. 1. Then policy J is the only policy which can possibly switch into E, D, K, M, N or O along \( f' = \phi \).

In the region \( F > \omega k_d \), \( k_d \) is positive along policy J and therefore \( \lambda_1 \) must be negative. That is, along any path of policy J which takes the economy to \( f' = \phi \) in the region \( f' > \omega \), \( \lambda_1 \) cannot equal 1 and must be greater than 1 at all points along the path in the region \( F \geq \omega k_d \). Therefore in the region \( F \geq \omega k_d \), policy A cannot switch into policy J. However, in the region \( f' < \omega \), \( \lambda_1 \) may become negative along policy J and policy J can switch into policy A.
In the region $F \geq wk_d$, $k_d$ is increasing along policy $J$ therefore $\lambda_1$ must be negative. That is, policy $J$ is only applicable in the region where $\lambda_1$ is negative or in the region $\rho(f' - \omega) > \lambda_2 \omega$, with $\rho \geq 1$. Similarly, along policy $A$ $k_d$ is decreasing and therefore $\lambda_1$ must be positive. That is, policy $A$ is only applicable in the region $(f' - \omega) < \lambda_2 \omega$. In other words, policy $J$ switches into policy $A$ at the boundary $(f' - \omega) = \lambda_2 \omega$.

Policies $D, E$ and $M$.

The policies $D, E$ and $M$ are internally consistent only if they lie along $f' = \phi$ in the region $f' > \omega$. On each of these policies consumption is zero and $k_d = \frac{F - wk_d}{F - wk_d}$ and $k_f$ is such as to move the economy along $f' = \phi$. On policy $D$ $k_f = -wk_f$, on policy $E$ $k_f > -wk_f$ and on policy $M$ $k_f < -wk_f$. Therefore the optimal policy will be determined by the slope of the $f' = \phi$ curve.

The slope of these policies is $\frac{k_f}{k_d} = \frac{x - wk_f}{F - wk_d}$, where $x$ is zero depending on the policy. The slope of $f' = \phi$ is $\frac{-f''}{f'' - \phi}$ (see Appendix No. 1). Then the optimal policy is the one along which $\frac{x - wk_f}{F - wk_d} = \frac{-f''}{f'' - \phi}$. It should be noted that the slope of $M$ > slope of $D$ > slope of $E$ and that switches can occur between these policies.
Appendix No. 3.

The Optimisation Problem

To minimise

\[ \int_0^T [U(c) - U(\dot{c})] \, dt , \]

where \( T \) is unspecified, along the path from any initial point to the fixed endpoint subject to

\[ f(k_d + k_f) - r(k_f)k_f - i_d - c \geq 0 \]
\[ k_d = i_d - \omega k_d \]
\[ k_f = i_f - \omega \]
\[ c \geq 0 \]
\[ i_d \geq 0 \]
\[ 0 \leq i_f \leq V . \]

It is also assumed that \( U'(c) > 0 \), \( U''(c) < 0 \) and \( U'(0) = 0 \).

The fixed endpoint is such that sustainable consumption per head is maximised. As derived in the Chapter V the endpoint \((k_d, k_f)\) is defined by \( f' = \phi = \omega \).

Applying Arrow's Method of Solution (Arrow [3]) again the Hamiltonian is

\[ H = -[U(c) - U(\dot{c})] + \lambda_1 [i_d - \omega k_d] + \lambda_2 [i_f - \omega k_f] \]

and the Lagrangian is

\[ L = H + p_d i_d + q_f i_f + \nu [V - i_f] + tc \]
\[ + \rho [f(k_d + k_f) - r(k_f)k_f - c - i_d] . \]

Necessary conditions for an optimal solution are:

(a) The shadow prices \( \lambda_1 \) and \( \lambda_2 \) are continuous functions of time given by
\[ \begin{align*}
\lambda_1 & = \frac{\partial L}{\partial k_d} = -[\rho f' - \lambda_1 \omega] \text{ and} \\
\lambda_2 & = -\frac{\partial L}{\partial k_f} = -[\rho (f' - \phi) - \lambda_2 \omega]. 
\end{align*} \]

(b) The constraint satisfies the Constraint Qualification at any point of time and the Hamiltonian is maximised subject to this constraint. That is

\[ \begin{align*}
\frac{\partial L}{\partial c} & = U'(c) + t - \rho = 0 \\
\frac{\partial L}{\partial d} & = \lambda_1 + p - \rho = 0 \\
\frac{\partial L}{\partial f} & = \lambda_2 + q - v = 0 \\
p & \geq 0; \quad p_i = 0 \\
q & \geq 0; \quad q_i = 0 \\
v & \geq 0; \quad v[V - i_f] = 0 \\
t & \geq 0; \quad t_c = 0 \\
\rho & \geq 0; \quad \rho[f - r k_f + c - i_d] = 0
\end{align*} \]

(c) The optimal value of the Hamiltonian is zero. That is,

\[ U(c) - U(c) = \lambda_1 [i_d - \omega k_d] + \lambda_2 [i_f - \omega k_f]. \]

These necessary conditions are also sufficient for an optimal solution because the Hamiltonian is a concave function of the state variables \( k_d, k_f \) given \( \lambda_1, \lambda_2 \) and \( t \). This sufficiency condition is given by Arrow [3], lecture 1, proposition 4.

From \( \frac{\partial L}{\partial c} = 0 \), \( \rho = U'(c) + t \). But \( t \geq 0 \) therefore \( \rho \geq U'(c) \).

Then all of net output is distributed between consumption and current domestic investment. As discussed in section I, any policy

---

along which consumption is zero is non-optimal. Therefore, the possible policy combinations are

<table>
<thead>
<tr>
<th>Policy</th>
<th>$i_d$</th>
<th>$c$</th>
<th>$i_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>$F$</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>$F$</td>
<td>$&gt;0$</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>$F$</td>
<td>$V$</td>
</tr>
<tr>
<td>D</td>
<td>$0&lt;i_d&lt;F$</td>
<td>$0&lt;i_c&lt;F$</td>
<td>0</td>
</tr>
<tr>
<td>E</td>
<td>$0&lt;i_d&lt;F$</td>
<td>$0&lt;i_c&lt;F$</td>
<td>$&gt;0$</td>
</tr>
<tr>
<td>F</td>
<td>$0&lt;i_d&lt;F$</td>
<td>$0&lt;i_c&lt;F$</td>
<td>$V$</td>
</tr>
<tr>
<td>G</td>
<td>$F-c$</td>
<td>$F$</td>
<td>$0$</td>
</tr>
<tr>
<td>H</td>
<td>$F-c$</td>
<td>$&lt;F$</td>
<td>$&gt;0$</td>
</tr>
<tr>
<td>I</td>
<td>$F-c$</td>
<td>$F$</td>
<td>$V$</td>
</tr>
</tbody>
</table>

where $F = f(k_d + k_f) - r(k_f)k_f$

Properties of policies

**Policy A**

$i_d = 0$ ; $c = F$ ; $i_f = 0$

$p \geq 0$ ; $v = 0$ ; $q \geq 0$ ; $\rho = U'(c)$

Therefore $\lambda_1 \leq U'(c)$ ; $\lambda_2 \leq 0$

and $\lambda_1 = -[\rho f' - \lambda_1 \omega]$ ; $\lambda_2 = -[\rho(f' - \phi) - \lambda_2 \omega]$.

**Policy B**

$i_d = 0$ ; $0 < i_f < V$ ; $c = F$

$p \geq 0$ ; $v = q = 0$ ; $t = 0$ ; $\rho = U'(c)$

Therefore $\lambda_1 \leq U'(c)$ ; $\lambda_2 = 0$

and $\lambda_1 = -[\rho f' - \lambda_1 \omega]$ ; $\lambda_2 = -U'(c)[f' - \phi]$.

But $\lambda_2 = 0$ therefore $\lambda_2 = 0$. Then policy B is only consistent in the region $f' < \omega$ and along $f' = \phi$.

**Policy C**

$i_d = 0$ ; $i_f = V$ ; $c = F$

$p \geq 0$ ; $q = 0$ ; $v \geq 0$ ; $t = 0$ ; $\rho = U'(c)$
Therefore $\lambda_1 \leq U'(c)$; $\lambda_2 \geq 0$

and $\lambda_1 = -[pf' - \lambda_1 \omega]$; $\lambda_2 = -[\rho(f' - \phi) - \lambda_2 \omega]$.

**Policy D**

\[
\begin{align*}
0 < i_d < F; & \quad i_f = 0; \quad 0 < c < F \\
p = 0; & \quad q \geq 0; \quad v = 0; \quad t = 0; \quad \rho = U'(c)
\end{align*}
\]

Therefore $\lambda_1 = U'(c)$; $\lambda_2 \leq 0$

and $\lambda_1 = -\lambda_1 [f' - \omega]$; $\lambda_2 = -[\lambda_1 (f' - \phi) - \lambda_2 \omega]$.

But $\lambda_1 = U'(c)$ so that

\[
\lambda_1 = U'(c) \hat{c}
\]

or $\hat{c} = -\frac{U'}{U''} [f' - \omega]$.

Along $f' = \omega$, $c = 0$. Therefore $c$ must equal $\hat{c}$ and this is policy G which will be discussed below.

Along policy D $\lambda_2 \leq 0$ with $\lambda_2 = -[\lambda_1 (f' - \phi) - \lambda_2 \omega]$. Then policy D can only be optimal in the region $f' < \phi$ so that $\lambda_2 > 0$.

**Policy E**

\[
\begin{align*}
0 < i_d < F; & \quad 0 < i_f < V; \quad 0 < c < F \\
p = 0; & \quad q = v = 0; \quad t = 0; \quad \rho = U'(c)
\end{align*}
\]

Therefore $\lambda_1 = U'(c)$; $\lambda_2 = 0$

and $\lambda_1 = -\lambda_1 [f' - \omega]$; $\lambda_2 = -[\lambda_1 (f' - \phi)]$.

Because $\lambda_2 = 0$, $\lambda_2 = 0$. Then this policy is only consistent along $f' = \phi$.

Also, as derived for policy D above,

$\hat{c} = -\frac{U'}{U''} [f' - \omega]$.

Then along policy E, $\hat{c} > 0$ in the region $f' > \omega$, $\hat{c} < 0$ where $f' < \omega$ and $\hat{c} = 0$ at $f' = \omega$. 
Policy F

\[ 0 < i_d < F ; \ i_f = V ; \ 0 < c < F \]
\[ p = 0 ; \ q = 0 ; \ \nu \geq 0 ; \ t = 0 ; \ \rho = U'(c) \]
Therefore \( \lambda_1 = U'(c) ; \ \lambda_2 \geq 0 \)
and \( \dot{\lambda}_1 = -\lambda_1 [(f' - \omega) - \lambda_2 \omega] ; \ \dot{\lambda}_2 = -[\lambda_1 (f' - \phi) - \lambda_2 \omega] \).

As derived for policy D, \( \dot{c} = -\frac{U'}{U''} (f' - \omega) \).

Along \( f' = \omega \), \( \dot{c} = 0 \). Therefore \( c = \dot{c} \) and this is policy L which will be discussed below.

Along policy F, \( \lambda_2 \geq 0 \) and \( \dot{\lambda}_2 = -[\lambda_1 (f' - \phi) - \lambda_2 \omega] \).
Then policy F can only be optimal in the region \( f' > \phi \) so that \( \lambda_2 < 0 \).

Policy G

\[ i_d = F - \hat{c} ; \ c = \hat{c} ; \ i_f = 0 \]
\[ p = 0 ; \ t = 0 ; \ \nu = 0 ; \ q \geq 0 ; \ \rho = U'(c) \]
Therefore \( \lambda_1 = U'(c) ; \ \lambda_2 \leq 0 \)
and \( \dot{\lambda}_1 = -U'(c) [f' - \omega] ; \ \dot{\lambda}_2 = -[\rho (f' - \phi) - \lambda_2 \omega] \).

As derived above, \( \dot{c} = -\frac{U'}{U''} (f' - \omega) \). Therefore policy G can only be optimal along \( f' = \omega \). But along \( f' = \omega \), \( \dot{k} = 0 \). That is, \( i_d \) must equal \( \omega \hat{k} \). The allocation from net output to current domestic investment is \( F - \hat{c} = r(k_f) \hat{k}_f - r(k_f) k_f + \omega \hat{k}_d \). Then policy G lies on \( f' = \omega \) only if
\[ \omega \hat{k} = r(k_f) \hat{k}_f - r(k_f) k_f + \omega \hat{k}_d \]
or if \( \omega \hat{k}_f = r(k_f) \hat{k}_f - r(k_f) k_f \).
But \( r(k_f) > r(\hat{k}_f) \) and \( k_f > \hat{k}_f \) therefore the right-hand side of the equation is negative and thus cannot equal \( \omega \hat{k}_f \). Then policy G does not lie along \( f' = \omega \) and is therefore non-optimal.
Policy H

\[ i_d = F - \hat{c}; \quad c = \hat{c}; \quad i_f > 0 \]
\[ p = 0; \quad t = 0; \quad v = q = 0; \quad \rho = U'(c) \]

Therefore \( \lambda_1 = U'(c); \quad \lambda_2 = 0 \).

Then this policy can only be optimal at the endpoint.

Policy I

\[ i_d = F - \hat{c}; \quad c = \hat{c}; \quad i_f = V \]
\[ p = 0; \quad t = 0; \quad q = 0; \quad v = 0; \quad \rho = U'(c) \]

Therefore \( \lambda_1 = U'(c); \quad \lambda_2 = 0 \)

and \( \lambda_1 = -U'(c)(f' - \omega); \quad \lambda_2 = -[\rho(f' - \phi) - \lambda_2 \omega] \).

As for policy G, this policy can only be optimal along \( f' = \omega \).

But along \( f' = \omega, \quad k = 0 \). Therefore this policy is only consistent if \( k_d = -k_f = -(V - \omega k_f) \). The amount of net output allocated to current domestic investment is \( F - \hat{c} \). Then

\[ k_d = F - \hat{c} - \omega k_d \], where \( \hat{c} = f(k) - r(\hat{k}_f)\hat{k}_f - \omega \hat{k}_d \). Then policy I is only optimal if

\[ -(V - \omega k_f) = [r(\hat{k}_f)\hat{k}_f - r(k_f)k_f] + \omega(\hat{k}_d - k_d) \],

that is, if

\[ -(V - \omega k_f) = [r(\hat{k}_f)\hat{k}_f - r(k_f)k_f] + \omega \hat{k}_d \].

The left-hand side of this equation is constant because \( k = \hat{k} \) along \( f' = \omega \) and \( V, \omega \) are given constants. The right-hand side is not constant but decreases along policy I. Therefore this condition for consistency is satisfied only at one point on \( f' = \omega \) and thus the policy is non-optimal.

Possible Policy Switches

It is known from the properties of the policies that \( \lambda_2 \) is zero along \( f' = \phi \) and \( \lambda_1 = U'(c) \) at the endpoint. Then using the
fact that the auxiliary variables are continuous functions of time and of the state variables and that the necessary conditions for optimality are

\[
\dot{\lambda}_1 = -[\rho f' - \lambda_1 \omega]; \quad \dot{\lambda}_2 = -[\rho (f' - \phi) - \lambda_2 \omega],
\]

all possible policy switches can be determined.

**Policy A**

\[\lambda_1 \leq U'(f); \quad \lambda_2 \leq 0\]

Policy A is only optimal in the region \( f' \leq \phi \) and \( \lambda_1 \leq U'(f) \).

As reasoned in Appendix No. 1 policy A switches into policy B at \( f' = \phi \).

\( \lambda_1 \) along policy D is \( U'(c) \) where \( c < F \). Then in the neighbourhood of \( \lambda_1 = U'(F) \), \( c \) is falling along D and \( \lambda_1 \) is increasing.

But in the limit, \( \lambda_1 \) on policy D must equal \( U'(F) \) at the boundary \( \lambda_1 = U'(F) \). That is \( \lambda_1 = U'(F) \) is the switching surface where policy A switches to policy D.

**Policy B**

\[\lambda_1 \leq U'(F); \quad \lambda_2 = 0\]

Policy B is optimal along \( f' = \phi \) in the region \( \lambda_1 \leq U'(f) \).

As discussed for policy A above policy B switches at \( \lambda_1 = U'(f) \) into policy E.

**Policy C**

\[\lambda_1 \leq U'(F); \quad \lambda_2 \geq 0\]

Policy C is only optimal in the region \( f' \geq \phi \) and \( \lambda_1 \leq U'(f) \).

As discussed in Appendix No. 1 policy C switches into policy B at \( f' = \phi \). Applying similar reasoning for policy F as for policy D in relation to policy A, policy C switches into policy
F at the "switching surface" \( \lambda_1 = U'(F) \).

Policy D

\[ \lambda_1 = U'(c); \quad \lambda_2 \leq 0 \]

If \( \lambda_2 = 0 \) along policy D in the region \( f' < \phi \) then policy D cannot switch into any other policy nor take the system to the endpoint. Therefore \( \lambda_2 \) must equal zero at \( f' = \phi \) and there policy D switches into policy E.

Policy E

\[ \lambda_1 = U'(c); \quad \lambda_2 = 0 \]

Policy E does not switch into any other policy but takes the system to the endpoint.

Policy F

\[ \lambda_1 = U'(c); \quad \lambda_2 \geq 0 \]

Using a similar reasoning here as for policy D policy F switches into policy E at \( f' = \phi \).

Shape of \( \dot{k}_d = 0 \) curve

At the endpoint \( \dot{k}_d = 0 \) and therefore this curve passes through the endpoint.

From the necessary conditions for optimality the Hamiltonian is zero. Therefore along \( \dot{k}_d = 0 \),

\[ U(\dot{c}) - U(c) = \lambda_2 [i_f - \omega_k] \].

Differentiating totally

\[ -U'(c) dc = -\lambda_2 \omega_k + \lambda_2 d_i_f \]
That is, along $k_d = 0$, $\frac{dc}{dk_f} < 0$ if $\lambda_2 < 0$

and $\frac{dc}{dk_f} > 0$ if $\lambda_2 > 0$, when $l_f$ is constant.

Along policy D, $\lambda_2 \leq 0$, $i_f = 0$ and $k_f < 0$ so that $k_d = 0$ curve cuts the path of policy D where $c$ is positive. As discussed under the properties of policy D, $c = -\frac{U'}{U''} (f' - \omega)$. Therefore $k_d = 0$ curve cuts the path of policy D somewhere in the region $f' > \omega$.

Similarly, for policy F $\lambda_2 \geq 0$, $i_f = V$ and $k_f > 0$, so that $k_d = 0$ curve cuts the path of policy F where $c$ is positive. That is, the path of policy F is cut somewhere in the region $f' > \omega$.

The exact slope of $k_d = 0$ is indeterminate but from the three properties above it takes the following shape:
Appendix No. 4.

Solution of the problem in section I

To minimise

$$\int_{0}^{w} (\dot{c} - c) dt$$

along the path from any initial point to the endpoint subject to

$$f(k_d + k_f) - r(Lk_f)k_f - i_d - c \geq 0$$

$$k_d = i_d - w_k_d$$

$$c \geq 0$$

$$i_d \geq 0 .$$

Applying Arrow's Method of Solution (Arrow [3]), the Hamiltonian is

$$H = - (\dot{c} - c) + \lambda_1 [i_d - w_k_d]$$

and the Lagrangian

$$L = H + \pi_i d + tc + \rho [f(k_d + k_f) - r(Lk_f)k_f - c - i_d] ,$$

where $$\lambda_1$$ is the shadow price of net domestic investment and $$\pi, t, \rho$$ are the Lagrange Multipliers.

Necessary conditions for an optimal solution are:

(a) $$\lambda_1$$ is a continuous function of time given by

$$\dot{\lambda}_1 = - \frac{\partial L}{\partial k_d} = - [\rho f' - \lambda_1 \omega] .$$

(b) The constraint satisfies the Constraint Qualification at any point of time (Arrow Hurwicz, Uzawa [4]) and the Hamiltonian is maximised subject to this constraint. That is

$$\frac{\partial L}{\partial c} = 1 + t - \rho = 0$$

$$\frac{\partial L}{\partial i_d} = \lambda_1 + \rho - \rho = 0$$

$$\frac{\partial L}{\partial k_f} = \rho [f'(k) - \Phi(Lk_f)] = 0$$

225.
\[ p \geq 0; \quad p_i = 0 \]
\[ t \geq 0; \quad t_c = 0 \]
\[ \rho \geq 0; \quad \rho[f(k) - r(k_f)k_f - c - i_d] = 0. \]

These necessary conditions can also be shown to be sufficient for optimality by using the theorem in Arrow [3], lecture 1, proposition 4.

Then from \( \frac{\partial L}{\partial c} = 0 \) and \( t \geq 0, \rho \geq 1 \). Therefore the budget constraint is satisfied with equality.

**Properties of the policies**

**Policy A**
\[ i_d = 0; \quad c = F. \]
Then \( p \geq 0; \quad t = 0; \quad \rho = 1 \).
Also \( \lambda_1 \leq 1 \)
and \( \lambda_1 = -[f' - \lambda_1 \omega] \).

**Policy B**
\[ i_d = F; \quad c = 0. \]
Then \( p = 0; \quad t \geq 0; \quad \rho \geq 1 \).
Also \( \lambda_1 \geq 1 \)
and \( \lambda_1 = -\lambda_1 [f' - \omega] \).

**Policy C**
\[ 0 < i_d < F; \quad 0 < c < F \]
Then \( p = 0; \quad t = 0; \quad \rho = 1 \).
Also \( \lambda_1 = 1 \)
and \( \lambda_1 = -[f' - \omega] \).

But \( \lambda_1 = 1 \) and therefore \( \lambda_1 = 0 \). Thus this policy is only consistent along \( f' = \omega \).
Solution of the problem in section II

To minimise

$$J = \int_0^a (\dot{c} - c) \, dt,$$

along the path from any initial point to the fixed endpoint subject to

$$f(k_d + k_f) - r(Lk_f)k_f - c - i_d \geq 0$$

$$k_d = i_d - \omega k_d$$

$$k_f = i_f - \omega k_f$$

$$c \geq 0$$

$$i_d \geq 0$$

$$0 \leq i_f \leq V(r).$$

Applying Arrow's Method of Solution again, the Hamiltonian is

$$H = -(\dot{c} - c) + \lambda_1 [i_d - \omega k_d] + \lambda_2 [i_f - \omega k_f]$$

and the Lagrangian is

$$L = H + p i_d + tc + q i_f + v[V(r) - i_f]$$

$$+ p[f(k_d)k_d] - r(Lk_f)k_f - c - i_d.$$ 

Necessary conditions for an optimal solution are:

(a) that the shadow prices $\lambda_1$ and $\lambda_2$ are continuous functions of time given by

$$\lambda_1' = -\frac{3L}{3k_d} = -[\rho f' - \lambda_1 \omega]$$

$$\lambda_2' = -\frac{3L}{3k_f} = -[\rho (f'(k) - \Phi(Lk_f)) - \lambda_2 \omega - vv' \Phi(Lk_f)]$$

(b) The constraint satisfies the Constraint Qualification at any point of time (Arrow, Hurwicz, Uzawa [4]) and the Hamiltonian is maximised subject to this constraint. That is

$$\frac{3L}{3c} = 1 + t - \rho = 0$$

$$\frac{3L}{3i_d} = \lambda_1 + p - \rho = 0$$

$$\frac{3L}{3i_f} = \lambda_2 + q - v = 0$$
\[ p \geq 0 \quad \text{and} \quad p_{id} = 0 \]
\[ q \geq 0 \quad \text{and} \quad q_{if} = 0 \]
\[ t \geq 0 \quad \text{and} \quad t_c = 0 \]
\[ v \geq 0 \quad \text{and} \quad v[V - i_f] = 0 \]
\[ \rho \geq 0 \quad \text{and} \quad \rho[f(k) - \Phi(Lk_f)k_f - c - i_d] = 0 . \]

From \( \frac{3L}{3c} = 0 \) and \( t \geq 0 \), \( \rho \geq 1 \). Therefore all of net output is allocated between \( i_d \) and \( c \). Then the possible policy combinations satisfying the given constraint are those listed in Chapter VII.

The properties of these policies and the switches between them are the same as for the policies of Appendix No. 1.
Appendix No. 5

Shape of $F' - \delta - r_f - K_f = 0$

Differentiating totally,

$$F''(dK_d + dK_f) - r_f' dK_f - [r_f''K_f + r_f']dK_f = 0.$$

Therefore

$$\frac{dK_f}{dK_d} = -\frac{F''}{F'' - r_f'}.$$

But assuming that $r_f'' < 0$,

$$\frac{dK_f}{dK_d} = -\frac{F''}{F'' - r_f'} < 0.$$

Shape of $F' - \delta - r_f = 0$

Differentiating totally

$$F''(dK_d + dK_f) = r_f' dK_f.$$

Therefore

$$\frac{dK_f}{dK_d} = -\frac{F''}{F'' - r_f'} < 0.$$

These two curves have negative slopes between 0 and -1 but the curve $F' - \delta = r_f$ is steeper and they intersect on the $K_f = 0$ axis.

The Optimisation Problem

To minimise

$$\int_0^T [U(\hat{c}) - U(c)]dt,$$

subject to

$$F(K_d + K_f) - F'(K)K_f + \tau[F'(K) - \delta]K_f - C - I_{d_1} - I_{d_2} \geq 0,$$

$$K_d = I_{d_1} + I_{d_2} - \delta K_d,$$

$$K_f = I_f - I_{d_2} - \delta K_f.$$
\( C > 0 \)
\[
\begin{align*}
I_d_1 & \geq 0 \\
I_d_2 & > 0 \\
0 & \leq I_f \leq (\alpha + \delta)K^2,
\end{align*}
\]
where \( \pi = \pi(I_d_2) \) and \( r = (1 - \tau)[F'(K) - \delta] \).

The control variables are now \( C, I_d_1, I_d_2, I_f \) and \( \tau \) and the state variables are \( K_d \) and \( K_f \).

Define the Hamiltonian
\[
H = -[U(\dot{C}) - U(C)] + \lambda_1[I_d_1 + I_d_2 - \delta K_d] + \lambda_2[I_f - I_d_2 - \delta K_f]
\]
and the Lagrangian
\[
L = H + p[F(K_d + K_f) - F'(K)K_f + \tau F'(K) - \delta] - C - \pi I_d_2 + \pi I_d_1
+ p I_d_1 + \pi I_d_2 + \pi C + \pi F(K_d) - I_f
+ z[r - r^2].
\]

Necessary conditions for an optimal solution are:

(a) The auxiliary variables \( \lambda_1, \lambda_2 \) are continuous functions of time given by
\[
\begin{align*}
\lambda_1 &= -\frac{\partial L}{\partial K_d} = -[\rho(F' - F''K_f + \tau F''K_f) - \lambda_1 \delta - \frac{\partial}{\partial \tau} F''] \\
\lambda_2 &= -\frac{\partial L}{\partial K_f} = -\rho(-F''K_f + \tau(F' - \delta) + \tau F''K_f) + \nu(\alpha + \delta)
+ z(1 - \tau)F'' - zr \dot{r}' - \lambda_2 \delta.
\end{align*}
\]

(b) The constraint satisfies the Constraint Qualification at any point of time (Arrow, Hurwicz, Uzawa [4]) and the Hamiltonian is maximised subject to this constraint. That is,
\[
\begin{align*}
\frac{\partial L}{\partial I_d_1} &= \lambda_1 - \rho + p = 0 \\
\frac{\partial L}{\partial I_d_2} &= \lambda_1 - \lambda_2 - \rho[\pi + I_d_2 \pi'] + m = 0 \\
\frac{\partial L}{\partial I_f} &= \lambda_2 + \dot{q} - \nu = 0
\end{align*}
\]
\[
\frac{\partial L}{\partial C} = U'(C) - \rho + n = 0
\]
\[
\frac{\partial L}{\partial T} = (F' - \delta)(\rho K_f - z) = 0
\]
\[
p \geq 0 ; \quad p I_{d_1} = 0
\]
\[
m \geq 0 ; \quad m I_{d_2} = 0
\]
\[
n \geq 0 ; \quad n C = 0
\]
\[
q \geq 0 ; \quad q I_f = 0
\]
\[
v \geq 0 ; \quad v[(a + \delta)K_f - I_f] = 0
\]
\[
\lambda \geq 0 ; \quad \lambda[r - r^2] = 0
\]
\[
\rho \geq 0 ; \quad \rho[F' K_f + \tau(F' - \delta)K_f - C - I_{d_1} - n I_{d_2}] = 0.
\]

These necessary conditions are also sufficient for optimality from Arrow's sufficiency theorem Arrow [3], lecture 1, page 10.

From \( \frac{\partial L}{\partial T} = 0 \), \( z = \rho K_f \) (\( F' \neq \delta \) as discussed earlier in part I of this chapter). Substituting for \( z \) in the equations for \( \lambda_1 \) and \( \lambda_2 \), they become

\[
\lambda_1 = -[\rho F' - \lambda_1 \delta] \quad \text{and} \quad \lambda_2 = -[\rho F(F' - \delta) + v(a + \delta) - zr^2 - \lambda_2 \delta].
\]

From \( \frac{\partial L}{\partial C} = 0 \), \( \rho = U'(C) + n > 0 \). Therefore the budget constraint is satisfied with equality and \( z \) is positive for all \( K_f > 0 \) so that at any point of time when \( K_f > 0 \), \( r = r^2 \). Then from the budget constraint all possible policies are

<table>
<thead>
<tr>
<th>Policy</th>
<th>( I_{d_1} )</th>
<th>( I_{d_2} )</th>
<th>( C )</th>
<th>( I_f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>0</td>
<td>f</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>0</td>
<td>f</td>
<td>( 0 &lt; I_f &lt; (a+\delta)K_f )</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>0</td>
<td>f</td>
<td>( (a+\delta)K_f )</td>
</tr>
<tr>
<td>D</td>
<td>( 0 &lt; I_{d_1} &lt; f )</td>
<td>0</td>
<td>( 0 &lt; C &lt; f )</td>
<td>0</td>
</tr>
</tbody>
</table>
### Properties of these policies

**Policy A**

\[
\begin{align*}
I_{d_1} &= 0; \\
I_{d_2} &= 0; \\
C &= f; \\
I_f &= 0
\end{align*}
\]

Then \( p \geq 0; \quad m \geq 0; \quad n = 0; \quad q \geq 0; \quad v = 0 \)

and \( \rho = U'(C) \).

Therefore \( \lambda_1 \leq U'(C) \) and \( \lambda_2 \leq 0 \).

Since \( I_{d_1} + I_{d_2} = 0 \), \( \kappa_d < 0 \). Therefore \( \lambda_1 \) must be positive.

Then policy A is only consistent in the region \( \rho F' - \lambda_1 \delta < 0 \), that is, in the region \( F' < \frac{\lambda_1 \delta}{U'(C)} < \delta \). But the only feasible region of the phase diagram is \( F' \geq \delta \) and therefore this policy is non-optimal.

Then using this same reasoning, it can be shown that policies B and
C are also non-optimal.

**Policy D**

\[ I_{d_1} > 0; \ I_{d_2} = 0; \ C > 0; \ I_f = 0. \]

Then \( p = 0; \ m \geq 0; \ n = 0; \ v = 0; \ q \geq 0 \)
and \( \rho = U'(C). \)

Therefore \( \lambda_1 = U'(C), \)
\[ \lambda_2 = -q \leq 0 \]
and \( \lambda_1 - \lambda_2 = \rho - m. \)

But \( \lambda_1 - \lambda_2 = U'(C) + q. \)

So that \( q = -m. \)

But this is only possible if \( q = m = 0, \) that is if \( \lambda_1 = U'(C) \)
and \( \lambda_2 = 0. \)
Because \( \lambda_2 = 0, \lambda_2 = 0. \) Therefore policy D is only optimal
along the curve
\[ \tau_p(F' - \delta) - zr_{\alpha'} = 0. \]

That is,
\[ T_{\text{opt.}} = \frac{K_f r_{\alpha'}}{(F' - \delta)}. \]

**Policy E**

\[ I_{d_1} > 0; \ I_{d_1} = 0; \ C > 0; \ 0 < I_f < (a + \delta)K^\alpha \]

Then \( p = 0; \ m \geq 0; \ n = 0; \ v = q = 0 \)
and \( \rho = U'(C). \)

Therefore \( \lambda_1 = U'(C) \) and \( \lambda_2 = 0. \)
Because \( \lambda_2 = 0, \lambda_2 = 0 \) and policy E is only consistent if
\[ \tau_p(F' - \delta) - zr_{\alpha'} = 0. \]

That is, policy E is only consistent if
\[ T_{\text{opt.}} = \frac{K_f r_{\alpha'}}{(F' - \delta)}. \]

Then policies E and D are only consistent if they lie along
\[ r_{\alpha'} = (1 - T_{\text{opt.}})(F' - \delta) \text{ i.e. along } F' - \delta = r_{\alpha'} + K_f r_{\alpha'}. \]
Policy F

\[ I_{d_1} > 0 ; \quad I_{d_2} = 0 ; \quad C > 0 ; \quad I_f = (\alpha + \delta)K \]

Then \( p = 0 ; \quad m \geq 0 ; \quad n = 0 ; \quad q = 0 ; \quad v \geq 0 \)

and \( \rho = U'(C) \).

Therefore \( \lambda_1 = U'(C) \) and \( \lambda_2 = v \geq 0 \).

Also \( \lambda_1 = -U'(C)[F' - \delta] \).

Therefore in the region \( F' > \delta, \lambda_1 < 0 \) and thus policy F is consistent only if \( K_d > 0 \).

Policy G

\[ I_{d_1} = 0 ; \quad I_{d_2} > 0 ; \quad C > 0 ; \quad I_f = 0 \]

Then \( p \geq 0 ; \quad m = 0 ; \quad n = 0 ; \quad q \geq 0 ; \quad v = 0 \)

and \( \rho = U'(C) \).

Therefore \( \lambda_1 \leq U'(C) \) and \( \lambda_2 \leq 0 \).

Because \( \lambda_1 \leq U'(C) \), \( \lambda_1 \) must be positive if policy G is to switch into any other policy or to take the economy to the endpoint. That is, \( \lambda_1 > 0 \) if \( F' < \frac{\lambda_1 \delta}{\rho} < \delta \).

Then policy G is non-optimal as the region \( F' < \delta \) is not a feasible area of the phase diagram.

Policy H

\[ I_{d_1} = 0 ; \quad I_{d_2} > 0 ; \quad C > 0 ; \quad 0 < I_f < (\alpha + \delta)K \]

Then \( p \geq 0 ; \quad m = 0 ; \quad n = 0 ; \quad v = q = 0 \)

and \( \rho = U'(C) \).

Therefore \( \lambda_1 \leq U'(C), \lambda_2 = 0 \) and \( \lambda_1 - \lambda_2 = U'(C)[\pi + \pi'I_{d_2}] \).

But \( \lambda_1 - \lambda_2 \leq U'(C) \). Therefore policy H is internally consistent only if \( [\pi + \pi'I_{d_2}] \) is less than one. However from the discussion
of \( \pi(I_{d_2}') \) when \( I_{d_2} > 0 \) and \( I_f > 0 \), \( \pi \) must be greater than unity. Therefore policy H is non-optimal.

**Policy I**

\[
I_{d_1} = 0; \quad I_{d_2} > 0; \quad C > 0; \quad I_f = (a + \delta)k\hat{f}
\]

Then \( p \geq 0; \quad m = 0; \quad n = 0; \quad q = 0; \quad v \geq 0 \)

and \( \rho = U'(C) \).

Therefore \( \lambda_1 = U'(C), \quad \lambda_2 > 0 \) and \( \lambda_1 - \lambda_2 = U'(C)[\pi + \pi'I_{d_2}'] \).

But \( \lambda_1 - \lambda_2 = U'(C) - p - v \leq U'(C) \). Therefore policy I is internally consistent only if \( \pi \leq 1 \). However from the discussion of the price \( \pi(I_{d_2}') \) when \( I_f = (a + \delta)k\hat{f} \) and \( I_{d_2} > 0 \), \( \pi \) must be \( > 1 \). Therefore this policy is non-optimal.

**Policy J**

\[
I_{d_1} > 0; \quad I_{d_2} > 0; \quad C > 0; \quad I_f = 0
\]

Then \( p = 0; \quad m = 0; \quad n = 0; \quad v = 0; \quad q \geq 0 \)

and \( \rho = U'(C) \).

Therefore \( \lambda_1 = U'(C) \) and \( \lambda_2 = -q \).

Because \( \lambda_2 < 0 \) and \( k_f < 0 \), \( \lambda_2 \) must be positive if it is to switch into any other policy or to take the system to the endpoint. Thus policy J can only switch into some other policy if 

\[
\rho(F' - \delta) - zr^\delta - \lambda_2 < 0.
\]

That is, policy J is optimal in the region \( \rho(F' - \delta) < K_f r^\delta \).

**Policy K**

\[
I_{d_1} > 0; \quad I_{d_2} > 0; \quad C > 0; \quad 0 < I_f < (a + \delta)k\hat{f}
\]

Then \( p = 0; \quad m = 0; \quad n = 0; \quad v = q = 0 \)

and \( \rho = U'(C) \).

Therefore \( \lambda_1 = U'(C), \quad \lambda_2 = 0 \) and \( \lambda_1 - \lambda_2 = U'(C)[\pi + \pi'I_{d_2}'] \).
But \( \lambda_1 - \lambda_2 = U'(C) \). Therefore policy K is consistent only if \( \pi + \pi' \lambda_2 = 1 \) and this occurs only at \( I_{d_2} = 0 \). Therefore policy K is non-optimal.

Since \( K_d > 0 \), \( \lambda_1 \) must be positive. Then policy K is non-optimal. It can be shown that \( \lambda_1 > 0 \) implies policy K is non-optimal.

**Policy L**

\[
I_{d_1} > 0; \quad I_{d_2} > 0; \quad C > 0; \quad I_f = (\alpha + \delta)k^a
\]

Then \( p = 0; \quad m = 0; \quad n = 0; \quad q = 0; \quad v \geq 0 \)

and \( \rho = U'(C) \).

Therefore \( \lambda_1 = U'(C), \lambda_2 > 0 \) and \( \lambda_1 - \lambda_2 = U'(C)[\pi + \pi' \lambda_2] \).

But \( \lambda_1 - \lambda_2 \leq U'(C) \). Therefore policy L is consistent only if \( \pi < 1 \). But from the discussion of the price \( \pi \) when \( I_{d_2} > 0 \) and \( I_f = (\alpha + \delta)k^a \), \( \pi \) must be > 1. Therefore policy L is non-optimal.

**Policy M**

\[
I_{d_1} = 0; \quad I_{d_2} < 0; \quad C > 0; \quad I_f = 0
\]

Then \( p \geq 0; \quad m = 0; \quad n = 0; \quad q \geq 0; \quad v = 0 \)

and \( \rho = U'(C) \).

Therefore \( \lambda_1 \leq U'(C) \) and \( \lambda_2 \leq 0 \).

Also \( \lambda_1 = -[\rho F' - \lambda_1 \delta] \).

Since \( K_d < 0 \), \( \lambda_1 \) must be positive. Then policy M can be consistent only in the region \( F' < \delta \). But this region is not part of the feasible area of the phase diagram. Therefore policy M is non-optimal.

**Policy N**

\[
I_{d_1} = 0; \quad I_{d_2} < 0; \quad C > 0; \quad 0 < I_f < (\alpha + \delta)k^a
\]

Then \( p \geq 0; \quad m = 0; \quad n = 0; \quad q = v = 0 \)

and \( \rho = U'(C) \).
Therefore $\lambda_1 \leq U'(C)$ and $\lambda_2 = 0$.

Also $\lambda_1 = -[\rho F' - \lambda_1^*]$.

Since $K_d < 0$, $\lambda_1$ must be positive. Then policy M is non-optimal for the same reason as policy N is non-optimal. It can be shown that for the same reason policy O is also non-optimal.

**Policy P**

\[ I_d > 0 ; I_d < 0 ; C > 0 ; I_f = 0 \]

Then $p = 0 ; m = 0 ; n = 0 ; q \geq 0 ; \nu = 0$

and $\rho = U'(C)$.

Therefore $\lambda_1 = U'(C), \lambda_2 \leq 0$ and $\lambda_1 - \lambda_2 = U'(C)[\pi + \pi I_d]$.

Since $\lambda_1 - \lambda_2 = U'(C) + q \geq U'(C), [\pi + \pi I_d]$ must be greater than one. But from the discussion of the price $\pi(I_d)$ when $I_f = 0$ and $I_d < 0$, $\pi$ must be less than unity. Therefore policy P is non-optimal.

**Policy Q**

\[ I_d > 0 ; I_d < 0 ; C > 0 ; 0 < I_f < (\alpha + \delta)K^\alpha \]

Then $p = 0 ; m = 0 ; n = 0 ; q = \nu = 0$

and $\rho = U'(C)$.

Therefore $\lambda_1 = U'(C), \lambda_2 = 0$ and $\lambda_1 - \lambda_2 = U'(C)[\pi + \pi I_d]$.

Since $\lambda_1 - \lambda_2 = U'(C)$, policy Q is consistent only if $[\pi + \pi I_d]$ equals one. But this is impossible unless $I_d = 0$. Therefore policy Q is non-optimal.

**Policy R**

\[ I_d > 0 ; I_d < 0 ; C > 0 ; I_f = (\alpha + \delta)K^\alpha \]

Then $p = 0 ; m = 0 ; n = 0 ; q = 0 ; \nu \geq 0$
and \( p = U'(C) \).

Therefore \( \lambda_1 = U'(C) \), \( \lambda_2 \geq 0 \) and \( \lambda_1 - \lambda_2 = U'(C)[\pi + \pi' I_{d_2}] \).

Since \( \lambda_1 - \lambda_2 \leq U'(C) \), \( [\pi + \pi' I_{d_2}] \) must be less than one. But from the discussion of \( \pi(I_{d_2}) \) when \( I_f = (a + \delta)K \) and \( I_{d_2} < 0 \), \( \pi \) must be > 1. Therefore policy R is non-optimal.

Slopes of policies D and E

If either policy D or E is to lie along the curve \( F' - \delta = r^k + K_F r^k' \), then the slope of the policy must equal that of the curve.

The slope of \( F' - \delta = r^k + K_F r^k' \) has been derived earlier in this appendix and is equal to \( \frac{F''}{1 - 2r^k K} \). That is, it is negative and lying between zero and minus one.

The slope of policy E is

\[ \frac{dK_F}{dK_d} = \frac{K_F}{I_d - \delta K_d} \]

Then \( I_f \) is positive and is such as to allow policy E to lie along \( F' - \delta = r^k + K_F r^k' \).

The slope of policy D is

\[ \frac{dK_F}{dK_d} = \frac{-\delta K_F}{I_{d_1} - \delta K_d} \]

The slope of policy D is steeper than that of policy E and D will be the optimal policy along that part of the curve \( F' - \delta = r^k + K_F r^k' \) which is too steep for policy E. It should be noted that policy D can switch into policy E and this will occur at a point along the curve where its slope is equal to \( \frac{-\delta K_F}{I_{d_1} - \delta K_d} \). It should be noted that by choosing \( I_{d_1} \) correctly, the curve \( F' - \delta = r^k + K_F r^k' \) is never too steep for policy D.

Possible policy switches

The auxiliary variables are continuous functions of time. From the
necessary and sufficient conditions for optimality,

\[ \lambda_1 = -(\rho F' - \lambda_1 \delta) \]

\[ \lambda_2 = -[\rho t(F' - \delta) + \nu(a + \delta) - zr^2 - \lambda_2 \delta]. \]

Using these relationships with the fact that \( \lambda_2 = 0 \) along

\[ F' - \delta = r^2 + r^2 K_e \]

it can be shown that the only possible policy

switches are policies F and J into either D or E.
Appendix No. 6.

Shapes of curves

A. Both countries completely specialized

\[ r^\alpha = R - \delta - r^\alpha \lambda_f + \frac{E_1 r^\alpha}{D^\alpha} \]

Differentiating totally,

\[ r^\alpha dK_f = R'(dK_d + dK_f) - \left( r^\alpha \lambda_f + r^\alpha \gamma_f \right) dK_f + \frac{E_1}{D^\alpha} r^\alpha dK_f . \]

Therefore

\[ \frac{dK_f}{dK_d} = \frac{R'}{2r^\alpha - R' - r^\alpha \lambda_f}, \quad \text{assuming } r^\alpha \lambda_f > 0 . \]

Thus

\[ -1 < \frac{dK_f}{dK_d} < 0 . \]

Since this curve is equivalent to

\[ r^\alpha = (1 - \tau)(R - \delta) \]

with \( \tau > 0 \), it lies completely below the curve \( r^\alpha = (R - \delta) \).

B. Home country completely specialized and foreign country incompletely specialized

\[ r^\alpha = R - \delta - \frac{E_1}{D^\alpha} \frac{\partial r^\alpha}{\partial \lambda_f} (1 - \nu r^\alpha) \]

\[ dr^\alpha = R'(dK_d + dK_f) + \frac{E_1}{D^\alpha} \frac{\partial r^\alpha}{\partial \lambda_f} \frac{\partial r^\alpha}{\partial \gamma_f} dK_f . \]

Therefore

\[ \frac{dK_f}{dK_d} = \frac{R'}{-R' + (r^\alpha)^2/D^\alpha}, \quad \text{as } dr^\alpha = 0 , \text{ and } \]

\[ \frac{dK_f}{dK_d} < -1 \quad \text{with } \frac{dK_f}{dK_d} , \quad \text{when good 1 is labour-intensive abroad, is greater than } \frac{dK_f}{dK_d} , \quad \text{when good 1 is capital-intensive abroad.} \]

If it is optimal for the home country to subsidise foreign investment, this curve lies above \( r^\alpha = R - \delta \). Otherwise it lies below.
the curve \( r^\alpha - R - \delta \).

C. Home country incompletely specialized and foreign country completely specialized

\[
r^\alpha = R - \delta - r^\alpha K_f + E_r \frac{r^\alpha}{dK_f}.
\]

Differentiating totally,

\[
r^\alpha dK_f = -(r^\alpha K_f + r^\alpha) dK_f + E_r \frac{r^\alpha}{dK_f} dK_f - \frac{1}{dK_f} (dK_d + dK_f)
\]

\[+ R'dp.\]

From the tariff formula,

\[
p - p^\alpha = \frac{-E}{dp}, \text{ so that}
\]

\[
dp = \frac{F'}{dp} (dk_d + dk_f) \text{ for given } p^\alpha.
\]

Therefore

\[
\frac{dK_f}{dK_d} = \frac{F'(R' - r^\alpha)}{-F_1'(R' - r^\alpha) + 2r^\alpha E_1 r^\alpha'.}
\]

If good 1 is labour-intensive at home, then \( 0 > \frac{dK_f}{dK_d} > 1 \).

If good 1 is capital-intensive at home, then \( 0 > \frac{dK_f}{dK_d} > -1 \) except when \( R' > r^\alpha \) in which case \( \frac{dK_f}{dK_d} < -1 \).

The curve \( R - \delta = 0 \)

Differentiating totally,

\[R'dp = 0.\]

From the optimal tariff formula, \( dp = \frac{F_1'}{dp} (dK_d + dK_f) \).

Therefore

\[
\frac{dK_f}{dK_d} = -1.
\]

The curve \( r^\alpha = (R - \delta) \)

Differentiating totally,

\[
r^\alpha dK_f = R'dp.
\]

From the optimal tariff formula,
\[
D_1^{\text{r}_1^0 \text{d}_K} = F_1^{' \text{R}'} (d_{K_d} + d_{K_f}).
\]

Therefore
\[
\frac{d_{K_f}}{d_{K_d}} = F_1^{-' \text{R}'} + D_1^{\text{R}'} \text{R}.
\]

If good 1 is capital-intensive at home \( R' > 0, F_1' > 0 \) and \( d_{K_f} \), \( \frac{d_{K_f}}{d_{K_d}} \) lies between 0 and -1.

If good 1 is labour-intensive at home \( R' < 0, F_1' < 0 \) and
\[
0 > \frac{d_{K_f}}{d_{K_d}} > -1.
\]
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