AUSTRALIAN NATIONAL UNIVERSITY

NON-LINEAR LAGRANGIANS IN GENERAL RELATIVITY

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Except where due acknowledgement has been made in the text, the work described in this thesis is the candidate's own.

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SUMMARY

The field equations of gravitation derived from second order non-linear invariants are considered. In Chapter I, it is shown that Robertson-Walker metric is compatible with the field equations derived from an arbitrary second order non-linear invariant and the stress-energy tensor $T_{\mu\nu}$ of a perfect fluid distribution. Considering the only interesting quadratic second-order invariant $R + \alpha R^2 - 2\Lambda$ in the context of the Robertson-Walker metric, a few special cosmological solutions are obtained. With an interest in examining the possibility of field equations which can admit Robertson-Walker type cosmological solution free from singularity at any time, the field equations derived from an invariant, a function of R, are considered. A special solution of the problem is obtained. In Chapter II, the Einstein-Infeld-Hoffmann method and Gregory's work on invariant $R + \frac{a^2}{6} R^2$ in the context of the problem of motion are reviewed. The derivation of different starting terms of the expansions of the field-quantities $\gamma_{\mu\nu}$ in the case of general relativity and that of the theory based on the invariant $R + \frac{a^2}{6} R^2$, respectively, is clearly described. This derivation is instructive if one wishes to apply the Einstein-Infeld-Hoffmann method to the problem of motion for a theory based on some other invariant. In Chapter III, the derivation of the static, spherically symmetric and asymptotically flat solution of the linearized empty field-equations, derived from the invariant $R + \frac{a^2}{6} R^2$, is discussed in the rectangular harmonic coordinates, and in the so called harmonic polar coordinates. A coordinate transformation transforming this solution into isotropic polar coordinates is obtained. The linearized field equations are derived and solved, directly, in the isotropic polar

coordinates, the condition that the solution be flat at infinity being imposed. It is shown that this solution precisely agrees with that obtained by successive coordinate transformations. In Chapter IV, the linearized non-empty field equations derived from the invariant $R + \frac{a^2}{6} R^2$ and the stress-energy tensor $T_{\mu\nu}$ of a perfect fluid distribution are considered. The meaning of such linearization is discussed. Considering a static, spherically symmetric uniform-density sphere of radius b, the interior solution in the isotropic polar coordinates is obtained. The condition that this solution be regular at the origin is imposed. Then by choosing suitable boundary conditions, the interior solution is joined to the asymptotically flat solution of the linearized empty field equations obtained in Chapter III. By considering a continuous density-distribution which, in the limit, tends to the above-mentioned sphere, as a parameter η tends to zero, a justification for the choice of the boundary conditions is obtained. In Chapter V, from purely a mathematical point of view, a source-term $T_\mu^{\ \nu}$ of a static, spherically symmetric point-particle located at the origin of either rectangular harmonic or isotropic polar coordinates, which fits with the linearized field equations derived from the invariant $R + \frac{a^2}{6} R^2$, is obtained. In Chapter VI, we consider the exact field equations of general relativity corresponding to a static, spherically symmetric point-source located at the origin of the standard polar coordinates. Expressing the source-term $T_\mu^{\ \nu}$ by a realistic δ-function, $\Delta(\vec{r},\epsilon)$, the field equations are integrated. It is shown that as parameter ϵ tends to zero, $T_\mu^{\ \nu}$ tends to Tangherlini's expression. Though the discussion is purely mathematical, it is instructive in telling us how to make use of the δ-function philosophy in the problem involving non-linear differential equations. Moreover, it gives a more rigorous derivation of Tangherlini's result.

CONTENTS

INTRODUCTION

In the theory of general relativity, the field equations of gravitation are formulated by coupling the stress-energy tensor $T_{\mu\nu}$ to the Einstein-tensor $G_{\mu\nu}$ on the basis of the so-called principle of equivalence. This coupling depends upon the following two facts. Firstly both of these tensors are symmetric and second-rank tensors such that the divergence of $G_{\mu\nu}$ vanishes identically while that of $T_{\mu\nu}$ vanishes due to the principle of conservation of energy and momentum. Secondly, since in the formulation of this theory, one seeks for a generalization of the Poisson-equation of the Newtonian theory of gravitation and since this equation involves the partial derivatives of the gravitational potential only up to second order, one demands that the field equations should be second order partial differential equations for the tensor $g_{\mu\nu}$. However, the Einstein-tensor $G_{\mu\nu}$ is the simplest symmetric second-rank tensor whose divergence vanishes identically and which involves the partial derivatives of $g_{\mu\nu}$ only up to second-order and consequently one is led to the coupling of $G_{\mu\nu}$ to $T_{\mu\nu}$. The tensor $G_{\mu\nu}$ can be obtained by varying the tensor $g_{\mu\nu}$ in the action-integral

$$A = \int R\sqrt{-g}\; d^4X \tag{1}$$

where R is the scalar curvature of the space-time, g is the determinant of $g_{\mu\nu}$ and d^4X is a four dimensional volume element. Here and in what follows in the present thesis, the signature of the space-time metric is supposed to be -2 . The quantity $-G^{\mu\nu}$ is called the Hamiltonian derivative of the invariant or Lagrangian R and is denoted by $\frac{hR}{hg_{\mu\nu}}$.

However, the Hamiltonian derivative of any invariant involving the components of tensor $g_{\mu\nu}$ and its partial derivatives up to second order is a symmetric and second-rank tensor whose divergence vanishes identically. (See Buchdahl [5], Eddington [6].) Consequently, it follows that the principle of conservation of energy and momentum is still maintained even if one couples the tensor $T_{\mu\nu}$ to the Hamiltonian derivative of any such invariant and thus constructs the field equations of gravitation, ignoring the requirement that the field equations should be second order and not higher order partial differential equations for $g_{\mu\nu}$. In most of the work in the present thesis, the more general invariants considered are those of degree m, meaning thereby that the invariant is a polynomial of degree m in the components of the Riemann tensor.

The motivation for studying the use of non-linear invariants to formulate the field equations of gravitation can be described as follows: The static, spherically symmetric solution of the field equations $R_{\mu\nu} = 0$ of the theory of general relativity certainly implies a singularity at the origin. The same situation prevails in classical electrodynamics. However, Born and Infeld overcome this difficulty by adding a non-linear term to the Lagrangian $\frac{1}{4}f_{\mu\nu}f^{\mu\nu}$ of classical electrodynamics. (See Born and Infeld [4].) The question arises whether one can resolve the similar difficulty in the theory of general relativity by adding a non-linear term to the Lagrangian R. Secondly the theory of general relativity does not admit any spherically symmetric cosmological solutions free from singularity at any time. Again the question arises whether one can obtain non-singular spherically symmetric cosmological solutions from the field equations obtained by adding a non-linear term to the Lagrangian R or by using some other non-linear Lagrangian. With the above-mentioned motivations in mind, in the present thesis, we study a few problems concerning gravitation by using the field equations derived from some non-linear second order Lagrangians. In what follows, first we shall briefly state the work done by others on the relevant topics and then summarize the work presented in this thesis.

Following upon Weyl's attempt to formulate a unified field theory based upon a gauge-invariant action integral, non-linear Lagrangians have been considered from time to time, for example by Bach, Eddington, Lanczos, Gregory, Buchdahl and others. Lanczos has investigated the possible quadratic invariants:-

$$
\begin{aligned}
I_1 &= R_{\mu\nu}R^{\mu\nu} \quad , \quad I_2 = R^2 \quad , \quad I_3 = R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} \\
K_1 &= (-g)^{-\frac{1}{2}}\epsilon^{\mu\nu\sigma\rho}R_{\mu\nu\alpha\beta}R_{\sigma\rho}{}^{\alpha\beta} \\
K_2 &= (-g)^{-1}\epsilon^{\mu\nu\sigma\rho}\epsilon^{\alpha\beta\xi\eta}R_{\mu\nu\alpha\beta}R_{\sigma\rho\xi\eta}
\end{aligned}
\tag{2}
$$

where the various symbols have the usual meaning. It has been shown that there remain only two effective quadratic invariants I_1 and I_2 as far as the construction of the field equations of gravitation is concerned. (See Bach [1], Lanczos [12].) Using the Einstein-Infeld-Hoffmann method of approximation, Gregory has studied the invariant $R + \frac{a^2}{6}R^2$ (where a^2 is a constant) in connection with the problem of motion and shown that the equations of motion of a system of p-particles interacting only under their own gravitational field follow from the field equations. (See Gregory [10].) Assuming the expression for the stress-energy tensor $T_\mu^{\ \nu}$ of a static, spherically symmetric point-source in terms of the δ-function, Pechlaner and Sexl have explored the linearized non-empty field equations deduced from the invariant $R + \frac{a^2}{6}R^2$, in the case of such a system. Considering the extended distribution corresponding to a static, spherically symmetric sphere with uniform density, they have concluded that the exterior solution for such a sphere differs from that for their model of a point-source. (See Pechlaner and Sexl [13].)

In Chapter I of the present thesis, we consider a few cosmological consequences of the non-empty field equations derived from various non-linear second order Lagrangians. In Section 1.3 it is shown that the Robertson-Walker metric is compatible with the field equations derived from an arbitrary second order non-linear Lagrangian. Then in Section 1.4 it follows as a corollary that the same result is true in the case of the invariants R^2 and $R_{\mu\nu}R^{\mu\nu}$. In Section 2.1 it is shown that, from the point of view of the Robertson-Walker cosmology, there exists only one interesting second order quadratic Lagrangian, $R + \alpha R^2 - 2\Lambda$, where α and Λ are constants. In Section 2.2, using the field equations derived from the Lagrangian $R^2 - 2\Lambda$ and $R + \alpha R^2 - 2\Lambda$ respectively, a few special Robertson-Walker type cosmological solutions are obtained. In Sections 3.1 and 3.2, the field equations derived from the non-linear second order Lagrangian, a function of R, are considered and an attempt is made to find whether such equations admit the Robertson-Walker type cosmological solution free from singularity at any time. The equation of state is assumed to be $\rho = 3p$. In the special case for which R is a small negative constant, the field equations do admit such a solution with the curvature constant $k = -1$, provided the Lagrangian is positive.

In Chapter II, we discuss a few known aspects of the Einstein-Infeld-Hoffmann method of successive approximations applied to the problem of motion. (See Einstein, Infeld and Hoffmann [9].) In Section 4.1 we briefly describe this method and a similar procedure by which Gregory has obtained the equations of motion up to the approximation stage $\ell = 2$, from the field equations derived from the invariant $R + \frac{a^2}{6} R^2$. (See Gregory [10].) However, Gregory has not explained the meaning of the new constant of integration r_{0k}, associated with the kth singularity, appearing in the equations of motion. In this work, matter is regarded as spherically symmetric singularities of the field. The work of Pechlaner and Sexl appears to suggest that the constant r_{0k} should be proportional to the mass $\overset{k}{m}$ of the kth singularity. However, this work depends upon a particular definition of a point-source. (See Pechlaner and Sexl [13].) In Section 5.1, we describe how the different expansions of the field-quantities $\gamma_{\mu\nu}$ used in the work of Einstein, Infeld and Hoffmann and that of Gregory, respectively, can be deduced. This description can guide us to obtain the suitable expansions of $\gamma_{\mu\nu}$ if we wish to apply the Einstein-Infeld-Hoffmann method to the problem of motion in the case of some other Lagrangian.

In Chapter III, we consider linearized empty field equations derived from the invariant $R + \frac{a^2}{6} R^2$, corresponding to a static, spherically symmetric point-particle located at the origin of the coordinate system. In Section 6.1, we describe how these equations can be solved in harmonic rectangular coordinates and how the solution can be transformed into Gregory's solution without disturbing the harmonic character of the coordinate system. In Section 7.1, we state how this solution can be transformed into so-called harmonic polar coordinates. Next, we obtain a coordinate transformation, transforming this solution into isotropic polar coordinates from harmonic polar coordinates. In Section 8.1, we consider the same problem as that of Section 6.1 but directly in isotropic polar coordinates. The solution obtained there precisely agrees with that obtained in Section 7.1 by successive coordinate transformations. The discussion brings out the simplicity of the isotropic polar coordinate system with respect to the problem.

In Chapter IV, we consider the linearized non-empty field equations derived from the invariant $R + \frac{a^2}{6} R^2$, corresponding to a static, spherically symmetric perfect fluid distribution in a sphere with radius b . We suppose that the matter inside the sphere is in an ordinary physical condition, meaning that the density ρ and the pressure p , both, are of order zero in the parameter $\epsilon = 1/c^2$, c being the velocity of light. Next we assume that the coupling constant $\chi = 8\pi G/c^2$, G being the Newtonian constant of gravitation, so that when $a = 0$ the field equations reduce to those of general relativity. Thus $\chi \sim \epsilon$. Using these assumptions, in Section 9.1, we obtain the linearized non-empty field equations in rectangular harmonic coordinates. It is observed that the pressure p disappears from the field equations. Hence if one solves the linearized non-empty field equations, it is possible to determine ρ but not p to order zero in ϵ . However, by linearizing the conservation equations we obtain a relation between ρ and p which can help us to determine p to order zero in ϵ . In Section 10.1, for the same physical situation and with the same assumptions, we obtain the linearized non-empty field equations in isotropic polar coordinates. Also by linearizing the conservation equations, we obtain a relation between ρ and p . Imposing the condition that the solution be regular at the origin and supposing ρ to be a constant, we solve the field equations. By imposing suitable boundary conditions at $\bar{r} = b$, we join the interior solution to the asymptotically flat exterior solution which is already known from Section 8.1 of Chapter III. Next, with a view to justifying the choice of the boundary conditions, we consider a density-distribution which is a function of $\bar{r}$ and a parameter η such that as η tends to zero, ρ tends to a non-zero positive constant or zero according as $\bar{r}$ is less or greater than b. When η is non-zero we have a continuous fluid distribution with positive density, filling the entire space. Using such a density-distribution and imposing the conditions that the solution be regular at the origin and asymptotically flat at infinity, we solve the field equations. We find that the solution is such that when η tends to zero, according as $\bar{r}$ is less or greater than b, it tends to the interior or exterior solution obtained before by imposing suitable boundary conditions. This method, in fact, helps us to find the correct boundary conditions. The exterior solution obtained there precisely agrees with that obtained by Pechlaner and Sexl by using Green's functions. (See Pechlaner and Sexl [13].)

In Chapter V we consider the linearized field equations derived from the invariant $R + \frac{a^2}{6} R^2$ and containing a source-term $T_\mu^{\ \nu}$ on the right-hand side. The source is supposed to be a static, spherically symmetric point-particle located at the origin of the coordinate system and the expression for $T_\mu^{\ \nu}$ contains the δ-function and its derivatives. Integrating the field equations, so formulated, by the method of Fourier transforms, we show that the solution precisely agrees with that obtained in Chapter III by solving the linearized empty field equations corresponding to a static, spherically symmetric point-particle located at the origin. The problem is discussed in rectangular harmonic as well as isotropic polar coordinates in Sections 11.1 and 12.1 respectively. The entire treatment of the problem is made from purely a mathematical point of view.

In general relativity, considering the exact field equations, Tangherlini has obtained the expression for $T_\mu^{\ \nu}$ of a static, spherically symmetric point-source located at the origin of the standard polar coordinate system, requiring that the Schwarzschild solution be valid even at the origin. (See Tangherlini [17].) In Chapter VI, we examine the same problem by using a realistic δ-function, $\Delta(\vec{r},\epsilon)$, to represent the source-term $T_{\mu}^{\ \nu}$. The exact equations so formed are solved and it is seen that in the limit as the parameter ϵ tends to zero, $T_{\mu}^{\ \nu}$ tends to the expression obtained by Tangherlini. Though the treatment of the problem is purely mathematical, it is rather instructive in telling us how to use the δ-function in problems involving non-linear differential equations. Moreover, it gives a more rigorous derivation of Tangherlini's result.

In the present thesis, the scalar curvature R has the dimension of reciprocal of length squared, so that the constant a appearing in the invariant $R + \frac{a^2}{6} R^2$ has the dimension of length. The symbols and notations used are explained wherever necessary otherwise they have the conventional meaning. The Latin indices take on only the spatial values 1,2,3 while Greek indices refer to both, space and time, running over the values 0,1,2,3 or 1,2,3,4 . The dummy indices mean summation unless otherwise stated. A comma followed by an index means partial differentiation whereas a semi-colon followed by an index denotes covariant differentiation. The expressions used for R, $R_{\mu\nu}$ and $R_{\mu\nu\sigma\rho}$ are according as "Riemannian Geometry" by L.P. Eisenhart (Princeton University Press, 1949). We normally abbreviate "Einstein-Infeld-Hoffmann" to "EIH", "Robertson-Walker" to "RW" and "static, spherically symmetric" to "sss".

CHAPTER I

COSMOLOGY

§1.1 The Field Equations Derived from Non-Linear Second Order Invariants

In this chapter we shall consider the field equations of gravitation obtained by coupling the Hamiltonian derivative of a second order invariant of degree m to the stress-energy tensor of a perfect fluid distribution in context of cosmology.

Without loss of generality one can restrict oneself to the case in which the invariant has the form

$$K = \Gamma^{\mu_1\nu_1\sigma_1\rho_1\cdots\rho_m} \prod_{s=1}^{m} R_{\mu_s\nu_s\sigma_s\rho_s} \tag{1.1}$$

where $\Gamma^{\mu_1\nu_1\sigma_1\rho_1\cdots\rho_m}$ is a tensor of rank $4m$ constructed from the metric tensor alone and $R_{\mu_s\nu_s\sigma_s\rho_s}$ is the covariant curvature tensor, given by

$$R_{\mu_s\nu_s\sigma_s\rho_s} = \tfrac{1}{2}(g_{\mu_s\rho_s,\nu_s\sigma_s} + g_{\nu_s\sigma_s,\mu_s\rho_s} - g_{\mu_s\sigma_s,\nu_s\rho_s} - g_{\nu_s\rho_s,\mu_s\sigma_s}) + $$
$$+ g_{\alpha\beta}(\Gamma^{\alpha}_{\nu_s\sigma_s}\Gamma^{\beta}_{\mu_s\rho_s} - \Gamma^{\alpha}_{\nu_s\rho_s}\Gamma^{\beta}_{\mu_s\sigma_s}) . \tag{1.2}$$

This is because the most general second order invariant is an arbitrary function of K_i, each of the latter being of the form (1.1). When the Hamiltonian derivatives of the K_i are known, that of the most general second order invariant follows at once. (See Buchdahl [5].)

Now, coupling the Hamiltonian derivative $P^{\mu\nu}$ of K to the stress-energy tensor $T^{\mu\nu}$ of a perfect fluid distribution, we obtain the following field equations of gravitation:

$$P^{\mu\nu} = \kappa T^{\mu\nu} \tag{1.3}$$

where κ is a coupling constant and $P^{\mu\nu}$ and $T^{\mu\nu}$ are expressed as follows:

$$P^{\mu\nu} = Z^{\mu\nu\sigma\rho}{}_{;\sigma\rho} - \tfrac{2}{3}Z^{\nu\alpha\sigma\rho}R^{\mu}{}_{\alpha\sigma\rho} + \tfrac{1}{2}g^{\mu\nu}K \tag{1.4}$$

where $Z^{\mu\nu\sigma\rho} = \partial K/\partial g_{\mu\nu,\sigma\rho}$ (see Buchdahl [5])

$$T^{\mu\nu} = (\rho+p)\ \frac{dx^{\mu}}{ds}\cdot\frac{dx^{\nu}}{ds} - pg^{\mu\nu} \tag{1.5}$$

where ρ and p are respectively the proper macroscopic density and pressure as they would be measured by a local observer at rest in the fluid, and the quantity dx^{μ}/ds is the macroscopic velicity of this fluid with respect to the coordinate system in use. In this chapter the units are so chosen that the velocity of light, c, is equal to unity. (See Tolman [18].)

§1.2 The Robertson-Walker Metric

If we assume that the space-time of the universe is Riemannian and four dimensional described by the coordinates x^{μ} and the metric-form

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} \tag{1.6}$$

with signature -2, then from two simple assumptions, it follows independently of the field equations of gravitation of a theory, that the metric-form

(1.6) is expressible in the form

$$ds^2 = (dx^4)^2 - \frac{S^2(x^4)}{\left[1 + \frac{k}{4} \sum_{i=1}^{3} (x^i)^2\right]^2} [(dx^1)^2 + (dx^2)^2 + (dx^3)^2] \qquad (1.7)$$

where $k = 1$, 0 or -1 and $S(x^4)$ is an arbitrary function of x^4 , the time coordinate. First of these assumptions is known as Weyl's postulate and the other one states that the metric-form (1.6) is spatially isotropic. By virtue of Weyl's postulate, the coordinate system in use becomes the so-called co-moving coordinate system. In polar coordinates (r,θ,ϕ,t) , the line-element (1.7) takes the form

$$ds^2 = dt^2 - \frac{S^2(t)}{\left(1 + \frac{k}{4} r^2\right)^2} (dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2) \qquad (1.8)$$

where $k = 1$, 0 or -1 and $S(t)$ is an arbitrary function of time t . The line-element (1.7) or (1.8) is known as Robertson-Walker line-element. (See Bondi [3].)

Since the above result is independent of the field equations of a theory, we observe that it remains valid even when we use the field equations (1.3), that is, the metric-form of the universe will still be reducible to the RW-metric.

§1.3 Compatibility of the Robertson-Walker Metric with the Field Equations of Section 1.1

In this section we shall show that the RW-metric (1.8) is compatible with the field equations (1.3) which can be written in the mixed form

$$P_\mu{}^\nu = \kappa T_\mu{}^\nu \; . \tag{1.9}$$

As seen in Section 1.2, the reduction of the line-element of the universe to the RW-metric involves the use of co-moving coordinate system. Therefore from (1.5) it follows that $T_\mu{}^\nu$ becomes diagonal with

$$T_1{}^1 = T_2{}^2 = T_3{}^3 \; . \tag{1.10}$$

Consequently, to prove the necessary result we should prove that

$$P_1{}^1 = P_2{}^2 = P_3{}^3 \tag{1.11}$$

$$P_\mu{}^\nu = 0 \quad \text{if} \quad \mu \neq \nu \tag{1.12}$$

for the RW-metric.

For (1.11) and (1.12) to hold, we first observe that the RW-metric is spherically symmetric. Therefore any quantity deduced from it will be spherically symmetric. Since the components of $P_\mu{}^\nu$ are deduced from the RW-metric, they are spherically symmetric and therefore the non-vanishing components of $P_\mu{}^\nu$ are $P_1{}^1$, $P_2{}^2$, $P_3{}^3$, $P_4{}^4$, $P_1{}^4$ and $P_4{}^1$ which are at most functions of r and t, such that

$$P_2{}^2 = P_3{}^3 \quad , \quad P_4{}^1 = g^{11} P_1{}^4 \tag{1.13}$$

(see Takeno [16]).

Next since we know that the divergence of $P_\mu{}^\nu$ vanishes identically

$$P_{\mu\,;\nu}^{\nu} = 0 \; . \tag{1.14}$$

Hence for the RW-metric, when $\mu = 1$,

$$(P_1{}^1)' + 2\left(\frac{1}{r} - \frac{\chi'}{\chi}\right)(P_1{}^1 - P_2{}^2) = 0 \tag{1.15}$$

where $\chi = 1 + \frac{k}{4} r^2$ and the overhead dash denotes differentiation with respect to r .

From (1.13) and (1.15) it follows that to obtain the necessary result we must prove

$$P_1{}^1 = \text{a function of time alone} \tag{1.16}$$

$$P_4{}^1 = 0 \quad . \tag{1.17}$$

We shall prove this by considering the expression for $P_\mu{}^\nu$. Since

$$P_\mu{}^\nu = Z_\mu{}^{\nu\sigma\rho}{}_{;\sigma\rho} - \tfrac{2}{3}Z^{\nu\alpha\sigma\rho}R_{\alpha\sigma\rho\mu} + \tfrac{1}{2}\delta_\mu{}^\nu K \tag{1.18}$$

where K is given by (1.1), to prove (1.16) we must show that, when $\mu = \nu = 1$, the right-hand side of (1.18) is a function of time alone and to prove (1.17) we must show that when $\mu = 4$ and $\nu = 1$, the right-hand side of (1.18) vanishes.

We first note that the RW-metric is a conformally flat metric and therefore

$$R_{\mu_s\nu_s\sigma_s\rho_s} = \tfrac{1}{2}(g_{\mu_s\rho_s}R_{\nu_s\sigma_s} + g_{\nu_s\sigma_s}R_{\mu_s\rho_s} - g_{\mu_s\sigma_s}R_{\nu_s\rho_s} - g_{\nu_s\rho_s}R_{\mu_s\sigma_s}) + \frac{1}{6}R(g_{\mu_s\sigma_s}g_{\nu_s\rho_s} - g_{\mu_s\rho_s}g_{\nu_s\sigma_s}). \tag{1.19}$$

Secondly, we note that for the RW-metric the surviving components of $R_\mu{}^\nu$ are such that

$$R_1{}^1 = R_2{}^2 = R_3{}^3 = \text{a function of time alone}$$

$$R_4{}^4 = \text{a function of time alone} \tag{1.20}$$

$$R_\mu{}^\nu = 0 \quad \text{if} \quad \mu \neq \nu \quad .$$

Looking at the structure of K as given by (1.1) and using (1.19) and (1.20) we infer that

$$K = \text{a function of time alone} \quad . \tag{1.21}$$

Next, as

$$Z^{\mu\nu\sigma\rho} = \tfrac{1}{2} \sum_{r=1}^{m} \Gamma^{\mu_1\nu_1 \ldots \rho_{r-1}\mu\nu\sigma\rho\mu_{r+1} \ldots \rho_m} \prod_{s=1}^{m}{}' R_{\mu_s\nu_s\sigma_s\rho_s} \tag{1.22}$$

where the overhead dash means the factor with suffix $s = r$ is to be omitted from the product, $Z^{\nu\alpha\sigma\rho}R_{\alpha\sigma\rho\mu}$ consists of terms of the form

$$U_\mu{}^\nu = \Gamma^{\nu_1\alpha_1\sigma_1\rho_1 \ldots \nu\alpha\sigma\rho \ldots \rho_m} \left(\prod_{s=1}^{m}{}' R_{\nu_s\alpha_s\sigma_s\rho_s} \right) R_{\alpha\sigma\rho\mu} \tag{1.23}$$

(see Buchdahl [5]).

We note that in (1.23) there are only two free indices μ and ν. Therefore, because of (1.19) $U_\mu{}^\nu$ is a sum of terms of the types

$$\delta_\mu{}^\nu R^{\cdot}_{\cdot} R^{\cdot}_{\cdot} \ldots R^{\cdot}_{\cdot}$$

$$R_{\cdot}{}^\nu R^{\cdot}_{\cdot} R^{\cdot}_{\cdot} \ldots R_\mu{}^{\cdot} \ldots R^{\cdot}_{\cdot} \tag{1.24}$$

$$R_\mu{}^\nu \ldots R^{\cdot}_{\cdot} \ldots R^{\cdot}_{\cdot}$$

From (1.20) we infer that the terms of the forms (1.24) can be either zero or a function of time alone according as $\mu \neq \nu$ or $\mu = \nu$ respectively. Therefore it follows that

$$\begin{aligned} Z^{\nu\alpha\sigma\rho} R_{\alpha\sigma\rho\mu} &= \text{a function of time alone} \quad \text{if} \quad \mu = \nu \\ &= 0 \qquad\qquad\qquad\qquad\qquad\quad \text{if} \quad \mu \neq \nu \ . \end{aligned} \tag{1.25}$$

Finally we consider the term $Z_\mu{}^{\nu\sigma\rho}{}_{;\sigma\rho}$ of $P_\mu{}^\nu$. We note that

$$Z_\mu{}^{\nu\sigma\rho}{}_{;\sigma\rho} = g^{\sigma\alpha}(Z_\mu{}^\nu{}_\sigma{}^\rho)_{;\alpha\rho} \quad . \tag{1.26}$$

From (1.19) and (1.22) the generic form of $Z_\mu{}^\nu{}_\sigma{}^\rho$ is given by

$$\begin{aligned} Z_\mu{}^\nu{}_\sigma{}^\rho = {} & g^{\nu\rho} g_{\mu\sigma} I_1 + \delta_\mu{}^\nu \delta_\sigma{}^\rho I_2 + \delta_\mu{}^\rho \delta_\sigma{}^\nu I_3 + g^{\nu\rho} A_{\mu\sigma} + g_{\mu\sigma} A^{\nu\rho} + \delta_\mu{}^\nu A_\sigma{}^\rho + \\ & + \delta_\mu{}^\rho A_\sigma{}^\nu + \delta_\sigma{}^\nu A_\mu{}^\rho + \delta_\sigma{}^\rho A_\mu{}^\nu + A_\mu{}^\nu A_\sigma{}^\rho + A_\mu{}^\rho A_\sigma{}^\nu + A^{\nu\rho} A_{\mu\sigma} \end{aligned} \tag{1.27}$$

where the I's are invariants and the A's are second-rank tensors.

Using (1.26) and (1.27) we obtain

$$\begin{aligned} Z_\mu{}^{\nu\sigma\rho}{}_{;\sigma\rho} = g^{\sigma\alpha} [& g^{\nu\rho} g_{\mu\sigma} (I_1)_{;\alpha\rho} + \delta_\mu{}^\nu \delta_\sigma{}^\rho (I_2)_{;\alpha\rho} + \delta_\mu{}^\rho \delta_\sigma{}^\nu (I_3)_{;\alpha\rho} + g^{\nu\rho} (A_{\mu\sigma})_{;\alpha\rho} + \\ & + g_{\mu\sigma} (A^{\nu\rho})_{;\alpha\rho} + \delta_\mu{}^\nu (A_\sigma{}^\rho)_{;\alpha\rho} + \delta_\mu{}^\rho (A_\sigma{}^\nu)_{;\alpha\rho} + \delta_\sigma{}^\nu (A_\mu{}^\rho)_{;\alpha\rho} + \\ & + \delta_\sigma{}^\rho (A_\mu{}^\nu)_{;\alpha\rho} + (A_\mu{}^\nu)_{;\alpha\rho} A_\sigma{}^\rho + A_\mu{}^\nu (A_\sigma{}^\rho)_{;\alpha\rho} + (A_\mu{}^\rho)_{;\alpha\rho} A_\sigma{}^\nu + \\ & + A_\mu{}^\rho (A_\sigma{}^\nu)_{;\alpha\rho} + (A^{\nu\rho})_{;\alpha\rho} A_{\mu\sigma} + A^{\nu\rho} (A_{\mu\sigma})_{;\alpha\rho}] \quad . \end{aligned} \tag{1.28}$$

Since the invariants I_1, I_2, I_3 can be functions of R only and since R is a function of time alone, we have $I_{,\alpha} = (0,0,0,\dot{I})$ where $\dot{I} = I_{,4}$

and I stands for I_1 , I_2 or I_3 . Therefore

$$I_{;\alpha\rho} = I_{,\alpha\rho} - \Gamma^4_{\alpha\rho}\,\dot{I} \; . \tag{1.29}$$

Using the values of $\Gamma^4_{\alpha\rho}$ for the RW-metric, from (1.29), the surviving components of $I_{;\sigma\rho}$ become

$$\begin{aligned}
I_{;11} &= -\Gamma^4_{11}\,\dot{I} = g_{11}\frac{\dot{S}}{S}\,\dot{I} \\
I_{;22} &= -\Gamma^4_{22}\,\dot{I} = g_{22}\frac{\dot{S}}{S}\,\dot{I} \\
I_{;33} &= -\Gamma^4_{33}\,\dot{I} = g_{33}\frac{\dot{S}}{S}\,\dot{I} \\
I_{;44} &= \ddot{I}
\end{aligned} \tag{1.30}$$

where the overhead dot denotes differentiation with respect to time t . Next we note that $A_\alpha{}^\beta$ is of the form

$$A_\alpha{}^\beta = R_\alpha{}^{\mu_1} R^{\cdot}_{\mu_1} \ldots R^\beta_{\mu_m} \; . \tag{1.31}$$

Therefore from (1.20) it follows that

$$\begin{aligned}
A_\alpha{}^\beta &= \text{a function of time alone} \quad \text{if } \alpha = \beta \\
&= 0 \quad \text{if } \alpha \neq \beta
\end{aligned} \tag{1.32}$$

and

$$(A_\alpha{}^\beta)_{;\sigma\rho} = (\text{a function of time alone})(R_\alpha{}^\beta)_{;\sigma\rho} \tag{1.33}$$

$$(A_{\alpha\beta})_{;\sigma\rho} = (\text{a function of time alone})\,g_{\alpha\eta}(R_\beta{}^\eta)_{;\sigma\rho} \tag{1.34}$$

$$(A^{\alpha\beta})_{;\sigma\rho} = (\text{a function of time alone})\,g^{\alpha\eta}(R_\eta{}^\beta)_{;\sigma\rho} \; . \tag{1.35}$$

Now putting $\mu = \nu = 1$ in (1.28) we obtain

$$Z_1{}^{1\sigma\rho}{}_{;\sigma\rho} = g^{\sigma\alpha}g^{1\rho}g_{1\sigma}(I_1)_{;\alpha\rho} + g^{\sigma\alpha}\delta_\sigma{}^\rho(I_2)_{;\alpha\rho} + g^{\sigma\alpha}\delta_1{}^\rho\delta_\sigma{}^1(I_3)_{;\alpha\rho} +$$
$$+ g^{\sigma\alpha}g^{1\rho}g_{\sigma\eta}(A_1{}^\eta)_{;\alpha\rho} + g^{\sigma\alpha}g_{1\sigma}g^{\rho\eta}(A_\eta{}^1)_{;\alpha\rho} + g^{\sigma\alpha}(A_\sigma{}^\rho)_{;\alpha\rho} +$$
$$+ g^{\sigma\alpha}(A_\sigma{}^1)_{;\alpha 1} + g^{1\alpha}(A_1{}^\rho)_{;\alpha\rho} + g^{\sigma\alpha}\delta_\sigma{}^\rho(A_1{}^1)_{;\alpha\rho} + g^{\sigma\alpha}(A_1{}^1)_{;\alpha\rho}A_\sigma{}^\rho +$$
$$+ g^{\sigma\alpha}A_1{}^1(A_\sigma{}^\rho)_{;\alpha\rho} + g^{\sigma\alpha}(A_1{}^\rho)_{;\alpha\rho}A_\sigma{}^1 + g^{\sigma\alpha}A_1{}^\rho(A_\sigma{}^1)_{;\alpha\rho} +$$
$$+ g^{\sigma\alpha}g^{\rho\eta}g_{\sigma\xi}(A_\eta{}^1)_{;\alpha\rho}A_1{}^\xi + g^{\sigma\alpha}g^{\rho\eta}g_{\sigma\xi}A_\eta{}^1(A_1{}^\xi)_{;\alpha\rho} \quad . \tag{1.36}$$

Because of (1.30) to (1.35), (1.36) reduces to

$$Z_1{}^{1\sigma\rho}{}_{;\sigma\rho} = g^{11}(I_1)_{;11} + (g^{11}I_{2;11} + g^{22}I_{2;22} + g^{33}I_{2;33} + g^{44}I_{2;44}) +$$
$$+ g^{11}(I_3)_{;11} + g^{11}(R_1{}^\eta)_{;\eta 1}f_1 + g^{\rho\eta}(R_\eta{}^1)_{;1\rho}f_2 + g^{\sigma\alpha}(R_\sigma{}^\rho)_{;\alpha\rho}f_3 +$$
$$+ g^{\sigma\alpha}(R_\sigma{}^1)_{;\alpha 1}f_4 + g^{11}(R_1{}^\rho)_{;1\rho}f_5 + g^{\rho\alpha}(R_1{}^1)_{;\alpha\rho}f_6 +$$
$$+ (g^{11}R_1{}^1{}_{;11}f_7 + g^{22}R_1{}^1{}_{;22}f_8 + g^{33}R_1{}^1{}_{;33}f_9 + g^{44}R_1{}^1{}_{;44}f_{10}) +$$
$$+ g^{\sigma\alpha}(R_\sigma{}^\rho)_{;\alpha\rho}f_{11} + g^{11}(R_1{}^\rho)_{;1\rho}f_{12} + g^{\sigma\alpha}(R_\sigma{}^1)_{;\alpha 1}f_{13} +$$
$$+ g^{\rho\eta}(R_\eta{}^1)_{;1\rho}f_{14} + g^{11}(R_1{}^\xi)_{;\xi 1}f_{15} \tag{1.37}$$

where the f's are functions of time alone. The right-hand side of (1.37) involves quantities of the following types in which the summation over dummy indices has been suspended.

$$\begin{array}{ll} g^{\mu\mu}(R_\mu{}^\mu)_{;\mu\mu} & \\ g^{\mu\mu}(R_\mu{}^\nu)_{;\nu\mu} & (\mu \neq \nu) \\ g^{\mu\mu}(R_\mu{}^\nu)_{;\mu\nu} & (\mu \neq \nu) \\ g^{\mu\mu}(R_\nu{}^\nu)_{;\mu\mu} & (\mu \neq \nu) \\ g^{\mu\mu}(I)_{;\mu\mu} & . \end{array} \tag{1.38}$$

Now using (1.30) it can be shown by direct substitution of the RW-metric that the quantities in (1.38) are functions of time alone. Consequently it follows that

$$Z_1{}^{1\sigma\rho}{}_{;\sigma\rho} = \text{a function of time alone} \quad . \tag{1.39}$$

Following the similar line of argument we can also show that

$$Z_4{}^{1\sigma\rho}{}_{;\sigma\rho} = 0 \quad . \tag{1.40}$$

Therefore from (1.18), (1.21), (1.25), (1.39) and (1.40), it follows that

$$\left.\begin{aligned} P_1{}^1 &= \text{a function of time alone} \\ P_4{}^1 &= 0 \quad . \end{aligned}\right\} \tag{1.41}$$

This proves that the RW-metric is compatible with the field equations (1.3).

§1.4 The Robertson-Walker Metric and the Invariants R^2 and $R_{\mu\nu}R^{\mu\nu}$

In this section we shall consider second order quadratic invariants as a particular example. It has been shown that there exist only two second order quadratic invariants R^2 and $R_{\mu\nu}R^{\mu\nu}$ which are of possible interest in formulating the field equations of gravitation. (See Bach [1], Lanczos [12].) Therefore in what follows we shall consider the field equations (1.3) with $P^{\mu\nu}$ derived from R^2 and $R_{\mu\nu}R^{\mu\nu}$ respectively and show that the RW-metric is compatible with these equations.

Case (i): The Invariant R^2

The tensor $P_\mu{}^\nu$ obtained from R^2 is

$$P_\mu{}^\nu = (Z_\mu{}^{\nu\sigma\rho})_{;\sigma\rho} - \tfrac{2}{3}Z^{\nu\alpha\sigma\rho}R_{\alpha\sigma\rho\mu} + \tfrac{1}{2}\delta_\mu{}^\nu K \tag{1.42}$$

where

$$K = R^2 \tag{1.43}$$

and

$$Z^{\mu\nu\sigma\rho} = (2g^{\mu\nu}g^{\sigma\rho} - g^{\mu\sigma}g^{\nu\rho} - g^{\nu\sigma}g^{\mu\rho})R \quad . \tag{1.44}$$

To obtain the necessary result, we know that we should prove

$$P_1{}^1 = \text{a function of time alone}$$

and

$$P_4{}^1 = 0 \quad ,$$

when the RW-metric is substituted in (1.42). First we note that for the RW-metric

$$R^2 = \text{a function of time alone} \quad . \tag{1.45}$$

Next by (1.44), for the RW-metric

$$Z^{1\alpha\sigma\rho}R_{\alpha\sigma\rho 1} = 3Rg^{11}g^{\sigma\rho}R_{1\sigma\rho 1} \quad . \tag{1.46}$$

Since the RW-metric is conformally flat, (1.46) becomes

$$Z^{1\alpha\sigma\rho}R_{\alpha\sigma\rho 1} = 3Rg^{11}g^{\sigma\rho}(\tfrac{1}{2}g_{\sigma\rho}R_{11} + \tfrac{1}{2}g_{11}R_{\sigma\rho} - \frac{1}{6}Rg_{\sigma\rho}g_{11})$$

$$= 6RR_1{}^1 - \tfrac{1}{2}R^2 \quad . \tag{1.47}$$

From (1.47) it follows, for the RW-metric, that

$$Z^{1\alpha\sigma\rho}R_{\alpha\sigma\rho 1} = \text{a function of time alone} \quad . \tag{1.48}$$

Following a similar procedure we can show that

$$Z^{1\alpha\sigma\rho}R_{\alpha\sigma\rho 4} = 0 \quad . \tag{1.49}$$

Next we consider $Z_1{}^{1\sigma\rho}{}_{;\sigma\rho}$. For the RW-metric, by (1.44) this becomes

$$Z_1{}^{1\sigma\rho}{}_{;\sigma\rho} = [g_{11}R(2g^{11}g^{\sigma\rho} - g^{1\sigma}g^{1\rho} - g^{1\sigma}g^{1\rho})]_{;\sigma\rho}$$

$$= 2(g^{22}R_{;22} + g^{33}R_{;33} + g^{44}R_{;44}) \quad . \tag{1.50}$$

We can easily see that for the RW-metric the right-hand member of (1.50) is a function of time alone. Therefore

$$Z_1{}^{1\sigma\rho}{}_{;\sigma\rho} = \text{a function of time alone} \quad . \tag{1.51}$$

Following a similar procedure we can show that

$$Z_4{}^{1\sigma\rho}{}_{;\sigma\rho} = 0 \quad . \tag{1.52}$$

Therefore from (1.42), (1.45), (1.48), (1.49), (1.51) and (1.52) it follows that

$$\begin{aligned} P_1{}^1 &= \text{a function of time alone} \\ P_4{}^1 &= 0 \quad . \end{aligned} \tag{1.53}$$

Hence the result follows.

<u>Case (ii)</u>: The Invariant $R_{\mu\nu}R^{\mu\nu}$

The tensor $P_\mu{}^\nu$ obtained from $R_{\mu\nu}R^{\mu\nu}$ is

$$P_\mu{}^\nu = (Z_\mu{}^{\nu\sigma\rho})_{;\sigma\rho} - \tfrac{2}{3}Z^{\nu\alpha\sigma\rho}R_{\alpha\sigma\rho\mu} + \tfrac{1}{2}\delta_\mu{}^\nu K \tag{1.54}$$

where

$$K = R_{\mu\nu}R^{\mu\nu} \tag{1.55}$$

and

$$Z^{\mu\nu\sigma\rho} = g^{\mu\nu}R^{\sigma\rho} + g^{\sigma\rho}R^{\mu\nu} - \tfrac{1}{2}g^{\mu\sigma}R^{\nu\rho} - \tfrac{1}{2}g^{\nu\sigma}R^{\mu\rho} - \tfrac{1}{2}g^{\mu\rho}R^{\nu\sigma} - \tfrac{1}{2}g^{\nu\rho}R^{\mu\sigma} \tag{1.56}$$

(see Buchdahl [5]). We require as before

$$P_1{}^1 = \text{a function of time}$$

and

$$P_4{}^1 = 0$$

when the RW-metric is substituted in (1.54). We note that, for the RW-metric,

$$\begin{aligned} R_{\mu\nu}R^{\mu\nu} &= (R_1{}^1)^2 + (R_2{}^2)^2 + (R_3{}^3)^2 + (R_4{}^4)^2 \\ &= \text{a function of time alone} \quad , \end{aligned} \tag{1.57}$$

and from (1.56),

$$\begin{aligned} Z^{1\alpha\sigma\rho}R_{\alpha\sigma\rho 1} &= (g^{1\alpha}R^{\sigma\rho} + g^{\sigma\rho}R^{1\alpha} - \tfrac{1}{2}g^{1\sigma}R^{\alpha\rho} - \tfrac{1}{2}g^{\alpha\sigma}R^{1\rho} - \tfrac{1}{2}g^{1\rho}R^{\alpha\sigma} - \tfrac{1}{2}g^{\alpha\rho}R^{1\sigma})R_{\alpha\sigma\rho 1} \\ &= g^{11}R^{\sigma\rho}R_{1\sigma\rho 1} + \ldots \end{aligned} \tag{1.58}$$

Considering the first term of the right-hand side of (1.58) for example and noting that the RW-metric is conformally flat we obtain

$$\begin{aligned} g^{11}R^{\sigma\rho}R_{1\sigma\rho 1} &= g^{11}R^{22}(\tfrac{1}{2}g_{11}R_{22} + \tfrac{1}{2}g_{22}R_{11} - \tfrac{1}{6}Rg_{11}g_{22}) + g^{11}R^{33}(\tfrac{1}{2}g_{11}R_{33} + \tfrac{1}{2}g_{33}R_{11} - \\ &\quad - \tfrac{1}{6}Rg_{11}g_{33}) + g^{11}R^{44}(\tfrac{1}{2}g_{11}R_{44} + \tfrac{1}{2}g_{44}R_{11} - \tfrac{1}{6}Rg_{11}g_{44}) \\ &= \tfrac{1}{2}(R_2{}^2)^2 + \tfrac{1}{2}R_1{}^1R_2{}^2 - \tfrac{1}{6}RR_2{}^2 + \tfrac{1}{2}(R_3{}^3)^2 + \tfrac{1}{2}R_1{}^1R_3{}^3 - \tfrac{1}{6}RR_3{}^3 + \\ &\quad + \tfrac{1}{2}(R_4{}^4)^2 + \tfrac{1}{2}R_1{}^1R_4{}^4 - \tfrac{1}{6}RR_4{}^4 \\ &= \text{a function of time alone} \quad . \end{aligned} \tag{1.59}$$

Similarly we can show that the remaining terms of the right-hand side of (1.58) are functions of time alone. Therefore it follows that

$$Z^{1\alpha\sigma\rho}R_{\alpha\sigma\rho 1} = \text{a function of time alone} \quad . \tag{1.60}$$

Following the procedure used above, we can show that

$$Z^{1\alpha\sigma\rho}R_{\alpha\sigma\rho 4} = 0 \quad . \tag{1.61}$$

Next consider $(Z_1{}^{1\sigma\rho})_{;\sigma\rho}$. For the RW-metric, by (1.56) this becomes

$$(Z_1{}^{1\sigma\rho})_{;\sigma\rho} = g^{\sigma\alpha}(R_\alpha{}^\rho)_{;\sigma\rho} + g^{\sigma\rho}(R_1{}^1)_{;\sigma\rho} - g^{\rho\alpha}(R_\alpha{}^1)_{;1\rho} - g^{\sigma\alpha}(R_\alpha{}^1)_{;\sigma 1} \quad . \tag{1.62}$$

Here the right-hand side involves terms of the following types in which the summation over dummy indices has been suspended:

$$\begin{array}{c} g^{\mu\mu}(R_\mu{}^\mu)_{;\mu\mu} \\ g^{\mu\mu}(R_\mu{}^\nu)_{;\nu\mu} \\ g^{\mu\mu}(R_\mu{}^\nu)_{;\mu\nu} \\ g^{\mu\mu}(R_\nu{}^\nu)_{;\mu\mu} \quad . \end{array} \tag{1.63}$$

It can be shown, by direct substitution of the RW-metric, that the terms in (1.63) are functions of time alone. Consequently

$$(Z_1{}^{1\sigma\rho})_{;\sigma\rho} = \text{a function of time alone} \quad . \tag{1.64}$$

Again using the same procedure, we can show that

$$(Z_4{}^{1\sigma\rho})_{;\sigma\rho} = 0 \quad . \tag{1.65}$$

Therefore (1.54), (1.57), (1.60), (1.61), (1.64) and (1.65) imply

$$\begin{aligned} P_1{}^1 &= \text{a function of time alone} \\ P_4{}^1 &= 0 \quad . \end{aligned} \tag{1.66}$$

Consequently the required result follows.

§2.1 The Robertson-Walker Metric and the Invariant $R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} - 2\Lambda$

In this section we shall study a few cosmological consequences of the field equations of gravitation deduced from the invariant $R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} - 2\Lambda$ where α , β and Λ are constants, in the context of the RW-metric.

By coupling the Hamiltonian derivative of the invariant $R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} - 2\Lambda$ to the stress-energy tensor of a perfect fluid distribution, the field equations of gravitation are:

$$\begin{aligned} &(R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}) + 2\alpha[R_{;\mu\nu} + RR_{\mu\nu} - g_{\mu\nu}(\Box R + \tfrac{1}{4}R^2)] + \\ &+ \beta[R_{;\mu\nu} - \Box R_{\mu\nu} + 2R_{\mu\xi\eta\nu}R_{\rho\sigma}g^{\xi\rho}g^{\eta\sigma} - \tfrac{1}{2}g_{\mu\nu}(\Box R + R_{\rho\sigma}R^{\rho\sigma})] = -\kappa T_{\mu\nu} \end{aligned} \tag{2.1}$$

where $\Box R = g^{\alpha\beta}R_{;\alpha\beta}$, κ is a coupling constant and the contravariant components $T^{\mu\nu}$ are given by (1.5). (See Buchdahl [5].) By virtue of the results obtained in Section 1.4 it follows that the RW-metric is compatible with the field equations (2.1). Using the RW-metric

$$ds^2 = dt^2 - \frac{S^2(t)}{\left(1 + \frac{k}{4} r^2\right)^2} (dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2) \tag{2.2}$$

where $k = 1$, 0 or -1 and $S(t)$ is an arbitrary function of time t and noting that the coordinate system is the co-moving coordinate system, from (2.1) we obtain the following expressions for the density ρ and pressure p, respectively,

$$\kappa\rho = 3\left(\frac{k + \dot{S}^2}{S^2}\right) + \frac{6(3\alpha + \beta)}{S^2}\left(2\dot{S}\,\dddot{S} - \ddot{S}^2 + \frac{2\dot{S}^2\ddot{S}}{S} - \frac{3\dot{S}^4}{S^2} - \frac{2k\dot{S}^2}{S^2} + \frac{k^2}{S^2}\right) - \Lambda$$

and

$$\begin{aligned} \kappa p = &-\left(\frac{k + \dot{S}^2 + 2S\ddot{S}}{S^2}\right) - \frac{2(3\alpha + \beta)}{S}\left(2\ddddot{S} + \frac{4\dot{S}\,\dddot{S}}{S} + \frac{3\ddot{S}^2}{S} - \frac{4k\ddot{S}}{S^2} - \right. \\ &\left. - \frac{12\dot{S}^2\ddot{S}}{S^2} + \frac{3\dot{S}^4}{S^3} + \frac{2k\dot{S}^2}{S^3} - \frac{k^2}{S^3}\right) + \Lambda \ . \end{aligned} \tag{2.3}$$

We infer from (2.3) that the Robertson-Walker cosmology is the same in the theories deduced from each of the invariants $R + \alpha R^2 - 2\Lambda$, $R + \beta R_{\mu\nu}R^{\mu\nu} - 2\Lambda$ and $R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} - 2\Lambda$. Next we infer that when $3\alpha = -\beta$, the theory based upon the invariant $R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} - 2\Lambda$ is equivalent to that based upon the invariant $R - 2\Lambda$ as far as the RW-cosmology is concerned and, in this sense the invariant $R + \alpha(R^2 - 3R_{\mu\nu}R^{\mu\nu})$ is equivalent to the invariant R of the theory of general relativity.

From the above considerations it follows that, from the point of view of the RW-cosmology, there exists only one interesting second order quadratic invariant, $R + \alpha R^2 - 2\Lambda$, where α and Λ are constants. Consequently in what follows we shall study this invariant.

§2.2 Robertson-Walker Type, Special Cosmological Solutions and the Invariants $R^2 - 2\Lambda$ and $R + \alpha R^2 - 2\Lambda$

In this section we shall merely state a few special RW-type cosmological solutions deduced from the field equations based upon the invariant $R + \alpha R^2 - 2\Lambda$. We shall divide our considerations into two separate cases in which the invariants used are $R^2 - 2\Lambda$ and $R + \alpha R^2 - 2\Lambda$ respectively. Moreover in each of these cases we shall divide the special cosmological solutions into two classes, of static and non-static solutions respectively. The solutions for which

$$\rho = \text{constant} > 0 \quad , \quad p = \text{constant} \geqq 0 \tag{2.4}$$

will be called static solutions and the solutions for which

$$\begin{aligned} &\rho = \text{a function of time} > 0 \\ \text{and} \quad &p = \text{a function of time} \geqq 0 \end{aligned} \tag{2.5}$$

will be called non-static solutions. The special non-static solutions which will be stated here are characterized by the equation of state

$$\rho(t) - 3p(t) = 0 \quad . \tag{2.6}$$

<u>Case (i)</u>: The Invariant $R^2 - 2\Lambda$

In this case the field equations become

$$2[R_{;\mu\nu} + RR_{\mu\nu} - g_{\mu\nu}(\Box R + \tfrac{1}{4}R^2)] + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu} \ . \tag{2.7}$$

The expressions for the macroscopic proper density and pressure, ρ and p respectively, corresponding to the field equations (2.7) are

$$\kappa\rho = \frac{18}{S^2}\left(2\dot{S}\,\dddot{S} - \ddot{S}^2 + \frac{2\dot{S}^2\ddot{S}}{S} - \frac{3\dot{S}^4}{S^2} - \frac{2k\dot{S}^2}{S^2} + \frac{k^2}{S^2}\right) - \Lambda$$

and (2.8)

$$\kappa p = -\frac{6}{S}\left(2\ddddot{S} + \frac{4\dot{S}\,\dddot{S}}{S} + \frac{3\ddot{S}^2}{S} - \frac{4k\ddot{S}}{S^2} - \frac{12\dot{S}^2\ddot{S}}{S^2} + \frac{3\dot{S}^4}{S^3} + \frac{2k\dot{S}^2}{S^3} - \frac{k^2}{S^3}\right) + \Lambda \,.$$

Some special static solutions corresponding to the equations (2.7) are as follows:

(i) $S = \text{constant} \;,\quad k = +1 \text{ or } -1$

$$\kappa\rho = \frac{18}{S^4} - \Lambda \;,\quad \kappa p = \frac{6}{S^4} + \Lambda \;.$$

If we assume that the radiant-energy has negligible density compared with the energy-density of matter and that the matter exerts no pressure, then we obtain

$$\Lambda = -\frac{6}{S^4} \;,\quad S = \text{constant} \;,\quad \kappa\rho = \frac{24}{S^4} \;,\quad p = 0 \;.$$

On the other hand if we assume that the model is filled only with radiation then we must put

$$\rho = 3p = \text{constant} > 0 \;.$$

Then we obtain

$$\Lambda = 0 \;,\quad S = \text{constant} \;,\quad \kappa\rho = 3\kappa p = \frac{18}{S^4} \;.$$

The case $k = +1$ corresponds to the static Einstein universe of general relativity while the case $k = -1$ gives a new possibility for the static universe.

$$\text{(ii)} \qquad S = e^{t/a} \quad (a = \text{constant}) \quad , \quad k = 0 \quad , \quad \Lambda = 0 \quad ,$$
$$\kappa\rho = \kappa p = 0 \quad .$$

Thus we obtain that approximation to the universe in which the effect of matter on the underlying metric is negligible. The nebulae are to be considered as the test-particles whose existence has no effect on the structure of the universe. The universe is empty and $\Lambda = 0$. This special solution is represented by the same metric as that of de-Sitter's universe, but the difference here is that $\Lambda = 0$ whereas in de-Sitter's solution Λ is positive.

$$\text{(iii)} \qquad S = a \cosh\left(\frac{t-\bar{t}}{a}\right) \quad , \quad (a, \bar{t} \equiv \text{constants}) \quad , \quad k = +1 \quad ,$$
$$\Lambda = 0 \quad , \quad \kappa\rho = \kappa p = 0 \quad .$$

$$\text{(iv)} \qquad S = a \sinh\left(\frac{t-\bar{t}}{a}\right) \quad , \quad (a, \bar{t} \equiv \text{constants}) \quad , \quad k = -1 \quad ,$$
$$\Lambda = 0 \quad , \quad \kappa\rho = \kappa p = 0 \quad .$$

Both of the special solutions (iii) and (iv) pass into the special solution (ii) as t tends to infinity. For a formal proof we put $t + a \log \frac{2\alpha}{a}$ in place of $t-\bar{t}$ and $\frac{r}{\alpha}$, $\frac{\theta}{\alpha}$, $\frac{\phi}{\alpha}$ in place of r, θ, ϕ respectively, α being a constant.

(v) $\quad S = b \sin\left(\frac{t-\bar{t}}{b}\right)$, $(b,\bar{t} \equiv \text{constants})$, $\quad k = -1$

$$\Lambda = 0 \quad , \quad \kappa\rho = \kappa p = 0 \quad .$$

(vi) $\quad S = at + b$, $(a,b \equiv \text{constants})$, $\quad k = \pm 1$

$$\Lambda = 0 \quad , \quad \kappa\rho = \kappa p = 0 \quad .$$

(vii) $\quad S = (at + b)^{\frac{1}{2}}$, $(a,b \equiv \text{constants})$, $\quad k = 0$

$$\Lambda = 0 \quad , \quad \kappa\rho = \kappa p = 0 \quad .$$

Next some special non-static solutions of equations (2.7) will be considered; because of the equation of state (2.6), from (2.8) we obtain

$$S^2\ddddot{S} + 3S\dot{S}\dddot{S} + S\ddot{S}^2 - 5\dot{S}^2\ddot{S} - 2k\ddot{S} - \frac{\Lambda S^3}{9} = 0 \quad . \tag{2.9}$$

We obtain the following special non-static solutions.

(i) $\quad S = at + b$, $(a,b \equiv \text{constants})$, $\quad k = \pm 1$

$$\Lambda = 0 \quad , \quad \kappa\rho = 3\kappa p = \frac{18(1 \pm 2a^2 - 3a^4)}{(at + b)^4}$$

where a is so chosen that $(1 \pm 2a^2 - 3a^4) > 0$ according as $k = \pm 1$.

(ii) $\quad S^2 = ae^{bt}(1 + ce^{-2bt})$, $(a,b,c \equiv \text{constants})$, $\quad k = 0$

$$\Lambda = 0 \quad , \quad \kappa\rho = 3\kappa p = \frac{-18b^4a^2c}{S^4} \quad , \quad (c < 0) \quad .$$

Case (ii): The Invariant $R + \alpha R^2 - 2\Lambda$

In this case the field equations become

$$(R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}) + 2\alpha[R_{;\mu\nu} + RR_{\mu\nu} - g_{\mu\nu}(\Box R + \tfrac{1}{4}R^2)] = -\kappa T_{\mu\nu} . \qquad (2.10)$$

The expressions for the macroscopic proper density and pressure, ρ and p respectively, deduced from (2.10) are

$$\kappa\rho = 3\left(\frac{k + \dot{S}^2}{S^2}\right) + \frac{18\alpha}{S^2}\left(2\dot{S}\dddot{S} - \ddot{S}^2 + \frac{2\dot{S}^2\ddot{S}}{S} - \frac{3\dot{S}^4}{S^2} - \frac{2k\dot{S}^2}{S^2} + \frac{k^2}{S^2}\right) - \Lambda$$

and (2.11)

$$\kappa p = -\left(\frac{k + \dot{S}^2 + 2S\ddot{S}}{S^2}\right) - \frac{6\alpha}{S}\left(2\ddddot{S} + \frac{4\dot{S}\dddot{S}}{S} + \frac{3\ddot{S}^2}{S} - \frac{4k\ddot{S}}{S^2} - \frac{12\dot{S}^2\ddot{S}}{S^2} + \right.$$
$$\left. + \frac{3\dot{S}^4}{S^3} + \frac{2k\dot{S}^2}{S^3} - \frac{k^2}{S^3}\right) + \Lambda \quad .$$

In the special solutions (i) to (v) of Case (i), if we keep $\Lambda \neq 0$ instead of zero, we obtain special static solutions for this case. These special static solutions are special static solutions of relativistic cosmology also. (See Robertson [14].) The special solution (vi) with $k = -1$ of Case (i) is a special static solution of the present case as well as that of the relativistic cosmology. (See Robertson [14].)

Whereas solution (vii) of Case (i) is a solution for the second invariant, (vi) of Case (i) is not. Next a special non-static solution of the field equations (2.10) will be considered. The equation of state (2.6), with (2.11) gives

$$(kS + S\dot{S}^2 + S^2\ddot{S}) + 6\alpha(S^2\ddddot{S} + 3S\dot{S}\dddot{S} + S\ddot{S}^2 - 5\dot{S}^2\ddot{S} - 2k\ddot{S} - \frac{\Lambda S^3}{9}) = 0 \quad . \qquad (2.12)$$

We obtain the following special non-static solution:

(i) $$S = (at + b)^{\frac{1}{2}} , \quad (a,b \equiv \text{constants}) , \quad k = 0$$

$$\Lambda = 0 , \quad \kappa\rho = 3\kappa p = \frac{3a^2}{4(at + b)^2} .$$

§3.1 <u>The Robertson-Walker Metric and the Invariant $\phi(R)$</u>

With a view to examining the possibility of the existence of field equations of gravitation which can admit cosmological solutions without any singularity, at any time, in this section we shall consider an invariant which is an arbitrary function of the scalar curvature R, say $\phi(R)$. By coupling the Hamiltonian derivative of $\phi(R)$ to the stress-energy tensor of a perfect fluid distribution, we obtain the following field equations:

$$(R_{;\mu}R_{;\nu} - g_{\mu\nu}R_{;\alpha}R^{;\alpha})\phi''' + (R_{;\mu\nu} - g_{\mu\nu}\Box R)\phi'' + R_{\mu\nu}\phi' - \tfrac{1}{2}g_{\mu\nu}\phi = -\kappa T_{\mu\nu} \tag{3.1}$$

where the overhead dash denotes ordinary differentiation with respect to R and κ is a coupling constant.

It follows from Section 1.3 that the RW-metric

$$ds^2 = dt^2 - \frac{S^2(t)}{\left(1 + \frac{k}{4}r^2\right)^2}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2) \tag{3.2}$$

is compatible with the field equations (3.1). Using the RW-metric (3.2), from (3.1) we obtain the following expressions for the density ρ and pressure p , respectively,

$$\kappa\rho = \frac{3\dot{S}\dot{R}}{S}\phi'' - \frac{3\ddot{S}}{S}\phi' + \tfrac{1}{2}\phi$$

$$\kappa p = -\dot{R}^2\phi''' - \left(\frac{2\dot{S}\dot{R}}{S} + \ddot{R}\right)\phi'' + \frac{(2k + 2\dot{S}^2 + S\ddot{S})}{S^2}\phi' - \tfrac{1}{2}\phi \tag{3.3}$$

where

$$R = \frac{6(k + S\ddot{S} + \dot{S}^2)}{S^2} \quad . \tag{3.4}$$

Hereinafter the overhead dot denotes ordinary differentiation with respect to time t .

§3.2 An Oscillating Robertson-Walker Metric and the Invariant $\phi(R)$

Now considering a simple case for which R is a constant, we shall show that there exist simple invariants from which one can deduce field equations which possess cosmological solutions free from singularity at any time, the equation of state being $\rho = 3p$. In this special case the field equations (3.1) reduce to

$$R_{\mu\nu}\phi' - \tfrac{1}{2}g_{\mu\nu}\phi = -\kappa T_{\mu\nu} \quad . \tag{3.5}$$

Suppose the fluid distribution in the universe to be such that the equation of state is

$$\rho(t) - 3p(t) = 0 \quad . \tag{3.6}$$

The expressions for the macroscopic proper density and pressure, ρ and p respectively, in this case reduce to

$$\kappa\rho = -\frac{3\ddot{S}}{S}\phi' + \tfrac{1}{2}\phi$$

$$\kappa p = \frac{(2k + 2\dot{S}^2 + S\ddot{S})}{S^2}\phi' - \tfrac{1}{2}\phi \quad . \tag{3.7}$$

Because of (3.6), on contracting the field equations (3.5) we obtain

$$R\phi' - 2\phi = 0 \tag{3.8}$$

which is an algebraic relation for the constant R . Now from (3.4), because R is a constant, we obtain

$$R = -3\omega^2 < 0 \tag{3.9}$$

where the constant ω is given by the relation

$$S^2 = A + a \sin \omega t = W \tag{3.10}$$

where $A = -2k/\omega^2$ and a is arbitrary. If we imagine that the fluid in the cosmological model oscillates between two non-zero radii, we must demand that the magnification factor $S^2(t)$ in (3.10) must oscillate between two non-zero positive values. Therefore we must have

$$A > |a| > 0 \quad .$$

So the curvature constant k can only have the value -1 . Therefore we obtain

$$A = \frac{2}{\omega^2} > |a| > 0 \quad . \tag{3.11}$$

Next since

$$R_{44} = \frac{3}{2}\frac{\ddot{W}}{W} - \frac{3}{4}\frac{\dot{W}^2}{W^2} \tag{3.12}$$

from (3.5) it follows that

$$\kappa\rho = -\frac{3}{4W^2}(2W\ddot{W} - \dot{W}^2)\phi' + \tfrac{1}{2}\phi \quad . \tag{3.13}$$

Substituting for W from (3.10), we obtain

$$\kappa\rho = \frac{(A^2 - a^2)}{2W^2}\phi = 3\kappa p \quad . \tag{3.14}$$

Since $A^2 > a^2$, to maintain the pressure and density as positive we have an additional inequality:

$$\phi > 0 \quad . \tag{3.15}$$

Thus we have shown that if the invariant $\phi(R)$ of the theory is suitably chosen so that the inequality (3.15) is satisfied and if the scalar curvature R is a constant then the field equations of gravitation of such a theory will admit a radiation-filled cosmological solution oscillating between two finite radii and free from singularity at any time. In what follows we shall consider a few simple examples.

Example (i): One can very easily verify that the invariants $\phi = R$ and $\phi = R - 2\Lambda$ where R and Λ are constants do not satisfy the necessary requirements.

Example (ii): Let $\phi(R) = R + \alpha R^2 - 2\Lambda$ where R, α and Λ are constants. Therefore from (3.8) and (3.9),

$$R = -3\omega^2 = 4\Lambda \quad . \tag{3.16}$$

Hence

$$\Lambda < 0 \tag{3.17}$$

and from (3.15) and (3.16) it follows that

$$1 + 8\alpha\Lambda < 0 \quad . \tag{3.18}$$

Therefore by (3.11) and (3.16),

$$12\alpha > A = \frac{2}{\omega^2} > |a| > 0 \quad . \tag{3.19}$$

Thus the necessary requirements are expressed by the relations (3.16), (3.18) and (3.19) and it is obvious that they can be satisfied.

Example (iii): Let $\phi(R) = \pm(R^2-\beta^2)^{\frac{1}{2}}$ where R and β are constants. Therefore by (3.8) and (3.9)

$$R = \pm\sqrt{2} \quad \beta = -3\omega^2 \quad . \tag{3.20}$$

For (3.15) to be satisfied

$$\beta > 0 \quad \text{if} \quad \phi(R) = +(R^2-\beta^2)^{\frac{1}{2}}$$

and

$$\beta < 0 \quad \text{if} \quad \phi(R) = -(R^2-\beta^2)^{\frac{1}{2}} \quad . \tag{3.21}$$

The necessary requirements are the relations (3.20) and (3.21) and it is obvious that they can be satisfied.

§3.3 The Difference Between the Field Equations of Section 3.2 and Those of General Relativity

We note that the field equations of gravitation considered in the above sections appear to be the same as the following equations of general relativity:

$$R_{\mu\nu} - \alpha g_{\mu\nu} = -\kappa T_{\mu\nu} \tag{3.22}$$

where α and κ are constants. However there is a difference between the equations of general relativity of the form (3.22) and those considered in this Chapter. As we know that the field equations of general relativity are derived from the invariant R or $R-2\Lambda$ where Λ is a constant. When R is also a constant these equations reduce in form to the equations (3.22). But as seen in Example (i) they do not admit a cosmological solution of the type considered, as invariant corresponding to these equations does not satisfy the requirement (3.15).

CHAPTER II

PROBLEM OF MOTION

§4.1 The Einstein-Infeld-Hoffmann Method and the Invariant $R + \frac{a^2}{6} R^2$

The problem of motion of a test-particle in general relativity was first attacked in 1927 by Einstein and Grommer who showed that the equations of motion of the test-particle follow from the field equations of the theory. Further it is a well-known fact that only in the case of a non-linear field theory, one can expect it to be possible to deduce the equations of motion of a system of interacting particles from the field equations of the theory. Because of this one can expect it to be possible to obtain the equations of motion of a system of interacting particles from the field equations of general relativity, which are non-linear. Following this idea, in 1938, using an ingenious method of approximation Einstein, Infeld and Hoffmann showed that the equations of motion of p interacting particles follow from the field equations of general relativity. (See Einstein and Grommer [7], Einstein and Infeld [8], Einstein, Infeld and Hoffmann [9].) By using the procedure similar to that of Einstein, Infeld and Hoffmann, Gregory has examined the problem of motion of a system of p interacting particles in a theory based on an invariant $R + \frac{a^2}{6} R^2$ where a^2 is a constant (see Gregory [10]). It has been shown that the equations of motion of such a system follow from the field-equations of the theory and, in the second approximation, they are such that if the inter-particle distances are sufficiently larger than the constant a then, provided a is positive, they reduce to the Newtonian equations of motion of such a system in classical mechanics, irrespective of the magnitude of a .

Now in what follows we shall first state the basic assumptions and procedure of the EIH-method of approximation which are also used in Gregory's work. In the procedure adopted by Einstein, Infeld and Hoffmann matter is regarded as consisting of a system of interacting point-particles. Each particle is represented by a singularity of the gravitational field along a time-like world-line. The gravitational field satisfies the empty field equations $R_{\mu\nu} = 0$ everywhere except where the point-particles of the system are situated and these equations determine the motion of such a system. The field-variables are supposed to be expandible in ascending powers of a small parameter $\lambda = 1/c$, c being the velocity of light. Thus one writes the metric-tensor $g_{\alpha\beta}$ in the form

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \quad , \quad h_{\alpha\beta} = \sum_{\ell=1}^{\infty} \lambda^{\ell} \underset{\ell}{h}_{\alpha\beta} \tag{4.1}$$

where $\underset{\ell}{h}_{\alpha\beta}$ is of order $\mathcal{O}$ in λ and $h_{\alpha\beta} \to 0$ as $r = (x^s x^s)^{\frac{1}{2}} \to \infty$. Moreover $\eta_{\alpha\beta} = \text{diag.}(1,-1,-1,-1)$.

Further the quantities $h^{*\alpha\beta}$ and $\gamma_{\alpha\beta}$ are defined by

$$g^{\alpha\beta} = \eta^{\alpha\beta} + h^{*\alpha\beta} \quad , \quad h^{*\alpha\beta} = \sum_{\ell=1}^{\infty} \lambda^{\ell} \underset{\ell}{h}^{*\alpha\beta} \tag{4.2}$$

$$(-g)^{-\frac{1}{2}} g_{\alpha\beta} = \eta_{\alpha\beta} + \gamma_{\alpha\beta} \quad , \quad \gamma_{\alpha\beta} = \sum_{\ell=1}^{\infty} \lambda^{\ell} \underset{\ell}{\gamma}_{\alpha\beta} \tag{4.3}$$

where $\underset{\ell}{h}^{*\alpha\beta}$ and $\underset{\ell}{\gamma}_{\alpha\beta}$ are of order $\mathcal{O}$ in λ . $h^{*\alpha\beta}$ and $\gamma_{\alpha\beta} \to 0$ as $r = (x^s x^s)^{\frac{1}{2}} \to \infty$ and $\eta^{\alpha\beta} = \eta_{\alpha\beta} = \text{diag}(1,-1,-1,-1)$. The reason for making the transition to the quantities $\gamma_{\alpha\beta}$ from the quantities $h_{\alpha\beta}$ is merely for the sake of simplicity of the calculations involved. The relations among

$h_{\alpha\beta}$, $h^{*\alpha\beta}$ and $\gamma_{\alpha\beta}$ are as follows:

$$(\eta_{\alpha\beta} + h_{\alpha\beta})(\eta^{\beta\sigma} + h^{*\beta\sigma}) = \delta_{\alpha}^{\ \sigma} \tag{4.4}$$

$$\gamma_{\alpha\beta} = h_{\alpha\beta} - \tfrac{1}{2}\eta_{\alpha\beta}\eta^{\sigma\rho}h_{\sigma\rho} + \text{non-linear terms} \tag{4.5}$$

$$h_{\alpha\beta} = \gamma_{\alpha\beta} - \tfrac{1}{2}\eta_{\alpha\beta}\eta^{\sigma\rho}\gamma_{\sigma\rho} + \text{non-linear terms} \quad . \tag{4.6}$$

Next the new temporal variable is introduced by the relation

$$\tau = \lambda x^0 \quad . \tag{4.7}$$

As one is supposed to be dealing with quasi-stationary fields, it is assumed that the derivatives of a quantity with respect to the time coordinate x^0 are small compared to the quantity itself or its derivatives with respect to the space coordinates x^i . In other words the quantity and its spatial derivatives are of the same order of magnitude as its derivative with respect to the auxiliary temporal variable τ . Differentiation with respect to (x^0,x^1,x^2,x^3) is denoted by a stroke while that with respect to (τ,x^1,x^2,x^3) is denoted by a comma. Thus the stroke-differentiation with respect to x^0 is equivalent to λ multiplied by the comma-differentiation with respect to τ . To express this explicitly we write

$$\phi_{\ldots|s} = \phi_{\ldots,s} \quad ; \quad \phi_{\ldots|0} = \lambda\phi_{\ldots,\underset{1}{0}} \quad . \tag{4.8}$$

Using the above procedure each function that appears in the field equations is developed in a power series in λ . Einstein, Infeld and Hoffmann expand $\gamma_{\mu\nu}$ as:

$$\gamma_{00} = \lambda^2 \underset{2}{\gamma}_{00} + \lambda^4 \underset{4}{\gamma}_{00} + \dots\;\dots$$
$$\gamma_{0m} = \lambda^3 \underset{3}{\gamma}_{0m} + \lambda^5 \underset{5}{\gamma}_{0m} + \dots\;\dots \tag{4.9}$$
$$\gamma_{mn} = \lambda^4 \underset{4}{\gamma}_{mn} + \lambda^6 \underset{6}{\gamma}_{mn} + \dots\;\dots$$

This expansion of $\gamma_{\mu\nu}$ is justified by them in the following heuristic way, as far as the starting terms of $\gamma_{\mu\nu}$ are concerned. Assuming the stress-energy tensor to be

$$T_{\mu\nu} = \rho \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \tag{4.10}$$

we have for a quasi-stationary field, approximately

$$\gamma_{00|ss} = -2\rho$$
$$\gamma_{0m|ss} = -2\rho \frac{dx^m}{d\tau} \lambda \tag{4.11}$$
$$\gamma_{mn|ss} = -2\rho \frac{dx^m}{d\tau} \cdot \frac{dx^n}{d\tau} \lambda^2 \cdot$$

Therefore it follows that

$$\gamma_{mn} \sim \lambda\gamma_{0m} \sim \lambda^2\gamma_{00} \tag{4.12}$$

and it is a matter of pure convention that one starts with power λ^2 for γ_{00}.

Similarly using the basic assumptions and procedure of the EIH-method, while dealing with the problem of motion in the theory based upon the invariant $R + \frac{a^2}{6} R^2$ where a^2 is a constant, Gregory assumes the following expansion of $\gamma_{\mu\nu}$:

$$
\begin{aligned}
\gamma_{00} &= \lambda^2 \underset{2}{\gamma}_{00} + \lambda^4 \underset{4}{\gamma}_{00} + \cdots \\
\gamma_{0m} &= \lambda^3 \underset{3}{\gamma}_{0m} + \lambda^5 \underset{5}{\gamma}_{0m} + \cdots \\
\gamma_{mn} &= \lambda^2 \underset{2}{\gamma}_{mn} + \lambda^4 \underset{4}{\gamma}_{mn} + \cdots
\end{aligned}
\tag{4.13}
$$

The reason for choosing the particular starting terms of this expansion of $\gamma_{\mu\nu}$ can also be justified in the same heuristic manner as that of Einstein, Infeld and Hoffmann, mentioned above. However, we note that this heuristic justification for the choice of the starting terms of the expansions of $\gamma_{\mu\nu}$ is obtained by using the non-empty field equations whereas only the empty field equations are used in dealing with the problem. Therefore it seems to be of interest to us to give a clearer derivation of the starting terms of the expansions (4.9) and (4.13). This derivation will be given in the next section. Before that we shall describe how, using the expansions (4.13) of $\gamma_{\mu\nu}$, Gregory has obtained the equations of motion of a system of p interacting particles from the field equations of the theory based upon the invariant $R + \frac{a^2}{6} R^2$.

In this case the empty field equations and the contracted equation obtained from them are respectively written as follows:

$$
\begin{aligned}
\Phi_{00} + 2\Lambda_{00} + 2 H_{00} &= \Phi_{00} + 2\Omega_{00} = 0 \\
\Phi_{0n} + 2\Lambda_{0n} + 2 H_{0n} &= \Phi_{0n} + 2\Omega_{0n} = 0 \\
\Phi_{mn} + 2\Lambda_{mn} + 2 H_{mn} &= \Phi_{mn} + 2\Omega_{mn} = 0
\end{aligned}
\tag{4.14}
$$

$$a^2 R_{|\ell\ell} - R = a^2 R_{|00} + a^2 R_{|\alpha}(\Gamma^\alpha_{\ell\ell} - \Gamma^\alpha_{00}) + a^2 h^{*\alpha\beta} R_{|\alpha\beta} - a^2 h^{*\alpha\beta} R_{|\sigma}\Gamma^\sigma_{\alpha\beta} \tag{4.15}$$

where

$$\begin{aligned} \Phi_{00} &= -\gamma_{00|ss} + \gamma_{\ell s|\ell s} \\ \Phi_{0n} &= -\gamma_{0n|ss} + \gamma_{0s|sn} \\ \Phi_{mn} &= -\gamma_{mn|ss} + \gamma_{ms|ns} + \gamma_{ns|ms} - \delta_{mn}\gamma_{\ell s|\ell s} \end{aligned} \tag{4.16}$$

$$\begin{aligned} 2\Lambda_{00} &= (L'_{00} + L'_{ss}) \\ 2\Lambda_{0n} &= \gamma_{ns|s0} - \gamma_{00|n0} + (2L'_{0n}) \\ 2\Lambda_{mn} &= -\gamma_{0m|0n} - \gamma_{0n|0m} + 2\delta_{mn}\gamma_{0s|0s} + \gamma_{mn|00} - \\ &\quad - \delta_{mn}\gamma_{00|00} + (2L'_{mn} - \delta_{mn}L'_{ss} + \delta_{mn}L'_{00}) \end{aligned} \tag{4.17}$$

$$\begin{aligned} 2\,\text{Ⓗ}_{00} &= \tfrac{1}{3}a^2 R_{|\ell\ell} + \tfrac{1}{3}R + \tfrac{1}{3}a^2 R_{|00} - \tfrac{1}{3}a^2(\Gamma^\alpha_{00} + \Gamma^\alpha_{\ell\ell})R_{|\alpha} + \frac{1}{6}a^2R^2 + \tfrac{1}{3}a^2(\Phi_{00} + 2\Lambda_{00})R - \\ &\quad - \tfrac{1}{3}\gamma_{00}R - \frac{1}{6}a^2\gamma_{00}R^2 \\ 2\,\text{Ⓗ}_{0n} &= \tfrac{2}{3}a^2 R_{|0n} - \tfrac{1}{3}\gamma_{0n}R - \tfrac{2}{3}a^2 R_{|\alpha}\Gamma^\alpha_{0n} - \frac{1}{6}a^2\gamma_{0n}R^2 + \tfrac{1}{3}a^2(\Phi_{0n} + 2\Lambda_{0n})R \\ 2\,\text{Ⓗ}_{mn} &= -\tfrac{1}{3}\delta_{mn}R - \tfrac{1}{3}a^2\delta_{mn}R_{|\ell\ell} + \tfrac{2}{3}a^2 R_{|mn} + \tfrac{1}{3}a^2\delta_{mn}R_{|00} - \tfrac{1}{3}\gamma_{mn}R - \\ &\quad - \tfrac{2}{3}a^2(\Gamma^\alpha_{mn} - \tfrac{1}{2}\delta_{mn}\Gamma^\alpha_{\ell\ell} + \tfrac{1}{2}\delta_{mn}\Gamma^\alpha_{00})R_{|\alpha} - \frac{1}{6}a^2\delta_{mn}R^2 + \tfrac{1}{3}a^2(\Phi_{mn} + 2\Lambda_{mn})R - \\ &\quad - \frac{1}{6}a^2\gamma_{mn}R^2 \end{aligned} \tag{4.18}$$

where $L'_{\mu\nu}$ represents the non-linear part of $R_{\mu\nu}$ and the scalar curvature R is explicitly given by

$$R = (-\tfrac{1}{2}h_{00|ss} + \tfrac{1}{2}h_{nn|ss} + 2h_{0s|0s} - h_{ss|00}) + (-\tfrac{1}{2}h^{*00}h_{00|ss} - \tfrac{1}{2}h^{*00}h_{ss|00} +$$
$$+ h^{*00}h_{0s|0s} - h^{*0n}h_{0n|ss} - \tfrac{1}{2}h^{*mn}h_{mn|ss} - h^{*mn}h_{0m|0n} + \tfrac{1}{2}h^{*mn}h_{mn|00} +$$
$$+ L'_{00} - L'_{nn}) + (h^{*00}L'_{00} + 2h^{*0n}L'_{0n} + h^{*mn}L'_{mn}) \quad . \tag{4.19}$$

Noting that Φ_{0n} and Φ_{mn} can be written as

$$\begin{aligned} \Phi_{0n} &= (-\gamma_{0n|\ell} + \gamma_{0\ell|n})|\ell \\ \Phi_{mn} &= (-\gamma_{mn|\ell} + \gamma_{m\ell|n} - \delta_{mn}\gamma_{\ell s|s} + \delta_{m\ell}\gamma_{ns|s})|\ell \end{aligned} \tag{4.20}$$

we obtain

$$\oint_S^k \Phi_{0n} \cos(x^n,N)\,dS = 0 \quad , \quad \oint_S^k \Phi_{mn} \cos(x^n,N)\,dS = 0 \tag{4.21}$$
$$(k = 1,2,\ldots,p)$$

where S is a closed surface not passing through any singularity but enclosing the kth singularity and (x^n,N) denotes an angle between the direction of x^n and the normal N to S. From (4.14) and (4.21) it follows that

$$\oint_S^k 2\Omega_{0n} \cos(x^n,N)\,dS = 0 \quad , \quad \oint_S^k 2\Omega_{mn} \cos(x^n,N)\,dS = 0 \tag{4.22}$$
$$(k = 1,2,\ldots,p) \quad .$$

Each of these integrals, being independent of the shape of the surface, can give a relation only between the coordinates of the singularity and their time derivatives, and therefore the $4p$ equations in (4.22) are called "the equations of motion of the p-particles". However if we directly apply the EIH-method of approximation to the field equations (4.14) and the equations of motion (4.22) we encounter difficulty because (4.22) gives 4p times infinite number of equations for only 3p unknown functions $\overset{k}{\xi}{}^{m}(\tau)$, the space coordinates of the p-particles. In order to avoid this difficulty, the field equations (4.14) are generalized and after the application of the approximation method, they are written as follows:

$$
\begin{aligned}
\underset{2\ell}{\Phi}{}_{00} + 2\underset{2\ell}{\Omega}{}_{00} &= -\sum_{k=1}^{p} \left(\underset{2\ell}{\overset{k}{C}}{}_{s} \overset{k}{V}{}_{|s} \right) \\
\underset{2\ell+1}{\Phi}{}_{0n} + 2\underset{2\ell+1}{\Omega}{}_{0n} &= -\sum_{k=1}^{p} \left(\underset{2\ell+1}{\overset{k}{C}}{}_{0} \overset{k}{V}{}_{|n} + \underset{2\ell}{\overset{k}{C}}{}_{n} \underset{1}{\overset{k}{V}}{}_{|0} + \underset{2\ell\ 1}{\overset{k}{C}}{}_{n|0} \overset{k}{V} \right) \\
\underset{2\ell}{\Phi}{}_{mn} + 2\underset{2\ell}{\Omega}{}_{mn} &= -\sum_{k=1}^{p} \left(\underset{2\ell}{\overset{k}{C}}{}_{m} \overset{k}{V}{}_{|n} + \underset{2\ell}{\overset{k}{C}}{}_{n} \overset{k}{V}{}_{|m} - \delta_{mn} \underset{2\ell}{\overset{k}{C}}{}_{s} \overset{k}{V}{}_{|s} \right)
\end{aligned}
\tag{4.23}
$$

$$(\ell = 1,2,\ldots,\infty)$$

where

$$
\begin{aligned}
&\overset{k}{V} = 1/\overset{k}{r} \ , \quad (\overset{k}{r})^2 = (x^s - \overset{k}{\xi}{}^{s})(x^s - \overset{k}{\xi}{}^{s}) \\
&\underset{2\ell+1}{\overset{k}{C}}{}_{0} = \underset{2\ell+1}{\overset{k}{C}}{}_{0}(\tau) \ , \quad \underset{2\ell}{\overset{k}{C}}{}_{m} = \underset{2\ell}{\overset{k}{C}}{}_{m}(\tau)
\end{aligned}
\tag{4.24}
$$

$$(\ell = 1,2,\ldots,\infty)$$

and $\overset{k}{\xi}{}^{s}(\tau)$ are the spatial coordinates of the kth singularity. From (4.23) we obtain the following 4p surface-integrals:

$$
\begin{aligned}
\frac{1}{4\pi} \oint_{S}^{k} 2\underset{2\ell+1}{\Omega}{}_{0n} \cos(x^n, N)\, dS &= \underset{2\ell+1}{\overset{k}{C}}{}_{0} - \tfrac{1}{3} \underset{2\ell}{\overset{k}{C}}{}_{s} \underset{1}{\overset{k}{\xi}}{}^{s}{}_{|0} \\
\frac{1}{4\pi} \oint_{S}^{k} 2\underset{2\ell}{\Omega}{}_{mn} \cos(x^n, N)\, dS &= \underset{2\ell}{\overset{k}{C}}{}_{m}
\end{aligned}
\tag{4.25}
$$

$$(\ell = 1,2,\ldots,\infty) \quad .$$

To solve the gravitational field equations (4.14) we must assume

$$\sum_{\ell=1}^{\infty} \lambda^{2\ell+1} \underset{2\ell+1}{\overset{k}{C}}{}_0(\tau) = 0 \; , \qquad \sum_{\ell=1}^{\infty} \lambda^{2\ell} \underset{2\ell}{\overset{k}{C}}{}_m = 0 \tag{4.26}$$

which represent $4p$ equations of motion.

One can show that it is possible to choose $\underset{2\ell+1}{\overset{k}{C}}{}_0$ equal to zero by the addition of simple harmonic functions to $\underset{2\ell}{\gamma}{}_{00}$ at each stage of approximation. (See Einstein and Infeld [8].) Moreover restricting the coordinate system by means of four non-tensor equations of the form

$$\begin{aligned} \underset{2\ell}{\gamma}{}_{ns|s} &= - \sum_{k=1}^{p} \underset{2\ell}{\overset{k}{C}}{}_n \overset{k}{V} \\ \underset{2\ell+1}{\gamma}{}_{0s|s} - \underset{2\ell}{\gamma}{}_{00|\underset{1}{0}} &= 0 \end{aligned} \tag{4.27}$$

we can write the field equations (4.23), the surface-integrals (4.25) and the equations of motion (4.26) in the following form:

$$\begin{aligned} \underset{2\ell}{\gamma}{}_{00|ss} &= 2\underset{2\ell}{\Omega}{}_{00} \\ \underset{2\ell+1}{\gamma}{}_{0n|ss} &= 2\underset{2\ell+1}{\Omega}{}_{0n} \\ \underset{2\ell}{\gamma}{}_{mn|ss} &= 2\underset{2\ell}{\Omega}{}_{mn} \end{aligned} \tag{4.28}$$

$$\frac{1}{4\pi}\oint_S^{k} 2\Omega_{\substack{mn\\2\ell}}\cos(x^n,N)\,dS = \overset{k}{C}_{\substack{m\\2\ell}}$$

$$\frac{1}{4\pi}\oint_S^{k} (2\Omega_{\substack{0n\\2\ell+1}} - \gamma_{\substack{ns|s0\\2\ell\;\;1}})\cos(x^n,N)\,dS = 0 \tag{4.29}$$

$$\sum_{\ell=1}^{\infty} \lambda^{2\ell} \overset{k}{C}_{\substack{m\\2\ell}}(\tau) = 0 \quad . \tag{4.30}$$

It is assumed that at every stage the equation (4.15) is satisfied. If we take successive approximations, then for the stage $\ell = q$ the approximate equations of motion are given by

$$\sum_{\ell=1}^{q} \lambda^{2\ell} \overset{k}{C}_{\substack{m\\2\ell}}(\tau) = 0 \quad . \tag{4.31}$$

Using (4.27), (4.28), (4.29) and (4.31) and following the same procedure as that of Einstein, Infeld and Hoffmann, Gregory has obtained the equations of motion in the approximations $\ell = 1$ and $\ell = 2$. In the case of $\ell = 1$, since $\overset{k}{C}_{\substack{m\\2}}$ vanishes identically, the motion is arbitrary. Gregory has obtained the following equations of motion for the approximation $\ell = 2$, calling it the Newtonian approximation,

$$\overset{k}{m}\,\overset{k}{\ddot{\xi}}{}^{n} + \sum_{j\neq k}^{p} (\overset{j}{m}\,\overset{k}{m}\,V_{jk} + \frac{a^2}{12}\, r_{0j}\, r_{0k}\, W_{jk})_{,n} = 0 \tag{4.32}$$

$$(k = 1,2,\ldots,p \;;\; n = 1,2,3)$$

where

$$V_{jk} = \frac{1}{r_{jk}} \quad , \quad W_{jk} = \frac{e^{-r_{jk}/a}}{r_{jk}}$$

$\overset{k}{\xi}$ = space coordinates of kth singularity

r_{jk} = distance between kth and jth singularities

$\overset{k}{m}$ = mass of kth singularity

r_{0k} = new constant of integration associated with kth singularity

and $(\ )_{,n}$ denotes partial differentiation with respect to $\overset{k}{\xi}{}^{n}$. However, Gregory has not explained the meaning of the constant r_{0k}. In the course of his work matter is represented by spherically symmetric singularities of the field. From the work of Pechlaner and Sexl it appears that r_{0k} should be proportional to the mass $\overset{k}{m}$. However, this work depends upon a particular definition of a point-source. From (4.32) it appears that if the inter-particle distances r_{jk} are sufficiently larger than the constant a and, if a is positive, then (4.32) reduces to the Newtonian equations of motion of such a system in classical mechanics, irrespective of the magnitude of the constant a.

§5.1 The Starting Terms of the Expansions of the Field-Quantities $\gamma_{\mu\nu}$ and the Invariants R and $R + \frac{a^2}{6} R^2$

In this section we shall discuss how the starting terms of the expansions (4.9) of $\gamma_{\mu\nu}$ can be obtained. Assuming that a test-particle, influenced only by the gravitational field, moves along a geodesic, we shall compare the equations of motion of the test-particle in the Newtonian theory of gravitation with those in general relativity. Using this procedure and applying the EIH-method to the empty field equations of general relativity, we shall derive the starting terms of the expansions (4.9) of $\gamma_{\mu\nu}$.

In the Newtonian theory of gravitation we have for the test-particle of mass Δm , influenced only by the gravitational potential $\phi(x^{\mu})$,

$$(-\Delta m)\ \delta \int_{\tau_1}^{\tau_2} (c^2 - \tfrac{1}{2}\dot{\xi}^s\dot{\xi}^s - \phi)d\tau = 0 \tag{5.1}$$

where $\dot{\xi}^s = d\xi^s/d\tau$ are the velocity components of the test-particle, $c\tau = \xi^0 = x^0$ is the time coordinate, $\xi^n = \xi^n(\tau)$ are the space coordinates of the test-particle at time τ and c is the velocity of light.

On the other hand the geodesic equations of motion of the test-particle in general relativity are

$$(-\Delta m)\ \delta \int_{\tau_1}^{\tau_2} \frac{ds}{d\tau}\, d\tau = 0 \tag{5.2}$$

where

$$ds^2 = (\eta_{\alpha\beta} + h_{\alpha\beta})d\xi^{\alpha}d\xi^{\beta} \tag{5.3}$$

and ξ^{α} have the same meaning as in (5.1) while $\eta_{\alpha\beta}$ and $h_{\alpha\beta}$ have the meaning as in Section 4.1. We note that $h_{\alpha\beta}$ are functions of x^{μ} alone. (See Infeld and Plebanski [11].)

Now from (5.3) we obtain

$$\left(\frac{ds}{d\tau}\right)^2 = c^2 - \dot{\xi}^s\dot{\xi}^s + c^2h_{00} + 2ch_{0n}\dot{\xi}^n + h_{mn}\dot{\xi}^m\dot{\xi}^n \quad . \tag{5.4}$$

Therefore from (5.2) we obtain

$$(-\Delta m)\delta \int_{\tau_1}^{\tau_2} (c^2 - \tfrac{1}{2}\dot{\xi}^s\dot{\xi}^s + \tfrac{1}{2}c^2h_{00} + ch_{0n}\dot{\xi}^n + \tfrac{1}{2}h_{mn}\dot{\xi}^m\dot{\xi}^n)d\tau = 0 \quad . \tag{5.5}$$

Comparing (5.1) with (5.5) and keeping in mind that ϕ and $h_{\mu\nu}$ are functions of x^{μ} alone, we infer that $\frac{1}{2}c^2 h_{00} \sim \lambda^0$, $ch_{0n}\dot{\xi}^n \sim \lambda$ and $h_{mn}\dot{\xi}^m\dot{\xi}^n \sim \lambda$. From this it follows that the $h_{\mu\nu}$ are expandible as

$$
\begin{aligned}
h_{00} &= \lambda^2 \underset{2}{h}_{00} + \lambda^3 \underset{3}{h}_{00} + \ldots\ldots \\
h_{0n} &= \lambda^2 \underset{2}{h}_{0n} + \lambda^3 \underset{3}{h}_{0n} + \ldots\ldots \\
h_{mn} &= \lambda \underset{1}{h}_{mn} + \lambda^2 \underset{2}{h}_{mn} + \ldots\ldots
\end{aligned}
\tag{5.6}
$$

Therefore by (4.5) and (5.6) it follows that the $\gamma_{\mu\nu}$ are expandible in the following manner:

$$
\begin{aligned}
\gamma_{00} &= \lambda \underset{1}{\gamma}_{00} + \lambda^2 \underset{2}{\gamma}_{00} + \ldots\ldots \\
\gamma_{0n} &= \lambda^2 \underset{2}{\gamma}_{0n} + \lambda^3 \underset{3}{\gamma}_{0n} + \ldots\ldots \\
\gamma_{mn} &= \lambda \underset{1}{\gamma}_{mn} + \lambda^2 \underset{2}{\gamma}_{mn} + \ldots\ldots
\end{aligned}
\tag{5.7}
$$

Now consider the empty field equations of general relativity and apply to them the EIH-method of successive approximations, using the expansions (5.7) of $\gamma_{\mu\nu}$ instead of (4.9). Then we can easily see that the coordinate conditions, the field equations, the surface-integrals and the equations of motion at the stage $\ell = q$ can, respectively, be written as follows:

$$
\begin{aligned}
\underset{\ell}{\gamma}_{ms|s} &= - \sum_{k=1}^{p} \underset{\ell}{\overset{k}{C}}_m \overset{k}{V} \\
\underset{\ell+1}{\gamma}_{0s|s} - \underset{\ell}{\gamma}_{00|\underset{1}{0}} &= 0
\end{aligned}
\tag{5.8}
$$

$$\begin{aligned} \underset{\ell}{\gamma}_{00|ss} &= 2\underset{\ell}{\Lambda}_{00} \\ \underset{\ell+1}{\gamma}_{0n|ss} &= 2\underset{\ell+1}{\Lambda}_{0n} \\ \underset{\ell}{\gamma}_{mn|ss} &= 2\underset{\ell}{\Lambda}_{mn} \end{aligned} \tag{5.9}$$

$$\begin{aligned} \frac{1}{4\pi}\oint_S^k 2\underset{\ell}{\Lambda}_{mn}\cos(x^n,N)\,dS &= \underset{\ell}{\overset{k}{C}}_m \\ \frac{1}{4\pi}\oint_S^k \left(2\underset{\ell+1}{\Lambda}_{0n} - \underset{\ell}{\gamma}_{ns|s\underset{1}{0}}\right)\cos(x^n,N)\,dS &= 0 \end{aligned} \tag{5.10}$$

$$\sum_{\ell=1}^{q} \lambda^{\ell}\, \underset{\ell}{\overset{k}{C}}_m(\tau) = 0 \tag{5.11}$$

At the stage $\ell = 1$, using (5.7) from (4.17) we obtain $2\underset{1}{\Lambda}_{mn} = 0$. Therefore by (5.10) $\underset{1}{\overset{k}{C}}_m$ vanishes identically. Consequently [1] the coordinate conditions (5.8) and the field equations (5.9), at this stage, become

$$\underset{1}{\gamma}_{ms|s} = 0 \quad , \quad \underset{2}{\gamma}_{0s|s} - \underset{1}{\gamma}_{00|\underset{1}{0}} = 0 \tag{5.12}$$

$$\begin{aligned} \underset{1}{\gamma}_{00|ss} = 0 \quad &, \quad \underset{1}{\gamma}_{mn|ss} = 0 \\ \underset{2}{\gamma}_{0n|ss} &= 0 \quad . \end{aligned} \tag{5.13}$$

Therefore

$$\underset{1}{\gamma}_{mn} = 0 \quad . \tag{5.14}$$

Next considering the relation (4.6) and taking the first approximation we obtain

$$2\underset{1}{h}_{00} = \underset{1}{\gamma}_{00} + \underset{1}{\gamma}_{\ell\ell} \quad . \tag{5.15}$$

Therefore by (5.6) and (5.15),

$$\underset{1}{\gamma}_{00} = 0 \quad . \tag{5.16}$$

From this, (5.12) and (5.13), it follows that

$$\underset{2}{\gamma}_{0s} = 0 \quad . \tag{5.17}$$

Consequently the expansions of the $\gamma_{\mu\nu}$ are:

$$\begin{aligned}
\gamma_{00} &= \lambda^{2}\underset{2}{\gamma}_{00} + \lambda^{3}\underset{3}{\gamma}_{00} + \ldots\ldots\ldots \\
\gamma_{0n} &= \lambda^{3}\underset{3}{\gamma}_{0n} + \lambda^{4}\underset{4}{\gamma}_{0n} + \ldots\ldots \\
\gamma_{mn} &= \lambda^{2}\underset{2}{\gamma}_{mn} + \lambda^{3}\underset{3}{\gamma}_{mn} + \ldots\ldots\ldots
\end{aligned} \tag{5.18}$$

Using (5.18), from (4.17) we obtain $2\underset{2}{\Lambda}_{mn} = 0$. Therefore by (5.10) $\overset{k}{\underset{2}{C}}{}_{m}$ vanishes identically. Consequently the coordinate conditions (5.8) and the field equations (5.9), at this stage, become

$$\underset{2}{\gamma}_{ms|s} = 0 \ , \quad \underset{3}{\gamma}_{0s|s} - \underset{2}{\gamma}_{00|\underset{1}{0}} = 0 \tag{5.19}$$

$$\begin{gathered}
\underset{2}{\gamma}_{00|ss} = 0 \ , \quad \underset{3}{\gamma}_{0n|ss} = 0 \\
\underset{2}{\gamma}_{mn|ss} = 0 \quad .
\end{gathered} \tag{5.20}$$

Hence

$$\underset{2}{\gamma}_{mn} = 0 \quad . \tag{5.21}$$

Again from the relation (4.6), at this stage we obtain

$$2\underset{2}{h}_{00} = \underset{2}{\gamma}_{00} + \underset{2}{\gamma}_{\ell\ell} \quad . \tag{5.22}$$

By (5.21), therefore

$$2\underset{2}{h}_{00} = \underset{2}{\gamma}_{00} \tag{5.23}$$

which is not necessarily equal to zero. Therefore it follows from (5.19) and (5.20) that $\underset{3}{\gamma}_{0n}$ is also not necessarily equal to zero. Consequently the expansions of the $\gamma_{\mu\nu}$ are:

$$\begin{aligned} \gamma_{00} &= \lambda^2 \underset{2}{\gamma}_{00} + \lambda^3 \underset{3}{\gamma}_{00} + \ldots\ldots \\ \gamma_{0n} &= \qquad\quad \lambda^3 \underset{3}{\gamma}_{0n} + \lambda^4 \underset{4}{\gamma}_{0n} + \ldots\ldots \\ \gamma_{mn} &= \qquad\quad \lambda^3 \underset{3}{\gamma}_{mn} + \lambda^4 \underset{4}{\gamma}_{mn} + \ldots\ldots \end{aligned} \tag{5.24}$$

Using (5.24), from (4.17) we obtain $2\underset{3}{\Lambda}_{mn} = 0$. Therefore from (5.10) it follows that $\underset{3}{C}{}^{k}_{m}$ vanishes identically. Consequently the coordinate conditions (5.8) and the field equations (5.9), at this stage become:

$$\underset{3}{\gamma}_{ms|s} = 0 \ , \quad \underset{4}{\gamma}_{0s|s} - \underset{3}{\gamma}_{00|\underset{1}{0}} = 0 \tag{5.25}$$

$$\underset{3}{\gamma}_{00|ss} = 0 \ , \quad \underset{4}{\gamma}_{0n|ss} = 0 \ , \quad \underset{3}{\gamma}_{mn|ss} = 0 \quad . \tag{5.26}$$

It follows that

$$\underset{3}{\gamma}_{mn} = 0 \quad . \tag{5.27}$$

Consequently the $\gamma_{\mu\nu}$ can be expanded in the following manner:

$$\begin{aligned}
\gamma_{00} &= \lambda^2 \underset{2}{\gamma}_{00} + \lambda^3 \underset{3}{\gamma}_{00} + \dots\dots\dots\dots \\
\gamma_{0n} &= \qquad\quad \lambda^3 \underset{3}{\gamma}_{0n} + \lambda^4 \underset{4}{\gamma}_{0n} + \dots\dots\dots \\
\gamma_{mn} &= \qquad\qquad\qquad \lambda^4 \underset{4}{\gamma}_{mn} + \lambda^5 \underset{5}{\gamma}_{mn} + \dots\dots
\end{aligned} \tag{5.28}$$

Using (5.28), from (4.17) we note that

$$2\underset{4}{\Lambda}_{mn} \neq 0 \quad . \tag{5.29}$$

Consequently the field equations (5.9) imply

$$\underset{4}{\gamma}_{mn|ss} = 2\underset{4}{\Lambda}_{mn} \neq 0 \tag{5.30}$$

and therefore we cannot conclude that $\underset{4}{\gamma}_{mn}$ vanishes. Consequently the required result follows.

Now, we shall discuss how the starting terms of the expansions (4.13) of $\gamma_{\mu\nu}$ can be obtained. Following the procedure of the first part of this Section ~~5.1~~, first we find that $\gamma_{\mu\nu}$ should be expanded in the manner indicated in (5.7). Now consider the empty field equations for this case and apply to them the EIH-method, using the expansions (5.7) of $\gamma_{\mu\nu}$ instead of (4.13). Then we can easily see that the coordinate conditions, the field equations, the surface integrals and the equations of motion, at the stage $\ell = q$, can respectively be written as follows:

$$\underset{\ell}{\gamma}_{ns|s} = -\sum_{k=1}^{p} \underset{\ell}{\overset{k}{C}}_{n} \overset{k}{V} \tag{5.31}$$

$$\underset{\ell+1}{\gamma}_{0s|s} - \underset{\ell}{\gamma}_{00|\underset{1}{0}} = 0$$

$$\underset{\ell}{\gamma}_{00|ss} = 2\underset{\ell}{\Omega}_{00}$$

$$\underset{\ell+1}{\gamma}_{0n|ss} = 2\underset{\ell+1}{\Omega}_{0n} \tag{5.32}$$

$$\underset{\ell}{\gamma}_{mn|ss} = 2\underset{\ell}{\Omega}_{mn}$$

$$\frac{1}{4\pi}\oint_{S}^{k} 2\underset{\ell}{\Omega}_{mn} \cos(x^{n},N)\,dS = \underset{\ell}{\overset{k}{C}}_{m}$$

$$\frac{1}{4\pi}\oint_{S}^{k} (2\underset{\ell+1}{\Omega}_{0n} - \underset{\ell}{\gamma}_{ns|s\underset{1}{0}})\cos(x^{n},N)\,dS = 0 \tag{5.33}$$

$$\sum_{\ell=1}^{q} \lambda^{\ell} \underset{\ell}{\overset{k}{C}}_{m}(\tau) = 0 \quad . \tag{5.34}$$

Besides these equations it is assumed that at every stage the equation (4.15) is also satisfied.

Now using (4.4) and (4.6) we obtain

$$\underset{1}{h}^{*00} = \underset{1}{h}_{00} = 0 \quad , \quad \underset{1}{h}^{*0m} = \underset{1}{h}_{0m} = 0 \quad , \quad \underset{1}{h}^{*mn} = -\underset{1}{h}_{mn} \neq 0$$

$$\underset{2}{h}^{*00} = -\underset{2}{h}_{00} \neq 0 \; , \quad \underset{2}{h}^{*0m} = \underset{2}{h}_{0m} \neq 0 \quad , \quad \underset{2}{h}^{*mn} \neq 0 \quad , \quad \underset{2}{h}_{mn} \neq 0 \tag{5.35}$$

$$\underset{3}{h}^{*00} = -\underset{3}{h}_{00} \neq 0 \quad , \quad \underset{3}{h}^{*0m} \neq 0 \quad , \quad \underset{3}{h}_{0m} \neq 0$$

$$\underset{1}{h}_{00} = \tfrac{1}{2}(\underset{1}{\gamma}_{00} + \underset{1}{\gamma}_{\ell\ell}) \quad , \quad \underset{1}{h}_{0m} = \underset{1}{\gamma}_{0m} \tag{5.36}$$

$$\underset{1}{h}_{mn} = \underset{1}{\gamma}_{mn} + \tfrac{1}{2}\delta_{mn}(\underset{1}{\gamma}_{00} - \underset{1}{\gamma}_{\ell\ell}) \quad .$$

By (5.6) and (5.35) it follows that $h^{*\mu\nu}$ are expandible as

$$h^{*00} = \lambda^2 \underset{2}{h}^{*00} + \lambda^3 \underset{3}{h}^{*00} + \ldots \ldots$$

$$h^{*0m} = \lambda^2 \underset{2}{h}^{*0m} + \lambda^3 \underset{3}{h}^{*0m} + \ldots \ldots \tag{5.37}$$

$$h^{*mn} = \lambda \underset{1}{h}^{*mn} + \lambda^2 \underset{2}{h}^{*mn} + \ldots \ldots$$

At the stage $\ell = 1$, because of (4.16), (4.17), (4.18) and (4.19), by using (5.6), (5.7) and (5.37) we obtain

$$2\underset{1}{\Lambda}_{mn} = 0 \quad , \quad 2\,\underset{1}{\textcircled{H}}_{mn} = 0 \quad . \tag{5.38}$$

Therefore by (5.33) it follows that $\underset{1}{C}^{k}_{m}$ vanishes identically. Consequently the coordinate conditions (5.31) become

$$\underset{1}{\gamma}_{ms|s} = 0 \quad , \quad \underset{2}{\gamma}_{0s|s} - \underset{1}{\gamma}_{00|\underset{1}{0}} = 0 \quad . \tag{5.39}$$

Further from (4.18), because of (4.16), (4.17), (4.19), (5.6), (5.7) and (5.37), we obtain

$$2\,\underset{1}{\textcircled{H}}_{00} = 0 \quad , \quad 2\,\underset{1}{\textcircled{H}}_{0m} = 0 \quad , \quad 2\,\underset{1}{\textcircled{H}}_{mn} = 0 \tag{5.40}$$

$$2\,\underset{2}{\textcircled{H}}_{0m} = 0 \quad .$$

Thus the field equations (5.32) become

$$\underset{1}{\gamma}_{00|ss} = 0 \quad , \quad \underset{1}{\gamma}_{mn|ss} = 0 \quad , \quad \underset{2}{\gamma}_{0n|ss} = 0 \quad . \tag{5.41}$$

From (5.39) and (5.41), we obtain

$$\underset{1}{\gamma}_{mn} = 0 \quad . \tag{5.42}$$

Now because of (5.36) and (5.42)

$$2\underset{1}{h}_{00} = \underset{1}{\gamma}_{00} \quad . \tag{5.43}$$

Therefore by (5.6)

$$\underset{1}{\gamma}_{00} = 0 \tag{5.44}$$

and hence by (5.39) and (5.41),

$$\underset{2}{\gamma}_{0m} = 0 \quad . \tag{5.45}$$

Consequently the $\gamma_{\mu\nu}$ should be expanded in the following manner:

$$\begin{aligned}
\gamma_{00} &= \lambda^2 \underset{2}{\gamma}_{00} + \lambda^3 \underset{3}{\gamma}_{00} + \ldots \ldots \ldots \\
\gamma_{0m} &= \qquad\quad \lambda^3 \underset{3}{\gamma}_{0m} + \lambda^4 \underset{4}{\gamma}_{0m} + \ldots \ldots \ldots \\
\gamma_{mn} &= \lambda^2 \underset{2}{\gamma}_{mn} + \lambda^3 \underset{3}{\gamma}_{mn} + \ldots \ldots \ldots
\end{aligned} \tag{5.46}$$

Now because of (5.46), at the stage $\ell = 2$, (4.17) implies

$$\underset{2}{2\Lambda_{mn}} = 0 \quad , \tag{5.47}$$

and since $\underset{1}{R} = 0$, (4.18) implies

$$2\underset{2}{H_{mn}} = -\tfrac{1}{3}\delta_{mn}\underset{2}{R} - \frac{a^2}{3}\delta_{mn}\underset{2}{R}_{|\ell\ell} + \frac{2a^2}{3}\underset{2}{R}_{|mn}$$

which, due to (4.15), reduces to

$$2\underset{2}{H_{mn}} = -\tfrac{2}{3}\delta_{mn}\underset{2}{R} + \frac{2a^2}{3}\underset{2}{R}_{|mn} \tag{5.48}$$

where

$$\begin{aligned} \underset{2}{R} &= \tfrac{1}{2}(\underset{2}{h_{\ell\ell}} - \underset{2}{h_{00}})_{|ss} \\ &= \tfrac{1}{2}(\underset{2}{\gamma_{00}} - \underset{2}{\gamma_{\ell\ell}})_{|ss} \quad . \end{aligned} \tag{5.49}$$

We note that in general $\underset{2}{R}$ does not vanish and hence in general $2\underset{2}{H_{mn}}$ does not vanish. Now the field equations (5.32) become

$$\begin{aligned} \underset{2}{\gamma_{00|ss}} &= \tfrac{2}{3}\underset{2}{R} \\ \underset{3}{\gamma_{0n|ss}} &= \frac{2a^2}{3}\underset{2}{R}_{|\underset{1}{0n}} \\ \underset{2}{\gamma_{mn|ss}} &= -\tfrac{2}{3}\delta_{mn}\underset{2}{R} + \frac{2a^2}{3}\underset{2}{R}_{|mn} \quad . \end{aligned} \tag{5.50}$$

Since $\underset{2}{R}$ does not vanish, from (5.50) we cannot infer that $\underset{2}{\gamma_{00}}$, $\underset{3}{\gamma_{0n}}$ or $\underset{2}{\gamma_{mn}}$ should vanish. Therefore we should start the expansions of $\gamma_{\mu\nu}$ in the manner indicated in (5.46).

CHAPTER III

LINEARIZED EMPTY SPACE FIELD EQUATIONS AND THE INVARIANT $R + \frac{a^2}{6} R^2$

§6.1 The Static, Spherically Symmetric Solution in Rectangular Harmonic Coordinates

In this chapter we shall consider the field of a sss-particle in its surrounding region, using the theory based upon the invariant $R + \frac{a^2}{6} R^2$, a^2 being constant. Because the field of such particle in the region around it can be described by the empty field equations, we shall here be concerned with only such equations. Moreover, since the exact empty field equations are mathematically too complicated to be solved, we shall suppose that the field of the particle is weak so that one can ignore the products, squares and higher powers of the field-quantities and consequently linearize the field equations.

Because the case of a static particle is a special case of the slowly moving particle, it follows that the procedure of Chapter II can be applied to linearize the field equations. Thus in the notation of Chapter II, the linear approximation means omitting the terms of order higher than that of $\lambda (= 1/c^2)$, from the field variables. Because of this, we put

$$\gamma_{00} = \underset{2}{\gamma_{00}} \ , \quad \gamma_{0m} = \underset{2}{\gamma_{0m}} = 0 \ , \quad \gamma_{mn} = \underset{2}{\gamma_{mn}} \tag{6.1}$$

where $\underset{2}{\gamma_{00}}$ and $\underset{2}{\gamma_{mn}}$ are functions of the space coordinates only. Moreover, it follows that the empty field equations, the contracted field equations and the harmonic coordinate conditions respectively, after linearization, become

$$\underset{2}{\gamma}_{00|ss} = \frac{a^2}{3}\underset{2}{R}_{|ss} + \tfrac{1}{3}\underset{2}{R}$$

$$\underset{2}{\gamma}_{mn|ss} = -\tfrac{1}{3}\delta_{mn}\underset{2}{R} - \frac{a^2}{3}\delta_{mn}\underset{2}{R}_{|ss} + \frac{2a^2}{3}\underset{2}{R}_{|mn} \tag{6.2}$$

$$a^2\underset{2}{R}_{|ss} - \underset{2}{R} = 0 \tag{6.3}$$

$$\underset{2}{\gamma}_{mn|n} = 0 \tag{6.4}$$

where

$$\underset{2}{R} = \tfrac{1}{2}(\underset{2}{h}_{\ell\ell} - \underset{2}{h}_{00})_{|ss}$$

$$= \tfrac{1}{2}(\underset{2}{\gamma}_{00} - \underset{2}{\gamma}_{\ell\ell})_{|ss} \quad . \tag{6.5}$$

Using the symbol Δ to denote the operator "$|ss$" and putting $(\underset{2}{h}_{00} - \underset{2}{h}_{\ell\ell}) = \underset{2}{h}$ and $(\underset{2}{\gamma}_{00} - \underset{2}{\gamma}_{\ell\ell}) = \underset{2}{\gamma}$ we can write the equations (6.2) to (6.5) in the following form:

$$\Delta\underset{2}{\gamma}_{00} - \frac{2a^2}{3}\Delta\underset{2}{R} = 0$$

$$\Delta\underset{2}{\gamma}_{sn} + \frac{2a^2}{3}\delta_{sn}\Delta\underset{2}{R} - \frac{2a^2}{3}\underset{2}{R}_{|sn} = 0 \tag{6.6}$$

$$a^2\Delta\underset{2}{R} - \underset{2}{R} = 0 \tag{6.7}$$

$$\underset{2}{\gamma}_{sn|n} = 0 \tag{6.8}$$

$$\underset{2}{R} = \tfrac{1}{2}\Delta\underset{2}{\gamma} = -\tfrac{1}{2}\Delta\underset{2}{h} \quad . \tag{6.9}$$

Therefore, to obtain the solution of the linearized empty field equations, we should solve the equations (6.6) subject to the conditions (6.7), (6.8) and (6.9). However, we shall be interested only in the spherically symmetric and asymptotically flat solution of these equations. Because of spherical symmetry Δ reduces to

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$$

where $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$. The required type of solution of the equation (6.7) is immediately seen to be

$$\underset{2}{R} = A.\,\frac{e^{-r/a}}{r} \tag{6.10}$$

where A is a constant of integration. Then solving the first equation in (6.6), we obtain

$$\underset{2}{\gamma_{00}} = \frac{2a^2}{3}\underset{2}{R} + \frac{B}{r} \tag{6.11}$$

where B is a constant of integration. Solving the second equation in (6.6) we obtain

$$\underset{2}{\gamma_{sn}} = -\delta_{sn}\cdot\frac{2a^2}{3}\underset{2}{R} + \frac{2a^4}{3}\underset{2}{R}_{|sn} + \frac{\alpha_{sn}}{r} + \beta_{sn}\left(\frac{1}{r}\right)_{|sn} \tag{6.12}$$

where $\alpha_{sn} = \alpha_{ns}$ and $\beta_{sn} = \beta_{ns}$ are constants of integration and the summation over ~~dummy~~ repeated indices has been dropped.

Now because of the coordinate conditions (6.8), (6.12) gives

$$-\frac{2a^2}{3}\underset{2}{R}_{|s} + \sum_{n=1}^{3}\frac{2a^4}{3}\underset{2}{R}_{|snn} + \sum_{n=1}^{3}\left(\frac{\alpha_{sn}}{r}\right)_{|n} + \sum_{n=1}^{3}\beta_{sn}\left(\frac{1}{r}\right)_{|snn} = 0 \tag{6.13}$$

where there is no sum on s . By (6.7), (6.13) reduces to

$$\sum_{n=1}^{3} \left(\frac{\alpha_{sn}}{r} \right)_{|n} + \sum_{n=1}^{3} \beta_{sn} \left(\frac{1}{r} \right)_{|snn} = 0 \tag{6.14}$$

where there is no sum on s . Writing explicitly, (6.14) gives

$$\left(\frac{\alpha_{s1}}{r} \right)_{|1} + \left(\frac{\alpha_{s2}}{r} \right)_{|2} + \left(\frac{\alpha_{s3}}{r} \right)_{|3} + \beta_{s1} \left(\frac{1}{r} \right)_{|s11} + \beta_{s2} \left(\frac{1}{r} \right)_{|s22} + \beta_{s3} \left(\frac{1}{r} \right)_{|s33} = 0 \tag{6.15}$$

where there is no sum on s .

(6.15) requires

$$\begin{aligned} \alpha_{sn} &= 0 \\ \beta_{s1} = \beta_{s2} = \beta_{s3} &= \beta_s \quad \text{say} \quad . \end{aligned} \tag{6.16}$$

Therefore (6.12) now becomes

$$\underset{2}{\gamma}_{sn} = -\delta_{sn} \frac{2a^2}{3} \underset{2}{R} + \frac{2a^4}{3} \underset{2}{R}_{|sn} + \beta_s \left(\frac{1}{r} \right)_{|sn} \tag{6.17}$$

where the summation over s has been dropped.

Since we require that $\gamma_{\mu\nu}$ be symmetric in μ and ν , from (6.17) it follows that

$$\beta_1 = \beta_2 = \beta_3 = C \quad \text{say} \quad . \tag{6.18}$$

Therefore (6.17) reduces to

$$\underset{2}{\gamma}_{sn} = -\delta_{sn}\cdot\frac{2a^2}{3}\underset{2}{R} + \frac{2a^4}{3}\underset{2}{R}_{|sn} + C\left(\frac{1}{r}\right)_{|sn} \quad . \tag{6.19}$$

Now putting $A = \frac{1}{r_0}$, $B = -4m$ and $\phi = -\frac{2m}{r}$ where r_0 and m are constants, we write the solution of the equations (6.6) as follows:

$$\begin{aligned}
&\underset{2}{\gamma}_{00} = 2\phi + \frac{2a^2}{3}\underset{2}{R} \ , \quad \underset{2}{\gamma}_{0n} = 0 \\
&\underset{2}{\gamma}_{sn} = -\delta_{sn}\cdot\frac{2a^2}{3}\underset{2}{R} + \frac{2a^4}{3}\underset{2}{R}_{|sn} + C\left(\frac{1}{r}\right)_{|sn} \\
&\underset{2}{R} = \frac{e^{-r/a}}{r_0 r} = \tfrac{1}{2}\Delta\underset{2}{\gamma} \ , \quad \phi = -\frac{2m}{r} \quad .
\end{aligned} \tag{6.20}$$

In terms of $\underset{2}{h}_{\mu\nu}$ the solution (6.20) becomes

$$\begin{aligned}
&\underset{2}{h}_{00} = \phi - \frac{a^2}{3}\underset{2}{R} \ , \quad \underset{2}{h}_{0n} = 0 \\
&\underset{2}{h}_{sn} = \delta_{sn}\phi + \delta_{sn}\frac{a^2}{3}\underset{2}{R} + \frac{2a^4}{3}\underset{2}{R}_{|sn} + C\left(\frac{1}{r}\right)_{|sn} \\
&\underset{2}{R} = \frac{e^{-r/a}}{r_0 r} = -\tfrac{1}{2}\Delta\underset{2}{h} \ , \quad \phi = -\frac{2m}{r} \quad .
\end{aligned} \tag{6.21}$$

Now consider a coordinate transformation of the type

$$x^{*\alpha} = x^{\alpha} + \epsilon\, v^{\alpha}(x^{\rho}) \tag{6.22}$$

which changes the coordinate values only by amounts which are proportional to the parameter $\epsilon = 1/c^2$. The inverse transformations are

$$x^{\alpha} = x^{*\alpha} - \epsilon\, v^{\alpha}(x^{\rho}) \sim x^{*\alpha} - \epsilon\, v^{\alpha}(x^{*\rho}) \tag{6.23}$$

up to quantities of first order in ϵ .

The components of the metric-tensor $g_{\mu\nu}$ transform according to the law

$$g^{*\mu\nu} = \frac{\partial x^{\alpha}}{\partial x^{*\mu}} \cdot \frac{\partial x^{\beta}}{\partial x^{*\nu}} g_{\alpha\beta} \sim (\delta^{\alpha}{}_{\mu} - \epsilon\, v^{\alpha}{}_{,\mu*})(\delta^{\beta}{}_{\nu} - \epsilon\, v^{\beta}{}_{,\nu*})(\eta_{\alpha\beta} + \epsilon\, \underset{2}{h}_{\alpha\beta})$$

$$\sim \eta_{\mu\nu} + \epsilon\, (\underset{2}{h}_{\mu\nu} - \eta_{\alpha\nu} v^{\alpha}{}_{,\mu*} - \eta_{\alpha\mu} v^{\alpha}{}_{,\nu*}) \qquad (6.24)$$

up to first order quantities. The transformation law for the quantities $\underset{2}{h}_{\mu\nu}$ is therefore

$$\underset{2}{h}{}^{*}_{\mu\nu} = \underset{2}{h}_{\mu\nu} - \eta_{\alpha\nu} v^{\alpha}{}_{,\mu*} - \eta_{\alpha\mu} v^{\alpha}{}_{,\nu*} \quad . \qquad (6.25)$$

Consequently the transformation law for the quantities $\underset{2}{\gamma}_{\mu\nu}$ is

$$\underset{2}{\gamma}{}^{*}_{\mu\nu} = \underset{2}{\gamma}_{\mu\nu} - \eta_{\alpha\nu} v^{\alpha}{}_{,\mu*} - \eta_{\alpha\mu} v^{\alpha}{}_{,\nu*} + \eta_{\mu\nu} v^{\rho}{}_{,\rho*} \quad . \qquad (6.26)$$

Now by carrying out the coordinate transformation (6.22) and consequently using the transformations (6.25) and (6.26), we can annihilate the constant C from the solutions (6.20) and (6.21), if we choose the functions v^{α} as follows:

$$v^{0} = 0$$
$$v^{s} = -\tfrac{1}{2}\left(\frac{C}{r}\right)_{|s} \qquad (6.27)$$

(see Bergmann [2]). Therefore the solution of the field equations finally reduces to

$$\underset{2}{\gamma}_{00} = 2\phi + \frac{2a^2}{3}\underset{2}{R} \,, \qquad \underset{2}{\gamma}_{0n} = 0$$

$$\underset{2}{\gamma}_{sn} = -\delta_{sn} \cdot \frac{2a^2}{3}\underset{2}{R} + \frac{2a^4}{3}\underset{2}{R}_{|sn} \tag{6.28}$$

$$\underset{2}{R} = \frac{e^{-r/a}}{r_0 r} = \tfrac{1}{2}\Delta\underset{2}{\gamma} \,, \qquad \phi = -\frac{2m}{r} \,.$$

In terms of $\underset{2}{h}_{\mu\nu}$, the solution (6.28) becomes

$$\underset{2}{h}_{00} = \phi - \frac{a^2}{3}\underset{2}{R} \,, \qquad \underset{2}{h}_{0n} = 0$$

$$\underset{2}{h}_{sn} = \delta_{sn}\phi + \delta_{sn}\frac{a^2}{3}\underset{2}{R} + \frac{2a^4}{3}\underset{2}{R}_{|sn} \tag{6.29}$$

$$\underset{2}{R} = \frac{e^{-r/a}}{r_0 r} = -\tfrac{1}{2}\Delta\underset{2}{h} \,, \qquad \phi = -\frac{2m}{r} \,.$$

The solution given by (6.28) or (6.29) is precisely what is used by Gregory in his work on the problem of motion. (See Gregory [10].)

We note that the coordinate conditions (6.8) are still satisfied by the solution (6.28). However, in what follows we shall use the solution of the field equations given by either (6.20) or (6.21) and transform into different coordinate systems.

§7.1 The Solution of Section 6.1 and Coordinate Transformations

Considering the $\underset{2}{h}_{\mu\nu}$ given by the solution (6.21) of the field equations, we write them explicitly as follows:

$$\underset{2}{h}_{00} = -\frac{2m}{r} - \frac{a^2 e^{-r/a}}{3r_0 r} , \quad \underset{2}{h}_{0n} = 0$$

$$\underset{2}{h}_{sn} = -\delta_{sn}\cdot\frac{2m}{r} + \delta_{sn}\cdot\frac{a^2 e^{-r/a}}{3r_0 r} + \frac{2a^4}{3r_0}\left(\frac{e^{-r/a}}{r}\right)_{|sn} + C\left(\frac{1}{r}\right)_{|sn} \tag{7.1}$$

where m, r_0 and C are three different constants of integration. Now if we put $-2f\psi_{\mu\nu} = \underset{2}{h}_{\mu\nu}$, $2m = M$, $\frac{a^2}{r_0} = M$ and $C = -\frac{2a^2 M}{3}$ in (7.1), we obtain

$$-2f\psi_{00} = -\frac{M}{r} - \frac{Me^{-r/a}}{3r} , \quad -2f\psi_{0n} = 0$$

$$-2f\psi_{sn} = -\delta_{sn}\cdot\frac{M}{r} + \delta_{sn}\cdot\frac{Me^{-r/a}}{3r} + \frac{2a^2}{3}\left(\frac{Me^{-r/a}}{r} - \frac{M}{r}\right)_{|sn} \tag{7.2}$$

where M is a constant.

We note that the solution (7.2) is precisely the solution obtained by Pechlaner and Sexl using the field equations with the source term which is expressed in terms of the δ-function. (See Pechlaner and Sexl [13].)

But if we put $2m = M$, $\frac{a^2}{r_0} = M'$ and $C = -\frac{2a^2}{3}M''$ in (7.1) then we obtain

$$\underset{2}{h}_{00} = -\frac{M}{r} - \frac{M'e^{-r/a}}{3r} , \quad \underset{2}{h}_{0n} = 0$$

$$\underset{2}{h}_{sn} = -\delta_{sn}\frac{M}{r} + \delta_{sn}\frac{M'e^{-r/a}}{3r} + \frac{2a^2}{3}\left(\frac{M'e^{-r/a}}{r} - \frac{M''}{r}\right)_{|sn} . \tag{7.3}$$

The solutions (7.1), (7.2) and (7.3) are expressed in rectangular harmonic coordinates. Now we define coordinates obtained from rectangular harmonic coordinates by making the following standard coordinate transformation

$$
\begin{aligned}
x^0 &= t \\
x^1 &= r \sin\theta \cos\phi \\
x^2 &= r \sin\theta \sin\phi \\
x^3 &= r \cos\theta
\end{aligned}
\tag{7.4}
$$

as harmonic polar coordinates. We shall transform the solutions (7.2) and (7.3) to harmonic polar coordinates by making the transformation (7.4). Carrying out this transformation, the line-elements corresponding to the solutions (7.2) and (7.3) take, respectively, the following diagonal forms:

$$
\begin{aligned}
ds^2 = {} & \left(1 - \frac{M}{r} - \frac{Me^{-r/a}}{3r}\right) dt^2 - \\
& - \left[1 + \frac{M}{r} + \frac{4a^2M}{3r^3} - e^{-r/a}\left(\frac{M}{r} + \frac{4aM}{3r^2} + \frac{4a^2M}{3r^3}\right)\right] dr^2 - \\
& - \left[1 + \frac{M}{r} - \frac{2a^2M}{3r^3} - e^{-r/a}\left(\frac{M}{3r} - \frac{2aM}{3r^2} - \frac{2a^2M}{3r^3}\right)\right] r^2 d\Omega^2
\end{aligned}
\tag{7.5}
$$

and

$$
\begin{aligned}
ds^2 = {} & \left(1 - \frac{M}{r} - \frac{M'e^{-r/a}}{3r}\right) dt^2 - \\
& - \left[1 + \frac{M}{r} + \frac{4a^2M'}{3r^3} - e^{-r/a}\left(\frac{M'}{r} + \frac{4aM'}{3r^2} + \frac{4a^2M'}{3r^3}\right)\right] dr^2 - \\
& - \left[1 + \frac{M}{r} - \frac{2a^2M''}{3r^3} - e^{-r/a}\left(\frac{M'}{3r} - \frac{2aM'}{3r^2} - \frac{2a^2M'}{3r^3}\right)\right] r^2 d\Omega^2
\end{aligned}
\tag{7.6}
$$

where

$$
d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2 \quad .
$$

Thus the solution of the field equations (6.6) is presented in the diagonal form in harmonic polar coordinates. The line-element (7.5) is precisely the same as that obtained by Pechlaner and Sexl (see Pechlaner and Sexl [13]).

Now we shall transform the line-elements (7.5) and (7.6) into isotropic polar coordinates by making the following coordinate transformations:

$$\bar{r} = r\left[1 - \frac{a^2 M}{3r^3} + e^{-r/a}\left(\frac{aM}{3r^2} + \frac{a^2 M}{3r^3}\right)\right] , \qquad \theta = \theta , \quad \phi = \phi , \quad t = t , \tag{7.7}$$

$$\bar{r} = r\left[1 - \frac{a^2 M'}{3r^3} + e^{-r/a}\left(\frac{aM'}{3r^2} + \frac{a^2 M'}{3r^3}\right)\right] , \qquad \theta = \theta , \quad \phi = \phi , \quad t = t \tag{7.8}$$

where $(\bar{r},\theta,\phi,t)$ are isotropic polar coordinates and (r,θ,ϕ,t) are harmonic polar coordinates.

The line-elements (7.5) and (7.6) respectively take the following diagonal forms when we carry out the transformations (7.7) and (7.8):

$$ds^2 = \left(1 - \frac{M}{\bar{r}} - \frac{Me^{-\bar{r}/a}}{3\bar{r}}\right)dt^2 - \left(1 + \frac{M}{\bar{r}} - \frac{Me^{-\bar{r}/a}}{3\bar{r}}\right)(d\bar{r}^2 + \bar{r}^2 d\Omega^2) \tag{7.9}$$

and

$$ds^2 = \left(1 - \frac{M}{\bar{r}} - \frac{M'e^{-\bar{r}/a}}{3\bar{r}}\right)dt^2 - \left(1 + \frac{M}{\bar{r}} - \frac{M'e^{-\bar{r}/a}}{3\bar{r}}\right)(d\bar{r}^2 + \bar{r}^2 d\Omega^2) . \tag{7.10}$$

To verify the validity of the above results, for example (7.10), apply the transformation (7.8). Then because of (7.8),

$$\frac{1}{\bar{r}} = \frac{1}{r}\left[1 + \frac{a^2M''}{3r^3} - e^{-r/a}\left(\frac{aM'}{3r^2} + \frac{a^2M'}{3r^3}\right)\right] \tag{7.11}$$

and

$$\frac{e^{-\bar{r}/a}}{\bar{r}} = \frac{e^{-r/a}}{r}\left[1 + \frac{a^2M''}{3r^3} + \frac{aM''}{3r^2} - e^{-r/a}\left(\frac{M'}{3r} + \frac{2aM'}{3r^2} + \frac{a^2M'}{3r^3}\right)\right] \tag{7.12}$$

up to first order terms in M , M' and M'' . Therefore it follows from (7.11) and (7.12) that up to first order in M , M' and M''

$$\frac{M}{\bar{r}} = \frac{M}{r}\ , \quad \frac{M'e^{-\bar{r}/a}}{3\bar{r}} = \frac{M'e^{-r/a}}{3r}\quad . \tag{7.13}$$

Next because of (7.8), up to first order in M, M' and M''

$$\bar{r}^2 = r^2\left[1 - \frac{2a^2M''}{3r^3} + e^{-r/a}\left(\frac{2aM'}{3r^2} + \frac{2a^2M'}{3r^3}\right)\right] \tag{7.14}$$

and

$$d\bar{r}^2 = \left[1 + \frac{4a^2M''}{3r^3} - e^{-r/a}\left(\frac{2M'}{3r} + \frac{4aM'}{3r^2} + \frac{4a^2M'}{3r^3}\right)\right] dr^2\ . \tag{7.15}$$

Substituting in (7.10) from (7.13), (7.14) and (7.15), we obtain up to first order in M , M' and M''

$$\begin{aligned} ds^2 = {} & \left(1 - \frac{M}{r} - \frac{M'e^{-r/a}}{3r}\right) dt^2 - \\ & - \left[1 + \frac{M}{r} + \frac{4a^2M''}{3r^3} - e^{-r/a}\left(\frac{M'}{r} + \frac{4aM'}{3r^2} + \frac{4a^2M'}{3r^3}\right)\right] dr^2 - \\ & - \left[1 + \frac{M}{r} - \frac{2a^2M''}{3r^3} - e^{-r/a}\left(\frac{M'}{3r} - \frac{2aM'}{3r^2} - \frac{2a^2M'}{3r^3}\right)\right] r^2 d\Omega^2 \end{aligned} \tag{7.16}$$

which is precisely the same as the line-element (7.6) in harmonic polar coordinates and hence the necessary result follows.

§8.1 The Static, Spherically Symmetric Solution in Isotropic Polar Coordinates

In the previous sections we have considered the field of a sss-particle using linearized empty field equations derived from the invariant $R + \frac{a^2}{6} R^2$, in rectangular harmonic coordinates and then we have transformed the solution to harmonic polar coordinates as well as to isotropic polar coordinates. But since the field of the particle is supposed to be sss, we can directly consider the problem in isotropic polar coordinates expecting that the form of the line-element corresponding to the solution to be obtained, should be sss, that is

$$ds^2 = e^{\nu}dt^2 - e^{\mu}(d\bar{r}^2 + \bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2\theta \, d\phi^2) \tag{8.1}$$

where $\nu = \nu(\bar{r})$ and $\mu = \mu(\bar{r})$.

According to this point of view, in this section we shall linearize the empty field equations in isotropic polar coordinates, solve them and show that the solution is precisely the same as that obtained in Section 7.1.

The exact field equations are as follows:

$$(R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu}) + \frac{a^2}{6}(2RR_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R^2 - 2g_{\mu\nu}g^{\alpha\beta}R_{;\alpha\beta} + 2R_{;\mu\nu}) = 0 \tag{8.2}$$

where the covariant and contravariant components of the metric-tensor, respectively, are as follows:

$$g_{00} = e^{\nu} \;, \quad g_{11} = -e^{\mu} \;, \quad g_{22} = -\bar{r}^2 e^{\mu} \;, \quad g_{33} = -\bar{r}^2 \sin^2\theta \, e^{\mu} \;,$$
$$g^{00} = (g_{00})^{-1} \;, \quad g^{11} = (g_{11})^{-1} \;, \quad g^{22} = (g_{22})^{-1} \;, \quad g^{33} = (g_{33})^{-1} \;. \tag{8.3}$$

Therefore to linearize the field equations we put

$$e^{\nu} = 1 + \epsilon\nu \;, \quad e^{\mu} = 1 + \epsilon\mu \tag{8.4}$$

where ϵ is a small parameter.

Using (8.3) and (8.4) we see that the field equations (8.2) reduce to the following linearized form:

$$R_0^{\;0} - \tfrac{1}{2}R + \frac{a^2}{3}\left(R'' + \frac{2R'}{\bar{r}}\right) = 0$$
$$R_1^{\;1} - \tfrac{1}{2}R + \frac{2a^2 R'}{3\bar{r}} = 0 \tag{8.5}$$
$$R_2^{\;2} - \tfrac{1}{2}R + \frac{a^2}{3}\left(R'' + \frac{R'}{\bar{r}}\right) = 0 \;.$$

Hereinafter the overhead prime denotes differentiation with respect to $\bar{r}$ and

$$R_0^{\;0} = -\left(\frac{\nu''}{2} + \frac{\nu'}{\bar{r}}\right)$$
$$R_1^{\;1} = -\left(\mu'' + \frac{\nu''}{2} + \frac{\mu'}{\bar{r}}\right)$$
$$R_2^{\;2} = -\left(\frac{\mu''}{2} + \frac{3\mu'}{2\bar{r}} + \frac{\nu'}{2\bar{r}}\right) \tag{8.6}$$
$$R = -\left(2\mu'' + \nu'' + \frac{4\mu'}{\bar{r}} + \frac{2\nu'}{\bar{r}}\right) \;.$$

The linearized field equations (8.5) are equivalent to the following three equations:

$$
\begin{aligned}
&R_0^{\ 0} - \tfrac{1}{2}R_1^{\ 1} - R_2^{\ 2} + \tfrac{1}{4}R = 0 && \text{(i)} \\
&R_1^{\ 1} - \tfrac{1}{2}R + \frac{2a^2R'}{3\bar{r}} = 0 && \text{(ii)} \\
&R - a^2\left(R'' + \frac{2R'}{\bar{r}}\right) = 0 \quad . && \text{(iii)}
\end{aligned}
\tag{8.7}
$$

In what follows we shall solve the equations (8.7). Considering the equation (iii) in (8.7), we obtain

$$\bar{r}^2 \frac{d^2R}{d\bar{r}^2} + 2\bar{r}\frac{dR}{d\bar{r}} - \frac{R}{a^2} = 0 \quad . \tag{8.8}$$

As we are interested only in the asymptotically flat solutions of the equations (8.7), the solution of (8.8) is

$$R = A\frac{e^{-\bar{r}/a}}{\bar{r}} \tag{8.9}$$

where A is a constant of integration.

Next considering the equation (i) of (8.7), we obtain

$$\mu'' - \nu'' + \frac{2\mu'}{\bar{r}} - \frac{2\nu'}{\bar{r}} = 0 \quad . \tag{8.10}$$

The asymptotically flat solution of this equation is

$$\mu - \nu = \frac{B}{\bar{r}} \tag{8.11}$$

where B is a constant of integration.

Finally considering the equation (ii) of (8.7), we obtain

$$(\mu' + \nu') - \frac{2a^2}{3}\left(2\mu'' + \nu'' + \frac{4\mu'}{\bar{r}} + \frac{2\nu'}{\bar{r}}\right)' = 0 \quad . \tag{8.12}$$

Eliminating μ from (8.12) with the help of (8.11), we obtain

$$\left(2\nu' - \frac{B}{\bar{r}^2}\right) - 2a^2\left(\nu'' + \frac{2\nu'}{\bar{r}}\right)' = 0 \quad . \tag{8.13}$$

The asymptotically flat solution of this equation is

$$\nu = -\frac{B}{2\bar{r}} + \frac{C\,e^{-\bar{r}/a}}{\bar{r}} \tag{8.14}$$

where C is a constant of integration.

Since we require

$$R = -2\mu'' - \nu'' - \frac{4\mu'}{\bar{r}} - \frac{2\nu'}{\bar{r}} \tag{8.15}$$

it follows that

$$C = -\frac{a^2}{3}A \quad . \tag{8.16}$$

Therefore the solution of the field equations (8.7) is as follows:

$$\begin{aligned} \mu &= \frac{B}{2\bar{r}} - \frac{a^2 A\,e^{-\bar{r}/a}}{3\bar{r}} \\ \nu &= -\frac{B}{2\bar{r}} - \frac{a^2 A\,e^{-\bar{r}/a}}{3\bar{r}} \\ R &= A.\,\frac{e^{-\bar{r}/a}}{\bar{r}} \quad . \end{aligned} \tag{8.17}$$

The corresponding linearized line-element is

$$ds^2 = \left(1 - \frac{B}{2\bar{r}} - \frac{a^2 A\, e^{-\bar{r}/a}}{3\bar{r}}\right) dt^2 - \left(1 + \frac{B}{2\bar{r}} - \frac{a^2 A\, e^{-\bar{r}/a}}{3\bar{r}}\right)(d\bar{r}^2 + \bar{r}^2 d\Omega^2) \tag{8.18}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2 \quad .$$

Now putting $\frac{B}{2} = M$ and $a^2 A = M'$ where M and M' are constants, from (8.18) we obtain the line-element

$$ds^2 = \left(1 - \frac{M}{\bar{r}} - \frac{M' e^{-\bar{r}/a}}{3\bar{r}}\right) dt^2 - \left(1 + \frac{M}{\bar{r}} - \frac{M' e^{-\bar{r}/a}}{3\bar{r}}\right)(d\bar{r}^2 + \bar{r}^2 d\Omega^2) \tag{8.19}$$

which is precisely the same as the line-element obtained in Section 7.1, by successive coordinate transformations of the corresponding line-element in rectangular harmonic coordinates.

CHAPTER IV

LINEARIZED NON-EMPTY FIELD EQUATIONS AND THE INVARIANT $R + \frac{a^2}{6} R^2$

§9.1 The Field Equations in Rectangular Harmonic Coordinates

In this chapter we shall consider the non-empty field equations of a theory based upon the invariant $R + \frac{a^2}{6} R^2$, in the case of a sss , perfect fluid distribution. Because the exact field equations are too complicated to be solved we shall consider only the linearized equations, and discuss the meaning of linearization. First, the problem will be discussed in rectangular harmonic coordinates and then in isotropic polar coordinates.

Since we shall be dealing with the static field, we expect that the line-element to be obtained will be time-symmetric. Therefore because x^0 is the time coordinate, the line-element will be of the form

$$ds^2 = g_{00}(dx^0)^2 + g_{ik}dx^i dx^k \quad . \tag{9.1}$$

In order to carry out the linearization of the problem, as it is suggested from Chapters II and III, we should put

$$\begin{aligned} g_{00} &= 1 + \epsilon h_{00} \\ g_{ik} &= -\delta_{ik} + \epsilon h_{ik} \end{aligned} \tag{9.2}$$

where ϵ is a small parameter equal to $1/c^2$, c being the velocity of light. Hereinafter we shall omit the suffix 2 written below the field-quantities to denote their linearity in the parameter $1/c^2$.

The exact non-empty field equations are as follows:

$$(R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu}) + \frac{a^2}{3}(R_{;\mu\nu} + RR_{\mu\nu} - g_{\mu\nu}g^{\alpha\beta}R_{;\alpha\beta} - \tfrac{1}{4}g_{\mu\nu}R^2) = -\chi T_{\mu\nu} \tag{9.3}$$

where χ is the coupling constant and $T_{\mu\nu}$ is the stress-energy tensor. In the case of a perfect fluid distribution, the contravariant components $T^{\mu\nu}$ are as follows:

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) v^{\mu}v^{\nu} - \frac{p}{c^2} g^{\mu\nu} \tag{9.4}$$

where ρ and p respectively are the proper density and pressure and the four-vector v^{μ} is to be interpreted as follows: The element of matter which occupies the point x^{μ} of the space-time has the motion $x^{\mu}(s)$ such that $\frac{dx^{\mu}}{ds} = v^{\mu}$. By virtue of the discussion made in Chapters II and III, we can show that if we linearize the non-empty field equations (9.3), using (9.2), then we obtain the following linearized non-empty field equations:

$$\begin{aligned} &\Delta h_{00} - \tfrac{1}{2}\Delta h + \frac{a^2}{3}\Delta.\Delta h = 2(\chi\ T_{00})_{\mathcal{L}} \\ &\Delta h_{sn} + \tfrac{1}{2}\delta_{sn}\Delta h - \frac{a^2}{3}\delta_{sn}\Delta.\Delta h + \frac{a^2}{3}\Delta h_{|sn} = 2(\chi\ T_{sn})_{\mathcal{L}} \end{aligned} \tag{9.5}$$

where $h = h_{00} - h_{\ell\ell}$ and suffix $\mathcal{L}$ denotes a linearized quantity. Hereinafter we regard the constant a^2 to be of order zero in the parameter $\epsilon = 1/c^2$.

The contracted field equations following from (9.5) are as follows:

$$a^2\Delta.\Delta h - \Delta h = 2(\chi T)_{\mathcal{L}}. \tag{9.6}$$

The linearized harmonic coordinate conditions become

$$h_{sn|n} + \tfrac{1}{2}h_{|s} = 0 \quad . \tag{9.7}$$

Now consider the stress-energy tensor for a perfect fluid distribution given by

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) v_\mu v_\nu - \frac{p}{c^2} g_{\mu\nu} \quad . \tag{9.8}$$

Since we are dealing with the static case, the matter is at rest with respect to the coordinate system in use. Therefore

$$v^\mu = (v^0,0,0,0) \quad , \quad \rho = \rho(x^i) \quad , \quad p = p(x^i) \quad .$$

Because

$$g_{\mu\nu}v^\mu v^\nu = 1 \quad , \quad v_\mu = g_{\mu\nu}v^\mu \tag{9.9}$$

to first order in ϵ , we obtain

$$\begin{aligned} (v^0)_{\mathcal{L}} &= 1 - \tfrac{1}{2}\epsilon h_{00} \quad , \quad (v^s)_{\mathcal{L}} = 0 \\ (v_0)_{\mathcal{L}} &= 1 + \tfrac{1}{2}\epsilon h_{00} \quad , \quad (v_s)_{\mathcal{L}} = 0 \quad . \end{aligned} \tag{9.10}$$

Moreover in the ordinary physical conditions of the fluid, we regard ρ and p as being of order zero in the parameter ϵ . Therefore ρc^2 is of order -1 and p is of order zero. Consequently under these conditions ρc^2 is much larger than p , that is, ρc^2 is not comparable with p . Thus we obtain

$$\rho \sim \epsilon^0$$
$$\frac{p}{c^2} \sim \epsilon \quad . \tag{9.11}$$

Now let us further assume that

$$\chi = \frac{8\pi G}{c^2} \sim \epsilon \tag{9.12}$$

where G is the Newtonian constant of gravitation.

Therefore, from (9.8), (9.10), (9.11) and (9.12), it follows that

$$(\chi\, T_{00})_{\mathcal{L}} = \frac{8\pi G}{c^2}\,\rho$$
$$(\chi\, T_{sn})_{\mathcal{L}} = 0 \quad . \tag{9.13}$$

Therefore

$$(\chi\, T)_{\mathcal{L}} = (\chi\, T_{00} - \chi\, T_{\ell\ell})_{\mathcal{L}} = \frac{8\pi G}{c^2}\,\rho \quad . \tag{9.14}$$

Substituting in (9.5) and (9.6), from (9.13) and (9.14), we obtain

$$\Delta h_{00} - \tfrac{1}{2}\Delta h + \frac{a^2}{3}\,\Delta.\Delta h = \frac{16\pi G}{c^2}\,\rho \qquad \text{(i)}$$
$$\Delta h_{sn} + \tfrac{1}{2}\delta_{sn}\Delta h - \frac{a^2}{3}\,\delta_{sn}\Delta.\Delta h + \frac{a^2}{3}\,\Delta h_{|sn} = 0 \qquad \text{(ii)} \tag{9.15}$$

$$a^2\Delta.\Delta h - \Delta h = \frac{16\pi G}{c^2}\,\rho \quad . \tag{9.16}$$

The equations (9.15) and (9.16) can be written in the following form:

$$\Delta h_{00} - \frac{a^2}{6}\,\Delta.\Delta h = \frac{8\pi G}{c^2}\,\rho \qquad \text{(i)}$$

$$\Delta h_{sn} + \frac{a^2}{6}\,\delta_{sn}\Delta.\Delta h + \frac{a^2}{3}\,\Delta h_{|sn} = \frac{8\pi G}{c^2}\,\delta_{sn}\rho \qquad \text{(ii)} \tag{9.17}$$

$$a^2\Delta.\Delta h - \Delta h = \frac{16\pi G}{c^2}\,\rho \quad . \tag{9.18}$$

We note that when the terms with the constant factors a^2 in equations (9.17) and (9.18) are omitted, we obtain the linearized non-empty field equations of general relativity, that is

$$\Delta h_{00} = \frac{8\pi G}{c^2}\,\rho \qquad \text{(i)}$$

$$\Delta h_{sn} = \frac{8\pi G}{c^2}\,\delta_{sn}\rho \qquad \text{(ii)} \tag{9.19}$$

$$\Delta h = -\frac{16\pi G}{c^2}\,\rho \quad . \tag{9.20}$$

If we put

$$h_{00} = h_{11} = h_{22} = h_{33} = -2V/c^2$$

$$h_{12} = h_{13} = h_{23} = 0 \tag{9.21}$$

then (9.19) and (9.20) reduce to a single equation

$$\Delta V = -4\pi G\rho \tag{9.22}$$

which is the well-known Poisson equation of the Newtonian theory of gravitation, V being the scalar potential.

In equations (9.17) and (9.18), ρ is a function of the space coordinates. However we do not have an equation for p which is also a function of the space coordinates. As we have assumed p to be of order zero in the parameter ϵ, to first order of approximation p does not vanish. However it has disappeared from the linearized field equations. To determine the pressure to zeroth order we shall linearize the conservation equations:

$$T_{\mu\ ;\nu}^{\ \nu} = 0 \quad . \tag{9.23}$$

Because

$$T_\mu^{\ \nu} = \left(\rho + \frac{p}{c^2}\right) v_\mu v^\nu - \frac{p}{c^2}\,\delta_\mu^{\ \nu} \tag{9.24}$$

by earlier discussion it follows that

$$(T_0^{\ 0})_{\mathcal{L}} = \rho \sim \epsilon^0 \quad , \quad (T_1^{\ 1})_{\mathcal{L}} = (T_2^{\ 2})_{\mathcal{L}} = (T_3^{\ 3})_{\mathcal{L}} = -\frac{p}{c^2} \sim \epsilon$$

$$(T_0^{\ s})_{\mathcal{L}} = (T_s^{\ 0})_{\mathcal{L}} = 0 \tag{9.25}$$

$$(T_s^{\ n})_{\mathcal{L}} = 0 \qquad \text{if} \quad s \neq n \quad .$$

Now writing the equations (9.23) explicitly,

$$\begin{aligned}
T_{0\ ,\nu}^{\ \nu} - \Gamma_{0\ \nu}^{\ \alpha} T_\alpha^{\ \nu} + \Gamma_{\nu\alpha}^{\nu}\, T_0^{\ \alpha} &= 0 \\
T_{1\ ,\nu}^{\ \nu} - \Gamma_{1\ \nu}^{\ \alpha} T_\alpha^{\ \nu} + \Gamma_{\nu\alpha}^{\nu}\, T_1^{\ \alpha} &= 0 \\
T_{2\ ,\nu}^{\ \nu} - \Gamma_{2\ \nu}^{\ \alpha} T_\alpha^{\ \nu} + \Gamma_{\nu\alpha}^{\nu}\, T_2^{\ \alpha} &= 0 \\
T_{3\ ,\nu}^{\ \nu} - \Gamma_{3\ \nu}^{\ \alpha} T_\alpha^{\ \nu} + \Gamma_{\nu\alpha}^{\nu}\, T_3^{\ \alpha} &= 0 \quad .
\end{aligned} \tag{9.26}$$

Linearizing the equations (9.26), in the static case we obtain

$$
\begin{aligned}
-(\Gamma_{0\ 0}^{\ 0})_{\mathcal{L}} T_0^{\ 0} + (\Gamma_{\nu\ 0}^{\ \nu})_{\mathcal{L}} T_0^{\ 0} &= 0 \\
(T_1^{\ 1}{}_{,1})_{\mathcal{L}} - (\Gamma_{1\ 0}^{\ 0})_{\mathcal{L}} T_0^{\ 0} &= 0 \\
(T_2^{\ 2}{}_{,2})_{\mathcal{L}} - (\Gamma_{2\ 0}^{\ 0})_{\mathcal{L}} T_0^{\ 0} &= 0 \\
(T_3^{\ 3}{}_{,3})_{\mathcal{L}} - (\Gamma_{3\ 0}^{\ 0})_{\mathcal{L}} T_0^{\ 0} &= 0
\end{aligned}
\tag{9.27}
$$

where $T_0^{\ 0} \sim \epsilon^0$.

Since in the static case

$$
\begin{aligned}
(\Gamma_{s\ 0}^{\ s})_{\mathcal{L}} &= 0 \\
(\Gamma_{s\ 0}^{\ 0})_{\mathcal{L}} &= \tfrac{1}{2}(h_{00})_{,s}
\end{aligned}
\tag{9.28}
$$

the equations (9.27) reduce to

$$
\begin{aligned}
(T_1^{\ 1}{}_{,1})_{\mathcal{L}} - \tfrac{1}{2}(h_{00})_{,1} T_0^{\ 0} &= 0 \\
(T_2^{\ 2}{}_{,2})_{\mathcal{L}} - \tfrac{1}{2}(h_{00})_{,2} T_0^{\ 0} &= 0 \\
(T_3^{\ 3}{}_{,3})_{\mathcal{L}} - \tfrac{1}{2}(h_{00})_{,3} T_0^{\ 0} &= 0
\end{aligned}
\tag{9.29}
$$

where $(h_{00})_{,s} \sim \epsilon$ and $T_0^{\ 0} \sim \epsilon^0$. These equations are equivalent to

$$
2p_{,s} + c^2\rho(h_{00})_{,s} = 0 \tag{9.30}
$$

where both of the terms on the left hand side are of order zero in the parameter $\epsilon = 1/c^2$. The equation (9.30) determines the pressure to zeroth order.

In order to obtain the exterior and interior solutions of a sphere with a sss , perfect fluid distribution, we should integrate the equations (9.17) within and without the sphere, choosing the proper boundary conditions. Outside the sphere, since $\rho = 0$, the equations to be satisfied are linearized empty field equations corresponding to the sss field as considered in Chapter III. Inside the sphere $\rho = \rho(x^1)$ and putting ρ = constant we can simplify the field equations (9.17) and try to solve them. But before doing this, we shall consider the problem in isotropic polar coordinates $(\bar{r},\theta,\phi,t)$, instead of rectangular harmonic coordinates x^i . The problem is considerably more simple in isotropic polar coordinates.

§10.1 The Field Equations in Isotropic Polar Coordinates, the Interior and Exterior Solutions and the Boundary Conditions

In this section we shall consider the problem discussed in Section 9.1 using isotropic polar coordinates instead of rectangular harmonic coordinates. Since the fluid distribution considered is sss , the line-element to be obtained will be of the form:

$$ds^2 = e^{\nu}dt^2 - e^{\mu}(d\bar{r}^2 + \bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2\theta \, d\phi^2) \tag{10.1}$$

where $\nu = \nu(\bar{r})$ and $\mu = \mu(\bar{r})$.

Using the line-element (10.1) and the procedure described in Chapter III, we linearize the non-empty field equations. Thus we obtain

$$(R_0{}^0 - \tfrac{1}{2}R) + \frac{a^2}{3}\left(R'' + \frac{2R'}{\bar{r}}\right) = -(\chi T_0{}^0)_{\mathcal{L}}$$

$$(R_1{}^1 - \tfrac{1}{2}R) + \frac{2a^2R'}{3\bar{r}} = -(\chi T_1{}^1)_{\mathcal{L}} \tag{10.2}$$

$$(R_2{}^2 - \tfrac{1}{2}R) + \frac{a^2}{3}\left(R'' + \frac{R'}{\bar{r}}\right) = -(\chi T_2{}^2)_{\mathcal{L}} = -(\chi T_3{}^3)_{\mathcal{L}}$$

where the overhead prime denotes differentiation with respect to $\bar{r}$ and the suffix $\mathcal{L}$ denotes the linear part in the parameter $\epsilon = 1/c^2$. Moreover the linearized quantities $R_0{}^0$, $R_1{}^1$, $R_2{}^2$ and R are described as follows:

$$R_0{}^0 = -\left(\frac{\nu''}{2} + \frac{\nu'}{\bar{r}}\right)$$

$$R_1{}^1 = -\left(\mu'' + \frac{\nu''}{2} + \frac{\mu'}{\bar{r}}\right)$$

$$R_2{}^2 = -\left(\frac{\mu''}{2} + \frac{3\mu'}{2\bar{r}} + \frac{\nu'}{2\bar{r}}\right) \tag{10.3}$$

$$R = -\left(2\mu'' + \nu'' + \frac{4\mu'}{\bar{r}} + \frac{2\nu'}{\bar{r}}\right) .$$

Now putting $\chi = 8\pi G/c^2$ and noting that the fluid distribution considered is static and perfect, we can apply the procedure of Section 9.1 to linearize the expression $(\chi\, T_\mu{}^\nu)$. Then equations (10.2) become

$$(R_0{}^0 - \tfrac{1}{2}R) + \frac{a^2}{3}\left(R'' + \frac{2R'}{\bar{r}}\right) = -\frac{8\pi G\rho}{c^2}$$

$$(R_1{}^1 - \tfrac{1}{2}R) + \frac{2a^2R'}{3\bar{r}} = 0 \tag{10.4}$$

$$(R_2{}^2 - \tfrac{1}{2}R) + \frac{a^2}{3}\left(R'' + \frac{R'}{\bar{r}}\right) = 0 .$$

Contracting the field equations (10.4), we obtain

$$R - a^2\left(R'' + \frac{2R'}{\bar{r}}\right) = \frac{8\pi G}{c^2}\rho \quad . \tag{10.5}$$

From (10.4) and (10.5) we can obtain the following three equations which are equivalent to the field equations (10.4):

$$\begin{aligned} -2R_0{}^0 + \frac{a^2}{3}\Delta R &= \frac{8\pi G}{c^2}\rho \\ \tfrac{1}{2}R - R_0{}^0 - \frac{a^2}{3}\Delta R &= \frac{8\pi G}{c^2}\rho \\ R_1{}^1 - \tfrac{1}{2}R + \frac{2a^2R'}{3\bar{r}} &= 0 \quad . \end{aligned} \tag{10.6}$$

We note that when the terms with the constant factor a^2 from (10.6) are omitted, we obtain the linearized non-empty field equations of general relativity, that is,

$$\begin{aligned} -2R_0{}^0 &= \frac{8\pi G}{c^2}\rho \\ \tfrac{1}{2}R - R_0{}^0 &= \frac{8\pi G}{c^2}\rho \\ R_1{}^1 - \tfrac{1}{2}R &= 0 \quad . \end{aligned} \tag{10.7}$$

Writing (10.7) explicitly in terms of μ and ν we obtain

$$\Delta\nu = \frac{8\pi G}{c^2}\rho$$

$$\Delta\mu = -\frac{8\pi G}{c^2}\rho \tag{10.8}$$

$$\mu' + \nu' = 0 \quad .$$

If we put

$$\nu = -\mu = -\frac{2V}{c^2} \tag{10.9}$$

then (10.8) reduces to a single equation

$$\Delta V = -4\pi G\rho \tag{10.10}$$

which is the well-known Poisson-equation of the Newtonian theory of gravitation, V being the scalar potential. Now we write the equations (10.6) explicitly in terms of μ and ν. They become

$$\begin{aligned}
\Delta\nu - \frac{a^2}{3}\Delta.\Delta(2\mu+\nu) &= \frac{8\pi G}{c^2}\rho \\
\Delta\mu - \frac{a^2}{3}\Delta.\Delta(2\mu+\nu) &= -\frac{8\pi G}{c^2}\rho \\
(\mu+\nu)' - \frac{2a^2}{3}[\Delta(2\mu+\nu)]' &= 0 \quad .
\end{aligned} \tag{10.11}$$

In (10.11) ρ but not p appears, both of them being functions of only $\bar{r}$. As we have assumed p to be of order zero, to the first order of approximation considered in the present discussion it does not vanish. To determine p to zeroth order we shall linearize the conservation equations:

$$T_{\mu\,;\nu}^{\ \nu} = 0 \quad . \tag{10.12}$$

Using the procedure of Section 9.1 we can show that in the sss case of a perfect fluid distribution considered here (10.12), when linearized, reduces to the following single equation:

$$(T_1{}^1)'_{\mathcal{L}} - \frac{\nu'}{2} T_0{}^0 = 0 \qquad (10.13)$$

where $(T_1{}^1)'_{\mathcal{L}} = -\frac{p'}{c^2} \sim \epsilon$, $T_0{}^0 = \rho \sim \epsilon^0$ and $\nu' \sim \epsilon$.

Equation (10.13), when written explicitly in terms of ρ and p , becomes

$$2p' + \nu'\rho c^2 = 0 \qquad (10.14)$$

where p' and ρ are of order zero in ϵ .

The above consideration clearly shows that though $T_1{}^1$, $T_2{}^2$ and $T_3{}^3$ do not appear in the linearized field equations, it does not mean that they vanish in the first approximation. In fact the dominant parts of $T_1{}^1 = T_2{}^2 = T_3{}^3 = - p/c^2$ are of order one in ϵ as assumed at the beginning of this chapter.

To obtain the exterior and interior solutions of a sphere with a sss perfect fluid distribution, we should integrate the equations (10.11) with $\rho \neq 0$ and $\rho = 0$ respectively under the restriction of suitable boundary conditions. Outside the sphere, since $\rho = 0$, the equations to be satisfied are linearized empty field equations. Inside the sphere $\rho = \rho(\bar{r})$ and putting $\rho =$ constant we simplify the field equations (10.11) and write them as follows:

$$\Delta\mu - \frac{a^2}{3}\Delta.\Delta(2\mu+\nu) = -A$$

$$\Delta\nu - \frac{a^2}{3}\Delta.\Delta(2\mu+\nu) = A \qquad (10.15)$$

$$(\mu'+\nu') - \frac{2a^2}{3}[\Delta(2\mu+\nu)]' = 0$$

where $A = \frac{8\pi G\rho}{c^2}$ = constant.

It is easy to verify that the solution of the above equations is

$$\mu_i = -\frac{A}{6}(\bar{r})^2 + \frac{A_1 e^{-\bar{r}/a}}{\bar{r}} + \frac{A_2 e^{\bar{r}/a}}{\bar{r}} + \frac{A_3}{\bar{r}} + A_4$$

$$\nu_i = \frac{A}{6}(\bar{r})^2 + \frac{A_1 e^{-\bar{r}/a}}{\bar{r}} + \frac{A_2 e^{\bar{r}/a}}{\bar{r}} - \frac{A_3}{\bar{r}} + A_5 \qquad (10.16)$$

where A_1, A_2, A_3, A_4 and A_5 are constants of integration and the suffix i denotes the interior solution. Demanding that the metric be regular everywhere inside the sphere, including the centre $\bar{r} = 0$, from (10.16) it follows that

$$A_3 = 0$$

$$A_1 + A_2 = 0 \quad . \qquad (10.17)$$

Therefore the solution (10.16) takes the following form:

$$\mu_i = -\frac{A}{6}(\bar{r})^2 + \alpha_1 \frac{\sinh(\bar{r}/a)}{(\bar{r}/a)} + \alpha_2$$

$$\nu_i = \frac{A}{6}(\bar{r})^2 + \alpha_1 \frac{\sinh(\bar{r}/a)}{(\bar{r}/a)} + \alpha_3 \qquad (10.18)$$

where α_1 , α_2 and α_3 are arbitrary constants.

In the region outside the sphere the field equations to be satisfied are the linearized empty field equations and consequently, as follows from Section 8.1 of Chapter III, the asymptotically flat solution of the field equations in this region is:

$$\mu_e = \beta_1 \frac{e^{-\bar{r}/a}}{(\bar{r}/a)} + \frac{\beta_2}{\bar{r}}$$
$$\nu_e = \beta_1 \frac{e^{-\bar{r}/a}}{(\bar{r}/a)} - \frac{\beta_2}{\bar{r}} \tag{10.19}$$

where β_1 and β_2 are arbitrary constants and the suffix e denotes the exterior solution. We write the interior solution (10.18) and the exterior solution (10.19) as

$$\mu_i = -\frac{A}{6}(\bar{r})^2 + \alpha_1 w + \alpha_2$$
$$\nu_i = \frac{A}{6}(\bar{r})^2 + \alpha_1 w + \alpha_3 \tag{10.20}$$

$$\mu_e = \beta_1 u + \frac{\beta_2}{\bar{r}}$$
$$\nu_e = \beta_1 u - \frac{\beta_2}{\bar{r}} \tag{10.21}$$

where α_1 , α_2 , α_3 , β_1 and β_2 are arbitrary constants and

$$u = \frac{e^{-\bar{r}/a}}{(\bar{r}/a)}$$
$$w = \frac{\sinh(\bar{r}/a)}{(\bar{r}/a)} \; . \tag{10.22}$$

Now we wish to join the interior solution (10.20) to the exterior solution (10.21) by choosing suitable boundary conditions. Since the differential equations involved in the problem are fourth-order linear differential equations, one might expect to impose boundary conditions that the functions μ and ν and their derivatives up to third order be continuous across the boundary. However by choosing $\bar{r} = b =$ constant as the boundary of the sphere and by using (10.20) and (10.21), one can easily show the impossibility of satisfying such boundary conditions, at least in the present coordinate system. However, if we impose boundary conditions that the functions μ and ν and their first derivatives as well as R and its first derivative are continuous across the boundary of the sphere, then it follows that such boundary conditions can be satisfied, and all the five constants appearing in the solutions (10.20) and (10.21) can be determined. This will be shown as follows: Using (10.20) and (10.21) and imposing the continuity of μ , ν and their first derivatives across the boundary of the sphere $\bar{r} = b$, we obtain

$$\begin{aligned} \mu_i &= -\frac{A}{6}(\bar{r})^2 - \frac{\beta(a+b)e^{-b/a}}{a}w + \left(\beta + \frac{Ab^2}{2}\right) \\ \nu_i &= \frac{A}{6}(\bar{r})^2 - \frac{\beta(a+b)e^{-b/a}}{a}w + \left(\beta - \frac{Ab^2}{2}\right) \end{aligned} \tag{10.23}$$

$$\begin{aligned} \mu_e &= \beta\frac{b^3}{a^3}n\left(\frac{b}{a}\right)u + \frac{Ab^3}{3\bar{r}} \\ \nu_e &= \beta\frac{b^3}{a^3}n\left(\frac{b}{a}\right)u - \frac{Ab^3}{3\bar{r}} \end{aligned} \tag{10.24}$$

where β is an arbitrary constant and

$$n\left(\frac{b}{a}\right) = \frac{a^2}{b^2}\left(\cosh\frac{b}{a} - \frac{a}{b}\sinh\frac{b}{a}\right) . \tag{10.25}$$

Denoting by $D(f)$ the quantity $(f_i - f_e)$ at $\bar{r} = b$ where f is any function of $\bar{r}$, for $D(R)$ we obtain

$$D(R) = -D(2\mu'' + \nu'') - \frac{2}{b} D(2\mu' + \nu') . \tag{10.26}$$

Because μ' and ν' are continuous at $\bar{r} = b$

$$D(2\mu' + \nu') = 0 . \tag{10.27}$$

Hence from (10.26)

$$D(R) = -D(2\mu'' + \nu'') \tag{10.28}$$

using (10.23) and (10.24), from (10.28) we obtain

$$D(R) = A + \frac{3\beta}{a^2} . \tag{10.29}$$

We require that R be continuous across the boundary $\bar{r} = b$. Therefore $D(R)$ must vanish. Consequently from (10.29)

$$\beta = -\frac{Aa^2}{3} \tag{10.30}$$

Next since, as it follows from (10.28)

$$D(R') = -D(2\mu''' + \nu''') \tag{10.31}$$

using (10.23), (10.24) and (10.30) we can easily show that

$$D(R') = 0 \quad . \tag{10.32}$$

Consequently the boundary condition that R' should be continuous across the boundary of the sphere $\bar{r} = b$ is also satisfied. Now using the value of β given by (10.30) we write (10.23) and (10.24) in the following form:

$$\mu_i = -\frac{A}{6}(\bar{r})^2 + \frac{Aa(a+b)e^{-b/a}}{3} w + \frac{A(3b^2-2a^2)}{6}$$

$$\nu_i = \frac{A}{6}(\bar{r})^2 + \frac{Aa(a+b)e^{-b/a}}{3} w - \frac{A(3b^2+2a^2)}{6} \tag{10.33}$$

$$\mu_e = -\frac{Ab^3}{3a} n\left(\frac{b}{a}\right) u + \frac{Ab^3}{3\bar{r}} \tag{10.34}$$

$$\nu_e = -\frac{Ab^3}{3a} n\left(\frac{b}{a}\right) u - \frac{Ab^3}{3\bar{r}} \quad .$$

The equations (10.33) and (10.34) describe the interior and exterior gravitational fields of the uniform-density sphere with radius $\bar{r} = b$. Putting $\frac{Ab^3}{3} = M$, it can easily be verified that the result obtained above precisely agrees with that of Pechlaner and Sexl obtained by using Green's functions (see Pechlaner and Sexl [13]).

With a view to justifying the above boundary conditions, we shall consider the non-empty linearized field equations with density ρ defined by

$$\frac{8\pi G\rho}{c^2} = A f_\eta(\bar{r}) \tag{10.35}$$

where

$$f_\eta(\bar{r}) = \tfrac{1}{2}\left[1 + \frac{(b-\bar{r})}{\{\eta^2+(b-\bar{r})^2\}^{\frac{1}{2}}} + \frac{\eta^2(b-\bar{r})}{2\{\eta^2+(b-\bar{r})^2\}^{3/2}}\right] \quad . \tag{10.36}$$

As η tends to zero, $8\pi G\rho/c^2$ tends to A or zero according as $\bar{r}$ is less or greater than the radius of the sphere. When η is non-zero we have a continuous fluid distribution depending upon $\bar{r}$ and η and extending over entire space. In this case the field equations become

$$\begin{aligned} \Delta\mu - \frac{a^2}{3}\Delta.\Delta(2\mu+\nu) &= -Af_\eta(\bar{r}) \\ \Delta\nu - \frac{a^2}{3}\Delta.\Delta(2\mu+\nu) &= Af_\eta(\bar{r}) \\ (\mu'+\nu') - \frac{2a^2}{3}[\Delta(2\mu+\nu)]' &= 0 \quad . \end{aligned} \tag{10.37}$$

These field equations are equivalent to the following two field equations:

$$\begin{aligned} \Delta(\mu-\nu) &= -2Af_\eta(\bar{r}) & \text{(i)} \\ (\mu'+\nu') - \frac{2a^2}{3}[\Delta(2\mu+\nu)]' &= 0 \quad . & \text{(ii)} \end{aligned} \tag{10.38}$$

Integrating equation (ii) in (10.38) once with respect to $\bar{r}$, we obtain

$$(\mu+\nu) - \frac{2a^2}{3}\Delta(2\mu+\nu) = \text{constant} \quad . \tag{10.39}$$

Since we are interested only in the asymptotically flat solution of the field equations the constant appearing in (10.39) must be zero. Therefore we have

$$\frac{2a^2}{3}\Delta(2\mu+\nu) - (\mu+\nu) = 0 \quad . \tag{10.40}$$

Using equation (i) in (10.38) we can write (10.40) as

$$\Delta(\mu+\nu) - \frac{1}{a^2}(\mu+\nu) = \frac{2A}{3} f_\eta(\bar{r}) \quad . \tag{10.41}$$

Let

$$\mu - \nu = \phi \quad , \quad \mu + \nu = \psi \quad . \tag{10.42}$$

Then the field equations (10.38) become

$$\begin{aligned} \Delta\phi &= -2Af_\eta(\bar{r}) \\ \Delta\psi - \frac{\psi}{a^2} &= \frac{2A}{3} f_\eta(\bar{r}) \quad . \end{aligned} \tag{10.43}$$

The solution of (10.43) which is regular at the origin and asymptotically flat at infinity is as follows:

$$\begin{aligned} \mu - \nu = \frac{A}{\bar{r}} \Bigg[& \frac{z^3}{6} + \frac{(z^2+\eta^2)^{3/2}}{6} - \frac{b}{2}(z^2+\eta^2) - \frac{bz}{2}(z^2+\eta^2)^{\frac{1}{2}} - \frac{\eta^2}{2}(z^2+\eta^2)^{\frac{1}{2}} - \\ & - \frac{\eta^2 z}{4} + \tfrac{1}{3}(b^2+\eta^2)^{3/2} + \frac{b^3}{3} + \frac{3b\eta^2}{4} \Bigg] \end{aligned} \tag{10.44}$$

$$\mu + \nu = \frac{a^2 A}{3\bar{r}} \left[\int_{\xi=0}^{\bar{r}} \sinh\left(\frac{\bar{r}-\xi}{a}\right) F\left(\frac{\xi}{a}\right) d\xi - \int_{\xi=0}^{\infty} e^{-\xi/a} \sinh\left(\frac{\bar{r}}{a}\right) F\left(\frac{\xi}{a}\right) d\xi \right] \tag{10.45}$$

where

$$z = b - \bar{r}$$

$$F\left(\frac{\xi}{a}\right) = \frac{\xi}{a}\left[1 + \frac{(b-\xi)}{\{\eta^2+(b-\xi)^2\}^{\frac{1}{2}}} + \frac{\eta^2(b-\xi)}{2\{\eta^2+(b-\xi)^2\}^{3/2}}\right] \quad . \tag{10.46}$$

Allowing η to tend to zero, from (10.44) and (10.45) we obtain the following solution of (10.43), according as $\bar{r}$ is less or greater than b :

$$(\mu)_{\bar{r}<b} = -\frac{A\bar{r}^2}{6} + \frac{Aa(a+b)e^{-b/a}}{3} w + \frac{A(3b^2-2a^2)}{6}$$

$$(\nu)_{\bar{r}<b} = \frac{A\bar{r}^2}{6} + \frac{Aa(a+b)e^{-b/a}}{3} w - \frac{A(3b^2+2a^2)}{6} \tag{10.47}$$

$$(\mu)_{\bar{r}>b} = -\frac{Ab^3}{3a}\, n\left(\frac{b}{a}\right) u + \frac{Ab^3}{3\bar{r}}$$

$$(\nu)_{\bar{r}>b} = -\frac{Ab^3}{3a}\, n\left(\frac{b}{a}\right) u - \frac{Ab^3}{3\bar{r}} \tag{10.48}$$

where

$$w = \frac{\sinh(\bar{r}/a)}{(\bar{r}/a)}$$

$$u = \frac{e^{-\bar{r}/a}}{(\bar{r}/a)} \tag{10.49}$$

$$n\left(\frac{b}{a}\right) = \frac{a^2}{b^2}\left(\cosh\frac{b}{a} - \frac{a}{b}\sinh\frac{b}{a}\right) .$$

It can easily be verified that at $\bar{r} = b$, μ and ν are continuous. We note that as η tends to zero the density ρ tends to a non-zero positive constant or zero according as $\bar{r}$ is less or greater than b . Thus it follows that the solution (10.47) is the same as the interior solution (10.33) and the solution (10.48) is the same as the exterior solution (10.34). However the solutions (10.33) and (10.34) were obtained by imposing suitable boundary conditions. Whereas the solutions (10.47) and (10.48) are obtained as a limit, when the parameter η tends to zero, of the solution given by (10.44) and (10.45) for the non-empty linearized field equations corresponding to the continuous distribution which, in the limit, tends to the uniform-density sphere with radius $\bar{r} = b$. From these considerations, the justification for the boundary conditions ensues.

CHAPTER V

STRESS-ENERGY TENSOR OF A STATIC, SPHERICALLY SYMMETRIC POINT-SOURCE AND THE INVARIANT $R + \frac{a^2}{6} R^2$

§11.1 The Source-Term $T_\mu^{\ \nu}$ in Rectangular Harmonic Coordinates

In this chapter, from purely a mathematical point of view, we shall discuss the problem of finding an expression for the stress-energy tensor $T_\mu^{\ \nu}$ of a point-source. By a point-source we shall mean a sss point-particle located at the origin of the coordinate system. The problem will be discussed by considering the linearized field equations derived from the invariant $R + \frac{a^2}{6} R^2$, in rectangular harmonic coordinates as well as in isotropic polar coordinates. We shall assume an expression for the tensor $T_\mu^{\ \nu}$ of the point-source, in terms of the δ-function and its derivatives. Integrating the field equations so obtained, we shall show that the solution precisely agrees with that obtained by solving the corresponding linearized empty field equations in Chapter III.

Pechlaner and Sexl have worked on the linearized field equations derived from the invariant $R + \frac{a^2}{6} R^2$. These equations contain on the right-hand side a source-term $T_\mu^{\ \nu}$ defined in terms of the δ-function. This source-term $T_\mu^{\ \nu}$ corresponds to a sss point-particle located at the origin of the coordinate system. However, in this work it is assumed that the expression for $T_\mu^{\ \nu}$ should not contain the derivatives of the δ-function, by arguing that such terms do not correspond to the positive definite mass-density. (See Pechlaner and Sexl [13].) But we note that when the exact line-element of a given space-time is diagonalized, we can immediately know which one of the four coordinates is time-like. The component of $T_\mu^{\ \nu}$ corresponding to this time-like coordinate denotes the mass-density and consequently that particular component of $T_\mu^{\ \nu}$ must be certainly positive-definite at the point of space-time considered. This argument seems to be acceptable as long as the exact line-elements are contemplated but in the case of linearized line-elements it seems to break down. In fact, one cannot accept the linearized line-element as valid in the neighbourhood of the point-source since the linearized field equations cannot be valid there. Consequently, from the solution of such field equations, it seems to be impossible to infer anything about the components of $T_\mu^{\ \nu}$ at the point where the source is located.

On the other hand we note that, in their work, Pechlaner and Sexl have assumed

$$T_\mu^{\ \nu} = m\delta(\underset{\sim}{X})\mathrm{diag}(1,0,0,0) \tag{11.1}$$

where m is the mass of the point-source and $\delta(\underset{\sim}{X})$ is the three dimensional δ-function, and solved the linearized field equations with this source-term. In the solution there finally appears a single constant $M = \chi m/4\pi$, χ being the coupling constant. However, as we have seen in Chapter III, if we solve the linearized empty field equations then there finally appear two different constants of integration in the solution. Obviously the following question arises: "What will be the expression for $T_\mu^{\ \nu}$ in the case of the sss point-source so that one obtains two different constants in the solution instead of only one?"

Guided by the above-mentioned considerations, we shall use an expression for $T_\mu^{\ \nu}$ which, though it depends upon the derivatives of the

δ-function, leads to two different constants of integration in the solution of the linearized field equations. Let us assume the following expression for $T_{\mu\nu}$:

$$T_{00} = m\delta(\underset{\sim}{X}) + a^2 n\Delta\delta(\underset{\sim}{X})$$

$$T_{\mu\nu} = 0 \qquad \text{if} \quad \mu \neq 0 \quad \text{or} \quad \nu \neq 0 \tag{11.2}$$

where m and n are two different constants, $\delta(\underset{\sim}{X})$ is a three dimensional spatial δ-function and Δ is a three dimensional spatial Laplacian operator. Now putting

$$h_{00} = \tfrac{1}{2}(\gamma_{00} + \gamma_{\ell\ell})$$

$$h_{0n} = \gamma_{0n} = 0 \tag{11.3}$$

$$h_{mn} = \gamma_{mn} + \tfrac{1}{2}\delta_{mn}(\gamma_{00} - \gamma_{\ell\ell})$$

where $h_{\mu\nu}$ and $\gamma_{\mu\nu}$ are of first order in the parameter $\epsilon = 1/c^2$, we obtain from the linearized non-empty field equations (9.5) of Chapter IV, the following equivalent field equations:

$$\Delta\gamma_{00} - \frac{a^2}{3}\Delta.\Delta\gamma = 2(\chi\, T_{00})_{\mathcal{L}}$$

$$\Delta\gamma_{s\ell} + \frac{a^2}{3}\delta_{s\ell}\,\Delta.\Delta\gamma - \frac{a^2}{3}\Delta\gamma_{|s\ell} = 2(\chi\, T_{s\ell})_{\mathcal{L}} \tag{11.4}$$

where $\gamma = \gamma_{00} - \gamma_{\ell\ell}$ and χ is the coupling constant while the suffix $\mathcal{L}$ indicates a linearized quantity. Putting $\chi = \frac{8\pi G}{c^2}$ and assuming that $T_{\mu\nu}$ as given by (11.2) is of order zero in the parameter $\epsilon = 1/c^2$, we obtain, from (11.4), the following linearized non-empty field equations for the point-source:

$$\Delta\gamma_{00} - \frac{a^2}{3}\Delta.\Delta\gamma = 2\chi(m\delta(\underset{\sim}{X}) + a^2 n\Delta\delta(\underset{\sim}{X}))$$

$$\Delta\gamma_{s\ell} + \frac{a^2}{3}\delta_{s\ell}\Delta.\Delta\gamma - \frac{a^2}{3}\Delta\gamma_{|s\ell} = 0 \quad . \tag{11.5}$$

Now contracting the field equations (11.5) we obtain

$$a^2\Delta.\Delta\gamma - \Delta\gamma = -2\chi(m\delta(\underset{\sim}{X}) + a^2 n\Delta\delta(\underset{\sim}{X})) \quad . \tag{11.6}$$

To integrate the equations (11.5) and (11.6) we shall make use of the theory of Fourier transforms. Therefore first we shall state a few necessary definitions and results from this theory. The Fourier transform $g(\xi,\eta,\zeta)$ of a function $f(x,y,z)$ is defined by the equations

$$\tilde{f} = F(f) = g(\xi,\eta,\zeta) = \iiint e^{-2i\pi(\xi x+\eta y+\zeta z)} f(x,y,z)\,dxdydz \tag{11.7}$$

$$\tilde{g} = F(g) = f(x,y,z) = \iiint e^{2i\pi(x\xi+y\eta+z\zeta)} g(\xi,\eta,\zeta)\,d\xi d\eta d\zeta \quad . \tag{11.8}$$

Putting

$$x = x^1 \quad , \quad y = x^2 \quad , \quad z = x^3$$

$$2\pi\xi = \xi^1 \quad , \quad 2\pi\eta = \xi^2 \quad , \quad 2\pi\zeta = \xi^3 \quad , \tag{11.9}$$

we write (11.7) and (11.8) as

$$\tilde{f} = F(f) = g(\xi,\eta,\zeta) = \iiint e^{-i(\xi^1x^1+\xi^2x^2+\xi^3x^3)} f(x^1,x^2,x^3)\,dx^1dx^2dx^3 \tag{11.10}$$

$$\tilde{g} = F(g) = f(x^1,x^2,x^3) = \frac{1}{(2\pi)^3}\iiint e^{i(x^1\xi^1+x^2\xi^2+x^3\xi^3)} g(\xi^1,\xi^2,\xi^3)\,d\xi^1d\xi^2d\xi^3. \tag{11.11}$$

Putting

$$\begin{aligned} x^1 &= r \sin\theta \cos\phi \\ x^2 &= r \sin\theta \sin\phi \\ x^3 &= r \cos\theta \end{aligned} \tag{11.12}$$

and

$$\begin{aligned} \xi^1 &= \rho \sin\alpha \cos\beta \\ \xi^2 &= \rho \sin\alpha \sin\beta \\ \xi^3 &= \rho \cos\alpha \end{aligned} \tag{11.13}$$

we obtain

$$\begin{aligned} r^2 &= (x^1)^2 + (x^2)^2 + (x^3)^2 \\ \text{and} \qquad \rho^2 &= (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 \ . \end{aligned} \tag{11.14}$$

Therefore for a function f which depends only upon the radial variable r , (11.11) gives

$$f(r) = \frac{1}{(2\pi)^3} \iiint g(\rho) e^{ir\rho\cos\alpha} \rho^2 \sin\alpha \, d\rho d\alpha d\beta \quad . \tag{11.15}$$

Now from the above-mentioned formulae we can obtain the following results:

$$\begin{aligned} (i\xi^k)^m \tilde{f} &= F\left(\frac{\partial^m f}{\partial (x^k)^m} \right) \\ (-i)^{-m} \frac{\partial^m \tilde{f}}{\partial (\xi^k)^m} &= F[(x^k)^m f] \end{aligned} \tag{11.16}$$

where $k = 1,2,3$ and m is a positive integer. Moreover, we shall use the following two formulae regarding the δ-function:

$$F[\delta(\underset{\sim}{X})] = F[\delta(x^1)\delta(x^2)\delta(x^3)] = 1 \tag{11.17}$$

$$(i\xi^k)^m = F\left[\frac{\partial^m \delta(\underset{\sim}{X})}{\partial (x^k)^m}\right] \tag{11.18}$$

(see L. Schwartz [15]).

Now consider equation (11.16). Taking the Fourier transform of this equation we obtain

$$\rho^2(a^2\rho^2 + 1)\tilde{\gamma} = -2\chi m + 2\chi a^2 n\rho^2 \quad . \tag{11.19}$$

Therefore

$$\tilde{\gamma} = \frac{-2\chi m}{\rho^2(a^2\rho^2+1)} + \frac{2\chi a^2 n}{(a^2\rho^2+1)} \quad . \tag{11.20}$$

Taking the inverse transform of (11.20),

$$\gamma = \frac{-\chi m}{2\pi r} + \frac{\chi(m+n)e^{-r/a}}{2\pi r} \quad . \tag{11.21}$$

Now considering the first equation in (11.5) and taking its Fourier transform,

$$-\rho^2\tilde{\gamma}_{00} = \frac{a^2}{3}\rho^4\tilde{\gamma} + 2\chi m - 2\chi a^2 n\rho^2 \quad . \tag{11.22}$$

Therefore

$$\tilde{\gamma}_{00} = -\frac{a^2}{3}\rho^2\tilde{\gamma} - \frac{2\chi m}{\rho^2} + 2\chi a^2 n \quad . \tag{11.23}$$

Taking the inverse-transform of (11.23),

$$\gamma_{00} = \frac{a^2}{3}\Delta\gamma - \frac{\chi m}{2\pi r} + 2\chi a^2 n\delta(\underset{\sim}{X}) \quad . \tag{11.24}$$

Substituting in (11.24) for γ, from (11.21) and observing that

$$\Delta\left(-\frac{1}{4\pi r}\right) = \delta(\underset{\sim}{X})$$

$$\Delta\left(-\frac{e^{-r/a}}{4\pi r}\right) = -\frac{e^{-r/a}}{4\pi r a^2} + \delta(\underset{\sim}{X}) \tag{11.25}$$

we obtain

$$\gamma_{00} = -\frac{\chi m}{2\pi r} + \frac{\chi(m+n)e^{-r/a}}{6\pi r} + \frac{4a^2}{3}\chi n\delta(\underset{\sim}{X}) \quad . \tag{11.26}$$

Finally considering the second equation in (11.5) and taking its Fourier transform,

$$\tilde{\gamma}_{s\ell} = \delta_{s\ell}\frac{a^2}{3}\rho^2\tilde{\gamma} - \frac{a^2}{3}\xi^s\xi^\ell\tilde{\gamma} \quad . \tag{11.27}$$

Taking the inverse-transform of (11.27),

$$\gamma_{s\ell} = -\delta_{s\ell}\frac{a^2}{3}\Delta\gamma + \frac{a^2}{3}\gamma_{|s\ell} \quad . \tag{11.28}$$

Substituting in (11.28) for γ from (11.21) and using (11.25),

$$\gamma_{s\ell} = -\delta_{s\ell}\frac{\chi(m+n)e^{-r/a}}{6\pi r} + a^2\chi\left[\frac{(m+n)e^{-r/a}}{6\pi r} - \frac{m}{6\pi r}\right]_{|s\ell} + \delta_{s\ell}\frac{2a^2\chi n}{3}\delta(\underset{\sim}{X}) . \tag{11.29}$$

Changing to $h_{\mu\nu}$ from $\gamma_{\mu\nu}$ and using (11.3), (11.21), (11.26) and (11.29), we obtain

$$h = \frac{\chi m}{2\pi r} - \frac{\chi(m+n)e^{-r/a}}{2\pi r}$$

$$h_{00} = -\frac{\chi m}{4\pi r} - \frac{\chi(m+n)e^{-r/a}}{12\pi r} + \frac{4a^2\chi n}{3}\delta(\underset{\sim}{X}) \tag{11.30}$$

$$h_{s\ell} = -\delta_{s\ell}\frac{\chi m}{4\pi r} + \delta_{s\ell}\frac{\chi(m+n)e^{-r/a}}{12\pi r} + a^2\chi\left[\frac{(m+n)e^{-r/a}}{6\pi r} - \frac{m}{6\pi r}\right]_{|s\ell} + \delta_{s\ell}\frac{2a^2\chi n}{3}\delta(\underset{\sim}{X}) \quad .$$

Now putting in (11.30)

$$\frac{\chi m}{4\pi} = M \quad , \quad \frac{\chi(m+n)}{4\pi} = M' \tag{11.31}$$

we obtain

$$\begin{aligned} h_{00} &= -\frac{M}{r} - \frac{M' e^{-r/a}}{3r} + \frac{16\pi a^2}{3}(M'-M)\delta(\underset{\sim}{X}) \\ h_{s\ell} &= -\delta_{s\ell}\frac{M}{r} + \delta_{s\ell}\frac{M' e^{-r/a}}{3r} + \frac{2a^2}{3}\left(\frac{M' e^{-r/a}}{r} - \frac{M}{r}\right)_{|s\ell} + \frac{8\pi a^2}{3}\delta_{s\ell}(M'-M)\delta(\underset{\sim}{X}) \end{aligned} \tag{11.32}$$

where M and M' are constants.

The result in (11.32) precisely agrees with that of Pechlaner and Sexl if we put $M' = M$, that is, $n = 0$.

Since we are interested in the exterior field of the static point-source we can ignore the δ-function from the solution (11.32). Then we obtain

$$\begin{aligned} h_{00} &= -\frac{M}{r} - \frac{M' e^{-r/a}}{3r} \\ h_{s\ell} &= -\delta_{s\ell}\frac{M}{r} + \delta_{s\ell}\frac{M' e^{-r/a}}{3r} + \frac{2a^2}{3}\left(\frac{M' e^{-r/a}}{r} - \frac{M}{r}\right)_{|s\ell} \quad . \end{aligned} \tag{11.33}$$

Considering (11.29) we can easily verify that the harmonic conditions

$$\gamma_{s\ell|\ell} = 0 \tag{11.34}$$

are satisfied.

As seen in Chapter III, the term containing the factor $(M/r)_{|s\ell}$ in (11.33) can be transformed away by making a suitable coordinate transformation and without disturbing the harmonic character of the coordinate system. Then it follows that the solution (11.33) precisely agrees with that of Gregory. Further as seen in Chapter III we can transform the solution to harmonic polar coordinates and isotropic polar coordinates.

§12.1 The Source-Term $T_\mu{}^\nu$ in Isotropic Polar Coordinates

In this section we shall consider the same problem as that of Section 11.1 but by using an isotropic polar coordinate system instead of a rectangular harmonic coordinate system.

We shall assume the following expression for $T_{\mu\nu}$:

$$
\begin{aligned}
T_{00} &= m\delta(\underline{\bar{r}}) + a^2 n\Delta\delta(\underline{\bar{r}}) \\
T_{\mu\nu} &= 0 \qquad \text{if} \quad \mu \neq 0 \quad \text{or} \quad \nu \neq 0
\end{aligned}
\tag{12.1}
$$

where m and n are constants and

$$
\Delta = \frac{\partial^2}{\partial\bar{r}^2} + \frac{2}{\bar{r}}\frac{\partial}{\partial\bar{r}} \quad , \quad \delta(\underline{\bar{r}}) = \frac{\delta(\bar{r})}{2\pi\bar{r}^2} \quad . \tag{12.2}
$$

The point-source is supposed to be located at the origin $\bar{r} = 0$. As seen in Chapter IV, the linearized non-empty field equations in isotropic polar coordinates become

$$
\begin{aligned}
R_0{}^0 - \tfrac{1}{2}R + \frac{a^2}{3}\left(R'' + \frac{2R'}{\bar{r}}\right) &= -(\chi T_{00})_{\mathcal{L}} \\
R_1{}^1 - \tfrac{1}{2}R + \frac{2a^2R'}{3\bar{r}} &= (\chi T_{11})_{\mathcal{L}} \\
R_2{}^2 - \tfrac{1}{2}R + \frac{a^2}{3}\left(R'' + \frac{R'}{\bar{r}}\right) &= (\chi T_{22})_{\mathcal{L}} = (\chi T_{33})_{\mathcal{L}}
\end{aligned}
\tag{12.3}
$$

where the linearized $R_0^{\ 0}$, $R_1^{\ 1}$, $R_2^{\ 2}$ and R are given by (10.3) in Chapter IV. Putting $\chi = 8\pi G/c^2$ and supposing that the components of $T_{\mu\nu}$ as described by (12.1) are of the order zero in the parameter $\epsilon = 1/c^2$, from (12.3) we obtain the following three equivalent equations:

$$R_0^{\ 0} - \tfrac{1}{2}R_1^{\ 1} - R_2^{\ 2} + \tfrac{1}{4}R = -\chi(m\delta(\bar{r}) + a^2 n\Delta\delta(\bar{r}))$$

$$R_1^{\ 1} - \tfrac{1}{2}R + \frac{2a^2R'}{3\bar{r}} = 0 \tag{12.4}$$

$$a^2\Delta R - R = -\chi(m\delta(\bar{r}) + a^2 n\Delta\delta(\bar{r})) \quad .$$

Since

$$R = -\Delta(2\mu+\nu) \tag{12.5}$$

putting

$$2\mu + \nu = \xi \quad , \tag{12.6}$$

$$R = -\Delta\xi \quad . \tag{12.7}$$

Therefore the last equation in (12.4) becomes

$$a^2\Delta\Delta\xi - \Delta\xi = \chi(m\delta(\bar{r}) + a^2 n\Delta\delta(\bar{r})) \quad . \tag{12.8}$$

Using the variables x^1 , x^2 , x^3 such that

$$(\bar{r})^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 \tag{12.9}$$

we can apply the theory of Fourier transforms to integrate the equation (12.8). Therefore taking the Fourier transform of the equation (12.8) we obtain

$$\tilde{\xi} = \frac{\chi m}{\rho^2(a^2\rho^2+1)} - \frac{\chi a^2 n}{(a^2\rho^2+1)} \quad . \tag{12.10}$$

Taking the inverse transform of (12.10) we obtain

$$\xi = \frac{\chi m}{4\pi\bar{r}} - \frac{\chi(m+n)e^{-\bar{r}/a}}{4\pi\bar{r}} \quad . \tag{12.11}$$

Therefore

$$2\mu + \nu = \frac{\chi m}{4\pi\bar{r}} - \frac{\chi(m+n)e^{-\bar{r}/a}}{4\pi\bar{r}} \quad . \tag{12.12}$$

Now consider the first equation in (12.4). This can be written as follows:

$$\Delta(\mu-\nu) = -2\chi(m\delta(\underset{\sim}{r}) + a^2 n\Delta\delta(\underset{\sim}{r})) \quad . \tag{12.13}$$

Taking the Fourier transform of (12.13) we obtain

$$\tilde{\mu} - \tilde{\nu} = \frac{2\chi m}{\rho^2} - 2\chi a^2 n \quad . \tag{12.14}$$

Taking the inverse transform of (12.14), we obtain

$$\mu - \nu = \frac{\chi m}{2\pi\bar{r}} - 2\chi a^2 n\delta(\underset{\sim}{r}) \quad . \tag{12.15}$$

Therefore using (12.12) and (12.15) we obtain

$$\begin{aligned} \mu &= \frac{\chi m}{4\pi\bar{r}} - \frac{\chi(m+n)e^{-\bar{r}/a}}{12\pi\bar{r}} - \frac{2\chi a^2 n}{3}\delta(\underset{\sim}{r}) \\ \nu &= -\frac{\chi m}{4\pi\bar{r}} - \frac{\chi(m+n)e^{-\bar{r}/a}}{12\pi\bar{r}} + \frac{4\chi a^2 n}{3}\delta(\underset{\sim}{r}) \quad . \end{aligned} \tag{12.16}$$

Now consider the second equation in (12.4). This can be written as follows:

$$\mu' + \nu' - \frac{2a^2}{3}\Delta(2\mu'+\nu') = 0 \quad . \tag{12.17}$$

Using (12.12) and (12.16) we can show that the equation (12.17) is satisfied provided we note that

$$\begin{aligned} \Delta\left(-\frac{1}{4\pi\bar{r}}\right) &= \delta(\underset{\sim}{\bar{r}}) \\ \Delta\left(-\frac{e^{-\bar{r}/a}}{4\pi\bar{r}}\right) &= -\frac{e^{-\bar{r}/a}}{4\pi\bar{r}a^2} + \delta(\underset{\sim}{\bar{r}}) \quad . \end{aligned} \tag{12.18}$$

Now putting in (12.16)

$$\frac{\chi m}{4\pi} = M \quad , \quad \frac{\chi(m+n)}{4\pi} = M' \tag{12.19}$$

we obtain

$$\begin{aligned} \mu &= \frac{M}{\bar{r}} - \frac{M'e^{-\bar{r}/a}}{3\bar{r}} - \frac{8\pi a^2}{3}(M'-M)\delta(\underset{\sim}{\bar{r}}) \\ \nu &= -\frac{M}{\bar{r}} - \frac{M'e^{-\bar{r}/a}}{3\bar{r}} + \frac{16\pi a^2}{3}(M'-M)\delta(\underset{\sim}{\bar{r}}) \end{aligned} \tag{12.20}$$

where M and M' are constants. Since we are interested in the solution of the field equations outside the source, we can ignore the δ-function appearing in (12.20). Thus the solution becomes

$$\begin{aligned} \mu &= \frac{M}{\bar{r}} - \frac{M'e^{-\bar{r}/a}}{3\bar{r}} \\ \nu &= -\frac{M}{\bar{r}} - \frac{M'e^{-\bar{r}/a}}{3\bar{r}} \end{aligned} \tag{12.21}$$

which precisely agrees with the solution obtained by solving the linearized empty field equations in Chapter III.

CHAPTER VI

STRESS-ENERGY TENSOR OF A STATIC, SPHERICALLY SYMMETRIC POINT-SOURCE IN GENERAL RELATIVITY

§13.1 On Tangherlini's Source-Term in General Relativity

Considering the exact non-empty field equations of general relativity in the standard polar coordinate system and continuing the Schwarzschild exterior solution of a sss point-source down into the source, Tangherlini has derived the expression for its stress-energy tensor. (See Tangherlini [17].) In this chapter, using the philosophy of the realistic δ-functions, we shall show that Tangherlini's result follows. The exact non-empty field equations of general relativity are

$$G_\mu^{\ \nu} = -8\pi T_\mu^{\ \nu} \tag{13.1}$$

where $G_\mu^{\ \nu}$ is the Einstein-tensor and $T_\mu^{\ \nu}$ is the stress-energy tensor, the units of mass and time being so chosen that the coupling constant $\chi = 8\pi$.

When one is dealing with the gravitational field of an isolated point-source in the region outside the source, obviously one puts $T_\mu^{\ \nu} = 0$ and consequently the field equations (13.1) read as

$$G_\mu^{\ \nu} = 0 \qquad \text{(for the region outside the source)} . \tag{13.2}$$

The solution of (13.2) describes the gravitational field of the source in the region around it.

Assuming that the gravitational field of the source is sss , one would expect that it must be describable by the sss line-element, expressed in the standard polar coordinate system,

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 \tag{13.3}$$

where $\nu = \nu(r)$, $\lambda = \lambda(r)$ and it is assumed that the source is located at the origin $r = 0$.

Calculating the components of the Einstein-tensor $G_\mu^{\ \nu}$ corresponding to the line-element, for the non-vanishing components we obtain:

$$\begin{aligned} G_0^{\ 0} &= -\frac{1}{r^2} + \frac{e^{-\lambda}}{r^2} - \frac{e^{-\lambda}\lambda'}{r} \\ G_1^{\ 1} &= -\frac{1}{r^2} + \frac{e^{-\lambda}}{r^2} + \frac{e^{-\lambda}\nu'}{r} \\ G_2^{\ 2} = G_3^{\ 3} &= e^{-\lambda}\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} + \frac{\nu'}{2r} - \frac{\lambda'}{2r} - \frac{\lambda'\nu'}{4}\right) \end{aligned} \tag{13.4}$$

where an overhead dash denotes differentiation with respect to r .

Substituting for $G_\mu^{\ \nu}$ in (13.2) , from (13.4), and integrating, one obtains the well-known Schwarzschild solution:

$$e^{\nu} = e^{-\lambda} = 1 - \frac{2m}{r} \qquad (r > 0) \tag{13.5}$$

where the constant m is equal to the gravitational mass of the particle.

Since the particle is located at the origin $r = 0$, the tensor $T_\mu^{\ \nu}$ must be non-zero at $r = 0$ and since there is no matter at a point outside the origin, the tensor $T_\mu^{\ \nu}$ must vanish if $r > 0$. Moreover, because we suppose that the particle has a finite non-vanishing gravitational mass it follows that the expression for $T_\mu^{\ \nu}$ of the particle must be expressible in terms of the δ-function. Noting that $G_\mu^{\ \nu}$ vanishes identically if $\mu \neq \nu$, we infer that the tensor $T_\mu^{\ \nu}$ of the particle must be zero if $\mu \neq \nu$. Consequently we write

$$\begin{aligned} T_0^{\ 0} &= a_0 m\Delta(\vec{r},\epsilon) \\ T_1^{\ 1} &= a_1 m\Delta(\vec{r},\epsilon) \\ T_2^{\ 2} &= T_3^{\ 3} = am\Delta(\vec{r},\epsilon) \end{aligned} \tag{13.6}$$

$$\lambda = \lambda(r,\epsilon) \quad , \quad \nu = \nu(r,\epsilon) \tag{13.7}$$

where ϵ is a positive parameter, a_0, a_1 and a are constants and $\Delta(\vec{r},\epsilon)$ is a three dimensional realistic δ-function defined by

$$\Delta(\vec{r},\epsilon) = \frac{\Delta(r,\epsilon)}{2\pi r^2} = \frac{\epsilon}{2\pi^2 r^2 (r^2+\epsilon^2)} \tag{13.8}$$

where $\Delta(r,\epsilon)$ is a one dimensional realistic δ-function.

Now consider the field equations (13.1). The $\langle 00 \rangle$ and $\langle 11 \rangle$ field equations are as follows:

$$\begin{aligned} -\frac{1}{r^2} + \frac{e^{-\lambda}}{r^2} - \frac{e^{-\lambda}\lambda'}{r} &= -8\pi T_0^{\ 0} \\ -\frac{1}{r^2} + \frac{e^{-\lambda}}{r^2} + \frac{e^{-\lambda}\nu'}{r} &= -8\pi T_1^{\ 1} \end{aligned} \tag{13.9}$$

where $\lambda = \lambda(r,\epsilon)$, $\nu = \nu(r,\epsilon)$ and $T_0{}^0$ and $T_1{}^1$ are given by (13.6).

Because of (13.6) and (13.8), the equations (13.9) become

$$\begin{aligned} -\frac{1}{r^2} + \frac{e^{-\lambda}}{r^2} - \frac{e^{-\lambda}\lambda'}{r} &= -\frac{4a_0 m\epsilon}{\pi r^2(r^2+\epsilon^2)} = -8\pi T_0{}^0 \\ -\frac{1}{r^2} + \frac{e^{-\lambda}}{r^2} + \frac{e^{-\lambda}\nu'}{r} &= -\frac{4a_1 m\epsilon}{\pi r^2(r^2+\epsilon^2)} = -8\pi T_1{}^1 \end{aligned} \tag{13.10}$$

where $\lambda = \lambda(r,\epsilon)$ and $\nu = \nu(r,\epsilon)$.

Now integrating the first of equations (13.10), between the limits zero and r , we obtain

$$r(1-e^{-\lambda}) - [r(1-e^{-\lambda})]_{r=0} = \frac{4ma_0}{\pi} \arctan\frac{r}{\epsilon} = 8\pi\int_{r=0}^{r} T_0{}^0 r^2 dr \quad . \tag{13.11}$$

Since in the δ-function philosophy we suppose that the function $\lambda(r,\epsilon)$ is continuous at the origin $r = 0$, we observe that $e^{-\lambda(r,\epsilon)}$ is finite at the origin $r = 0$. Consequently we have

$$[r(1-e^{-\lambda})]_{r=0} = 0 \quad . \tag{13.12}$$

Because of (13.12), (13.11) reduces to

$$r(1-e^{-\lambda}) = \frac{4ma_0}{\pi} \arctan\frac{r}{\epsilon} = 8\pi\int_{r=0}^{r} T_0{}^0 r^2 dr \quad . \tag{13.13}$$

Therefore

$$e^{-\lambda} = 1 - \frac{4ma_0}{\pi r} \arctan\frac{r}{\epsilon} = 1 - \frac{8\pi}{r}\int_{r=0}^{r} T_0{}^0 r^2 dr \quad . \tag{13.14}$$

We observe that as r tends to zero we obtain

$$\lim_{r\to 0} e^{-\lambda} = 1 - \frac{4ma_0}{\pi\epsilon} = \text{a finite quantity} \quad . \tag{13.15}$$

Therefore $e^{-\lambda}$ is regular at the origin as required. Finally taking the limit as ϵ tends to zero in equation (13.14) we obtain

$$\lim_{\epsilon\to 0} e^{-\lambda} = 1 - \frac{2ma_0}{r} \quad . \tag{13.16}$$

Equation (13.16) shows that we must put

$$a_0 = 1 \tag{13.17}$$

so that as ϵ tends to zero we obtain the Schwarzschild solution given by (13.5). Since

$$T_0{}^0 = a_0 m \frac{\epsilon}{2\pi^2 r^2 (r^2+\epsilon^2)} \tag{13.18}$$

when $a_0 = 1$,

$$T_0{}^0 = \frac{m}{2\pi r^2} \cdot \frac{\epsilon}{\pi(r^2+\epsilon^2)} \quad . \tag{13.19}$$

We observe that when ϵ tends to zero we obtain

$$T_0{}^0 = \frac{m}{2\pi r^2} \delta(r) \quad . \tag{13.20}$$

Now subtracting the first of equations (13.10) from the second and putting $a_0 = 1$, we obtain

$$\frac{e^{-\lambda}}{r}(\lambda'+\nu') = (1-a_1)\frac{4m\epsilon}{\pi r^2(r^2+\epsilon^2)} \quad . \tag{13.21}$$

Therefore substituting for $e^{-\lambda}$ from (13.14) with $a_0 = 1$,

$$\lambda' + \nu' = \frac{(1-a_1)\left[\dfrac{4m\epsilon}{\pi r(r^2+\epsilon^2)}\right]}{\left[1 - \dfrac{4m}{\pi r}\arctan\dfrac{r}{\epsilon}\right]} \quad . \tag{13.22}$$

This can be written as

$$\lambda' + \nu' = \frac{(a_1-1)\left[\dfrac{-4m\epsilon}{\pi(r^2+\epsilon^2)}\right]}{\left[r - \dfrac{4m}{\pi}\arctan\dfrac{r}{\epsilon}\right]} \quad . \tag{13.23}$$

Putting $\arctan\frac{r}{\epsilon} = z$, we obtain

$$d(\lambda+\nu) = (a_1-1)\frac{dz}{z + A\tan z} \tag{13.24}$$

where $A = -\pi\epsilon/4m$.

Integrating the equation (13.24) between the limits zero and r ,

$$(\lambda+\nu)_r - (\lambda+\nu)_{r=0} = (a_1-1)\int_{z=0}^{\arctan r/\epsilon}\frac{dz}{z + A\tan z} \tag{13.25}$$

where the integral on the right is divergent.

Equation (13.25) shows that $(\lambda+\nu)$ cannot be a regular function at $r = 0$ unless we put $a_1 = 1$. Therefore putting $a_1 = 1$ in the $\langle 11\rangle$ field equation in (13.10) we obtain

$$-\frac{1}{r^2} + \frac{e^{-\lambda}}{r^2} + \frac{e^{-\lambda}\nu'}{r} = -\frac{4m\epsilon}{\pi r^2(r^2+\epsilon^2)} = -8\pi T_1{}^1 \quad . \tag{13.26}$$

Substituting for $e^{-\lambda}$ from (13.14) with $a_0 = 1$ and solving for ν' we obtain

$$\nu' = \frac{\left[\frac{4m}{\pi r^2}\arctan\frac{r}{\epsilon} - \frac{4m}{\pi r}\frac{\epsilon}{(r^2+\epsilon^2)}\right]}{\left[1 - \frac{4m}{\pi r}\arctan\frac{r}{\epsilon}\right]} \quad . \tag{13.27}$$

Integrating (13.27) between the limits zero and r, we obtain

$$\nu(r,\epsilon) - \nu(0,\epsilon) = \left[\log\left(1 - \frac{4m}{\pi r}\arctan\frac{r}{\epsilon}\right)\right]_{r=0}^{r} \quad . \tag{13.28}$$

Taking the limits as r tends to zero, we observe that

$$\nu(0,\epsilon) = \log\left(1 - \frac{4m}{\pi\epsilon}\right) \quad . \tag{13.29}$$

This shows that $\nu(r,\epsilon)$ is regular at the origin $r = 0$. From (13.28) and (13.29) we obtain

$$\nu(r,\epsilon) = \log\left(1 - \frac{4m}{\pi r}\arctan\frac{r}{\epsilon}\right) \quad . \tag{13.30}$$

Therefore

$$e^{\nu(r,\epsilon)} = 1 - \frac{4m}{\pi r}\arctan\frac{r}{\epsilon} \quad . \tag{13.31}$$

From (13.14) with $a_0 = 1$ and (13.31), it follows that

$$e^{-\lambda(r,\epsilon)} = e^{\nu(r,\epsilon)} = 1 - \frac{4m}{\pi r}\arctan\frac{r}{\epsilon} \quad . \tag{13.32}$$

Now since

$$T_1{}^1 = a_1 m \frac{\epsilon}{2\pi^2 r^2 (r^2+\epsilon^2)} \tag{13.33}$$

when we put $a_1 = 1$

$$T_1{}^1 = \frac{m}{2\pi r^2} \frac{\epsilon}{\pi(r^2+\epsilon^2)} \quad . \tag{13.34}$$

We observe that as ϵ tends to zero

$$T_1{}^1 = \frac{m}{2\pi r^2} \delta(r) \tag{13.35}$$

From (13.20) and (13.35) we obtain

$$T_0{}^0 = T_1{}^1 = \frac{m\delta(r)}{2\pi r^2} \quad . \tag{13.36}$$

Since the $\langle 22 \rangle$ field equation is the same as the $\langle 33 \rangle$ field equation, finally we consider the $\langle 22 \rangle$ field equation in (13.1):

$$G_2{}^2 = -8\pi T_2{}^2 \quad . \tag{13.37}$$

This can be written as

$$e^{-\lambda}\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} + \frac{\nu'}{2r} - \frac{\lambda'}{2r} - \frac{\lambda'\nu'}{4} \right) = -8\pi \text{ am } \frac{\epsilon}{2\pi^2 r^2 (r^2+\epsilon^2)} \tag{13.38}$$

where $\lambda = \lambda(r,\epsilon)$ and $\nu = \nu(r,\epsilon)$.

Using the result (13.32), from (13.38) we obtain

$$e^{\nu}\left(\frac{\nu''}{2} + \frac{\nu'^2}{2} + \frac{\nu'}{r} \right) r^2 = -8\pi \text{ am } \frac{\epsilon}{2\pi^2 (r^2+\epsilon^2)} \quad . \tag{13.39}$$

Integrating (13.39) between the limits zero and r , we obtain

$$\frac{1}{2}\int_{r=0}^{r} (e^{\nu}\nu''r^2 + e^{\nu}\nu'^2r^2 + 2e^{\nu}\nu'r)\,dr = -\frac{4am}{\pi}\int_{r=0}^{r} \frac{\epsilon\, dr}{(r^2+\epsilon^2)} \quad . \qquad (13.40)$$

Therefore

$$\frac{1}{2}\int_{r=0}^{r} d(e^{\nu}\nu'r^2) = -\frac{4am}{\pi}\int_{r=0}^{r} \frac{\epsilon\, dr}{(r^2+\epsilon^2)} \quad . \qquad (13.41)$$

Therefore

$$\tfrac{1}{2}(e^{\nu}\nu'r^2)_{r=0}^{r} = -\frac{4am}{\pi}\arctan\frac{r}{\epsilon} \quad . \qquad (13.42)$$

Because $e^{\nu}\nu'$ is regular at the origin $r = 0$, $(e^{\nu}\nu'r^2)_{r=0}$ is equal to zero. Therefore

$$e^{\nu}\nu'r^2 = -\frac{8am}{\pi}\arctan\frac{r}{\epsilon} \, . \qquad (13.43)$$

Substituting for $e^{\nu}\nu'$ from (13.27), we obtain

$$\frac{4m}{\pi}\arctan\frac{r}{\epsilon} - \frac{4mr}{\pi}\frac{\epsilon}{(r^2+\epsilon^2)} = -\frac{8am}{\pi}\arctan\frac{r}{\epsilon} \quad . \qquad (13.44)$$

Taking the limits as ϵ tends to zero we obtain

$$2m = -4am \quad . \qquad (13.45)$$

Therefore

$$a = -\tfrac{1}{2} \quad . \qquad (13.46)$$

Now since

$$T_2^{\ 2} = T_3^{\ 3} = am\,\frac{\epsilon}{2\pi^2r^2(r^2+\epsilon^2)} \qquad (13.47)$$

when we put $a = -\frac{1}{2}$

$$T_2^{\ 2} = T_3^{\ 3} = -\frac{m}{4\pi r^2}\ \frac{\epsilon}{\pi(r^2+\epsilon^2)} \quad . \tag{13.48}$$

We note that as ϵ tends to zero

$$T_2^{\ 2} = T_3^{\ 3} = -\frac{m}{4\pi r^2}\,\delta(r) \quad . \tag{13.49}$$

From the above considerations it follows that the expression for $T_\mu^{\ \nu}$ of the sss point-source located at the origin of the standard polar coordinate system is

$$T_\mu^{\ \nu} = \frac{m}{2\pi r^2}\,\delta(r)\,\mathrm{diag}(1,1,-\tfrac{1}{2},-\tfrac{1}{2}) \tag{13.50}$$

where $\delta(r)$ is one dimensional radial δ-function.

Tangherlini has obtained the same result requiring the Schwarzschild solution as valid also at the origin where the source is situated. The above discussion justifies Tangherlini's assumption.

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