AN IMPROVED UPPER BOUND FOR THE ERROR IN THE
ZERO-COUNTING FORMULAE FOR DIRICHLET L-FUNCTIONS
AND DEDEKIND ZETA-FUNCTIONS

T. S. TRUDGIAN

Abstract. This paper contains new explicit upper bounds for the number of
zeroes of Dirichlet \( L \)-functions and Dedekind zeta-functions in rectangles.

1. Introduction and Results

This paper pertains to the functions \( N(T, \chi) \) and \( N_K(T) \), respectively the num-
ber of zeroes \( \rho = \beta + i\gamma \) of \( L(s, \chi) \) and of \( \zeta_K(s) \) in the region \( 0 < \beta < 1 \) and \( |\gamma| \leq T \).
The purpose of this paper is to prove the following two theorems.

**Theorem 1.** Let \( T \geq 1 \) and \( \chi \) be a primitive nonprincipal character modulo \( k \). Then
\[
\left| N(T, \chi) - \frac{T}{\pi} \log \frac{kT}{2\pi e} \right| \leq 0.317 \log kT + 6.401.
\]
In addition, if the right side of (1.1) is written as \( C_1 \log kT + C_2 \), one may use the
values of \( C_1 \) and \( C_2 \) contained in Table 1.

**Theorem 2.** Let \( T \geq 1 \) and \( K \) be a number field with degree \( n_K = [K : \mathbb{Q}] \) and
absolute discriminant \( d_K \). Then
\[
\left| N_K(T) - \frac{T}{\pi} \log \left( \frac{T}{2\pi e} \right)^{n_K} \right| \leq 0.317 \{ \log d_K + n_K \log T \} + 6.333n_K + 3.482.
\]
In addition, if the right side of (1.2) is written as \( D_1 \{ \log d_K + n_K \log T \} + D_2n_K +
D_3 \), one may use the values of \( D_1, D_2 \) and \( D_3 \) contained in Table 2.

Theorem 1 and Table 1 improve on a result due to McCurley [3, Thm 2.1];
Theorem 2 and Table 2 improve on a result due to Kadiri and Ng [2, Thm 1]. The
values of \( C_1 \) and \( D_1 \) given above are less than half of the corresponding values in
[3] and [2]. The improvement is due to Backlund’s trick — explained in \( \S 3 \) — and
some minor optimisation.

Explicit expressions for \( C_1 \) and \( C_2 \) and for \( D_1, D_2 \) and \( D_3 \) are contained in
(4.11) and (4.12) and in (5.11) and (5.12). These contain a parameter \( \eta \) which,
when varied, gives rise to Tables 1 and 2. The values in the right sides of (1.1) and

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(1.2) correspond to \( \eta = \frac{1}{4} \) in the tables. Note that some minor improvement in the lower order terms is possible if \( T \geq T_0 > 1 \); Tables 1 and 2 give this improvement when \( T \geq 10 \).

Table 1. \( C_1 \) and \( C_2 \) in Theorem 1 and in \([3]\) for various values of \( \eta \)

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>McCurley ([3])</th>
<th>When ( T \geq 1 )</th>
<th>When ( T \geq 10 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.506</td>
<td>16.989</td>
<td>0.248</td>
</tr>
<tr>
<td>0.10</td>
<td>0.552</td>
<td>13.202</td>
<td>0.265</td>
</tr>
<tr>
<td>0.15</td>
<td>0.597</td>
<td>11.067</td>
<td>0.282</td>
</tr>
<tr>
<td>0.20</td>
<td>0.643</td>
<td>9.606</td>
<td>0.300</td>
</tr>
<tr>
<td>0.25</td>
<td>0.689</td>
<td>8.509</td>
<td>0.317</td>
</tr>
<tr>
<td>0.30</td>
<td>0.735</td>
<td>7.641</td>
<td>0.334</td>
</tr>
<tr>
<td>0.35</td>
<td>0.781</td>
<td>6.929</td>
<td>0.351</td>
</tr>
<tr>
<td>0.40</td>
<td>0.827</td>
<td>6.330</td>
<td>0.369</td>
</tr>
<tr>
<td>0.45</td>
<td>0.873</td>
<td>5.817</td>
<td>0.386</td>
</tr>
<tr>
<td>0.50</td>
<td>0.919</td>
<td>5.370</td>
<td>0.403</td>
</tr>
</tbody>
</table>

Table 2. \( D_1 \), \( D_2 \) and \( D_3 \) in Theorem 2 and in \([2]\) for various values of \( \eta \)

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>Kadiiri and Ng ([2])</th>
<th>When ( T \geq 1 )</th>
<th>When ( T \geq 10 )</th>
</tr>
</thead>
<tbody>
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<td>16.95</td>
<td>7.663</td>
</tr>
<tr>
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<td>13.163</td>
<td>7.663</td>
</tr>
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<td>11.029</td>
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<td>9.567</td>
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<td>0.25</td>
<td>0.689</td>
<td>8.471</td>
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</tr>
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<td>0.30</td>
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<td>7.663</td>
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<td>0.781</td>
<td>6.891</td>
<td>7.663</td>
</tr>
<tr>
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</tr>
<tr>
<td>0.45</td>
<td>0.873</td>
<td>5.778</td>
<td>7.663</td>
</tr>
<tr>
<td>0.50</td>
<td>0.919</td>
<td>5.331</td>
<td>7.663</td>
</tr>
</tbody>
</table>

Explicit estimation of the error terms of the zero-counting function for \( L(s, \chi) \) is done in \( \S2 \). Backlund’s trick is modified to suit Dirichlet \( L \)-functions in \( \S3 \). Theorem 1 is proved in \( \S4 \). Theorem 2 is proved in \( \S5 \).

The Riemann zeta-function, \( \zeta(s) \), is both a Dirichlet \( L \)-function (albeit to the principal character) and a Dedekind zeta-function. The error term in the zero counting function for \( \zeta(s) \) has been improved, most recently, by the author \([7]\). One can estimate the error term in the case of \( \zeta(s) \) more efficiently owing to explicit bounds on \( \zeta(1 + it) \), for \( t \gg 1 \). It would be of interest to see whether such bounds for \( L(1 + it, \chi) \) and \( \zeta_K(1 + it) \) could be produced relatively easily — this would lead to an improvement of the results in this paper.
2. Estimating $N(T, \chi)$

Let $\chi$ be a primitive nonprincipal character modulo $k$, and let $L(s, \chi)$ be the Dirichlet $L$-series attached to $\chi$. Let $a = (1 - \chi(-1))/2$ so that $a$ is 0 or 1 according as $\chi$ is an even or an odd character. Then the function

$$
(2.1) \quad \xi(s, \chi) = \left( \frac{k}{\pi} \right)^{(s+a)/2} \Gamma \left( \frac{s+a}{2} \right) L(s, \chi),
$$

is entire and satisfies the functional equation

$$
(2.2) \quad \xi(1-s, \overline{\chi}) = \frac{i a k^{1/2}}{\tau(\chi)} \xi(s, \chi),
$$

where $\tau(\chi) = \sum_{n=1}^{k} \chi(n) \exp(2\pi i n/k)$.

Let $N(T, \chi)$ denote the number of zeroes $\rho = \beta + i\gamma$ of $L(s, \chi)$ for which $0 < \beta < 1$ and $|\gamma| \leq T$. For any $\sigma_1 > 1$ form the rectangle $R$ having vertices at $\sigma_1 + iT$ and $1 - \sigma_1 \pm iT$, and let $C$ denote the portion of the boundary of the rectangle in the region $\sigma \geq \frac{1}{2}$. From Cauchy’s theorem and (2.2) one deduces that

$$
N(T, \chi) = \frac{1}{\pi} \Delta_C \arg \xi(s, \chi).
$$

Thus

$$
(2.3) \quad N(T, \chi) = \frac{1}{\pi} \left\{ \Delta_C \arg \left( \frac{k}{\pi} \right)^{(s+a)/2} + \Delta_C \arg \Gamma \left( \frac{s+a}{2} \right) + \Delta_C \arg L(s, \chi) \right\}
$$

$$
= \frac{T}{\pi} \log \frac{k}{\pi} + \frac{2}{\pi} \Im \log \Gamma \left( \frac{1}{4} + \frac{a}{2} + iT \right) + \frac{1}{\pi} \Delta_C \arg L(s, \chi).
$$

To evaluate the second term on the right-side of (2.3) one needs an explicit version of Stirling’s formula. Such a version is provided in [4, p. 294], to wit

$$
(2.4) \quad \log \Gamma(z) = (z - \frac{1}{2}) \log z - z + \frac{1}{2} \log 2\pi + \frac{\theta}{6|z|}.
$$

which is valid for $|\arg z| \leq \frac{\pi}{2}$, and in which $\theta$ denotes a complex number satisfying $|\theta| \leq 1$. Using (2.4) one obtains

$$
\Im \log \Gamma \left( \frac{1}{4} + \frac{a}{2} + iT \right) = \frac{T}{2} \log \frac{T}{2e} + \frac{T}{4} \log \left( 1 + \frac{(2a+1)^2}{4T^2} \right)
$$

$$
(2.5) \quad + \frac{2a-1}{4} \tan^{-1} \left( \frac{2T}{2a+1} \right) + \frac{\theta}{3|\frac{1}{4} + \frac{a}{2} + iT|}.
$$

Denote the last three terms in (2.5) by $g(a, T)$. Using elementary calculus one can show that $|g(0, T)| \leq g(1, T)$ and that $g(1, T)$ is decreasing for $T \geq 1$. This, together with (2.3) and (2.5), shows that

$$
(2.6) \quad \left| N(T, \chi) - \frac{T}{\pi} \log \frac{kT}{2\pi e} \right| \leq \frac{1}{\pi} |\Delta_C \arg L(s, \chi)| + \frac{2}{\pi} g(1, T).
$$

All that remains is to estimate $\Delta_C \arg L(s, \chi)$. Write $C$ as the union of three straight lines, viz. let $C = C_1 \cup C_2 \cup C_3$, where $C_1$ connects $\frac{1}{2} - iT$ to $\sigma_1 - iT$; $C_2$ connects $\sigma_1 - iT$ to $\sigma_1 + iT$; and $C_3$ connects $\sigma_1 + iT$ to $\frac{1}{2} + iT$. Since $L(\pi, \chi) = \overline{L(s, \chi)}$,
a bound for the integral on $C_3$ will serve as a bound for that on $C_1$. Estimating the contribution along $C_2$ poses no difficulty since

$$|\arg L(\sigma_1 + it, \chi)| \leq |\log L(\sigma_1 + it, \chi)| \leq \log \zeta(\sigma_1).$$

To estimate $\Delta_{C_3} \arg L(s, \chi)$ define

$$(2.7) \quad f(s) = \frac{1}{2} \{L(s + iT, \chi)^N + L(s - iT, \overline{\chi})^N\},$$

for some positive integer $N$, to be determined later. Thus $f(\sigma) = \Re L(\sigma + iT, \chi)^N$. Suppose that there are $n$ zeroes of $\Re L(\sigma + iT, \chi)^N$ for $\sigma \in \left[\frac{1}{2}, \sigma_1\right]$. These zeroes partition the segment into $n + 1$ intervals. On each interval $\arg L(\sigma + iT, \chi)^N$ can increase by at most $\pi$. Thus

$$|\Delta_{C_3} \arg L(s, \chi)| = \frac{1}{N} |\Delta_{C_3} \arg L(s, \chi)^N| \leq \frac{(n + 1)\pi}{N},$$

whence (2.6) may be written as

$$(2.8) \quad \left| N(T, \chi) - \frac{T}{\pi} \log \frac{kT}{2\pi e} \right| \leq \frac{2}{\pi} \{\log \zeta(\sigma_1) + g(1, T)\} + \frac{2(n + 1)}{N}.$$

One may estimate $n$ with Jensen's Formula.

**Lemma 1** (Jensen's Formula). Let $f(z)$ be holomorphic for $|z - a| \leq R$ and non-vanishing at $z = a$. Let the zeroes of $f(z)$ inside the circle be $z_k$, where $1 \leq k \leq n$, and let $|z_k - a| = r_k$. Then

$$(2.9) \quad \log \frac{R^n}{|r_1 r_2 \cdots r_n|} = \frac{1}{2\pi} \int_{0}^{2\pi} \log f(a + Re^{i\phi}) \, d\phi - \log |f(a)|.$$

This is done in §4.

3. **Backlund's Trick**

For a complex-valued function $F(s)$, and for $\delta > 0$ define $\Delta_+ \arg F(s)$ to be the change in argument of $F(s)$ as $\sigma$ varies from $\frac{1}{2}$ to $\frac{1}{2} + \delta$, and define $\Delta_- \arg F(s)$ to be the change in argument of $F(s)$ as $\sigma$ varies from $\frac{1}{2}$ to $\frac{1}{2} - \delta$.

Backlund’s trick is to show that if there are zeroes of $\Re F(\sigma + iT)^N$ on the line $\sigma \in \left[\frac{1}{2}, \sigma_1\right]$, then there are zeroes on the line $\sigma \in \left[1 - \sigma_1, \frac{1}{2}\right]$. This device was introduced by Backlund in [1] for the Riemann zeta-function.

Following Backlund’s approach one can prove the following general lemma.

**Lemma 2.** Let $N$ be a positive integer and let $T \geq T_0 \geq 1$. Suppose that there is an upper bound $E$ that satisfies

$$|\Delta_+ \arg F(s) + \Delta_- \arg F(s)| \leq E,$$

where $E = E(\delta, T_0)$. Suppose further that there exists an $n \geq 3 + \lfloor NE/\pi \rfloor$ for which

$$n \pi \leq |\Delta_{C_3} \arg F(s)^N| < (n + 1)\pi.$$

Then there are at least $n$ distinct zeroes of $\Re F(\sigma + iT)^N$, denoted by $\rho_v = a_v + iT$ (where $1 \leq v \leq n$ and $\frac{1}{2} \leq a_n < a_{n-1} < \cdots < \sigma_1$), such that the bound $|\Delta \arg F(s)^N| \geq \nu \pi$ is achieved for the first time when $\sigma$ passes over $a_v$ from above.

In addition there are at least $n - 2 - \lfloor NE/\pi \rfloor$ distinct zeroes $\rho_v' = a_v' + iT$ (where $1 \leq \nu \leq n - 2$ and $1 - \sigma_1 \leq a_1' < a_2' < \cdots < \frac{1}{2}$).
Moreover

\( a_\nu \geq 1 - a'_\nu, \) for \( \nu = 1, 2, \ldots, n - 2 - \lfloor NE/\pi \rfloor, \)

and, if \( \eta \) is defined by \( \sigma_1 = \frac{1}{2} + \sqrt{2}(\eta + \frac{1}{2}) \), then

\[
\prod_{\nu=1}^{n} |1 + \eta - a_\nu| \prod_{\nu=1}^{n-2-\lfloor NE/\pi \rfloor} |1 + \eta - a'_\nu| \leq \left( \frac{1}{2} + \eta \right)^{2n-2-\lfloor NE/\pi \rfloor}.
\]  

**Proof.** It follows from (3.1) that \(| \arg F(s)^N | \) must increase as \( \sigma \) varies from \( \sigma_1 \) to \( \frac{1}{2} \). This increase may only occur if \( \sigma \) has passed over a zero of \( \Re F(s)^N \), irrespective of its multiplicity. In particular as \( \sigma \) moves along \( C_3 \)

\[ |\Delta \arg F(s)^N| \geq \pi, 2\pi, \ldots, n\pi. \]

Let \( \rho_\nu = a_\nu + it \) denote the distinct zeroes of \( \Re F(s)^N \) the passing over of which produces, for the first time, the bound \(|\Delta \arg F(s)^N| \geq \nu \pi \). It follows that there must be \( n \) such points, and that \( \frac{1}{2} \leq a_n < a_{n-1} < \ldots < a_2 < a_1 \leq \sigma_1 \). Also if \( \frac{1}{2} + \delta = a_\nu \) then

\[
|\Delta_+ \arg F(s)^N| \geq (n - \nu)\pi.
\]

For (3.4) is true when \( \nu = n \) and so, by the definition of \( \rho_\nu \), it is true for all \( 1 \leq \nu \leq n \).

By the hypothesis in Lemma 2,

\[
|\Delta_+ \arg F(s)^N + \Delta_- \arg F(s)^N| \leq NE.
\]

When \( \frac{1}{2} + \delta \geq a_\nu \), (3.4) and (3.5) show that

\[
|\Delta_+ \arg F(s)^N| \geq (n - \nu - NE/\pi)\pi,
\]

for \( 1 \leq \nu \leq n - 2 - \lfloor NE/\pi \rfloor \). When \( \frac{1}{2} + \delta = a_\nu \) and \( \nu = n - 2 - \lfloor NE/\pi \rfloor \), it follows from (3.6) that \(|\Delta_- \arg F(s)^N| \geq \pi \). The increase in the argument is only possible if there is a zero of \( \Re F(s)^N \) the real part of which is greater than \( \frac{1}{2} - \delta = 1 - a_{n-2-\lfloor NE/\pi \rfloor} \). Label this zero \( \rho'_{n-2-\lfloor NE/\pi \rfloor} = a_{n-2-\lfloor NE/\pi \rfloor} + i\tau \).

Repeat the procedure when \( \nu = n-3-\lfloor NE/\pi \rfloor, \ldots, 2, 1 \), whence (3.2) follows. This produces a positive number of zeroes in \( [1 - \sigma_1, \frac{1}{2}] \) provided that \( n \geq 3 + \lfloor NE/\pi \rfloor \).

For zeroes \( \rho_\nu \) lying to the left of \( 1 + \eta \) one has

\[ |1 + \eta - a_\nu||1 + \eta - a'_\nu| \leq (1 + \eta - a_\nu)(\eta + a_\nu), \]

by (3.2). This is a decreasing function for \( a_\nu \in [\frac{1}{2}, 1 + \eta] \) and so, for these zeroes

\[
|1 + \eta - a_\nu||1 + \eta - a'_\nu| \leq \left( \frac{1}{2} + \eta \right)^2.
\]

For zeroes lying to the right of \( 1 + \eta \) one has

\[ |1 + \eta - a_\nu||1 + \eta - a'_\nu| \leq (a_\nu - 1 - \eta)(\eta + a_\nu). \]

This is increasing with \( a_\nu \) and so, for these zeroes

\[
|1 + \eta - a_\nu||1 + \eta - a'_\nu| \leq \sigma_1^2 - \sigma_1 - \eta(1 + \eta).
\]

The bounds in (3.7) and (3.8) are equal\(^1\) when \( \sigma_1 = \frac{1}{2} + \sqrt{2}(\eta + \frac{1}{2}) \). Thus (3.3) holds for \( \sigma_1 = \frac{1}{2} + \sqrt{2}(\eta + \frac{1}{2}) \). For the unpaired zeroes one may use the bound

\[ |1 + \eta - a_\nu| \leq \frac{1}{2} + \eta, \]

whence (3.3) follows.

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\(^1\)McCurley does not use Backlund's trick. Accordingly, his upper bounds in place of (3.7) and (3.8) are \( \frac{1}{2} + \eta \) and \( \sigma_1 - 1 - \eta \). These are equal at \( \sigma_1 = \frac{1}{2} + 2\eta \), which is his choice of \( \sigma_1 \).
3.1. Applying Backlund’s Trick. Apply Jensen’s formula on the function $F(s)$, with $a = 1 + \eta$ and $R = r(\frac{1}{2} + \eta)$, where $r > 1$. Assume that the hypotheses of Lemma 2 hold. If $1 + \eta - r(\frac{1}{2} + \eta) \leq 1 - \sigma_1$ then all of the $2n - 1 - \lfloor NE/\pi \rfloor$ zeroes of $\Re F(\sigma + iT)^N$ are included in the contour. Thus the left side of (2.9) is

$$\log \frac{\{r(\frac{1}{2} + \eta)\}^{2n-2-\lfloor NE/\pi \rfloor}}{|1 + \eta - a_1| \cdots |1 + \eta - a_n||1 + \eta - a'_1| \cdots |1 + \eta - a'_{n-2-\lfloor NE/\pi \rfloor}|} \geq (2n - 2 - \lfloor NE/\pi \rfloor) \log r,$$

by (3.3). If the contour does not enclose all of the $2n - 2 - \lfloor NE/\pi \rfloor$ zeroes of $\Re F(\sigma + iT)^N$, then the following argument, thoughtfully provided by Professor D.R. Heath-Brown, allows one still to make a saving.

To a zero at $x + it$, with $\frac{1}{2} \leq x \leq 1 + \eta$ one may associate a zero at $x' + it$ where, by (3.2), $1 - x \leq x' \leq \frac{1}{2}$. Thus, for an intermediate radius, zeroes to the right of $\frac{1}{2}$ yet still close to $1 + \eta$ will have their pairs included in the contour. Let $X$ satisfy $1 + \eta - r(\frac{1}{2} + \eta)/r < X < \min\{1 + \eta, r(\frac{1}{2} + \eta) - \eta\}$. Since $r > 1$, this guarantees that $X > \frac{1}{2}$. For a zero at $x + it$ consider two cases: $x \geq X$ and $x < X$.

In the former, there is no guarantee that the paired zero $x' + it$ is included in the contour. Thus the zero at $x + it$ is counted in Jensen’s formula with weight

$$\log \frac{r(\frac{1}{2} + \eta)}{1 + \eta - x} \geq \log \frac{r(\frac{1}{2} + \eta)}{1 + \eta - X}.$$

Now, when $x < X$, the paired zero at $x'$ is included in the contour, since $1 + \eta - r(\frac{1}{2} + \eta) < 1 - X < 1 - x \leq x'$. Thus, in Jensen’s formula, the contribution is

$$\log \frac{r(\frac{1}{2} + \eta)}{1 + \eta - x} + \log \frac{r(\frac{1}{2} + \eta)}{1 + \eta - x'} \geq \log \frac{r(\frac{1}{2} + \eta)}{1 + \eta - x} + \log \frac{r(\frac{1}{2} + \eta)}{\eta + x} = \log \frac{r^2(\frac{1}{2} + \eta)^2}{(1 + \eta - x)(\eta + x)}.

The function appearing in the denominator of (3.11) is decreasing for $x \geq \frac{1}{2}$. Thus the zeroes at $x + it$ and $x' + it$ contribute at least $2 \log r$.

Suppose now that there are $n$ zeroes in $[\frac{1}{2}, \sigma_1]$, and that there are $k$ zeroes the real parts of which are at least $X$. The contribution of all the zeroes ensnared by the integral in Jensen’s formula is at least

$$k \log \frac{r(\frac{1}{2} + \eta)}{1 + \eta - X} + 2(n - k) \log r = k \log \frac{(\frac{1}{2} + \eta)}{r(1 + \eta - X)} + 2n \log r \geq 2n \log r,$$

which implies (3.9)

3.2. Calculation of $E$ in Lemma 2. From (2.1) and (2.2) it follows that

$$\Delta_+ \arg \xi(s, \chi) = - \Delta_- \arg \xi(s, \chi).$$

Since $\arg(\pi/k)^{-\frac{i\pi}{2\pi}} = -\frac{1}{2} \log(\pi/k)$ then $\Delta_+ (\pi/k)^{-\frac{i\pi}{2\pi}} = 0$, whence

$$|\Delta_+ \arg L(s, \chi) + \Delta_- \arg L(s, \chi)| = |\Delta_+ \arg \Gamma(\frac{s+a}{2}) + \Delta_- \arg \Gamma(\frac{s+a}{2})|.$$ Using (2.4) one may write

$$|\Delta_+ \arg \Gamma \left(\frac{s+a}{2}\right) + \Delta_- \arg \Gamma \left(\frac{s+a}{2}\right)| \leq G(a, \delta, t),$$

where $G(a, \delta, t)$ is a function of $\delta$ and $t$.
On \( J \) since \( \chi \) of due to Rosser \([6]\). Write \( L \) applied.

Now suppose that \( |t| \) for it follows that \( \delta \) (3.14)

\[
E = G(0, \sqrt{2}(\frac{1}{2} + \eta), t_0),
\]

for \( t \geq t_0 \).

4. PROOF OF THEOREM 1

First, suppose that \( |\Delta c, \arg L(s, \chi)N| < 3 + |NE/\pi| \). Thus (2.6) becomes

\[
(4.1) \quad \left| N(T, \chi) - \frac{T}{\pi} \log \frac{kT}{2\pi e} \right| \leq \frac{2}{\pi} \left\{ \log \zeta(\sigma_1) + g(1, T) + E \right\} + \frac{6}{N}.
\]

Now suppose that \( |\Delta c, \arg L(s, \chi)N| \geq 3 + |NE/\pi| \), whence Lemma 2 may be applied.

To apply Jensen’s formula to the function \( f(s) \), defined in (2.7), it is necessary to show that \( f(1 + \eta) \) is non-zero: this is easy to do upon invoking an observation due to Rosser [6]. Write \( L(1 + \eta + iT, \chi) = Ke^{i\psi} \), where \( K > 0 \). Choose a sequence of \( N \)'s tending to infinity for which \( N \psi \) tends to zero modulo \( 2\pi \). Thus

\[
(4.2) \quad \frac{f(1 + \eta)}{|L(1 + \eta + iT, \chi)|^N} \to 1.
\]

Since \( \chi \) is a primitive nonprincipal character then \( f(s) \) is holomorphic on the circle. It follows from (2.9) and (3.9) that

\[
(4.3) \quad n \leq \frac{1}{4\pi \log r} J - \frac{1}{2\log r} \log |f(1 + \eta)| + 1 + \frac{NE}{2\pi},
\]

where

\[
J = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \log |f(1 + \eta + r(\frac{1}{2} + \eta)e^{i\phi})| d\phi.
\]

Write \( J = J_1 + J_2 \) where the respective ranges of integration of \( J_1 \) and \( J_2 \) are \( \phi \in [-\pi/2, \pi/2] \) and \( \phi \in [\pi/2, 3\pi/2] \). For \( \sigma > 1 \)

\[
(4.4) \quad \frac{\zeta(2\sigma)}{\zeta(\sigma)} \leq |L(s, \chi)| \leq \zeta(\sigma),
\]

which shows that

\[
(4.5) \quad J_1 \leq N \int_{-\pi/2}^{\pi/2} \log \zeta(1 + \eta + r(\frac{1}{2} + \eta) \cos \phi) d\phi.
\]

On \( J_2 \) use

\[
\log |f(s)| \leq N \log |L(s + iT, \chi)|,
\]
and the convexity bound [5, Thm 3]

\begin{equation}
|L(s, \chi)| \leq \left( \frac{k|s+1|}{2\pi} \right)^{(1+\eta-\sigma)/2} \zeta(1+\eta),
\end{equation}

valid for \(-\eta \leq \sigma \leq 1+\eta\), where \(0 < \eta \leq \frac{1}{2}\), to show that

\begin{equation}
J_2 \leq \pi N \log \zeta(1+\eta) + N \frac{r(\frac{1}{2} + \eta)}{2} \int_{\pi/2}^{3\pi/2} \left( -\cos \phi \right) \log \left( \frac{kTw(T, \phi, \eta, r)}{2\pi} \right) d\phi,
\end{equation}

where

\begin{equation}
w(T, \phi, \eta, r)^2 = \frac{1 + 2r(\frac{1}{2} + \eta) \sin \phi + r^2(\frac{1}{2} + \eta)^2 + (2 + \eta)^2 + 2r(\frac{1}{2} + \eta)(2 + \eta) \cos \phi}{T^2}.
\end{equation}

For \(\phi \in [\pi/2, \pi]\), the function \(w(T, \phi, \eta, r)\) is decreasing in \(T\); for \(\phi \in [\pi, 3\pi/2]\) it is bounded above by \(w^*(T, \phi, \eta, r)\) where

\begin{equation}
w^*(T, \phi, \eta, r)^2 = 1 + \frac{r^2(\frac{1}{2} + \eta)^2 + (2 + \eta)^2 + 2r(\frac{1}{2} + \eta)(2 + \eta) \cos \phi}{T^2},
\end{equation}

which is decreasing in \(T\).

To bound \(n\) using (4.3) it remains to bound \(-\log |f(1+\eta)|\). This is done by using (4.2) and (4.4) to show that

\[-\log |f(1+\eta)| \to -\pi N \log |L(1+\eta+iT)| \leq -\pi N \log \zeta(2+2\eta)/\zeta(1+\eta)].

This, together with (2.8), (4.1), (4.3), (4.5), (4.7) and sending \(N \to \infty\), shows that, when \(T \geq T_0\)

\begin{equation}
\left| N(T, \chi) - \frac{T}{\pi} \log \frac{kT}{2\pi e} \right| \leq \frac{r(\frac{1}{2} + \eta)}{2\pi \log r} \log kT + C_2,
\end{equation}

where

\begin{equation}
C_2 = \frac{2}{\pi} \left\{ \log \zeta(\frac{1}{2} + \sqrt{2}(\frac{1}{2} + \eta)) + g(1, T) + \frac{E}{2} \right\} + \frac{3}{2 \log r} \log \zeta(1+\eta)
- \frac{\log \zeta(2+2\eta)}{\log r} + \frac{1}{2 \log r} \int_{-\pi/2}^{\pi/2} \log \zeta(1 + \eta + r(\frac{1}{2} + \eta) \cos \phi) \, d\phi
+ \frac{r(\frac{1}{2} + \eta)}{4\pi \log r} \left\{ -2 \log 2\pi + \int_{-\pi/2}^{\pi/2} (-\cos \phi) \log w(T_0, \phi, \eta, r) \, d\phi
+ \int_{-\pi}^{3\pi/2} (-\cos \phi) \log w^*(T_0, \phi, \eta, r) \, d\phi \right\}.
\end{equation}

4.1. A small improvement. Consider that what is really sought is a number \(p\) satisfying \(-\eta \leq p < 0\) for which one can bound \(L(p + it, \chi)\), provided that \(1+\eta - r(\frac{1}{2} + \eta) \geq p\). Indeed the restriction that \(p \geq -\eta\) can be relaxed by adapting the convexity bound, but, as will be shown soon, this is unnecessary.

The convexity bound (4.6) becomes the rather ungainly

\begin{equation}
|L(s, \chi)| \leq \left( \frac{k|s+1|}{2\pi} \right)^{(1/2-p)(1+\eta-\sigma)/2} \zeta(1-p)^{1+\eta-\sigma}\zeta(1+\eta)^{\sigma-p}\right)^{1/(1+\eta-p)},
\end{equation}
valid for \(-\eta \leq p \leq \sigma \leq 1 + \eta\). Such an alternation only changes \(J_2\), whence the coefficient of \(\log kT\) in (4.10) becomes
\[
\frac{r(\frac{1}{2} + \eta)(\frac{1}{2} - p)}{\pi(1 + \eta - p) \log r}.
\]
This is minimised when \(r = (1 + \eta - p)/(1/2 + \eta)\), whence (4.10) becomes
\[
(4.11) \quad \left| N(T, \chi) - \frac{T}{\pi} \log kT \right| \leq \frac{\frac{1}{2} - p}{\pi \log \left(\frac{1 + \eta - p}{1/2 + \eta}\right)} \log kT + C_2,
\]
where
\[
(4.12) \quad C_2 = \frac{2}{\pi} \left\{ \log \zeta(\frac{1}{2} + \sqrt{2}(\frac{1}{2} + \eta)) + g(1, T) + \frac{G(0, \sqrt{2}(\frac{1}{2} + \eta), T_0)}{2} \right\}
\]
\[
+ \frac{1}{\log \left(\frac{1 + \eta - p}{1/2 + \eta}\right)} \frac{3}{2} \log \zeta(1 + \eta) - \log \zeta(2 + 2\eta) + \frac{1}{\pi} \log \frac{\zeta(1 - p)}{\zeta(1 + \eta)}
\]
\[
+ \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \log \zeta(1 + \eta + (1 + \eta - p) \cos \phi) \, d\phi + \frac{1}{2} - \frac{p}{2\pi} \left( -2 \log 2\pi 
\right)
\]
\[
+ \int_{\pi/2}^{\pi} (-\cos \phi) \log w(T_0, \phi, \eta, r) \, d\phi + \int_{\pi/2}^{3\pi/2} (-\cos \phi) \log w^*(T_0, \phi, \eta, r) \, d\phi \right\},
\]
in which \(g(1, T), G(a, \delta, T_0), w \) and \(w^* \) are defined in (2.5), (3.13), (4.8) and (4.9).

The coefficient of \(\log kT\) in (4.11) is minimal when \(p = 0\) and \(r = \frac{1 + \eta}{1/2 + \eta}\). One cannot choose \(p = 0\) nor should one choose \(p\) to be too small a negative number lest the term \(\log \zeta(1 - p)/\zeta(1 + \eta)\) become too large. Choosing \(p = -\eta/7\) ensures that \(C_2\) in (4.11) is always smaller than the corresponding term in McCurley’s proof.

Theorem 1 follows upon taking \(T_0 = 1\) and \(T_0 = 10\). One could prove different bounds were one interested in ‘large’ values of \(kT\). In this instance the term \(C_2\) is not so important, whence one could choose a smaller value of \(p\).

5. The Dedekind zeta-function

This section employs the notation of §§2-3. Consider a number field \(K\) with degree \(n_K = [K : \mathbb{Q}]\) and absolute discriminant \(d_K\). In addition let \(r_1\) and \(r_2\) be the number of real and complex embedlings in \(K\), whence \(n_K = r_1 + 2r_2\). Define the Dedekind zeta-function to be
\[
\zeta_K(s) = \sum_{a \in \mathcal{O}_K} \frac{1}{(N(a))^s},
\]
where \(a\) runs over the non-zero ideals. The completed zeta-function
\[
(5.1) \quad \xi_K(s) = s(s - 1) \left( \frac{d_K}{\pi^{n_K} 2^{2r_2}} \right)^{s/2} \Gamma(s/2)^{r_1} \Gamma(s)^{r_2} \zeta_K(s)
\]
satisfies the functional equation
\[
(5.2) \quad \xi_K(s) = \xi_K(1 - s).
\]
Let $a(s) = (s - 1)\zeta_K(s)$ and let
\begin{equation}
(5.3) \quad f(\sigma) = \frac{1}{2} \left\{ a(s + iT)^N + a(s - iT)^N \right\}.
\end{equation}

It follows from (5.1) and (5.2) that
\begin{equation}
(5.4) \quad \left| \Delta_\pm \arg a(s) + \Delta_\pm \arg a(s) \right| \leq F(\delta, t) + n_K G(0, \delta, t),
\end{equation}
where $F(\delta, t) = 2 \tan^{-1} \frac{1}{2\pi} - \tan^{-1} \frac{1}{2 + \delta} - \tan^{-1} \frac{1}{2 - \delta}$, and $G(0, \delta, t)$ is defined in (3.13).

Thus, following the arguments in §§2-4, one arrives at
\begin{equation}
(5.5) \quad \left| N_K(T) - \frac{T}{\pi} \log \left\{ d_K \left( \frac{T}{2\pi e} \right)^{n_K} \right\} \right| \leq \frac{2(n + 1)}{N} + \frac{2n_K}{\pi} \{ \| g(0, T) \| + \log \zeta(\sigma_1) \} + 2,
\end{equation}
where $n$ is bounded above by (4.3), in which $f(s)$ is defined in (5.3). Using the right inequality in
\begin{equation}
(5.6) \quad \frac{\zeta_K(2\sigma)}{\zeta_K(\sigma)} \leq |\zeta_K(s)| \leq \{ \zeta(\sigma) \}^{n_K},
\end{equation}
onc
one can show that the corresponding estimate for $J_1$ is
\begin{equation}
(5.7) \quad J_1/N \leq \pi \log T + \int_{-\pi/2}^{\pi/2} \left\{ \log \tilde{w}(T, \phi, \eta, r) + n_K \log \zeta(1 + \eta + r(\frac{1}{2} + \eta) \cos \phi) \right\} d\phi
\end{equation}
where
\begin{equation}
(5.8) \quad \tilde{w}(T, \phi, \eta, r)^2 = 1 + \frac{2r(\frac{1}{2} + \eta) \sin \phi}{T} + \frac{\eta^2 + 2r(\frac{1}{2} + \eta) \cos \phi}{T^2}.
\end{equation}
For $\phi \in [0, \pi/2]$, the function $\tilde{w}(T, \phi, \eta, r)$ is decreasing in $T$; for $\phi \in [-\pi/2, 0]$ it is bounded above by $\tilde{w}^*(T, \phi, \eta, r)$ where
\begin{equation}
(5.9) \quad \tilde{w}^*(T, \phi, \eta, r)^2 = 1 + \frac{r^2(\frac{1}{2} + \eta)^2 + \eta^2 + 2r(\frac{1}{2} + \eta) \cos \phi}{T^2},
\end{equation}
which is decreasing in $T$.

The integral $J_2$ is estimated using the following convexity result.

\textbf{Lemma 3.} Let $-\eta \leq p < 0$. For $p \leq 1 + \eta - r(\frac{1}{2} + \eta)$ the following bound holds
\begin{equation}
|a(s)|^{1+\eta-p} \leq \left( \frac{1 - p}{1 + p} \right)^{1+\eta-\sigma} \zeta_K(1 + \eta)^{\sigma-p} \zeta_K(1 - p)^{1+\eta-\sigma} |1 + s|^{1+\eta-p} \times \left\{ d_K \left( \frac{|1 + s|}{2\pi} \right)^{n_K} \right\}^{(1+\eta-\sigma)(1/2-p)}.
\end{equation}

\textit{Proof.} See [5, §7]. When $p = -\eta$ the bound reduces to that in [5, Thm 4]. \hfill \Box
Using this it is straightforward to show that
\[
J_2/N \leq \frac{2r(\frac{1}{2} + \eta)}{1 + \eta - p} \left\{ \log \frac{\zeta_K(1 - p)}{\zeta_K(1 + \eta)} + \log \frac{1 - p}{1 + p} + (1/2 - p) \log \frac{d_K}{(2\pi)^{n_K}} \right\} + \pi \log \zeta_K(1 + \eta) + \log T \left( \pi + \frac{2rn_K(\frac{1}{2} + \eta)(\frac{1}{2} - p)}{1 + \eta - p} \right)
\]
\[\quad + \int_{\pi/2}^{3\pi/2} \log w(T_0, \phi, \eta, r) \left( 1 + \frac{n_Kr(\frac{1}{2} + \eta)(\frac{1}{2} - p)(-\cos \phi)}{1 + \eta - p} \right) d\phi \]
\]
\[(5.10)\]

The quotient of Dedekind zeta-functions can be dispatched easily enough using
\[\frac{-\zeta'_K}{\zeta_K}(\sigma) \leq n_K \left\{ -\frac{c'}{\zeta}(\sigma) \right\} \]
to show that
\[\log \frac{\zeta_K(1 - p)}{\zeta_K(1 + \eta)} = \int_{1-p}^{1+\eta} -\frac{\zeta'_K}{\zeta_K}(\sigma) d\sigma \leq n_K \int_{1-p}^{1+\eta} -\frac{\zeta'}{\zeta}(\sigma) d\sigma \leq n_K \log \frac{\zeta(1 - p)}{\zeta(1 + \eta)}.\]

Finally the term \(-\log |f(1 + \eta)|\) is estimated as in the Dirichlet L-function case — cf. (4.2). This shows that
\[\log |f(1 + \eta)| \geq N \log \frac{\zeta_K(2 + 2\eta)}{\zeta_K(1 + \eta)} + \frac{N}{2} \log(\eta^2 + T^2) + o(1).\]

This, together with (5.5), (5.7), (5.8), (5.9) and (5.10) and sending \(N \to \infty\), shows that, when \(T \geq T_0\),
\[\]
\[
\left| N_K(T) - \frac{T}{\pi} \log \left\{ d_K \left( \frac{T}{2\pi e} \right)^{n_K} \right\} \right| \leq \frac{r(\frac{1}{2} + \eta)(\frac{1}{2} - p)}{\pi \log r(1 + \eta - p)} \left( \log d_K + n_K \log T \right)
\]
\[\quad + \left( C_2 - \frac{2}{\pi} [g(1, T) - |g(0, T)|] \right) n_K + D_3, \]
\]
\[(5.11)\]

where \(C_2\) is given in (4.12) and
\[D_3 = 2 + \frac{r(\frac{1}{2} + \eta)}{\pi \log r(1 + \eta - p)} \log \left( \frac{1 - p}{1 + p} \right) + \frac{1}{\pi} F(\sqrt{2}(\frac{1}{2} + \eta), T_0)
\]
\[\quad + \frac{1}{2\pi \log r} \left( \int_{-\pi/2}^{0} \log \tilde{w}^*(T_0, \phi, \eta, r) d\phi + \int_{0}^{\pi/2} \log \tilde{w}(T_0, \phi, \eta, r) d\phi
\]
\[\quad + \int_{\pi/2}^{\pi} \log w(T_0, \phi, \eta, r) d\phi + \int_{\pi/2}^{3\pi/2} \log w^*(T_0, \phi, \eta, r) d\phi \right) \]
\]
\[(5.12)\]

If one chooses \(p = -\eta/T\), to ensure that the lower order terms in (5.11) are smaller than those in [2], one arrives at Theorem 2. One may choose a smaller value of \(p\) if one is less concerned about the term \(D_2\).

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References


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