PRICE UNCERTAINTY, PRODUCTION AND PROFIT

being a thesis submitted by

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for admission to the Degree of Doctorate of Philosophy
in The Australian National University.

This thesis presents my own ideas. It is not a joint study.

Clem Lindell
My motive for doing this analysis was to examine some of the effects of price uncertainty and instability upon production and profit in the hope that this would give some insight into agricultural price policy. So many theoretical problems were involved that it soon became apparent that it would be necessary to concentrate upon them rather than upon the policy side of things. Consequently, this thesis presents my current thoughts upon the theory of production and profit under conditions of price uncertainty and under pure competition. However, there is much more to be done at the group level, i.e. at the industry and economy level. Also, more allowance should be made for the element of time.

I thank Professor F. H. Gruen who has supervised my work at the Australian National University. He has always encouraged me to form my own ideas, express them and to learn techniques to develop them. Nevertheless, he has not failed to express such criticism as he felt was warranted. I am also grateful to Professor T. W. Swan for his constructive criticism. Discussions with Dr. J. Dillon, Dr. A. Fowell and other members of the University of Adelaide have been stimulating and useful. I am also indebted to Professor W. P. Hogan, P. W. Sherwood and other members of Newcastle University College who helped to create my interest in research and inquiry.

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CLEM TISDELL.
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CHAPTER I: Uncertainty is important in economic life because actions limit future possibilities. In retrospect, one is always liable to be disappointed by an action if it is chosen under conditions of uncertainty. We can rarely avoid the possibility of this disappointment but we can allow for it in deciding upon our actions.

CHAPTER II: We suppose that an individual acts rationally if he selects from his set of alternative acts, as known by him, the one whose possible outcomes are most preferred by him. This formulation of rational decision theory raises the question of what is meant by possible. We suppose the possibilities to be subjective and to express the individual's considered opinion. They need not accord with "the" objective set of possibilities.

Sometimes an individual's preference ordering indicates that his rational behaviour is in accordance with a simple rule or criterion. We discuss some criteria by dividing them into a group which is dependent upon cardinal probabilities and one which is not. Account is taken of Simon's objections to "orthodox" decision theory and it is suggested that "satisficing" and maximising are not inconsistent. The concept of a decision set is then introduced. The decision set is any set of values to which the firm applies a criterion in order to decide upon its act.

CHAPTER III: In this chapter, the Wald, maxmax, Hurwicz and expected profit criterion are applied to the determination of the purely competitive firm's level of production. An underlying assumption of this chapter and of all others
except VII is that the firm makes its output decision for \( t \) in \( t-n \) and that this output decision is unalterable \( n \geq 1 \).

The principle of maximum equivalence is introduced. The basic idea is that although the firm may not decide upon its output by consciously maximising an imputed profit function upon the basis of some price vector, nevertheless its actual output maximises profit for some price vector. Sometimes it is easy to identify the appropriate price vectors for different criteria and for different modes of behaviour.

**CHAPTER IV:** By using the principle of maximum equivalence, it is possible to apply Hicks' production theorems of Value and Capital in a different context to his. It can be applied to predict the effect upon a firm's output of a change of its criterion or of a change of its decision set of possibilities.

**CHAPTER V:** Two models are introduced. The first model shows that if price uncertainty exists increased price instability need not increase a firm's average profit. The publication of this model lead Oi to revise an assertion of his and to state: "So long as price instability contains a systematic component, greater price instability will lead to higher expected profits". By using a second model, we show that even if the correlation between actual price and shadow ("predicted") price is positive and constant, increased price instability can decrease the firm's average profit.

A "general" average profit function based upon a linear marginal cost function is introduced and some theorems are derived for the effect of price uncertainty and price instability upon average profit. We show that a recent
model of Nelson's involves special restrictions upon this average profit function.

**CHAPTER VI:** In this chapter two models are used to express doubts about Baumol's hypothesis that increased price uncertainty increases the probability of a firm adopting a flexible technique, and to support Stigler's conclusion that, under certainty, increased price instability increases the probability of the adoption of the flexible technique. The first model is based upon a general cost function but makes restrictive assumptions about the relationship between actual and shadow prices whereas the second is based upon a quadratic total cost function but allows a more general relationship between shadow and actual prices.

**CHAPTER VII:** A model is introduced which permits actual output to diverge from plan at extra cost. The model is based upon a quadratic cost function and enables us to reconsider Oi's revised conclusion and other theorems upon the effect of price uncertainty and instability upon average profit. Also, it enables us to see the relevance of Hart's notion of flexibility for technique choice.

**CHAPTER VIII:** The possible effect of price uncertainty upon industry profit (at a point of time) is briefly examined. If price prediction errors lead to a monopoly-like restriction of output, price uncertainty can raise industry profit above its level under certainty. This possibility has been overlooked by Nelson in a recent article but has been foreseen by Knight.
CHAPTER IX: We assume the economy's production function to be convex and consider the effect upon aggregate production of different shadow price ratios on the part of different firms in the economy. We then put the proposition that forward price schemes can feasibly raise aggregate production and consumption above their free competition level even if their operation involves increased surpluses of commodities.

APPENDIX I: This appendix illustrates the idea that it is possible to combine the "satisficing" and maximising approach. The firm is seen as wishing to maximise expected profit subject to the achievement of a satisfactory security level.
CHAPTER I
Uncertainty and Economic Actions

"We live only by knowing something about the future, while the problems of life, or of conduct at least, arise from the fact that we know so little. This is as true of business as of other spheres of activity. The essence of the situation is action according to opinion, of greater or less foundation and value, neither entire ignorance nor complete and perfect information, but partial knowledge. If we are to understand the workings of the economic system we must examine the meaning and significance of uncertainty..."1 (F. H. Knight)

"We live in a world full of contradiction and paradox, a fact of which perhaps the most fundamental illustration is this: that the existence of the problem of knowledge depends on the future being different from the past, while the possibility of the solution of the problem depends on the future being like the past.2 (F. H. Knight)

Man has less than perfect knowledge of his universe.

He lives and acts in the twilight. In his partial state of knowledge he is always liable to be incorrect in his predictions and to be dissatisfied, ex post, with his actions. His state of knowledge is important to him because it shapes his actions and these in turn affect the pattern of the universe. If he lacks full knowledge of the consequences of any action, his adopted acts may bring results which are contrary to his intention, or which do not satisfy his ultimate aims to the fullest possible extent. Therefore, it is not surprising that much of mankind's time has been directed towards judging the outcomes of different actions.3

2 Ibid. p. 313.
Economists and economic decision makers are continually wrestling with the problem.

We are rarely able to perfectly predict the consequences of any action. Yet our existence requires us to act at every point of time. In acting at any point of time, we can only accept our imperfect state of knowledge even though we can improve it over time.

Changes in our state of knowledge can affect the optimality of an act. An act which is optimal upon the basis of the information which is available at one period can appear to be less than optimal upon the basis of the information available at a later period. A difference between a realised act and an alternative one which subsequently appears to be optimal can arise because of the initial existence of uncertainty. If uncertainty surrounds the outcome of an act, the individual is liable to feel disappointed about his behaviour when further information comes to hand. A variable is uncertain if its value cannot be predicted with zero probability of error. The probability of this error may be imaginary or real, i.e., it may be subjective or objective. Some of the intricacies which are connected with the distinction between subjective and objective probabilities will be discussed in Chapter II. For the time being, we suppose that probability of the error to be real.

Then, an ordinary economic agent can make errors in predicting the value of a variable e.g., a producer can make

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4 According to this definition, any variable which has a known probability distribution with positive probabilities attaching to more than one value is uncertain. Uncertainty according to our definition includes risk and uncertainty in Knight's sense. See Knight, *op. cit.*
errors in predicting the value of price,
(i) because he sees the variable as being dependent upon an incorrect (general) relationship;
(ii) because, even though his general relationship is correct, he cannot obtain sufficient information to specify all of the values which enter into it,
(iii) because the value of the predicted variable depends upon random influences, i.e., at least one random variable.

It is debatable whether (iii) should be considered as a separate possibility from (i) and (ii). It is included as a concession to the view that non-deterministic (random) factors can arise in nature (in reality). But irrespective of whether they do, possibilities (i) and (ii) do appear to be frequent causes of errors.

Economic theory can take account of uncertainty in two distinct senses. First, it can take account of the uncertainty of economic agents. Secondly, it can allow for predictive inadequacies in the theory itself. This allowance may be necessary even when all economic agents are acting under certainty.

This analysis only takes account of uncertainty in the first sense. It deals with the effect of price uncertainty upon production and profit under conditions of pure competition. Since the theory of the firm's production decisions under price uncertainty can be considered as a particular application of the general theory of decision making under uncertainty, it seems best to consider this general theory before applying it to production. The general theory of decision making under uncertainty is considered in Chapter II.

In Chapter III some of the criteria which are outlined in Chapter II are applied to the determination of optimal levels
of output for the firm. In Chapter IV, we consider how these optima change as the firm changes its criterion and as its set of price possibilities change.

In dealing with these matters, we introduce and apply shadow price vectors which reduce production behaviour to simulated maximisation behaviour. The basis idea is that although the firm may not decide upon its output by consciously maximising anticipated profit upon the basis of a price vector, its actual output maximises profit for some shadow price vector. As these shadow price vectors change, so also does the firm's output. The firm's shadow price vector can change because the firm changes its criterion or because it changes its predictions. For some criteria the relevant shadow prices can be specifically identified and changes of production for changes of criteria and predictability can be described by using Hicks' production theorems. 5

The maximum equivalence theorem is useful for generalising the effect of price uncertainty and instability upon average profit. It enables the influence of these factors to be considered under diverse conditions of behaviour and predictability. In dealing with the effect of price uncertainty and instability upon the firm's average profit in Chapter V, we first develop a model which relies upon a general cost function but assumes special predictability assumptions. Then we introduce a model which relies upon the assumption of a linear marginal cost function but allows for a wider range

of relations between shadow and actual prices. A similar procedure is followed in Chapter VI for discussing the effect of price uncertainty and instability upon the difference in average profit from alternative techniques, and consequently, their possible impact upon technique choice. In Chapter VII, the static decision making assumption which is used in Chapters V and VI is relaxed. In discussing the topics of Chapters V, VI and VII we take account of the views of Baumol, Hart, Oi, Marschak, Nelson and Stigler.

As already pointed out, we sometimes assume the existence of quadratic total cost functions in Chapters V, VI and VII. Given pure competition, the latter assumption implies that any firm's profit function is quadratic. Since the average value of a variable which is dependent upon a quadratic function can be expressed in terms of the mean, variance, and covariance values of the independent variables of the function, the assumption of quadraticity simplifies any analysis of the average value of the dependent variable. Accordingly, it will simplify our analysis of changes in average profit.

The analysis is restricted to the level of the firm until Chapter VIII is reached. In Chapter VIII, the effect upon industry profit of (i) divergence between average shadow price in the industry and equilibrium price, and (ii) changes in the dispersion of shadow price between firms is taken into account. Finally, in Chapter IX, the impact upon aggregate production of divergent shadow prices is specified for a simple multi-product model, and
some influences of forward price policies upon aggregate production and consumption are noted. It is suggested that forward price policies can increase aggregate consumption.

These last two chapters are very sketchy.

Hence, the analysis is developed by proceeding from the firm to the industry level and from there to the economy level. In turn we deal with the influence of price uncertainty upon (i) the production of the individual firm (ii) the firm's profit, (iii) the industry's profit, and (iv) the "economy's" aggregate output.

In these explorations, we shall not cover (nor could we) the whole terrain of the economic theory of uncertainty nor reach a pinnacle high enough to survey it generally. Our path is only one of the many possible ones. Different paths may lead us to suspect that the general lay of the land is different. But every sortie brings to light new information which enables us to push just a little closer to the pinnacle.
CHAPTER II

Some Theories of Decision Making.

A. A Rational Decision Theory.

In order to construct a theory of rational choice under uncertainty, we make the following assumptions:

(i) At any point of time, the individual knows of a set, \( A^0 \), of all the possible acts open to him. If \( A \) represents the set of alternative acts available to the individual, \( A^0 \subseteq A \).

(ii) The individual associates a set of possible outcomes, \( O^0 \), with the set of possible acts, \( A^0 \). With each act he associates a set of outcome possibilities. We represent the set of outcome possibilities attaching to the \( i \)-th act by \( O_i^0 \). The elements of \( O_i^0 \) might consist of ordered pairs which indicate the probability of the various levels of profit which the individual associates with the act \( A_i^0 \). Where \( \Pi \) represents profit and \( \rho \) represents probability, \( O_i^0 \) might contain the elements \(( \Pi_1, \rho_1 ), ( \Pi_2, \rho_2 ), \cdots, ( \Pi_m, \rho_m )\). With each act the individual associates a number of variables which, in his opinion, have a positive probability of occurrence. For the act \( A_i^0 \), these variables form a set \( V_i \) and for the set of acts, \( A^0 \), they form a set \( V \).

(iii) The individual has a preference ordering over the \( O_i^0 \) sets and these meet the usual consistency requirements of transitivity and symmetry.

An individual's action is rational if he adopts an act, \( A_k^0 \), whose set of possible outcomes, \( O_k^0 \), is no less preferred by him than the set for any act contained in \( A^0 \). If there are \( i = 1, \cdots, m \) alternative acts contained in \( A^0 \), the individual
only acts rationally if he selects an act, $A_k^0$, such that
$$O_k^0 \geq O_i^0, \ i = 1, \ldots, m.$$  

B. The Meaning of Possible.

This formulation of decision theory raises the question of what is meant by the set of possible outcomes. The above possibilities, $O^0$, are subjective and express the individual's considered opinion. Given his existing knowledge, the individual considers it to be correct (i.e., to be "rational") to entertain these possibilities. In specifying the possibilities, $O^0$, the individual may attach weights (probabilities) to the elements of the $V$ sets. The individual's subjective formulation of possibilities need not accord with "the" objective set. There are two different conceptions of what constitutes the objective set. These are (i) the logical probability and (ii) the relative frequency views.

The logical probability view is that given the knowledge to which the probability of a conclusion is to be related, then it is rational to entertain a certain degree of belief, and no other degree of belief, as to the truth of the conclusion. It is held that the appropriate rational degree of belief and, therefore, the appropriate level of objective probability can be deduced by logical methods. This formulation is held to be independent of the individual and to be true "to the outside world". J. M. Keynes\(^1\) and R. Carnap\(^2\) have developed this view.

The relative frequency view is that the objective probability of an event, under a given set of conditions, is the

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relative frequency of its occurrence as the number of repetitions of the given set of conditions tends to infinity. Von Mises \(^3\) and Reichenbach \(^4\) are exponents of this view. Since actual sequences are finite it is at most only possible to approximate this objective probability distribution empirically. Unlike the logical probability concept, the relative frequency concept of necessity attaches cardinal weights (i.e., relative frequencies) to the possible values of the variables.

C. The Relationship between Criteria and the Rational Theory of Behaviour.

Sometimes an individual's preference ordering indicates that his rational behaviour is in accordance with a simple rule or criterion. A rule or criterion is in accordance with the individual's rational behaviour if it always leads to the selection of an act, \(A_k\), such that \(C_k^0 \geq C_i^0, i = 1, \ldots, m\). The rule as applied to the set \(C_0^0\) may be a multi-stage one. For example, its first stage may involve an operation on \(C_0^0\) to construct a new set, \(\hat{C}_0^0\). Then an operation (also, a rule) may be applied to the set \(\hat{C}_0^0\) to select the rational act. Some criteria which can be applied to the set \(C_0^0\) can also be a stage in other rules. For instance, the maximin criterion, which will be discussed later, can be applied to \(C_0^0\) to select an act. It is a second stage in the following criterion, "Maxmin' upon the basis of possible profit values which have a probability greater than 0.001. This rule involves the following two stages:

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(i) Select from the sets of possible profit values associated with each act, those profit values which have a probability of greater than 0.001. Let these values form a set $\Pi$.

(ii) From the set $\Pi$ solve for the maxmin level of profit. In introducing some criteria, we shall suppose them to be complete i.e., to apply to the set $O^0$, and not to be a stage in a more general criterion.

It is possible to divide the criteria into two broad groups. Group (a) includes all criteria which are dependent upon cardinal probabilities and group (b) includes those which are not. Group (a) encompasses all criteria which are dependent upon some moment of the choice variables. Group (a) includes

(i) expected profit maximisation;

(ii) expected utility maximisation;

(iii) criteria relying upon a preference ordering of first and second moments of choice variables, and even higher moments;

(iv) the maximisation of a preference functional over the total shape of the probability functions attaching to the variables, and

(v) some probability of loss criteria.

Group (b) includes

(i) Shackle's psychological theory,

(ii) the maxmin,

(iii) the Hurwicz and

(iv) the Savage regret criterion.

Let us consider each of these criterion.
D. Criteria which are dependent upon Cardinal Probabilities

(i) The expected profit maximisation criterion. If $\rho_{ij}$ is the
$j$-th state of nature and, if $a_{ij}$ is the money value attaching
to the outcome of the $i$-th act or strategy when the $j$-th
state of nature prevails, then the expected value of the $i$-th
strategy is

$$E_i = \sum_{j=1}^{n} \rho_{ij} a_{ij} \quad (i = 1, \ldots, m)$$

The optimal act is the one which maximises this expression.

This criterion is sometimes referred to as Bayes', but
Thomas Bayes' expression of it is at most implicit. This
criterion attaches no importance to moments other than the
first. It also implies that utility varies linearly with the value
of the money prize.

(ii) The expected utility maximisation criterion. Assuming utility
to be cardinal, let $u_{ij}$ be the utility attaching to the $i$-th act
when the $j$-th state of nature occurs. The expected utility
of the $i$-th act is then

$$E(u_i) = \sum_{j=1}^{n} \rho_{ij} u_{ij} \quad (i = 1, \ldots, m)$$

The optimal act is the one yielding the maximum level of
expected utility.

5 Marschak calls it Bayes. See J. Marschak: "Probability
in the Social Sciences", pp. 166-215 in Mathematical
Thinking in the Social Sciences, edited by P. L. Lazarsfeld,

6 Thomas Bayes: "An Essay Towards Solving a Problem
in the Doctrine of Chances" reprinted in Biometrika
Vol. 45, parts 3 and 4, pp. 293-315.
When Daniel Bernouilli\(^7\) advanced this criterion in 1733, he assumed cardinality of utility. Ramsey\(^8\), von Neumann and Morgenstern\(^9\) have been able to show that cardinality of utility follows from the assumption of a small number of simple axioms. Von Neumann and Morgenstern demonstrate that, if these axioms\(^10\) are fulfilled then utility is measurable up to a linear transform. Savage\(^11\) and Markowitz\(^12\) have shown that if the individual accepts the Neumann and Morgenstern utility axioms then to be consistent he must act so as to maximise expected utility. Logically expected utility maximisation is the only criterion consistent with these utility axioms. In consequence a number of writers reject such criteria as the maxmin gain, maxmin regret and Hurwicz upon the grounds of inconsistency. But this assumes that the utility axioms are both relevant and adequate. Indeed, their


\(^10\) For a statement of these axioms see H. Markowitz: Portfolio Selection, John Wiley and Sons, New York, 1959, pp. 229-234 or Appendix I of this thesis.


\(^12\) Markowitz: Op. cit., pp. 235-242
adequacy is questionable. In particular, the utility axiom that there is always some combination of probabilities each between zero and unity which will make an individual indifferent about participating in any lottery is open to objection. The individual may be unprepared to participate in a lottery in which a possible outcome could involve him in starvation, bankruptcy or a tremendous fall in social status. Therefore, as I have argued elsewhere, minmax and other criteria should not be rejected just because they fail to satisfy the Neumann and Morgenstern utility axioms. In those circumstances in which the maxmin gain criterion is inconsistent with the Neumann and Morgenstern utility index, one must reject one or the other in order to preserve consistency. The necessity for this choice will become apparent when we consider the maxmin criterion.

(iii) Preference orderings over the moments of payoff. These preference orderings may refer to any number of moments. They may refer to the mean alone, to the mean and variance, or to all possible moments. From the set of moments available to him, the individual selects an act which gives him his most preferable moment combination. Marschak.

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has considered the firm as having a preference ordering over all of the moments of the probability distribution of such variates as profits and the value of assets. The firm is seen as attempting to maximise its utility functional for these moments subject to its technical restrictions.

(iv) The maximisation of a preference functional over the total shape of the probability functions attaching to the payoffs.

Tintner\textsuperscript{15} has suggested this rule. A drawback of this rule is that it requires a great deal of knowledge about preference orderings.

(v) Probability of loss criteria. Several economists have argued that negative returns should enter specifically into some criteria. These negative values have been allowed for in several ways. Domar and Musgrave\textsuperscript{16} have suggested that the most preferable policy might be selected from a preference ordering over the expected total gains and expected losses of different policies. This approach ignores the dispersion of the losses. A. D. Roy\textsuperscript{17} argues that 'safety first' is the guiding principle of the firm, and consequently, that it wishes to minimise the probability of a disaster level of income.

\begin{itemize}
\item \textsuperscript{17} A. D. Roy: "Safety First and the Holding of Assets", Econometrica, Vol. 20, 1952, pp. 431-449.
\end{itemize}
have suggested that some firms may wish to maximise their expected profit subject to the condition that the probability of their profit being below a pre-set level be not less than a satisfactory value.

E. Criteria which do not rely on Cardinal Probabilities

Let us now deal with those criteria which do not require the formulation of cardinal probabilities for their application.

(i) Shackle's psychological theory. G. L. S. Shackle criticises the application of cardinal probability estimates to decision making under uncertainty. He argues that it is impossible to calculate relative frequencies for many economic occurrences because they are unique and non-repetitive. Although this is so, it may still be the case that decision makers do assign weights to the possible outcomes. We may if we wish, call these weights cardinal probabilities. Shackle doubts the widespread use of such weights. He suggests that the individual assesses the uncertainty of the possible outcomes in terms of potential surprise. The potential surprise function represents ordinally the degree to which an individual would be surprised by various outcomes. By considering the potential surprise function along with a function representing the degree of "interestingness", focus values and a gambler indifference system, Shackle formulates his rule for choice under uncertainty. The degree of interestingness function depends

18 C. Tisdell, op. cit.

ordinally on potential surprise and the value of the possible outcomes. The primary focus values for any scheme (act) are determined from the points of tangency of the potential surprise function and the contours of the degree of interestingness function. These primary focus values are standardised and are transferred to the gambler's indifference map. From this map the optimal scheme (act) is determined. This scheme is the one having the most preferred set of standardised focus values in the set of attainable values.

Mostly Shackle's model yields two focus values. While it is agreed that the individual may only give active consideration to a restricted set of values, it seems unduly restrictive to suppose him to focus his attention upon two. While Shackle's model seems to have a psychological counterpart it is, nevertheless, fundamentally indeterminate for it does not yield unique standardised focus values. If both the degree of potential surprise and the degree of interestingness are taken as pure rankings, then the potential surprise function and the degree of interestingness function fail to take up unique positions. In consequence, the focus and standardised values are not unique. Shackle's situation is analogous to trying to maximise an ordinal function (in this case the degree of interestingness) subject to an ordinal restriction (in this case the potential surprise function) - the solution is indeterminate. ²⁰

(ii) The maxmin criterion. According to this criterion, the

individual wishes to ensure himself of a payoff which is no less than the maximum of the minimum possible payoffs for the acts in $A^0$. Let the payoffs be in terms of money values. Then, if $a_{ij}$ represents the individual's money payoff for the $i$-th act when the $j$-th state of nature prevails, the maxmin act is one which corresponds

$$\max_i \min_j a_{ij}$$

The rule stated here is for pure and not mixed strategies.

A slightly different rule is the maxmin expected gain rule. The optimal act according to this rule is one which ensures the agent of at least the maximum of the least possible expected gains for different strategies. This criterion can involve mixed strategies. If the firm adopts a mixed strategy it runs the risk of obtaining less than the maximum of its least possible gains. With mixed strategies the rationale for a maxmin strategy i.e. to make sure of at least a certain minimum gain, disappears. If any individual is prepared to run a risk by adopting a mixed strategy, then we would not be surprised if he really wished to maximise expected gain (or utility) because the security aspect becomes almost insignificant in the minmax expected gain approach.

Sometimes an analogy is drawn between situations in which the minmax expected gain criterion is applied and situations involving zero sum two person games. The economic agent (firm, statistician, etc.) is assumed to be opposed by a fictitious player, Nature. Nature has a

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number of strategies available to it and the economic agent has a number of acts open to it. The outcome for both parties will be determined by their simultaneous strategies. The analogy cannot be pressed very far because complete correspondence requires (i) the utility gains of the economic agent to be equal to the utility losses of Nature and (ii) Nature to be an active opponent minimising its maximum level of expected loss. In addition, the economic agent must attempt to maximise its minimum level of expected gain.

A stable solution for the game will either exist for unmixed or mixed strategies. In these circumstances there is no inconsistency between the utility axioms and the use of the maxmin criterion because, if Nature always adopts its minmax loss strategy, the firm will always maximise its expected utility by adopting its maxmin act. One can object to the above formulation because it assumes Nature to be an active opponent. If Nature is a passive opponent, use of the maxmin criterion can be inconsistent with the Neumann and Morgenstern utility axioms. If the economic agent has some knowledge of the probabilities with which Nature "plays" its strategies, then its expected utility can be greatest for a non-maxmin act. Hence, a divergency arises between the maxmin rule and the expected utility maxim. If one is going to apply the maxmin rule in such circumstances, it is necessary to reject the applicability of the Neumann and Morgenstern utility axioms and the associated utility index.

The maxmin gain criterion (also called the Wald or
minmax loss criterion)\textsuperscript{22} is extremely conservative. It is consistent with the preference ordering of an individual who is not prepared to take a risk. It is so security biased that we rarely expect it to be applied to the set of possibilities, \( O^0 \).

(iii) The maxmax criterion. The maxmax stands at the opposite end of the gambling spectrum to the minmax. The individual bases his decision entirely upon the best possible payoff. The agent's optimal maxmax policy is the one for

\[
\max_i \max_j a_{ij}
\]

When applied to the set \( O^0 \) it implies an extreme type of gambling behaviour.

(iv) The Hurwicz criterion. The Hurwicz criterion\textsuperscript{23} is based upon a weighting of the greatest and least possible gain for each act. A fixed number, \( \beta \), which reflects the agent's pessimism, is attached to the least possible gain for each act and an optimism weight, \( 1 - \beta \), is attached to the greatest possible gain for each act. Given that \( 0 \leq \beta \leq 1 \), an index

\[
H_i = \beta \min_j a_{ij} + (1 - \beta) \max_j a_{ij}
\]

is computed for each act. The optimal Hurwicz act is the one which maximises this index i.e., the one for which


\[ \text{Max} \left[ \beta \text{Min} a_{ij} + (1 - \beta) \text{Max} a_{ij} \right] \]

occurs. If \( \beta = 1 \) this criterion is equivalent to the Wald and if \( \beta = 0 \), it is equivalent to the maxmax. The Hurwicz criterion focuses attention upon the best and worst possible outcome for each act to the exclusion of all others. It is possible to develop criteria which besides giving special weights to the extremes also give weights to other values.

(v) The Savage Regret Criterion. Prior to his presentation of the personalistic probability theory of expected utility maximisation, Savage suggested a regret criterion \(^24\) which allowed for the alternative which is possibly foregone by selecting one act in preference to another. Originally this criterion was in terms of utilities. Let \( u_{ij} \) be the utility for the outcome of the \( i \)-th act and the \( j \)-th state of nature. Then a regret matrix, \( r_{ij} \) with the elements

\[ r_{ij} = \text{Max}_j u_{ij} - u_{ij} \]

indicates the possible regrets. The firm's aim is to ensure that its regret does not exceed the minimum of the maximum possible regret values for the acts. Its optimal act corresponds to

\[ \text{Min}_i \text{Max}_j r_{ij} \]

It is by no means apparent that regret ought to be measured by differences in utility nor that it should vary linearly with such differences \(^25\). Since the minmax regret criterion is inconsistent


with the Neumann and Morgenstern utility index, we replace the $u_{ij}$ by the money values, $a_{ij}$.

Before discussing Simon's criticism of most of the above criteria, we make the following two points:

(i) Since preference orderings differ between individuals, and differ for the same individual in different circumstances, a wide range of criteria can be consistent with rational behaviour.

(ii) In any situation which is objectively the same, any two individuals using the same criterion may act differently even if they act rationally. This can occur if their set of known acts i.e., their $A_0$ acts differ, or if their sets of possible outcomes, $O_0$, differ.

F. Simon's Objections to "Orthodox" Decision Theory.

Simon objects to all of the previous decision making theories (with the possible exception of Shackle's) upon the grounds that they imply that (i) the individual is more rational that he actually is, and (ii) they require knowledge of him which is in excess of the capacity of the human mind. This is not only true of the traditional theories just examined but is also true of such criterion as the Wald since they require a knowledge of the outcomes for all possible acts and states of nature. "The capacity of the human mind for formulating and solving complex problems is very small compared with the size of the problems whose solution is required for objectively rational behaviour in the real world - or even for a reasonable approximation to

such objective rationality", 27

Simon's objection seems to be applicable to those decision theories in which the individual is supposed to have sufficient information to calculate "the" set of objective outcomes and to discover all the alternative acts which are open to him. These theories assume that the individual by using his thought processes does discover the objective set of possible outcomes and all of his possible acts. As a description of reality such theories appear unsound. The rational decision theory which was formulated at the beginning of this chapter makes more modest assumptions. The individual is not assumed to discover "the" objective set of possible outcomes nor to know the set of all of his possible alternative acts. Nevertheless, one suspects that Simon would object to this modest rational decision theory. Some possible objections are:

(i) The theory assumes that the individual acts after considering the whole of his preference ordering. In practice, the individual may focus his attention upon only a small part of his ordering (for emotional reasons) and this may lead him to act irrationally as previously defined.

(ii) It assumes that the individual does have a consistent ordering over the possible outcomes. In fact, the individual may act without formulating that ordering. Simon seems to be very critical of criteria which are based upon extrema or some combination of extrema problems. He is of the opinion that the solution of such problems exceeds

27 Simon: Models of Man, p. 198.
the mental capacity of most "ordinary" individuals. This is an empirical question, but it seems possible that a lot of ordinary people can consciously or subconsciously maximise variables. In accepting this view, one need not reject Simon's satisfier approach. In some circumstances, the satisfier model is probably a close description of actual behaviour.

G. Simon's Satisfier Theory.

According to Simon the individual divides outcomes into those which are satisfactory and into those which are unsatisfactory according to whether they meet his aspiration level or not. In attempting to attain his aspiration level, the individual at first considers a restricted subset of acts, $A^0$, of the total class of possible acts, $A$. If there is no act within the set $A^0$ which permits the individual to achieve his aspiration level, then depending upon his persistence, he may adjoin new acts (this will involve effort and cost) or he may reduce his aspiration level. Once the individual attains his aspiration level he is satisfied and does not wish to attain a minimum or maximum solution.

Simon's theory has consistency implications. If the individual does not select an act which meets his aspiration level then he acts inconsistently with the criterion. This theory is incomplete in so far as it does not indicate what determines the aspiration level. While Simon rejects maximisation or minimisation criteria they are not inconsistent with a satisfier approach. As I have suggested elsewhere, the firm may wish to maximise one variable subject to satisfactory levels being achieved in others.

Such an approach differs from Simon's in so far as it supposes maximisation or minimisation to be important. Such maximisation or minimisation may take place for outcome values attached to a subset of all possible acts.

H. A Theory for Consistent but not Necessarily Rational Behaviour

We can conceive of theories of behaviour which are different to the one outlined at the beginning of this Chapter but which, nevertheless, rely on some consistency postulates. For instance, a theory satisfying the following assumptions does:

(i) At any point of time, the individual knows of a set, \( A^0 \), of all the possible acts, \( A \), open to him, \( A^0 \subseteq A \).

(ii) The individual associates with (assigns to) the \( i \)-th act of the set \( A^0 \) a set of variables. We represent this set by \( D_i \) and assume all such sets assigned to \( A^0 \) to form a set \( D \). The elements of the \( D_i \) are called decision variables and may include weights (probability) values. The \( D_i \) sets may or may not accord with the \( O_i^0 \) sets and may be obtained by operations on the \( O_i^0 \) sets.

(iii) To the set of decision variables, the individual applies a rule or criterion, e.g. a minmax, expected profit or maxmax rule, to select the set from all the \( D_i \) which satisfies this rule. Let us represent this set by \( D_h \).

(iv) The individual adopts the act which corresponds to the \( D_i \) set satisfying this rule or criterion. He adopts act \( h \).

An act adopted in accordance with this theory need not accord with the rational act as defined in the earlier theory i.e. act \( h \) need not accord with the rational act \( k \) as previously defined.
In special cases it will.

By using the above theory, which is dependent upon "the" decision set of possibilities, we shall develop the formal conditions of production which must be satisfied if the firm's output decision is to be in accordance with some of the above criteria. In so doing, we must remember that the decision set of possibilities need not be identical with the set of possibilities, $O^0$, nor with "the" set of objective possibilities. Also criteria applied to $D$ can result in acts which are security biased in one direction when referred to $D$ and security biased in the opposite direction when referred to $O^0$.

In suggesting the latter behaviour model, I do not wish to imply that all behaviour is in accordance with this theory. It is likely

(i) that decision sets are not always specified,
(ii) that the individual sometimes acts without a clearcut aim in mind,
(iii) that, if he has a clearcut aim and decision set, he sometimes makes mistakes in selecting his act.

The above rational and consistent types of behaviour theory are in accordance with the view that behaviour is diverse, and that aims differ from individual to individual and for the same individual in different circumstances. But behaviour is even more diverse than may appear from these theories because individual's can act inconsistently with their aims.
CHAPTER III

Static Production Decisions

A. Introduction

We shall develop some formal production conditions which hold if the firm consistently applies different criteria to a decision set of price possibilities. We shall assume that the decision set of price possibilities (i) contains more than one possible price and (ii) is independent of the criterion used by the firm. Assumption (i) is made in order to retain an essential element of price uncertainty viz: that no one price appears certain. Assumption (ii) is made for notational convenience. It enables us to use the one set throughout the discussion but it is not essential for a formal statement of conditions. We can formally apply criteria to different decision sets e.g. the minmax gain criterion can be applied to some set D' and the maxmax to some set D''.

This chapter gives substance to the following statements:

(i) Given the production model of this chapter, the firm's actual production can always be obtained as the production combination which maximises an imputed profit function which is based upon a shadow price vector.

(ii) For some criteria these shadow price vectors can be readily identified. The solution of the appropriate simulated maximisation problem for any criterion is identical to the production level arising from the criterion's consistent application. The shadow price vector of the simulated maximisation problem is called 'simple' if it is an element of the firm's decision set. The conversion of the solution of a criterion problem to the solution of a simulated maximisation problem is called "simple" if a simple price vector occurs in the simulated problem. The use of shadow prices
effects a simplification (i) because it reduces some decision problems to readily identified maximisation ones, and (ii) because it enables us to treat differences in production as arising merely from differences in shadow prices (shadow price ratios).

The shadow prices enable us to treat diverse behaviour within a manageable framework.

Unless otherwise stated, it is assumed that price is the only variable which is subject to uncertainty. As far as the individual firm is concerned price is assumed not to vary with its output, and also the individual firm is assumed to make static plans. Once static plans are acted upon, they are put into practice without revision. In some period, t-n, the firm plans to produce a specific quantity of output for t, and begins to take steps to achieve this goal. If output is a controlled variable, then subsequently, and irrespective of what happens after t-n, the firm will produce the output in t which is planned in t-n. With dynamic decision making the plan is subject to revision after operations have started. ¹ The assumption of static decision making drastically simplifies the theory of decision making under uncertainty since it eliminates the possibility of variation of the controlled variables after a decision is made. While this assumption is retained for our first approximations, it is relaxed for later ones.

B. Assumptions

Before commencing detailed analysis, let us be specific about the assumptions. The following assumptions are made in relation to the individual firm's production:

¹ The above distinctions between static and dynamic decision making are made by Theil, H. Theil: Economic Forecasts and Policy, North-Holland Publishing Company, 2nd ed; Amsterdam, 1961, Chapter VII "Forecasts and Policy: Problems and Tools".
(i) Prices are subject to uncertainty and the firm's action (decision) set of prices and probabilities can be taken as datum.

(ii) The production function is certain at t-n for production relating to the output of t.

(iii) The outputs for t and the inputs relating to the production of t must be decided prior to t in some period t-n.

(iv) The decision of t-n is unalterable.

(v) The decision of t-n is independent of other decisions and the output of t is solely dependent upon the decision made in t-n.

(vi) The discount rate is zero. This assumption is made as a simplification since interest rates are not essential to the theme of the analysis. Both inputs and outputs are controlled variables. ²

Before beginning technical analysis, we ought to be familiar with some situations in which a decision of t-n exactly determines the output possibilities of t and the combination of inputs which made it possible. Putting legal and institutional factors temporarily to one side, let us concentrate upon physical relationships.

It is possible to conceive of a case in which the output of t is solely dependent upon the input of t-n, and is independent of the inputs of other periods. In this case a unique lag between inputs and outputs occurs and substitution is only possible at t-n. The time points at which substitution is possible can vary. In some industries the decisions will take place in all time intervals,

but in other industries substitution is only possible at spaced intervals. For instance, in agriculture, decisions upon inputs might only be made during the planting season which may occur at spaced intervals. In this theoretical example, it is the lack of substitutability of factors and products at times other than t-n which dictates that they be decided upon at t-n and that the factors be applied at t-n with a view to the resultant output of t.

However, this supposes no factor hiring and no factor disposal lags. In the absence of these lags the prices of the factors hired and disposed of at t-n will normally be certain. But frequently there are lags for institutional, technical and administrative reasons. Because of these lags input prices can be uncertain if a prior decision must be reached for the employment or disposal of factors at future market prices. Allowing for these employment and disposal lags, it might be more realistic to imagine a case in which the output of t is physically and solely dependent upon the inputs of t-o. The inputs of t-o are decided in t-n (n > o) because of the existence of a lag of n-o. In this case, factors and products of t are not substitutable after t-n.

There are other theoretical examples which accord with the above set of assumptions. For instance, the output of t may be dependent upon the inputs of several periods. Yet, if this is the case, all of these inputs must be decided at t-n. This could be done for a wide range of different reasons. Our requirement is that all of the values of the controlled variables which influence the output of t be dediced in t-n.
C. A One Product Model

Before dealing with the n-commodity production problem, it may be best to introduce a simple one good problem. The firm is assumed to produce one product, and the combination of factors which is required for that product is decided in t-n. Although all the factors may not be applied at t-n our general assumptions require that their quantities and time of application be decided then. For this example only, factor prices are assumed to be certain.

The firm's decision problem can be considered as one of choosing an optimal level of output for t if factors are optimally combined by the decision of t-n. In making its decision of t-n, the firm will be uncertain of its product's price in t. In making this decision, we suppose the firm to act upon some identifiable set of price possibilities. This decision set may or may not include all prices which the firm anticipates as possible for its product in t.

In terms of some current language, each price of the firm's decision set represents a "possible" state of nature. Against these possible states of nature the firm has a number of strategies. In this case, the firm's strategies are the levels of output which it can plan in t-n. The firm's strategies and the decision set of prices (states of nature) determine the outcomes i.e. the "possible" profit values which are relevant for the firm's decision. Although the firm's actual profit outcome will depend upon its strategy and the prevailing state of nature, this outcome may not be included in the preceding set of decision possibilities. Despite this, we shall refer to price and profit values of the decision set as possible values. This method of reference will be used until a different usage is specified.
Taking the set of possible prices as datum, let us derive the optimal production levels for some different criteria. In the decision problem, the set of possible prices, \( p \), may consist of a finite or infinite number of price values but these values are within finite bounds. In addition, the firm has a number of possible levels of output, \( x \). The set of possible prices refer to the price possibilities for the output of \( t \) and the output possibilities are the values which can be planned at \( t-n \) for period \( t \). Where the arguments \( x \) and \( p \) signify the possible levels of output and price respectively, the possible levels of profit can be represented by the functional equation

\[
\phi = \phi (x, p) = px - C(x)
\]

where \( C(x) \) represents total cost. If the price possibilities \( p_j, j = 1 \ldots m \), and the output possibilities \( x_i, i = 1 \ldots n \), are finite in number, the possible levels of profit can be shown in matrix form. To the \( j \)-th possible price and the \( i \)-th level of output, there corresponds a profit outcome

\[
a_{ij} = p_j x_i - C(x_i),
\]

Hence, a matrix of profit outcomes

\[
A = [a_{ij}],
\]

can be constructed.

The problem is to determine the firm's optimal level of output for period \( t \) if this must be decided in period \( t-n \) on the basis of \( \phi (x, p) \). A solution only exists if the firm's aim is suitably specified for the problem. Assuming that there is more than one value of price in the decision set (this is a reasonable assumption if price uncertainty exists), we notice immediately that the aim of profit maximisation cannot be used to select the optimal level of output. If marginal cost is not
perfectly inelastic, different optimal values of output will attach to the different possible price values. In the ex-ante situation at \( t-n \), there is nothing in the profit maximisation criterion which enables an optimal output to be selected. Without going into details, let us consider the application of some other criteria to this decision problem.

(i) As generally used, the maximisation of anticipated profit criterion requires that the firm anticipate the occurrence of one particular price. If the marginal cost function is differentiable, output is optimal when the anticipated price equals marginal cost when marginal cost is increasing and average variable cost is covered. Yet this criterion is inadequate for application to conditions of uncertainty. Under uncertainty, the firm will generally anticipate more than one price as possible. If the firm does anticipate a number of price possibilities, the anticipated profit criterion does not enable us to predict the firm's action. The anticipated profit criterion can only be used when the anticipated value of the price parameter is certain.

(ii) If the firm adopts the Wald or maxmin criterion, it wishes to assure itself of a gain no less than the maximum of the minimum gains for all acts. It should produce the level of output for which

\[
\max_x \min_p \ b(x, p) \quad 3.4
\]

occurs. In the matrix case in which the elements \( a_{ij} \) represent profit for the \( i \)-th act when the \( j \)-th state of nature prevails, the firm should select the output level which gives the maximum of the minimum profits in each
row, i.e. it should select the output value for

$$\max_i \min_j a_{ij}. \quad 3.5$$

The Wald criterion, like those to be subsequently considered, is consistent with the fundamental condition for price uncertainty, i.e. with the possibility of more than one price.

(iii) If the firm adopts the maxmax criterion, it aims to maximise its maximum possible level of profit. It should produce the output for

$$\max_x \max_p \phi(x, p). \quad 3.6$$

In the matrix case, the firm should select the output level which gives a maximum of the row maxima of profits, i.e. the output level for

$$\max_i \max_j a_{ij}. \quad$$

(iv) Both the Wald and the Maxmax are special cases of the Hurwicz criterion. The firm adopting the Hurwicz criterion aims to maximise an index consisting of the weighted sum of the maximum and minimum possible profit for each output level. A fixed pessimism weight, $\beta$, is assigned to the lowest profit outcome for each level of profit and, an optimism weight, $(1-\beta)$, is assigned to the highest profit outcome for each output level. The firm wishes to maximise the index

$$\beta \min_p \phi(x, p) + (1-\beta) \max_p \phi(x, p).$$

Its optimal level of output is the output value for

$$\max_x (\beta \min_p \phi(x, p) + (1-\beta) \max_p \phi(x, p)). \quad 3.7$$

The matrix optimal level of output is the value in the rows.
for

$$\text{Max} \left( \beta \text{Min} a_{ij} + (1 - \beta) \text{Max} a_{ij} \right) .$$

When $\beta = 1$ the Hurwicz solution is the same as the Wald and when $\beta = 0$ it is the same as the Maxmax.

(v) If the firm adopts the Bayes criterion it wishes to produce an output to maximise its expected profit. Showing mathematical expectation by $E$, the firm wishes to produce the output for

$$\text{Max}_x E \left[ \phi (x, p) \right].$$

But

$$\text{Max}_x E \left[ \phi (x, p) \right] = \text{Max}_x \phi (x, E(p))$$

$$= \text{Max}_x \left[ E(p) x - C(x) \right]$$

since $x$ is not subject to variation after $t-n$. If the cost function is differentiable, then expected profits will be at a maximum when

$$E(p) = C'(x),$$

$$C''(x) > 0,$$ and average variable cost is increasing.

For a maximum, production must be such that expected price is equal to marginal cost when marginal cost is increasing. In addition a net surplus must be realised on current production.

In the previous example, the optimal conditions for the different criteria have not been stated in detail. We shall now consider an $n$-commodity model and state the solutions of some criteria in detail. In stating the solutions, one of our most important tasks will be to find the price vectors and problems which give equivalent maximum solutions.
D. Conversions to Maxima

If the firm's production possibilities form a convex set and if its production is always such that it is impossible for it to increase its output of any commodity without increasing the input or decreasing the output of another commodity, then there is at least one imputed price vector which ensures that the maximum of its associated imputed profit function occurs for the firm's production bundle. There is always at least one hyperplane which passes through any point on the extremal (limits) of a convex set and which does not pass through its convex hull. Any point upon the extremal of a convex production set corresponds to the extremum of some imputed profit function because under pure competition the firm's imputed profit functions form families of hyperplanes in commodity space. The above assumptions not only limit the firm's production to values upon the extremal of its set of production possibilities but further limit it to particular values upon the extremal. All of these extremal values are associated with the maxima of imputed profit functions. The position can be made clearer by assuming that the firm's production function is at least twice differentiable.

Suppose that

\[ f(x_1, x_2, \ldots, x_q) = 0 \]

is the firm's production function for \( q \) commodities, and that
its profit function. We adopt the Hicksian convention of treating factors as negative products. Assuming specific price values, the necessary conditions for a maximum of $V$ subject to $f(x_1^*, x_2^*, \ldots, x_q^*) = 0$ are

\[ p_r = \lambda f_r \]

\[ f(x_1^*, x_2^*, \ldots, x_q^*) = 0 \]

where $\lambda$ is a Lagrange multiplier. The necessary conditions can also be expressed as

\[ \frac{p_1}{f_1} = \frac{p_2}{f_2} = \ldots = \frac{p_{q-1}}{f_{q-1}} = \frac{p_q}{f_q} \quad 3.10 \]

The sufficient condition is that the change of profit should be negative for all variations of output which satisfy the production function. This, as we shall show later, is satisfied if $d^2f$ is positive definite subject to $df = \sum_r f_r \, dx = 0$.

Consider any commodity vector satisfying $f(x_1^*, x_2^*, \ldots, x_q^*) = 0$. Let us represent it as $[x_1^*, x_2^*, \ldots, x_q^*]$. Since the $p$ values can be varied independently of one another, it is obvious that there is at least one combination of $p$ values such that

\[ \frac{p_1}{f_1(x_1^*, \ldots, x_q^*)} = \frac{p_2}{f_2(x_1^*, \ldots, x_q^*)} = \ldots = \frac{p_q}{f_q(x_1^*, \ldots, x_q^*)} \quad 3.11 \]

Let $[\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_q]$ be a set of price values which satisfies the equations. Then, any scalar multiple of $[\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_q]$ satisfies them and no other sets of price values satisfy them.

The values

\[ \frac{a_1 \hat{p}_1}{f_1(x_1^*, \ldots, x_q^*)}, \frac{a_2 \hat{p}_2}{f_2(x_1^*, \ldots, x_q^*)}, \ldots, \frac{a_q \hat{p}_q}{f_q(x_1^*, \ldots, x_q^*)} \]
are only equal if $a_1 = a_2 \ldots = a_{q-1} = a_q$.

This implies that there is only one hyperplane in commodity space which satisfies conditions 3.11.

If we maximise imputed profit upon the basis of the shadow prices $[\hat{p}_1, \ldots, \hat{p}_q]$ we should find that the imputed profit function reaches a maximum at $[x_1, x_2, \ldots, x_q]$.

In view of equations 3.10 and 3.11, the necessary conditions for an extremum are satisfied at $[x_1, \ldots, x_q]$. The assumption that $d^2f$ is positive definite ensures that this is the only extremum commodity bundle for any scalar multiple of $[\hat{p}_1, \ldots, \hat{p}_q]$, and that the extremum is in fact a maximum.

E. Some Simple Shadow Prices for Criteria in a Multi-Commodity Model.

Let us now try to identify the simple shadow price vectors in simulated maximisation problems for some criteria.

Let $X$ represent the possible commodity bundles or vectors one of which must be chosen at $t-n$. Let $P$ represent the firm's decision set of price vectors which apply to its determination of $X$ for $t$ in $t-n$. Individual price vectors will be denoted by subscripts to $P$, e.g. $P_o = (p_{10}, p_{20}, \ldots, p_{q0})$ where $p_{ro}$ is the zero-th possible price of the $r$-th commodity, and similarly, $X_o = (x_{10}, \ldots, x_{q0})$. In the forthcoming problems, except when specially mentioned, the $X$ values may be continuous or discontinuous and their domain may be limited by inequalities or other restrictions. The possible level of profit is a function of the available input-output possibilities and the price possibilities, and can be represented by

$$\phi = \phi (X, P).$$
Unless stated otherwise, the relation will be interpreted in a
general way - discontinuities and restrictions can underlie it.

Our course is now the following one: Taking a
decision set of price vectors, $P$, we obtain the optimal
commodity vectors for the Wald, maxmax, Hurwicz and Bayes
criteria. Also, we specifically identify some price vectors
which yield the Wald, maxmax, Hurwicz and Bayes commodity
optima as the solution of a maximum problem. Let us concen-
trate on each criterion in turn.

The Wald Solution

The optimal Wald solution is

$$\max_{X} \min_{P} b(X, P).$$

Suppose that an optimal Wald solution occurs for the commodity
vector $X = X_0$ and the price vector $P = P_0$, i.e.

$$\max_{X} \min_{P} b(X, P) = b(X_0, P_0).$$

Our aim is to find the condition for

$$\max_{X} b(X, P_0) = \min_{X} b(X, P).$$

An identical maximum solution involving $P_0$ need not occur for

$$\max_{X} b(X, P_0) \geq b(X_0, P_0).$$

If a saddle point exists for function $b$ at $(X_0, P_0)$ then
the equality of expression 3.15 will be satisfied and the maximum
solution will be identical to the minimax one. Using von Neumann
and Morgenstern's definition $^3 (X_0, P_0)$ is a saddle point of $b$ if
at the same time $b(X, P_0)$ assumes its maximum at $X = X_0$ and
$b(X_0, P)$ assumes its minimum at $P = P_0$. If a saddle point

---

$^3$ J. von Neumann and O. Morgenstern: *Theory of Games and
Economic Behaviour*, Princeton University Press, Princeton,
1944.
exists at \((X_0, P_0)\),

\[
\max_X \phi (X, P) = \min_P \phi (X_0, P) \quad \text{3.17}
\]

But

\[
\min_P \phi (X_0, P) = \phi (X_0, P_0) = \max_X \min_P \phi (X, P) \quad \text{3.18}
\]

Hence, from equation 3.17 and 3.18, the existence of a saddle point for \(\phi\) at \((X_0, P_0)\) ensures that the maxmin solution can be replaced by an identical one found by maximising \(\phi (X, P_0)\) with respect to \(X\). The prices of the price vector \(P_0\) are treated as constants.

But going further than this - only if a saddle point exists at \((X_0, P_0)\) can the maxmin solution be converted to an equivalent maximum one of the specific form, \(\max_X \phi (X, P_0)\). Proof: If a saddle point does not exist at \((X_0, P_0)\), then \(\phi (X_0, P_0)\) is not simultaneously the minimum of \(\phi (X_0, P)\) and the maximum of \(\phi (X, P_0)\). If \(\phi (X_0, P_0)\) is the minimum of \(\phi (X_0, P)\) then, if a saddle point does not exist at \((X_0, P_0)\), \(\phi (X_0, P_0)\) is not the maximum of \(\phi (X, P_0)\). If a saddle point does not exist at \((X_0, P_0)\) then \(\max_X \phi (X, P_0)\) occurs for some value other than \(X_0\). This maximum point, say \(\phi (X_\alpha, P_0)\), cannot be another saddle point value, for if a number of saddle points exist they have the same profit value. If \(\phi (X_\alpha, P_0)\) and \(\phi (X_0, P_0)\) are both saddle points

\[
\phi (X_\alpha, P_0) = \phi (X_0, P_0) \quad \text{3.19}
\]

and,

\[
\max_X \phi (X, P_0) \quad \text{occurs for} \quad (X_\alpha, P_0) \quad \text{and} \quad (X_0, P_0).
\]

Consequently, if a saddle point does not exist at \((X_0, P_0)\) then

4 [Ibid, p. 95.]
Given the state \( P_o \), the maximising strategy, \( X_{\alpha} \), gives greater profit than does the firm's Wald strategy. But if \((X_{\alpha}, P_o)\) is not a saddle point then obviously

\[
\min_{P} \phi(X_{\alpha}, P) < \min_{P} \phi(X_o, P) \tag{3.21}
\]

and the firm does not achieve its maxmin strategy by producing \( X_{\alpha} \). If a saddle point exists for the possible profit function \( \phi(X, P) \), then, and only then, does the Wald solution convert to an identical maximisation one based upon one of the possible states of nature. Of course, even though the solution need not be simple, the Wald solution will always convert to a maximisation one if the production function satisfies the required second order differentiability conditions which were mentioned in section D.

As a particular instance of the existence of a saddle point let us consider dominance. Where \( P^1 \) is one price vector and \( \bar{P}^1 \) represents all other possible price vectors, the vector \( P^1 \) is dominant for the Wald solution if

\[
\phi(X, P^1) < \phi(X, \bar{P}^1). \tag{3.22}
\]

It immediately follows that

\[
\min_{P} \max_{X} \phi(X, P) = \max_{X} \phi(X, P^1). \tag{3.23}
\]

Also,

\[
\max_{X} \min_{P} \phi(X, P) = \max_{X} \phi(X, P^1) \tag{3.24}
\]

because

\[
\min_{P} \phi(X, P) = \phi(X, P^1) \tag{3.25}
\]

for any value of \( X \).
Hence, given dominance,

\[
\min \max_b (X, P) = \max \min_b (X, P)
\]

which implies that a saddle point exists. The dominance case is a special one in which a saddle point always exists.

Turning our attention to special dominance cases, let \( p_{rj} \) be the \( j \)-th possible price for the \( r \)-th factor if \( r \leq m \), and, for the \( r \)-th product if \( m + 1 \leq r \leq q \). For the \( j \)-th price of the \( r \)-th commodity the firm's net profit function is

\[
V = \sum_{r=m+1}^{q} p_{rj} x_r + \sum_{r=1}^{m} p_{rj} x_r
\]

Here we are following the Hicksian convention and are treating factors as negative products.

The values of \( x \) are subject to the production function

\[
f(x_1, x_2, \ldots, x_q) = 0.
\]

If \( p_{r1} \) represents the lowest possible price for the \( r \)-th commodity and \( p_{rn} \) the highest, then a dominance situation exists if the vector \( \{ p_{1n}, p_{2n}, \ldots, p_{mn}; p_{m+1}, \ldots, p_{q1} \} \), is possible.

Profits for any commodity vector \( X^* = \{ x_1^*, x_2^*, \ldots, x_q^* \} \) and the price vector \( \{ p_{1n}, p_{2n}, \ldots, p_{mn}; p_{m+1}, \ldots, p_{q1} \} \) are

\[
V^* = \sum_{r=m+1}^{q} p_{r1} x_r^* + \sum_{r=1}^{m} p_{r1} x_r^*
\]

Profits are at a maximum for every possible output vector when the price vector \( \{ p_{1n}, p_{2n}, \ldots, p_{mn}; p_{m+1}, \ldots, p_{q1} \} \) occurs because \( x_r \leq 0 \) for \( r \leq m \) and \( x_r \geq 0 \) from \( m + 1 \leq r \leq q \). If the highest possible price of each factor can occur along with the

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lowest price for each product, dominance exists for the vector of highest factor and lowest product prices. Considering this vector possibility further by supposing that the production function is at least twice differentiable, let us pursue the equivalent Wald solution which is found by maximising

$$ V = \sum_{r=1}^{m} p_{rn} x_{r} + \sum_{r=m+1}^{q} p_{rl} x_{r} $$  \hspace{1cm} 3.30

subject to the production function

$$ f(x_1, x_2, \ldots, x_q) = 0 $$  \hspace{1cm} 3.31

The solution can be found by maximising the function

$$ z = \sum_{r=1}^{m} p_{rn} x_{r} + \sum_{r=m+1}^{q} p_{rl} x_{r} - \lambda f(x_1, x_2, \ldots, x_q) $$  \hspace{1cm} 3.32

where $\lambda$ is a Lagrange multiplier. To avoid double summations let

$$ q \sum_{r=1}^{m} p_{ro} x_{r} = \sum_{r=1}^{q} p_{rn} x_{r} + \sum_{r=m+1}^{q} p_{rl} x_{r} $$  \hspace{1cm} 3.33

Then, the necessary conditions for

$$ \max_{x} \{ \sum_{r=1}^{q} p_{ro} x_{r} - \lambda f(x_1, x_2, \ldots, x_q) \} $$  \hspace{1cm} 3.34

are

$$ p_{ro} = \lambda f_{x_{r}} \quad (r = 1, \ldots, q) $$  \hspace{1cm} 3.35

and

$$ f(x_1, x_2, \ldots, x_q) = 0 $$  \hspace{1cm} 3.36

This gives $q + 1$ equations in $q + 1$ unknowns.

The sufficient conditions for a maximum will be satisfied if $d^2 z$ is negative for all variations of output which satisfy the production function.

Now,

$$ d^2 z = - \lambda d^2 f $$  \hspace{1cm} 3.36

and

$$ d^2 f = \sum_{r} \left( \sum_{s} f_{x_{r}} d_{x_{r}} d_{x_{s}} \right) $$  \hspace{1cm} 3.37
where \( s = 1 \ldots q \).

Since \( \lambda > 0 \), \( d^2 z \) will be negative definite if \( d^2 f > 0 \) for all variations of output which satisfy the production function. Therefore, if \( d^2 f \) is positive definite subject to

\[
d f = \sum_r f_r \, d x_r = 0,
\]

\( d^2 z \) will be negative definite subject to the variations set by the production function, and the sufficient conditions for a maximum will be satisfied. \( d^2 f \) will be positive definite subject to the restrictions of equation 3.38 if the following bordered principal minors of the production function and its bordered discriminant are all negative:

\[
\begin{vmatrix}
0 & f_1 \\
1 & f_{11}
\end{vmatrix} < 0,
\begin{vmatrix}
0 & f_1 & f_2 \\
1 & f_{11} & f_{12}
\end{vmatrix} < 0, \ldots,
\begin{vmatrix}
0 & f_s \\
r & f_{rs}
\end{vmatrix} < 0
\]

where \( r = s = 1 \ldots q \).

Denoting any two commodities by the subscripts \( r \) and \( s \), the marginal conditions of 3.35 imply that

\[
\frac{d x_s}{d x_r} = \frac{f_r}{f_s} = \frac{p_{ro}}{p_{go}}
\]

In this dominance case, the marginal requirements for a Wald optimum are that

(a) the ratio of the lowest possible prices for any two products \((r > m \text{ and } s > m)\) be equal to the marginal rate of substitution between the products in production;

(b) the ratio of the highest prices for any two factors \((r \leq m \text{ and } s \leq m)\) be equal to their marginal technical rate of substitution;
(c) the ratio between the highest price for any factor \((r \leq m)\) and the lowest price for any product \((s > m)\) be equal to the technical rate of transformation between the factor and the product.

For a stable solution the second order conditions must be satisfied. These second order conditions imply that the firm's production possibilities form a strictly convex set. \(^6\) The second order conditions will only be met if the isoquants are convex (to the origin) and, if production is subject to decreasing returns to scale. The fulfilment of these conditions is essential for the existence of a true extremum.

But even if these marginal and second order conditions are satisfied, an absolute maximum may not exist for the values which fulfill them. If we only attend to these conditions, we shall sometimes fail to find the true Wald solution even though it exists for \(\max \ \beta (X, P_o)\). This is solely because the differential calculus does not determine the true maximum if an absolute maximum exists at the boundary values of \(x_r = 0 (r = 1, \ldots, q)\).

This is similar to the perfect competition case in which marginal conditions are met but an operating loss is incurred. \(^7\) For any combination satisfying the marginal and second order conditions, each product or group of products must yield a surplus so that it does not pay to abandon the production of any product or group of products. It follows from the relationship of average and marginal values that if any product or group of products yields

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\(^7\) J. R. Hicks: Value and Capital, pp. 87-88.
an operating surplus and satisfies the marginal and second order conditions that average variable cost is increasing. Hence, if the marginal and second order conditions are satisfied for a combination in which average variable cost is increasing for each product and group of products, then a non-boundary maximum occurs.

The dominance case in which

\[
\{ p_{1n}, p_{2n}, \ldots, p_{mn}, p_{m+1,1}, \ldots, p_{qn} \}
\]

can occur has been pursued at length because the above conditions and procedures apply with minor modification to other Wald situations. Similar marginal, second order and absolute value conditions apply \textit{mutatis mutandis} to any Wald solution which can be converted into a maximum one. This is so provided that the production function satisfies the differentiability requirements. In dominance and non-dominance cases where a saddle point exists the first order conditions (of 3.40) may be based upon a set of prices which contain some values each of which is not the highest possible for a factor, or the lowest possible price of a product.

If, as assumed previously, the firm produces only one good and this is subject to price uncertainty and if its costs are certain, this is a \underline{dominance case} in which the highest possible price of the factors can occur along with the lowest possible price for the product. We have deduced that the Wald solution always converts to a simple maximum one in this case. Therefore, if \( p_1 \) is the lowest possible price for the product and if the cost function is at least twice differentiable, then the Wald optimum occurs for

\[
p_1 = C'(x)
\]
provided that $C''(x) > 0$ and average variable cost is increasing at this output level.

To conclude: If a saddle point exists and profit is maximised upon the basis of the price vector which corresponds to the saddle point, then the firm attains its maxmin solution.

Only if a saddle point exists can a maxmin solution be attained by maximising upon the basis of one of the possible states of nature. A saddle point exists if dominance is possible. If the highest possible price of factors can occur along with the lowest possible price for products, we have a special but nevertheless relevant case of dominance. In this case the maxmin solution can be attained by maximising profit upon the basis of the price vector of the highest possible price for factors and the lowest possible prices for products. Naturally, if the price of any commodity is certain, its lowest and highest possible price coincide.

The Maxmax Solution

Where $b(X, P)$ represents the profit possibilities of the firm, let the firm’s optimal maxmax solution be

$$\begin{align*}
\text{Max}_X \; \text{Max}_P b &= b(X_1, P_1).
\end{align*}$$

We shall show that

$$\begin{align*}
\text{Max}_X \; \text{Max}_P b (X, P) &= \text{Max}_X b (X, P_1).
\end{align*}$$

Any two maxmax values commute since any two have the same characteristic property of being a maximum of $b(X, P)$.

Commutativity implies that

$$\begin{align*}
\text{Max}_X \; \text{Max}_P b (X, P) &= \text{Max}_F \; \text{Max}_X b (X, P).
\end{align*}$$
Further, if \((X_1, P_1)\) yields an absolute maximum of the function \(\phi\) which it does since \(\phi(X_1, P_1)\) is a maxmax value, then \(\phi(X, P_1)\) assumes its maximum at \(X = X_1\).

Proof:

\[
\max_X \max_P \phi(X, P) = \max_P \phi(X_1, P) = \phi(X_1, P_1), \quad 3.43
\]

\[
\max_P \max_X \phi(X, P) = \max_X \phi(X, P_1) = \phi(X_1, P_1). \quad 3.44
\]

Hence,

\[
\max_X \max_P \phi(X, P) = \max_X \phi(X, P_1) = \phi(X_1, P_1). \quad 3.43
\]

If \((X_1, P_1)\) is the only maxmax point for \(\phi(X, P)\) then the maximum solution gives only one value in \(X\). If there is more than one maxmax value, they are equal. Each maxmax value will have a corresponding maximum solution. Every maxmax solution has a simple maximum equivalent in which profits are maximised upon the basis of one of the vectors of the set of possible price vectors.

If the lowest possible price of each factor can occur simultaneously with the highest possible price for each product then the optimal maxmax solution occurs for

\[
\max \phi(X, P_1)
\]

where

\[
P_1 = (P_{11}, P_{21}, \ldots, P_{m1}; P_{m+1}, n', \ldots, P_{qn}).
\]

The state \(P_1\) is a dominant one since for any combination of commodities profits are highest for the vector \(P_1\). In this dominance case, we can maximise

\[
V = \sum_{r=1}^{m} P_{r1} x_r + \sum_{r=m+1}^{q} P_{rn} x_r \quad 3.45
\]
subject to the production function

\[ f(x_1, x_2, \ldots, x_q) = 0 \]

and obtain the maxmax solution. For a maxmax solution in this dominance case:

(a) The ratio of the highest possible prices for any two products must equal the technical rate of substitution between the products.

(b) The ratio of the lowest possible prices for any two factors must be equal to the technical rate of substitution of the factors.

(c) The ratio between the lowest price for any factor and the highest price for any product must be equal to the technical rate of transformation between the factor and the product. The second order conditions are once again applicable. Also, boundary values must be checked.

If the maxmax solution occurs for a vector other than

\[ p_1^* = (p_{11}, p_{21}, \ldots, p_{ml}; p_{m+1}, n, \ldots, p_{qn}) \]

(because this state is impossible) then, given the required differentiability conditions, a set of marginal conditions based upon the price vector for the maxmax solution can be stated by substituting the new set of prices in the above expressions.

If the firm produces one product and costs are certain, dominance always occurs for the maxmax solution, and this dominance is of the type previously discussed at length.

Hence, if \( p_n^* \) is the highest possible price for the product then, the maxmax solution can be found by maximising

\[ p_n^* x - C(x) \]

If this function is at least twice differentiable,
the maxmax optimum occurs for

$$p_n = C'(x)$$

if \( C''(x) > 0 \) and average variable cost is increasing at this output level.

Unlike the minimax solution, the maxmax solution always converts into a simple and equivalent maximum solution in which one of the possible states of nature is treated as though it is certain. The Hurwicz solution does not always have a simple maximum equivalent.

**The Hurwicz Solution**

Let the Hurwicz solution be

$$\text{Max} \left\{ \beta \text{Min } \phi (X, P) + (1 - \beta) \text{Max } \phi (X, P) \right\}$$

$$= \beta \phi (X^*, P_o) + (1 - \beta) \phi (X^*, P_1). \tag{3.47}$$

Then,

$$\text{Max} \left\{ \beta \phi (X, P_o) + (1 - \beta) \phi (X, P_1) \right\} \geq \beta \phi (X^*, P_o) + (1 - \beta) \phi (X^*, P_1) \tag{3.48}$$
because \( X \) is open on the left hand side of the expression.

But if dominance arises, the Hurwicz solution can be converted to a maximum one such that

$$\text{Max} \left\{ \beta \phi (X, P_o) + (1 - \beta) \phi (X, P_1) \right\}$$

$$= \beta \phi (X^*, P_o) + (1 - \beta) \phi (X^*, P_1). \tag{3.49}$$

If

$$\phi (X, P_o) \leq \phi (X, P) \tag{3.50}$$

and, \( \phi (X, P_1) \geq \phi (X, P) \tag{3.50} \)
then "Hurwicz" dominance occurs. Given relationship

$$\text{Min } \phi (X, P) = \phi (X, P_o). \tag{3.51}$$
and \( \max_{P} \beta(X, P) = \beta(X, P_1) \).

Therefore,
\[
\max_{X} \left\{ \beta \min_{P} \beta(X, P) + (1 - \beta) \max_{P} \beta(X, P) \right\}
= \max_{X} \left\{ \beta \beta(X, P_0) + (1 - \beta) \beta(X, P_1) \right\}
= \beta \beta(X^*, P_0) + (1 - \beta) \beta(X^*, P_1).
\]

As a particular case of "Hurwicz" dominance, suppose the vectors
\[
P_1 = \{P_{11}, P_{21}, \ldots, P_{mn}, P_{m+1,1}, \ldots, P_{q,1}\}
\] and
\[
P_n = \{P_{11}, P_{21}, \ldots, P_{ml}, P_{m+1,n}, \ldots, P_{q,n}\}
\]
to be possible. Then the Hurwicz solution can be found by maximizing
\[
m \Sigma (\beta p_{rm} + (1 + \beta) p_{r1}) x_r + q \Sigma (\beta p_{rl} + (1 - \beta) p_{rn}) x_r.
\]
subject to the production function. If the differentiability conditions are satisfied the marginal conditions involve the price "indices" \( \beta p_{rm} + (1 + \beta) p_{r1} \) for \( r \leq m \) and \( \beta p_{rl} + (1 - \beta) p_{rn} \) for \( r > m + 1 \). When \( \beta = 0 \) these first order conditions are the same as those for the Wald criterion and when \( \beta = 1 \) they are the same as for the maxmax. As before, second order conditions and boundary requirements must be met.

"Hurwicz dominance" always occurs in the one product case, if costs are certain. We know from the above theorem that the Hurwicz output can be obtained by maximizing
\[
[\beta p_{1} + (1 - \beta) p_{n}] x - C(x).
\]
If this function is at least twice differentiable, then a Hurwicz optimum occurs if
\[
C'(x) = \beta p_{1} + (1 - \beta) p_{n}.
\]
and if average variable cost is increasing at the point of equality.

The Bayes Solution

The Bayes solution occurs for

$$\max_x E \phi (x, P)$$

Since $X$ is not subject to variation after $t-n$, this becomes

$$\max_x \phi (x, E(P))$$

where

$$E(P) = \{ E(P_1), E(P_2), \ldots, E(P_q) \}$$

i.e. $E(P)$ is a vector consisting of the expected prices of the $q$ commodities. The Bayes solution is found by maximising

$$V = \sum_{r=1}^{q} E(P_r) x_r$$

subject to

$$f(x_1, x_2, \ldots, x_q) = 0.$$ 

If the production function is differentiable, we obtain a set of marginal conditions in terms of expected prices and rates of technical transformation which are similar to the perfect competition ones. The previous second order and boundary statements are also relevant.

It will be recalled that all of the previous analysis is based upon the assumption of an identifiable decision set. In postulating this decision set, we did not postulate its relation to actual prices or to objectively possible prices. Neither did we postulate any special relationship between actual output and the optimal output under certainty, or between maximum equivalent prices under uncertainty and actual prices.
Obviously, the firm's output under uncertainty is the joint result of its predictions and criterion, and from the point of view of its actual profit both factors have an influence.

From the point of view of comparing profit under uncertainty and under certainty both the firm's criterion and the quality of its prediction are important. When we come to consider the firm's actual average profit in later chapters we must take account of both factors.
CHAPTER IV

Static Plans and Differences of Production

A. Introduction

Differences of production can occur

(i) because of different criteria,
(ii) because of different decision sets, and
(iii) because of different forms of "irrational" behaviour.

If the firm makes static plans, different levels of production can be compared by utilising the concept of imputed "maximum" prices which was introduced in the last chapter. Comparisons of production both under certainty and uncertainty can be reduced to comparisons of constrained "profit" maxima. Each production value can be conceived of as a value which maximises an imputed profit function and changes of production can be viewed as resulting from changes in the shadow prices when the imputed profit function is maximised subject to the production function. By treating differences of production in this way, we are able to reinterpret many of the mathematical theorems which Hicks introduces for the perfect competition model. While there is mathematical similarity between the present production theorems and some of those for perfect competition, they are different in their economic interpretation and generality.

In this chapter, we shall state a few of Hicks' theorems and then show their relevance to the effect upon production of (i) changes of criteria when the decision set is given and (ii) changes of the decision set when the criterion is given. In comparing these particular changes, we should not lose sight of the general idea that no matter why production changes we
can treat the change as arising from a change in the shadow price vector of an imputed profit maximisation problem. This treatment is possible if the second derivative of the production function is positive definite subject to the first derivative being zero. In imputing the maximisation problem, we do not imply that the firm actually determines its output by some maximisation process.

B. The Production Effects of Price Vector Changes in the Constrained Maximum Problem

Before dealing with differences in production levels under uncertainty, we can profitably re-familiarise ourselves with the effects upon production of a change in the price vector if profit is maximised subject to a profit function. We should re-familiarise ourselves with the effect upon the maximising output vector of a change in the price vector when

\[ V = \sum_{r=1}^{q} p_r x_r \]

is maximised subject to

\[ f(x_1, x_2, \ldots, x_q) = 0. \]

If the production function is assumed to be differentiable, this will enable comparisons to be made of finite differences in production if the results of the differential calculus are extended by the theorem of the mean. Mathematically, the problem is identical to the perfect competition one of finding the effect

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1 P. A. Samuelson: *Foundations of Economic Analysis*, Harvard University Press, Cambridge, 1947, pp. 46-52. For an introductory mathematical survey of the theorems of the mean -


upon production of a change in a certain price vector when profit is constrained by the preceding production function.

For the price vector, \( \{ p_1, p_2, \ldots, p_q \} \), the firm's constrained profit function is

\[
z = \sum_{r=1}^{q} p_r x_r - \lambda f(x_1, x_2, \ldots, x_q)
\]

where \( \lambda \) is the Lagrange multiplier. The necessary conditions for a maximum of this function are

\[
\begin{align*}
p_r &= \lambda f_r \\
\frac{\partial z}{\partial \lambda} &= f(x_1, x_2, \ldots, x_q) = 0
\end{align*}
\]

If the price vector \( \{ p_1', p_2', \ldots, p_q' \} \) changes, our assumption implies that the firm will respond by altering its combination of resources in such a way that the marginal conditions continue to be satisfied. Therefore, the problem is to discover the change in production when prices change and the marginal conditions of expression 4.1 continue to be met. To obtain this change we differentiate the marginal conditions totally, and obtain

\[
\begin{align*}
dp_r - \lambda \sum_{r=1}^{s} f_{rs} - f_r d \lambda &= 0 \\
\sum_{r=1}^{s} f_s dx_s &= 0.
\end{align*}
\]

Rearranging,

\[
\begin{align*}
f_r d \lambda + \lambda \sum_{r=1}^{s} f_{rs} dx_s &= dp_r \\
\sum_{r=1}^{s} f_s dx_s &= 0.
\end{align*}
\]

Equations 4.3 expressed differently are
\[ 0 \, d \lambda + f_1 \, dx_1 + f_2 \, dx_2 + \ldots + f_q \, dx_q = 0 \]

\[ f_1 \, d \lambda + \lambda [ f_{11} \, dx_1 + f_{12} \, dx_2 + \ldots + f_{1q} \, dx_q ] = dp_1 \]

\[ f_q \, d \lambda + \lambda [ f_{q1} \, dx_1 + f_{q2} \, dx_2 + \ldots + f_{qq} \, dx_q ] = dp_q \]

or

\[
\begin{bmatrix}
0 & f_s \\
f_r - \lambda f_{rs}
\end{bmatrix}
\begin{bmatrix}
d\lambda \\
dx_r
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
dp_r
\end{bmatrix}
\tag{4.4}
\]

where \( r = s = 1, \ldots, q \).

Let

\[
G = 
\begin{bmatrix}
0 & f_s \\
\lambda f_{rs}
\end{bmatrix} = 
\begin{bmatrix}
g_{ij}
\end{bmatrix}
\tag{4.5}
\]

where \( r = s = 1, \ldots, q \)

and \( i = j = 1, \ldots, q + 1 \).

Then, by Cramer's Rule,

\[
dx_s = \frac{\sum_{r=1}^{q} \, dp_r \, G_{rs}}{G} \tag{4.6}
\]

where \( s = 1, \ldots, q \) and \( G \) is the cofactor of element \( g_{ij} = g_{r+1, s+1} \) in the matrix, \( [g_{ij}] \). If

\[
F = 
\begin{bmatrix}
0 & f_s \\
f_r - f_{rs}
\end{bmatrix} \tag{4.7}
\]

and, if \( F_{rs} \) represents the cofactor of \( f_{rs} \), then equation 4.6 reduces to

\[
dx_s = \frac{\sum_{r=1}^{q} \, F_{rs} \, dp_r}{\lambda F} \tag{4.8}
\]

The change in the output (or input) of the \( s \)-th commodity is found by summing the "technological" cofactors, \( F_{rs} \) for the
r commodities, and then dividing by the determinant $F$ times
the scalar multiple $\lambda$.

In the absence of special knowledge it is impossible to
place a sign upon equation 4.8. However, if the price of the
$r$-th resource alone increases then, if it is a product its supply
increases and if it is a factor its input decreases. This can be
shown as follows: Let $dp_r = 0$ for all $r \neq s$ and $dp_r > 0$ for
$r = s$. Then,

$$dx_s = \frac{\sum_{r=1}^{q} F_{rs} dp_r}{\lambda F} = \frac{F_{rr} dp_r}{\lambda F}.$$  \hspace{1cm} 4.9

Therefore,

$$\frac{\partial x_r}{\partial p_r} = \frac{F_{rr}}{\lambda F},$$ \hspace{1cm} 4.10

Since $r + r$ is even, $F_{rr}$ is equal to the minor of $f_{rr}$ with a
positive sign attached. The minor of the $q$-th commodity must
be negative if the stability conditions are to be met. Since any
commodity including the $r$-th commodity, can occupy the $q$-th
place in the bordered matrix, the minor, $\Delta_{rr}$, must be
negative if the stability condition of the last bordered principal
minor is to be satisfied. Hence, the cofactor, $F_{rr} = + (\Delta_{rr})$
is negative and $F$ is negative by the stability conditions. Therefore,

$$\frac{\partial x_r}{\partial p_r} = \frac{F_{rr}}{\lambda F} > 0,$$ \hspace{1cm} 4.11

because $\lambda > 0$. For $m + 1 \leq r \leq q$, $\frac{\partial x_r}{\partial p_r} > 0$, and, since
factors are treated as negative products, the change in the input
of a factor whose price alone rises is of opposite sign.

The effect upon the input and output of all
commodities of a rise in the price of a single commodity, all
other prices remaining unchanged, is
\[ dx_s = \left( \frac{F_{sr}}{\lambda F} \right) dp_r \quad (s = 1, \ldots, q) \]

where the price of the \( r \)-th commodity increases. From the previous argument it is known that \( dx_r > 0 \), but the effect upon the input or output of the other commodities is unknown. Only if all the cofactors \( F_{sr} \) where \( s = 1, \ldots, m \) are negative and, if the cofactors \( F_{sr}^{m+1}, s = m + 1, \ldots, q \), are positive will an increase in the price of product \( r \) lead to an increase in the employment of all factors and to an increase in the output of all products. In this case, complementarity is dominant both on the product and factor side. Complementarity dominates if constant returns to scale exist but, with decreasing returns, substitute and regressive relations become more important.

Let us now consider the effect of a rise in the price of a group of products. For simplicity suppose that all product prices increase. Then the change in the output or input of commodity \( s \) is

\[ dx_s = \frac{\sum_{r=1}^{q} F_{sr} dp_r}{\lambda F} \quad \text{for} \quad s = m + 1, \ldots, q, \]

where \( s = 1, \ldots, q \).

Unless all of the cofactors \( F_{rs} < 0 \) for \( r \leq m \) and all of the cofactors \( F_{rs} > 0 \) for \( r = m + 1, \ldots, q \), a rise in the price of one or more products can lead to a decrease in the output or input of some commodities. If the relative frequency of negative product cofactors is low, then there is a low probability of a rise in the price of any product causing a decline in the output of any other randomly chosen product. However, in the absence of further knowledge, the effect upon the output of any particular

2 Hicks: *Value and Capital*, pp. 97-98.
product is unknown. Similar conclusions hold for factors relations.

Comparisons of optimal aggregate output are complicated by the existence of substitute or regressive relations. A rise in the price of one or more products, and a fall in the price of one or more inputs can lead to an increase in the output of some products and to a decrease in the output of others. In consequence, comparison of aggregate output involve an index problem which can only be avoided in special instances.

Having considered some theorems for a change of a constrained maximum, we shall now apply them in two different ways. First, we shall apply them to the effects upon production of a change in the set of decision prices when the firm's criterion remains unchanged. Secondly, we shall apply them to the effects upon production of a change in criterion when the decision remains unchanged. For the purposes of this analysis, the assumptions of the previous chapter will be assumed to hold. This means that the foregoing analysis applies to conditions of static decision making.

C. The Effect Upon Production of a Change of the Decision Price Set.

Turning to the first application, consider an initial set of price possibilities \( P \) and a subsequent one \( P' \), where the elements of both sets are price vectors. As before, individual price vectors are shown by subscripts. Beginning with the minmax criterion, let us compare the optimal minmax production levels for both decision sets. Suppose that saddle points exist for \( (X_0, P_0) \) and \( (X'_0, P'_0) \). Then for the first decision set
Max Min \( b(X, P) = b(X, P_j) = \max_X b(X, P_j) \)

and for the second decision set

Max Min \( b(X, P_1) = b(X, P_1) = \max_X b(X, P_1) \).

To compare the two optimal levels of output \( X \) and \( X_1 \), we make use of maximum equivalence. The difference in these two optimal output levels is equivalent to the change in production when profit is first maximised upon the price vector \( P_j \) and then upon the vector \( P_1 \).

Given that the production function is at least twice differentiable the previous maximisation theorems can be applied. If the minmax price vector, \( P_1 \), contains one product price which exceeds its corresponding value in the vector, \( P_j \), then, other vector prices remaining constant, the output of the higher priced product will increase. Similarly, if the vector \( P_1 \) contains one factor price which is lower than its corresponding value in \( P_j \), then a change to decision set \( P_1 \) will increase the employment of the lower "priced" factor if all other imputed prices remain unchanged. If the vector \( P_1 \) is a scalar multiple of \( P_j \), optimal production will be the same for both decision sets.

The change in the input or output of each commodity for a change in the minmax price vector can be found by extending equation 4.13. Where the change of the price vector is finite the relevant partial derivatives which form the basis for this equation must be evaluated at an intermediate point so as to accord with the mean value theorem. There is no need to state this result specifically because the qualitative conditions of equation 4.13 carry over the finite case since we have assumed
the required monotonicity condition. If \( P_0' > P_0 \) because one
or more product prices are higher in \( P_0' \), the effect upon the
employment of factors and the output of products will depend
upon whether complementarity is dominant, the effect upon
aggregate production of a change in the minmax price vector
from \( P_0 \) to \( P_0' \) (where \( P_0' > P_0 \)) is not predictable in the
absence of further information.

If the firm adopts other criteria, such as the Bayes
or maxmax criterion, then by taking the relevant maximisation
price vectors, we can once more apply the preceding theorems
mutatis mutandis. Since the application is obvious
there is no need to give an account of it.

Index problems arise in the comparison of production
in the multi-product case. However, they do not arise in the
one product case if factor prices are certain. In this one
product case, the minmax solution always converts to a simple
maximum one. Therefore, a rise in the least possible price,
which is the basis of the maxmin solution, leads to an increase
in optimal output if average variable costs are covered and
marginal cost is not perfectly inelastic. Under the same
production conditions, output will expand for the Bayes
criterion and the maxmax criterion if the expected price and
the greatest possible price respectively increase. Insofar
as uncertainty output solutions convert to maximum ones, the
supply function for a change in the maximum price component
is identical \textit{in form} to the supply function under certainty.

D. \textbf{The Effect Upon Production of a Change of Criterion.}

Turning to the second comparative problem, viz; the
comparison of optimal levels of output for different criterion
when the decision set of prices, \( P \), remains constant, let a saddle point exist for \( (X_o', P_o') \). Then, the minmax solution is

\[
\max X \min P \quad (X, P) = (X_o', P_o') = \max X \min P \quad (X, P).
\]

Also, suppose the Bayes solution to be

\[
\max X \quad (E(P), X) = (E(P), X_E),
\]

and the maxmax solution to be

\[
\max X \quad (X, P) = (X_1, P_1) = \max X \quad (X, P).
\]

Our problem is to determine the relationship between the commodity vectors \( X_o', X_E', \) and \( X_1 \). This will depend upon the relationship between \( P_o', E(P), \) and \( P_1 \). If factor prices are certain and if product prices are uncertain, and, if the lowest price of any product can occur along with the lowest price of every other, and if the highest price of all products can occur simultaneously, then the vector relationships are \( P_o' < E(P) < P_1 \). If the price for one product only is highest in the vector \( P_1 \), lower in the vector \( E(P) \) and lowest in the vector \( P_0 \), the optimal output of that product will be highest for the maxmax criterion, lower for the Bayes criterion and lowest for the Wald criterion. This follows at once for the discussion of equation 4.10. However, this result does not indicate the effect of the change upon aggregate output. Only if complementarity is dominant will the maxmax, Bayes and Wald criterion lead to aggregate product outputs which are of descending magnitude if \( P_o' < E(P) < P_1 \). For the multi-good firm, the output of products will not necessarily be greatest for the maxmax criterion, next highest for the Bayes criterion and least for the Wald criterion because the
output of some products will increase and the output of others will decrease if substitutability is important. Similarly, the aggregate employment of factors need not be least for the Wald criterion higher for the Bayes criterion and highest for the maxmax. But these ambiguities of comparison only arise in the multi-product case.

If the price of the output of a one product firm is uncertain and if its costs are certain, the firm's output will be highest for the maxmax criterion, lower for the Bayes and lowest for the Wald criterion. This will be so provided that marginal cost is not perfectly inelastic and average variable cost is covered. To compare the Wald maxmax level of optimal output in this simple case, let \( \{ p_1, p_2, \ldots, p_n \} \) be the product decision set of prices and suppose that \( p_n > p_{n-1} > \ldots > p_1 \). The firm has a profit function

\[
\Pi = px - C(x)
\]

and, a supply function

\[
x = g(p),
\]

where \( g(p) \) is the inverse of the function \( p = C'(x) \) and \( \left( \frac{C(x)}{x} \right)' \geq 0 \).

By the inverse rule of differentiation,

\[
g'(p) = \frac{dx}{dp} = \frac{1}{C''(x)}
\]

If \( 0 < C''(x) < \infty \), then \( \frac{dx}{dp} > 0 \). A change from the Wald to the maxmax criterion involves a change from

\[
\text{Max} \left[ p_1 x - C(x) \right] \text{ to Max} \left[ p_n x - C(x) \right].
\]

This change results in an increase of production which is equivalent to that of a price increase of \( \Delta p = p_n - p_1 \) in the maximisation case. Such a price change leads to a
change in production of $\Delta x = g(p_n) - g(p_1)$. By the theorem of the mean, $\Delta x = g(p_n) - g(p_1) = \int_{p_1}^{p_n} g'(p) \, dp = g'(\bar{p}) \Delta p$

where $\bar{p} = p_1 + \theta(p_n - p_1)$ and, $0 < \theta < 1$. If $g'(p)$ is positive everywhere in the interval $p_1 \leq p \leq p_n$, it will be positive for $p = \bar{p}$. Since our assumption is that $C''(x) > 0$ everywhere within the required interval, $g'(\bar{p}) > 0$. Hence, if $\Delta p > 0$, $\Delta x > 0$. Therefore, a change from the Wald to the maxmax criterion leads to an increase of output if the decision set consists of more than one price value. Similarly, in the above case, the Bayes optimal level of output can be shown to lie between the Wald and maxmax level.

From the above analysis and under the static decision assumptions the supply functions for the one product firm with certain costs are as follows:

Under certainty, the supply function is

$$\begin{cases} x = S = g(p) & \text{for } p \geq \min A.V.C. \\ x = S = 0 & \text{for } p < \min A.V.C. \end{cases}$$

For the Wald criterion, it is

$$\begin{cases} x = S = g(p_1) & \text{for } p_1 \geq \min A.V.C. \\ x = S = 0 & \text{for } p_1 < \min A.V.C. \end{cases}$$

For the Bayes criterion, it is

$$\begin{cases} x = S = g(E(p)) & \text{for } E(p) \geq \min A.V.C. \\ x = S = 0 & \text{for } E(p) < \min A.V.C. \end{cases}$$

Finally, for the maxmax criterion it is

$$\begin{cases} x = S = g(p_n) & \text{for } p_n \geq \min A.V.C. \\ x = S = 0 & \text{for } p_n < \min A.V.C. \end{cases}$$

In this static decision case, all of the supply functions are of identical form - supplies only differ insofar as they depend
(or, can be made to depend) upon different price values.

E. Comparisons of Production Under Certainty and Uncertainty.

Although levels of output for different criteria under uncertainty have been compared, these levels have not been compared with output under certainty. Let us make a few tentative comments upon such comparisons. Consider the above one product model and suppose that in an interval of time the prices $p_j, j = 1, \ldots, n$, occur with the respective relative frequencies $p_j, j = 1, \ldots, n$. Then, the firm's average output under certainty is

$$\sum_{j=1}^{n} p_j g(p_j)$$

if it maximises profit. We suppose that each of these prices exceed minimum average variable cost, and that marginal cost is not perfectly inelastic.

Under uncertainty we suppose that the prices $p_j$ occur with the random probabilities $p_j$ and that the probability distribution does not change in time. Random price values are treated as being independent. Under these assumptions the average Wald optimal level of output for uncertainty is less than under certainty since

$$g(p_1) < \sum_{j=1}^{n} p_j g(p_j)$$

because

$$g'(p) = \frac{1}{C''(x)} > 0$$

Similarly, the average maxmax level of output is higher than the average level under certainty. The situation is more complicated for Bayes criterion. The average Bayesian level of optimal output under uncertainty is
\[
g(E[p]) = g(\sum_{j=1}^{n} \rho_j p_j) = 1\]

Now,
\[
g(\sum_{j=1}^{n} \rho_j p_j) \geq \sum_{j=1}^{n} \rho_j g(p_j)
\]

accordingly as
\[
g''(p) \leq 0.
\]

Applying the inverse differentiation rule,
\[
g''(p) > 0,
\]

accordingly as
\[
C''(x) \leq 0.
\]

If marginal cost increases at an increasing rate, then, for the Bayes criterion, average output will be less under uncertainty than under certainty.

E. Conclusions

The above relationship between the decision set of prices under uncertainty and actual prices is an extremely special one. Nevertheless, similar relationships have been assumed in economic analysis of this sort. However, the assumption appears so narrow that its use in the absence of positive empirical evidence suggesting applicability should be restricted. In practice, the minmax price value need not be equal to \( p_1 \), and could bear a wide range of relationships to actual price. Similarly, it may be possible to discover a wide range of relationships for the price values of other criteria. Therefore, the use of this particular assumption will be kept to a minimum and in the later part of the forthcoming analysis will not be used at all.

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On the whole, considerations of changes of production and production optima have been restricted to the minmax, maxmax and Bayes criteria. It can be said that these criteria respectively signify a security, gambling and average income bias on the part of the firm. However, I do not wish to suggest that these three criteria are the ones which are most frequently used under uncertainty. The frequency of the use of different criteria is an empirical question which is not to be settled without observation. Indeed, one of my hypothesis is that behaviour is diverse under uncertainty and that it is unlikely that it can be represented by a few finite criteria. However, even if it is diverse, it can often be reduced to some common index problem.

In the above analysis, it is the conversion of diverse behaviour results to equivalent maximisation results which make the effects of diverse behaviour readily comparable. This comparability is possible for diverse criteria and for diverse price forecasts, and will enable us to comprehend a number of general theorems. In considering the conversion of apparent non-extremum problems into extremum ones, Samuelson concludes that "it is well to emphasize that the conversion of a problem whose economic context does not suggest only human, purposive, maximizing behaviour into a maximum problem is to be regarded as merely a technical device for the purpose of quickly developing the properties of that equilibrium position". Although the conversion is technical, at the same time it is not to be disparaged. The

conversion establishes a general ordering device for different types of behaviour under the static decision making assumption.

Until there is strong empirical evidence to suggest that firm’s adopt only a limited number of criteria, it seems to be the correct course for a priori theories to embrace as wide a set of possibilities as may feasibly exist. Similarly, until further empirical evidence is forthcoming, a wide range of possible relations between decision prices and actual prices should be assumed.  

6 In developing the formal arguments of Chapters III and IV the following have been some of the more useful references.

(i) J. R. Hicks: Value and Capital, Chapters IV and VII and their appendices.


(iv) R. G. D. Allen, Mathematical Economics, pp. 613-618


CHAPTER V

Price Instability and the Firm's Average Profit

It has been asserted that increased price instability increases expected profit. I contend that for some feasible error patterns increased price instability can lead to a decrease of average profit. In order to support this contention I shall discuss two models. The first relies upon a general cost function but utilises special assumptions about the distribution of actual prices and shadow prices, and the second relies upon a quadratic cost function but involves general assumptions about the distribution of actual prices and shadow prices.

A. A Simple Model

W. Oi claims in *Econometrica*, Vol. 29, No. 1 (January 1961)\(^1\) that his analysis shows that "price instability is a virtue and not a vice".\(^2\) A virtue in this sense means that expected profit is greater the greater is the degree of price instability. This conclusion is based upon two explicit assumptions: (a) that firm's maximise short run profit during each period, and (b) that the marginal cost function is upward sloping over the relevant range. In addition, Oi implicitly assumes that production can be adjusted to price changes without any additional cost for error. Given the explicit assumptions and this implicit assumption, Oi's thesis holds. Oi's thesis also holds if instead

---


of assuming adjustment at no additional cost for errors, it is assumed that entrepreneurs always forecast price correctly. Neither the assumption of adjustment at no additional cost for errors nor the assumption of perfect prediction seem very likely to be satisfied in actual practice. In some cases - e.g., in the situation facing most agricultural producers under free competition - such assumptions seem inappropriate. Let us see if the non-fulfilment of these conditions could be of any consequence for Ol's generalisation. For this purpose let us use a simple model in which adjustment is ruled out for physical or cost reasons.

The following assumptions are made:

(i) The firm wishes to maximise expected profit.

(ii) The marginal cost curve increases over the relevant range and cost is certain.

(iii) The prices \((1 + a) p\) and \((1 - a) p\) each occur with a relative frequency of 0.5 under price certainty.

(iv) Under uncertainty the possible prices have a probability distribution which is the same as the relative frequency distribution of price under certainty. Under uncertainty, the producer knows that \((1 + a) p\) and \((1 - a) p\) each occur randomly, and that each has a probability of 0.5.

(v) Output is planned in period \(t - n\), \(n > 0\), for period \(t\) and is unalterable for physical or cost reasons.

We assume that expected price is stationary, that the cost function is the same through time, and that the decisions of each period are independent of those made in others. Decisions are independent but are unalterable once made in \(t - n\). Given the above assumptions, a comparison can be made between
the expected profit for a stable price of $p$ and that for a situation of price instability where $(1 + a) p$ and $(1 - a) p$ are two prices each of which occurs with a probability 0.5. From this comparison, propositions which conflict with Oi's judgment can be shown to hold. If $II$ represents profit, if $C(x)$ represents total cost and, if $a$ and $b$ are parameters such that $0 < a < 1$ and $0 < b < 1$, the profit functions when price is $(1 - a) p$, $p$, and $(1 + a) p$ are:

\[ II_1 = (1 - a) px - C(x) = f(x). \]  
\[ II_2 = px - C(x) = v(x). \]  
\[ II_3 = (1 + a) px - C(x) = h(x). \]

We assume the function $f(x)$ to reach a maximum at $x_1$, $v(x)$ to reach a maximum at $x_2$, and $h(x)$ to reach a maximum at $x_3$. Given that $0 < C'(x) < \infty$ for $x_1 \leq x \leq x_3$ then $x_1 < x_2 < x_3$ and $II_1 < II_2 < II_3$. Using these relations and two others to be specified later, it is possible to prove a number of propositions about expected profit.

**Proposition (1).** If price is uncertain and unstable, action to maximise profit at each instant leads to a lower level of expected profit than when price is stable at $p$. This will be so if the firm makes mistakes of a sufficient frequency and magnitude in its price forecasts. Suppose that with a probability of 0.25 the firm mistakenly predicts $(1 + a) p$ and that with the same probability it mistakenly predicts $(1 - a) p$. Then, if it acts on these predictions, its expected profit

\[ 0.25 \left[ f(x_1) + f(x_3) + h(x_1) + h(x_3) \right], \]  
will be less than that for the stable price, $p$, i.e.,

\[ 0.25 \left[ f(x_1) + f(x_3) + h(x_1) + h(x_3) \right] < v(x_2). \]

**Proof:** From equations 5.1, 5.2 and 5.3,
L.H.S. of expression 5.4 = 0.25 \left[ (1 - a) px_1 - C(x_1) + (1 - a) px_3 - C(x_3) + (1 + a) px_1 - C(x_1) + (1 + a) px_3 - C(x_3) \right] \\
\hspace{1cm} = 0.5 \left[ px_1 - C(x_1) + px_3 - C(x_3) \right] \\
\hspace{1cm} = 0.5 \left[ v(x_1) + v(x_2) \right]. \hspace{1cm} 5.5

But \( v(x) \) reaches a unique maximum at \( x_2 \). Hence,

\[ 0.5 \left[ v(x_1) + v(x_3) \right] < v(x_2). \hspace{1cm} 5.6 \]

Q.E.D.

If the firm is just as likely to be right as wrong in forecasting the actual price of \( t \), and, if it adjusts to its forecasts in the hope of maximising profit, an increase in the range of possible price will decrease expected profit. The firm's expected profit will be at a maximum when price is stable at \( p \).

**Proposition (ii).** As the range of price variation increases expected profit declines, if the firm attempts to maximise profit at each instant, and, if its distribution of prediction errors is of the same form as the previous one.

Although this proposition can be deduced from the previous argument, it can be easily supported by supposing that the prices \( (1-a-b)p \) and \( (1+a+b)p \) each occur with a probability of 0.5. The relevant profit functions are

\[ \Pi_o = (1 - a - b) px - C(x) = e(x) \hspace{1cm} 5.7 \]

and,

\[ \Pi_q = (1 + a + b) px - C(x) = m(x) \hspace{1cm} 5.8 \]

Let these functions reach a maximum at \( x_o \) and at \( x_q \), respectively. If the firm mistakenly acts upon \( (1 - a - b)p \) with a probability of 0.25 and mistakenly acts upon \( (1 + a + b)p \) with the same probability, its expected profit,
Proposition (iii). If the firm constantly produces the output, \( x_2 \), which maximises profit for the average price, \( p \), then its expected profit will be greater than if it adjusts to its price forecasts. This is supposing that expected price is stationary, and that the firm's forecast errors have a similar distribution to the one mentioned previously. Expected profit from producing \( x_2 \) is \( 0.5 \left[ f(x_2) + h\left( x_2 \right) \right] \) and the proposition asserts that

\[
0.5 \left[ f(x_2) + h(x_2) \right] > 0.25 \left[ f(x_1) + f(x_3) + h(x_1) + h(x_3) \right].
\]  

Proof:

\[
0.5 \left[ f(x_2) + h(x_2) \right] = 0.5 \left[ (1-a)px_2 - C(x_2) + (1+a)px_2 - C(x_2) \right] = px_2 - C(x_2) = v(x_2).
\]
From expression 5.9,
\[ v(x_2) > 0.25 \left[ f(x_1) + f(x) + h(x_1) + h(x) \right]. \]

Q.E.D.

It follows as a corollary of expression 5.12 that there are a number of output levels in the neighbourhood of \( x_2 \) which, although second best, yield an expected profit in excess of that which is earned by adjusting to forecasts with the above probability distribution of errors. Briefly, this is so because
\[ 0.5 \left[ f(x_2) + h(x_2) \right] = v(x_2) \] and this value exceeds
\[ 0.25 \left[ f(x_1) + f(x_3) + h(x_1) + h(x_2) \right] \] by some positive amount
and \( v(x) \) is continuous.

Proposition (iv). If the firm constantly produces the output, \( x_2 \), which maximises profit for the average price, \( p \), its expected profit will be at a maximum. Where \( x_\alpha \) is any level of output, the expected profit associated with it is
\[ 0.5 \left[ f(x_\alpha) + h(x_\alpha) \right] \] and the above proposition asserts that
\[ 0.5 \left[ f(x_2) + h(x_2) \right] \geq 0.5 \left[ f(x_\alpha) + h(x_\alpha) \right]. \]

Proof:
\[ 0.5 \left[ f(x_\alpha) + h(x_\alpha) \right] = 0.5 \left[ (1-a)px_\alpha - C(x_\alpha) + (1+a)px_\alpha - C(x_\alpha) \right] \]
\[ = px_\alpha - C(x_\alpha) = v(x_\alpha). \]

Since \( v(x) \) reaches a maximum at \( x_2 \), \( v(x_\alpha) \) reaches a maximum at \( x_\alpha = x_2 \). Q.E.D.

If expected price is stationary in time, expected profit will be maximised by holding output constant at the level which maximises profit for the average price. Even if the probability of forecast error is lower than 0.25 for each price, it can still be more profitable for the firm to hold its production constant at an intermediate level rather than to adjust its output marginally in
accordance with its price forecasts. This follows from
expression 4.4, i.e.,
\[ 0.25 \left[ f(x_1) + f(x_3) + h(x_1) + h(x_3) \right] < v(x_2). \]

Proposition (v). The expected profit to be earned by
constantly producing any level of output, \( x_\alpha \), is equal to the
profit when price is stable at \( p \) because
\[ 0.5 \left[ f(x_\alpha) + h(x_\alpha) \right] = v(x_\alpha). \]

Proposition (vi). If production is held constant and
if the mean price is stationary, expected profit does not decline
as the range of price variation increases. In fact, expected
profit remains constant. This proposition is contrary to Oi's
generalisation and contrasts with the forecast adjustment case,
i.e., proposition (i). Proposition (vi) asserts that
\[ 0.5 \left[ f(x_\alpha) + h(x_\alpha) \right] = 0.5 \left[ e(x_\alpha) + m(x_\alpha) \right]. \]

Proof:
\[ 0.5 \left[ f(x_\alpha) + h(x_\alpha) \right] = 0.5 \left[ (1-a)p x_\alpha - C(x_\alpha) + (1+a)p x_\alpha - C(x_\alpha) \right]. \]
\[ = px_\alpha - C(x_\alpha). \]
\[ 0.5 \left[ e(x_\alpha) + m(x_\alpha) \right] = 0.5 \left[ (1-a-b)p x_\alpha - C(x_\alpha) + (1+a+b)p x_\alpha - C(x_\alpha) \right]. \]
\[ = px_\alpha - C(x_\alpha). \]

Q.E.D.

Proposition (v) to (vi) inclusive can be generalised for any
probability distribution of price with a stationary value of
expected price. Let \( E(p) \) be the expected price of such a
probability distribution. Then, under uncertainty, expected
profit is
\[ E \left( II \right) = E \left[ px - C(x) \right] \]
\[ = E(p)x - C(x) \]
because the output of t-n is unalterable. Let the maximum of
this function occur for $x = \bar{x}$. Now, if price is stable at $\bar{p} = E(p)$, profit will be

$$\Pi = \bar{p}x - C(x).$$  
5.19

But since $\bar{p} = E(p)$ expression 5.19 equals expression 5.18.

Hence,

$$\text{Max } E(\Pi) = \bar{p} \bar{x} - C(\bar{x})$$

$$= \text{Max } [\bar{p}x - C(x)].$$  
5.20

In this particular model certainty equivalence exists, and, if expected price is constant, changes in the probability distribution of price do not affect the level of maximum expected profit and the level of optimal output. If expected price remains constant and if production is held constant at any level, increased price instability will leave expected profit unchanged. The expected profit from any level of output, $x_\alpha$, is

$$E(\Pi) = E(p) \times x_\alpha - C(x_\alpha).$$  
5.21

If both $E(p)$ and $x_\alpha$ are constant while the price variance increases, $E(\Pi)$ is constant.

From the above model it is clear that Oi's hypothesis that increased price instability leads to an increase in expected profit requires qualification. If forecast errors are of sufficient frequency and magnitude Oi's generalisation is false. Under the conditions of this model, increased price instability does not lead to a greater level of expected profit than is earnable with a stable price, and to maximise expected profit, it is necessary to hold production constant if expected price is constant. There are also a number of constant outputs in the neighbourhood of the optimum for maximising expected
profit which, although 'second best', enable higher expected profit to be earned than by varying production in accordance with one's imperfect forecasts. The desirability of constancy arises from the constancy of expected price and the fact that expected profits will be decreased by attempting to forecast and adjust to random movements.

It seems then, that Oi's conclusion for the perfectly competitive case, viz; that "instability is a virtue, and not a vice", is subject to severe limitation in the purely competitive case. Oi is correct that price instability leads to greater expected profit than stability, given either an assumption of perfect knowledge or that errors can be corrected without additional cost. When prediction errors are of sufficient frequency and when planned output can only be changed at considerable cost, Oi's hypothesis does not hold.

An assumption underlying the above analysis is that the firm at least knows the objective level of expected price. In practice, the firm will be forced to act upon a subjective estimate of expected price which may or may not accord with the objective level. If the subjective estimate remains stationary and the firm acts upon it to maximise subjective expected profit, then its level of optimal output will remain stationary, but its objective level of expected profit will increase, decrease or remain stationary accordingly as the objective level of expected price increases, decreases or remains stationary. It is obvious then that the underlying assumption of non-divergence between the subjective and objective level of expected price can limit

3 This assumption is relaxed to obtain proposition (ii).
the practical application of the above expected profit theorems.

In the concluding model of this chapter, we shall drop this assumption.

B. A Measure of Price Uncertainty.

Before dealing with the next production model, let us discuss some possible measures of price uncertainty.

There seems to be no agreed unique measure of this phenomenon, but let us consider the following measures as possibilities:

(i) \(- \text{Cov}(p, \hat{p}) = -E\left(\left[p - E(p)\right]\left[\hat{p} - E(p)\right]\right)\).

(ii) \(- R = -\frac{\text{Cov}(p, \hat{p})}{\sigma_p \sigma_{\hat{p}}}\).

(iii) \(1 - R^2\).

(iv) \(E(\left(p - \hat{p}\right)^2)\).

E represents expected or average values, R is the correlation coefficient, p represents actual price and \(\hat{p}\) shadow price, and \(\sigma\) and Cov represent the standard deviation and covariance respectively.

\(- \text{Cov}(p, \hat{p})\) is variable for proportional changes of price. If the p values increase by the fraction \(\lambda\) and the \(\hat{p}\) increase by the fraction \(\theta\), \(- \text{Cov}(p, \hat{p})\) changes to \(- \lambda \theta \text{Cov}(p, \hat{p})\).

Also, the value of \(- \text{Cov}(p, \hat{p})\) can be arbitrarily influenced by the units of measurement. These defects can be corrected by expressing \(\text{Cov}(p, \hat{p})\) in standard deviation units. If we do this, our measure of uncertainty becomes

\(- R = -\frac{\text{Cov}(p, \hat{p})}{\sigma_p \sigma_{\hat{p}}}\)

which is invariable for proportional changes of p and \(\hat{p}\) values.

The measure \(1 - R^2\) is invariable for proportional price changes but it is unsatisfactory as a general measure of
price uncertainty because it implies that price uncertainty is the same for positive and negative R values of equal absolute magnitude. It implies that uncertainty is zero if $R = -1$ but differences between shadow and actual prices certainly occur in this case.

The measure $E (p - \hat{p})^2$ is the mean of the square of the deviations of shadow price from actual price. More fully,

$$M = E [(p - \hat{p})^2]$$

$$= [E(p) - E(\hat{p})]^2 + \sigma_p^2 - 2 \text{Cov}(p, \hat{p}) + \sigma_{\hat{p}}^2$$

where $\sigma_{p-\hat{p}}^2$ is the variance of $p - \hat{p}$. This measure is not independent of proportional changes of price nor of changes in the units of measurement. The mean square of relative deviations of actual price from shadow price,

$$E \left( \left[ \frac{p - \hat{p}}{p} \right]^2 \right) = E \left( 1 - \frac{2\hat{p}}{p} + \frac{\hat{p}^2}{p^2} \right)$$

$$= 1 - E[2\hat{p}]E[\frac{1}{p}] - 2 \text{Cov}(\hat{p}, \frac{1}{p}) + E[\frac{\hat{p}^2}{p^2}] + \text{var}[\frac{\hat{p}}{p}]$$

is, in this respect, a more satisfactory measure, but unlike $M$, it is algebraically difficult to apply it to profit analysis.

4 In some formulations, uncertainty is measured by the entropy formula. If $\rho_j$ is the probability of the price $p_j$, $j = 1, \ldots, n$ and if $\sum_j \rho_j = 1$, then the entropy value is

$$H(\rho_1, \rho_2, \ldots, \rho_n) = -\sum_{j=1}^{n} \rho_j \text{lg} \rho_j$$

Increases in $H(\rho_1, \rho_2, \ldots, \rho_n)$ indicate increases in uncertainty. Some implications of the formula are outlined by A. I. Khinchin: *Mathematical Foundations of Information Theory*, Dover Publications, New York, 1959. The formula has been applied to economic decision making by J. Marschak in "Remarks on the Economics of Information", *Cowles Foundation Discussion Paper*, No. 70, 1959.
C. An Average Profit Model which allows more generality in the Distribution of Actual and Shadow Price Values.

Our assumptions for this model are:

(i) The firm's output of t equals its output planned for t in t-n.

(ii) The output of any period is independent of that of any other.

(iii) Costs are certain and stationary.

(iv) Costs are quadratic.

The firm's profit function is of the quadratic form
\[ \Pi = px - (ax^2 + bx + c). \]

Its quadraticity ensures that average profit is dependent upon only the first and second moments of actual and shadow prices.

Since
\[ x^* = x = \frac{\hat{p} - b}{2a}, \]

\[ \Pi = p (\frac{\hat{p} - b}{2a}) - [a (\frac{\hat{p} - b}{2a})^2 + b (\frac{\hat{p} - b}{2a}) + c] \]

where \( x^* \) represents planned output and \( \hat{p} > b \). Consequently, average profit, i.e., profit averaged over several periods, is

\[ E(\Pi) = \frac{E(p)E(\hat{p}) + Cov(p,\hat{p}) - bE(p) - c}{2a} \]

\[ - b \left[ \frac{E(\hat{p}) - b}{2a} \right] - \frac{E(\hat{p})^2 + \text{var} \hat{p} - 2 b E(p) + b^2}{4a} \]

\[ = \frac{2 E(p)E(\hat{p}) + 2 Cov(p,\hat{p}) - 2 bE(p) - E(\hat{p})^2 - \text{var} \hat{p} + b^2}{4a} - c. \]

---

If the average value of actual price is constant,

\[ \Delta E(\Pi) = \frac{1}{4a} \left\{ [2E(p) - 2E(\hat{p})] \Delta E(p) + 2 \Delta \text{Cov}(p, \hat{p}) - \Delta \text{var} \hat{p} \right\} \]

\[ = -\frac{1}{4a} \left\{ [ -2E(p) + 2E(\hat{p})] \Delta E(\hat{p}) - 2 \Delta \text{Cov}(p, \hat{p}) + \Delta \sigma_p^2 \right\} \] 5.33

If \( E(p) \) is constant,

\[ \Delta M = [ -2E(p) + 2E(\hat{p})] \Delta E(\hat{p}) + \Delta \sigma_p^2 - 2 \Delta \text{Cov}(p, \hat{p}) + \Delta \sigma_p^2 \hat{p} \] 5.34

Consequently, if \( E(p) \) and \( \sigma_p^2 \) are constant,

\[ \Delta E(\Pi) = -\frac{1}{4a} \Delta M. \] 5.35

(1) If the average value and variance of actual price are constant, an increase in the mean square of deviations of shadow price from actual decreases average profit by the reciprocal of twice the rate of change of marginal cost times the change of \( M \). If \( E(p) \) and \( \sigma_p^2 \) are constant,

\[ \Delta E(\Pi) \leq 0 \]

as \( \Delta M < 0. \)

Now consider the effect upon average profit of an increase in the variance of actual price if \( E(p) \) and \( M \) remain constant. By assumption,

\[ \Delta \sigma_p^2 \]

\[ \Delta M = [ -2E(p) + 2E(\hat{p})] \Delta E(\hat{p}) + \Delta \sigma_p^2 \]

\[ - 2 \Delta \text{Cov}(p, \hat{p}) + \Delta \sigma_p^2 = 0. \] 5.36

Therefore,

\[ [ -2E(p) + 2E(\hat{p})] \Delta E(\hat{p}) - 2 \Delta \text{Cov}(p, \hat{p}) + \Delta \sigma_p^2 = -\Delta \sigma_p^2. \] 5.37

Hence, if \( \Delta E(p) = 0 \) and if \( \Delta M = 0 \),

\[ \Delta E(\Pi) = \frac{1}{4a} \Delta \sigma_p^2. \] 5.38
(2) If average price and the mean square of deviations of shadow price from actual are constant, an increase in the price variance increases average profit by the reciprocal of twice the rate of change of marginal cost times the change of the variance of price.

If \( E(p) \) and \( M \) are constant,

\[
\Delta E(\Pi) < 0
\]

as

\[
\Delta \sigma_p^2 > 0.
\]

D. A Note Upon Oi's Revised Assertion

In a rejoinder to my Econometrica article of April 1963, Oi briefly mentions the possibility of relating actual and predicted prices (in our case shadow prices) by means of regression analysis.

He concludes: "If the explained variance (of actual price) is positive, then it can be shown that greater systematic fluctuations in prices must increase expected profits. Space precludes a full proof of this proposition... So long as price instability contains a systematic component, greater price instability will lead to higher expected profits". Since Oi states that the explained represents the systematic component his conclusion re-stated is: So long as price instability contains a positive explained variance, greater price instability will lead to higher expected profit.

Given even the most liberal interpretation, this conclusion appears to be incorrect. To see this, suppose


\[ z = \gamma_o + \gamma_1 \hat{p} \]  

5.39

to be the least squares regression of \( p \) on \( \hat{p} \). Then, the "explained" variance of \( p \) is

\[ \sigma_z^2 = \gamma_1^2 \sigma_p^2. \]

5.40

Also,

\[ \sigma_z^2 = \sigma_p^2. \]

5.41

Consequently,

\[ \sigma_z^2 = \frac{R^2 \sigma_p^2}{\gamma_1^2} = \frac{\sigma_z^2}{\gamma_1^2}. \]

5.42

\[ \text{Cov}(p, \hat{p}) = R^2 \frac{\sigma_p^2}{\gamma_1^2} \]

5.43

If we substitute equations 5.42 and 5.43 into the average profit function, we find, if \( \text{E}(p), \text{E}(\hat{p}) \) and \( \sigma_z^2 \) are positive and do not change with changes of the variance of actual price, that average profit remains constant as the price variance increases.

But let us suppose that Oi meant to imply that, if \( R \) is positive and constant, increased price instability leads to an increase of average profit if the regression line is maintained unchanged.

Then, even this proposition is false.

This can be shown as follows:

Assume that \( d R^2 = 0, \ d \text{E}(p) = 0, \) and \( d \text{E}(\hat{p}) = 0. \) Then, using equations 5.42 and 5.43,
\[
\frac{d E(\Pi)}{d \sigma^2_p} = R^2 \left( \frac{2 \gamma_1}{2 \sigma^2_p} - \frac{R^2}{4 \sigma^2_p} \right)
\]

5.44

\[
= \frac{R^2 \left( 2 \gamma_1 - 1 \right)}{4 \sigma^2_p}.
\]

5.45

If \( R^2 > 0 \),

\[
\frac{d E(\Pi)}{d \sigma^2_p} \geq 0
\]

as \( \gamma_1 = \frac{\text{Cov}(p, \hat{\sigma})}{\sigma^2_p} \geq \frac{1}{2} \).

Since

\[
\gamma_1 = R \frac{\sigma_{p}}{\sigma_{\hat{\sigma}}},
\]

5.46

equation 5.45 can be rewritten as

\[
\frac{d E(\Pi)}{d \sigma^2_p} = R^2 \left( \frac{2 R \sigma_{p}}{\sigma^2_{\hat{\sigma}}} - 1 \right)
\]

5.47

Hence, if \( d \sigma^2_p \)

\[
\frac{d E(\Pi)}{d \sigma^2_p} \geq 0
\]

as

\[
\sigma_{\hat{\sigma}} > 2 R \sigma_{p}.
\]

(3) If the correlation coefficient, average actual price and

average shadow price are constant, average profit is decreased

by an increase of the standard deviation of price if the standard

deviation of shadow price exceeds the standard deviation of

actual price times twice the correlation coefficient. Even if

\( R > 0 \), it is possible for \( \frac{1}{2} \sigma_{\hat{\sigma}} > R \sigma_{p} \) and for average profit

to decrease as the variance of price increases.
If average price and average shadow price are constant and if price is certain, it is true (except if marginal cost is completely inelastic) that increased price instability increases average profit. However, it is incorrect to suggest that increased price instability always increases average profit or never decreases it. If price is uncertain, increased price instability can reduce average profit even if the correlation between "predicted" and actual price is positive and constant.

E. Restrictions Upon the Relationship between the Moments of the Average Profit Function.

It is possible to place various restrictions upon the relationship between the moments of equation 5.32, i.e. the general average profit equation. In section D, we restricted this relationship by assuming a constant regression of p or \( \hat{p} \) of

\[ z = \gamma_0 + \gamma_1 \hat{p}. \]

Given this restriction, equation 5.32 becomes

\[
E(\Pi) = \frac{2E(p)E(\hat{p}) + \frac{2R^2}{\gamma_1} \sigma_p^2 - \frac{2bE(p) - E(\hat{p})}{\gamma_1}}{4a} - \frac{2R^2}{\gamma_1} \sigma_p^2 + \frac{b^2}{p}. \]

We note the following theorems:

(i) If \( E(p) \), \( E(\hat{p}) \) and \( \sigma_p^2 \) are constant,

\[
\frac{dE(\Pi)}{dR^2} = \frac{2}{\gamma_1} - \frac{1}{\gamma_1} \frac{\sigma_p^2}{4a} \sigma_p^2.
\]

and if \( \sigma_p^2 > 0 \) and \( 2a < \infty \),

\[
\frac{dE(\Pi)}{dR^2} \geq 0 \text{ as } R^2 \rightarrow \infty.
\]
\[ \gamma_1 = \frac{\text{Cov}(p, \hat{p})}{\sigma_p^2} < \frac{1}{2}. \]

But, given these conditions, an increase of \( \sigma_p^2 \) does not imply a decrease of \( M \). Under these conditions

\[ \frac{dM}{d\sigma_p^2} = \left( -\frac{2}{\gamma_1} + \frac{1}{\gamma_1^2} \right) \sigma_p^2. \]

If \( \sigma_p^2 > 0 \), \( \frac{dM}{d\sigma_p^2} \leq 0 \) as \( \gamma_1 < \frac{1}{2} \).

Theorem (i) implies that, if \( \gamma_1 < \frac{1}{2} \), the firm increases its average profit by reducing \( \sigma_p^2 \) to zero if \( E(p) \), \( E(\hat{p}) \) and \( \sigma_p^2 \) are constant. Even if \( R > 0 \), it can be more profitable for the firm not to attempt to predict variations of price but to rigidly predict the same price or produce the same quantity of output over a stretch of time.

(ii) If \( E(p) \), \( E(\hat{p}) \) and \( \sigma_p^2 \) is constant, 

\[ \frac{dE(\hat{p})}{d\sigma_p^2} = \frac{2}{\gamma_1} - \frac{1}{\gamma_1^2} R^2, \]

and if \( R^2 > 0 \) and if \( 2a < \infty \),

\[ \frac{dE(\hat{p})}{d\sigma_p^2} \geq 0 \]

as \( \gamma_1 < \frac{1}{2} \).

Given these conditions, an increase of \( \sigma_p^2 \) rarely leaves

\[ M = [E(p) - E(\hat{p})]^2 + \sigma_p^2 - \frac{2R^2}{\gamma_1} \sigma_p^2 + \frac{\gamma^2 \sigma_p^2}{\gamma_1^2} 5.50 \]

unchanged.
F. Nelson's Model as a Special Case.

If we suppose, as Nelson has done, that \( \hat{p} \) is an unbiased estimate of \( p \), the regression of \( p \) on \( \hat{p} \) becomes

\[
z = \hat{p},
\]

\[E(p) = E(\hat{p}),\]

and equation 5.49 specialises to Nelson's basic equation.\(^{10}\)

\[
E(\Pi) = \frac{R^2 \sigma^2}{4a} + \left[ E(p) - b \right]^2 - c.
\]

From this equation we obtain Nelson's following theorems:

(i) If \( E(p) \) and \( \sigma_p^2 \) are constant,

\[
\frac{dE(\Pi)}{dR^2} = \frac{1}{4a} \frac{\sigma^2}{p}.
\]

If \( 2a < \infty \) and if \( \sigma_p > 0 \), \( \frac{dE(\Pi)}{dR^2} > 0 \), and \( \frac{dM^2}{dR^2} < 0 \). In this special case, \( M \) increases with increases of \( 1 - R^2 \).

(ii) If \( E(p) \) and \( R^2 \) are constant,

\[
\frac{dE(\Pi)}{d\sigma_p^2} = \frac{1}{4a} R^2.
\]

If \( R^2 > 0 \) and if \( 2a < \infty \), \( \frac{dE(\Pi)}{d\sigma_p^2} > 0 \). In this case,

\[
\frac{dM^2}{d\sigma_p^2} \geq 0 \text{ as } R^2 \leq 0. \quad \frac{dM^2}{d\sigma_p^2} = 1 - R^2.
\]

---


10 Equation 5.52 corresponds to equation (3.7a) of Marschak and Nelson: op. cit., p. 52.
In Nelson's model, an increase in the price variance increases average profit if both $R^2$ and average price are constant, but in the general linear regression model an increase in the price variance can decrease average profit even if $R > 0$. We should also bear in mind that $1 - R^2$ seems to be only of special significance as a measure of price uncertainty in Nelson's particular model.
APPENDIX TO CHAPTER V.

Average Profit and a General Cost Model

A conflict arises in dealing with the effect of price instability and uncertainty upon average profit. The more general is the cost function the more difficult it is to analyse the effect upon the firm's average profit of changes in the distribution of actual and shadow prices. However, under general cost conditions it is possible to derive the effect upon average profit of special changes in the distribution of actual and shadow prices. In order to obtain the effect of a special change, we shall use the relationship between maximum profit and price.

Let \( \psi(p) \) represent profit as a function of price when output is adjusted to price so that a global maximum of profit exists. The function \( \psi(p) \) is obtained in the following way:

If

\[
\Pi = px - C(x)
\]

represents the firm's general profit function and, if its production ceases when average variable cost is not covered, its supply function is

\[
\begin{align*}
x &= g(p) \\
&= 0
\end{align*}
\]

for \( p \geq \text{min } A.V.C. \)

for \( p < \text{min } A.V.C. \)

Expression 2 is the inverse of

\[
\begin{align*}
p &= C'(x) \\
0 &= x
\end{align*}
\]

for \( p \geq \text{min } A.V.C. \)

for \( p < \text{min } A.V.C. \)

Substituting expression 2 into 1, maximum profit as a function of price is
\[ \psi (p) = p g(p) - C(g(p)) \quad \text{for} \quad p \geq \min A.V.C. \]
\[ \psi (p) = -K \quad \text{for} \quad p < \min A.V.C. \]

where \( K \) represents fixed cost. The rate of change of \( \psi (p) \) for \( p > \min A.V.C. \) is

\[ \frac{d\psi}{dp} = p \frac{dg}{dp} + g(p) - \frac{dC}{dg} \frac{dg}{dp} \]
\[ = \frac{dg}{dp} (p - \frac{dC}{dg}) + g(p) \]
\[ = g(p) \]

because profit maximisation requires that

\[ p = \frac{dC}{dx} = \frac{dC}{dg(p)}. \]

By the inverse rule of differentiation the slope of the maximum profit function is

\[ \frac{d\psi}{dp} = g(p) = \frac{1}{C'(x)} \quad \text{for} \quad p > \min A.V.C. \]
\[ = 0 \quad \text{for} \quad p < \min A.V.C. \]

The "curvature" of the profit function is

\[ \frac{d^2 \psi}{dp^2} = g'(p) = \frac{1}{C''(x)} \quad \text{for} \quad p > \min A.V.C. \]
\[ = 0 \quad \text{for} \quad p < \min A.V.C. \]

Since \( C''(x) > 0 \) for all output values in excess of minimum average variable cost \( \frac{d^2 \psi}{dp^2} > 0 \) for \( p > \min A.V.C. \). For price values in excess of minimum average variable cost, the maximum profit function increases at an increasing rate.

Because the function, \( \psi (p) \), is convex for \( p > \min A.V.C. \), then, if \( p > \min A.V.C. \) and if all values of \( p \) are not equal, 1

Expression 8 implies (i) that if a firm has price certainty, its average profit is less for a stable price than for unstable prices having an average value equal to the stable price and (ii) if the firm's shadow prices are equal to the average of actual price values, its average profit, if prices are not all of the same value, is less than if it is certain of actual prices. This assumes that the production conditions of Chapter V apply mutatis mutandis.
CHAPTER VI

Flexibility, Average Profit and Technique Choice

A. Introduction

It is commonly asserted that increased price uncertainty encourages the adoption of more flexible techniques. While there seems to be little need to doubt this hypothesis if flexibility is defined so as to accord with Hart's usage, this is not so if flexibility is defined so as to accord with the Baumol's apparent usage.

For Hart, flexibility refers to the ability to modify plans in time. A plan is flexible if it is possible to diverge from the planned values at a date subsequent to their acceptance. If the firm plans a particular value of output (a point value) its plan is flexible if it can subsequently produce an output different to the planned. In Theil's terminology, inflexible plans involve static decision making and flexible plans involve dynamic decision making.

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3 Hart: op. cit.

It seems that for both Stigler and Baumol flexibility refers to the rate of change of marginal cost when static decision making is assumed. Under this condition, a technique is considered to be more inflexible the larger is its second derivative of total cost. 5

B. Baumol's Approach to Changes in Technique Choice.

Baumol argues that "the existence of uncertainty will lead to the (increased) use of equipment whose scale of operation is flexible." 6 His argument is based upon a figure which is identical to figure 6.1. In this figure, the average cost function of Baumol's inflexible technique is shown by \(AC_I\) and that of the flexible is shown by \(AC_F\). Figure 6.1 is identical to the one which J.G. Stigler uses in putting forward his hypothesis that under certainty increased price instability increases the likelihood of the firm adopting the flexible technique. 7 In figure 6.1 the second derivative of average cost

Figure 6.1

Average Cost

Output


6 Baumol: op. cit.

of technique I exceeds that of technique F for all values of output.

Baumol's argument goes as follows: Given complete price certainty, suppose that the firm finds an output between 0m and 0q to be optimal. Then it will minimise its cost by adopting technique I. If the firm becomes uncertain of its price, then it is possible for an output greater than 0q to be optimal, and, in that event, technique F will minimise the firm's costs. Upon this basis, Baumol concludes that "the existence of uncertainty will lead to the use of equipment whose scale of operation is flexible".

Baumol's argument is insufficient to support his conclusion. Using what appears to be Baumol and Stigler's essential condition i.e., differences of the rate of change of marginal cost, we shall present two models for which (i) Baumol's hypothesis does not hold, and (ii) Stigler's hypothesis does hold. For these two models (i) "increased price uncertainty" increases the likelihood of the firm's adopting the technique with the greatest rate of change of marginal cost, i.e., the inflexible technique in Baumol's apparent usage and (ii) if "price uncertainty" is constant, increased price instability increases the probability of the firm's adopting the technique with the least rate of change of marginal cost. Under certainty increased price instability increases the probability of the firm's adopting the technique with the least rate of change of marginal cost i.e., the flexible technique according to Stigler's apparent usage.
The first model is an introductory one which is put forward because it relies upon a general cost relationship and not a quadratic one. The second model relies upon the assumption of a linear marginal cost function but enables us to treat the uncertainty aspect of the problem more generally.

C. Model I - A Simple Model.

In model I we consider how the difference in the comparative profitability of two techniques changes as price changes from certain to probable. We attempt to isolate under general cost conditions a cost factor which has an important bearing upon the direction of technique choice as price changes from certain to probable.

The following assumptions are made for model I:

(i) Under uncertainty the probability distribution of price is the same as the relative frequency distribution of price under certainty. The probability distribution is stationary in time.

(ii) The firm wishes to maximise expected profit.

(iii) Under uncertainty, the firm has perfect predictability of expected price.

(iv) The two techniques to be considered have the same length of life of n periods.

(v) Only one technique can be chosen for the n periods.

(vi) All of the n periods are of equal length.

(vii) The marginal cost functions of every period are equal for each technique. This is also the case for the average cost functions.

(viii) After reaching their minimum, the marginal cost functions increase monotonically.
Also, the static decision making assumptions of previous chapters are maintained:

(ix) Output decisions for each period must be made prior to that period, and once made are unalterable.

(v) The output decisions which are made at different points of time are independent.

The analysis of the importance of the rate of change of marginal cost for the choice of technique as price changes from certain to probable will be facilitated if we derive the function which expresses the difference in the maximum profitability of the two techniques as a function of price. From the appendix of the previous chapter, various properties of maximum profit as a function of price are already known. For both techniques these functions are constant up to the level of minimum average variable cost and then increase at an increasing rate. Using the subscripts 1 and 2 to indicate techniques one and two, technique one's maximum profit function is

\[ \psi_1(p) = p g_1(p) - C(g_1(p)) \quad \text{for } p \geq \min A.V.C.1 \]

\[ = -K_1 \quad \text{for } p < \min A.V.C.1 \]

and

\[ \psi_1'(p) = \frac{1}{C_1'(x)} \quad \text{for } p > \min A.V.C.1', \]

and

\[ \psi_1''(p) = \frac{1}{C_2''(x)} > 0 \quad \text{for } p > \min A.V.C.1 \]

8 The same symbols are used for the maximum profit function as were used in the appendix to the previous chapter.
The function \( \psi (p) \) is convex and is at least twice differentiable for all values of price in excess of minimum average variable cost. If \( K_1 > 0 \) and if \( C'_1 (x) > 0 \) for all output values in excess of minimum average variable cost, then the function can be illustrated as in figure 6.2. Similar conditions hold for the maximum profit function of technique two.

**Figure 6.2**

The difference between these two maximum profit functions is

\[
W(p) = \psi_1 (p) - \psi_2 (p)
\]

To keep to the essentials of the analysis, let us only consider the properties of this function for price values in excess of the greatest of the minima of the average costs of the two techniques, i.e., for

\[ p > \text{Max} \{ \min \text{AVC}_1, \min \text{AVC}_2 \} \]

For these values of \( p \),

\[
W'' (p) = \psi_1 '' - \psi_2 '' = \frac{1}{c_1 ''(x)} - \frac{1}{c_2 ''(x)}.
\]
Since stability requires that $C_1''(x) > 0$ and $C_2''(x) > 0$,
\[ W''(p) \geq 0 \]  
accordingly as
\[ C_1''(x) \leq C_2''(x). \]  
Under certainty, the difference in the expected profitability of two techniques is
\[ \sum_j r_j W(p_j) = \sum_j r_j \psi_1(p_j) - \sum_j r_j \psi_2(p_j) \]  
and, under price uncertainty the difference is
\[ W \left( \sum_j r_j p_j \right) = \psi_1 \left( \sum_j r_j p_j \right) - \psi_2 \left( \sum_j r_j p_j \right) \]
where $r_j$ represents the relative frequency of the $j$-th price.

If (i) $p > \max [\min AV_1, \min AV_2]$, if (ii) all prices are not equal, and if (iii) the same inequality relationship between $C_1''(x)$ and $C_2''(x)$ holds for all relevant values of output,
\[ \sum_j r_j W(p_j) \leq W \left( \sum_j r_j p_j \right) \]  
accordingly as
\[ C_1''(x) \leq C_2''(x). \]  
This is because $W''(p) \geq 0$, accordingly as $C_1''(x) \leq C_2''(x)$.

If no price less than $\max [\min AV_1, \min AV_2]$ can occur, then the excess profitability of technique one under certainty is greater than, equal to, or less than that under uncertainty accordingly as the rate of change of marginal cost for technique one is less than, equal to, or greater than the rate of change of technique two's marginal cost. The relationship between expression 6.9 and 6.10 implies that uncertainty increases the

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9 For a proof of the relevant theorem concerning the expectation of values defined on a convex function see G. H. Hardy, J. E. Littlewood, G. Polya: Inequalities, Cambridge University Press, 1934, pp. 74-75, theorem 90.
excess profitability of the technique with the greatest rate of change of marginal cost. Given the above conditions, if there is a change from one technique to another as a result of the occurrence of price uncertainty, it can only be away from the technique with the least rate of change of marginal cost and in favour of the technique with greatest rate of change of marginal cost.

This point can be illustrated by the example shown in figure 6.3. The function $W(p)$ is shown for $p > \max \{ \min AVC_1, \min AVC_2 \}$. Two prices, $p_1$ and $p_2$, are assumed to be the only possible ones, and $C_1''(x)$ is assumed to be less than $C_2''(x)$ for all the relevant values of output. Hence, $W''(p) > 0$ for the relevant domain of $p$ values. The difference function is a convex one. The excess expected profit of technique one is

$$
\sum_{j=1}^{2} r_j W(p_j) \quad \text{under certainty and,} \quad W \left( \sum_{j=1}^{2} r_j p_j \right) \quad \text{under uncertainty.}\n$$

From an inspection of figure 6.3 it is obvious that where $p_1$ and $p_2$ differ and where $r_1$ and $r_2$ are positive,

$$
\sum_{j=1}^{2} r_j W(p_j) > W \left( \sum_{j=1}^{2} r_j p_j \right).
$$

![Figure 6.3](image-url)
Arbitrary values of \( p_j \) and \( r_j \) are used in figure 6.3 to illustrate a case in which technique one is chosen under certainty and technique two (the technique with the greatest rate of change of marginal cost) is chosen under uncertainty.

The main point of the model is that if there is no price which makes a shutdown of production optimal, then uncertainty increases the probability of the adoption of the technique with the greatest rate of change of marginal cost. This result is not in accordance with Baumol's hypothesis. Also the rate of change of marginal cost and not the absolute level of marginal cost, i.e., the rate of change of total cost, is the significant factor determining changes in the direction of technique choice as uncertainty changes. This is because the rate of change of total cost has no influence upon the curvature of the difference function, \( W(p) \).

This can be seen from equation 6.4. The curvature of \( W(p) \) is of crucial importance for determining the direction of the change of technique choice as price uncertainty changes. This curvature is dependent upon the rate of change of marginal cost.

It also follows from inequalities 6.9 and 6.10 that under certainty the firm is more likely to adopt the technique with the least rate of change of marginal cost as price instability increases. The "excess" expected profit of the technique which has the least rate of change of marginal cost is greater for a series of unstable prices than for a stable price equal to the average of the unstable prices. This result is in accordance with Stigler's hypothesis.

Assumption (i), (ii), and (iii) limit the generality of the preceding model. To relax them, we shall, as in the second model of Chapter V, make the following assumptions:

(i) The firm's output of \( t \) equals its output planned for \( t \) in \( t-n \).

(ii) The output of any period is independent of that of any other.

(iii) Costs are certain and stationary.

(iv) Total cost is quadratic.

In addition, we suppose assumptions (iv), (v), (vi) and (vii) of the last model to hold.

We do not suppose as Marschak and Nelson do that

(i) the firm aims to maximise expected profit for each of the sub periods of the techniques' length of life \( T = \sum_{t=1}^{n} t \), and to maximise average profit for \( T \),

(ii) knows at the date of choosing its technique the relative frequency distribution of the "true" expected price values, and

(iii) knows at each production decision point the true expected price of the output dependent on that decision.

We shall suppose (i) that the firm may adopt any combination of criteria or behaviour pattern for the sub periods of \( T \) but for its actual pattern for \( T \) it wishes to adopt the technique which yields the greatest average profit, and (ii) that at its technique decision date it holds an anticipated relative frequency distribution of actual prices and its shadow prices for \( T \).

Suppose that the firm is uncertain of the relative frequency distribution of \( p \) and \( \hat{p} \) values which it will realise for \( T \). If \( E_e(\Pi) \) represents the average profit which the firm anticipates for \( T \) and if \( E(\Pi) \) represents average profit given the actual \( p \) and \( \hat{p} \) values of \( T \),

\[
\Pr \left\{ \left| E_e(\Pi) - E(\Pi) \right| > 0 \right\} > 0.
\]

The firm at its technique decision date acts upon \( E_e(\Pi) \) in choosing its technique. For any technique which has a cost function \( C = ax^2 + b + c \) anticipated average profit is

\[
E_e(\Pi) = \frac{1}{4a} \left( 2 E_e(p) E_e(\hat{p}) + 2 \text{Cov}_e(p, \hat{p}) - 2b E_e(p) \right) - \frac{E_e(\hat{p})^2}{\text{var}_e \hat{p}} + \frac{b^2}{c} - c.
\]

Obviously the theorems for the last model of Chapter V can be reinterpreted mutatis mutandis so as to apply only to anticipated values.

However, in this Chapter, we shall further develop the last model of Chapter V. We shall develop theorems which take account of changes in the average profit of techniques given the actual \( p \) and \( \hat{p} \) values. In order to apply these theorems to technique choice, it is supposed that there is a positive relationship between anticipated and actual average profit values and anticipated and actual differences of average profit values for the techniques.

The respective profit functions for technique one and two are

\[
\Pi_1 = px - (a_1 x^2 + b_1 x + c_1)
\]

6.11
\[ \Pi_2 = px - (a_2 x^2 + b_2 x + c_2). \]

Each of these functions can be expressed in terms of actual and shadow prices. Consequently, the average profit from the techniques is

\[
E(\Pi_1) = \frac{E(p) E(\hat{p}) + \text{Cov}(p, \hat{p}) - b_1 E(p)}{2a_1} - c_1
\]

\[
= \frac{E(\hat{p}) - b_1}{2a_1} - \frac{E(p)^2 + \text{var} \hat{p} - 2b_1 E(p) + b_1^2}{4a_1}
\]

and

\[
E(\Pi_2) = \frac{E(p) E(\hat{p}) + \text{Cov}(p, \hat{p}) - b_2 E(p)}{2a_2} - c_2
\]

\[
= \frac{E(\hat{p}) - b_2}{2a_2} - \frac{E(p)^2 + \text{var} \hat{p} - 2b_2 E(p) + b_2^2}{4a_2}
\]

The "excess" average profit from technique one is

\[
E[W] = E(\Pi_1) - E(\Pi_2).
\]

From the argument surrounding the derivation of equation 5.35, it is obvious that, if \(E(p)\) and \(\text{var} \, p\) are constant,

\[
\Delta [W] = - \left[ \frac{1}{2a_1} - \frac{1}{4a_2} \right] \Delta M
\]

where \(M\) represents the mean of the squared deviations of actual price from shadow price. Hence, if \(E(p)\) and \(\text{var} \, p\) are constant and if \(\Delta M > 0\),

\[
\Delta E[W] < 0
\]

accordingly as

\[
2a_1 > 2a_2.
\]
(1) If the average value and variance of actual price are constant, an increase in the mean of squared deviations of shadow price from actual leads to an increase in the "excess" average profit of the technique with the greatest rate of change of marginal cost. Consequently, under the above conditions, an increase in the mean of the squared deviations of shadow price from actual increases the probability of the firm adopting the technique which has the greatest rate of change of marginal cost. This conclusion does not accord with Baumol's.

Now consider the effect upon technique choice of an increase in the variance of price if the mean square of deviations of shadow from actual price is constant and if average actual price is constant. If $\Delta E(p) = 0$ and if $\Delta M = 0$, then by extension of the argument used for deriving equation 5.38,

$$\Delta E[W] = \left[ \frac{1}{4a_1} - \frac{1}{4a_2} \right] \Delta \sigma^2_p.$$  

Hence, if $E(p)$ and $M$ are constant, and if $\Delta \text{var } p > 0$,

$$\Delta E[W] \leq 0$$

accordingly as

$$2a_1 > 2a_2$$

(2) If the average value of price and the mean square of deviations of shadow price from actual are constant, an increase in the price variance increases the "excess" average profit of the technique which has the least rate of change of marginal cost. Consequently, under the above conditions, an increase in the variance of price increases the probability of the firm's adopting the technique with the least rate of change of marginal cost. Under price certainty, $E(p)$ and $M = 0$ are constant. Hence, proposition (2) is in accordance with Stigler's hypothesis.
In a recent article Marschak and Nelson, by considering Nelson's model which implies that

\[
E(\Pi) = \frac{R^2 \sigma_p^2 + [E(p) - b]^2}{4a} - c
\]

come to the conclusion that "the relative desirability of the flexible plant increases as:

(a) the variation in market price (as measured by variance) increases,

(b) the ability to predict market price before making an output decision increases".  

Since they use Nelson's profit model, they fail to support their hypothesis by a general model. Furthermore, they fail to point out that \( R^2 \) need not be a reasonable measure of the ability to predict if predicted price is not an unbiased estimate of the market price, \( p \).

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11 Ibid.

12 Ibid: p. 50. I had independently reached similar conclusions by early 1962.
CHAPTER VII
Dynamic Decision Making, Average Profit and Technique Choice.

A. Introduction

The firm has been assumed to make static and point output plans. Such plans can arise because great additional cost attaches to a divergence from planned output. It is desirable to relax this assumption which implies complete inflexibility in Hart's sense. We shall relax this assumption and show that (i) my hypothesis that increased price instability can decrease average profit if \( R > 0 \) is not completely dependent upon the assumption that output plans are completely inflexible, (ii) other things equal, "increased price uncertainty" increases the probability of the firm's adopting the technique with the least rate of change of marginal cost of divergence of actual output from planned, i.e., possibly the flexible plant in Hart's usage, and (iii) Stigler's hypothesis that uncertainty increased price instability increases the probability of the firm's adopting the technique with the least rate of change of marginal cost still holds.

B. Average Profit.

In order to simplify the analysis, it is necessary to restrict our attention to quadratic cost functions and to make some special dependence assumptions. The following assumptions are made:

(i) The firm makes only two decisions upon its output for any period, \( t \). The first is made in period \( t-n, n \geq 1 \), and the second is made in period \( t \). For any given output, costs can be increased by a difference between
the first and second decision.

(ii) The planned outputs of different periods are independent and so also are the actual outputs of different periods.

(iii) The cost function is stationary and is known.

(iv) This function is the quadratic

\[ \sum_{i=0}^{2} a_i x_i^2 + \sum_{i=0}^{2} b_i (x - x^*)^i \]

where \( b_0 = 0 \) if \( x - x^* = 0 \), and \( b_o \geq 0 \) if \( |x - x^*| > 0 \), and where \( x^* \) represents planned output, \( x \) represents actual output, \( a_2 > 0 \) and \( b_2 > 0 \).

In order to reduce the algebra we suppose that \( a_1 = 0, b_o = 0 \) and \( b_1 = 0 \). Consequently, the cost function simplifies to

\[ C(x, x^*) = \alpha_0 + \alpha_1 x^2 + \alpha_2 (x - x^*)^2. \]

The firm's profit function is

\[ \Pi = px - \left\{ \alpha_0 + \alpha_1 x^2 + \alpha_2 x^2 - 2 \alpha_2 xx^* + \alpha_2 x^* \right\}. \]

Since

\[ x^* = \frac{\hat{p}}{2 \alpha_1} \]

and

\[ x = \frac{p + 2 \alpha_2 x^*}{2 \alpha_1 + 2 \alpha_2} \]

\[ = \frac{p + 2 \alpha_2}{2 \alpha_1 + 2 \alpha_2} \hat{p} \]

\[ = \frac{2 \alpha_2}{2 \alpha_1 + 2 \alpha_2} \hat{p} \]

\[ = \frac{2 \alpha_2}{2 \alpha_1 + 2 \alpha_2} \hat{p} \]

\[ E(\Pi) = E \left\{ \frac{p + 2 \alpha_2 \hat{p}}{2 \alpha_1 + 2 \alpha_2} - \alpha_0 - \left( \alpha_1 + \alpha_2 \right) \left( \frac{p + 2 \alpha_2 \hat{p}}{2 \alpha_1 + 2 \alpha_2} \right)^2 \right\} \]
\[
\begin{align*}
\frac{2\alpha_2}{2\alpha_1} + \frac{2\alpha_2}{2\alpha_1} \left( \frac{p + \frac{2\alpha}{2\alpha_1} \hat{p}}{2\alpha_1 + 2\alpha_2} \right) - \alpha_2 \left( \frac{p}{2\alpha_1} \right)^2 \\
E(p)^2 + \text{var } p + \frac{2\alpha_2}{2\alpha_1} \left[ E(p) E(\hat{p}) + 2 \text{ Cov } (p, \hat{p}) \right] - \frac{2\alpha_2}{2\alpha_1} \left[ E(\hat{p})^2 + \text{var } \hat{p} \right]
\end{align*}
\]

7.6

\[
\begin{align*}
\frac{4\alpha_1 + 4\alpha_2}{4\alpha_1 + 4\alpha_2}
\end{align*}
\]

7.7

Hence, if \( E(p) \) and \( \text{var } p \) are constant,

1. To derive equation 7.7 we express equation 7.6 as

\[
E(\hat{p})^2 + \text{var } \hat{p} + \frac{2\alpha_2}{2\alpha_1} \left[ E(p) E(\hat{p}) + \text{Cov } (p, \hat{p}) \right]
\]

\[
E(\hat{p})^2 + \text{var } \hat{p} + \frac{4\alpha_1 + 4\alpha_2}{4\alpha_1 + 4\alpha_2}
\]

\[
\frac{4\alpha_2}{2\alpha_1} \left[ E(p) E(\hat{p}) + \text{Cov } (p, \hat{p}) \right] + \frac{4\alpha_2}{4\alpha_1} \left[ E(\hat{p})^2 + \text{var } \hat{p} \right]
\]

\[
+ \frac{2\alpha_2}{2\alpha_1} \left[ E(\hat{p})^2 + \text{var } \hat{p} \right]
\]

\[
- \alpha_2 \left[ E(\hat{p})^2 + \text{var } \hat{p} \right]
\]

\[
\begin{align*}
\frac{4\alpha_1 + 4\alpha_2}{4\alpha_1 + 4\alpha_2}
\end{align*}
\]

\( (i) \)

\[
\begin{align*}
E(p)^2 + \text{var } p + \frac{4\alpha_2}{2\alpha_1} \left[ E(p) E(\hat{p}) + \text{Cov } (p, \hat{p}) \right] + \frac{4\alpha_2}{4\alpha_1} \left[ E(\hat{p})^2 + \text{var } \hat{p} \right]
\end{align*}
\]

\[
= \frac{4\alpha_1 + 4\alpha_2}{4\alpha_1 + 4\alpha_2}
\]

\[
- \alpha_2 \left[ E(\hat{p})^2 + \text{var } \hat{p} \right]
\]

\( (ii) \)

Multiplying the last term of (ii) by \( \left( \frac{1}{\alpha_1} + \frac{\alpha_2}{\alpha_1^2} \right) \) and collecting terms, we obtain equation 7.7.
\[ \Delta E(\Pi) = \frac{2 \alpha_2}{\frac{2 \alpha_1}{1 + \frac{4 \alpha_2}{4 \alpha_1 + 4 \alpha_2}}} \left\{ \left[ 2 E(p) - 2 E(\hat{p}) \right] \Delta E(\hat{p}) + 2 \Delta \text{Cov}(p, \hat{p}) - \Delta \text{var} \hat{p} \right\} \]

\[ = - \frac{2 \alpha_2}{2(2 \alpha_1 + 2 \alpha_2)} \Delta M \]

\[ = \left[ \frac{1}{2} \left( \frac{1}{2 \alpha_1 + 2 \alpha_2} \right) - \frac{1}{4 \alpha_1} \right] \Delta M \]

where \( \Delta M \) represents the change of the mean of squared

2. The coefficient of equation 7.10 is obtained as follows:

\[ \frac{2 \alpha_2}{2 \alpha_1} = \frac{\alpha_2}{\frac{4 \alpha_2}{4 \alpha_1 + 4 \alpha_2}}. \quad (i) \]

\[ \frac{1 + \frac{2 \alpha_1}{2 \alpha_2}}{2 (2 \alpha_1 + 2 \alpha_2)} = \frac{\alpha_1 + \alpha_2}{\frac{4 \alpha_2}{4 \alpha_1 + 4 \alpha_2}}. \quad (ii) \]

\[ = \frac{1}{4 \alpha_1}. \quad (iii) \]

Therefore,

\[ \frac{2 \alpha_2}{2 \alpha_1} = \frac{1}{4 \alpha_1} - \frac{\alpha_1}{4 \alpha_1 \left( \frac{\alpha_1 + \alpha_2}{4 \alpha_2} \right)} \]

\[ = \frac{1}{4 \alpha_1} - \frac{1}{2 \left( 2 \alpha_1 + 2 \alpha_2 \right)}. \quad (iv) \]
deviations of shadow from actual price. If average price and
the variance of price are constant, an increase in $M$ decreases
average profit by $\Delta M$ times the reciprocal of twice the rate
of change of the marginal cost of output less the reciprocal of
twice the rate of change of the marginal cost of optimally planned
output. The decrease is greater the greater is the rate of change
of the marginal cost of the divergence of actual output from plan
i.e., the greater is $2 \alpha_2$. If $E(p)$ and $\text{var } p$ are constant, if
$2 \alpha_2 > 0$, and if $2 \alpha_1 < \infty$,

$$\Delta E(\Pi) > 0$$

as

$$\Delta M \leq 0.$$  

If average price and $M$ are constant, the change of
average profit for a change of the variance of price is

$$\Delta E(\Pi) = \frac{1 + \frac{2 \alpha}{2 \alpha_1}} {2 (2 \alpha_1 + 2 \alpha_2)} \Delta \text{var } p$$  \hspace{1cm} 7.11

$$= \frac{1}{4 \alpha_1} \Delta \text{var } p$$  \hspace{1cm} 7.12

because, if $\Delta M = 0$ and $\Delta E(\hat{p}) = 0$,

$$[-2E(p) + 2E(\hat{p})] \Delta E(\hat{p}) - 2 \Delta \text{Cov } (p, \hat{p}) + \Delta \text{var } \hat{p} = - \Delta \text{var } p.$$  

If average price and the mean of squared deviations of shadow
from actual price are constant, an increase in the variance of
price increases average profit by the reciprocal of twice the
rate of change of the marginal cost of optimally planned output
times the increase of the variance. If $\Delta E(p) = 0$, if $\Delta M = 0$,
and if $2 \alpha_1 < \infty$,

$$\Delta E(\Pi) \geq 0$$
as

\[ \Delta \text{var} p < 0. \]

C. A Further Note Upon Oil's Rejoinder.

Even if the correlation coefficient between shadow and actual price is positive, an increase in the variance of price can still decrease average profit under dynamic decision making conditions. To see this, assume as in Chapter V, that the regression of \( \hat{p} \) on \( p \) is

\[ z = \alpha_0 + \alpha_1 \hat{p} \]

and that this regression is maintained. Then, as before,

\[ \frac{\sigma_z^2}{\sigma_p^2} = \frac{R^2}{\alpha_1} \]

and

\[ \text{Cov}(p, \hat{p}) = \frac{R^2}{\alpha_1} \sigma_\hat{p}^2. \]

Substituting these values into equation 7.7, and differentiating it with respect to \( \sigma_p^2 \) while holding \( E(p) \) and \( E(\hat{p}) \) constant, the change of average profit for a change of the variance of price is

\[ \frac{dE(\Pi)}{d\sigma_p^2} = 1 + \frac{2 \alpha_2}{2 \alpha_1} \left[ \frac{2 R^2}{\alpha_1} - \frac{R^2}{\alpha_1^2} \right] \frac{2 (2 \alpha_1 + 2 \alpha_2)}{2 (2 \alpha_1 + 2 \alpha_2)} \]

\[ 7.13 \]

Because

\[ \gamma_1 = \frac{R \sigma}{\sigma_\hat{p}}, \]

equation 7.13 reduces to
\[
\frac{\mathrm{d}E(\Pi)}{\mathrm{d}\sigma^2} = \frac{1 + \frac{2\alpha_2}{2\alpha_1} \frac{\sigma^2}{\sigma_p} [2R - \frac{\sigma}{\sigma_p}]}{2(2\alpha_1 + 2\alpha_2)}
\]

If \( \frac{\sigma}{\sigma_p} > 2R \), i.e. if \( \sigma > 2R \sigma_p \), and if \( \frac{2\alpha_2}{2\alpha_1} \frac{\sigma^2}{\sigma_p} \) is large enough, \( \frac{\mathrm{d}E(\Pi)}{\mathrm{d}\sigma^2} \) can be negative even if \( R > 0 \). It is more likely to be negative, the greater is the ratio of the rate of change of marginal cost of divergence to the rate of change of marginal cost of optimally planned output, i.e. the greater is \( \frac{2\alpha_2}{2\alpha_1} \). Of course, if \( 2\alpha_2 = 0 \) an increase in the variance of price increases average profit if \( 2\alpha_1 < \infty \). In fact, the change of average profit is the same as occurs under certainty. However, if \( 2\alpha_2 > 0 \), it is possible for an increase in the price variance to lead to a decrease of average profit even if \( R > 0 \). In his rejoinder, Oi claims that there is short run responsiveness of supply in agriculture and implies that my earlier conclusion is inapplicable because it is based upon a static decision making assumption. But as just demonstrated, an increase in price instability can decrease average profit even if the firm makes dynamic production decisions and if the correlation between shadow and actual price is positive and constant. It seems that Oi's revised conclusion is misleading in both the flexible and inflexible (dynamic and static) decision cases.

---

D. A Special Model.

Let us suppose a regression of \( p \) on \( \hat{p} \) of

\[
z = \alpha_0 + \alpha_1 \hat{p}.
\]

Then, equation 7.7 becomes

\[
E(\hat{p}) = \frac{E(p)^2 + \text{var } p + \frac{2}{2} \frac{\alpha_2}{\alpha_1} \left[ 2E(p)E(\hat{p}) + \frac{2R^2}{\alpha_1} \sigma_p^2 \right]}{4 \alpha_1 + 4 \alpha_2}
\]

\[
= \frac{2}{4} \frac{\alpha_2}{\alpha_1} \left[ E(p)^2 + \frac{R^2}{\alpha_2} \sigma_p^2 \right] - \frac{1}{4} \frac{\alpha_2}{\alpha_1} - \alpha_0. \tag{7.15}
\]

and

\[
M = \left[ E(p) - E(\hat{p}) \right]^2 + \sigma_p^2 - \frac{2R^2}{\alpha_1} \sigma_p^2 + \frac{R^2}{\alpha_1} \sigma_p^2. \tag{7.16}
\]

Let us specialise this relationship further by supposing that the shadow prices are unbiased estimates of actual price.

This implies that

\[
z = \hat{p},
\]

\( \gamma_0 = 0 \) and \( \gamma_1 = 1 \). It also implies that \( E(p) = E(\hat{p}) \). Therefore, in this special case,

\[
E(\hat{p}) = \frac{2 \alpha_2}{2 \alpha_1} \frac{E(p)^2 + \text{var } p + \frac{2}{2} \frac{\alpha_2}{\alpha_1} R^2 \sigma_p^2}{4 \alpha_1 + 4 \alpha_2} - \alpha_0. \tag{7.17}
\]

\[
= \frac{1}{4} \frac{\alpha_1}{\alpha_1} E(p)^2 + \frac{1}{2} \left( \frac{1}{\alpha_1 + 2 \alpha_2} \right) \left( 1 - R^2 \right) \sigma_p^2 + \frac{1}{4} \frac{\alpha_1}{\alpha_1} R^2 \sigma_p^2. \tag{7.18}
\]
For this special case, we obtain the following theorems:

(i) If \( \mathbb{E}(p) \) and \( \sigma_p^2 \) are constant,

\[
\frac{d \mathbb{E}(\Pi)}{d(1 - R^2)} = \left[ \frac{1}{2(2 \alpha_1 + 2 \alpha_2)} \right] \sigma_p^2.
\]

If the rate of change of marginal cost of divergence from plan, \( 2 \alpha_2 \), exceeds zero, and if \( \sigma_p^2 > 0 \),

\[
\frac{d \mathbb{E}(\Pi)}{d(1 - R^2)} < 0. \quad \text{If } 2 \alpha_2 = 0, \frac{d \mathbb{E}(\Pi)}{d(1 - R^2)} = 0.
\]

(ii) If \( \mathbb{E}(p) \) and \( R^2 \) are constant,

\[
\frac{d \mathbb{E}(\Pi)}{d \sigma_p^2} = \frac{1}{2(2 \alpha_1 + 2 \alpha_2)} (1 - R^2) + \frac{1}{4 \alpha_1} R^2.
\]

If \( 2 \alpha_1 \to \infty \), \( \frac{d \mathbb{E}(\Pi)}{d \sigma_p^2} > 0 \).

D. Technique Choice.

The previous analysis provides an opportunity to extend our discussion to the effect of price uncertainty upon the choice of technique. In the analysis of Chapter VI, it was assumed that the firm's actual output did not diverge from its planned. Now it is possible to allow for flexibility in Hart's sense. We

\[ 4 \text{ We obtain the coefficients of } \sigma_p^2 \text{ in equation 7.19 as follows:} \]

The coefficient of \( \sigma_p^2 \) in equation 7.18 is

\[
1 + \frac{2 \alpha_2}{2 \alpha_1} R^2 = \frac{1}{2(2 \alpha_1 + 2 \alpha_2)} - \frac{2 \alpha_1}{4 \alpha_1 + 4 \alpha_2} R^2 \quad (i)
\]

\[
= \frac{1}{2(2 \alpha_1 + 2 \alpha_2)} - \frac{1}{2(2 \alpha_1 + 2 \alpha_2)} R^2 + \frac{1}{4 \alpha_1} R^2 \quad (ii)
\]

\[
= \frac{1}{2(2 \alpha_1 + 2 \alpha_2)} (1 - R^2) + \frac{1}{4 \alpha_1} R^2. \quad (iii)
\]

To obtain (ii), we use the results of footnote 2 of this Chapter.
suppose that (i) the firm has a choice of technique A or B for a period T consisting of n sub-periods, (ii) the techniques have equal resale (scrap) value at the end of T, and (iii) the production conditions which were previously outlined in this chapter hold.

The excess average profit of technique A is

$$E[W] = E(\Pi_A) - E(\Pi_B)$$  \hspace{1cm} 7.22

where $E(\Pi_A)$ and $E(\Pi_B)$ are similar in form to equation 7.6.

It follows from equation 7.9, that if the average value and variance of price are constant, an increase in the mean square of deviations of shadow price from actual changes the "excess" average profit of technique A by

$$\Delta E[W] = \left\{ - \frac{2\alpha_2, A}{2(2\alpha_1, A + 2\alpha_2, A)} + \frac{2\alpha_2, B}{2(2\alpha_1, B + 2\alpha_2, B)} \right\} \Delta M.$$  \hspace{1cm} 7.23

This can also be expressed in terms of the simplification of equation 7.10. Consequently, if $\Delta E(p) = 0$, if $\Delta \text{var } p = 0$ and if $\Delta M > 0$, then

(i) if $2\alpha_2, A = 2\alpha_2, B'$

$$\Delta E[W] \geq 0$$

as

$$2\alpha_1, A \geq 2\alpha_1, B'$$

and (ii) if $2\alpha_1, A = 2\alpha_1, B'$

$$\Delta E[W] \leq 0$$

and

$$2\alpha_2, A \geq 2\alpha_2, B'$$
as

\[ 2 \alpha_{2, A} < 2 \alpha_{2, B} \]

We suppose that for its anticipated production behaviour, the firm chooses the technique which maximises its anticipated average profit, and that its anticipated average profit rises as the average profit for its actual \( p \) and \( \hat{p} \) values rises. Also, assume average price and the variance of price to be constant. Then (i) if the rate of change marginal cost of divergence from planned output is the same for both techniques, an increase in \( M \) increases the likelihood of the firm's adopting the technique which has the greatest rate of change of marginal cost of optimally planned output, and (ii) if the rate of change of the marginal cost of optimally planned output is the same for both techniques, an increase in \( M \) increases the probability of the firm's adopting the technique which has the least rate of change of marginal cost of divergence from plan. If average price and the variance of price are constant, an increase in \( M \) increases the probability of the firm's adopting a technique which has the greatest rate of change of marginal cost of optimally planned output and the lowest rate of change of marginal cost of divergence of output from plan. Other things equal, an increase in \( M \) increases the probability of the firm's adopting the most flexible plant in Hart's (possible) usage.

If average price and \( M \) are constant, the change of the "excess" average profit of technique \( A \) for a change of the price variance is
\[ \Delta E[W] = \left[ \frac{1}{4\alpha_{1,A}} - \frac{1}{4\alpha_{1,B}} \right] \Delta \text{var } p. \] 7.24

\[ \Delta E[W] \geq 0 \]
as

\[ 2 \alpha_{1,A} \leq 2 \alpha_{1,B} \]

If \( E(p) \) and \( M \) are constant, an increase in the price variance increases the probability of the firm's adopting the technique with the greatest rate of change of marginal cost of optimally planned output.

Suppose that at the date of choosing its technique the firm knows its distribution \( \hat{p} \) values for \( T \) and that each of these is an unbiased estimate of \( p \). Then, equation 7.19 is applicable. We note the following points for this case:

(i) If \( E(p) \) and \( \sigma_p^2 \) are constant,

\[ \frac{dE(\Pi)}{d(1 - R^2)} \]

is less negative the greater is \( 2 \alpha_{1} \) and the smaller is \( 2 \alpha_{2} \). In these circumstances, an increase in \( 1 - R^2 \) increases the probability of the firm's adopting any technique which has the greatest rate of change of marginal cost of optimally planned output and the lowest rate of change of marginal cost of divergence of output from plan.

(ii) If \( E(p) \) and \( R^2 \) are constant \( \frac{dE(\Pi)}{d\sigma_p^2} \) is more positive the smaller is \( 2 \alpha_{1} \) and \( 2 \alpha_{2} \). In this case, an increase in the price variance increases the probability of the firm's adopting the technique which has both the least rate of change of marginal cost of optimally planned output and the least rate of change of marginal cost of divergence.
CHAPTER VIII
Price Uncertainty and Industry Profit.

It is commonly believed that price uncertainty must decrease the profit of a closed industry. In other words, it is believed that the existence of forecasting errors in a closed industry implies that the aggregate profit of that industry is less than under certainty. But this postulate is false. Given that plans are not completely flexible, group (industry) errors can increase the profit of the industry. While the profit of an individual firm decreases if its errors increase relative to a given group position, the profits of the industry do not necessarily decrease if the errors of the group increase.

To substantiate this hypothesis, let us consider the profit effects of possible errors at a point of time. It will be taken for granted that firms in an industry can adopt different criteria, that some firms may adopt no conscious criterion at all, that some may act inconsistently and without deliberation, and their forecasts may differ. The immediate aim is to compare industry profit under certainty with the possible level of profit under uncertainty by assuming that actual demand and cost functions are the same in both cases. The position will only be considered for the case in which plans involve short term inflexibility. Such a model is a simple first approximation for the qualitative effects of errors when additional costs must be assigned to their elimination.

Comparisons are made under the following assumptions:
i. Every firm makes its output decision for \( t \) in \( t-n \) and this output decision is unalterable.

ii. The only dependence in production occurs between the decision of \( t-n \) and the output \( n \) periods later.

iii. The marginal cost functions of all firms are linear and identical.

iv. Each firm produces only one good.

v. Demand is a monotonically decreasing function of price and marginal cost is an increasing function of planned (equals actual) output.

The assumption of linear marginal cost functions is made so as to resolve the effect of the dispersion of output between firms into the effect of the variance alone. However, this assumption is a simplification which does not alter the essentials of the analysis.

Let

\[
P_D = f(X)
\]

be the industry demand function where \( X \) represents industry output, and let the costs of the \( i \)-th firm be \( ax_i^2 + bx_i + c \) where \( x_i \) represents the \( i \)-th firm's output. Then, if there are \( k \) firms in the industry, industry profit is

\[
\psi = f(X) X - \sum_{i=1}^{k} (ax_i^2 + bx_i + c).
\]

Where \( E \) indicates that the industry value is averaged over all firms,

\[
\psi = f(X) X - k \left( E (ax_i^2) + b E (x_i) + c \right)
\]

\[
= f(X) X - k \left( a E (x_i)^2 + a \text{var} x_i + b E (x_i) + c \right)
\]

\[
= f(X) X - k \left( a \left( \frac{X}{k} \right)^2 + a \text{var} x + b \left( \frac{X}{k} \right) + c \right)
\]
\[ \frac{d}{dX} \Gamma(X) = f(X) + \frac{df}{dX} - \frac{dC}{dX} = 0 \]

and

\[ \frac{d^2 \Gamma}{dX^2} < 0. \]

Since the industry demand function is assumed to decrease monotonically, \( \frac{df}{dX} \) is negative for all values of \( X \), and so the marginal revenue function is less than the average revenue function for \( X > 0 \) because \( f(X) > f(X) + \frac{df}{dX} \) for \( X > 0 \). Since the industry marginal cost function is assumed to increase at less than an infinite rate,

\[ \frac{dC}{dX} = f(X) + X \frac{df}{dX} \quad \text{before} \quad \frac{dC}{dX} = f(X). \]
If the maximum of $\Gamma(X)$ occurs for $X_0$, then $\Gamma(X_0) > \Gamma(\tilde{X})$ where $X_0 < \tilde{X}$. If we suppose $\Gamma(X)$ to be concave upward, then $\Gamma(X) > \Gamma(\tilde{X})$ for $X_0 \leq X < \tilde{X}$. If $\Gamma(X)$ is concave upward, then there are a number of output levels in the neighbourhood of $X_0$ for which industry profit is greater than under certainty. Industry profit under uncertainty is

$$\psi = \Gamma(X) - k \text{var } X$$

and

$$\max_X \psi = \Gamma(X_0) - k \text{var } x.$$  

Since $\Gamma(X)$ increases continuously to its maximum, and then decreases continuously,

$$\Gamma(X_0 + \eta) - \delta > \Gamma(\tilde{X})$$

where $\eta$ lies between sufficiently small negative and positive values and $\delta$ is sufficiently small. There are values of output in the neighbourhood of $X_0$ for which profit is greater than under certainty even if the variance of output between firms is positive. Hence, uncertainty can lead to an increase in industry profit if it leads to a restriction of output. There is no good a priori reason for excluding this case as an empirical possibility.

Nevertheless, if var $x$ is large enough or if aggregate output is high enough or low enough profit can be lower under uncertainty.

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2 Where $R$ represents total revenue, $\frac{d^2 R}{dX^2}$ is negative for all values of output if $\frac{d^2 R}{dX^2} \leq 0$, because

$$\frac{d^2 \Gamma}{dX^2} = \frac{d^2 R}{dX^2} - \frac{d^2 C}{dX^2}$$

and $\frac{d^2 C}{dX^2} > 0$, $\frac{d^2 R}{dX^2}$ will be negative if the elasticity of the demand function decreases with increases of output.
than under certainty. For instance, if industry output under uncertainty is greater than \( X \), or if industry output is just less than \( X \) and \( \text{var} x \) is large, industry output will be less than under certainty. This is because \( \frac{d \psi}{d \text{var} x} = -ka < 0 \) and \( \frac{d \psi}{d x} < 0 \) for \( X > X_0 \).

Given a closed industry it is not impossible for industry profit to be higher under uncertainty than under certainty. Due to their mistakes, firms may bring about a monopoly-like restriction of output. Over time, such restrictions may alternate with overproduction. It is clearly possible for average profit to be less than, or greater than under certainty depending upon whether there is tendency to produce monopoly like levels of output or to overproduce.

From the above analysis it can be seen that the difference between industry profit under uncertainty and under certainty depends upon

(a) the spread of output between firms, and,

(b) the difference between aggregate output under uncertainty and equilibrium output under certainty.

Profit can also be expressed in terms of maximum equivalent (shadow) prices. If \( b < \infty \) and if \( 0 < 2a < \infty \), there is always some value of \( p \) such that the \( i \)-th firm's actual output \( x_i \) equals 
\[
\frac{p_i - b}{2a}.
\]
Let this value be \( \hat{p}_i \), and let us describe it as the "shadow" price. Then industry profit as a function of the firms' shadow prices is found by substituting
\[
x_i = \frac{\hat{p}_i - b}{2a}, \quad i = 1, \ldots, k,
\]
into expression 8.2. This gives
\[ \psi = P_D X - k \left( E \left[ a \left( \frac{\hat{p}}{2a} - b \right) \right]^2 + b \left( \frac{\hat{p}_i - b}{2a} \right) \right) \]

\[ = P_D X - k \left\{ \frac{E(\hat{p})^2 + \text{var} \hat{p}}{4a} - 2 E(\hat{p}) \frac{b}{2a} + \frac{b E(\hat{p}) - b^2}{2a} + c \right\} \]

where \( X = k \left( \frac{E(\hat{p}) - b}{2a} \right) \) and \( \text{var} \hat{p} \) is the variance of shadow price between firms in the industry. If the average shadow price in the industry is constant, an increase in the variance of the shadow price leads to a decrease in industry profit of

\[ \Delta \psi = - \frac{k}{4a} \Delta \text{var} \hat{p} \]

\[ = - k \left( \frac{1}{4a} \right) \Delta \text{var} \hat{p}. \]

In this case, an increase in the variance of shadow prices between firms decreases industry profit by \( k \) times the reciprocal of twice the slope of any firm's marginal cost function times the change of the variance. While \( \text{var} \hat{p} \geq 0 \) under uncertainty, \( \text{var} \hat{p} = 0 \) under certainty. Profit under uncertainty is less than or equal to that under certainty if the average shadow price is the same in both situations. But if average shadow price differs, industry profit can be greater under uncertainty than under certainty.

As the predictive ability of firms as a group increases, industry profit need not increase. As the group ability of firms to predict increases, the average predicted price for the group will approach the equilibrium price value, and industry output will approach \( \hat{X} \). If prior to the general increase in predictive ability, the industry had been producing the neighbourhood of \( X_0 \) and, if \( \text{var} x \) remains unchanged, an increase in industry output which brings it closer to \( X \) will decrease industry profit. While an increase in the predictive ability of a firm, the
predictive ability of other firms unchanged, increases its profit, profit for the whole industry can decrease if there is a general increase in predictive ability. Many fail to see this possibility because they equate individual and group experience.

The above analysis gives some ground for doubting an assertion made by Nelson. Nelson asserts that:

"Since a rise in profits will draw new firms into the industry, a general increase in skill in predicting will, by increasing profit, tend to shift the long-run supply curve to the right and cause a fall in average quantity produced".

Nelson also states that the economic logic of the problem indicates that relaxing the assumptions of his model will not change this qualitative conclusion. My disagreement is with Nelson's premise that a general increase in skill in predicting will (always) increase industry profit. As previously argued, an increase in group predictive ability can decrease industry profit if uncertainty has led to a restriction of industry output.

If the aggregate profit of the industry is greater under uncertainty than under certainty, it is still possible for the profits of some firms with inferior predictive ability to be less than under certainty. If output is less than X,

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4 Ibid, p. 62

5 Ibid.
and if industry profit is less than under certainty, the profits of some firms with superior predictive ability can be greater than under certainty. But if industry output exceeds the equilibrium level, $X$, then it is certain that the profits of all firms will be less than under certainty.

It is important to realise that it is fallacious to assume that for the same set of demand and cost functions (neither of which is completely elastic) that the prices of uncertainty can occur under perfect knowledge. To each supply and demand situation there is only one possible certainty price, i.e., the equilibrium price. If a price different to this equilibrium level occurs under uncertainty, then it cannot occur under certainty. Arguments of the form "if the prices which occur under uncertainty are certain, then..." can be misleading in the group context because the uncertainty prices may not be compatible with certainty. If we suppose that the prices of uncertainty can always occur under certainty inconsistency occurs at the basis of our argument for we imply that it is possible for non-equilibrium prices to occur under conditions in which all firms have perfect knowledge. In an uncontrolled market and under certainty this is impossible.

The analysis has been concerned with the possible industry profit situation at a point of time. Over time, profit may sometimes be above its certainty level and sometimes below it. My hypothesis is that (in a closed industry) there are some situations in which average profit under uncertainty is above that under certainty. The former situation may be a long run one due to the continuous effects of uncertainty.
There seems to be no theoretical justification for excluding the possibility of price uncertainty causing a rise in industry profit.
APPENDIX TO CHAPTER VIII

Upon the Relationship between the Preceding Theory and Knit's Theory of Profit.

Knight argues that profit is zero under conditions of certainty and that "profit arises from the fact that entrepreneurs contract for productive services in advance at fixed rates, and realize upon their use by the sale of the product in the market after it is made. Thus the competition for productive services is based upon anticipations. The prices of the productive services being the costs of production, changes in conditions give rise to profit by upsetting anticipations and producing a divergence between costs and selling price, which would otherwise be equalised by competition". ¹ Thus Knight has foreseen the possibility of profit being greater under price uncertainty than under certainty.

He also puts forward the following propositions:

"The condition, then, under which entrepreneurs as a group will realize a profit is that they underestimate the prospects of their business relatively to their dispositions to venture. If, on the contrary, they overestimate their prospects (considering the degree of conviction necessary to move their wills), they will in aggregate suffer loss, and if they estimate correctly on the whole neither will occur. If the estimates are a matter of pure chance it would seem that the variations in the two directions would be equal, the average correct, and the general level of pure profit zero". ²

² Ibid., pp.363-364.
When applied to industry profit some of these propositions are untrue. At least this is so if the analysis of Chapter VIII is applicable. Although the underestimation of price is necessary for an increase of industry profit above its certainty level, it is not sufficient for it. Again, if marginal cost increases with increases of output, errors of estimation do not cancel out. If marginal cost is linear and increasing, industry profit decreases as the variance of "predicted" (shadow) prices in the industry increases and so, if the industry's mean shadow price is equal to the equilibrium price and if all firms do not predict the mean price, industry profit is less than under certainty.
CHAPTER IX

Errors and Aggregate Output

In order to satisfactorily consider social questions, it is necessary to have some way of comprehending the economy as a whole. It should also be possible to make this way explicit by a model the implications of which can be specifically derived and checked against actualities. The model used in this analysis overlooks many aspects of aggregate production, and one can only hope that excluded factors are not so important as to make the conclusions misleading.

A. Errors and Aggregate Output.

The simple matter which I wish to consider first is the effect upon aggregate output of the occurrence of different shadow price ratios for the same period. Consider the effect under the following assumptions:

(i) The economy produces two (or more) products.

(ii) Each firm must decide upon its output $n$ periods in advance of its actual output, $n > 0$. The decision for any period is inflexible and independent of other decisions to be made in the future.

(iii) The distribution of inputs between firms is fixed and the aggregate level of these inputs is constant.

(iv) Each firm's technical transformation function (production possibility frontier) for its fixed amount of inputs is convex.

The above assumptions do not imply that all factors are fully employed, and in developing our thoughts we need not

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1 For instance, it does not allow for external economies.
suppose that the input and distribution of factors is actually fixed.
Let us consider the economy's production possibilities with any
given employed quantity of factors. For any given quantity of
these factors and distribution of them, the economy has a convex
production possibility frontier for its products. For any given
distribution of inputs between firms, this frontier is comprised
of only those bundles of products such that it is impossible to
increase the output of one product without decreasing the output
of any other. However, if the technical transformation functions
of all firms are convex, it is well known that for its employed
resources the economy's aggregate production falls below its
attainable level unless the rates of technical substitution between
products are equal for all firms. If these rates are unequal,
it is possible to increase the output of one or more products
without decreasing the output of any other and without employing
a greater quantity of factors than is already employed. If
firms "predict" different price ratios, and if their behaviour
is equivalent to profit maximisation upon the basis of these
price ratios, then aggregate production will fall below the
frontier which is attainable with the economy's employed
ratios
resources. If the shadow price of firms are different
production falls below the economy's frontier.
The extent to which output falls below the economy's
production frontier depends upon the form of firms' technical

2 This proposition is discussed by O. Lange in "The Foundations
of Welfare Economics", Econometrica, Vol. 10, 1942,
pp. 215-228. Footnote 14, p. 225 is particularly relevant.
3 The principle of maximum equivalence was discussed in
Chapters III and IV.
substitution functions and the relative frequency distribution of their shadow price ratios. Since the technical substitution functions of firms are convex, and so can be approximated by quadratic functions with negative second derivatives, there is a tendency for aggregate output to decrease as the variance of price ratio predictions increase. In the special case illustrated in the appendix, aggregate output depends functionally upon the average of shadow price ratios in the industry and their variance at any point of time, and decreases as this variance decreases.

If the rates of technical substitution are not equal everywhere, there are a number of equalised rates which for the same employment of factors increase the output of at least one product without decreasing the output of any other. The range of such rates increases with the variance of shadow price ratios. However, every equalised rate of technical substitution does not increase aggregate output. Consider the possibilities in terms of figure 9.1. In that figure, we suppose the economy to produce two products, X and Y. For some distribution and employment of resources, and given relative frequency distribution of (unequal) shadow price ratios, we observe that the aggregate output \((X_1, Y_1)\) occurs. For this distribution of resources, \((X_1, Y_1)\) falls below the economy's technical transformation frontier, \(Y = F(X)\). Any equalised transformation rate, \(\frac{dF(X)}{dX}\), will increase the output of at least one product without decreasing that of another if \(X_1 \leq X \leq X_2\). If the equalised rate of transformation can be represented by the slope of a tangent at \(F(X_1)\), \(F(X_2)\) or at any intermediate
point on the function, then aggregate output is unequivocally increased by the equalisation. However, if the equalised rate $\frac{dy}{dx}$ occurs for $X < X_1$ or $X > X_2$ then the output of at least one product is decreased.

Nevertheless, if different rates of transformation occur at the one time, there is a "range" potential for increasing the output of one product without decreasing the output of any other, and without employing any more factors than are already in employment. In a free market, shadow price ratios will normally differ, and hence, a potential arises for increasing the economy's aggregate output without increasing its usage of factors. Let us briefly consider the possibility of realising it.

B. Forward Price Schemes

The government may be able to realise this potential if it sets forward prices. Consider the case in which the economy's products are perishable (or not stored). Under

4 For a definition of forward prices see D. G. Johnson, Forward Prices for Agriculture, The University of Chicago Press, Chicago, Illinois, 1947, pp. 10-12. The two main aspects of forward prices are that they are certain and are set sufficiently far in advance for producers to adjust to them.
these circumstances, what conditions must be satisfied if the
government's forward price policy is to increase the aggregate
consumption of the community?

We can obtain an approximate answer by supposing
that the preceding assumptions apply and that the consumption
of the economy's products is a function of real aggregate income
and the products' price ratios. If this is so, then for any
particular level of aggregate real income and set of price
ratios, there is just one bundle of products which is demanded.
The government is able to regulate consumption by its taxation
and forward price policies. Since, in this case, the consumption
of any product cannot exceed its current output, the government's
forward price ratios must be such as to increase the output of
at least one product without decreasing that of any other. In
figure 9.2 aggregate output under free market conditions is
shown as \((X_o, Y_o)\). In this case, the government's forward price
ratio of \(Y\) to \(X\) must be such that the marginal rate of transformat-
tion occurs for \(X_o \leq X \leq X_1\). But this is only a necessary
condition for an increase in the consumption of at least one
product without a decrease in the consumption of any other.

At the government's forward price ratios, income
may be such that the demand for at least one product exceeds
its supply. In figure 9.2, we suppose the forward price ratio
\(\frac{OA}{OB}\) to be set. At this price ratio, aggregate output is
\((X_1, Y_1)\) and aggregate income in terms of one product is \(OA\).

---

5 This dependence is obviously a special case. The economy's
aggregate consumption can also depend upon the distribution
of income.
At this price ratio and income level, the aggregate income consumption function may not pass through the point, \((X_1', Y_1')\). If it does not, then the price ratio and income conditions are such that the demand for at least one product exceeds its supply.

If the income consumption line at the set forward price ratio passes through the area in which the output of one product is increased and none decreased (at the set forward price), then the government will be able to regulate income so that all demands can be met out of current supplies and so that the consumption of at least one product is higher without that of any other being lower. In figure 9.2, we enquire if the income consumption line passes through the rectangle

\[ [(X_o', Y_o'), (X_o', Y'_1), (X'_1, Y_1'), (X'_1, Y_o')] \]

If it does not then for the price ratio \(\frac{OA}{O_2'}\), the government is unable to regulate income so as to assure an increase in the consumption of at least one product without a decrease in the consumption of any other. But if the income consumption line passes through the rectangle at points other than \((X_o', Y_o')\) such an increase is possible. In figure 9.2 the income consumption line, \(WZ\), for

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6 Strictly, we require the income consumption line to pass through the rectangle at points other than \((X_o', Y_o')\). Also, in the limit in which the price ratios are equivalent to

\[
\frac{dF(X_o)}{dX} \quad \text{and} \quad \frac{dF(X_o')}{dX}
\]

the income consumption line must pass through the intervals \[ [(X_o', Y_o'), (X_o', Y_2')] \]

and \[ [(X_o', Y_o'), (X_2', Y_o')] \] at points other than \((X_o', Y_o')\).
the price ratio $\frac{OA}{OB}$ passes through the required rectangle. Any tax greater than $AA'$ and less than $AA''$ will ensure that at the price ratio $\frac{OA}{OB}$ all demands can be met from available supplies and that the quantity consumed will exceed $(X_0, Y_0)$ in the vectorial sense. After imposing any of these tax rates, the government will have a surplus in at least one of the products. This quantity may be destroyed or donated to separate economies. The existence of surpluses which are destroyed or donated is not proof that aggregate income or consumption is lower than it would be in the absence of forward prices.

It is similarly possible to increase aggregate consumption by forward prices if products are storable or enter into international trade. Even when products are storable, different shadow price ratios can arise under free market conditions. By eliminating these through forward prices, it is possible to increase aggregate output, and if required,
increase the quantity of stocks. Even after increased taxation and storage, consumption can be greater than under free market conditions. The existence of buffer stocks increases the area over which the government's taxation and forward price policy can increase aggregate consumption. Similarly, if the products enter into external trade, home producers may hold different shadow price ratios. If the government controls the country's international trade and fixes forward price ratios at home, and sells surplus production on the overseas market, and buys as required to meet internal demand, it may use its overseas funds (instead of buffer stocks) to increase domestic consumption. By setting forward price ratios, the country can increase the aggregate quantity of products which it has for exchange.

C. Forward Prices and Welfare Implications.

In the above analysis I have not tried to show that forward price schemes can improve the welfare of society. Since it seems likely that there are at least as many social welfare functions in existence as there are people, and since these functions may all be in a process of change, and

Buffer stock and forward price schemes involving international trade are discussed at further length in a forthcoming article of mine in the Economic Record. This article is entitled "Price Uncertainty and Pareto Optimality".

since some of these involve intertemporal and interpersonal
conflict, it is unlikely that a unique social welfare function can
be defined. Yet the economy through compromises and coercion
may work to some particular ordering of its possibilities.
According to some orderings, changes in production which
decrease the output of one or more products and increase the
output of others can be preferable to some changes which
increase the output of all products. The more "substitutability"
do these orderings allow between products, the greater is the
range of forward price policies which can place the economy
in a "preferred" position.

It is also possible for forward price schemes to have
an income distribution effect. They will equalise the profit of
firms operating under identical cost conditions. Under forward
price schemes no additional profit will arise from superior
predictive ability of the near future. The income redistribution
effect may have social welfare implications. Other factors may
also be affected which are important for social welfare functions,
e.g. the amount of government control is increased by forward
price schemes.

However, from the above model, we obtain the following
information for feeding into social welfare functions: Forward
price schemes can lead to a level of aggregate consumption
which exceeds that under free market conditions. They can do
this even when they involve the destruction of some production
or an increase in the level of stocks. But the total change in any
social welfare function as a result of a forward price scheme
is unlikely to depend just upon the scheme's
effect on aggregate consumption. The scheme's effect on other factors may also influence the change so that it cannot be gauged from the change in consumption alone. While this analysis does not imply that any forward price scheme will increase the level of aggregate consumption, it does suggest that there is some elasticity in the matter.
This appendix provides a simple illustration of the effect at any point of time of different shadow price ratios on the economy's level of aggregate output. We make the following assumptions in addition to those made at the beginning of the Chapter:

(a) Every firm's production possibility frontier for the products X and Y is the same.

(b) The i-th firm's production possibility frontier is quadratic and can be written explicitly as

\[ y_i = \beta_o + \beta_1 x_i - \beta_2 x_i^2 \quad \left\{ (x, y) \mid x \geq 0, \ y \geq 0 \right\} \]

where \( y \) and \( x \) respectively represent the firm's output of Y and X and \( \beta_o > 0, \beta_1 \geq 0 \) and \( \beta_2 > 0 \).

If there are \( k \) firms in the economy, the output of commodity Y for the whole economy at any point of time is

\[ Y = \sum_{i=1}^{k} y_i = \frac{1}{k} \sum_{i=1}^{k} \left( \beta_o + \beta_1 x_i - \beta_2 x_i^2 \right) \]

This output averaged over \( k \) firms in the industry is

\[ Y/k = E(y) = \frac{1}{k} \sum_{i=1}^{k} \left( \beta_o + \beta_1 x_i - \beta_2 x_i^2 \right) \]

\[ = \beta_o + \beta_1 E(x) - \beta_2 E(x)^2 - \beta_2 \text{var} \ x, \]

where \( E \) represents values averaged for all firms and \( \text{var} \ x \) is the variance of the outputs of individual firms from the average output of X in the industry, i.e. the variance from

\[ \frac{1}{k} \sum_{i=1}^{k} x_i. \]

Given expression 4, the economy's aggregate output of Y is
\[ Y = k E(y) = k \left\{ \beta_0 + \beta_1 E(x) - \beta_2 E(x)^2 - \beta_2 \text{var } x \right\} \]

\[ = k \left\{ \frac{\beta_0 + \beta_1 X}{k} - \frac{\beta_2}{k} \text{var } x - \beta_2 \frac{X^2}{k^2} \right\} \]

\[ = k \beta_0 + \beta_1 X - \frac{\beta_2}{k^2} X^2 - k \beta_2 \text{var } x \quad 5 \]

where \( X = \sum x_i \). Under certainty \( \text{var } x = 0 \) and the economy's technical transformation function is

\[ Y = k \beta_0 + \beta_1 X - \frac{\beta_2}{k} X^2. \quad 6 \]

If \( \text{var } x > 0 \), as we expect it to do under price uncertainty, then from equation 5, the output of \( Y \) for any given value of \( X \) is reduced and the economy's aggregate output falls below its production possibility frontier which is given by function 6.

This occurs because the coefficient of \( \text{var } x \) in equation 5 is negative. This is negative because

\[
\frac{\partial^2 y_i}{\partial x_i^2} = -2 \beta_2
\]

i.e. because increasing costs occur. We note that for any given level of \( X \) and given value of \( \text{var } x \), that the reduction of output brought about by a positive variance value is greater the speedier is increasing cost. Also if \( \text{var } x \) is constant and positive we obtain a "transformation function" which lies below the economy's production possibility frontier.

So far we have described what happens to the level of aggregate output if the variance of \( x \) increases. This does not explicitly indicate what happens to output if the variance of shadow price ratios is positive. Let us explicitly consider this matter. If the production possibility function is differentiable, and if \( \frac{p_Y}{p_x} \) is the \( i \)-th firm's shadow price ratio,
\[ \frac{dy_i}{dx_i} = \left( \frac{p_y}{p_x} \right)_i \]

Letting \( Q_i \) represent \( \left( \frac{p_y}{p_x} \right)_i \), and differentiating equation 1, equation 7 becomes

\[ \pm \beta_1 - 2 \beta_2 x_i = -Q_i \]

Therefore, the \( i \)-th firm's output of \( X \) is

\[ x_i = \frac{Q_i + \beta_1}{2 \beta_2} \]

The \( i \)-th firm's output of product \( Y \) is

\[ y_i = \beta_0 + \beta_1 (Q_i + \beta_1)/2 \beta_2 - \beta_2 \left[ (Q_i + \beta_1)/2 \beta_2 \right]^2 \]

Hence, the aggregate output of \( Y \) at any point of time is

\[ Y = \sum_{i=1}^{k} y_i = k E(y) \]

\[ = k \beta_0 + k \beta_1 \frac{E[Q] + \beta_1}{2 \beta_2} - k \beta_2 \left\{ \frac{E[Q] + \beta_1}{2 \beta_2} \right\}^2 - \frac{k \beta_2 \text{var} Q}{2 \beta_2} \]

where the last term reduces to \(-\frac{1}{2} k \text{var} Q\). Since

\[ X = k E(x) = k \frac{E[Q_i] + \beta_1}{2 \beta_2} \]

the aggregate output of \( X \) is constant if the average shadow price ratio, \( E[Q] \), is constant. Consequently, for any given value of \( E[Q] \) and hence, \( X \), it can be seen from equation 11 that the output of \( Y \) will decrease as the variance of shadow price ratios, \( \text{var} Q \), increases. If \( \text{var} Q \) is positive, the economy's output falls below its production possibility frontier and if \( \text{var} Q \) is positive and constant, we obtain a "technical transformation function" for uncertainty which lies below the community's attainable frontier.
APPENDIX I

Decision Making and the Probability of Loss.

In this appendix, while it is realised that various decision criteria are admissible in the determination of a firm's product-mix, a criterion is developed in terms of the maximisation of expected income subject to the satisfaction of a security restriction. Through the development of a new inequality this criterion is made to depend, for its operational content, upon the mean and the semi-variance. It is argued that for our purposes this inequality is more relevant than Cherbyshev's inequality which has been used by A. D. Roy¹ to illustrate the principle of "safety-first". Instead of assuming, as does Roy, that the firm attempts to minimise the probability of a disaster level of income, we see the firm as attempting to maximise its expected income subject to a restriction that the probability of its income being equal to, or below, some pre-set level be less than, or equal to, a specified probability. This security restriction is of the satisficer kind.²

I

While the variance is a well-known measure of variability, the semi-variance is not. The variance is a measure of the absolute dispersion of values of the variable

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on both sides of its mean. It is a weighted average of squared deviations from the mean, with each deviation weighted by its relative frequency or probability of occurrence. On the other hand, the semi-variance is a one-sided measure of variability—it may be upper or lower and can be calculated about any value, \( \alpha \). In the special case where \( \alpha \) is equal to the mean, the upper semi-variance is the expected value of squared positive deviations from the mean, while the lower semi-variance is the expected value of squared negative deviations from the mean. We shall be concerned with the lower semi-variance about the mean. This we shall designate by \( S \) and, unless qualified, use the term "semi-variance" to apply to \( S \).

Hence, where \( x \) represents the random variable; \( \mu \), the mean and \( (\cdot)^- \) indicates that only negative deviations are to be taken, the semi-variance is:

\[
S = E \left\{ \left[ (x - \mu)^- \right]^2 \right\}.
\]

It is only when the probability distribution is symmetrical about the mean that the semi-variance is equal to one half of the variance.

The two measures are distinct. If loss is of particular concern to the firm, e.g. as it is in Roy's analysis, then the semi-variance is more pertinent than the variance for allocation decisions. This is so since the semi-variance concentrates on negative deviations alone, whereas the variance incorporates both positive and negative deviations.
In his recent work, Markowitz attaches special significance to the mean, variance, and semi-variance in order to justify his efficient-set analysis. He shows that if the firm is to act consistently with the set of plausible axioms which he lists, then it must act to maximise its expected utility. Markowitz suggests that if the firm wishes to maximise expected utility, and if it experiences diminishing marginal utility from increasing levels of returns, i.e., has a concave utility function when viewed from the returns' axis, then, the mean and the variance or the mean and the semi-variance are

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4 This analysis is in terms of the mean and variance (or semi-variance). From the possible mean-variance values it involves the selection of the efficient values. Efficient values are all those possible values which cannot be dominated. In other words, the efficient set is comprised of mean and variance values for each of which there is no possible value with a lower variance for the same mean or a higher mean for the same variance.

5 Markowitz, *op. cit.*, pp. 229-234. These axioms can be stated symbolically where "" > "" designates "preferred to"" and "" = "" designates "indifferent to"".

**Axiom Ia:** If P and Q are two probability distributions of outcomes than either P > Q, or Q >P, or P = Q ("comparability").

**Axiom Ib:** If P ≥ Q and Q ≥R, then P ≥ R ("transitivity").

**Axiom II:** If P > Q and R is a probability distribution of an outcome then, aP + (1 - a) R >aQ + (1 - a) R, given that a >0.

**Axiom III:** If P > Q, and Q > R then there is a number c such that cP + (1 - c) R = Q, where 0 < c <1.

These axioms imply the measurability of utility up to a linear transform. They do not imply that utility can be added for different individuals, but merely provide the individual with a numerical ordering system.
relevant to the firm's diversification decision. But Markowitz goes beyond this - "Portfolios selected on the basis of expected loss, expected absolute deviation, or probability of loss are not to be trusted".

Markowitz's defence of the variance and semi-variance as measures of variability depends upon the existence of a measurable utility function which is concave and, furthermore, upon the possibility of approximating the function by a quadratic or semi-quadratic function. If the utility function exists and is concave and continuous then, by the application of Taylor's Theorem, it can be approximated by a quadratic utility function. Once a quadratic utility function is derived, it can be expressed in terms of the mean and variance of returns alone. Similarly, a semi-quadratic function (a better approximation than the quadratic to Markowitz's utility function) can be derived and expressed in terms of the mean and the semi-variance alone.

We must not conclude from this argument that the semi-variance and the variance are always the most pertinent measures of variability for the firm. The universal proposition is unacceptable. First, the firm's utility function may not be

6 Ibid., pp. 279-297
7 Ibid., p. 297
8 A semi-quadratic function is quadratic for the lower portion of its domain and is linear for the remainder.
10 Markowitz, op. cit., p. 122 for an explanation.
an adequate approximation. Secondly, even when a quadratic or semi-quadratic approximates to a concave utility function, it generally does so only over some finite range. In order to obtain closer approximations, polynomials of higher degree should be taken. If polynomials of a higher degree are taken, utility will be affected by moments of a higher order than that of the variance (or semi-variance).

So far we have assumed the utility function to exist and have objected that it need not have the form which Markowitz ascribes to it. Upon this basis and the second ground above, we conclude that the utility function cannot always be expressed in terms of the mean and variance (or semi-variance) alone. But we may go further than this, and doubt whether firms are always willing to accept Markowitz's utility axioms. In particular, axiom III may not be accepted.\(^{11}\) There are some situations, even if rare, in which \(P\) is preferred to \(Q\) and \(Q\) is preferred to \(R\) (i.e. \(P > Q > R\)) but there is no probability, \(c\), greater than zero but less than unity for which the firm would be indifferent between \(Q\) and a probability of \(P\) and \(R\). There may be no value of \(c\) such that \(0 < c < 1\) and \(Q = cP + (1 - c)R\). The firm will always prefer \(Q\) to taking any chance of \(P\).\(^{12}\) In consequence, the utility of \(Q\) cannot be calculated by the Markowitz method. Furthermore, if axiom III is not satisfied, Markowitz's rejection of the maximum loss rule loses its force since the rejection is based upon an inconsistency between the rule and the utility axioms.

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11 See footnote 4 above.
12 For instance the outcome \(P\) might involve the possibility of bankruptcy, starvation, or a tremendous fall in social status.
Markowitz presses the normative implications of his analysis. If the firm accepts his axioms, then it ought to act in accordance with his efficient-set analysis which involves the mean and variance (or semi-variance). But as has been pointed out, this does not follow precisely since it involves special restrictions upon the utility function and assumes that quadratic or semi-quadratic functions are adequate approximations to the utility functions. Again, the utility axioms may not be acceptable to the firm. In these circumstances, Markowitz's analysis loses its normative force. Of course, whether or not Markowitz's expected utility maxim and efficient-set analysis is of descriptive relevance is an empirical question. Although this question is outside the scope of this paper, the universal relevance of the analysis is doubtful.

II

Let us consider an approach which, although it differs from that of Markowitz, depends upon the mean and the variance, or upon the mean and the semi-variance, to give it operational content. Let us suppose that the firm wishes to maximise its expected profit subject to the restriction that the probability of income being below, or equal to, some particular amount be less than, or equal to, some specified probability. If the firm knows only the mean, variance and semi-variance of income, then it is impossible for it to determine the precise probability of income below any pre-set level. All that it can do is place an upper bound on this probability. This can be done by using inequalities such as Cherbyshev's. Cherbyshev's inequality is
as follows: let \( x \) be a random variable with a mean, \( \mu = E(x) \), and a variance, \( \sigma^2 = \text{var}(x) \). Then, for any value of \( t > 0 \),

\[
\Pr\left\{ \left| x - \mu \right| \geq t \right\} \leq \frac{\sigma^2}{t^2}
\]

i.e. the probability that a random variable deviates from its mean by \( t \), or greater, is less than, or equal to, its variance divided by \( t^2 \). This probability depends both upon the mean and the variance of the variable and the inequality holds under quite general conditions. In fact, the inequality holds provided that a probability distribution of the random variable exists at all.

Applying this theorem to income and treating income as a random variable, we obtain, by straightforward substitution in equation (1), the probability that income will deviate from its mean by an amount \( t = \mu - x_1 \) or greater, given that \( x_1 \) is less than \( \mu \). We obtain that

\[
\Pr\left\{ \left| x - \mu \right| \geq (\mu - x_1) \right\} \leq \frac{\sigma^2}{(\mu - x_1)^2}
\]

i.e. the probability that income will be less than, or equal to, \( x_1 \) or, greater than, or equal to \( \mu + (\mu - x_1) \) is less than, or equal to the variance divided by \( (\mu - x_1)^2 \). This expression is dependent upon deviations both below and above the mean. However, it may not be appropriate because the firm may not wish to reduce variability in both directions. Its prime concern may be with deviations below the mean and with the risk of income falling to, or below, a particular level, \( x_1 \).

At first, it might seem that a satisfactory expression for the upper bound of this probability could be obtained from expression 2. Whilst an upper-bound probability can be obtained from equation 2 it is not as satisfactory as the upper-bound probability which can be obtained by using a semi-variance inequality. To make this clear let us establish from equation 2 an expression for \( \Pr(x < x_1 \mid x_1 < \mu) \). Equation 2 can be rewritten as

\[
\Pr(x < x_1) + \Pr(x \geq \mu + x_1 - x_1) \leq \frac{\sigma^2}{t^2}.
\]

Rearranging equation 3 we obtain

\[
\Pr(x < x_1) \leq \frac{\sigma^2}{t^2} - \Pr(x \geq 2\mu - x_1).
\]

However, the value of \( \Pr(x \geq 2\mu - x_1) \) is unknown. Since \( \sigma^2 / t^2 > 0 \) and, by the probability rules, \( \Pr(x \geq 2\mu - x_1) \geq 0 \), we can only obtain the following upper-bound probability from equation 4:

\[
\Pr(x < x_1) \leq \frac{\sigma^2}{t^2}
\]

i.e. the probability of income being less than or equal to \( x_1 \) is less than or equal to \( \sigma^2 / t^2 \).

However, the upper-bound value in equation 5 can be reduced if an inequality based upon the semi-variance is used.

The semi-variance inequality theorem is as follows: \(^{14}\) let \( x \) be a random variable with a mean, \( \mu = E(x) \), and a semi-variance, \( S \). Then, for any value of \( t > 0 \)

\(^{14}\) In the appendix this inequality is shown to be a special case of a more general inequality.
\[ \Pr \left\{ \left| (x - \mu) \right| \geq t \right\} \leq \frac{S}{t^2} \]

i.e. the probability that a random variable will be below its mean by an amount \( t \), or greater, is less than, or equal to, its semi-variance divided by \( t^2 \). By substitution in equation 6, we obtain an expression for the probability of income being below or equal to \( x_1 \). We obtain

\[ \Pr \left( x \leq x_1 \mid x_1 < \mu \right) \leq \frac{S}{(\mu - x_1)^2} \cdot \]

In words, the probability that income will be less than, or equal to, \( x_1 \) is less than, or equal to, the semi-variance divided by \( (\mu - x_1)^2 \). This probability depends both upon the semi-variance and the mean. Inequality 7 is more powerful than inequality 5 since the semi-variance is always less than the variance. Hence, the upper-bound probability \( S/t^2 \) must always be less than \( \sigma^2/t^2 \). Put differently,

\[ \Pr \left( x \leq x_1 \mid x_1 < \mu \right) \leq \frac{S}{(\mu - x_1)^2} < \frac{\sigma^2}{t^2} \cdot \]

The semi-variance inequality is more powerful than Chebyshev's if we are concerned with "one-sided" deviations. Also, the semi-variance inequality only requires information about deviations below the mean whereas those based upon the Chebyshev inequality, since they involve the variance, require information about deviations both sides of the mean. As with Chebyshev's inequality, the semi-variance inequality holds so long as the random variable has a probability distribution at all.
Not only do these inequalities apply generally to probability distributions but they also apply whatever the nature of the laws of production. To each product-mix there corresponds a mean level of income, a variance and a semi-variance. This is sufficient to calculate the value of the above inequalities for every product-mix.

We now apply the semi-variance inequality to the problem of allocating resources in accordance with the criterion that the firm maximises expected income subject to a security restriction. Let the firm have a given value of resources which it may distribute among different products in different ways. These "ways" constitute the acts open to it. To each of these acts there corresponds a mean income and a semi-variance, and to the set of available acts there corresponds a set of mean values of income and semi-variances. The relationship between the mean and the semi-variance will depend upon the laws of production and the correlation between returns from different goods. We shall not give the mean and semi-variance set any particular form. Obviously, however, the set can be shown on a two-dimensional diagram with the mean on one axis and the semi-variance on the other.

Given the firm's set of possibilities in terms of the mean and the semi-variance, the problem is to divide these possibilities into those which meet its security restriction and those which do not. Let the firm desire that the probability of its income being below or equal to \( x_1 \) be less than or equal to \( \rho \). Since our information is limited, we can
at best determine a subset of all the satisfactory values. We find this satisfactory subset by applying the semi-variance inequality.

All the values (possibilities) which satisfy the inequality

\[ \frac{S}{(\mu - x_1)^2} \leq \rho \]

are satisfactory. This must be so since

\[ \Pr (x \leq x_1 \mid x_1 < \mu) \leq \frac{S}{(\mu - x_1)^2} \]

Therefore, all of those combinations of the mean and semi-variance upon or below the positive branch of the parabola \( S = \rho (\mu - x_1)^2 \) will be satisfactory.

This parabola is shown arbitrarily in figure 1. It reaches its minimum at \( x_1 \). The negative branch is of no interest to us since we have assumed that \( x_1 \) is less than the mean. Let us suppose that the separation parabola intersects the available possibilities of the firm which are shown as the shaded area BDC in figure 1. Then, some of the possible values of the mean and semi-variance meet the firm's security restriction. From the values meeting this restriction, we select the one which yields the highest expected profit. This we derive by finding the highest straight line "indifference" curve which is "tangential" to these restricted values. In figure 1, the optimal available value is \( C \), and the optimal product-mix is the one corresponding to \( C \). In this case, the firm maximises expected profit and the security restriction is not effective. If, however, the parabola
passes below C (it will do this if \( \rho \) is small enough) but continues to cut the possibility set, then the nature of the optimum is changed. In this case the security restriction becomes effective and results in the non-maximisation of expected profit.

Figure 1.

\[
S = \rho \left( \mu - x_1 \right)^2
\]

Sometimes there may be no optimal solution in terms of the firm's immediate aims. This would be the case if the positive branch of the parabola, \( S = \rho \left( \mu - x_1 \right)^2 \), is everywhere below the possibility set of the mean and the semi-variance. If the firm's initial security restriction cannot be satisfied, then, once it realises this, it may act to minimise the probability of its income being below \( x_1 \).

This, however, is not the only possibility.

Although Roy's analysis depends upon Cherbychev's inequality, the separation process is more efficient when the principle of "safety-first" is based upon the semi-variance inequality. In particular, this is so if the probability

\[\text{On average we would be closer to the true minimum probability of disaster if we used the semi-variance inequality rather than Cherbychev's. This is because the former is more efficient in recording deviations below the mean.}\]
distributions are skewed. Roy's view can be re-interpreted in terms of the semi-variance analysis as follows: let $x_1$ correspond to the disaster level of income for the firm. Then from its possible acts the firm should select an act such that $\Pr(x \leq x_1)$ is minimised. If the firm only knows the mean and the semi-variance, and if $\Pr(x \leq x_1) = r$, then it should select from the possible semi-variance and mean values that pair which corresponds to the least value of $r$. Thus in terms of figure 1, we let $r$ decrease in the expression $S = r(\mu - x_1)^2$ until the positive branch of the parabola just touches the lower side of the set BDC. The optimal act corresponds to the point of "tangency" of the positive branch of the parabola with the possible semi-variance and mean values. Roy's approach can be interpreted as a special case in which the firm desires the probability of a disaster level of income to be zero. When this is unattainable the firm minimises the probability of the disaster level.

III

The view taken above is not an unqualified satisficer one. The approach is one of maximisation or minimisation subject to restraints or inequalities of the satisficer kind. Although it was assumed that the firm knew all of the acts available to it, this can be, and ought to be, relaxed. When this is relaxed, the firm can search for and adjoin new acts in an attempt to satisfy its inequalities. Further, it was assumed above that the firm knew with certainty all of the mean, semi-variance and variance parameters for each of its possible acts. These will not generally be known. It
will have to act upon its subjective estimates of the true parameters. While computations must necessarily be based upon subjective estimates of the parameters, it is the true parameters which will determine the final outcome.

In the above inequalities, income has been treated as a random variable. There are three possible grounds for this approach: (a) there are disturbing factors which lead to irregular and unpredictable values of income; (b) the laws which determine future income are so complicated that it is impossible to forecast income exactly; and (c) it is unprofitable to incur a cost sufficient to predict income precisely. An important characteristic of a random variable is the impossibility, relative to some circumstance, of predicting the actual value which the variable will assume. In case (a) it is impossible to predict perfectly because the process is inherently random. In case (b) the deterministic laws are not precisely known or the techniques available for applying them are incapable of yielding answers prior to their date of application, e.g., the results of lengthy computations or the collection of data may not be known until it is too late to apply them. In case (c) the firm is unwilling to incur a cost sufficient to ensure perfect prediction even if it were possible. There would seem to be many cases in which at least one of the above conditions is satisfied. In such circumstances, income ought to be treated as a random variable and be given theoretical attention in this light.

The treatment of income as a random variable does not imply that income can take any value; nor that it can take every value with an equal probability; nor that future probability distributions are unrelated to past events or selected by a random process - in other words, it does not exclude prediction completely. What it does exclude is *precise* prediction of the future values which will be actually realised.

Markowitz's, Roy's and the present analysis are all capable of explaining product diversification under constant returns. However, none of the solutions imply that product diversification is always optimal under conditions of uncertainty. Indeed, in these circumstances specialisation may sometimes be optimal. Furthermore, these analyses always select product combinations which are efficient in terms of the mean and the variance or, the mean and the semi-variance. 17 From the efficient set, however, they may select different values as being optimal.

**IV**

Decisions based solely upon the mean and variance (or semi-variance) can be justified in three ways. First, by Markowitz's view that the utility function can be approximated by a quadratic (or a semi-quadratic) function. Another justification relies on lack of knowledge - i.e. the mean and the variance (or semi-variance) are the only known parameters of the probability distributions of

17 This is easily shown for Roy's analysis. Relative to any member of the inefficient set it is possible to find a member of the efficient set which gives a smaller value for the relation \( \sigma^2 / (\bar{\mu} - x_i)^2 \). Therefore, the minimum of \( \sigma^2 / (\bar{\mu} - x_i)^2 \), in view of our definition in footnote 4, must occur for a member of the efficient set.
outcomes. Finally, if decisions are based on expected profit and probable loss, then the mean and the semi-variance provide a convenient summary of the probability distribution which can be used to estimate boundary probabilities for losses. Therefore, if the firm adopts a probability-of-loss criterion similar to the one developed in this paper, we can justify the application of the semi-variance inequality of the text either on the grounds of lack of knowledge or of convenience. But, depending on the circumstances, we do not deny that the firm's decision can be improved by the use of parameters additional to the mean and variance (or semi-variance).
APPENDIX TO DECISION MAKING & PROBABILITY OF LOSS.

The purpose of the appendix is to prove inequality (6) of the text and to present a more general semi-variance inequality theorem.

Inequality (6) follows from a general inequality, namely, the lower semi-variance inequality theorem. This theorem can be stated as follows: Letting x be a random variable and a be any value, the lower semi-variance $S_a$ about a is

$$S_a = E\left\{ (x - a)^- \right\}.$$ Then, for any value of $t > 0$,

$$\Pr \left\{ \left| (x - a)^- \right| \geq t \right\} \leq \frac{S_a}{t^2}$$

i.e., the probability that x will be below a by t, or greater, is less than, or equal to, the lower semi-variance about the point a divided by $t^2$.

Proof:

The lower semi-variance is defined about the point a by a series comprised solely of positive terms, since it consists of squared negative deviations from a multiplied by their probability of occurrence. The deletion of values for which $\left| (x - a)^- \right| < t$ cannot increase the value of the lower semi-variance series. Denoting the sum of the series after deletion as $\gamma^* \left[ (x - a)^- \right]^2$ and their probability of $f(x^*)$,

$$S_a \geq \gamma^* \left[ (x - a)^- \right]^2 f(x^*).$$

Moreover, since $\left| (x - a)^- \right| \geq t$, we have

$$\gamma^* \left[ (x - a)^- \right]^2 f(x^*) \geq \gamma^* t^2 f(x^*) = \gamma^* f(x^*).$$

Hence,

$$S_a \geq t^2 \gamma^* f(x^*)$$
i.e.

\[ S_a \geq t^2 \Pr \left\{ \left| (x - a)^- \right| \geq t \right\}. \]

\[ \therefore \Pr \left\{ \left| (x - a)^- \right| \geq t \right\} \leq \frac{S_a}{t^2}. \]

In the special case where \( a = \mu = E(x) \) we obtain from (9) that

\[ \Pr \left\{ \left| (x - \mu)^- \right| \geq t \right\} \leq \frac{S_{\mu}}{t^2} \]

i.e. the probability that \( x \) will be below the mean by \( t \), or greater, is less than, or equal to, the lower semi-variance about the mean divided by \( t^2 \). This is equivalent to inequality (6) in the text.

Similarly, an upper semi-variance inequality theorem can be shown to hold.
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