THE SPLITTING OF A FINITE SOLUBLE GROUP

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CONTENTS

Introduction 11

CHAPTER 1 Groups and modules 1
1.1 Basic group theory 1
1.2 Operator groups and modules 5
1.3 Soluble groups 11

CHAPTER 2 Splitting extensions 17
2.1 Split extensions 17
2.2 Four useful basic results 21
2.3 A selection of results to 1961 22

CHAPTER 3 The theory of formations 26
3.1 Formations 26
3.2 Locally defined formations 30
3.3 Chief factors and formations 36

CHAPTER 4 Splitting with formations 41
4.1 Complements of formation residuals 41
4.2 Complements of normal subgroups 48
4.3 Conjugacy of the complements 51

BIBLIOGRAPHY 56

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INTRODUCTION

One of the main lines of research in finite group theory over the last decade or so has been the search for a complete classification of finite simple groups (those with no non-trivial proper normal subgroups), and it is by no means over. See the last two chapters of Gorenstein's book [5] for a description.

In the end, the motivation for this is the general classification problem for finite groups. Every finite group has a composition series, a maximally refined normal series; if \( G \) is the group, and

\[
G = G_0 \triangleright G_1 \triangleright \ldots \triangleright G_{n-1} \triangleright G_n = 1
\]  

(\#)

is a composition series, the composition factors \( G_i/G_{i+1} \) are simple groups. In this sense, every finite group is built up from finite simple groups.

But it is clearly not enough to have classified the finite simple groups: one must also know how they may be put together in a composition series. The "extension problem": the problem of determining for given groups \( N \) and \( K \) all possible extensions of \( N \) by \( K \) (i.e. groups \( G \) such that \( N \) is normal in \( G \) and \( G/N \cong K \)); arises here in the following way. If one has a composition series (\#), then \( G_1 \) is an extension of \( G_{i+1} \) by \( G_i/G_{i+1} \), so if an answer to the extension problem were available, along with a classification of all finite simple groups, we could by repeated applications list all possible composition series with the simple group \( G_{n-1} \) at the bottom. All finite groups would occur, as the starting simple group were varied. (See [12], pp. 127-128).

Two restrictions on this general idea make it more open to investigation.

First, we restrict ourselves to a special kind of extension, namely "split" extensions. The split extensions of a group \( N \) by a
group K are determined by a well-known construction, the semi-direct products of N by K corresponding to the possible homomorphisms from K to the automorphism group of N. Here the split extension can be described given a knowledge of K and the automorphism group of N. (Schenkman ([13], p. 143) is wrong in claiming that a knowledge of the latter is sufficient; his description of a split extension there is incorrect).

Second, we consider only the finite soluble groups, which have composition factors of prime order. This means that the finite simple soluble groups have been classified for us: they are the cyclic groups of prime order. The further advantage of these composition factors is that they are abelian. Then a chief factor of a soluble group, being a characteristically simple soluble group, and a direct product of isomorphic simple soluble groups, is elementary abelian, and can be regarded naturally as a vector space over which the group may be represented.

The only obstacle is knowing when an extension is a split extension; when is a normal subgroup N of a group G (thereby an extension of N) such that G is a split extension of N? This question is rephrased in the essay as a search for conditions on a normal subgroup for it to be complemented.

It is this problem that the essay surveys.

After presenting in Chapter 1 the finite group theory I need, I describe in Chapter 2 split extensions and summarise some conditions on a normal subgroup which guarantee splitting. The last few of these foreshadow more recent developments presented in the final chapter (4), for which the tools of formation theory are needed; these are prepared in Chapter 3. The main result of section 4.2 is logically independent of Chapter 3, but relies heavily on its ideas.
CHAPTER ONE: GROUPS AND MODULES

In this chapter I summarise the finite group theory which is needed in the following chapters. I include a proof for some elementary results which do not occur in the standard texts, at least not in a suitable form.

From now to the end, I use the term "group" to mean "finite group". This will not be confusing since I do not deal with infinite groups, though some of the statements may well be true for infinite groups as well.

The material in this chapter can be found in most books on the subject: see, for instance, [8], [5], [6], [11], [15], [12], and the "bible" [10]. Only [10] and [11], however, deal with the objects of the last sub-section.

1.1 BASIC GROUP THEORY

I assume such a familiarity with the following concepts that I do not even give their first properties:

inverse, subgroup, order, abelian, coset, conjugacy, normal subgroup, characteristic subgroup, factor (quotient) group, homomorphism, isomorphism etc., inner automorphism, direct product, centre, Sylow subgroup, normalizer, centralizer, commutator, field, vector space, group algebra.

Throughout this essay, an upper case Roman letter, especially G, is used to refer to a group, and a lower case Roman letter to refer to a group element. With a few noted exceptions, the group operation is written multiplicatively and 1 denotes ambiguously the identity and the trivial subgroup consisting of the identity of whatever group is under consideration.

*) An integer in square brackets refers to the bibliography at the end.
That $H$ is a subgroup of $G$ is denoted by $H \leq G$, a proper subgroup
by $H < G$, and a normal subgroup by $H \trianglelefteq G$. The order of $G$ is written $|G|$, the
index of a normal subgroup $N$ in $G$ $[G:N]$, the quotient group of cosets
of $N \triangleleft G$ $G/N$ or $\frac{G}{N}$, the commutator $g^{-1}h^{-1}gh [g,h]$, and that the integer $p$
divides the integer $n$ $p \mid n$. Sometimes $[G,G]$ is written $G'$, and con-
jugacy is written in the exponential manner: $h^g = g^{-1}hg$. Mappings and
coglate operations are written on the right. The end of a proof is
marked by $\Box$.

If $x$ is a set of prime numbers, an integer $n$ is called a $x$-number
when it is divisible only by primes in $x$, that is, the prime factor-
ization of $n$ only primes in $x$ (but not necessarily all of them). A
group is a $x$-group if its order is a $x$-number. In particular, if $x = \{p\}$,
we drop the braces and talk of $p$-groups, groups of order a power of $p$.
The complementary set of all primes not in $x$ is denoted by $x'$, and we
write $p'$ for $\{p\}'$.

If $H$ and $K$ are normal $x$-subgroups of the group $G$, then so is $HK$,
as $|HK|$ divides $|H| |K|$. Thus it makes sense to talk of the maximal
normal $x$-subgroup of $G$, $O_x(G)$. For convenience, $O_{\{p\}}(G)$ is written as
$O_{p}(G)$, and $O_{p'}(G)$ as $O_{p'}(G)$. It is easy to check that $O_x(G)$ is a character-
istic subgroup of $G$. Finally, $O_{p'}(G)$ is defined contextually by

$$\frac{O_{p'}(G)}{O_{p}(G)} = O_{p'}\left(\frac{G}{O_{p}(G)}\right).$$

(1.1) PROPOSITION

If $G$ is a group with subgroups $H$ and $K$, $H \leq G$, and $H \cap K = 1$,
then $O_{p}(KH/H) = O_{p}(K)H/H$.

Proof: First, $KH/H \trianglelefteq K/H \trianglelefteq K$. Second, $O_{p}(K)H/H \ntrianglelefteq O_{p}(K)$.

Hence $O_{p}(KH/H) \trianglelefteq O_{p}(K) \ntrianglelefteq O_{p}(K)H/H$. ($\ast$)

Also, $O_{p}(K)$ is normal in $K$, and $H$ is normal in $G$, so we have
$O_{p}(K)H$ normal in $KH$, and $O_{p}(K)H/H$ is a normal $p'$-subgroup of $KH/H$.,
so \( o_P(k)H \leq o_P(kH) \). But these two subgroups have the same order, by (a).

Hence they are identical. \( \square \)

In the "n" terminology, a Sylow p-subgroup may be defined as a p-subgroup with index a p'-number. I repeat here for future reference the famous theorems on Sylow subgroups.

(1.2) SYLOW'S THEOREMS

If \( G \) is a group, and \( p \) a prime number, then

(a) \( G \) has a Sylow p-subgroup,

(b) any two Sylow p-subgroups are conjugate in \( G \),

(c) any p-subgroup of \( G \) is contained in some Sylow p-subgroup of \( G \).

Proof: See, for instance, [6], pp. 44-45.

The following two results on Sylow subgroups have applications later.

(1.3) PROPOSITION

Let \( G \) be a group with normal subgroup \( N \). Then every Sylow p-subgroup of \( G/N \) has the form \( P/N \), where \( P \) is a p-subgroup of \( G \).

Proof: Let \( S/N \) be a Sylow p-subgroup of \( G/N \), where \( S \leq G \). Let \( P \) be a Sylow p-subgroup of \( S \). Then \( P \leq S \) and \( N \leq S \), so \( P/N \leq S/N \) and \( P/N \) is a subgroup of \( S/N \).

Consider the integer \( |S/N| = |S|/|P/N| = |S|/|P\cap N| = |S|/|P| \cdot |N/|P\cap N| \).

Now \( |S|/|P| \) is a p'-number, since \( P \) is a Sylow p-subgroup;

and \( |N/|P\cap N| \) is an integer since \( P\cap N \leq N \). Hence \( |S/N|/|P/N| \) is a p'-number.

But it is also a p-number, since \( S/N \) is a p-group. The only possibility for it is 1, so \( |S/N| = |P/N| \) and \( S/N = P/N \). \( \square \)
(1.4) **PROPOSITION**

Let $G$ be a group with Sylow $p$-subgroup $P$, where $p$ is a prime.

If $P$ is abelian, then

$$P \cap Z(G) \cap G' = 1.$$  

**Proof:** See the proof in [10] (IV.2.2, pp. 416-417) which uses the special form of the "transfer" from $G$ to $P/P' = P$.

**ABELIAN GROUPS**

The well-known structure theorem for abelian groups: "A group is abelian if and only if it is the direct product of cyclic groups of prime-power order" provides the basis for defining a particular sort of abelian group, namely when the direct product mentioned is quite simple. A group is **elementary abelian** if it has a decomposition by the fundamental theorem into cyclic groups of the same prime order, say $p$. Thus, an elementary abelian $p$-group is isomorphic to

$$C_p \times C_p \times \ldots \times C_p.$$  

If it has order $p^n$, there are $n$ factors. That this is a proper definition is shown by the uniqueness theorem (3.3.2) of [6],

An elementary abelian $p$-group of order $p^n$ is isomorphic in a natural way to a vector space of dimension $n$ over the field $GF(p)$ of $p$ elements. (For a proof of this, see [5], Theorem 3.2, p. 10).

**NILPOTENT GROUPS**

The **lower central series** of a group $G$ is a sequence of subgroups defined recursively by

$$L_i(G) = G_i, \quad L_{i+1}(G) = [L_i(G), G], \quad i \in \mathbb{N}.$$  

Each $L_i(G)$ is a characteristic subgroup of $G$.

A group is **nilpotent** if its lower central series terminates at $1$, that is, if $L_i(G) = 1$ for some non-negative integer $i$.

For each prime $p$, every $p$-group is nilpotent. In fact a group is
nilpotent if and only if it is the direct product of its Sylow subgroups, so a nilpotent group is the direct product of groups of orders the power of distinct primes. (See [5], Theorem 3.2, p. 23). So nilpotent groups form a class wider than that of abelian groups, and in some senses generalize abelian groups.

A group $G$ is called $p$-nilpotent if $p$ is a prime, and $G$ has a normal $p'$-subgroup $K$ such that $G/K$ is a $p$-group. Clearly $K = O^p(G)$, and $G = O_{p'}(G)$; thus $O_{p'}(H)$ is sometimes called the maximal $p$-nilpotent normal subgroup of the group $H$. Further, it can be checked that $K$ complements every Sylow $p$-subgroup of $G$, so it is sometimes called a normal $p$-complement of $G$.

A nilpotent group is $p$-nilpotent for each prime $p$ - this follows directly from the structure theorem mentioned above.

1.2 OPERATOR GROUPS AND MODULES

In the next section the behaviour of the elements of a group under the inner automorphisms of the group will be classified to some extent. A large part of this work can be done in the more general context where, instead of the group of inner automorphisms, we consider any set of endomorphisms, and then a group of endomorphisms of an abelian group. This amounts to the study of operator groups, and of modules over groups. We pursue this for the moment in this section, with a view to its later applications.

OPERATOR GROUPS

Let $G$ be a group, and $S$ a set. Assume we are given a mapping $\alpha: S \rightarrow \text{End } G$, so that with each $s \in S$ there is associated a homomorphism of $G$ into $G$. Then $G$ is called an $S$-group, suppressing the nature of $\alpha$.

*) More usually in the literature, the symbol $\mathcal{L}$ is used.
We usually write $g^s$ for $g(sa)$, where $g \in G$ and $s \in S$.

A subgroup $H$ of $G$ is called an **S-subgroup** if $H^s \leq H$ for all $s \in S$. If $N$ is a normal subgroup of $G$, then we make the quotient group $G/N$ into an $S$-group by defining $(gN)^s = g^sN$ for all $s \in S$. (This is well-defined, since if $g_1N = g_2N$, then $g_1g_2^{-1} \in N$ and $(g_1g_2^{-1})^s = g_1^s(g_2^s)^{-1} \in N$ because $N$ is invariant under $S$).

An **irreducible** $S$-group is one with no $S$-subgroups other the trivial subgroup or the whole group.

The **centralizer of $S$ in $G$**, $C_G(S)$, is defined by

$$C_G(S) = \{g \in G \mid g^s = g \text{ for all } s \in S\}.$$

Clearly it is an $S$-subgroup of $G$. If the centralizer of $S$ in $G$ is trivial, we say that $S$ acts **fixed-point-freely** (**fpf** on $G$, for then every non-identity element of $G$ is shifted by some element of $S$.

A mapping $\Theta$ from an $S$-group $G_1$ to an $S$-group $G_2$ is an **$S$-homomorphism** if $\Theta: G_1 \rightarrow G_2$ is a group homomorphism, and for all $g \in G_1$,

$$(g^s)\Theta = (g\Theta)^s \text{ for all } s \in S.$$

An **$S$-isomorphism** is a group isomorphism which is also an $S$-homomorphism.

An **$S$-series** for the $S$-group $G$ is a series $\langle G_1 \mid i=1, \ldots, n \rangle$ (properly, a sequence) of $S$-subgroups of $G$, with each normal in the preceding one:

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n = 1.$$

If each $G_{i+1}$ is a maximal $S$-subgroup of $G$ contained in $G_i$ ($i=0, \ldots, n-1$), or, what is equivalent, if each factor $G_i/G_{i+1}$ is an irreducible $S$-group, then the $S$-series is called an **$S$-composition series**, and the factors $G_i/G_{i+1}$ are then called **$S$-composition factors**. Clearly any $S$-group, having only a finite number of subgroups, has an $S$-series, and any $S$-series can be refined by inserting suitable $S$-subgroups in it to give an $S$-composition series.

The degree to which an $S$-composition series and its $S$-composition
factors is unique is the subject of the following form of the Jordan-Hölder theorem due to W. Krull (1925-26) and E. Noether.

(1.5) JORDAN-HÖLDER THEOREM FOR OPERATOR GROUPS

Let $G$ be an $S$-group, and let $<G_i | i=1, \ldots, n>$ and $<H_j | j=1, \ldots, m>$ be two $S$-composition series for $G$. Then $m=n$, and the $S$-composition factors $G_{i+1}/G_i$ and $H_{i+1}/H_i$ are, after a possible renumbering of the $H_j$'s, $S$-isomorphic to each other.

Proof: See [6], Theorem 8.4.4, pp. 126-127, or [10], I.11.5, pp. 63-64.

This allows us to talk of the $S$-composition factors of an $S$-group: strictly, we mean $S$-isomorphism classes of $S$-composition factors.

Let $N$ be an $S$-subgroup of the $S$-group $G$. A $S$-composition factor of $G$ is said to lie above $N$ if it is $S$-isomorphic to some factor whose "denominator" contains $N$ out of some $S$-composition series for $G$ which has $N$ as one of its terms.

The following result will be used quite often in special cases later, and often without comment, since it is suggested by the form of an $S$-subgroup lattice.

(1.6) PROPOSITION


MODULES

An important special case of an operator group occurs when the set of operators is itself a group, and the operator group is abelian. We change the use of symbols, so that $G$ will now often be used for the operators.
Let \( M \) be a \( G \)-group, where \( G \) is a group, and \( M \) is an additively written abelian group. Then \( M \) is called a (right) \( G \)-module if we also require that \( a : G \to \text{End } M \) respect the group structure of \( G \); that is,

\[
(gh)a = ga \cdot ha
\]

(composition of mappings in \( \text{End } M \))

\[
la = l_M
\]

(identity mapping on \( M \))

for all \( g, h \in G \). This is equivalent to the condition that \( m^1 = m \), and \( m^{gh} = (m^g)^h \) for all \( m \) in \( M \).

It follows immediately that each \( ga \ (g \in G) \) is an invertible epimorphism of \( M \), with inverse \((g^{-1})a\). Then \( Ga \) is a group of automorphisms of \( M \), where the group operation is composition of mappings, and is called the group of \( G \)-automorphisms of \( M \), \( \text{Aut}_G(M) \). Clearly \( \text{Aut}_G(M) \cong \frac{G}{\ker a} \), and

\[
\ker a = \{g \in G \mid ga = l_M\}
\]

\[
= \{g \in G \mid m(ga) = m \text{ for all } m \in M\}
\]

which is, by definition, the centralizer of \( M \) in \( G \), \( C_G(M) \). If the centralizer of \( M \) in \( G \) is trivial, we say that \( M \) is a faithful \( G \)-module, for then \( \text{Aut}_G(M) \cong G \). It is easy to see that \( C_G(M) \trianglelefteq G \).

By definition, every non-identity element of \( \text{Aut}_G(M) \) has non-identity action on \( M \). This is rephrased in the following proposition. Note that if \( N \) is a subgroup of \( G \) contained in the centralizer of \( M \), we can define an action of \( G/N \) on \( M \) naturally by \( m^{gN} = m^g \) for all \( g \in G \) and \( m \in M \); \( N \trianglelefteq C_G(M) \) makes it well-defined.

(1.7) PROPOSITION

If \( M \) is a \( G \)-module for the group \( G \), then it is a faithful \( G/C_G(M) \)-module.

Let \( G \) be a group, and \( K \) a field, and suppose the additively written abelian group \( M \) has the extra structures of both a \( G \)-module and a finite-dimensional vector space over \( K \). If these two structures are compatible:
\[(\beta m)^g = \beta(m^g) \quad \text{for all } \beta \in K, g \in G \text{ and } m \in M\]

then we can define an action of the group algebra \(KG\) on \(M\) by recursion from:

\[m^{g+b} = m^a + m^b; \quad m \in M, \quad g \in G, \quad \beta \in K, \quad a, b \in KG.\]

Then \(M\) is called a \(KG\)-module.

We will be interested in the group algebra \(GF(p)G\), where \(G\) is a group and \(GF(p)\) the (Galois) field of \(p\) elements, for a prime \(p\).

(1.8) Proposition

If \(G\) is a group, and \(M\) is an irreducible \(GF(p)G\)-module, then \(G/C_G(M)\) has no non-trivial normal \(p\)-subgroups, i.e. \(O_p(G) = C_G(M)\).

Proof: The proof proceeds by showing that every normal \(p\)-subgroup of \(G\) has trivial action on \(M\). See [2], Theorem (27.28), p. 189, or [10], V.5.17, p. 485.

(1.9) Corollary

If \(H\) is a normal subgroup of the group \(G\), \(H\) has normal \(p\)-complement \(N, M\) is an irreducible \(GF(p)G\)-module and \(N \leq C_G(M)\), then \(H \leq C_G(M)\).

Proof: Since \(N \leq C_G(M)\), we can consider \(M\) as an irreducible \(GF(p)G/N\)-module.

Let \(P\) be a Sylow \(p\)-subgroup of \(H\), so that \(H = NP\), and \(H/N = PN/N\) is a normal subgroup of \(G/N\), and \(PN/N \cong P/N \times P = P\) is a \(p\)-group. Hence by (1.8) \(PN/N\) acts trivially on \(M\), that is, \(PN/N = H/N\) is contained in \(C_{G/N}(M)\), and so \(H\) is contained in \(C_G(M)\).

I shall also have use for two classical theorems of representation theory, couched in terms of modules. But before I present them, I need some definitions.

A \(G\)-subgroup of a \(G\)-module is called a \(G\)-submodule; similarly a \(KG\)-submodule. A \(KG\)-module is completely reducible if it is the direct sum of irreducible \(KG\)-submodules; equivalently, if for every \(KG\)-submodule \(M_i\)
there is a KG-submodule $M_2$ so that the whole KG-module is $M_1 \oplus M_2$.

If $H$ is a subgroup of $G$, and $M$ is a KG-module, then clearly $M$ is also a KH-module, which we denote by $M_H$. If $N$ is a normal subgroup of $G$, and $M_1$ and $M_2$ are KN-modules, then we say $M_1$ and $M_2$ are conjugate if there is a vector space isomorphism $\lambda: M_1 \rightarrow M_2$ and an element $g \in G$ such that

$$(\lambda^h)\lambda = (\lambda g)^{-1}h g, \quad \text{for all } h \in H \text{ and } m \in M_1.$$ 

(1.10) MASCHKE'S THEOREM

Let $G$ be a group, and $K$ a field of characteristic $p$ (for example, $GF(p)$). If $p \nmid |G|$, then every KG-module is completely reducible.

Proof: See [2], Theorem (15.6), p. 88; [5], Theorem 3.1, p. 66; or [10], I.17.7, p. 123.

(1.11) CLIFFORD'S THEOREM

Let $G$ be a group with normal subgroup $N$, and $K$ a field. Let $M$ be an irreducible KG-module. Then $M_N$ is a completely reducible KN-module, and any two irreducible KN-submodules of $M_N$ are conjugate.

Proof: See [2], Theorem (49.2), pp. 343-344; or [10], V.17.3, pp. 565-567.

(1.12) COROLLARY

Let $G$ be a group and $K$ a field, and suppose $M$ is an irreducible KG-module. If $N$ is a normal subgroup of $G$ which does not act trivially on $M$ ($N \not\in C_G(M)$), then $N$ acts fixed-point-freely on $M$.

Proof: By Clifford's theorem, there exist conjugate irreducible $K_N$-submodules of $M_i$, $i = 1, \ldots, n$ say, such that $M = \bigoplus_{i=1}^n M_i$.

If $N$ has a non-zero fixed point in $M$, then it has one in at least one of these direct summands, $M_k$ say. Then $C_K(N)$ is a non-zero KN-submodule of the irreducible KN-module $M_k$, so it must be all of $M_k$. 
Therefore every element of $N$ acts trivially on every element of $M$. This is equivalent to $C_N(M) = N$. Since the $M_i$ are all conjugate, it is easy to show that the $C_N(M_i)$ must be conjugate subgroups of $G$ — in this case, conjugate to $N$. But $N$ is normal in $G$, so $C_N(M_i) = N$ for all $i$, and hence $N \leq C_G(M)$, a contradiction. □

1.3 SOLUBLE GROUPS

CHIEF SERIES

As mentioned at the beginning of the last section, we now consider $G$ the special case of an $S$-group where the elements of $S$ correspond to the inner automorphisms of $G$, and all inner automorphisms are represented. In this case there is a natural identification of $S$ and $G$, where the endomorphism corresponding to $g \in G$ is conjugation by $g$, so we call $G$ in this case a G-group. We call a G-composition series a chief series for $G$, and the G-composition factors chief factors of $G$.

A G-subgroup of $G$ is one invariant under all inner automorphisms of $G$, that is, a normal subgroup of $G$, so a chief series for $G$ is a sequence $<G_i: i=1,\ldots,n>$ of normal subgroups of $G$,

$$G = G_0 > G_1 > \ldots > G_n = 1, \quad G_i \triangleleft G$$

with $G_{i+1}$ a normal subgroup of $G$ maximal with respect to being properly contained in $G_i$, or, equivalently, with $G_i/G_{i+1}$ a minimal normal subgroup of $G/G_{i+1}$ for $i=0,\ldots,n-1$.

SOLUBLE GROUPS

A soluble group is one with abelian chief factors. Soluble groups may also be defined as those groups $G$ whose derived series $<G^{(i)}>\end{inline}$ ends at 1, or as those groups whose composition factors all have prime order. But showing that the definitions are equivalent is straightforward: see, for instance, [6] Theorem 9.2.5, p. 140.
I now list some basic properties of soluble groups, which can be proved directly from the definition. They will be used quite often, and mostly without mention.

(a) subgroups and homomorphic images (quotient groups) of soluble groups are soluble,

(b) nilpotent (and, clearly, abelian) groups are soluble, (the lower central series is a series of normal subgroups with abelian factors, so any chief series refined from it has abelian factors),

(c) a minimal normal subgroup \( M \) of a soluble group \( G \) is an elementary abelian \( p \)-group, for some prime \( p \), \( (M \) is the last term in some chief series for \( G \), and so is abelian; if it were not elementary abelian, it would have a non-trivial proper characteristic subgroup which would then be normal in \( G \), a contradiction).

Let \( G \) be a soluble group with chief series \( \langle G_i \mid i=1, \ldots, n \rangle \). By (a) above, \( G/G_{i+1} \) is soluble \((i=0, \ldots, n-1)\), and has minimal normal subgroup \( G_i/G_{i+1}' \) which must by (c) above be elementary abelian. Hence each chief factor \( G_i/G_{i+1} \) of \( G \) may be written as the direct product

\[
G_i/G_{i+1} \cong C_p^{(1)} \times C_p^{(2)} \times \cdots \times C_p^{(m)}
\]

of \( m \) copies of the cyclic group of order \( p \), where \( p \) is some prime, called the characteristic of \( G_i/G_{i+1} \), and \( m \) is some integer, called the rank of \( G_i/G_{i+1}' \) and both of which depend on \( G_i/G_{i+1} \). In this case \( G_i/G_{i+1} \) would be called a \( p \)-chief factor of rank \( m \). As mentioned on page 4 above, it is a vector space of dimension \( m \) over \( GF(p) \).

Now \( G_i/G_{i+1} \) is a \( G \)-module as well, and if \( \bar{\beta} \in G_i/G_{i+1} \),

\[
(\beta \bar{\beta})^h = \beta (\bar{\beta}^h) \quad \text{for all } \beta \in GF(p) \text{ and } h \in G
\]

and the \( G \)-module structure fits in with the vector space structure in the required manner to make \( G_i/G_{i+1} \) an irreducible \( GF(p)G \)-module.

Note that the characteristic of a chief factor divides its order,
and thus the order of its group. Therefore, Maschke's theorem (1.10) cannot be used directly.

The following equivalence will be used without comment.

(1.13) PROPOSITION

If $G$ is a soluble group, and $H/K$ a chief factor of $G$ (so $H/K$ is a $G$-module), then for any subgroup $F$ of $G$,\
\[
F \subseteq C_G(H/K) \iff [F, H] \subseteq K.
\]

Proof: Let $f \in F$. Then\

\[
f \in C_G(H/K) \iff (hK)^f = hK \quad \text{for all } h \in H
\]
\[
\iff f^{-1}hfhK = hK \quad \text{for all } h \in H
\]
\[
\iff f^{-1}hfh^{-1} \in K \quad \text{for all } h \in H
\]
\[
\iff [f, h^{-1}] \in K \quad \text{for all } h \in H
\]
\[
\iff [f, H] \subseteq K.
\]

The following terminology will prove to be a good abbreviation: if $G$ is a soluble group with subgroup $F$ and chief factor $H/K$, then $F$ is said to cover $H/K$ if $FK \geq H$, and avoid $H/K$ if $F \cap H < K$. The reason for the choice of terms is evident on drawing a conventional subgroup lattice.

It follows that $F$ cannot both cover and avoid $H/K$: if it were to, $FK = FH$ and $F \cap H = F \cap K$, so $\frac{H}{F \cap H} \cong \frac{FH}{F} = \frac{FK}{F} \cong \frac{K}{F \cap K} = \frac{K}{F \cap H}$ which implies that $|H| = |K|$, and this is absurd. However, $F$ may fail to cover or avoid $H/K$.

πA-GROUPS

This is an appropriate stage to introduce a special class of soluble groups which will come up again in Chapter 4. We have already seen a hint of them in (1.4) above.

Let $\pi$ be a set of prime numbers. A $\pi$A-group is a soluble group
such that each of its Sylow $p$-subgroups is abelian, for every prime $p$ in $\pi$. Naturally, $pA$-group is written for $\{p\}A$-group; and if $\pi$ contains all prime numbers, a $\pi A$-group is known as an $A$-group.

(1.14) PROPOSITION

If $G$ is a $\pi A$-group with normal subgroup $N$, then $G/N$ is a $\pi A$-group.

Proof: Let $p \in \pi$, and $S/N$ be a Sylow $p$-subgroup of $G/N$. By (1.3) on page 3 above, there is a $p$-subgroup of $G$, $P$ say, such that $S/N = PN/N$. Now $PN/N \cong P/N \cap P$, and $P$, being contained in a Sylow $p$-subgroup of $G$ by the third Sylow theorem (1.20), is abelian. Hence $S/N$ is abelian. □

HALL SUBGROUPS

An important generalization of Sylow $p$-subgroups, where the prime $p$ is replaced by a set of primes $\pi$, is especially relevant to soluble group theory. It was introduced by Philip Hall in 1937, and the generalized Sylow subgroups have come to bear his name.

A Hall $\pi$-subgroup of a group is a $\pi$-subgroup with index a $\pi'$-number. Clearly a Hall $\{p\}$-subgroup is a Sylow $p$-subgroup. The Sylow theorems also generalize, but only for the class of soluble groups.

(1.15) HALL THEOREMS

If $G$ is a soluble group, and $\pi$ a set of prime numbers, then

(a) $G$ has a Hall $\pi$-subgroup,
(b) any two Hall $\pi$-subgroups of $G$ are conjugate in $G$,
(c) any $\pi$-subgroup of $G$ is contained in some Hall $\pi$-subgroup of $G$.

Proof: See, for instance, [5], Theorem 4.1, p. 231, or [11], Theorem 11.1.1, pp. 185-187.

In fact, Hall showed that this is a characterization of soluble groups, since the converse of (1.15a) holds: if a group has Hall
subgroups for all sets of primes, then it is soluble.

**SYSTEM NORMALIZERS**

Armed with the Hall theorems for soluble groups, we proceed to identify a certain class of subgroups of any soluble group, called the "system normalizers". These subgroups turn out to be special cases of objects to be defined in Chapters Three and Four, but they can be defined now, whereas the generalizations rely on constructions yet to be presented.

For the rest of this section, \( G \) is a soluble group.

Assume we are given for each prime \( p \) dividing \( |G| \) a Hall \( p' \)-subgroup \( \mathcal{X}^p \) of \( G \), sometimes called a **Sylow \( p \)-complement** of \( G \). This is possible by Hall's first theorem (1.15a). The set
\[
\{ \mathcal{X}^p \mid p \text{ divides } |G| \}
\]
is called a **complete set of Sylow complements** of \( G \).

Then, define for any set \( \pi \) of primes dividing \( |G| \) the subgroup
\[
\mathcal{X}^\pi = \bigcap_{p \in \pi} \mathcal{X}^p,
\]
\( \mathcal{X}^\varnothing = G \) if \( \pi \) is empty.
A calculation of orders shows that \( \mathcal{X}^\pi \) is a Hall \( \pi' \)-subgroup of \( G \).
(See [11], pp. 20 and 190). The set \( \mathcal{E} \) of all possible intersections of collections of Sylow complements
\[
\mathcal{E} = \{ \mathcal{X}^\pi \mid \pi \text{ a set of primes dividing } |G| \}
\]
is called a **Sylow system** of \( G \) corresponding to the given complete set of Sylow complements.

Further, corresponding to each Sylow system \( \mathcal{E} \) of \( G \), there is the subgroup of \( G \) consisting of those elements which normalize every member of \( \mathcal{E} \); the **system normalizer** \( N(\mathcal{E}) \) of \( G \) corresponding to \( \mathcal{E} \). Since all the groups of \( \mathcal{E} \) are obtained by taking intersections of certain Sylow complements, an element \( g \) of \( G \) belongs to \( N(\mathcal{E}) \) if and only if
\[
g^{-1} \mathcal{X}^p g = \mathcal{X}^p \quad \text{for each } p \text{ dividing } |G|.
\]
This shows that we need only worry about the Sylow complements in \( \mathcal{E} \),
and that \( N(\Sigma) = \bigcap_{p \mid |G|} N_G(\Sigma^p) \).

System normalizers were defined and studied by Philip Hall [7]. Among other things, he showed that any two system normalizers of \( G \) are conjugate in \( G \) ([7] Theorem 3.1, p. 510, [11] 11.2.3, p. 192) and that if \( N \) is a normal subgroup of \( G \) and \( D \) is a system normalizer of \( G \), then \( DN/N \) is a system normalizer of \( G/N \) ([7] Theorem 7.3, pp. 523-524, [11] 11.4.3, p. 198).

He also introduced a slight generalization of the concept in the case when \( G \) is contained in a larger (not necessarily, but usually soluble) group \( H \). The relative system normalizers of \( G \) in \( H \), \( N_H(\Sigma) \), which consist of those elements of \( H \) which transform, by conjugation, the Sylow systems \( \Sigma \) into themselves. (See [7] Section 4, pp. 514 ff.)
CHAPTER TWO : SPLITTING EXTENSIONS

In this chapter I introduce the concept of the "split extension", show the various guises in which it can occur, and prove four lemmas which will be useful later on. I finish with a brief survey of some of the history of the search for "naturally occurring" split extensions: results which show that certain groups are split extensions. General references for this chapter are [5] Section 2.5 (p. 25), [13] Section III.4 (pp. 90-92), [10] (p. 89), and [15].

2.1 SPLIT EXTENSIONS

EXTENSIONS

If \( N \) and \( K \) are groups, an extension of \( N \) by \( K \) is a group \( G \) with normal subgroup \( N^* \) such that \( N^* \cong N \) and \( G/N^* \cong K \). Roughly, \( G \) can be thought of as a "product" of \( G/N^* \) and \( N^* \). The "extension problem" is given two groups \( N \) and \( K \), determine up to isomorphism all the possible extensions of \( N \) by \( K \).

COMPLEMENTS

If \( G \) is a group with normal subgroup \( N \), we say that \( N \) is complemented by the subgroup \( K \) of \( G \), or that \( K \) complements \( N \) in \( G \), if \( G = NK \) and \( N \cap K = 1 \). The term arises because this is just lattice-theoretic complementation in the subgroup lattice of \( G \), where the "join" of two subgroups is just the subgroup generated by them (their product if one is normal), and the "meet" is their intersection. It can be represented by the diagram

```
     G
    /  
   /    
 N  \  / 
  \ /   
   \    
    K
```

If \( G = NK \), but we do not require \( N \cap K = 1 \), then \( K \) is called a
supplement to $N$ in $G$. If $G = NK$, but $NL < G$ for any $L < K$, that is, $K$ supplements $N$ but any proper subgroup of $K$ fails to, then $K$ is called a minimal supplement to $N$ in $G$.

It is now clear why the normal $p$-complement (page 5) and the Sylow $p$-complement (page 15) have the names they do.

SPLIT EXTENSIONS

A group $G$ is a split extension of its subgroup $N$ by its subgroup $K$ if $N$ is normal in $G$ and $K$ complements $N$ in $G$, and $G$ is said to split over $N$. Sometimes the notation $G = [N]K$ is used (e.g. [13]). Note that $G/N = NK/N \cong K/NK = K$, so the terminology is consistent: a split extension is a particular sort of extension.

SEMI-DIRECT PRODUCTS

Given two groups $N$ and $K$, a group $G$ can be constructed such that $G$ is a split extension of an isomorphic copy of $N$ by an isomorphic copy of $K$: namely the (external) direct product $N \times K$, where

$$N \cong \{(n,1) \mid n \in N\}, \quad K \cong \{(1,k) \mid k \in K\}.$$

This has a generalization in the semi-direct product, described in most books on the subject. As well as $N$ and $K$ we are given an "action" of $K$ on $N$: a group homomorphism $\alpha: K \to \text{Aut} N$. (Aut $N$ is the group (under composition) of automorphisms of $N$.) At least one such action always exists, namely the trivial one $k \mapsto 1_N$ (where $1_N$ is the identity on $N$), and the semi-direct product corresponding to the trivial action is the direct product.

In general, the semi-direct product of $N$ with $K$ corresponding to $\alpha$ is the group with underlying set

$$G = \{(n,k) \mid n \in N, k \in K\}$$

and binary operation

$$(n_1,k_1)(n_2,k_2) = (n_1(k_2\alpha) n_2, k_1 k_2).$$
It is routine to check that this does in fact define a group, that
\[ N^* = \{(n,1) \mid n \in N\}, \quad K^* = \{(1,k) \mid k \in K\} \]
as subgroups identifiable with \( N \) and \( K \) respectively, and that
\[ N^* \triangleleft G, \quad N^* \cap K^* = 1, \quad N^*K^* = G. \]
Hence \( G \) is a split extension of \( N^* \) by \( K^* \). It depends on the action \( a \) as well as \( N \) and \( K \).

Conversely, if \( G \) is a split extension of \( N \) by \( K \), then there is an action of \( K \) on \( N \) (namely conjugation) such that \( G \) is isomorphic to the semi-direct product of \( N \) with \( K \). This is the virtue of the semi-direct product construction. (See \[5\], pp. 25-26).

**SPLIT SHORT EXACT SEQUENCES**

Just as in other categories, a sequence of groups and homomorphisms
\[ G_1 \xrightarrow{\theta_1} G_2 \xrightarrow{\theta_2} \cdots \xrightarrow{\theta_{i-1}} G_i \xrightarrow{\theta_i} \cdots \]
is exact at \( G_m \) if the image \( G_i \theta_i \) coincides with the kernel of \( \theta_{i+1} \).

Suppose we have the short exact sequence of groups and homomorphisms
\[ 1 \rightarrow N \xrightarrow{a} G \xrightarrow{\beta} K \rightarrow 1 \]
which is such a sequence exact at each group not at either end, and where \( 1 \rightarrow N \) is the embedding of the identity in \( N \), and \( K \rightarrow 1 \) is the trivial endomorphism of \( K \). Then \( a \) must be a monomorphism, and \( \beta \) an epimorphism; \( \ker \beta = Na \); \( K \cong G/\ker \beta = G/Na \), and \( N \cong Na \). Hence \( G \) is an extension of \( N \) by \( K \).

If there is a homomorphism \( \theta: K \rightarrow G \) such that
\[ 1 \rightarrow N \xrightarrow{a} G \xrightarrow{\beta} K \xrightarrow{\theta} 1 \]
commutes, then the short exact sequence is said to **split**.

Let \( K^* = K\theta \). Then \( \beta \) maps \( K^* \) isomorphically onto \( K \), so
\[ K^* \cap \ker \beta = 1. \]

But \( \ker \beta = Na \), so \( Na \cap K^* = 1 \), and \( G \) is an extension of \( Na(GN) \) by \( K^*(\mathbb{ZK}) \).
This is a generalization of the definition of split extensions to the case where \( N \) and \( K \) have isomorphic copies in \( G \), but are not necessarily subgroups.

Conversely, if \( G \) is a split extension of \( N \) by \( K \), let \( \alpha: N \to G \) be the embedding map, and \( \pi: G \to K \) (\( G/N \)) be the composition of the natural projection \( G \to G/N \) and the isomorphism \( G/N \cong K \). In this case it is easy to see that

\[
1 \to N \xrightarrow{\alpha} G \xrightarrow{\pi} K \to 1
\]

is a split short exact sequence: the embedding \( K \to G \) does for the left inverse of \( \pi \).

For this approach, see [15] Chapter 9 (pp. 209-254), or [12] (pp. 133-147) where it is given a setting in homology theory (Chapter 10, pp. 207-234).

**COSET REPRESENTATIVES**

An interesting way of thinking of a complemented normal subgroup \( N \) of a group \( G \), is that a set of coset representatives of \( N \) may be chosen such that they form a group, a subgroup of \( G \).

Certainly, if this can be done, giving a subgroup \( K \), then clearly \( N \cap K = 1 \) (the representative of the identity coset), and \( NK = G \) (since any group element can be written as the product of a coset representative and an element of \( N \)).

Conversely, if \( N \triangleleft G \), \( K \triangleleft G \), and \( G = NK \), \( N \cap K = 1 \), then \( G/N = NK/N \cong K/N \cap K = K \), so we have a one-to-one correspondence between cosets of \( N \) and elements of \( K \). Also, \( K \cap gN \) is non-empty for any coset \( gN \) of \( N \) in \( G \), since \( G = NK \). Hence \( K \cap gN \) is a singleton for each coset \( gN \), by an application of the pigeon-hole principle to the partition of \( G \) given by the cosets.
2.2 **FOUR USEFUL BASIC RESULTS**

In what follows, the complemented-normal-subgroup approach to split extensions will be used: we have then only to work with subgroups of a given group.

(2.1) **PROPOSITION**

If $A$ is an abelian normal subgroup of the group $G$, and the subgroup $B$ of $G$ supplements $A$, then $A \triangleleft B$ is normal in $G$.

**Proof:** Let $a \in A \triangleleft B$, and $g \in G$. Then $g = c b$ for some $c \in A$ and $b \in B$. Hence $g^{-1}ag = b^{-1}(c^{-1}a)c b = b^{-1}ab$ since $A$ is abelian

$\begin{align*}
\{ \epsilon A & \quad \text{since } A \triangleleft G \\
\{ \epsilon B & \quad \text{since } b, a \in B.
\end{align*}$

(2.2) **PROPOSITION**

A minimal normal subgroup of a soluble group is complemented only by maximal subgroups.

**Proof:** Let $M$ be a minimal normal subgroup of the soluble group $G$, so $M$ is abelian (remark (c), page 12). Let $K$ be a complement for $M$ in $G$, and let $K < L \triangleleft G$. We want to show that $L = G$.

Clearly $L$ supplements $M$, so by (2.1) above, $L \triangleleft M$ is a normal subgroup of $G$, contained in the minimal normal subgroup $M$.

If $L \cap M = 1$, then $K \not\triangleleft K/M = G/M = LM/M \not\triangleleft L$, contradicting $K < L$.

Hence $L \cap M = M$, and $M \subseteq L$, so $G = LM = L$.

(2.3) **PROPOSITION**

If $G$ is a group with subgroup $K$ which covers each chief factor of $G$ lying above the normal subgroup $N$ in a chief series for $G$ through $N$, then $K$ supplements $N$. Conversely, if $K$ supplements $N$, then it covers all chief factors lying above $N$ in any chief series through $N$. 
Proof: Let \( N = G_n < G_{n-1} < \cdots < G_0 = G \) be that part of the chief series above \( N \). We have that \( K G_i \geq G_{i-1}, \ i=1,\ldots,n, \) and prove by induction that \( K G_i = G, \ i=0,\ldots,n. \)

\[ i = 0: \ K G_0 = K G = G. \]

\[ i > 0: \ i < n \Rightarrow K G_i \geq G_{i-1} \text{ since } K \text{ covers } G_{i-1}/G_i; \]

so \( K G_i \geq K G_{i-1} = G \) by the inductive hypothesis. But \( K \not\leq G, \ G_i \leq G. \)

Hence \( K G_i = G. \)

In particular, when \( i = n, \ K G_n = K N = G. \)

Conversely, if \( K N = G, \) and \( H \geq N, \) then \( K H = G. \)

(2.4) PROPOSITION

If \( G \) is a group with subgroup \( K \) which supplements the normal subgroup \( N \) of \( G \), and avoids each chief factor of \( G \) below \( N \) in a chief series for \( G \) through \( N \), then \( K \) complements \( N \). Conversely, if \( K \) complements \( N \), then \( K \) avoids all chief factors lying below \( N \) in any chief series through \( N. \)

Proof: Let \( N = G_m > G_{m-1} > \cdots > G_0 = 1 \) be that part of the chief series below \( N \). We have that \( K \cap G_{i+1} \leq G_i \) for \( i=1,\ldots,m \) and prove by induction that \( K \cap G_i = 1, \ i=0,\ldots,m \) in a manner dual to the above proof.

The particular case \( i = m \) gives \( K \cap N = 1. \)

Conversely, if \( K \cap N = 1, \) and \( H \geq N, \) then \( K \cap H \leq \ K \cap N = 1. \)

2.3 A SELECTION OF RESULTS TO 1961

Various conditions have been found which when placed on a group and one of its normal subgroups guarantee that the group splits over the normal subgroup. In this section I survey some of the results which emphasize conditions on the normal subgroup, and which were discovered prior to the introduction of the theory of formations in 1963. Attention can be given to finding conditions on a group so that it splits over certain sorts of subgroups, and conditions which imply that a group splits
when one of its normal subgroups does. But I am concerned with conditions on the normal subgroup.

The following famous splitting theorem is attributed to Schur by Zassenhaus (on page 162 of his book), to Zassenhaus by Huppert, and to both Schur and Zassenhaus by most authors.

(2.5) SCHUR-ZASSENHAUS THEOREM

Let $G$ be a group with normal subgroup $N$. If the order of $N$ is coprime with its index in $G$, then $N$ is complemented in $G$. If $N$ or $G/N$ is soluble*, then any two complements of $N$ are conjugate in $G$.

Since $N$ has order coprime with its index, it must be a Hall $\pi$-subgroup of $G$ for some set $\pi$ of prime numbers. Further, a complement to $N$ has order $[G:N]$ and index $|N|$, so it must be a Hall $\pi'$-subgroup of $G$. Thus the theorem may be re-worded:

If $N$ is a normal Hall $\pi$-subgroup of the group $G$, then $G$ has a Hall $\pi'$-subgroup which complements $N$, and any two Hall $\pi'$-subgroups are conjugate in $G$ if $N$ or $G/N$ is soluble*.

Proof: See [5] Theorem 2.1, pp. 221-224, or [13] Theorem IV.7.c, pp. 144-145. For soluble groups, the theorem is a restatement of the first two Hall theorems (1.15a,b) on page 14.

Wolfgang Gaschütz [3] investigated the more manageable case where the normal subgroup is assumed to be abelian. First, he gave two theorems for reducing the problem:

(2.6) REDUCTION THEOREM I

Let $G$ be a group with abelian normal subgroup $A = \prod_{i=1}^{n} A(p_i)$, where each $A(p_i)$ is a direct product of cyclic groups of $p_i$-power order. Let $A'(p_i) = A(p_i) \times \cdots \times A(p_{i-1}) \times A(p_{i+1}) \times \cdots \times A(p_n)$. Then $G$ splits over $A$ if and only if $G/A'(p_i)$ splits over $A/A'(p_i)$ for all $i$.

*) This hypothesis may be omitted, by the Feit-Thompson theorem.
(2.7) REDUCTION THEOREM II

Let $A$ be an abelian normal subgroup of the group $G$, and suppose $A$ is a $p$-group for a prime $p$. Then $G$ splits over $A$ if and only if each Sylow $p$-subgroup of $G$ splits over $A$.

Proof: See [3], pp. 98 and 100 respectively. For the second theorem, see also [13] Theorem IV.8.b, p. 145, which is stated and proved for $G$ possibly infinite.

An application of the reduction theorems was made by Gaschütz in the same paper to prove the following well-known theorem.

(2.8) THEOREM ([5] Theorem 1, p. 99)

Let $G$ be a group with abelian normal subgroup $A$. Suppose that $A \leq B \leq G$, and that the order of $A$ is coprime with the index of $B$ in $G$. If $A$ has a complement in $B$ then it has one in $G$, and if the complements to $A$ in $B$ are conjugate in $B$ then the complements in $G$ are conjugate in $G$.


The corollary of (2.8) in the case $A = B$ is a special case, known as Schur's theorem, of the Schur-Zassenhaus theorem (2.5).

The following theorems show that a further sort of abelian normal subgroup is complemented: the ones that arise as the penultimate term in certain normal series.

The lower nilpotent series $\langle N_i(G) \rangle$ of a group $G$ is defined by

\[
N_0(G) = G,
\]

\[
N_i(G) = \text{the smallest normal subgroup of } N_{i-1}(G) \text{ such that } N_{i-1}(G)/N_i(G) \text{ is nilpotent, } i > 0.
\]

If $k$ is the smallest integer such that $N_k(G) = 1$, then $k$ is called the nilpotent length of $G$: a group of nilpotent length $k$ has $N_{k-1}(G)$ nilpotent.
Certainly a soluble group has a nilpotent length. For other groups, such as non-abelian simple groups, the lower nilpotent series may not terminate at 1.

(2.9) ([9] Theorem 3, p. 458)

Let $G$ be a group with nilpotent length $k$. If $N_{k-1}(G)$ is abelian, then it is complemented in $G$ and any two complements are conjugate.

This theorem is the form a result of Graham Higman's takes when a certain equivalence relation on the prime numbers is assumed to be discrete.

The case $k = 2$ was, in fact, shown the previous year (1955) by Eugene Schenkman. He dealt with $G^* = \bigcap \limits_{i=0}^{\infty} L_i(G)$, the last term of the lower central series of $G$ (page 4), which can readily be shown to be the smallest normal subgroup of $G$ giving a nilpotent factor group, that is, $N_1(G)$. His result is as follows.

(2.10) THEOREM ([14], Theorems 1 and 2, pp. 287-288)

If $G$ is a group with $G^*$ abelian, then $G^*$ is complemented in $G$ (by a nilpotent subgroup), and any two complements of $G^*$ are conjugate in $G$.

In turn, a particular case of (2.10), when $G$ is also assumed to be an $A$-group was proved by D.R. Taut in 1949 ([18] Theorem (3.4), p. 28), for in an $A$-group $G$, $N_1(G) = G^{(1)}$ the 1-th term of the derived series. Taut showed that if $G$ were an $A$-group and $N_1(G)$ were abelian, then the relative system normalizers of $N_{i-1}(G)$ in $G$ complement $N_i(G)$, and in particular that the system normalizers of $G$ complement $G' = N_1(G) = G^*$ ([18] Theorem (4.5), p. 30).

This classification of the complements becomes a special case of Roger Carter's: he showed that, when $G$ is soluble, the complements of $N_{k-1}(G)$ in (2.9) are the relative system normalizers of $N_{k-2}(G)$ in $G$, ([0] Theorem 2, p. 91).
CHAPTER THREE: THE THEORY OF FORMATIONS

A class of groups is a collection of groups which contains a group of order 1, and all isomorphic copies of any group in the collection. Thus any group-theoretic property which is possessed by the trivial group can be used to define a class of groups, so examples of classes include: all nilpotent groups, all soluble groups, all p-groups (for some prime p), all abelian groups.

In this chapter a special sort of class is introduced: the "formation", first defined and studied by Wolfgang Gaschütz in 1963 [4]. Most common classes of groups are in fact formations as well, and properties of formations can be derived just from the definition of formation which tell us something about those particular classes which are formations; in this sense formations are a unifying concept. They are important in this essay for they provide a language for some generalizations of the results of section 2.3 above; this is done in the next chapter. For the moment formations themselves will be studied, and properties needed later will be presented.

3.1 FORMATIONS

A formation \( \mathcal{F} \) is a (non-empty) class of groups which is \( G \)-closed and \( R_0 \)-closed, that is

a) if \( N \) is a normal subgroup of a group \( G \) in \( \mathcal{F} \) then \( G/N \) is in \( \mathcal{F} \)

b) if \( N_1 \) and \( N_2 \) are normal subgroups of a group \( G \), and \( G/N_1 \) and \( G/N_2 \) are in \( \mathcal{F} \), then \( G/(N_1 \cap N_2) \) is in \( \mathcal{F} \).

All the classes mentioned above are formations. Some are represented by standard symbols: \( \mathcal{N} \) is the formation of nilpotent groups, \( \mathcal{F}_p \) the formation of p-groups, for the prime \( p \), \( \mathcal{N}_k \) the formation of soluble groups of nilpotent length (page 24) at most \( k \), for the non-negative integer \( k \).
An important feature of the two rules a) and b) for adding groups to a formation is that they introduce no new chief factors in the sense that every chief factor of a group obtained is operator-isomorphic to some chief factor of the group (or groups in the case of b) already known to be in the formation.

For a) this fact is a restatement of Proposition (1.6) (page 7).

For b), the following proof shows the required group isomorphism, which can be checked to be a $G/(N_1 \cap N_2) -$ isomorphism:

Let $G/N_1 \cap N_2$ be the sub-direct product of $G/N_1$ and $G/N_2$, and take a chief series for $G/N_1 \cap N_2$ which passes through $N_1/N_1 \cap N_2$:

$$N_1 \cap N_2 = G_0 < G_1 < \ldots < G_k = N_1 < G_{k+1} < \ldots < G_{k+h} = G.$$  

For $i \leq k + 1$, the chief factor $G_i/N_i \cap N_2$ of $G/N_1 \cap N_2$ is $G_i/N_i \cap N_1$, a chief factor of $G/N_1$.

For $i \leq k$, $G_{i}/N_{i}/G_{i-1}N_{2} = G_{i}/G_{i-1}N_{2}/G_{i-1}N_{2} \cong G_{i}/(G_{i} \cap G_{i-1}N_{2})$, and $G_{i-1} \subseteq (G_{i} \cap G_{i-1}N_{2}) \subseteq G_{i}$, $G_{i} \cap G_{i-1}N_{2} \unlhd G$.

Since $G_{i}/N_{i}/G_{i-1}N_{2}$ is a chief factor, there are no normal subgroups of $G/N_1 \cap N_2$ properly contained between the "numerator" and the "denominator" so it must be that $G_{i-1} = G_{i} \cap G_{i-1}N_{2}$ or else $G_{i} = G_{i} \cap G_{i-1}N_{2}$. If the latter holds, $G_{i} \subseteq G_{i-1}N_{2}$, so if $g \in G_{i}$, there is some $f \in G_{i-1}$ and $n \in N_{2}$ such that $g = fn$. It follows that

$$n = gf^{-1} \in G_{i}G_{i-1} = G_{i} \subseteq N_{1}$$

$$n \in N_{1} \cap N_{2} \subseteq G_{i-1}$$

$$hn = g \in G_{i-1} \Rightarrow G_{i} \subseteq G_{i-1}$$ which is absurd.

Hence $G_{i}N_{2}/G_{i-1}N_{2} \cong (G_{i}/N_{1} \cap N_{2})/ (G_{i-1}/N_{1} \cap N_{2})$, while $N_{2} \subseteq G_{i-1}N_{2} \subseteq G_{i-1}N_{2} \subseteq G$, $G_{i-1}N_{2} \unlhd G$, $G_{i}N_{2} \unlhd G$, so we have shown that $G_{i}N_{2}/G_{i-1}N_{2}$ is a chief factor of $G/N_{2}$. ☐
FORMULATION RESIDUALS

Corresponding to every group $G$ and formation $\mathcal{F}$, there is a subgroup $G^\mathcal{F}$ of $G$ called the $\mathcal{F}$-residual of $G$, the smallest normal subgroup of $G$ which gives a factor group in $\mathcal{F}$. In other words, $G^\mathcal{F}$ is the minimal element of

$$Q = \{ N \mid N \leq G, G/N \in \mathcal{F} \}.$$ 

Certainly $Q$ is non-empty, since $\mathcal{F}$ contains groups of order 1 and thus $G$ is in $Q$. And if $N_1$ and $N_2$ are in $Q$, then $N_1 \cap N_2$ is also in $Q$ by the $R_0$-closure of $\mathcal{F}$. So it is clear that

$$G^\mathcal{F} = \bigcap \{ N \mid N \leq G, G/N \in \mathcal{F} \}$$

and $G^\mathcal{F} = 1$ if and only if $G$ is in $\mathcal{F}$. Further, $G$ is a normal, in fact a characteristic, subgroup of $G$.

The following properties of the formation residual will be used without comment.

(3.1) PROPOSITION

If $G$ is a group with normal subgroup $N$, and $\mathcal{F}$ is a formation contained in the formation $\mathcal{F}$, then

a) $G/N \in \mathcal{F} \implies G^\mathcal{F}/N \leq N$

b) $G/N \simeq G^\mathcal{F}/N$

c) $G^\mathcal{F} \leq G^\mathcal{F}$.

Proof: a) If $G/N$ is in $\mathcal{F}$, then $G^\mathcal{F}$ is contained in $N$ by definition. Conversely, if $G^\mathcal{F} \leq N$, then $G/N \cong (G/G^\mathcal{F})/(N/G^\mathcal{F})$ is in $\mathcal{F}$ since $(G/G^\mathcal{F})$ is in $\mathcal{F}$ and $\mathcal{F}$ is $Q$-closed.

b) $G^\mathcal{F}/N = \bigcap \{ M \leq G \mid G/M \in \mathcal{F} \}/N$

= $\bigcap \{ MN \leq G \mid G/M \in \mathcal{F} \}/N$

= $\bigcap \{ L \leq G \mid N \leq L, G/L \in \mathcal{F} \}$

= $\bigcap \{ L/N \leq G/N \mid (G/N)/(L/N) \in \mathcal{F} \} = (G/N)^\mathcal{F}$.

c) $G^\mathcal{F} = \bigcap \{ N \leq G \mid G/N \in \mathcal{F} \} \leq \bigcap \{ N \leq G \mid G/N \in \mathcal{F} \} \text{ for if } G/N \in \mathcal{F}, G/N \in \mathcal{F}$

$\leq G$. 

\[\Box\]
Some well-known subgroups turn up as formation residuals. If $G$ is a group, and if

1) $\mathcal{N}$ is the formation of all abelian groups, then $G^a = G'$, the commutator subgroup of $G$. (See [5] Theorem 2.1(vi), p. 18).

2) $\mathcal{N}$ is the formation of all nilpotent groups, then $G^\mathcal{N} = G^*$, the last term of the lower central series of $G$, as mentioned above (page 25).

For a proof, see [10] III.2.5(d), p. 261.

3) $\mathcal{N}_k$ is the formation of all soluble groups of nilpotent length at most $k$, then $G^{\mathcal{N}_k} = N_{k-1}(G)$, the penultimate term of the lower nilpotent series for $G$ when $G$ is soluble (page 24) of nilpotent length $k$.

FORMATION SUBGROUPS

In a group $G$ there may exist corresponding to the formation $\mathcal{F}$ a collection of subgroups called the $\mathcal{F}$-subgroups, or sometimes the $\mathcal{F}$-covering subgroups, of $G$. A subgroup $F$ of $G$ is an $\mathcal{F}$-subgroup if

a) $F \leq \mathcal{F}$

b) $F \leq H \leq G \Rightarrow FH = H$.

In particular, (when $H = G$) each $\mathcal{F}$-subgroup of $G$ supplements the $\mathcal{F}$-residual of $G$.

In contrast to the formation residual, formation subgroups may not exist, may not be unique, and are usually not normal subgroups.

The following properties of formation subgroups are basic.

(3.2) PROPOSITION ([4], Lemmas 2.1-2.3, p. 301)

Let $G$ be a group and $\mathcal{F}$ a formation, and let $N$ be a normal subgroup of $G$. Then

a) if $F$ is an $\mathcal{F}$-subgroup of $G$, and $F \leq K \leq G$, then $F$ is an $\mathcal{F}$-subgroup of $K$;

b) if $F$ is an $\mathcal{F}$-subgroup of $G$, then $FN/N$ is an $\mathcal{F}$-subgroup of $G/N$;

c) if $H/N$ is an $\mathcal{F}$-subgroup of $G/N$, and $F$ one of $H$, then $F$ is an $\mathcal{F}$-subgroup of $G$. 
Proof: a) If \( F \leq H \leq K \), then \( F \leq H \leq G \), so \( FH = H \).

b) First \( FN/N \cong F/N \triangleright F \in \mathcal{F} \) since \( F \in \mathcal{F} \).

Then, let \( FN/N \leq K/N \leq G/N \), so \( F/N \triangleleft (K/N) \cong (F/N)(K/N/N) = FK\cdot N/N = KN/N \) since \( FK = K \). Then since \( N \leq K \), \( KN/N = K/N \).

Also, see [10] VI.7.9(b), p. 699.


When Gaschütz introduced formations, he showed that several well-known classes of subgroups of a group were special cases of classes of \( \mathcal{F} \)-subgroups. Examples include (see [4], section 4)

1) the Sylow \( p \)-subgroups of a group, which are precisely the \( \mathcal{F}_p \)-subgroups;
2) the Hall \( \pi \)-subgroups of a soluble group, which are the \( \mathcal{F}_\pi \)-subgroups, where \( \pi \) is a set of primes and \( \mathcal{F}_\pi \) is the formation of \( \pi \)-groups;
3) the "Carter subgroups" (self-normalizing nilpotent subgroups) of a soluble group, which are the \( \mathcal{N} \)-subgroups.

This classification alone is an achievement of formation theory.

3.2 **Locally Defined Formations**

From now on we consider only soluble groups, and formations of soluble groups. The main benefit of this restriction is the knowledge this gives us of the chief factors of the groups being considered. We present a special sort of formation which emphasizes the nature of chief factors, and which is the one used in Chapter Four.

Let \( \pi \) be a set of prime numbers. We say that \((f, \pi)\) is a **formation function with support** \( \pi \) if with each \( p \in \pi \) there is associated a formation \( f(p) \): \( f \) may be thought of as a function from the set \( \pi \) to the collection of all formations.

Given a formation function \((f, \pi)\), we define "locally" the class \( \mathcal{F} \) of soluble groups by making the soluble group \( G \) be in \( \mathcal{F} \) if and only if \( |G| \) is a \( \pi \)-number, and \( A_g(c) \in f(p) \) for each \( p \)-chief factor \( c \) of \( G \).
The Jordan-Hölder theorem (1.5), and the fact that each $f(p)$ is a class show that it is immaterial which chief series $\mathcal{C}$ is chosen from. Also, since the two rules in the definition of a formation (page 26) add no new chief factors (page 27), it follows that $\mathcal{J}$ is a formation, called the formation locally defined by the formation function $(f, \pi)$, or just a local formation.

This definition has an economical rephrasing in terms of the subgroup $F_p(G)$ of the group $G$. If $G$ is soluble, and $p$ is a prime number dividing $|G|$, we define

$$F_p(G) = \bigcap \{C_G(\mathcal{C}) \mid \mathcal{C} \text{ is a } p\text{-chief factor of } G\},$$

the subgroup consisting of those elements of $G$ which centralize every $p$-chief factor of $G$. The connexion comes through the fact (page 9) that

$$A_G(\mathcal{C}) \cong G/C_G(\mathcal{C}).$$

Then $A_G(\mathcal{C}) \in f(p)$ for all $p$-chief factors $\mathcal{C}$

$$\iff G/C_G(\mathcal{C}) \in f(p) \text{ for all } p\text{-chief factors } \mathcal{C}$$

$$\iff G/\bigcap C_G(\mathcal{C}) \in f(p) \iff G/F_p(G) \in f(p),$$

using the $Q$- and $R_p$-closure of $f(p)$. So the alternative definition of the formation $\mathcal{J}$ locally defined by the formation function $(f, \pi)$ is:

$$G \in \mathcal{J} \iff |G| \text{ is a } \pi\text{-number, and } G/F_p(G) \in f(p) \text{ for all } p \mid |G|.$$  

This is all the more useful because of:

(3.3) PROPOSITION

If $G$ is a group and $p$ is a prime which divides the order of $G$, then

$$F_p(G) = \mathcal{O}_p'(G) \text{ (the maximal } p\text{-nilpotent normal subgroup of } G).$$


EXAMPLE

Consider the particular formation function $(n, \Pi)$, where $\Pi$ is the set of all prime numbers, and $n(p)$ is the class of groups of order 1, for each prime $p$. Then this defines locally the formation of all nilpotent groups, for a group is nilpotent if and only if it is $p$-nilpotent for every prime $p$, that is, by (3.3), if and only if it is
a group $G$ with the property $G = F_p(G)$ for all primes $p$ dividing $|G|$. (Clearly if $p$ does not divide the order of a group, then the group is straightaway $p$-nilpotent).

It should now be clear that if the support of a formation function $(f, \pi)$ contains all prime numbers, then $(f, \pi)$ locally defines a formation containing the formation of nilpotent groups (for any class of groups contains all the groups of order 1).

UNIQUENESS OF THE FORMATION FUNCTION

If $\mathcal{F}$ is a locally defined formation, there is no reason as yet to believe that $\mathcal{F}$ cannot be locally defined by two different formation functions $(f_1, \pi_1)$ and $(f_2, \pi_2)$.

However, if $p \not\in \pi_1$ then $f_1(p)$ contains, at least, groups of order 1. Hence $C_p$, the cyclic group of order $p$, which has just one chief factor a trivial $C_p$-module, must have

$$A_{C_p}(C_p) = 1 \in f_1(p)$$
so $C_p \in \mathcal{F}$, and hence $A_{C_p}(C_p) \in f_2(p)$, so $p \in \pi_2$. Similarly, $\pi_2 \subseteq \pi_1$, so $\pi_1 = \pi_2$, = $\pi$ say, and we may define the support of the formation $\mathcal{F}$ to be $\pi$, this common set of primes,

$$\pi = \text{support of } \mathcal{F} = \{p \mid p \text{ a prime, } C_p \in \mathcal{F}\}.$$

There are two non-restrictive conditions which we can impose on formation functions such that we can prove a uniqueness theorem. They arise out of the following two propositions.

(3.4) PROPOSITION

If $\mathcal{F}$ is a local formation, then it can be locally defined by a formation $(f, \pi)$ satisfying $f(p) \subseteq \mathcal{F}$ for all $p \in \pi$.

Proof: Suppose $\mathcal{F}$ is locally defined by $(f^*, \pi)$. Let $f(p) = f^*(p) \cap \mathcal{F}$ for each prime $p$ in $\pi$, so clearly $(f, \pi)$ satisfies the condition of the proposition. So we must prove that
$(f, \pi)$ in fact locally defines $\mathcal{J}$.

If $A_G(\mathcal{L}) \in f(p) = f^*(p) \cap \mathcal{J}$ for each $p$-chief factor $\mathcal{L}$ of $G$, then clearly $A_G(\mathcal{L}) \in f^*(p)$, so $G$ is in $\mathcal{J}$.

Conversely, if $G$ is in $\mathcal{J}$, $A_G(\mathcal{L}) \in f^*(p)$ for each $p$-chief factor $\mathcal{L}$ of $G$. And since $G \in \mathcal{J}$, $A_G(\mathcal{L}) \in G/C_G(\mathcal{L}) \in \mathcal{J}$. Hence $A_G(\mathcal{L}) \in f^*(p) \cap \mathcal{J} = f(p)$ and $G$ is in the formation locally defined by $(f, \pi)$. □

If $(f, \pi)$ is a formation function which locally defines the formation $\mathcal{J}$, and additionally $f(p) \subseteq \mathcal{J}$ for all $p$ in $\pi$, then $(f, \pi)$ is called an integrated formation function. What we have shown is that every formation which is locally defined can be defined by an integrated formation function.

(3.5) PROPOSITION

If $\mathcal{J}$ is a local formation, then it can be defined by a formation function $(f, \pi)$ satisfying $\bigcup_p f(p)^\mathcal{J} = f(p)$ for all $p \in \pi$, and further $(f, \pi)$ may also be required to be integrated.

Proof: Suppose $\mathcal{J}$ is locally defined by $(f^*, \pi)$.

Let $f(p) = \bigcup_p f^*(p)$ for each prime $p$ in $\pi$, so $(f, \pi)$ satisfies the required condition since $\bigcup_p f_p = f_p$. So if we prove that $(f, \pi)$ in fact locally defines $\mathcal{J}$, we are finished the first part.

Since $f^*(p) \subseteq f(p)$, it is immediate that $\mathcal{J}$ is contained in the formation locally defined by $(f, \pi)$. Conversely, if $A_G(\mathcal{L}) \in \bigcup_p f^*(p)$ for all $p$-chief factors $\mathcal{L}$ of $G$, then $A_G(\mathcal{L}) \in f^*(p)$ since $\mathcal{L}$ is an irreducible $\text{GF}(p)G$-module and has by (1.8) no non-trivial normal $p$-subgroups; and so $G$ is in $\mathcal{J}$.

If further we have that $(f^*, \pi)$ is integrated, which by (3.4) is no

†) If $\mathcal{A}$ and $\mathcal{B}$ are classes of groups, $\mathcal{A}\mathcal{B}$ is the class of all extensions (page 17) of a group in $\mathcal{A}$ by a group in $\mathcal{B}$.
restriction, then \((f, \pi)\) as defined is also integrated.

For if \(G \ni f(p) = \bigcup_p f^*(p)\), then \(G\) has a normal \(p\)-subgroup \(N\) such that \(G/N \in \mathcal{F}(p) \subseteq \mathcal{F}\), so all \(q\)-chief factors \(\mathcal{L}\) of \(G/N\) have

\[ A_{G/N}(\mathcal{L}) \in f^*(q) \text{ for all primes } q \text{ dividing } |G/N|, \]

that is, \(A_{G}(\mathcal{L}) \in f^*(q)\) for any \((q-)\)chief factor \(\mathcal{L}\) of \(G\) lying above \(N\).

And if \(\mathcal{L}\) is a chief factor of \(G\) lying below \(N\), it must be a \(p\)-chief factor. Since by (1.8) \(A_q(\mathcal{L})\) has in this case no non-trivial normal \(p\)-subgroups, it must be that \(N \subseteq C_G(\mathcal{L})\). But \(G/N \in f^*(p)\), so

\[ A_q(\mathcal{L}) = G/C_G(\mathcal{L}) \subseteq (G/N)/(C_G(\mathcal{L})/N) \in f^*(p). \]

Hence \(G\) is in \(\mathcal{F}\), and so \((f, \pi)\) is integrated. \(\square\)

If \((f, \pi)\) is a formation function for which

\[ \bigcup_p f(p) = f(p) \text{ for all } p \text{ in } \pi \]

then \((f, \pi)\) is called a full formation function. What we have now shown is that every local formation can be locally defined by a full, integrated formation function.

We now have the setting we need for

(3.6) **UNIQUENESS THEOREM**

If \((f_1, \pi_1)\) and \((f_2, \pi_2)\) are two full, integrated formation functions defining the same formation \(\mathcal{F}\), then \(\pi_1 = \pi_2\) and

\[ f_1(p) = f_2(p) \text{ for all } p \text{ in } \pi_1 = \pi_2. \]

**Proof:** We have already (page 32) shown that \(\pi_1 = \pi_2\).

I sketch the proof of the second part which may be found in [1] (Theorem 2.2, p. 178). Suppose there is a prime \(p\) such that \(f_1(p) \neq f_2(p)\), so that there is, without loss of generality, a group \(J\) such that \(J \in f_1(p)\) but \(J \notin f_2(p)\).

Let \(H = J^{f_2(p)}\), so \(H \neq 1\), and we can choose \(K \triangleleft J\) such that \(H/K\) is a chief factor of \(J\). If we now let \(G = J/K\), and \(\mathcal{U} = C_p\ wr G\), we can prove that \(\mathcal{U}\) is both in \(\mathcal{F}\) and not in \(\mathcal{F}\), a contradiction. \(\square\)
From now on, when I use the term "local formation" or something similar, I will mean a formation of soluble groups locally defined by a full, integrated formation function, which (as (3.6) shows) is uniquely determined; references to its support, or one of its local "component" formations are then well-defined.

FORMATION NORMALIZERS

I now define the generalization of system normalizers promised on page 15. Their importance will be evident in the next section.

Let \( \mathcal{F} \) be a local formation with support \( \pi \), and \( G \) a soluble group with Sylow system \( \Sigma \). Let \( E_{\pi} = \Sigma' \), the Hall \( \pi \)-subgroup in \( G \). Then

\[
E_{\pi} \cap \bigcap_{p \in \pi} N_G(G^f_p) \cap \Gamma^p
\]

is called the \( \mathcal{F} \)-normalizer of \( G \) corresponding to \( E \), and is clearly a subgroup of \( G \). There is an \( \mathcal{F} \)-normalizer of a soluble group for each Sylow system of the group.

Note the way in which this generalizes the system normalizers. If we put for \( \mathcal{F} \) the formation of nilpotent groups, we can use the full, integrated formation function \( f(p) = \varphi_p \) for each prime \( p \) (as can be checked using (1.8)). And for a soluble group \( G \), \( G^f_p \) must contain every \( p' \)-subgroup of \( G \), so it contains \( \Gamma^p \), and \( G^f_p \cap \Gamma^p = \Gamma^p \). It follows that the nilpotent normalizers of the group \( G \) are

\[
\bigcap_{p} N_G(\Gamma^p),
\]

and if \( p \) does not divide \( |G| \), \( \Gamma^p = G \) and \( N_G(\Gamma^p) = G \), so the intersection may as well be over the primes dividing \( |G| \). Then we have a system normalizer, as on page 16.

Formation normalizers were introduced by Roger Carter and Trevor Hawkes in 1967 [1] in the case of a formation with support consisting of all the primes (i.e. a formation containing the formation of nilpotent groups), and the definition for arbitrary support was given
by Gary Seitz and Charles Wright in 1970 [16]. Most of Carter and Hawkes' proofs carry over to the more general case with only minor modifications.

We shall need the following two properties, which I present without proof.

(3.7) PROPOSITION ([16], Proposition 1.1 and Corollary 1.2, p. 140)

Let \( \mathcal{F} \) be a local formation locally defined by the formation function \((f, \pi)\). Let \( \mu \subseteq \kappa \). If \( \mathcal{H} \) is the formation locally defined by \((h, \mu)\), and \( h(p) = f(p) \) for all \( p \) in \( \mu \), then a Hall \( \mu \)-subgroup of an \( \mathcal{F} \)-normalizer of a soluble group is one of its \( \mathcal{H} \)-normalizers.

(3.8) PROPOSITION ([1], Corollary 2 to Theorem 4.1, pp. 185-186)

Let \( \mathcal{F} \) be a local formation, \( G \) a soluble group with normal subgroup \( N \), and \( F \) an \( \mathcal{F} \)-normalizer of \( G \). Then \( FN/N \) is an \( \mathcal{F} \)-normalizer of \( G/N \).

3.3 CHIEF FACTORS AND FORMATIONS

Given a local formation and a soluble group, we classify the chief factors of the group into two sorts. The formation normalizers respect this classification - they behave in two distinct ways in relation to chief factors, corresponding to the two-way classification. This distinction is exploited in proofs of complementation theorems.

Let \( \mathcal{F} \) be a local formation locally defined by the formation function \((f, \pi)\), and let \( G \) be a soluble group.

A \( p \)-chief factor \( C \) of \( G \) is called \( \mathcal{F} \)-central when \( A_G(C) \subseteq f(p) \), and otherwise \( \mathcal{F} \)-eccentric. This enables another rephrasing of the local definition (page 30) to say that a group is in the formation \( \mathcal{F} \) if and only if every chief factor is \( \mathcal{F} \)-central.

The origin of the term "\( \mathcal{F} \)-central" can be found in the special
case when \( \mathcal{F} \) is the formation of all nilpotent groups, \( \mathcal{N} \). Then
\[ f(p) = \mathcal{F}_p \] locally defines \( \mathcal{N} \) so a p-chief factor \( H/K \) of a soluble group \( G \) is \( \mathcal{N} \)-central if and only if
\[ A_G(H/K) \in \mathcal{F}_p. \]

But (1.8) shows that this as good as requiring that \( A_G(H/K) = 1 \), since
\[ A_G(H/K) \cong G/G_G(H/K) \] can have no normal p-subgroups. And
\[ A_G(H/K) = 1 \iff G = C_G(H/K) \]
\[ \iff H/K \leq Z(G/K), \]
that is, \( H/K \) is central in \( G/K \).

(3.9) THEOREM

Let \( \mathcal{F} \) be a local formation, and \( G \) a soluble group. Then

a) every chief factor of \( G \) lying above \( \mathcal{F} \) in a chief series for \( G \)
through \( G^{\mathcal{F}} \) is \( \mathcal{F} \)-central;

b) every chief factor lying immediately below \( \mathcal{F} \) in \( \mathcal{G} \)-chief series
for \( G \) through \( G^{\mathcal{F}} \) is \( \mathcal{F} \)-eccentric.

Proof: a) Let \( H/K \) be a p-chief factor of \( G \) with \( G^{\mathcal{F}} \leq K \). Then
by (1.6) \( H/K \) is a p-chief factor of \( G/G^{\mathcal{F}} \), and \( G/G^{\mathcal{F}} \in \mathcal{F} \) by definition, so \( H/K \) must be \( \mathcal{F} \)-central.

b) If \( G^{\mathcal{F}} = 1 \), the statement is vacuously true. Assume that \( G \neq 1 \), and that \( G/K \) is a chief factor of \( G \).

If \( G/K \) were \( \mathcal{F} \)-central, then every chief factor of \( G/K \) (being either a chief factor of \( G/G^{\mathcal{F}} \) or \( G/K \) itself) is \( \mathcal{F} \)-central, so \( G/K \) is a group in \( \mathcal{F} \). But this would mean that \( G \leq K \), which is absurd, since \( K \leq G^{\mathcal{F}} \). \( \square \)
COVERING AND AVOIDANCE

The usefulness of formation normalizers lies in the applications of the following theorem, usually in proving complementation. I follow it with another application which gives a partial link between formation subgroups and formation normalizers to be used later.

(3.10) COVERING AND AVOIDANCE THEOREM

Let \( \mathcal{F} \) be a local formation, and \( G \) a soluble group. In any chief series for \( G \), each \( \mathcal{F} \)-normalizer of \( G \) covers every \( \mathcal{F} \)-central chief factor, and avoids every \( \mathcal{F} \)-eccentric chief factor.

Proof: "Cover" and "avoid" were defined on page 13.

The theorem was proved by Carter and Hawkes in the case where the support of \( \mathcal{F} \) contains all the prime numbers \([1] \) Theorem 4.1, p. 185). They used a different definition of \( \mathcal{F} \)-normalizer, but proved it equivalent to the one I use \([1] \) Lemma 3.1, p. 182).

Seitz and Wright point out \([16], \) p. 141) that (3.7) above allows us to carry over Carter and Hawkes' proof to arbitrary supports.

Alternatively, the theorem can be proved as for the corresponding property of system \( \mathcal{U} \)-normalizers, for which see Philip Hall's paper where this was first done \([7] \) Theorems 6.2 and 7.2, pp. 521-523), or \([10] \) VI.11.10, p. 728.

Given this theorem, and (3.9a),(2.3), and (2.4), we have immediately

(3.11) THEOREM

Let \( \mathcal{F} \) be a local formation, and \( G \) a soluble group. Then \( G \) is \( \mathcal{F} \)-supplemented by the \( \mathcal{F} \)-normalizers of \( G \), and furthermore \( G \) is complemented by the \( \mathcal{F} \)-normalizers of \( G \) if and only if every chief factor of \( G \) lying below \( G \) in a chief series for \( G \) through \( G \) is \( \mathcal{F} \)-eccentric.
FORMATION SUBGROUPS AND FORMATION NORMALIZERS

The similarity between these two classes of subgroups is shown partly by the following two results.

(3.12) PROPOSITION ([1] Corollary 1 to Theorem 4.1, and Theorem 5.6) Let \( \mathcal{F} \) be a local formation, and \( G \) a soluble group. Suppose \( F \) is an \( \mathcal{F} \)-normalizer of \( G \). Then

a) \( F \) is in \( \mathcal{F} \)

b) if \( G \) is nilpotent, then \( F \) is an \( \mathcal{F} \)-subgroup of \( G \)

Proof: a) Only a sketch of the proof is given.

The intersection with \( F \) of a chief series for \( G \) can be shown to give

a chief series for \( F \), with the chief factors that do not collapse

operator-isomorphic. Since \( F \) covers the \( \mathcal{F} \)-central chief factors, and

avoids the others, it is easy to check that \( |F| \) is the product of the

orders of the \( \mathcal{F} \)-central chief factors of \( G \). And \( |F| \) is also the product

of the orders of its own chief factors. So every chief factor of \( F \) is

\( \mathcal{F} \)-central, so \( F \) is in \( \mathcal{F} \).

b) The proof for this part is rather different from that of

Carter and Hawkes. It is adapted from one of Wright's ([19] Theorem 8,
p. 278).

Denote by \( F(G) \) the maximal nilpotent normal subgroup of \( G \). Since

\( G \) is assumed nilpotent, \( G \mathcal{F} \subseteq F(G) \) and \( G/F(G) \) is \( \mathcal{F} \)-central. Hence every chief

factor of \( G \) lying above \( F(G) \) is \( \mathcal{F} \)-central, so by (2.3) and (3.10)

\( G = F(G)F \).

Since \( F(G) \) is nilpotent, it is \( p \)-nilpotent for every prime \( p \), and

thus \( F(G) \mathcal{F} \leq \mathcal{O}_p(F(G)) = \mathcal{F}_p \) by (3.3) (page 31), for every prime \( p \).

Hence \( F(G) \) centralizes every chief factor of \( G \). In particular, if \( M \)
is a minimal normal subgroup of \( G \) it is centralized by \( F(G) \). Using

\( G = F(G)F \), we have that \( M \) is an irreducible \( F \)-module as well as an

irreducible \( G \)-module, that is, \( M \) is also a minimal normal subgroup of \( FM \).
By (3.8), \( \mathcal{F} \)-normalizer of \( G/M \). Since \( (G/M) = G/M/M \) is isomorphic to \( G/M \cap G \) and this is nilpotent if \( G \) is, we may assume inductively that \( F/M \) is an \( \mathcal{F} \)-subgroup of \( G/M \). If we now show that \( F \) is an \( \mathcal{F} \)-subgroup of \( FM \), we are finished on applying (3.2c) (page 29).

First, \( F \) is in \( \mathcal{F} \) from part a).

Second, if \( M \leq F \), then \( FM = F \), so \( F \) is clearly an \( \mathcal{F} \)-subgroup of \( FM \).

Otherwise \( M \not\leq F \), and \( F \) does not cover the chief factor \( M \), so \( M \) must be \( \mathcal{F} \)-socentric, and \( F \) avoids it. Hence \( F \cap M = 1 \), and \( FM/M = F \circ \mathcal{F} \), so \( (FM)^\mathcal{F} \leq M \). Now \( (FM)^\mathcal{F} \) is a normal subgroup of \( FM \) contained in the minimal normal subgroup \( M \) of \( FM \), which forces \( (FM)^\mathcal{F} = 1 \) or \( (FM)^\mathcal{F} = M \). If \( (FM)^\mathcal{F} = 1 \), \( FM \circ \mathcal{F} \) and then \( M \) is a chief factor of a group in \( \mathcal{F} \), so it must be \( \mathcal{F} \)-central. But we have shown it is \( \mathcal{F} \)-socentric. Hence \( (FM)^\mathcal{F} = M \), a minimal normal subgroup of the soluble group \( FM \) complemented in \( FM \) by \( F \).

This means that \( F \) is a maximal subgroup of \( FM \), by (2.2) (page 21).

Hence the obvious equalities \( FF^\mathcal{F} = F \) and \( F(FM)^\mathcal{F} = FM \) are all that is needed to show that \( F \) is an \( \mathcal{F} \)-subgroup of \( FM \). \( \square \)

The second result of the last proposition may be restated as follows: "If \( \mathcal{F} \) is a local formation and \( G \in \mathcal{N}_\mathcal{F} \), then the \( \mathcal{F} \)-normalizers of \( G \) are \( \mathcal{F} \)-subgroups of \( G \)." In fact, the \( \mathcal{F} \)-normalizers are precisely the \( \mathcal{F} \)-normalizers in this case, but I do not need this.

This also gives a sufficient condition for the formation subgroups of a group to exist. In fact, they exist for any local formation ([4], Theorem 2.1, p. 301, [10] VI.7.10, p. 700 - these proofs rely on a completely different definition of a local formation, which needs the "Labeseder theorem" [10] VI.7.5 and VI.7.25 to show is equivalent to the one I use).
CHAPTER FOUR: SPLITTING WITH FORMATIONS

This chapter provides a justification for the development of the theory of formations in the last chapter, for it uses it to show that certain classes of soluble groups split over a normal subgroup, and generalizes the results of section 2.3 along the way. Beyond this, some recent work of Charles Wright which avoids explicit use of formations is presented. The chapter ends with two theorems which guarantee that all the complements to the normal subgroup over which the group splits are conjugate in the group.

4.1 COMPLEMENTS OF FORMATION RESIDUALS

Using the language of formations, Schenkman's and Higman's results mentioned in section 2.3 may be rephrased as follows.

(4.1) THEOREM

Let $G$ be a soluble group, and $\mathfrak{N}$ be $\mathfrak{N}_0$ or $\mathfrak{N}_1$. If $G$ is abelian, then it is complemented in $G$, and all complements are conjugate in $G$.

Proof: See (2.10) and (2.9) respectively, and the other remarks on page 25; and 2) and 3) on page 29. □

As one might conjecture, this theorem can now be generalized to any local formation. We aim at first for an even more general result.

(4.2) THEOREM

Let $\mathfrak{N}$ be a local formation and $G$ be a soluble group. Suppose that $G$ is a minimal counterexample to the proposition:

"If $\mathfrak{H}$ is a $\mathfrak{N}$-closed class of groups, $H$ a soluble group, and $H \in \mathfrak{H}$, then $H$ is complemented by the $\mathfrak{N}$-normalizers of $H$".

Then $G$ has a unique minimal normal subgroup properly contained in $G$ which is the only $\mathfrak{N}$-central chief factor of $G$ lying below $G^\mathfrak{N}$.
Proof: This proof was inspired by the first part of the proof of Theorem 2.1 of [16] (p. 143).

The minimal counterexample $G$ is a soluble group with $G^+$ in $\mathcal{K}$, but with an $\mathcal{F}$-normalizer $F$ of $G$ which does not complement $G^+$.

Since $FG^+ = G$ by (3.11), the complementation fails because $F \cap G^+ > 1$. Hence it cannot be true that each chief factor below $G$ is $\mathcal{F}$-eccentric, that is, there is a prime $p$ in the support of $\mathcal{F}$ and a $p$-chief factor below $G^+$ which is $\mathcal{F}$-central.

Since $F \cap G^+ > 1$, $G \not\triangleleft 1$ and has minimal normal subgroups. Given one, $M$, say, note that the hypotheses of the proposition are satisfied by $G/M = G^+ M/M \supseteq G^{+} M/M \triangleleft G^+ \supseteq G \subseteq \mathcal{K}$ if $G^+ \in \mathcal{K}$, and $|G/M|$ is less than $|G|$, so the conclusion of it follows for $G/M$; so the $\mathcal{F}$-normalizers of $G/M$ complement $(G/M)^+ = G^+ M/M$. Hence the $\mathcal{F}$-normalizers of $G/M$ avoid every chief factor of $G/M$ below $G^+ M/M$, that is, every chief factor of $G$ between $M$ and $G M$ is $\mathcal{F}$-eccentric.

Now each chief factor of $G$ above $G^+$ is $\mathcal{F}$-central by (3.9a), so we must have $G^+ M = G$, and $M \triangleleft G^+$.

Each chief factor between $M$ and $G^+$ is thus $\mathcal{F}$-eccentric, while we have above that there is an $\mathcal{F}$-central $p$-chief factor below $G^+$. The only remaining possibility for it in a chief series through $M$ and $G^+$ is $M$ itself.

If $N$ is a minimal normal subgroup of $G$, and $M \not\triangleleft N$, then $M \cap N = 1$ and $M \times N \not\triangleleft MN \triangleleft G$. Further, $MN/M (\cong N)$ is a chief factor of $G$ between $M$ and $G^+$, so it is $\mathcal{F}$-eccentric. But $N$, since like $M$ it is a minimal normal subgroup, is $\mathcal{F}$-central. This contradiction shows that $M = N$, that is, $M$ is unique.

Finally, $M \triangleleft G^+$. For otherwise every chief factor is either above $G^+$ or $M$ itself, and is $\mathcal{F}$-central in either case, so $G \in \mathcal{F}$, $G^+ = 1$, contrary to $M \triangleleft G^+$.
Theorem (4.2) is the basic formation-theoretic tool used in proving the next two theorems. The first generalizes (4.1).

(4.3) THEOREM

Let $G$ be a soluble group, and $\mathcal{F}$ a local formation. If $G^\mathcal{F}$ is abelian, then it is complemented in $G$ by the $\mathcal{F}$-normalizers of $G$.

This result was first proved in 1966 by Ernest E. Shult [17] in terms of $\mathcal{F}$-subgroups, and simultaneously by Roger Carter and Trevor Hawkes [1] (Theorem 5.15, pp. 197-198) in terms of $\mathcal{F}$-normalizers. However, in view of (3.12) and its converse, that distinction is superficial. But the proofs read quite differently; Carter and Hawkes' is more heavily dependent on formation theory. The following proof begins like Shult's, and ends like Carter and Hawkes'.

Proof: Let $G$ be a minimal counterexample, and $F$ an $\mathcal{F}$-normalizer of $G$ which fails to complement $G^\mathcal{F}$, so $F \cap G^\mathcal{F} > 1$, and $F$ is also an $\mathcal{F}$-subgroup of $G$, by (3.12b).

We can assume that $F \cap G^\mathcal{F}$ is a proper subgroup of $G^\mathcal{F}$, for otherwise $F \cap G^\mathcal{F} = G^\mathcal{F}$, so $F$ is $\mathcal{F}$, giving $G = FG^\mathcal{F} = F \in \mathcal{F}$ and $G^\mathcal{F} = 1 = F \cap G^\mathcal{F}$.

Since the class of abelian groups is closed under the taking of quotient groups, we can use (4.2). Let, then, $M$ be the unique minimal normal subgroup of $G$. We know that $M \triangleleft G^\mathcal{F}$ and that $M$ is the only $\mathcal{F}$-central chief factor of $G$ lying below $G^\mathcal{F}$.

I now proceed to make a number of deductions, which eventually lead to a contradiction, showing that no counterexample can exist.

$F \cap G^\mathcal{F} = M$: First, $F \cap G^\mathcal{F} \not\triangleleft G$ by (2.1), since $G^\mathcal{F}$ is abelian.

Second, the premises of the theorem hold for $G/M$, and by (3.2b) $FM/M$ is an $\mathcal{F}$-subgroup of $G/M$, so $(G/M)^\mathcal{F}$ and $FM/M$ intersect trivially, that is $FM \cap FM \cong M$, and so

$$1 \neq F \cap G^\mathcal{F} \triangleleft (F \cap G^\mathcal{F})M \triangleleft G^\mathcal{F} \cap FM \triangleleft M,$$

and the result follows by the minimality of $M$. 
\( F \) is a maximal subgroup of \( G \): Assume that \( F \not\leq H < G \).

Then \( G \cong H^F \cong FG^F = G \), so \( H^G = G \), and \( H/H \cap G^F \cong H^G/G^F = G/G^F \cong \mathcal{F} \),

that is, \( H \cong H \cap G^F \leq G^F \) which is abelian. Also, by (3.2a) \( F \) is an \( \mathcal{F} \)-subgroup of \( H \). Hence the theorem applies to \( H \), of order less than \(|G|\), to give \( F \cap H^F = 1 \).

Now \( H^F \) is a normal subgroup of \( G \), for if \( g \in G \), then \( g = kf \) for some \( k \in H \) and \( f \in G^F \), so for all \( h \in H^F \),

\[
g^{-1}hg = (kf)^{-1}h(kf)
\]

\[
= f^{-1}(k^{-1}hk)f = k^{-1}hk
\]
since \( H^F \) is a normal subgroup of \( H \) and an abelian subgroup of \( G^F \).

Finally, \( k^{-1}hk \in H^F \) since \( H^F \) is normal in \( H \).

If \( H^F \neq 1 \), then \( M \leq H^F \), so \( M \not\leq F \cap H^F = 1 \), which is absurd.

Hence \( H^F = 1 \), that is, \( FH^F = H = F \).

\( G/F \) is a chief factor of \( G \): First, \( M \) and \( G^F \) are both normal subgroups of \( G \), and \( M < G^F \). Now we show that any normal subgroup \( N \) of \( G \) lying between \( M \) and \( G^F \) must in fact be \( M \) or \( G^F \), so \( G/F \) is a minimal normal subgroup of \( G/M \) and hence is a chief factor.

We have \( F \leq FN \leq FG^F = G \), so \( F = FN \) or \( FN = G \) by the maximality of \( F \).

If \( F = FN \), then \( N \leq F \), so \( N \not\leq F \cap G^F = M \), that is, \( M = N \).

If \( FN = G \), \( G/N = FN/N \cong F/F \cap N = F/M = F/F \cap G^F \cong FG^F/G^F = G/G^F \), so

\( |N| = |G|/|G^F| \), and \( N = G^F \).

\( G^F \) is an (abelian) \( p \)-group, for some prime \( p \): We know that there are primes \( p \) and \( q \) such that \( M \) is an (elementary abelian) \( p \)-group, and \( G^F/M \) is an (elementary abelian) \( q \)-group. Then \( G^F \) is an abelian \( \{p,q\} \)-group.

If \( p \neq q \), the Sylow \( q \)-subgroup of \( G^F \) is characteristic in \( G^F \) and has trivial intersection with \( M \) (the Sylow \( p \)-subgroup); that is, it is a non-trivial normal subgroup of \( G \) not contained in \( M \). This is impossible since \( M \) is the unique minimal normal subgroup. Hence \( p = q \), and \( G^F \) is a \( p \)-group.
From now to the end of the proof, I shall use the symbol $P$ to refer to $G$, for ease of writing, and as a reminder that it is a $p$-group.

We have established so far that $FP = G$, $F \cap P = M$, $F$ is a maximal subgroup, and $P/M$ is a $p$-chief factor of $G$, and have from (4.2) that $P/M$ is $G$-eccentric and that $M$ is $G$-central.

Let $(f, \pi)$ be the full, integrated formation function locally defining $G$. Since $(f, \pi)$ is integrated, $P \leq G^f(p)$.

$G^f(p)/P$ is $p$-nilpotent: Since $G/P \leq G/F$, $F/(G/F) \leq f(p)$, and so $(G/P)f(p) = G^f(p)/P \leq F/(G/P)$, which is $p$-nilpotent by (3.3). It is easy to check that this means that $G^f(p)/P$ is also $p$-nilpotent.

Let $Q$ be a $p'$-subgroup of $G$ such that $Q/(G^f(p)/P)$, that is, $QP/P$ is a normal $p$-complement of $G^f(p)/P$.

$QP/C_G(P/M)$: Since $P/M$ is $G$-eccentric, $G/C_G(P/M) \not\leq f(p)$, and $G^f(p) \not\leq C_G(P/M)$. If it were the case that $QP \leq C_G(P/M)$, then the irreducible $G/F(p)G$-module $P/M$ is also an irreducible $G/F(p)/P$-module with $G^f(p)/P$, a normal subgroup of $G/P$, having a normal $p$-complement $QP/P$ which centralizes $P/M$. By (1.9) (page 9), we have immediately that $G^f(p)/P$ is contained in the centralizer of $P/M$ in $G/P$, so $G^f(p) \not\leq C_G(P/M)$, a contradiction.

$[Q, M] = 1$: Since $M$ is $G$-central, $G/C_G(M) \leq f(p)$ and $G^f(p) \leq C_G(M)$.

Since $QP/P$ is a subgroup of $G^f(p)/P$, $Q$ is a subgroup of $G^f(p)$.

Hence $Q \leq C_G(M)$, and $[Q, M] = 1$ by (1.13) (page 13).

We now distinguish two cases for $P$, and gain a contradiction in each.

If $P$ is elementary abelian: then it is a $G/F(p)Q$-module, and thus completely reducible by Maschke's theorem (1.10) (page 10) since $|Q|$ is a $p'$-number. Now $M$ is a $G/F(p)Q$-submodule of $P$, so it is a direct factor (from the equivalent definition of complete reducibility on
pages 9 and 10); and \( P = M \times N \), where \( N \) is a \( G_0(P)Q \)-module.

Define \( T = [Q, P] \). Then \( T = [Q, P] \), since \( P \) is an abelian normal subgroup, so \( T = [Q, P] = [q, M \times N] = [q, N] \), since \( [q, M] = 1 \), and \( [q, N] \leq N \), since \( N \) is a \( Q \)-module.

Also \( QP \) is a characteristic subgroup of \( G_0(P) \) (since \( QP/P \) is a normal \( p \)-complement of \( G_0(P)/P \)), and \( G_0(P) \) is a normal subgroup of \( G \), so \( QP \) is normal in \( G \), and so \( T = [QP, P] \) is normal in \( G \). Also, \( T \) is contained in \( N \), and so properly contained in \( P \). This forces \( P \) to be either \( 1 \) or \( M \). But in the latter case, \( T = M \leq N \), which is impossible. In the former, \( [QP, P] = 1 \), so \( QP \leq C_0(P/M) \), contradicting an earlier deduction.

If \( P \) is not elementary abelian, then \( P^P = \{ g^P \mid g \in P \} \) is non-trivial, otherwise every element of \( P \) has order either \( 1 \) or \( p \), and \( P \) is elementary abelian. Further, \( P^P \) is a characteristic subgroup of \( P \), and \( P \) is normal in \( G \), so \( P^P \) is normal in \( G \). Hence, \( M \leq P^P \).

The mapping \( P \rightarrow P^P; g \mapsto g^P \) is not injective, since \( M^P = 1 \), and so cannot be onto \( P \); that is, \( P^P < P \). This forces \( M = P^P \).

So \( [Q, M] = 1 = [Q, P^P] \), and

\[
[q, g^P] = 1 \quad \text{for all } q \in Q \text{ and } g \in P.
\]

Further, \( [q, gh] = [q, h][q, g]^g = [q, g][q, h] \) since \( [Q, P] \leq P \), so \( g \) commutes with \( [q, g] \). Similarly, by induction, \( [q, g^P] = [q, g]^P \), so \( [Q, P]^P = 1 \) and \( [Q, P] \leq M \), since it contains the elements of \( P \) of order \( p \) or \( 1 \). This is equivalent to \( Q \leq C_0(P/M) \). Clearly \( P \leq C_0(P/M) \), so \( QP \leq C_0(P/M) \), contradicting an earlier deduction. □

The next theorem and its corollary are a further generalization.

For the first time since the Schur-Zassenhaus theorem was mentioned (2.5), page 23), splitting over a non-abelian normal subgroup is considered.
(4.4) THEOREM

Let $G$ be a soluble group, and $\mathcal{F}$ a local formation with support $\pi$. If $G^\mathcal{F}$ is a $\mu\alpha$-group, then it is complemented in $G$ by the $\mathcal{F}$-normalizers of $G$.

The essence of this theorem is contained in (4.2) and (1.4), given that we have proved (4.3), for I prove the theorem by reducing it to the case where $G^\mathcal{F}$ is abelian. This is different from Seitz and Wright's self-contained method ([16], second part of proof of Theorem 2.1).

**Proof:** Let $G$ be a minimal counterexample. Since the class of $\mu\alpha$-groups is $\mathcal{Q}$-closed by (1.14) (page 14), we may apply (4.2).

Let $M$ be the minimal normal subgroup of $G$. Since $M$ is $\mathcal{F}$-central, and is a $p$-chief factor of $G$ for some prime $p$ in $\pi$,

$$G/C_M(M) \triangleleft f(p) \leq \mathcal{F}, \quad (f, \pi) \text{ the formation function for } \mathcal{F}. $$

Hence $G^\mathcal{F} \leq C_G(M)$, that is, $M \leq Z(G^\mathcal{F})$.

Also, $M$ is contained in every Sylow $p$-subgroup $P$ of $G^\mathcal{F}$, since it is normal.

If $(G^\mathcal{F})' > 1$, then $M \leq (G^\mathcal{F})'$, and in this case

$$M \leq P \cap Z(G) \cap (G^\mathcal{F})' = 1 \quad \text{by (1.4)}$$

which is absurd. Hence $(G^\mathcal{F})' = 1$, and $G^\mathcal{F}$ is abelian. In this case (4.3) takes over, and shows that $G^\mathcal{F}$ is complemented by the $\mathcal{F}$-normalizers of $G$, a contradiction to the assumption that $G$ is a counterexample. $\square$

(4.5) COROLLARY ([16], Corollary 2.4, p. 144)

Let $G$ be a soluble group, and $\mathcal{F}$ a local formation.

Let $\pi = \{ p \mid p \text{ a prime number dividing } [G:G^\mathcal{F}] \}$. If $G^\mathcal{F}$ is a $\mu\alpha$-group, then it is complemented in $G$ by a Hall $\mu$-subgroup of an $\mathcal{F}$-normalizer of $G$.

**Proof:** Let $\pi$ be the support of $\mathcal{F}$, and $(f, \pi)$ be the formation function locally defining $\mathcal{F}$. Since $G/G^\mathcal{F} \triangleleft \mathcal{F}$, every chief factor of $G/G^\mathcal{F}$ has characteristic in $\pi$; so $\mu \leq \pi$. 

Let $\mathcal{H}$ be the formation locally defined by $(h, \mu)$, where

$$h(p) = f(p) \quad \text{for all } p \in \mu,$$

so $\mathcal{H}$ is contained in $\mathcal{J}$.

Since $G/\mathcal{H} \in \mathcal{J}$, and every chief factor of $G/\mathcal{H}$ is a $p$-chief factor for some prime $p$ in $\mu$, we have $G/\mathcal{H} \in \mathcal{H}$, and $G/\mathcal{H} \in \mathcal{J}$ (3.1c).

Conversely, $G/\mathcal{H} \in \mathcal{H} \subseteq \mathcal{J}$, so $G/\mathcal{H} \in \mathcal{J}$.

Hence $G = G/\mathcal{H}$, and we can apply the theorem to the formation $\mathcal{H}$, and conclude that $\mathcal{J}(\mathcal{H})$ is complemented by the $\mathcal{H}$-normalizers of $G$.

But a Hall $\mu$-subgroup of an $\mathcal{J}$-normalizer is an $\mathcal{H}$-normalizer, by (3.7).

A special case of the corollary occurs when $\mathcal{J}$ has order prime to its index in $G$. Then no prime in $\mu$ divides $|G|$, so $\mathcal{J}$ is vacuously a $\mu\mathcal{A}$-group. In this case, $\mathcal{J}$ is complemented as a result of the corollary if its order is prime to its index: a special case of the Schur-Zassenhaus theorem (2.5).

Attempts at a further generalization by Seitz and Wright ([16], Theorem 2.2, p. 143), in which the Sylow subgroups were allowed to be modular for odd primes and quaternion-free for the prime 2, were later shown by them to be in vain, for they showed that the Sylow subgroups would then be abelian anyway. (This is mentioned in [20], p. 128).

### 4.2 COMPLEMENTS OF NORMAL SUBGROUPS

In this section, I present the basis of Wright's recent work in [20]. His aim is to eliminate an obstacle in the application of the theory of the previous section. That difficulty is that to apply the theory to a group $G$ with normal subgroup $H$ it is necessary to know that $H = \mathcal{J}$ for some local formation $\mathcal{J}$. This means, at least, that $H$ must be characteristic in $G$, but Wright points out that "to date, no really usable intrinsic characterization of those subgroups which are $\mathcal{J}$-residuals has been given". ([20], p. 125).
Let $G$ be a soluble group, and $H$ a normal subgroup of $G$; assume that these remain fixed for the remainder of this section.

Let $\pi = \{p \mid p$ is a prime dividing $[G:H]\}$; and define the subgroup $\frac{P_H}{H}$ of $G$ for each prime number $p$ by

$$\frac{P_H}{H} = \gamma^p_G\left(\frac{G}{H}\right).$$

Then the usual condition imposed on $H$ is that for each $p$ in $\pi$, $P_H$ centralizes no $p$-chief factor of $G$ lying below $H$; this is written $H \in G$, and assumes that $H$ is a normal subgroup of $G$. The advantage of this relation is that a calculation in a given group will tell whether or not it holds.

A complement for $H$ is found among the $H$-normalizers of $G$, defined by

$$\Sigma_{\pi} \cap \bigcap_{p \in \pi} N_G(P_H \cap \gamma^p_G), \quad \Sigma \text{ a Sylow system for } G.$$ It is reasonable to conjecture from this definition that each $H$-normalizer plays a role similar to that of an $\mathcal{F}$-normalizer of $G$, and $P_H$ similar to $G^f(p)$, at least when $H = H^f$ (for some local formation $\mathcal{F}$ with formation function $(f, \pi)$). In fact,

(4.6) PROPOSITION

Let $G$ be a soluble group, and $\mathcal{F}$ a formation locally defined by $(f, \mu)$. Then $G^f(p) \leq P^G_G$ for each prime $p$ dividing $[G:G^f]$.\[
\text{Proof: Since } G/G^f \in \mathcal{F} \text{ and } p \mid [G:G^f], f(p) \text{ contains the group } \]

$$\frac{G/G^f}{P^G_P(G/G^f)} = \frac{(G/G^f)/G^p(G/G^f)}{(P^G_P(G/G^f))} = \frac{(G/G^f)/(P^G_P(G/G^f))}{P^G_P(G/G^f)} \cong \frac{G/P^G_G}{G/G^f}$$

which implies that $G^f(p) \leq P^G_G$ for $(f, \mu)$ is full, and so $G^f(p) \leq P_G$.

The $H$-normalizers also have a cover and avoidance property, analogous to (3.10).
(4.7) COVERING AND AVOIDANCE THEOREM

Let $G$ be a soluble group with normal subgroup $H$, and let $\pi$ be the set of primes dividing $[G:H]$. In any chief series for $G$, each $H$-normalizer of $G$ covers every $p$-chief factor, for $p \in \pi$, which is centralized by $^pH$, and avoids every other chief factor.

Proof: Just as for (3.10).

Using (4.7) we can now prove the basic splitting theorem.

(4.8) THEOREM ([20], Proposition 1.4, p. 127)

If $G$ is a soluble group, and $H \leq G$, then $H$ is complemented in $G$ by the $H$-normalizers of $G$.

Proof: First, we show that $^pH$ centralizes every $p$-chief factor of $G$ lying above $H$:

for, if we consider in particular a chief series for $G$ passing through $H$ and $^pH$, then clearly every $p$-chief factor above $H$ lies, in fact, above $^pH$, since $^pH/H$ is a $p'$-group. And any chief factor above $^pH$ is centralized by $^pH$. Hence, using the Jordan-Hölder theorem, the $p$-chief factors above $H$ in any chief series are centralized by $^pH$.

This is true for every prime dividing $[G:H]$, so any $H$-normalizer covers every chief factor above $H$, by (4.7) above, and so supplements $H$, by (2.3).

Second, $^pH$ does not centralize any $p$-chief factor of $G$ below $H$ (since $H \leq G$) for each prime $p$ dividing $[G:H]$; and an $H$-normalizer avoids these chief factors, and any others below $H$, by (4.7); so each $H$-normalizer complements $H$, by (2.4) (page 22).

The connexion with the theorems of the last section is shown by the next theorem, which shows when a formation residual satisfies Wright's new condition "e".
(4.9) THEOREM ([20], Proposition 1.3, p. 126)

Let \( G \) be a soluble group, and \( \mathcal{F} \) a local formation. Suppose that there are no \( \mathcal{F} \)-central chief factors of \( G \) lying below \( G^\mathcal{F} \). Then \( G^\mathcal{F} \in G \).

Proof: Let \( p \) be a prime dividing \( |G/G^\mathcal{F}| \), so that \( p \) is in the support \( \mu \) of \( \mathcal{F} \). Let \( (f, \mu) \) be the formation function defining \( \mathcal{F} \), and \( L/M \) be a \( p \)-chief factor of \( G \) lying below \( G^\mathcal{F} \).

Since \( L/M \) is \( \mathcal{F} \)-eccentric, \( G/C_G(L/M) \neq f(p) \), and, equivalently, \( G^f(p) \neq C_G(L/M) \). But \( G^f(p) \neq P(G^\mathcal{F}) \) by (4.6). Hence \( P(G^\mathcal{F}) \neq C_G(L/M) \), that is, \( P(G^\mathcal{F}) \) does not centralize \( L/M \).

This is true for any such \( L/M \), and \( p \mid [G:G^\mathcal{F}] \), so, by definition, \( G^\mathcal{F} \in G \). \( \square \)

Combining (4.9) and (4.8), rather like (3.11) (page 38) follows: "If \( \mathcal{F} \) is a local formation, and \( G \) a soluble group, and all chief factors of \( G \) lying below \( G^\mathcal{F} \) are \( \mathcal{F} \)-eccentric, then \( G^\mathcal{F} \) is complemented in \( G \) by the \( G^\mathcal{F} \)-normalizers of \( G \)."

It is now easy to prove various sufficient conditions for \( H \in G \), such as Corollary 2.2 of [20] (p. 127) which requires that \( H \) be a \( \mathcal{M} \)-group and that \( \overline{P}H \) centralize no \( p \)-chief factor of \( G \) immediately below \( H \) for each \( p \) in \( \mathcal{M} \); this generalizes (4.5). In fact (4.5) is a corollary in view of (3.9b).

4.3 CONJUGACY OF THE COMPLEMENTS

In this section I show that in most of the above theorems where a normal subgroup is shown to be complemented, all the complements are conjugate. We know that there must be an isomorphism between any two complements; proving that this must be an inner automorphism of the whole group is in the tradition of the Sylow theorems and the Hall theorems (1.2b) and (1.15b).

Shult's method of showing that the complements in Theorem (4.3)
are conjugate was to prove that every complement must be a formation normalizer, and then to use the result of Gaschütz ([4], Theorem 2.1, p. 301) that the formation subgroups are conjugate. (In his case, the formation subgroups and formation normalizers coincide). This method is much the same as that used by Seitz and Wright ([16], Theorem 3.1, p. 144); they assume the result of Carter and Hawkes [1] that the formation normalizers are conjugate.

Instead I adopt the formation-free approach of Wright [20]; its only disadvantage is that it does not show that the formation normalizers of any soluble group are conjugate, a result of interest in itself.

(4.10) THEOREM ([20], Theorem 3.1, p. 129)

Let \( G \) be a soluble group with normal subgroup \( H \). Let \( \pi \) be the set of primes dividing \([G:H]\). Suppose that \( H \) is complemented in \( G \) by \( K \triangleleft G \), and that for all \( p \in \pi \), \( O_p(K) \) acts fpf on each \( p \)-chief factor of \( G \) lying below \( H \). Then all complements to \( H \) in \( G \) are conjugate in \( G \).

Proof: Let \( G \) be a minimal counterexample to the theorem; let \( K \) and \( L \) be complements to \( H \) such that \( K \) is not conjugate to \( L \) in \( G \). In particular, \( H \not\triangleleft L \), otherwise \( K = G = L \).

Let \( M \) be a minimal normal subgroup of \( G \) contained in \( H \). Since \(|G/M| < |G|\), and \( G/M \) satisfies the hypotheses of the theorem (it is soluble, \( H/M \) is a normal subgroup of \( G/M \) which is complemented by \( KM/M \), and for all \( p \in \pi = \{ p \mid p \text{ divides } [G/M:H/M]\} \), \( O_p(KM/M) \cong O_p(K) \) acts fpf on each \( p \)-chief factor of \( G/M \) lying below \( H/M \)), the conclusion of the theorem holds for \( G/M \): \( KM/M \) is conjugate to \( LM/M \) in \( G/M \); that is

\[
(KM/M)^{\sigma M} = (KM)^{G/M} = K^{G/M} = LM/M
\]

for some \( \sigma \in G \), and so \( K^{G/M} = LM \). If we now prove that \( K^G \) and \( L \) are conjugate in \( G \), so must \( K \) and \( L \) be, so we assume without loss of generality that \( KM = LM \).
We now prove that $KM = G (= LM)$.

Let $M$ have characteristic $p$. If $M$ had order prime to its index $[KM:M] = |K|$ in $KM$, then the Schur-Zassenhaus theorem's second part would apply, and all complements of $M$, including $K$ and $L$, would be conjugate in $MK$, and a fortiori in $G$. But this is not the case: so $p$ must divide $|K| = [G:H]$, that is, $p \equiv 1 \pmod{p}$. Hence $M$ is a chief factor of $G$ which is acted on fpf by $O^p(K)$. Maschke's theorem applies to the $GF(p)O^p(K)$-module $M$ to give the decomposition $M = \bigoplus_{i=1}^{t} M_i$, where each $M_i$ is an irreducible $GF(p)O^p(K)$-module. If $O^p(K)$ acts with a fixed point on some $M_i$, then that fixed point would also occur as one of $M_i$; so $O^p(K)$ acts fpf on each $M_i$. The $p$-chief factors of $KM$ lying below $M$ are just the composition factors of $M$ as a $K$-module (since $M$ acts trivially on each $p$-chief factor). Given a composition series of the $K$-module $M$, it can be refined to give a composition series of the $O^p(K)$-module $M$ (since $O^p(K)$ is a subgroup of $K$), and the composition factors thereby obtained must be $O^p(K)$-isomorphic to those obtained from the previous decomposition $M = \bigoplus_{i=1}^{t} M_i$, by the Jordan-Hölder theorem. So $O^p(K)$ acts fpf on each one. Hence $O^p(K)$ acts fpf on each composition factor of the unrefined composition series for $M$ as a $K$-module.

Further, $M$ is normal in $MK$, and $K$ and $L$ complement $M$ in $MK = ML$, and are not conjugate in $MK$: so $MK$ is a counterexample to the theorem. Hence $KM = G$ by the minimality of $G$.

Since $O^p(K)$ is normal in $K$, $K \trianglelefteq N_G(O^p(K))$. If $G = N_G(O^p(K))$, then $O^p(K)$ would be normal in $G$. Using the fact that $M$ is normal in $G$ as well,

$$[O^p(K), M] \trianglelefteq O^p(K)/M = 1$$

so $O^p(K)$ centralizes $M$ (1.13). But it is also assumed to act fpf on $M$, and $M$ is non-zero ($M > 1$). This contradiction shows that each normalizer $N_G(O^p(K)), N_G(O^p(L))$ is proper in $G$, while $K$ and $L$ are maximal subgroups of $G$ by (2.2), so we must have $K = N_G(O^p(K))$ and $L = N_G(O^p(L))$. 
Further, \(O^p(G/M) = O^p(MK/M) = O^p(K)M/M\) by (1.1) (page 2), and also \(O^p(G/M) = O^p(L)M/M\) by a similar argument, so \(O^p(K)M = O^p(L)M\).

Now \(M\) has order prime to its index \(|O^p(K)|\) in \(O^p(K)M\), so its complements, including \(O^p(K)\) and \(O^p(L)\), must be conjugate by the second part of the Schur-Zassenhaus theorem. It is now a straightforward proof from the definition of the normalizer to show that this makes \(N_G(O^p(K))\) and \(N_G(O^p(L))\), that is, \(K\) and \(L\), conjugate. This contradicts our original assumption. \(\square\)

(4.11) THEOREM ([20], Corollary 3.2, p. 129)

Let \(G\) be a soluble group with normal subgroup \(H\), and let \(\pi\) be the set of primes dividing \([G:H]\). Suppose that \(H \leq G\), and \(H\) has a normal \(p\)-complement for all \(p \in \pi\). Then all complements to \(H\) in \(G\) are conjugate.

Proof: Theorem (4.8) gives immediately that \(H\) has a complement, say \(K\), in \(G\). Let \(L/M\) be a \(p\)-chief factor of \(G\) lying below \(H\), for a prime \(p\) in \(\pi\).

\(H \leq C_G(L/M)\): Let \(N\) be a normal \(p\)-complement for \(H\), then \(N\) is a normal subgroup of \(H\), and \(L\) is too, so \([L,N] \leq L \cap N\).

Since \(L/M\) is a \(p\)-group, and \(NM/M \cong N/N \cap M\) is a \(p'\)-group,
\(L/M \cap NM/M = (L/NM)/M\) is trivial; that is, \(L/NM = M\), and \(L \cap N \leq M\).
Hence \([L,N] \leq M\), and \(N \leq C_G(L/M)\). Applying (1.9) (page 9), we get it.

\(O^p(K)\) acts on \(L/M\): Since \(P_H/H = O^p(G/H)\) is normal in \(G/H\), \(P_H \triangleleft G\).

By assumption, \(P_H \not\leq C_G(L/M)\). Hence by (1.12) (page 10), \(P_H\) acts fixed-point-freely on \(L/M\).

Now \(P_H/H = O^p(G/H) = O^p(HK/H) = O^p(K)H/H\) by (1.1), so \(P_H = O^p(K)H\).

If \(O^p(K)\) had a fixed point on \(L/M\), there would be an element \(\bar{M} \in L/M\) such that \((\bar{M})^k = \bar{M}\) for all \(k \in O^p(K)\). Hence
\[(\bar{M})^{kh} = ((\bar{M})^k)^h = (\bar{M})^k = \bar{M}\] for all \(k \in O^p(K)\) and \(h \in H\), since \(H\) centralizes \(L/M\). In other words, \((\bar{M})^g = \bar{M}\) for all \(g \in P_H\).
But $P_H$ acts fpf on $L/M$, so $LM$ must be the identity $M$, and $G^P(K)$ too acts fpf on $L/M$.

In this case Theorem (4.10) applies, giving immediately that all complements to $H$ are conjugate in $G$. □

This corollary is relevant to (4.3) and (4.4): it applies to complements of formation residuals.

(4.12) COROLLARY

Let $G$ be a soluble group, and $\mathcal{F}$ a local formation. If $G^\mathcal{F}$ is nilpotent and complemented by an $\mathcal{F}$-normalizer of $G$, then all complements of $G^\mathcal{F}$ are conjugate in $G$.

Proof: Since $G^\mathcal{F}$ is nilpotent, it is $p$-nilpotent, and so has a normal $p$-complement for all primes $p$.

Since $G^\mathcal{F}$ is complemented by an $\mathcal{F}$-normalizer of $G$, there are by (3.11) no $\mathcal{F}$-central chief factors of $G$ below $G^\mathcal{F}$, and so $P(G^\mathcal{F})$ centralizes no $p$-chief factor of $G$ below $G^\mathcal{F}$ for all primes $p$ dividing $[G:G^\mathcal{F}]$, that is $G^\mathcal{F} \leq G$ by (4.9).

Hence the theorem applies. □

This result occurs as Theorem 3.1 of [16], though the proof there is different (as noted on page 52). It shows that the complements to $G^\mathcal{F}$ in (4.3) are conjugate (since an abelian group is nilpotent), and that if $G \in \mathcal{N}\mathcal{F}$, the complements to $G^\mathcal{F}$ in (4.3) are conjugate.
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