Slow Maths—a metaphor of connectedness for school mathematics

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Declaration by author

This thesis is composed of my original work, and contains no material previously published or written by another person except where due reference has been made in the text. I have clearly stated the contribution by others to jointly-authored works that I have included in my thesis.

I have clearly stated the contribution of others to my thesis as a whole, including statistical assistance, survey design, data analysis, significant technical procedures, professional editorial advice, and any other original research work used or reported in my thesis. The content of my thesis is the result of work I have carried out since the commencement of my research higher degree candidature and does not include a substantial part of work that has been submitted to qualify for the award of any other degree or diploma in any university or other tertiary institution. I have clearly stated which parts of my thesis, if any, have been submitted to qualify for another award.

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ABSTRACT

This dissertation proposes a metaphor of connectedness for school mathematics that honours the discipline of mathematics and engages students in authentic mathematical activity. It is located at the intersection of mathematics, education and philosophy, providing a fresh reading and synthesis of established ideas. It is thus an argument about what really counts in school mathematics, rather than a presentation of new empirical research.

Part 1 of the thesis establishes the personal and political imperative for the thesis. It highlights the mismatch between dominant pedagogies of mathematics education and an alternative view that foregrounds students as mathematicians. I argue that the political debates that have punctuated and permeated school mathematics education over several decades are less about achievement levels or effective teaching than they are about epistemology. I examine dominant metaphors of education, arguing that they are locked into what Heidegger terms the technological enframing, casting students as products of, rather than participants in, the educational process. I describe the origins of slow food as a protest against a one-size-fits-all philosophy, and introduce the metaphor of *Slow Maths* as an alternative to these metaphors of education. I then re-examine the philosophical and epistemological underpinnings of mathematics education, arguing that absolutist and relativist philosophies of mathematics, rather than being binary opposites, arise from viewing mathematics from the outside and the inside, or far away and close-at-hand, respectively.

Part 2 of the thesis presents extensive evidence to support three dimensions of connectedness that underpin *Slow Maths*: mathematical connectedness, cultural connectedness and contextual connectedness. I show that mathematics is legitimated through a knowledge mode, providing evidence that mathematics "speaks for itself". I show that it is continuously developing as a field, and

describe the cultural context that promotes this development. I describe the reflexive relationship between mathematics and the world, providing evidence that mathematics both models the world and develops in response to the world. For each of these dimensions I use evidence from the discipline of mathematics itself, from the work of mathematicians and their personal narratives and from influential and contemporary mathematics education research. This thesis is unique in synthesising evidence from these three different sources in support of a philosophical position.

Part 3 of the thesis moves from theory to practice. I give a practical example of a unit of work in secondary mathematics that moves beyond mechanistic solutions methods with contrived pseudo real world applications to one that has strong and rich mathematical, cultural and contextual connections. I argue for a new dimension of mathematical knowledge for teaching that I term cultural and contextual knowledge of mathematics, and provide reflections from a preservice teacher education course that affirm its value.

The thesis does not purport to provide ready-made solutions. Rather it asserts the centrality of connectedness as a critical aspect of school mathematics and promotes an attitude of slowness as a way of engaging with both the discipline of mathematics and the activity of doing mathematics.

Keywords

slow, connectedness, culture, context, mathematics, education, metaphor, philosophy

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ANZSRC code: 130313, Teacher Education and Professional Development of Educators, 20%

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FoR code: 1302, Curriculum and Pedagogy, 60%

FoR code: 1303, Specialist Studies in Education, 20%

FoR code: 2202, History and Philosophy of Specific Fields, 20%

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FOREWORD AND ACKNOWLEDGEMENTS

"It's just a PhD, only the beginning". This is the well-meaning advice of numerous colleagues who have encouraged me to "get it done". But this is neither just a PhD, nor is it the beginning. In a sense it is the end of a forty-year journey in mathematics education. Like G.H. Hardy who wrote *A Mathematician's Apology* at a time when his career as a research mathematician was nearing an end, I write this near the end of a lengthy career as a mathematics teacher in schools and as a teacher of mathematics teachers in universities. Like Hardy, I seek to reflect on what it all means and to provide some personal insights into the nature of mathematics, education, schools and our relationship to knowledge.

The journey began at around the age of 15, when I first realised that I was a mathematician. I knew then that mathematics made sense and that I did not need to learn a multitude of facts and skills. I expected to get all the mathematical questions that I was asked to answer correct, not because I was particularly brilliant, but because mathematics made sense, and it could not be otherwise. This expectation was, of course, a response to the lucid mathematical explanations of my wonderful high school teachers, Lois Dippy and Nick Olijnyk, who presented mathematics as first and foremost a sense-making activity. My university lecturers such as the late Ren Potts inspired me to continue that relationship with mathematics.

Whether or not I was talented enough to pursue a career as a research mathematician I will never know, for I chose to pursue an alternative path as a teacher. Here I encountered mathematics in a different way, and experienced the joy of thinking about elementary mathematics as a creative endeavour. For this I am eternally grateful to inspiring mentors such as the late John Gaffney. My students at school were quick to let me know whether what I was teaching and how I was teaching worked. I am grateful to the hundreds of teenagers who

sat before me in mathematics classes—I enjoyed every minute of the 20 or so years I taught school mathematics.

During the past 20 years I have been privileged to create and debate problems as part of the *Mathematics Challenge for Young Australians*. The mix of gifted mathematicians and teachers on this committee is possibly unique in Australian education. It is a cauldron of mathematical problem-solving, bubbling over with a heady mixture of creativity, rigour and simple enjoyment of mathematics. My friend and colleague Peter Taylor was generous enough to offer me a position with the Australian Mathematics Trust, and I acknowledge him and all the other members of the Challenge Committee who have provided the stimuli for so many adventures with mathematics problems.

The mathematics teaching and more recently education research communities have been my "intellectual home" throughout most of these forty years. I acknowledge the countless members of AAMT and MERGA, who have presented little gems of insight that have contributed in sometimes visible but often invisible ways to the picture of school mathematics presented in these pages. The sudden moments of enlightenment when someone has made me adopt a totally different perspective have added up to what I hope is now a coherent and integrated view of school mathematics. I particularly acknowledge Will Morony, who for many years has been a valued colleague and friend and has engineered many opportunities for me to be involved in interesting projects.

This thesis would not, of course, have been possible without the ongoing support and encouragement of my supervisory panel. Although the current PhD looks nothing like the former version commenced with my good friend and colleague Robyn Zevenbergen, the time spent reading and reflecting on socially critical mathematics pedagogy was instrumental in shaping the mathematical world-view that I have now called *Slow Maths*. I acknowledge

Robyn's graciousness in accepting that neither the time nor the situation was right for me to complete my PhD at Griffith University.

Margaret Kiley persuaded me that the PhD needed to be written and has remained a constant source of encouragement throughout my time at the Australian National University. She had the good sense to recruit Alan Carey and Will Morony to the panel to provide the mathematician's and mathematics educator's expertise. Margaret, Alan and Will have given me the space to do things my way, for which I am extremely grateful.

None of the things I have learned during the past forty years would have been possible without the love and support of my family. My late father Ed and my mother Marj created a home environment where academic excellence was valued but not forced, and my brother Andrew was a role model. My former wife Chris was beside me for thirty years and encouraged me to become involved in the wider mathematics education community. My current partner Ann has been a constant source of support during the past five years. She has given me new ways to see the world, which has given me a sense of balance and perspective. It has been a privilege to be part of the lives of my own children, Michelle and Les, and more recently Ann's children, Enya, Toby and Sebastian, watching and hopefully helping them, to grow into beautiful young adults. I have learned much from their experiences, good and bad, of learning mathematics at school, and they have provided the motivation for me to try to make a difference.

I am not sure what the future holds or where it leads. However I hope that I will have more and more time to enjoy the company of my grandchildren, Carla and Luke. They are the present and the future, and I hope that in some small way this thesis helps to make their world and, through them, our world, a little better. I hope that at least a few people will read this thesis and that in

some small way it challenges the way they see things and helps to make school mathematics a more joyous and stimulating experience for their students.

Of one thing I am certain. No matter where I go or what I do, when asked on census, immigration or other forms for my occupation, I will proudly say "mathematics teacher".

For Carla and Luke

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LIST OF ABBREVIATIONS

Abbreviation	Meaning	Page
AAMT	Australian Association of Mathematics Teachers	20
ACARA	Australian Curriculum and Assessment Reporting Authority	10
ACM	Australian Curriculum: Mathematics	36
CCSS	Common Core State Standards	13
MERGA	Mathematics Education Research Group of Australasia	135
MKT	Mathematical knowledge for teaching	157
NAPLAN	National Assessment Plan for Literacy and Numeracy	10
NNS	National Numeracy Strategy	16
PCK	Pedagogical content knowledge	159
PISA	Program for International Student Assessment	29
RME	Realistic Mathematics Education	122
TIMSS	Trends in International Mathematics and Science Study	2

PUBLICATIONS ARISING FROM THE THESIS

Atweh, B., Miller, D., & Thornton, S. (2012). The Australian curriculum: mathematics—world class or déjà vu. In B. Atweh, M. Goos, R. Jorgensen & D. Siemon (Eds.), *Engaging the Australian curriculum mathematics: perspectives from the field* (pp. 1-18). Online publication: Mathematics Education Research Group of Australasia.

Thornton, S. (2008). *Speaking with Different Voices: Knowledge Legitimation Codes of Mathematicians and Mathematics Educators*. Paper presented at the 31st Annual Conference of the Mathematics Education Research Group of Australasia. Navigating Currents and Charting Directions, Brisbane. 523-530

Thornton, S. (2010). *Towards a better understanding of how knowledge is produced and legitimated in mathematics education research*. Paper presented at the 34th Conference of the International Group for the Psychology of Mathematics Education. Mathematics in different settings, Belo Horizonte, Brazil. Vol 4, 225-232

Thornton, S. (2011a). In Search of Uncertainty. *Curriculum Perspectives 31*(1), 74-76

Thornton, S. (2011b). *Towards an Epistemological Map of Curriculum in School Mathematics*. Paper presented at epiSTEME 4—International Conference to Review Research on Science, Technology and Mathematics Education, Mumbai India, 81-86

Thornton, S. (2014). The technological enframing of mathematics education. In J. Anderson, M. Cavanagh & A. Prescott (Eds.). *Curriculum in focus: Research guided practice (Proceedings of the 37th annual conference of the Mathematics Education Research Group of Australasia)* pp. 613–620. Sydney: MERGA.

NOTES FOR THE READER

The use of literature and data in this thesis

As an essentially reflective and theoretical argument this thesis does not contain copious amounts of new data or research, although I have used original data where appropriate. Nevertheless the studies reported in the thesis, particularly those relating to mathematics education research, are rich in data. In most cases these studies are well known. This is intentional; to create a convincing argument it is important to cite established research.

At first reading it may appear that the thesis is a statement of personal beliefs about the teaching of mathematics with literature selectively recruited to support a preconceived position. However, the process has been by no means unidirectional. My reading of the literature has profoundly influenced my beliefs about the purposes of education and in particular the teaching of mathematics, and my experience as a teacher has profoundly influenced my understanding of the literature. I am aware that in all cases the research is much richer than reported in this thesis, however I touch lightly on the research, using it in fresh ways to show that the central ideas of the thesis are, in fact, congruent with and supported by the literature.

The structure of the thesis

The thesis is in three parts. Part 1 establishes a rationale and philosophical basis for the thesis. Just as I touch lightly on research, I touch lightly on philosophy—my goal is to establish a foundation for the thesis. Part 2 fleshes out the argument, giving the details of the proposal for *Slow Maths* contained in the thesis. Part 3 provides some possibilities for enacting the ideas of the thesis in schools and universities. Parts 1 and 2 are each introduced by a narrative from my own experience. These narratives set the scene and provide

a point of reference for the discussion in the subsequent chapters. I encourage the reader to revisit the narratives having read the discussion.

Some key themes that permeate the thesis

A number of key themes permeate the thesis. I introduce them briefly here, and encourage the reader to keep them in mind in reading the thesis, even when they are not made explicit.

Slowness

Slowness is not only about time, although doing something slowly has its own rewards. Rather it is about doing something thoroughly and becoming immersed in the culture and context of the activity. In effect, slowness is the glue that holds the thesis together.

Connectedness

Connectedness concerns not only the three dimensions of connectedness discussed in Chapters 4 to 7, but more generally an orientation towards knowledge and life that sees the world holistically rather than as isolated fragments.

The technological enframing of education

Heidegger's notion of the technological enframing is introduced in Chapter 2. I use it to argue that we have become so consumed with questions of efficiency and effectiveness that we are no longer able to ask questions of purpose and worth.

Inside/outside views of mathematics

Philosophies of mathematics are discussed briefly in Chapter 3. Here I introduce the idea of inside/outside views of mathematics, or mathematics seen from near at hand or far away. I argue that school mathematics needs to encompass both views.

The goals and purposes of school mathematics education

Articulating the goals and purposes of school mathematics is the focus of Parts 1 and 2 of the thesis. It would have been convenient to find a succinct phrase that I could use throughout the thesis to capture the essence of those goals and purposes, but any such phrase inevitably becomes glib and ultimately loses meaning. School mathematics involves students in both becoming immersed in the culture and discipline of mathematics, and using mathematics critically in the world. It is thus simultaneously traditional, maintaining rigour within established academic traditions, and progressive, pointing to new ways of acting in society.

There is a secret bond between slowness and memory, between speed and forgetting.

Consider this utterly commonplace situation: a man is walking down the street. At a certain moment, he tries to recall something, but the recollection escapes him. Automatically, he slows down.

Meanwhile, a person who wants to forget a disagreeable incident he has just lived through starts unconsciously to speed up his pace, as if he were trying to distance himself from a thing still too close to him in time.

The degree of slowness is directly proportional to the intensity of memory; the degree of speed is directly proportional to the intensity of forgetting.

Milan Kundera, Slowness *(1996, p. 34)*

PART 1: SLOWNESS

A narrative of slowness: *The Mathematics Challenge for Young Australians*

For many years I have been a member of the problems committee of the *Mathematics Challenge for Young Australians*, conducted by the Australian Mathematics Trust. The Challenge epitomises slowness. It is slow in time, students having three weeks to complete a small number of problems. It is also slow in intent in that our aim in setting the problems is for students to engage deeply with important mathematical ideas. We value variability: this might be in the form of alternative solutions, but we also encourage it by providing extensions.

<u>The question: RIFTWIBs</u>

For many years I had asked students in year 8 to investigate writing positive integers as the sum of two or more consecutive integers. For example, 11 can be written as 5 + 6, 20 can be written as 2 + 3 + 4 + 5 + 6. Students quickly discover that all odd numbers can be written as the sum of two consecutive integers. They also find that some integers cannot be written as the sum of consecutive integers: 1, 2, 4 and 8 cannot be written in this way. They conjecture that no power of 2 can be written as the sum of consecutive integers. They also find that some integers, such as 9, can be written as the sum of consecutive integers in more than one way (4 + 5, or 2 + 3 + 4) and that some integers, such as 28, can be written as the sum of consecutive integers commencing with 1. These latter cases are called triangular numbers as they can be drawn as a triangular array of dots.

There is a standard method for proving these results using sums of arithmetic series. However, a visualisation using arrays stimulated me to find an alternative proof using the median of an odd number of consecutive integers. It

also suggested a question that we might ask for the *Mathematics Challenge for Young Australians*. I proposed a problem to the committee that asked students to draw trapezia using dots to represent certain integers such as 35, to explain why all odd numbers could be written as the sum of consecutive integers and to explain why a power of 2, such as 64, could not be written in this way.

Rather than presenting the problem as one involving numbers, we decided to make it visual. After much discussion regarding the need to eliminate possible interpretations such as non-integers or arrangements such as 2 + 4 + 6, we decided to draw them on a grid, as shown in Figure 1. In this figure the total number of dots on the boundary of or inside the outlined trapezium is 18, and can be seen as the sum of the dots in each row, 3 + 4 + 5 + 6. We needed a name for these special numbers, and after a very slow process called them "RIFTWIBs", or right forty five degree trapezia with integer base. After much further discussion we posed some related questions for students at two levels. We asked students in years 7 and 8 three questions:

1. Draw a diagram to show that 35 is a RIFTWIB number.
2. Show that all odd numbers greater than 3 are RIFTWIB numbers.
3. Find all the ways of representing 198 as a RIFTWIB number. Explain how you know there are no more.

For students in years 9 and 10 we asked an additional question:

4. Show that 128 is not a RIFTWIB number.

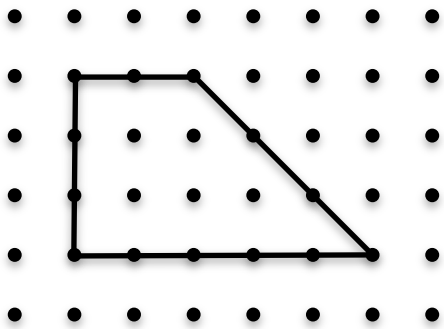

Figure 1 Diagram of the number 18, drawn as a RIFTWIB.

As in most mathematics the relative clarity of the end product obscures the experimentation, discussion and negotiation in its development. In all problems in the Challenge we try to encourage the slow progression towards rigour in the way we structure the problem. A relatively easy introduction to the problem allows students space to experiment by examining a specific case. Since there are three possible representations variety is allowed and encouraged. The justification that all odd numbers are RIFTWIBs encourages students to begin to generalise. The final question provides students the opportunity to test some other generalisations that we hope they have made using a larger number. We expected that most students would deduce that if a number has an odd factor smaller than its square root it could be written as a RIFTWIB. For example, since $198 = 3 \times 66$ it can be written as $65 + 66 + 67$. This also gives the representations 9×22 ($18 + 19 + \ldots + 25 + 26$) and 11×18 ($13 + 14 + \ldots + 22 + 23$). However it is much harder to find the representations $48 + 49 + 50 + 51$, or $11 + 12 + \ldots + 21 + 22$ that each has an even number of terms. For older students we then asked for an impossibility proof.

The question thus intentionally provides the inside/outside view of mathematics I discuss in Chapter 3. It encourages initial exploration, the formulation of hypotheses, proof of the hypotheses and application to a harder case. It also values the qualities of slowness discussed in Chapter 2 as it takes time to become immersed in the mathematics, and encourages students to use familiar ingredients, such as their knowledge of factors, in creative ways to solve a new problem.

However for members of the committee, in particular, the problem has inherent mathematical interest and links to historical episodes in number theory. The obvious question is whether there are integers that can be written as the sum of consecutive integers, but that can only be written as triangular numbers. For

example $6 = 1 + 2 + 3$ and $10 = 1 + 2 + 3 + 4$ are examples. We posed this more general problem as an extension.

Several members of the committee investigated the problem, each in their own way. After much playing with the idea, I deduced that a given integer can be represented as the sum of consecutive integers commencing with a number greater than 1 iff it is not:

- A power of 2;
- A number such as 6 or 28, which is of the form $2^n(2^{n+1} - 1)$ where $2^{n+1} - 1$ is prime; or
- A number such as 10 or 136, which is of the form $2^n(2^{n+1} + 1)$ where $2^{n+1} + 1$ is prime.

Hence there is a special relationship between these numbers, powers of 2, perfect numbers, and Mersenne and Fermat primes. We posed this as an extension to the problem in the hope that some students would explore the idea and be prompted to learn about the history and culture of number theory.

This narrative serves two purposes. One is to illustrate slowness in the process of setting a question, which normally takes six full days spread over a calendar year. The other is to provide a point of reference for the discussion of the philosophy of mathematics from the outside and inside in Chapter 3.

CHAPTER 1: "PLUS ÇA CHANGE…"

> …if these videos and data represent fairly normal current practice in these countries (and the teachers involved and others say that they do), then there are a lot of pretty boring, artificial, low-level, irrelevant, mentally stifling lessons being delivered round the globe in the name of Year 8 mathematics, and it is not surprising that so many adults don't want to know anything more about mathematics after they leave school. (McIntosh, 2003, p. 106)

Synopsis

A personal perspective: My reflection on many years of teaching mathematics and of educating preservice teachers to teach mathematics raises questions about why classrooms in which students are interested, engaged and think and act as mathematicians still seem to be outliers.

A historical and political perspective: The ongoing and almost universal antagonism between mathematicians and mathematics educators articulated in the *Math Wars* raises questions about the priorities of school mathematics education and how classroom materials and practices might better prepare students as future mathematicians.

Assertion: These questions are much less about how to teach well or what works, and much more about what counts as worthwhile. That is, these are fundamentally philosophical and epistemological, rather than pedagogical debates.

After more than thirty years of involvement in mathematics education I know too much. I also know too little. I know too much to write a thesis that is detached and dispassionate. I have seen too many aspects of school mathematics change, yet stay the same. I know too little about why they stay

the same, despite the goodwill and hard work of a multitude of intelligent and passionate educators, among whose number I include myself. As a result this thesis is written from the perspective of someone who is not only *in* and *for* the mathematics education community, but as someone who is *with* the mathematics education community (Moustakas, 1995). Like my friend and colleague Alistair McIntosh, writing above about the Year 8 mathematics lessons on which he was invited to comment as part of the Australian report of the *Trends in International Mathematics and Science Study* (TIMSS) video study of mathematics classrooms around the world, I continue to be dismayed at how little professional development, new resources or new curricula impact on the daily activity in most mathematics classrooms, particularly in the early years of high school. As Beeby (cited in Griffiths & Howson, 1974) points out about the role of teachers in curriculum reform, "the average teacher has a very great capacity for going on doing the same thing under a different name" (p. 143).

The first part of this chapter is unashamedly self-indulgent, yet without it the thesis has neither grounding nor raison d'être. I set out to tell my own story both as a student and as a teacher. In this story I highlight key aspects of my relationship with mathematics, teaching, learning and epistemology that I learned from each of the various settings in which I have studied and worked. Although it is arranged in themes it is essentially chronological. The themes are not meant to imply that the issues were somehow completely resolved at that time, but rather to highlight the critical issues that arose for me and hence this research.

It is not a complete story as it highlights only selected incidents and recollections that seem to me to impinge on that relationship. Nor is it finished as that relationship is ongoing and developing. However it is important to the thesis as it aims to show that no matter how valuable and important preservice teacher education, ongoing professional development and curriculum resources

may be, the most crucial and pressing issue in mathematics education is epistemology. I maintain that this is deeper than described in the extensive literature on beliefs about mathematics (e.g. Leder, Pehkonen, & Törner, 2002); it is essentially a relationship with knowledge and mathematics. This relationship lies at the heart of the various debates about school mathematics curriculum evident in the so-called *Math Wars* (D. Klein, 2007) or the well-documented difficulties of implementing pedagogical change through professional or curriculum development (e.g. Cuban, 1993; Ross, McDougall, & Hogaboam-Gray, 2002). It is the reason why in many ways in education "plus ça change, plus c'est la même chose".

1.1 Personal knowledge and developing tensions

1.1.1 Discovering mathematical connectedness

I enjoyed mathematics at school. I enjoyed the logic and connections. I enjoyed the feeling of understanding a concept as part of a coherent whole. I cannot recall remembering long lists of formulae in the hope of reproducing them in an examination, but I can vividly recall my personal pride in refusing to ask my teacher for assistance, preferring to work it out for myself. Boaler (2003), after Pickering (1995) would term this the "agency of the discipline". I term it a relationship with knowledge. Without being able to articulate and describe this relationship with knowledge, I somehow saw that ultimately it was mathematics itself that determined correctness, and in turn I was determined to understand the mathematics that I was learning deeply enough that I did not need to rely on an external arbiter of correctness.

I carried this relationship with knowledge into my undergraduate years, but in some ways it did not serve me well. For me the logic of mathematics became obscured by being asked to learn too much too quickly. Mathematics became a set of disconnected techniques to answer questions I had no particular interest in answering. Perhaps this is the experience of many, if not most, children at

school. It was not until my Honours year when faced with the problem of writing a program in Fortran to solve a complex system of linear equations (which can now be accomplished instantly with readily available software or even a graphics calculator) that I rediscovered a relationship with knowledge in which logic was paramount, and in which external agents did not judge correctness. Significantly, this was a slow process, taking many months and many attempts.

It was at about this time that I also discovered a latent talent for communication. In particular I discovered that I greatly enjoyed the weekly tutorials that I was asked to conduct for first year students, and that somehow I was able to explain things in a way that engaged these students and enabled them to understand what was previously obscure. Although I was not conscious of it at the time, I am sure that, as in all acts of teaching, my own understanding was also called into question and deepened as a result. Thus I decided that I had a vocation as a teacher and enrolled in a Graduate Diploma of Teaching.

Suddenly logic was a thing of the past. There were no right or wrong answers, although in many cases things that I thought were right turned out to be wrong, at least in a pragmatic sense. I recall developing and delivering creative presentations to my peers of an idea from science or mathematics, perhaps using role-plays or other relatively innovative techniques. Yet I also recall the disappointment and frustration when similar presentations proved completely ineffective in engaging a class of year 9 students during my teaching practicum. I reverted to doing it "the teacher's way", which consisted of writing notes on the board for the students to copy, usually followed by silent answering of textbook questions. Already I was faced with the tension between a personal view of mathematics and science as creative pursuits in which people actively pursue knowledge, and an education system that promotes conformity and passivity. As an eternal optimist, however, I knew that such tensions would largely disappear when I became a teacher with my own class

with whom I could share the inherent elegance and logic of mathematics and the pure sciences.

Or so one thinks when one is young! While a chronological account of a teaching career may be personally therapeutic, or may even make good reading, that is not the purpose of this chapter. Rather, I want to sketch out some of the defining moments and events that have made it imperative for me to attempt to investigate, analyse and document an alternative way of thinking about school mathematics. By doing this, I hope to highlight the tensions that inevitably arise from competing views of knowledge.

1.1.2 Seeking pedagogical and epistemological synergy

A group of curriculum leaders in mathematics from several schools in the South East of South Australia meet[1] for a regular professional development discussion. Although I have been teaching for only two years, as the only teacher with mathematics qualifications in a small country school, I am invited to attend. On this occasion a guest, Bill Fitzgerald, is visiting from Michigan State University. Bill leads us through some materials that he and his colleagues have written entitled *The Mouse and the Elephant*[2]. The materials investigate the effect of enlargements on surface area and volume by looking at how many mouse coats are needed to make an elephant coat or how many mouse pellets are needed to feed an elephant, as part of a trip into space. We use small ceramic tiles to make two-dimensional shapes and linking cubes to make solids. We investigate conceptually rich questions about area, perimeter, surface area and volume in a way that has never occurred to me before. I think to myself, "Wow! Can you really teach mathematics like this?"

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[1] I write this paragraph in present tense to convey the sense of immediacy and revelation that accompanied the incident.

[2] This has since become part of the unit *Covering and Surrounding* in the *Connected Mathematics Project* materials (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1996).

Although I may not have been able to articulate it at the time, *The Mouse and the Elephant* provided a synergy between a personal view of knowledge that had been forming over twenty years, and a pedagogy that was creative and engaging. What was particularly appealing to me was the way in which the materials promoted making connections, and hence the development of an understanding that was much deeper than skills and processes. However, on many occasions I seemed unable to convince my students that striving for understanding was more important than mastering techniques and obtaining correct answers as found in the back of the book. There seemed to be a mismatch between my approach to mathematics and that of my students.

It was at about this time that I first became aware of Skemp's (1976) notion of relational and instrumental understanding. Skemp introduces the idea of *faux amis* or false friends; words which mean one thing to one person but something quite different to someone else. He highlights two faux amis in the mathematics classroom, one of which is understanding, the other is mathematics itself, and discusses the difficulties that arise when the teacher has one view of understanding or of mathematics and the students have a different view. There is an obvious miscommunication arising from a lack of shared understanding about what it means to learn mathematics and, indeed, about the nature of mathematics itself. Thus I became acutely aware of the tensions between my personal beliefs about mathematics and those of my students.

After four years in a small country school I realised that some of the tensions mentioned above had not actually disappeared. It was time to move. A larger city school, with a philosophy that espoused student-centredness, flexibility and community involvement beckoned. This was a new secondary school, building from the ground up. An initially small school, it gave me the opportunity to shape the school mathematics program in any way I wished, and to adopt whatever approach I wished in my own classes. It gave me the freedom to take risks.

I tried to introduce important mathematical ideas *through* problem-solving. I felt that I achieved at least some measure of success, yet my approach was somewhat derailed by the arrival of a new and highly respected mathematics faculty head. The faculty head emphasised mathematics *for* problem-solving or teaching *about* problem-solving (Van de Walle, Karp, & Bay-Williams, 2010), expecting each class to complete a set of somewhat random problems once per week, generally on a Friday afternoon. To me this was problem-solving as an add-on to mathematics, rather than as an integral part of mathematics. It was the antithesis of what I felt I had being trying to achieve in my own teaching. Thus, while convinced that my approach to teaching and learning was both effective and remained faithful to the activity in which mathematicians actually engage, I was dismayed that such a view was not shared by my colleagues.

1.1.3 Excursions into meta-mathematics

During these years I was fortunate to be given six months paid leave to undertake personal study in emerging areas of mathematics at the University of South Australia. Areas of mathematics such as exploratory data analysis and dynamic programming were much less clearly defined than those I had studied in my previous school and university education, and showed that mathematics could be emergent and dynamic.

I also had the opportunity to read critiques of the fixed and exact view of mathematics (Davis & Hersh, 1981, 1986) so characteristic of school mathematics curricula. I began to see mathematics as having inherent and unresolvable uncertainties and ambiguities (e.g. Gödel, 1992; Skovsmose, 2009). I saw mathematics as being both produced from a world-view and, in turn, shaping a particular world-view. If this was the case then the study of mathematics was a social contract between the teacher and the learners (Davis, 1988), rather than the presentation and learning of a set of facts and relationships. For me, such a view called into question many of the taken-for-

granted elements of mathematics education, such as the objectivity of test results and the hierarchical structure of the school syllabus, with the consequent need for a clearly defined developmental sequence of ideas. Rather than being a search for truth, mathematics education became a search for meaning, enabling us to "live in a mathematised world and to contribute to this world with intelligence" (Davis, 1988, p. 1027).

Yet at the back of my mind there was, and remains, a nagging sense that mathematical knowledge does embody certainty. There is a rigour to mathematics that is not present in any other field, with the possible exception of computer programming and logic. Once one accepts some fundamental axioms of a system and the rules of mathematical logic, then results within the system are guaranteed. This is what gives mathematics its internal consistency. I was thus faced with the problem of simultaneously holding what seemed to be two opposing view of mathematics: one as something dynamic and creative, subject to debate and revision; the other as an established and consistent body of knowledge.

1.1.4 Leading change where results are paramount

Uncertainty and ambiguity are, of course, in sharp contrast to the pragmatic imperative of succeeding in high stakes external exams. This was exactly the position in which I found myself in a new position as Head of Mathematics at a large, established, independent school. A clash of ideologies occurred within weeks of my acceptance of the position.

From the outset I tried to create a shared view of mathematics and learning with my year 8 class. Using the handshakes lesson described at the beginning of Part 2 of the thesis as a springboard for students to investigate ideas about triangular, square and other figurate numbers, I tried to make my pedagogy and epistemology explicit (Sullivan, Zevenbergen, & Mousley, 2003). I did not give the students a test to normalise and group them! I was very quickly

questioned by concerned parents whose experience of mathematics was very different from that which I was providing to their children and for whom results in tests were paramount.

I arranged a family maths evening, to which I invited the parents of my year 8 class and other siblings. Almost all attended, which may have been because they had paid considerable money for their children's education and hence were interested, but it may also have been because of their shared concern at this non-conventional approach to mathematics. Their perceptions were challenged. They formed groups! They talked about a problem I gave them! They used hands-on materials! They read out their results to the other people present! They had fun and learnt some new mathematics! In the words of one parent, whom I later discovered was a primary school principal, "It is so refreshing to see a high school maths teacher who uses primary school methodology". I still consider this to be one of the nicest compliments I have received as a teacher.

Over time I found that a classroom environment that emphasised process over product, and meaning over accuracy, could be effective. Students' results in external activities such as competitions or high stakes examinations in senior secondary[3] were at worst unaffected, and possibly improved. It was possible to change pedagogy even when the need for results remained paramount. We abolished common tests. We abolished strict streaming of students in years 8 and 9. We introduced investigations as a regular part of the mathematical activity in classrooms. We embraced discussion, group work and the use of materials or technology where they helped to create meaning and develop conceptual understanding. We gave ourselves permission to adopt a somewhat emergent and flexible approach to curriculum.

[3] This was prior to the introduction of system-wide external assessments in years 3, 5, 7 and 9.

Yet I am deeply conscious that not everyone shares my epistemology. Many of my colleagues had, and continue to have, views that are different, even diametrically opposed. As many of my preservice secondary mathematics teachers have expressed, the security of the rightness or wrongness of answers is one of the attractions of mathematics and one of the motivations to become a teacher of school mathematics. Reconciling the apparent certainty of mathematics with the uncertainty of its development and learning is the key issue in this thesis.

1.1.5 Becoming a teacher educator and educational researcher

The subsequent 15 years in a university environment have reinforced the sense that there is an epistemological divide between the mathematics education research community, many mathematicians and many teachers of mathematics. Teachers are caught in the middle of this philosophical and epistemological divide (White-Fredette, 2009). Beliefs about mathematics are notoriously hard to shift (Handal & Herrington, 2003), and despite all my efforts to challenge the dominant paradigms in mathematics teaching, most lessons that I have seen while watching preservice teachers have followed a model of students reproducing a set of relatively low-level, disconnected skills that the teacher has demonstrated. Indeed, as one of my preservice teachers reported following a critique of his lesson by a teacher, to do things differently would be "messing with students' minds". I cannot help but wonder what the role of teachers is if it is not "messing with students' minds"!

Yet I have found a home in the mathematics education research community. Good research encourages questioning and uncertainty. I have discovered a number of different research paradigms that serve different purposes, and are valuable in different contexts. I have been able to read, reflect and collaborate; activities that are often far removed from the daily activity of teaching in schools. This thesis is very much part of that journey of reading, reflection and

collaboration. Through it I hope to make sense of the philosophical and epistemological divides I have encountered in 20 years of teaching in schools and 15 years of work with teachers and preservice teachers. I hope to pull together some of the various ways of seeing the world to which I have been introduced in this research environment, and to add a little towards our understanding of why not everyone sees the world as questions.

Like most of my work, the thesis itself has taken a number of twists and turns, all of which have added to the rich tapestry of thought that sits behind it. It is influenced by ideas as diverse as applied linguistics and discourse analysis (Fairclough, 1995; Gee, 1989; Halliday & Martin, 1993), funds of knowledge (Moje et al., 2004; Moll, Amanti, Neff, & Gonzalez, 1992), sociology of education (Bernstein, 2000), studies of mathematical tasks (Watson, 2004), sociomathematical norms (Yackel & Cobb, 1996) and studies of knowledge in higher education (Becher & Trowler, 2001). Each of these has been instrumental in sharpening the focus towards what really counts in mathematics education and research. For I maintain that questions about why mathematics teaching seems to continue in much the same way as it has for the past one hundred or so years do not concern curriculum, resourcing, assessment, preservice education or professional development. Rather they are deep epistemological questions about how we see the world, and particularly about how we see knowledge in mathematics and mathematics education.

1.2 Historical and political tensions

The personal tensions and frustrations described above play out on a larger scale in the political arena, where proponents of progressive approaches to curriculum and pedagogy are often opposed by those arguing for retention of a more conservative set of curriculum content coupled with system-wide assessment approaches. The debate is evident in the so-called *US Math Wars* (D. Klein, 2007); in controversies surrounding the introduction of a UK national curriculum and associated testing (Cooper, 1994); in questions about

the value of *Realistic Mathematics Education* in the Netherlands (van den Heuvel-Panhuizen, 2010); in debates surrounding the development of national curriculum statements in Australia in the early 1990s (Ellerton & Clements, 1994); and in ongoing debates about the *Australian Curriculum: Mathematics*[4] (Australian Curriculum and Assessment Reporting Authority [ACARA], 2013) and the Australian *National Assessment Plan for Literacy and Numeracy* [NAPLAN] (Ladwig, 2010). Just as the questions that arise for me from my personal reflections as a long-term teacher of mathematics are fundamentally about our relationship with knowledge, so I suggest that the historical and political debates about mathematics curriculum, assessment and pedagogy are less about what is effective and more about philosophy and epistemology. As Hiebert (1999) said about particular teaching methods and the development of the *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000), "Debates about what the research says will not settle the issue; only debates about values and priorities will be decisive" (p. 5).

I have argued elsewhere that underlying differences in the way communities such as mathematics education researchers and mathematicians view knowledge are at the heart of much of the long-standing debate surrounding school mathematics curriculum and pedagogy, suggesting that "the debate over what counts in mathematics education and the school curriculum is, in effect, a battle for control of the epistemic device" (Thornton, 2008, p. 524). Or, as Sfard (1997) writes:

> On the one hand, there is the paradigm of mathematics itself where there are simple, unquestionable criteria for distinguishing right from wrong and correct from false. On the other hand, there is the paradigm of the

[4] As I refer to the *Australian Curriculum: Mathematics* frequently in this thesis, for ease of reading I do not reference it every time.

> social sciences where there is no absolute truth any longer; where the idea of objectivity is replaced with the concept of intersubjectivity, and where the question about correctness is replaced by the concern for usefulness (p. 491).

I suggest that the battle between these paradigms continues, and is played out in curriculum authorities, universities and school classrooms throughout the Western world.

1.2.1 An international perspective

Seldom, if ever, is a new curriculum introduced without some controversy. In some cases the debates surrounding new curricula have been relatively benign, involving little more than discussions of where content should be placed relative to school levels or the clarification of points of ambiguity; while in others debate has escalated to the political arena, involving accusations of secret meetings and biased committees (Battista, 1999; Becker & Jacob, 1999). The brief discussions that follow give an indication of the universality of such debates, but more importantly shed light on the differing values and philosophies of various protagonists in these debates. This creates a case for a re-examination of the philosophy of mathematics education to seek common ground.

The most extreme debate surrounding school mathematics curriculum was located in the United States in the late 20th century and early years of the 21st century, where proponents of a reform curriculum and those of a traditional curriculum engaged in the *US Math Wars* (D. Klein, 2007). In this debate reformists (the *mathematically sane*) accused traditionalists (the *mathematically correct*) of a reductionist, back-to-basics approach that subjugated the process of learning mathematics to a set of well-defined procedures. On the other hand, those who claimed to be mathematically correct accused reformists of being *fuzzy*, of valuing any method so long as it worked,

and of allowing students to work everything out for themselves. Reformists argued for a more socially just, progressive curriculum, while traditionalists argued that social inequities could only be addressed by providing basic skills for all. D. Klein (2007) concluded that "the math wars are unlikely to end until programmes espoused by progressives incorporate the intellectual content demanded by parents of school children and mathematicians" (p. 32). However one might equally suggest that the *Math Wars* are unlikely to end until programmes espoused by traditionalists recognise that uncertainty and hesitancy are a natural part of human learning.

Despite the assertions from both sides of the debate that their concern was to promote excellence in school mathematics, at heart the *Math Wars* were fundamentally about philosophies of mathematics and the related question of what was valued in school mathematics. Battista (1999) claimed that the arguments of the anti-reformists lacked understanding of both "the essence of mathematics and of scientific research on how students learn mathematics" (p. 425). The University mathematicians Haimo and Milgram (2000) responded to assertions that the *California Mathematics Standards*, developed following the critique of the reform standards, were a return to the past by asking: "Why not call the curriculum internationally benchmarked mathematics? Or, as we prefer to call it, *real* mathematics?" (p. 146, italics added). Along with emotive statements in the media and commentary on websites such as *Mathematically Correct* (archived at Siebert, n.d.) and *Mathematically Sane* (2011), statements such as these moved the debate beyond effective teaching and learning into the realm of who should control school mathematics curriculum, pedagogy and assessment and hence what would be valued.

The National Council of Teachers of Mathematics attempted to address many of the concerns arising in the Math Wars through the release of their *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence* (National Council of Teachers of Mathematics, 2006).

The *Focal Points* detailed the concepts, skills, and procedures considered essential at each grade to provide students with the depth of knowledge required for progress to more advanced mathematics. They emphasised fewer important topics per year, with an emphasis on connections between mathematical concepts, particularly in number. Similarly the most recent curriculum development in the USA, the *Common Core State Standards* (CCSS), are intended to "address the problem of a curriculum that is 'a mile wide and an inch deep'" (Common Core State Standards Initiative, 2010, p. 3). They represent a shift away from individual state standards such as those of California and Massachusetts, which were an outcome of the *Math Wars* (Stotsky, 2007), and a move towards a nationally agreed set of standards for English, language arts and mathematics. Wu (2013), one of the most vocal critics of the so-called *New-New Math* during the California Math Wars, proclaims the CCSS as a gigantic leap from textbook mathematics to "correct, coherent, precise, and logical" mathematics (p. 8).

On the other hand the *Common Core State Standards* are not without their critics. Much of this criticism focuses on the extent to which, in comparison with what are perceived as highly regarded state or international standards, they prepare students for higher levels of mathematics (Stotsky & Wurman, 2010). Such criticism focuses almost exclusively on the content of the CCSS and the relative placement of different topics throughout the years of schooling. A more rigorous analysis (Porter, McMaken, Hwang, & Yang, 2011) links content with level of cognitive demand, creating a matrix of some 1,085 cells in a two-dimensional framework constructed by crossing 217 mathematical topics with five levels of cognitive demand. This analysis suggests that the CCSS have higher levels of cognitive demand than state standards, yet less emphasis on procedural fluency than expected in curricula in countries such as Singapore, Finland and Japan. While Porter et al. question whether this implies that the CCSS ought to place greater emphasis on the teaching and learning of

standard procedures, Cobb and Jackson (2011) criticise the atomistic cell-by-cell analysis created by the intersections of topic and cognitive demand, and suggest that a more holistic analysis would examine key topics to show the development of cognitive demand.

What is significant about these critiques, and similarly about critiques of documents such as the *Australian Curriculum: Mathematics*, is the almost exclusive focus on whether or not the documents contain the appropriate mathematics content and sequencing supposedly required for higher levels of mathematical study. Discussion of the purpose and philosophy of school mathematics is relatively minor, or often conspicuously absent. Three notable exceptions relate to the Standards' lack of real world connection, to their failure to promote a critical view of problem-solving, and to their emphasis on college readiness rather than mathematics for social good.

Garfunkel and Mumford (2011) argue for a greater emphasis on mathematics for solving real life problems, suggesting that traditional topics in mathematics such as algebra, geometry and calculus be replaced by a sequence such as finance, data and basic engineering. They argue that such an approach would not devalue abstract thinking, but rather that mathematics has flourished over the centuries because of its connection to real life problems. Wiggins (2012) argues that the mathematics standards are a step backwards, failing to present students with non-routine, authentic and ill-defined problems, and that they fail to encourage students to find problems rather than merely to solve well-formed, teacher-posed problems. In May 2014 the Chicago Teachers' Union House of Delegates voted to oppose the introduction of the *Common Core.* The rationale was stated thus:

> The authors of the Common Core view the purpose of education as college and career readiness. We view the purpose of public education as a means for educating a populace of critical thinkers who are capable of

> shaping a just and equitable society in order to lead good and purpose-filled lives. (Cody, 2014, para. 6)

Although the heated debate and vitriolic rhetoric of the *US Math Wars* of the late 1990s has largely disappeared, debates surrounding mathematics curriculum and pedagogy have recently surfaced in the Netherlands. This debate was prompted by mathematicians expressing concerns about the ability of Dutch children to calculate answers to written arithmetic problems, which they claimed had diminished following the widespread adoption of *Realistic Mathematics Education* (RME). Van den Heuvel-Panhuizen (2010) suggests that such debates are a universal outcome of mathematics education reform movements, concluding her discussion by listing some lessons for any country engaging in reform of mathematics education. Among these are that Mathematics Wars are based on emotion rather than facts; that such debates are geographically and temporally universal; and that there is always a political element to the debates. Debate in the Netherlands is ongoing.

Perceptions of low literacy and numeracy standards also led to the development of a national curriculum in the United Kingdom. This commenced with the Callaghan Labour government of the mid 1970s and gained momentum under the Thatcher era of the 1980s (M. Brown, Bibby, & Johnson, 2000). The national curriculum enterprise was marked by tensions between the political preference for a conservative curriculum stressing basic skills and testing, and the emphasis on understanding and applying mathematics and problem-solving recommended by the Cockcroft report (1982). The *UK National Curriculum* (Department of Education and Science and the Welsh Office, 1989) was also developed with a backdrop of nationally administered standardised tests, which arguably drew attention away from debates about the content and aims of the curriculum itself (Noss, 1990). The *Curriculum* was thus received largely passively by the mathematics education

community who, like teachers in the USA, focused on issues of coherence or placement of content rather than substantive issues of purpose.

On the other hand, Dowling and Noss (1990), in their critique of the *UK National Curriculum*, focused attention on deeper issues of purpose, philosophy and the nature of mathematics. They criticised the very nature of the statements of achievement in the document, arguing that "the NC has effectively emptied the mathematics curriculum of mathematics" (p.1). Citing the Working Party Report that laid the groundwork for the *National Curriculum*, Noss argued that hierarchical assumptions about the nature of mathematics both distort pedagogical practice and give a false impression of the nature of mathematics itself. He critiqued statements such as:

> Mathematics is the most abstract of subjects. Attainment targets in mathematics have to be very tightly defined to avoid ambiguity, and the degree of precision required gives a very clear indication of the "content, skills and processes" associated with the targets. (DES, 1998 cited in Noss, 1990, p. 18)

Far from being statements that promoted a view of mathematics as a way of making sense of the world, or as a medium to express generality or structure, these attainment targets were reduced to a set of tools that could be applied to often artificial economic or industrial problems, or were politicised towards an emphasis on basic methods of computation, including long division. As argued by S. Ball (1999), rather than being a call to raise standards, this might be considered a form of "cultural restoration", which has widespread political and popular appeal due to its discourse of nostalgia. It would seem that the situation has changed little, with the Education Secretary, Michael Gove, in a speech to the Royal Society (2011) calling for a renewed emphasis on basic skills—including long division—that will reverse the decline in the academic performance of children in the UK.

Noss (1990) concluded his discussion of the *UK National Curriculum* by proposing that the relative silence around the content of the curriculum (long multiplication and division notwithstanding) actually created a space in which teachers could reintroduce mathematics into the mathematics curriculum. Rather than maintaining the low priority of content vis-à-vis assessment, Noss recommended raising it to a high priority and bringing students into contact with substantive ideas that show mathematical thought as a powerful means of understanding the world and of promoting social change. Similarly, I argue that rather than focusing attention on the mechanics of where content is placed, on how it is sequenced or on how it articulates with national assessment regimes, Australian mathematics educators ought to have an increased focus on what the *Australian Curriculum: Mathematics* is saying about the nature of mathematics itself, and on how the content descriptions enable students to engage with substantial mathematical concepts and ways of thinking.

Moves to specify content via a UK National Curriculum were followed in the mid-1990s by moves to specify pedagogic approaches. Again concerned with relatively low standards of numeracy compared to other countries, the Conservative Major government, followed by the Labour Blair government, commissioned reports into teaching standards and methods (M. Brown et al., 2000). This resulted in the development of the *National Numeracy Strategy* (NNS) (Department for Education and Employment, 1999), which mandated:

- An increased emphasis on number and on calculation;
- A three-part template for daily mathematics lessons, including sustained periods of whole-class teaching and discussion;
- Detailed planning using a suggested week-by-week framework of detailed objectives; and
- A systematic and standardised national training programme.

Like the mechanical and utilitarian emphasis of the *National Curriculum*, such a Strategy addresses surface issues in the teaching of mathematics, rather than addressing deeply-held beliefs about the nature of learning in mathematics, or indeed about mathematics itself (Handal & Herrington, 2003; Pratt, 2006). In their analysis of the impact of the NNS Brown, Askew, Millett and Rhodes (2003) assert that "[w]hen the beliefs of the teachers about how children should learn and be taught numeracy…and the way that teachers interact with children, are examined, it appears that in almost no cases have 'deep' changes taken place" (p. 668). Despite the surface changes in classrooms effected through the NNS, students continued to be positioned as passive receivers of knowledge, rather than actively engaging in mathematical thinking and talking (Pratt, 2006).

1.2.2 The Australian context

The curriculum and pedagogical debates in the USA, the UK and the Netherlands briefly outlined above are mirrored to varying degrees in Australia. On the one hand, mathematics educators, in particular university-based teacher educators and mathematics education researchers, have called for a mathematics curriculum that is responsive to a changing society, that values and incorporates the use of technology and that recognises the hesitant way in which students construct knowledge; while on the other politically active groups, including some University mathematicians, have called for a more conservative approach that highlights fundamental skills. This has most recently occurred, as in the UK and the USA, against a backdrop of standardised testing.

Arguably the most virulent debates centred around the development and introduction of *A National Statement on Mathematics for Australian Schools* (Australian Education Council, 1991) and its associated document *Mathematics – a curriculum profile for Australian schools* (Australian

Education Council, 1994) in the early 1990s. The mathematics education community and the mathematics community were united in their concern over the process by which the documents were produced, citing lack of adequate consultation in their development and the apparent determination of the writing team to pursue a particular agenda. Indeed the process surrounding the attempt to develop an Australian national curriculum was dubbed a *National Curriculum Debacle* (Ellerton & Clements, 1994). Mathematicians and mathematics educators also expressed concern over the content of the documents, though these concerns had very different bases.

Mathematics educators were concerned that "reductionist behaviourist approaches to teaching and learning mathematics…give rise to atomistic approaches to curriculum development and encourage methods of teaching and learning that fail to assist the development of a holistic view of mathematics" (Ellerton & Clements, 1994, p. 10). It was stated that a behaviourist approach was contrary to the view of leading national and international educators, who throughout the 1980s, had argued for a curriculum that promoted relational understanding (Skemp, 1976). While also being concerned about atomistic approaches to curriculum, mathematicians condemned the *Statement* and *Profile* for a lack of quality of mathematical thinking. "(I)f the documents do not faithfully reflect the history of mathematics and do not represent quality contemporary mathematical thinking, then the school mathematics programs engendered by these documents will inevitably be less than satisfactory" (Ellerton & Clements, 1994, p. 10). Mathematicians expressed concern at the omission of important topics in mathematics and at the lack of rigour expected of teachers and students in the pointers contained in the *Profile*.

In the late 1990s and early 2000s several Australian states and territories moved away from traditional subject boundaries and discipline-based curriculum documents, adopting what was purportedly a more holistic and futures-oriented approach, focusing on outcomes that equip students for living

and working in "new times" (Education Queensland, 2000, p. 6). Such calls for reform focused on ways of being in the world, blurring subject boundaries and embedding traditional discipline areas such as mathematics in "clusters of real world and futures-oriented practices, and their affiliated skills and knowledges" (Education Queensland, 2000, p. 42). These approaches to curriculum acknowledged that children in the twenty-first century bring knowledge from an increasing number of sources, including technology and the media. They claimed that students face serious issues about identity, family structures, poverty and social dislocation, and that curriculum change and behaviour management need to be understood within this context. Accordingly the *New Basics* project developed clusters of practices focusing on life pathways, multiliteracies, active citizenship and environments and technologies. The claim was that a curriculum focusing on these critical, global issues would better equip students to meet the economic and social challenges of the twenty-first century.

However, the act of embedding mathematics in global attributes, or of calling much of what has traditionally been regarded as mathematics "numeracy" risks casting mathematics as little more than practical skills for living. Within the *New Basics* technical paper (Education Queensland, 2000) the term *mathematics* is mentioned just three times, each time with reference to existing Key Learning Area curriculum documents, while *numeracy* or *numeracies* are mentioned fourteen times, each in reference to skills needed for new times. One might argue that this reflects changing priorities in education and society; however, opponents of such an approach argue that it loses the essence of what it means to be mathematical—that is, it deemphasises distinctively mathematical ways of thinking such as abstraction, generalisation and proof. Critics of holistic approaches to mathematics curriculum might well ask, "Where is the maths?"

On the other hand, in an argument ostensibly informed by mathematicians, Donnelly (2007) called for a more rigorous curriculum, arguing against constructivist approaches, against "outcomes-based and politically correct" education and against "fuzzy maths" (p. 55). One might argue that this honours the precision and rigour of mathematical knowledge; however, opponents may argue that in such an approach mathematics is cast as absolutist, formalised knowledge, with little or no account taken of its socially constructed origins. Hesitancy, uncertainty and emergence of ideas, all of which are characteristic of how mathematics develops, are lost in the push for rigour. Reformists might well ask of traditionalists, "Where is the student?"

As in the USA and the UK curriculum debates have been enacted alongside debates about pedagogy. The *Queensland School Reform Longitudinal Study* (Lingard et al., 2002) examined aspects of pedagogy such as intellectual quality, connectedness, supportive classroom environment and recognition of difference in 975 classes from years 6, 8 and 11 in Queensland, approximately one quarter of them in mathematics. The study found that coding scores for the intellectual quality dimension of pedagogy were consistently lower in mathematics and science than in other areas of the curriculum. Such research has led to calls for pedagogy that encourages more student intellectual risk-taking, and that is more connected to the world outside the classroom.

At the same time others have argued that a student-centred curriculum weakens academic standards (Donnelly, 2007), calling for a rediscovery of teacher-directed pedagogy that emphasises fundamental skills and extensive practice of basic facts and skills. In a study of the effectiveness of a direct instruction mental mathematics programme Farkota (2003) found that students who engaged in fifteen minutes per day of mental computation developed greater self-efficacy in mathematics, and that gaps in achievement were reduced. She concluded that a "competently designed, properly implemented teacher-

directed approach is ideally suited (to the acquisition of basic mathematical skills)" (p. xi).

While there is much less acrimony over the introduction of the *Australian Curriculum: Mathematics*[5] than over previous attempts to reform curriculum and pedagogy, it is not without its critics. Like similar debates in the USA and UK much of this relates to issues of implementation in particular states (Nayler, 2011) or debates about overcrowding of content (Browne, 2010). Of some 700 contributions to the Australian Association of Mathematics Teachers (AAMT) email list posted between January 2010 and February 2012, 210 made direct reference to the *Australian Curriculum: Mathematics*. Of these, 52 referred to the placement of content across year levels or the development of programmes, 35 related to errors in the glossary, 19 related to teacher support or materials such as textbooks, 16 related to the availability of textbooks, 12 related to the use of technology, while only three called for "passionate debate" about the teaching of mathematics in the era of the *Australian Curriculum*[6]. It would appear then, that the view of mathematics underpinning the goals of teaching and learning mathematics as articulated in the *Australian Curriculum* have been accepted somewhat unproblematically by much of the Australian mathematics education community.

More substantive criticism relates to the research base of the curriculum development process. Reid (2005) argues that previous attempts to introduce a national curriculum in Australia have failed because they have failed to

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[5] However deferring the implementation of the curriculum to states and territories has created a space for compromise, and we have yet to see the impact of changes to senior secondary courses.

[6] This survey was conducted using the archived AAMT-L email list postings available at http://www.lists.esa.edu.au/lists/lists/viewarchive?list=aamt-l. It is intended to be indicative of the concerns of those contributing to the email list, most of whom are teachers in schools, rather than an in-depth analysis of the content of the list postings.

establish an adequate rationale to develop a rigorous theoretical base, and to take account of what is known about curriculum change. As Siemon (2011) argues, the Australian curriculum development process made promises of having a sound research base, and of having a sound rationale built around the big ideas of mathematics, but that this has been overtaken by management concerns and tensions between teachers, mathematics educators, mathematicians, curriculum authorities, politicians and the public. She maintains that what was intended to be a forward-looking curriculum has reverted to something familiar that reproduces the fragmentation of previous documents. In comparing the research base of the current Australian curriculum development with that of the *Common Core State Standards* Watt (2009) claims that "Australia is in a weaker position with respect to developing high quality, internationally benchmarked content descriptions and elaborations, which meet all the criteria for invention, design and construction" (p. 50).

In one of the few critiques of the rationale for the *Australian Curriculum: Mathematics*, Atweh and Goos (2011) argue that the document omits an important aim for school mathematics: that of engaging with and changing the world. In their reading of the document mathematical concepts are largely decontextualised and disconnected from the students' world.

> …the decontexualised knowledge of school mathematics is not sufficient guarantee that it will contribute to development of informed citizenship. Seen from this perspective, the development of an appreciation of mathematics for its beauty and elegance, and developing mathematics that is useful for careers, jobs and further study, are seen as secondary to the development of mathematics that has the capacity to transform aspects of the life of the students, both as current and future citizens (pp. 217-218).

However, Polster and Ross (2010)—writing from the perspective of mathematicians concerned about the state of Australian mathematics education—lament the emphasis on real world examples that they read into the document, claiming that such an emphasis is boring and ignores the richness of mathematical culture. They claim that the document constitutes "unbalanced, ugly, bitsy, pseudo-applied mathematics" (para. 10), with little sense of the fun and beauty of mathematics. From a very different starting point, I have also argued (Thornton, 2011) that although the *Australian Curriculum: Mathematics* claims to promote mathematics as an enjoyable area of study, the verbs in the preamble to the document largely reflect acquisition and understanding of mathematics, rather than acknowledging that the development of mathematical knowledge is inevitably marked by hesitancy and uncertainty. I argue that for students to discover the beauty and joy of mathematics, they need to engage in thinking that resembles that of mathematicians in their endeavours to develop new knowledge.

1.3 …plus c'est la même chose

I began the chapter with a quote from Alistair McIntosh following his viewing of videos of Year 8 mathematics classrooms around the world. He continues:

> I have a feeling that if people in 100 years time view these videos, they will wonder how such rubbish was allowed to continue for so long. (Another chilling corner of my mind suspects that they are much more likely to say with relief "Ah, nothing has changed!") (McIntosh, 2003, p. 106).

It is this deep and somewhat depressing sense that "the more things change the more they stay the same", that provides the imperative for this research. It is my hope that it will enable me to contribute something towards a better understanding of the deep epistemological beliefs that inhibit or allow change. In the process I can be *with* the wider community of mathematics educators, challenging such beliefs in a spirit of mutual engagement and advocacy.

This chapter has served to provide a personal and historical motivation and context for the research in this thesis. It is borne out of a deep personal commitment to mathematics and to children's learning of significant and powerful mathematical ideas moderated, often frustratingly, by the realisation that teacher change is both deep and problematic. It is situated in a political context marked by often-heated debates about curriculum and pedagogy in mathematics and mathematics education that are both geographically and temporally pervasive. Rather than providing an extensive historical narrative of curriculum change in school mathematics, I have tried to highlight that, at the heart of the personal frustrations and political debates, lie deep questions about the nature of mathematics itself. As I have argued, ever-present differences of opinion over mathematics curriculum and pedagogy all boil down to this central issue. It raises the crucial question that forms the remainder of this thesis: What is a coherent conceptualisation of school mathematics that respects mathematics, its development as a discipline, its place in the world and how it is learned?

In Chapters 2 and 3 I introduce philosophical arguments concerning the purposes of schooling and the nature of mathematics, arguing that we need to adopt life-affirming metaphors for education and acknowledge mathematics as both a creative discipline and a coherent body of knowledge. In Chapters 4 to 6 I discuss three dimensions of connectedness that underpin the metaphor I have termed *Slow Maths*. In Chapters 7 and 8 I discuss approaches to pedagogy in school mathematics and preservice teacher education that might embody and enact this metaphor.

CHAPTER 2: CHALLENGING THE METAPHOR OF EDUCATION AS A RACE

> Often when one works at a hard question, nothing good is accomplished at the first attack. Then one takes a rest, longer or shorter, and sits down anew to the work. During the first half-hour, as before, nothing is found, and then all of a sudden the decisive idea presents itself to the mind. It might be said that the conscious work has been more fruitful because it has been interrupted and the rest has given back to the mind its force and freshness. (Poincaré, 1908/2000, p. 90)

Synopsis

Hegemon: Dominant metaphors of education narrow it to an instrumental agenda within a technological enframing. Excessive emphasis is placed on the timely achievement of predefined sequences of skills or concepts perceived to have economic importance.

Alternative: The slow movement provides an alternative metaphor that sees education as a process of becoming rather than achieving. Rather than, "How should we do this?" the primary question in slow education is, "Should we do this?"

Enactment: Slow Maths places connectedness to mathematics, person and context at the heart of mathematics education, inviting students to become mathematical through their immersion in mathematical practices that have deep philosophical and ethical dimensions.

Metaphors shape the way we think and act (Lakoff & Johnson, 1980). They are not mere words, but rather they are concepts that deeply affect how we view the world and the ways we interact with the world. They have both a reflective and generative quality in that they hide certain aspects of a concept and highlight others, in the process creating roles and realities that shape the way we act. For example, Lakoff and Johnson show how the structural metaphor

"argument is war" is embodied in language such as attacking or defending a position, the demolition or shooting down of an argument, and how the protagonists in the argument become winners and losers.

> How we think metaphorically matters. It can determine questions of war and peace, economic policy, and legal decisions, as well as the mundane choices of everyday life…Because we reason in terms of metaphor, the metaphors we use determine a great deal about how we live our lives. (Lakoff & Johnson, 1980, pp. 216-217)

In this chapter I briefly discuss some metaphors for education and how they have shaped the educational debate. I focus on what I argue has become one of the dominant metaphors for education: that of education as a race. This metaphor positions the curriculum as a one-dimensional track, assessment as the generation of a single number that is valued above all else, teachers as coaches detached from the participants, and students as runners striving to reach the only end-point that matters. I describe the beginnings of the slow movement as a protest to the one-size-fits-all approach of fast food, and challenge the metaphor of education as a race by proposing the generative metaphor of education as slow food. This has the potential to reposition curriculum, pedagogy, assessment, teachers and students as constituents and actors in a process and product steeped in history and culture, and in which diversity is an attribute to be valued rather than minimised. I apply this metaphor to the specific activity of school mathematics and suggest how *Slow Maths* might generate an approach that positions school mathematics as an activity intimately connected to life, culture and the discipline of mathematics itself.

2.1 Metaphors of education

The role of metaphor is largely unexamined in educational discourse. In the school environment metaphor is most often studied within poetry or literature studies as a language device that is largely decorative or ornamental (Postman,

2011). Rarely are metaphors studied as interactive devices, shaping the way that we see the world and how we structure reality. It is equally rare to see the role of metaphor examined in the discourse about education where its role in shaping teachers' and, more generally, society's views of the goals of education is seldom examined. Yet metaphors profoundly affect aspects of education such as policy, pedagogy, the role of the teacher and student, the nature and purpose of curriculum and the nature of the school as an educational institution (Botha, 2009).

Arguably the two dominant root metaphors of education since the advent of compulsory schooling have been the metaphors of *education as production* and *education as a cure* (Cook-Sather, 2003). To these I add the relatively recent metaphor of education as a journey, which taken to its extreme becomes *education as a race.* Each of these metaphors generates a set of associated metaphors that together ascribe particular roles to teachers and students, purposes to the curriculum and overarching goals for schooling. Despite their apparent differences, each of these root metaphors is underpinned by a common set of values and assumptions that dehumanises students and teachers, relegating them to receivers and implementers of an external agenda.

The *education as production* metaphor casts schools as factories, a conception that has dominated much educational discourse since the mid nineteenth century, and remains prevalent today (Darling-Hammond & Friedlaender, 2008). Within the "school as factory" students are conveyed from one site of production to another: in the primary school setting at the end of each year, and in the secondary school setting often at the end of each 50-minute period. In the secondary setting the school timetable is by and large sacrosanct, with the efficient operation of the school relying on strict adherence to times. Within the school as factory teachers are workers or managers, students are products, and the curriculum is the common production line along which students are progressed. Efficiency, compliance and quality control, exercised in the form

of standardised tests of achievement, are valued, while diversity, critique and initiative are marginalised. A "culture of performance and development" enshrined in documents such as professional standards for teachers (Australian Institute for Teaching and School Leadership, 2012) and an accountability agenda replete with the language of school improvement (Flint & Peim, 2012) derive from the quality control processes of a factory.

The *education as a cure* metaphor casts schools as clinics that cure not only the ills of children, but through that, the ills of society. Its initial conception was in the very first religious schools that aimed to cure the innately sinful and depraved nature of humanity (Cook-Sather, 2003), particularly among children from the lower classes of colonial society. The school as clinic focuses on identifying the individual needs of each child as she moves towards a state of health captured by an idealised image of the educated person. The curriculum becomes a prescription, differentiated for each patient on the basis of testing by the teacher, who is both diagnostician and therapist. Within the education as a cure metaphor educational research is dominated by evaluation of the effectiveness of various interventions as measured by their effect-size (Hattie, 2013), a construct borrowed primarily from the psychological literature (Huberty, 2002). Educational discourse is replete with the need for "evidence-based practice" (e.g. Department for Education and Child Development, 2013); a term that is used to validate positivist genres of educational research at the expense of holistic paradigms (Flint & Peim, 2012). In the Australian context the education as a cure metaphor is most obvious in the view of teacher education embodied in the clinical practice model at the University of Melbourne, in which prospective teachers are taught to embrace teaching as a clinical practice profession. "There is growing recognition that teachers need to be able to 'diagnose' individual student learning and provide appropriate 'prescriptions' for improvement i.e., to be clinical, evidence-based,

interventionist practitioners in the manner of health professionals" (Dinham, 2012, p. 35).

Every metaphor has the potential to highlight or obscure aspects of the concept it seeks to illuminate. The education as production metaphor highlights efficiency, while the education as a cure metaphor highlights effectiveness. Efficiency and effectiveness are worthy characteristics of a system where there is a clearly defined process and goal, but they are responsive rather than generative. The metaphors of education as production or cure offer neither space to question philosophical bases of education, nor to ask *why* or *whether* rather than *how*. Furthermore, as Cook-Sather (2003) argues, both obscure the individual subjectivity of the people that matter most in education: students. While the education as a cure metaphor appears at face value to add a human dimension to the education as production metaphor, "their underlying premises—that students are quantifiable products to be packaged or diseased beings in need of remedy—…disable and control those within their constructs" (Cook-Sather, 2003, p. 947). Hence, students are dependent on the factory worker or clinician in their journey towards a common standard or state of health, rather than creating their own destiny within a culture where diversity is valued not feared.

Yet another metaphor for education has, of course, appeared in the above sentence—that of education as a journey. Indeed, it is hard to write about education without using a journey metaphor. Students *progress* through levels of schooling where they may be assigned to different *tracks* according to whether they are *ahead of* or *behind* their peers. The National Education Agreement between the Australian Commonwealth and states and territory governments mandates that "[school] reports will give an accurate and objective assessment of the student's *progress* and include assessment of the student's achievement *relative* to the student's peer group" (Council of Australian Governments, 2009, p. 11, italics added). Indeed, the very terms

curriculum and *course* have their origins in the Latin verb *currere*[7], meaning "to run". Like the production and cure metaphors of education, the journey metaphor highlights worthwhile aspects of education, but it offers equally little space to question the value or goals of the journey itself.

Taken to its extreme the journey metaphor becomes a metaphor of *education as a race*, which I suggest has become dominant in political rhetoric in the Western world. In this metaphor what matters above all else is where a student, or even an entire education system, is placed relative to others. Being left behind is to be avoided at all costs. In the United States the Bush administration's *No Child Left Behind* legislation (Bush, 2001) was replaced by the Obama administration's *Race to the Top* agenda (Obama, 2009). Although one focuses on students and the other on the system, they are flip sides of the same root metaphor of education as a race. Similarly in Australia former Prime Minister Kevin Rudd's (2008) apology to Australia's Indigenous Peoples undertook to "halve the *widening gap* in literacy, numeracy and employment outcomes and opportunities for Indigenous Australians" (p. 3, italics added), while former Minister for Education, Employment and Workplace Relations and subsequently Prime Minister, Julia Gillard, warned that Australia was 'in danger of *losing the education race*' (Franklin, 2012, para. 1, italics added) and introduced reforms that aimed to see Australia "back in the *top five* by 2025" (Tovey & McNeilage, 2012, para. 11, italics added)[8].

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[7] Perhaps tellingly the word alphabetically preceding *curriculum* in the Oxford Dictionary of English (Stevenson, 2008) is the word *curricle*. A curricle was a horse-drawn vehicle in Edwardian England, used for example by Mr. Darcy in Jane Austen's *Pride and Prejudice* (1813/2008). Curricles were designed for competition in races, were almost exclusively owned by well-to-do males and were almost always single-person modes of transportation.

[8] Perhaps paradoxically at the time of making final editorial changes to the thesis I have received an email alert stating that Shanghai, the highest scoring province in the Programme for International Student Assessment (PISA), may no longer be concerned with being "number

Like the production and cure metaphors for education, the race metaphor effectively silences any discussion of whether or not the race is worth competing in or how winners or losers will be determined. In a race there is no time to stop and admire the scenery or to take a diversion to somewhere that might be more interesting. There is no opportunity for the runners (students) or their coaches (teachers) to question the course to be run (curriculum) or to determine an alternative destination. In a race metaphor for education, rigour and challenge are easily equated with the early introduction of demanding topics, such as suggested in the UK Government's review of curricula in high performing jurisdictions. The review of mathematics noted that "Singapore, Hong Kong, Massachusetts and Finland sequence more demanding content earlier in the domains of fractions and decimals, covering the majority of this sub-domain by the end of primary" and that in secondary algebra "Singapore is by far the most challenging…by covering significantly more demanding content at an earlier stage" (Department for Education, 2012, pp. 61-62). Equating challenge in the curriculum with the early introduction of difficult mathematical topics both misrepresents the nature of mathematical rigour, and directly contradicts convincing evidence spanning almost 100 years showing that the early introduction of formal algorithms is counterproductive to developing conceptual understanding of arithmetic operations (e.g. Benezet, 1935; McIntosh, 2002) and even to enabling procedural fluency. Yet such is the impact of adopting a metaphor of education as a race.

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one". A Chinese newspaper article reports: "it is clear that Shanghai officials have acknowledged that PISA does not give them what they want. Its narrow definition of education quality as test scores obscures other aspects of education that are much more important" (Zhao, 2014).

2.2 Heidegger and the technological enframing[9]

The factory, clinic and race metaphors for education share one common attribute: standardisation. Whether the teacher is a producer, clinician or coach, the role is one of developing a common set of standard outcomes for students. "In many Western countries that have Anglo-Saxon origins, governments and schools have rigid control structures in place and schools are driven by standardized curricula with tests and targets to ensure uniform outcomes. The emphasis is on the outcome not on the process" (Slow Movement, 2013b, para. 3). Teachers tend to see their role primarily as that of enabling as many students as possible to achieve a specified level of performance on a standardised test such as the Australian *National Assessment Program in Literacy and Numeracy* [NAPLAN] (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2011).

Standardised testing is a relatively recent phenomenon (Neyland, 2010), arising in the post-industrial neoliberal era. It both validates and is validated by metaphors of education as production, cure or race within a society that sees a well-educated[10] population as the key to the nation's economic prosperity (Office of the Chief Scientist, 2013). Although political strategies for realising accountability and efficiency may differ according to political taste, the need for standardisation remains the same and overshadows questions of purpose.

While one might wish that political debate and educational debate be idealistic or visionary, the reality is that they form part of the wider fabric of societal expectations that have arisen in a time of economic rationalism. They are

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[9] I use Heidegger's ideas selectively rather than subscribing to his entire metaphysical agenda. The technological enframing is useful in highlighting the dehumanising impact of metaphors for education such as production, cure or race.

[10] I use the term "educated" advisedly to reflect a vague notion of the aim and result of universal schooling. Of course it has a multitude of possible meanings and much associated baggage, but for the purposes of this discussion I use it in a common rather than critical sense.

deeply rooted in a technological approach to life that the German philosopher Martin Heidegger (1977) terms *Gestell*, or *the enframing*. It is important to note that the use of the term *technological* in this context is not associated with technology per se, nor indeed does its manifestation in society rely on or imply a particular form of technology. Rather it is used to indicate a means-end rationality that underpins much of how Western society has come to see the world. Heidegger's argument is that our whole way of thinking has become technological, colonising the world with an associated focus on issues such as accountability, standards, improvement and managerialism.

> Technology treats everything with "objectivity". The modern technologist is regularly expected, and expects himself, to be able to impose order on all data, to "process" every sort of entity, nonhuman and human alike, and to devise solutions for every kind of problem. He is forever getting things under control. (Lovitt, in Heidegger, 1977, p. xxvii)

Heidegger introduces two important terms in his discussion of the enframing. One is *Dasein*, which has no direct translation into English. Roughly speaking Dasein is "an openness for Being", or "the Being for whom being is the question". I interpret it as simultaneously singular, plural and conceptual, representing a particular individual reciprocally related to society as a whole, immersed in the question of being itself. This term is significant in an educational sense since current educational policy is focused very much on one aspect of Dasein: Dasein as individual[11]. Curriculum is "differentiated" to meet "individual needs" through "personalised learning plans". I am not arguing that we should not encounter each student as an individual, but rather that we should also encounter them as Dasein in its totality; that is, as an individual

[11] I discuss the extent to which current educational policy is located within the technological enframing in Thornton (in press), in which I analyse the text of a current system numeracy strategy document (Department for Education and Child Development, 2013).

acting in society in the act of being. The broad goal of education is then one of becoming, rather than one of achieving. In the terms I used in Chapter 1, it is about developing a relationship with knowledge, or as Biesta (2009a) terms it coming into the world. In the current educational climate Heidegger may well claim that the focus on capacities and achievements that are located within the technological enframing dehumanise the individual and limit the capacity of Dasein to fully express herself in relation to the world.

The second important term Heidegger introduces is *Bestand*, or loosely translated "standing reserve". As standing reserve individuals are relegated to a potential pool of energy to be called on as and when needed.

> …things are not even regarded as objects, because their only important quality has become their readiness for use. Today all things are being swept together into a vast network in which their only meaning lies in their being available to serve some end that will itself also be directed toward getting everything under control. (Lovitt, in Heidegger, 1977, p. xxix)

The purpose of education, particularly in science and mathematics, then becomes one of the nation becoming more economically competitive, despite the evidence for such a link being somewhat contested, or at the very least much more complex than assumed in the rhetoric (West, 2011). Within the standing reserve the results of individuals are aggregated to give a picture of the performance of the school, system or nation, and policies are put in place to attempt to improve performance. Students are thus rendered as part of Bestand, important primarily because of their potential in the immediate goal of improving performance and the longer-term goal of enhancing economic competitiveness. Again, the effect is to emphasise the measurability of achievement rather than valuing education as an act of coming into the world.

The enframing identified by Heidegger is so pervasive that it has become largely invisible as a taken-for-granted way of seeing the world. Yet this

invisibility is what gives it its power in shaping the world, resulting in humanity becoming increasingly estranged from itself. "But we are delivered over to it in the worst possible way when we regard it as something neutral; for this conception of it, to which today we particularly like to do homage, makes us utterly blind to the essence of technology" (Heidegger, 1977, p. 4). The enframing manifests itself in virtually all areas of schooling, from the governance of childhood through the school improvement agenda, to teacher education and educational research (Flint & Peim, 2012). It is particularly evident in regimes of standardised testing, where the backwash effect on school curriculum and classroom activity is profound. As much as 44% of the total time spent in class by English students during April and May is taken up with preparation for the national test (Mansell, 2007), while submissions to the recent Senate Enquiry into the effectiveness of NAPLAN suggest that similar effects are being felt in Australia (Senate Standing Committee on Education Employment and Workplace Relations, 2013).

Hence in the current educational climate the question asked by the teacher is how do I best prepare/cure/coach students to achieve the best outcomes? Biesta (2009a, p. 15) terms this preoccupation a *language of learnification*. He argues that broader educational purposes, such as becoming an active, critical subject in the world, have been displaced by an emphasis on learning a set of predefined skills for the purposes of achieving better outcomes. Yet, as Nathan (2009) points out, "[t]he hardest questions aren't on the test" (title). In an era when both sides of politics are preoccupied with "winning" the education race, I argue that the most critical challenge is to ask these "hardest questions". Instead of asking "How should we learn this?" the question becomes "Why should we learn this?" or even "Should we learn this?" Just as Heidegger's essential concern was the question of being, so it is precisely this question of purpose that I address in this thesis.

With Biesta (2009a) then, I maintain that neither subject-centred education nor child-centred education move beyond questions of how best to help children learn predetermined sets of skills or concepts or learn about themselves and their feelings. The language of learnification fails to address more fundamental questions of purpose, and fails to enable students to come into the world as active, critical subjects. Like Biesta (2012) I suggest that education needs to create a space of both connection and resistance—connection between the child and the world and the resistance that is encountered when one tries to engage deeply with questions that matter. Living with the inevitable frustrations of a complex and uncertain middle ground takes time and requires a different view of the purpose of schools.

> It is through engagement with the experience of resistance that our worldly existence in the world and thus the existence of the world itself become possible…school thus needs to give resistance a place, which also means that it needs to resist the all too simple demands for personalisation, flexibility and a customer orientation if such demands are aimed at taking the essential difficulty out of the educational process (p. 101).

I argue, therefore, that we need alternative metaphors for education that reaffirm students as active participants not only in the process of learning, but also in the process of deciding what to learn. These are "life affirming metaphors that cast [teachers] but also, and more importantly, students as active creators not only of their education but also…of themselves" (Cook-Sather, 2003, p. 952). One such metaphor that directly challenges the metaphor of education as a race is that of education as *slow food* preparation. Slowness provides a space for teachers and students alike to ask the "hardest questions". It addresses not only technical questions of efficiency and effectiveness, but also more fundamental questions of education as coming into the world.

2.3 Slowness

In 1986 a McDonald's restaurant was opened at the Piazza di Spagna in Rome. Journalist Carlo Petrini wondered why, if there was fast food, there could not also be slow food, and organised a demonstration in which he and his followers brandished bowls of penne as weapons of protest (Honoré, 2004). This was the start of an international *Slow Food Movement*, which has since spawned offshoots such as slow travel, slow living, slow cities, slow books, and slow parenting.

Despite its name, the Slow Movement is not, first and foremost, a movement against speed itself. Rather, it is a philosophy that rejects the one-size-fits-all approach to life that emphasises uniformity, predictability and measurability. In essence it rejects the technological enframing. A core philosophy of this one-size-fits-all approach is to minimise product variability. Rather than expecting people who operate their business to reinvent the wheel, the McDonald's company simply expects them to "make it turn faster" (n.d., p. 3). To maintain quality and uniformity, each restaurant in the McDonald's fast food chain must follow set formulas and specifications for menu items, set methods of operation, inventory control, bookkeeping, accounting and marketing, and set concepts for restaurant design and layout. These philosophies typify *fast*. "Fast is busy, controlling, aggressive, hurried, analytical, stressed, superficial, impatient, active, quantity-over-quality" (Honoré, 2004, p. 14).

On the other hand Peter Gilmore, chef at Quay, a well-known and highly regarded Sydney restaurant, states:

> Over the years my food philosophy has evolved into a personal style that celebrates being a cook in Australia. It embraces nature's diversity and seeks to achieve a sense of balance and purity through produce, technique, texture, flavour and composition. (Quay Restaruant, n.d., para. 10)

Although not advertising itself as a slow restaurant, Quay epitomises a *slow* philosophy. "Slow is ... calm, careful, receptive, still, intuitive, unhurried, patient, reflective, quality-over-quantity. It is about making real and meaningful connections—with people, culture, work, food, everything" (Honoré, 2004, p. 14).

To elaborate further upon the concept of slowness it is helpful to look at some key principles that underpin the slow food movement.

> First, it expresses a definite *philosophical position*—that life is about more than rushed meals. Second, it draws upon *tradition and character*—eating well means respecting culinary knowledge and recognizing that eating is a social activity that brings its own benefits. A respect for tradition also *honors complexity*—most sauces have familiar ingredients, but how they are combined and cooked vitally influences the result. And third, slow food is about *moral choices*—it is better to have laws that allow rare varieties of cheese to be produced, it is better to take time to judge, to digest, and to reflect upon the nature of "quiet material pleasure" and how everyone can pursue it. (M. Holt, 2002, p. 267, emphases in original)

I would add to these two further principles: that uncertainty is inherent in the process of creation, and that variability is a quality to be treasured rather than feared.

One could substitute the word "education" or "mathematics" for each of the words pertaining to food in the above description, and the paragraph would make almost perfect sense. As this thesis sets out, *Slow Maths* has a clearly articulated philosophical basis; it values culture and tradition; it blends established techniques and fresh ideas in an environment where uncertainty is encouraged and complexity valued; it values variability rather than uniformity; and it has an ethical dimension with which to judge what is good and worthwhile. It is first and foremost about connections—to people, to culture, to

history, to knowledge, to learning, to the world, and to self (Slow Movement, 2013a).

There are isolated examples of what might be termed slow schools (Biesta, 2012; M. Holt, 2002). These include schools such as those based on a Steiner philosophy (e.g. Gidley, 1998), the Boston Academy of the Arts (Nathan, 2009) and Big Picture Schools (Washor & Mojkowski, 2006). Despite having very different origins and underlying beliefs, each of these bases its education on a well-articulated philosophy that is about much more than covering curriculum content. They share a common commitment to working with the interests and motivations of students, using extended periods of teaching, learning and apprenticeship to examine deeply issues that matter in students' lives. They emphasise connectedness between the child and the world.

However, the approach to mathematics, even in these schools, continues to be driven by the goal of requiring students to complete content (J. Hogan, personal communication, 2012) in order to run the curriculum race. They are hesitant to take alternative approaches to mathematics lest they fail to enable students to meet all of the 278 content descriptions in the *Australian Curriculum: Mathematics* or similar. It is little wonder that, as discussed in Chapter 1, teachers' comments in online discussions relating to the *Australian Curriculum: Mathematics* focus almost exclusively on issues related to the placement of content rather than passionately debating the purpose or teaching of mathematics in the era of the ACM (Atweh, Miller, & Thornton, 2012). Clearly there is a strong argument to slow down by dramatically reducing the number of content descriptions in the school mathematics curriculum.

But *Slow Maths* is not simply about taking longer to learn the same set of skills and concepts for the same purposes, even though there is ample evidence that students do learn traditional content better by learning it more slowly and deeply (e.g. Boaler, 1997; Nathan, 2009; Riordan & Noyce, 2001). Rather, it is

about a fundamentally different approach to, and mindset about, school mathematics—one that emphasises connections between the student, the mathematics and the world. It does, of course, involve learning skills and concepts as in traditional approaches; it does engage students in the process of constructing mathematical knowledge as in progressive approaches; but it goes further. It creates a space which foregrounds mathematics as a way of being and acting in the world.

For example, rather than seeing the Golden Ratio as an example and application of the concept of ratio, ratio might be taught because it is needed to understand and investigate the Fibonacci sequence and the Golden Ratio[12]. It might also be taught because it is needed to understand issues of social importance, such as relative wealth or the social implications of the body image conveyed by Barbie dolls (Mukhopadhyay, 2005). Such examples are not new. However they typically form an add-on to the curriculum, to be explored after the acquisition of skills and concepts. The premise of *Slow Maths* is that these and similar examples should form the core of the curriculum, and that the teaching of skills and concepts should be immersed in the culture, traditions and contexts of mathematics.

In a *Slow Maths* curriculum, then, what matters most is not the description of a set of skills and concepts that might be able to be applied to solve problems, but rather the articulation of a bank of mathematically, culturally and contextually connected situations through which skills and concepts are developed. Such a bank might come from several sources, described in more detail in Chapter 7:

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[12] The lack of connection to history or culture in the *Australian Curriculum: Mathematics* is apparent in that there is no mention of either Fibonacci or the Golden Ratio, nor Pascal's triangle, anywhere in the document, and the only mathematician named is Pythagoras.

- Sources in contemporary mathematics such as the "living and connected view of mathematics" described in the *Klein Project* (Barton, 2008, p. 16), or applications of mathematics in computer science transformed into classroom use described in *Computer Science Unplugged* (T. Bell, Alexander, Freeman, & Grimley, 2009);
- Sources in the history of mathematics such as those described in the 10th *ICMI Study in Mathematics Education* (Fauvel & van Maanen, 2000) that have the potential to humanise mathematics, make it more interesting, understandable, and approachable, and that give insight into concepts, problems, and problem-solving (Fried, 2001; Povey, 2013); or
- Sources from the world that require mathematics to be informed and act critically such as those described in the critical mathematics literature (Gutstein, 2003) and discussions of contemporary applications of mathematics in society (Garfunkel & Malkevitch, 1994).

Enacting a curriculum such as this clearly requires time and serious engagement with deep mathematics. Yet the slow, creative problem-solving process of mathematicians is hardly ever reflected in school mathematics classrooms. In his seminal reflective paper written just after the turn of the 20th century Poincaré (1908/2000) describes how much of the work of creative problem-solving is unconscious, taking place over time. Studies by mathematicians highlight the need for periods of thoughtful orientation and exploration, relaxation to allow the creative mind to work, and rigorous reflection and verification of the solution (Hadamard, 1945; Pólya, 1945). This process is encapsulated in the *Mathematics Challenge for Young Australians* outlined in the Narrative of Slowness introducing Part 1 of this thesis. The message is clear: it takes time to think creatively and solve higher order problems. This time is rarely provided in school classrooms, even in environments advocated in current research.

A pedagogic approach informed by a metaphor of *slowness* is also open to the diversions and uncertainties encountered whenever students engage in serious study of significant mathematical problems. As Neyland (2010) says:

> Creative problem-solving uses the *slow* mode of the undermind. It is less purposeful and less clear-cut. It thrives on curiosity, playfulness, a love of experimentation, and a willingness to take imaginative risks. It is open to metaphorical thinking and aesthetic features, and it is characterized by mental flexibility. Importantly, if divergent thinking of this sort is required, it is necessary to allow the undermind to work. This requires keeping our self-conscious, anxious, deliberative selves in check while the playful undermind allows ideas to ferment (p. 182, italics added).

This creative problem-solving is marginalised in a school environment located within the technological enframing. As Neyland (2010) suggests too much emphasis on target-minded, deliberative and instrumental thinking leaves little room for imaginative and creative ways of thinking. He maintains that the careful planning and performance targets of the closely managed school environment leave little or no room for the uncertainties of the creative imagination, that the sense of the aesthetic cannot be developed in an environment that is aligned solely to the achievement of specific learning outcomes, and that a school culture that accounts for every moment of a child's time in school, demanding that more must be done at an earlier age, leaves little time for creative thinking. On the other hand the principles of slowness, which include taking time to think creatively, valuing tradition and culture, embracing variability, and including an aesthetic dimension that allows one to make judgements about what counts, allow creative problem-solving as practised by mathematicians throughout the centuries to thrive.

2.4 Conclusion

I have argued that dominant metaphors of education as production, cure or race position students as products rather than participants in the educational

endeavour. They provide no space to ask hard questions about what really matters in education. Students, except for the rare few who see through school mathematics and catch a glimpse of what mathematicians actually do, fail to engage in the resistance and connections that are a necessary part of coming into and acting in the world. In order to challenge those metaphors I have built on the slow food movement to introduce the concept of *Slow Maths*. *Slow Maths* has a clearly articulated philosophical basis; it values culture and tradition; it blends established techniques and fresh ideas in an environment where uncertainty is encouraged; it values variability rather than uniformity; and it has an ethical dimension with which to judge what is good and worthwhile.

I began the chapter with a quote from Henri Poincaré's *Mathematical Creation*, in which he discusses his own mathematical thought processes. He describes in detail the process of incubation, central to a *slow* approach to education, and its relation to the aesthetic.

> Now we have seen that mathematical work is not simply mechanical, that it could not be done by a machine, however perfect. It is not merely a question of applying rules, of making the most combinations possible according to certain fixed laws. The combinations so obtained would be exceedingly numerous, useless and cumbersome. The true work of the inventor consists in choosing among these combinations so as to eliminate the useless ones or rather to avoid the trouble of making them, and the rules which must guide this choice are extremely fine and delicate. It is almost impossible to state them precisely; they are felt rather than formulated…
>
> Among the great numbers of combinations blindly formed by the subliminal self, almost all are without interest and without utility; but just for that reason they are also without effect upon the esthetic sensibility. Consciousness will never know them; only certain ones are harmonious, and, consequently, at once useful and beautiful. They will be capable of

> touching this special sensibility of the geometer of which I have just spoken, and which, once aroused, will call our attention to them, and thus give them occasion to become conscious. (Poincaré, 1908/2000, pp. 91-92)

This thesis asserts that mathematical, cultural and contextual connectedness is at the heart of mathematics. These three dimensions are embedded in the discipline of mathematics itself; they are central to the work of mathematicians; and they are evident in the classroom situations advocated by mathematics education researchers. Together they provide a credible conceptualisation of school mathematics that is simultaneously dynamic in process and connected in product. I argue that rather than propelling students along a narrow track of predetermined outcomes, school mathematics education needs to build on such a conceptualisation by valuing the traditions and culture of the discipline of mathematics and by providing students with the knowledge and experiences that enable them to judge the usefulness and beauty of mathematical ideas in an environment where uncertainty is encouraged and where variability is valued rather than feared. In Chapter 3 I suggest that such a conceptualisation also has the potential to bring together the absolutist and relativist philosophies that lie at the heart of the *Math Wars* discussed in Chapter 1, and hence generate greater levels of consensus within the mathematics and mathematics education communities.

CHAPTER 3: COMPETING PHILOSOPHIES OF MATHEMATICS—INSIDE AND OUTSIDE PERSPECTIVES

> Those who belong to OWWAAB[13] emphasize four adjectives to designate some of the most important qualities of the entity they invoke: OWWAAB, for them, is outside, unified, inanimate and its workings are undisputable. But if we dig a little further, we fall upon a strange and apparently contradictory attribute: OWWAAB is simultaneously out and beyond, yes, but also inside tiny networks of practice that seem necessary to access it and that are called 'scientific disciplines.' Every time we designate a feature of the 'natural world' that has some of the properties of OWWAAB, we are also asked to follow the path of a knowledge producing procedure. Our sight goes simultaneously far away and close at hand focusing on two opposite places at once. As if there was a tension between the exteriority and the interiority of this entity: as a set of results, OWWAAB is outside, "untouched by human hands"; as a process of production, the same OWWAAB resides inside conduits where many human hands with the help of much paraphernalia are busy making it an outside reality. (Latour, 2013, p. 15)

Synopsis

Review: Neither absolutist nor relativist philosophies of mathematics education have delivered on promises of higher standards or engagement in school mathematics. The failure of reform and non-reform movements in mathematics education is due as much to philosophical issues as it is to questions of implementation.

Critique: Relativist philosophies of mathematics education such as Ernest's social constructivism confound ontology, epistemology and pedagogy and fail to honour the distinctive nature of mathematical

[13] OWWAAB is Bruno Latour's term *Out Of Which We Are All Born*, which I explain at the end of the chapter. Quotations from Latour's unpublished work are used with permission.

knowledge. While absolutist philosophies value mathematical certainty, they ignore questions of epistemology and pedagogy.

Assertion: Relativist and absolutist philosophies of mathematics, rather than being binary opposites, represent differences in representational grainsize, arising from viewing mathematics from the inside and outside, or near at hand and far away, respectively. Kitcher's mathematical naturalism provides a coherent account of both how mathematics develops and the nature of knowledge in mathematics.

Proposition: Slow Maths, as an enactment of mathematical naturalism, incorporates ontology in that it takes account of the objects and structure of mathematics, epistemology in that it takes account of the nature of mathematical activity, and pedagogy in that it focuses on knowledge and learning. It is simultaneously near and far away, intimately connecting mathematics, culture and context.

More than 20 years ago Ernest (1991) argued that philosophies of mathematics education profoundly influence the way curriculum is constructed and enacted. He claimed that the hierarchical content-driven view of mathematics prevailing in schools was both a product and a reification of absolutist philosophies of mathematics, whereas more progressive, learner-centred approaches built on and suggested relativist philosophies. Others have critiqued Ernest's social constructivist agenda, arguing that it fails to value the objectivist nature of mathematics as a discipline (Rowlands, Graham, & Berry, 2001). As discussed in Chapter 1 these two seemingly dichotomous views on the nature of mathematics have been, if not at the heart of the so-called *Math Wars* (D. Klein, 2007), at least significant contributing factors. Teachers, myself included, are caught between the two (White-Fredette, 2009); on the one hand promoting mathematics as a coherent, connected whole where results are

certain, and on the other promoting it as a human activity with its accompanying hesitancy and openness.

In this Chapter I suggest that within the context of formal education neither absolutist nor relativist philosophies of mathematics alone can account for the nature of mathematical discovery and the connectedness of mathematical knowledge. I commence with a brief review of absolutist and relativist philosophies of mathematics, and discuss the limitations that each viewpoint imposes on curriculum and pedagogy. I recap arguments from Chapter 1 and argue that the apparent agreement reached with the publication of documents such as the *Common Core State Standards* (Common Core State Standards Initiative, 2010) or the *Australian Curriculum: Mathematics* (Australian Curriculum and Assessment Reporting Authority [ACARA], 2013) may, in fact, be an "uneasy peace".

I critique both absolutist and relativist philosophies, showing that neither on its own can faithfully reflect both the distinctive nature of mathematical truth and the human and cultural aspects of the development of mathematical knowledge. I suggest that absolutist and relativist philosophies, rather than representing conflicting views of mathematics, may be more productively seen as differences in representational grainsize arising from external and internal views of mathematics respectively. I conclude that both are necessary if we are to fully appreciate the human and disciplinary dimensions of mathematics. I outline the elements of Kitcher's (1988) mathematical naturalism, and argue that it provides an account of both the development of mathematics as a human endeavour and the distinctive nature of knowledge in mathematics.

I then revisit *Slow Maths* as an enactment of mathematical naturalism and argue that Kitcher's ideas provide a viable philosophy for school mathematics. Above all else, slowness is about making real and meaningful connections between knowledge, humanity and the world. It therefore captures both inside

and outside perspectives on mathematics. I describe three aspects of connectedness that characterise *Slow Maths*: mathematical, cultural and contextual connectedness.

3.1 Philosophical perspectives on mathematics and mathematics education

3.1.1 Absolutist philosophies of mathematics

While it is beyond the scope of this thesis to give a detailed account of the crisis in the foundations of mathematics at the beginning of the twentieth century (Quinn, 2012) and the resulting paradigm shifts in the philosophy of mathematics, it is important to discuss how philosophies of mathematics have impacted on school mathematics education. Davis and Hersh (1981) suggest that "the typical working mathematician is a Platonist on weekdays and a formalist on Sundays" (p. 359). That is, while engaged in the activity of doing mathematics, most mathematicians do not question whether or not the objects with which they are working have an independent existence or model reality. There is an assumption that the work on which they are engaged is, albeit in many cases somewhat removed, about discovering the properties of some external reality. This is even truer of the school situation where elementary mathematical ideas are explained through models or diagrams that are familiar in students' everyday lives.

However, when asked to give a philosophical justification for their work, mathematicians may be confronted with the inconsistencies in the Platonist position and are likely to resort to formal justifications to try to explain the foundations of mathematics. They may explain their work as a game of symbol manipulation that does not necessarily relate to any reality beyond the realm of mathematics. The *New Math* experiment of the 1960s, in which set theory and logic replaced much of the traditional arithmetic of the primary school, was built on a belief that logical foundations could be the basis of all learning in

mathematics. Despite the apparent failure of the *New Math* experiment (Ernest, 1985) a formalist approach is still advocated by Quinn (2012) who argues that school education has failed to take into account the formalist philosophical changes in academic mathematics in the early twentieth century. He suggests, for example, that learning fractions through models and physical experiences is ultimately dysfunctional and hence counterproductive, recommending instead that fractions be introduced with "a precise, formal definition that looks obscure at first but that can be internalized by working with it" (p. 37), leading to far more effective learning later.

The formalist approach to curriculum and pedagogy is, perhaps, most succinctly summarised by Wu (2011), who enumerates five fundamental principles of mathematics that, in his opinion, should underpin all instruction in mathematics:

- Every concept is precisely defined, and definitions furnish the basis for logical deductions;
- Mathematical statements are precise. At any moment, it is clear what is known and what is not known;
- Every assertion can be backed by logical reasoning;
- Mathematics is coherent; it is a tapestry in which all the concepts and skills are logically interwoven to form a single piece;
- Mathematics is goal oriented, and every concept or skill has a purpose (pp. 379-380).

On the other hand, Ernest (1985) argues strongly that "what is a virtue in foundational studies, namely explicitness in the treatment of sets and laws, is a vice in teaching" (p. 605). He highlights three "excesses" in mathematics education that he attributes to absolutist philosophies of mathematics. At the primary school level he points to set theory, which was encapsulated in the *New Math* movement of the 1960s. He suggests that in high school an undue

emphasis on Euclidean geometry can be traced directly to logicist philosophies of mathematics, and at tertiary level he identifies the cyclic form of "definition, lemma, theorem, corollary" that prevails in undergraduate mathematics courses. Ernest argues that each of these emphases at various levels of schooling devalues sense making in mathematics, in favour of a form of argument that is often counterintuitive and certainly alien to most students. Furthermore they fail to give a sense of how mathematics is "created, grasped or used" (Davis, 2013, p. 18).

While most contemporary approaches to school mathematics are more moderate in their adherence to mathematical formality, mathematical text written for school students continues to be structured so that definitions and worked examples are the beginning of instruction. Often these supposedly fundamental skills, concepts and procedures are highlighted in shaded boxes or by cartoon characters emphasising their importance (e.g. Haese, Haese, & Humphries, 2013). The consequence is that mastery of fundamental skills is seen by teachers and students alike as a necessary prerequisite to application to more complex problems or investigation of further mathematical ideas. This emphasis was evident in a study of school mathematics texts written for year 8 students in Australia (Vincent and Stacey, 2008), in which the majority of problems presented to students were of low complexity and emphasised the use of procedures rather than more complex reasoning. Many were repetitive; few required applications of knowledge; and almost none emphasised proof or logical deduction. As I have argued in Chapter 2, it is not that fundamental skills are unimportant, but the act of highlighting them and requiring them to be learned prior to problem-solving provides them status as the essence of mathematics. In the process it relegates applications and further investigation of mathematical ideas to being add-ons, only for a select few students.

Thus one can argue that, regardless of the particular form they take, absolutist philosophies of mathematics fail as a basis for school mathematics. Formalist

philosophies fail, as Ernest (1991) points out, because they contain a form of reasoning that is obscure and alien to students. Platonist philosophies fail as they emphasise definitions and skills rather than more complex reasoning and problem-solving. The prevailing beliefs produced among students are that all mathematics problems have a correct answer; that all have a correct method that needs to be explained step-by-step; and that all questions can be answered in a short space of time (Schoenfeld, 1992). Ernest (1991) therefore claims that only relativist philosophies of mathematics can adequately account for the genesis of mathematical knowledge through social processes of linguistic interactions and negotiation of objectivity, and introduces his social constructivist philosophy of mathematics education.

3.1.2 Relativist philosophies of mathematics

Invoking results such as the development of non-Euclidean geometries through the questioning of Euclid's fifth axiom or Godel's incompleteness theorem (Nagel & Newman, 2001), relativists argue that the Platonists' reliance on what they perceive as physical reality and the formalists' reliance on absolute certainty in the foundations of mathematics is ultimately misdirected.

> I think the Platonist philosophy of mathematics that is currently claimed to justify set theory and mathematics more generally is thoroughly unsatisfactory, and that some other philosophy grounded in inter-subjective human conception will have to be sought to explain the apparent objectivity of mathematics. (Feferman cited in Davis, 2013, p. 19)

Rather than being prescriptive of a set of axioms, rules and concepts that constitute mathematics, relativist philosophies of mathematics are descriptive of mathematics as a human activity, subject to negotiation and renegotiation, and inseparable from its history, philosophy and sociology. Such a view derives in part from the work of Lakatos' influential *Proof and Refutations*

(1976), in which he described the fallible and changing nature of the mathematical knowledge developing through hypothetical interactions between a teacher and a group of (extremely perceptive) students. Influenced by Popper's (1962) writing on the philosophy of science, Lakatos argued that mathematics is fundamentally a product of human activity, characterised by intuition, argumentation and reformulation.

Hersh (1997) articulates four fundamental principles on which a relativist philosophy of mathematics rests:

1. Mathematics is human. It's part of and fits into human culture.
2. Mathematical knowledge isn't infallible. Like science, mathematics can advance by making mistakes, correcting and recorrecting them.
3. There are different versions of proof and rigour, depending on time, place and other things.
4. Mathematical objects are a distinct variety of social-historical objects. They're a special part of culture (p. 41)

Taken to their extreme, relativist philosophies of mathematics assert that mathematical truth is therefore "fallible and corrigible, and can never be regarded as beyond revision and correction" (Ernest, 1991, p. 18). For the extreme relativist it is then a short step to suggest that the discipline of mathematics as a whole is fallible.

However neither the development of non-Euclidean geometry nor the recognition of the incompleteness of any system of mathematical results leads to the rejection of the certainty of the logic of mathematics. Rather, they actually *arise* from an acceptance of mathematical argument as a basis for determining mathematical truth. In each of these situations as in many others, the certainty of mathematical argument has enabled the development of new knowledge both within and beyond the system of mathematics itself. The mistake in invoking such results in defence of a relativist philosophy is "the

failure to distinguish between establishing *certainty within* a system with establishing the *certainty of* a system (Rowlands et al., 2001, p. 227, italics in original).

Within the field of mathematics education the most prominent advocate of a relativist philosophy of mathematics is Ernest (1991, 1998). Embracing a radical constructivist view of mathematics (Von Glasersfeld, 1990) that denies the existence of an objective, ontological reality, he describes his social constructivist philosophy of mathematics education in which:

- The personal theories which result from the organization of the experiential world must 'fit' the constraints imposed by physical and social reality;
- They achieve this by a cycle of theory-prediction-test-failure-accommodation-new theory;
- This gives rise to socially agreed theories of the world and social patterns and rules of language use;
- Mathematics is the theory of form and structure that arises within language. (Ernest, 1998)

Ernest claims that he can then account for both the apparent certainty and objectivity of mathematical knowledge and the "unreasonable effectiveness of mathematics in the natural sciences" (Wigner, 1960). He draws heavily on Wittgenstein's (1978) notion of mathematics as a language game with socially agreed rules that refines and extends written language. Students are active participants in the knowledge production process within this language game through scaffolded discussion, argumentation, and refutations in a social context.

Given the widespread influence of Ernest's (1991, 1998) social constructivist philosophy of mathematics education and its unproblematic adoption in much of the mathematics education community, including recent "research-based"

reports into effective mathematics teaching in Australia (Sullivan, 2011) it is important to outline the basis of this philosophy and to examine more closely its fundamental tenets.

(1) An individual possesses subjective knowledge of mathematics. (Ernest, 1991, p. 43)

Ernest argues that the mathematical thought of an individual results in "unique subjective representations of mathematical knowledge". However, is knowledge or its representation "unique"? While I agree that our understanding is, more or less, unique, this is not true of knowledge, or more particularly propositional knowledge in mathematics. For example, we may have different depths of understanding of the concept of an angle, the definition of a degree, the concept of a triangle, the number 180, the concept of sum, the notion of equality, the idea of a theorem, or the laws of deduction and logic that are used in proof, but there *is* a common piece of "knowledge"—the sum of the angles in a triangle is 180 degrees.

Some may understand this very deeply—they understand angles as amounts of turn and see them statically and dynamically; they understand that a degree is a historically-produced convention defined in terms of a revolution, they understand 180 as half of 360 and hence the angle measure of a straight line; or they realise that a theorem is a result that can be deduced logically from other axioms and theorems and hence understand that the result is restricted to triangles in Euclidean geometry. For others this piece of knowledge may be poorly understood as their understanding of the concept of angle may be very limited, or they may not be able to make the logical deductions necessary to justify the result, as their reasoning is immature. However, I argue that this does *not* mean that the knowledge is different or subjective.

Given appropriate scaffolding we can, as Ernest suggests, use this knowledge to "construct [our] own, unique mathematical production, the creation of new

subjective mathematical knowledge" (p. 43). However we can also use it to make logical deductions that are correct beyond the individual. Even if our understanding of the subtleties of geometric argument is limited, we can still deduce that if two angles of a triangle are 30 and 60 degrees, then the third must be 90 degrees. I argue that this is *not* subjective knowledge—our understanding of what we are doing might be very different to that of others, but the knowledge itself is the same, and the deduction is correct.

(2) Publication is necessary (but not sufficient) for subjective knowledge to become objective mathematical knowledge. (Ernest, 1991, p. 43)

Contrary to Ernest's assertion, I have deduced many mathematical results that I know are true, purely by virtue of the internal logic of mathematics. One of these is the result relating to RIFTWIB numbers outlined in the Narrative of Slowness introducing Part 1 of the thesis. Using an argument based on the sum of consecutive integers being equal to the product of the median integer and the number of integers in the sum I was able to deduce that only numbers of the form 2^n, $2^n(2^{n+1} - 1)$ where $2^{n+1} - 1$ is prime, or $2^n(2^{n+1} + 1)$ where $2^{n+1} + 1$ is prime cannot be written as RIFTWIB numbers. When I first deduced this result I was excited to be able to make connections with other areas of mathematics.

At the time of deducing it I was unaware of whether or not it had ever been published[14]. It is elementary mathematics, so I did not deem it of sufficient interest to publish in a mathematics journal. However the fact that at the time of posing the problem, this result had not to my knowledge been published did not prevent it from being objective knowledge. I knew that the result was true, and hence objective, by virtue of the laws of logic that underpin mathematics, and I knew that anyone with a rudimentary knowledge of number theory could

[14] It was, in fact, published by Jones and Lord (1999), however this was not until some twelve months after we had set the problem as part of the *Mathematics Challenge for Young Australians*.

have verified it. However, the correctness of the result is independent of anyone else accepting it via review. As discussed later, in Maton's (2000) terms mathematics has a knowledge mode of legitimation.

(3) Through Lakatos' heuristic published knowledge becomes objective knowledge of mathematics. (Ernest, 1991, p. 44)

As Ernest says published mathematics is subject to scrutiny and criticism by others, which may result in its reformulation and acceptance. His claim is that knowledge always remains open to challenge, however the distinctive feature of mathematical proof is that, as long as one accepts the axioms and constraints of the system in which it is located, it is not open to challenge. Rather it remains open to clarification and elaboration. To use the example of RIFTWIB numbers I argue that, regardless of whether or not the result I deduced had been published elsewhere, the result would not have been open to challenge. My proof, if it were to be published, may be subject to criticism as being not well expressed or containing gaps, and it is certainly not an enumeration of the complete set of non-RIFTWIB numbers as there is no known complete enumeration of Mersenne or Fermat primes. Thus it is subject to clarification and refinement as more Mersenne or Fermat primes, if they exist, are found. However it *is* a complete algebraic formulation that can be justified using the logic of mathematics, and as such is not open to challenge.

(4) This heuristic depends on objective criteria. (Ernest, 1991, p. 44)

As Rowlands et al. (2001) point out there is contained within this statement an inherent contradiction that mathematics cannot be separated from its social origins, yet at the same time that it has a logical necessity that is independent of its origin. It is curious that those aspects of mathematics that are most clearly socially constructed and Western in origin—the criteria and rules of logic underpinning mathematical argument—are called "objective" by Ernest.

(5) The objective criteria for criticising published mathematical knowledge are based on objective knowledge of language, as well as mathematics. (Ernest, 1991, p. 44)

While, as I discuss in Chapter 5, language is undoubtedly a key aspect of learning and communicating in mathematics, there is a wealth of evidence (e.g. Hadamard, 1945) that many mathematicians develop and become convinced of mathematical truth through visual images that may or may not include language. For example, the book *Proofs Without Words* (Nelson, 1993) contains a large number of visual proofs of mathematical results from which I can become immediately convinced of the truth of a piece of mathematics. Whether or not such visualisations are acceptable as legitimate and complete proofs is debatable; however J. R. Brown (2008) and others argue that non-linguistic proofs make a unique contribution to our understanding of mathematics.

(6) Subjective knowledge of mathematics is largely internalised, reconstructed objective knowledge. (Ernest, 1991, p. 44)

As I have argued above it is not the knowledge that is subjective, but rather our understanding of, or as I prefer to term it our relationship to mathematical knowledge that is largely internalised, reconstructed objective knowledge. I suggest that mathematical objects reside beyond the individual, and it is our relationship to these objects that is individual.

(7) Individual contributions can add to, restructure or reproduce mathematical knowledge. (Ernest, 1991, p. 44)

Ernest argues that individuals make potential contributions to the pool of objective knowledge on the basis of their subjective knowledge. However, I suggest that this is only possible because mathematical objects reside beyond the individual, and indeed beyond the social. Such mathematical objects can be abstractions of the world (e.g. triangles), they may be properties of such objects

(e.g. angles), they may be measurements relating to those properties (numbers), they may be relationships (sums of angles), or they may even be pure inventions (RIFTWIB numbers). My own individual contribution was to define a new type of number, and to deduce some properties of these numbers. These numbers were independently described as trapezoidal numbers by Jones and Lord (1999), and the identical properties deduced. However, even if they had not been described by me or by Jones and Lord, I suggest that RIFTWIB numbers would still have existed. They are a type of number, just as mathematically valid and well-defined, although arguably not as useful or interesting, as even or odd numbers.

Thus while Ernest's (1991, 1998) social constructivist philosophy of mathematics education captures the process underlying the development, refinement, publication and reproduction of mathematical knowledge, I suggest that it fails to adequately account for the ontology of mathematical objects and of the logic criteria that underpin the distinctive nature of mathematical proof. Ernest argues that valuing logic and proof above all else within an absolutist philosophy of mathematics at school and university has inevitably led to a transmission model of teaching. He claims that it has caused mathematics to be portrayed as the pinnacle of human reasoning, a product of a Western value system, exacerbating social divides, and therefore argues that no one view of mathematics should be privileged over another. He argues instead for his social constructivist philosophy of mathematics that he claims addresses social inequities by valuing the mathematics that has arisen from a variety of cultural traditions.

I suggest that this is a confusion of ontology, epistemology, pedagogy, and even axiology. Even if we accept that mathematics arises differently in different social constructs we do not need to accept that all are equally robust in either explaining the world or in their own internal logic. Nor is there any logical necessity for an absolutist view of mathematics such as that prevailing

in schools and universities to imply a "transmission" model of teaching. Indeed a transmission model of teaching, with students unquestioningly accepting the authority of an external person, actually obscures the logic and structure of mathematics. The need for proof is much more acute when mathematical arguments are debated and negotiated by the collective, such as in the classroom described by Lakatos (1976).

Nor need social divides necessarily be exacerbated by valuing the reasoning of Western mathematics above that of less formal systems arising in other cultures. As Rowlands and Carson (2002) argue, rather than being labelled as part of the oppressive system of Western knowledge that has served to marginalise minority groups, formal, academic mathematics should be democratised. As I shall show in Chapter 6 this is exactly the conclusion reached by Dowling (1998) from a sociological perspective: by focusing on everyday examples rather than mathematics that is more strongly classified with respect to content and expression, we disenfranchise the already powerless and cast them as dependents rather than apprentices.

Just as absolutist philosophies of mathematics have led to the curriculum excesses described by Ernest (1991), I suggest that within the discourse of learning informed by a social constructivist philosophy of mathematics education teachers are naturally positioned as "facilitators of learning" (Zaslavsky, 2005) rather than as experts representing the wider academic community. Their "role becomes one of accompaniment, facilitation, mentoring, support and guidance in the service of learners' own efforts to access, use and ultimately create knowledge" (Commission of the European Communities 1998, p. 9, cited in Biesta, 2009a, p. 17). Of course, the role of teachers does include mentoring and supporting, but it is also one of inducting students into the world of academic mathematics, a world that I argue cannot be created ex nihilo by students through investigation and discovery. Thus I suggest that extreme relativism represents an unattainable shift in responsibility

from teachers to learners who (ultimately futilely) attempt to "construct their own meanings from, and for, the ideas, objects and events which they experience" (Australian Education Council, 1991, p. 16). Indeed, the authors of the report *Everybody Counts: A report to the nation on the future of mathematics education*, go so far as to state that "in reality no one can *teach* mathematics" (National Research Council, 1989, p. 58, italics in original).

Extreme relativist philosophies of mathematics lie at the heart of excesses such as the so-called "process maths" of the 1980s (Westwood, 2003) in which what matters is the process rather than the product and knowledge of fundamental mathematical facts takes a back seat to experimentation and discovery. The remnants of this approach can be seen in the use of the term *investigate* in the *Australian Curriculum: Mathematics*. In the curriculum document the verb *investigate* occurs 81 times, accounting for some 6% of the total number of verbs used. It is the second most frequent verb after *use*. In contrast the verb *deduce* occurs just once, and the verb *prove* five times. Again, it is not that investigation and discovery are unimportant, but taken to their extreme, relativist philosophies of mathematics education deemphasise mathematics as a discipline with an established body of knowledge and distinctive ways of reasoning.

I argue, then, that neither an absolutist nor a relativist philosophy of mathematics alone can account for both the nature of mathematics as a discipline and for the human aspects of the development of mathematical knowledge. Each also leads to excesses in pedagogy of one form or another. Writing about educational policy in general, Luke (2003) points out:

> (S)ixty years after (Dewey) the binary divide in epistemology, methodology and educational policy debates remains. Their ghosts are sustained by a persistent strain of dialectics: quantitative versus qualitative, child-centred versus behaviourist, progressive/constructivist

> versus direct instruction, implicit versus explicit pedagogy, project-based work versus skills orientation" (p. 92).

To this I would add that the binary divide between absolutist and relativist philosophies of mathematics remains.

So I suggest that debate about philosophies of mathematics is as important now as it was for Hersh (1979), who wrote 40 years ago that "controversies about high school teaching cannot be resolved without confronting problems about the nature of mathematics" (p. 34). Until such debate occurs the relative agreement over the introduction of new curriculum documents such as the *Common Core State Standards* and the *Australian Curriculum: Mathematics* may well represent an uneasy peace.

3.2 Representational grainsize and inside/outside views

Rather than presenting absolutist and relativist philosophies of mathematics as logically inconsistent, I suggest that each represents an aspect of mathematics when viewed from a particular perspective. Davis and Hersh (1981) use the analogy of a cube looked at from different angles. They argue that

> [M]athematics is not one single thing. The Platonist, formalist and constructivist[15] views of it are believed because each corresponds to a certain view of it, a view from a certain angle, or an examination with a particular instrument of observation…Since they represent views of the same thing they are compatible (pp. 396, 397).

As in the introductory quote from Latour's (2013) analysis of the nature and development of knowledge in science, I suggest that looked at from far away

[15] Constructivist in the sense of the mathematical philosophy of constructivism as described, for example, by Brouwer (Bridges & Mines, 1984), in which the only mathematical objects that are said to exist are those that can be constructed by finite means. This is not to be confused with constructivism as a theory of learning or with social constructivism as a philosophy of mathematics education.

mathematics appears outside, unified, inanimate and undisputable. Yet looked at from close at hand in the act of doing mathematics it is internal, multiple, animate and disputable. Absolutist and relativist philosophies should thus not be seen as mutually exclusive, but rather as essentially differences in representational grainsize (after Manders, n.d. cited in J. R. Brown, 2008), or outside and inside views of mathematics respectively. Just as diagrams, text or equations can each represent the same mathematical object, absolutist and relativist philosophies refer to the same activity and discipline. Just as diagrams, text or equations each reveal or obscure particular aspects of mathematical objects, so absolutist and relativist philosophies reveal or obscure certain aspects of mathematics. Absolutist philosophies, like diagrams, have larger grainsize in that they are highly effective in showing the big picture and part-whole relationships. They are mathematics seen from the outside. However, they obscure the fine details revealed in the development of mathematics, just as diagrams can obscure details such as exact lengths of line segments (J. R. Brown, 2008). On the other hand relativist philosophies reveal the inherent uncertainties in the development of mathematics, but obscure the big picture of the logical interweaving of mathematical concepts. They are mathematics seen from the inside.

The differences in points of view are vividly revealed in Burton's (1998) interviews with 70 practising research mathematicians in the UK. She found that it was "impossible...to speak about mathematics as if it is one thing, mathematical practices as if they are uniform and mathematicians as if they are discrete from both of these" (p. 141). Many of the mathematicians in her study used metaphors of geography or jigsaw pieces to refer to the task of locating their work within a big picture. For some this provided a personal dilemma, as they were caught between wanting to subscribe to this big picture image of mathematics as a discipline and the socially constructed research activity in which they were engaged.

> Mathematicians do mostly subscribe to an absolute view of mathematics and what I have called The Big Picture lies not far away from where they position themselves. On the other hand, they speak in very different voices about the nature of the enterprise upon which they are engaged making clear how integral to it is both their own person-ness and the nature of their professional interactions (Burton, 1998, p. 140).

The big picture, or as I have termed it the external view, dominated these mathematicians' approach to teaching, as I suggest it does for most teachers of mathematics in schools. Few would argue that obtaining a big picture of mathematics is unimportant—indeed, without a big picture it would seem that there is little point in seeking mathematical knowledge through social activity. However, Burton found that the fine-grained or internal view of mathematics as a human activity was "entirely lost in the 'objective' mathematics they, as teachers, thrust towards reluctant learners" (p. 140).

The inside and outside views of mathematics are very much in evidence in the Narrative of Slowness introducing Part 1. By exploring numbers that could or could not be represented as trapezia I was able to make and test some tentative hypotheses. What was interesting about those hypotheses was their connection to other mathematical ideas, a connection that was not immediately obvious from the context of the problem. I wanted to understand these connections, so set about proving the result. This was the human endeavour of doing mathematics. However, at all times I was conscious of the big picture of mathematics. If I had not expected the pieces to fit together I would not have sought a solution to the problem, nor a proof of the result. The personal satisfaction gained from the problem was thus a result of both internal and external views of mathematics: without engaging in the act of mathematical exploration, discovery and explanation the result would have been of little interest, but without an elegant result that drew together several mathematical ideas the exploration would have been fruitless.

Hence I argue that we require a philosophy for school mathematics that adequately reflects both mathematics as a human activity and mathematics as a coherent body of knowledge. One such philosophy is Kitcher's (1988) mathematical naturalism. Kitcher emphasises the historical development of mathematics as a process that builds on existing knowledge and leads to new knowledge. This occurs through what he terms rational interpractice transitions, where a mathematical practice has five essential components:

> a language employed by mathematicians whose practice it is, a set of statements accepted by those mathematicians, a set of questions that they regard as important and currently unsolved, a set of reasonings that they use to justify the statements they accept, and a set of mathematical views embodying their ideas about how mathematics should be done, the ordering of mathematical disciplines, and so forth" (p. 299).

While at first glance Ernest's (1991, 1998) social constructivism may appear to be a variant of mathematical naturalism, I suggest that there are several important differences. Firstly mathematical naturalism does not deny the existence of an external reality—indeed, it assumes that early mathematical practices were based firmly in a desire to understand the world. Secondly mathematical naturalism emphasises the rationality of mathematics in that mathematical practices change through a rational process characterised by proof and rigour. It explicitly values this rationality, which may be individual or collective, as the process through which epistemic or external ends are met. Truth is seen as "what rational inquiry will produce, in the long run" (p. 314). Thirdly mathematical naturalism does not deny the certainty of mathematics. Rather it suggests that increasing certainty is the ultimate end of mathematical enquiry.

3.3 *Slow Maths* as an enactment of mathematical naturalism

In Chapter 2 I introduced the term *Slow Maths* as a metaphor to challenge dominant views of education as production, as a cure, or as a race. Building on

the slow food movement *Slow Maths* emphasises the culture and traditions of mathematics, taking existing ingredients and blending them in new ways. Rather than being concerned only with learning a set of facts or skills, the goal of *Slow Maths* is the induction of students into the discipline of mathematics. It is thus primarily concerned with the rational interpractice transitions described by Kitcher (1988).

Slow Maths, therefore, addresses internal and external views, or the near-at-hand and far away, by valuing equally the activity and culture of mathematics and the end product of mathematical enquiry. It is grounded in the world of mathematics; it respects the tradition and culture of the discipline of mathematics and it describes a pedagogical approach that apprentices students into the world of mathematics. It is captured below in three dimensions of connectedness that I have termed mathematical, cultural and contextual, and which form the subject of Chapters 4 through 6.

Mathematical connectedness

As argued above mathematics is a coherent system in which each concept or result can be derived logically from more fundamental concepts. There is a logical necessity to mathematical conclusions that is abstract, independent of context. It is this abstraction that gives mathematics its elegance of reasoning (Dudley, 2000), its power to model and solve difficult problems and its value as a way of understanding the world (Wigner, 1960). The logic of mathematics allows mathematical knowledge to stand on its own and speak for itself. That is, decisions about whether or not a piece of mathematics is correct stand or fall on the basis of its inherent logic and consistency with other mathematics, rather than being solely dependent on social acceptance.

Through mathematical connectedness *Slow Maths* values both the skills and concepts that make up the rich tapestry of mathematics and the inherent logic, elegance and efficiency of mathematical reasoning.

Cultural connectedness

Mathematics has both its own culture and develops in cultural contexts. It is a human endeavour (Davis & Hersh, 1981, 1986; Jacobs, 1970), evolving and growing as any other field of knowledge. "The pursuit of mathematics is a continuing activity that attracts a wide variety of delightful, individualistic, and devoted men and women" (Gunning, in Cook, 2009, p. 9). In the formal culture of academic mathematics enquiry just as in the classroom, debate and negotiation of bounds and meaning are paramount. As a result research activity in mathematics in Australia and internationally continues to grow at a rapid rate (Cohen, 2006).

By acknowledging and valuing the cultural-historical roots of mathematics, *Slow Maths* has the potential to empower rather than marginalise, and to simultaneously shape, interpret and challenge the world.

Contextual connectedness

Galileo is reported to have said that "mathematics is the language with which god wrote the universe". However mathematics not only explains the world, but it grows in response to the need to make sense of the world. Despite this connection to the world, apart from the relatively small number of academics involved in research in pure mathematics, perhaps the only area where mathematical knowledge does not grow from a desire to make sense of and solve problems is in schools. I argue that mathematics should not only be learned *for* solving problems, but also be learned *through* solving problems so that students come to see mathematics *as* solving problems.

Through contextual connectedness *Slow Maths* enables students both to use mathematical reasoning to solve problems in context more effectively, and to develop a richer understanding of mathematics itself through the context.

3.4 Conclusion

Critics of absolutist philosophies of mathematics such as formalism or Platonism point to their limitations when recontextualised as pedagogical practice (Ernest, 1991). Critics of relativist philosophies of mathematics point to their limitations in failing to reveal the rigour of mathematics, its connectedness as a body of knowledge or its capacity to explain the world (Rowlands et al., 2001). However, as I have argued there is no logical necessity for an absolutist view of mathematics to imply a transmission model of teaching. Neither is there any logical reason why a model of teaching based on negotiation of meaning and truth within a social environment cannot develop an appreciation of the logic and coherence of mathematics—indeed it could be argued that unless there is a coherent and agreed body of knowledge as an endpoint, such negotiation of meaning and truth itself is empty and meaningless. That is, the very purpose of adopting an inside, multiple, animate and disputable approach to mathematics pedagogy is that it points to the development of an outside, unified, inanimate and indisputable view of the discipline of mathematics itself.

Recent curriculum documents such as the *Common Core State Standards* and the *Australian Curriculum: Mathematics* have attempted to highlight the human and disciplinary aspects of mathematics through the inclusion of both proficiencies and content descriptions, however I contend that they have failed to articulate a clear philosophical basis. Hence the valued reforms run the risk of being derailed by political or partisan agendas, particularly those arising from perceived inadequacies in national and international tests. I have argued that it is possible, indeed necessary, to subscribe simultaneously to absolutist and relativist philosophies of mathematics as external and internal views whose differing grainsizes have the capacity to reveal different aspects of the discipline that mathematicians and mathematics educators cherish so dearly. Viewed from the "inside" mathematics appears hesitant, open to question and

full of potential dead-ends or conflicts. This is mathematics with small grainsize. However, it is only when viewed from the "outside", with the benefit of hindsight, that the structure and interconnectedness of mathematics takes shape. This is mathematics with large grainsize.

I began the chapter with a quote from Bruno Latour's Gifford Lectures (2013), in which he addresses the dual qualities of nature. He uses the term OWWAAB, "Out Of Which We Are All Born", in preference to terms such as nature, science or religion, which carry preconceived meanings. He proceeds to analyse the ontology and epistemology of science and nature, describing an imaginary debate between protagonists of the interior and exterior views. He concludes that for a holistic view of the nature of science eight adjectives are necessary: OWWAAB is simultaneously outside/inside, unified/multiple, inanimate/animate, indisputable/disputable.

> When taken as practices, scientific disciplines, launched in the hard step by step process of reaching the invisible and the far away, have to encounter, one after the other, each of the new and surprising agents composing a world that is not yet unified, not yet undisputed, and not yet outside. This is why the scientific way of life is simultaneously so slow, so diverse, so exciting — and also why it's so frustrating and often so controversial. To call something "scientific" is not a guarantee of certain success but the warning that a risk has been taken that may thus end up in failure (p. 43).

Latour emphasises that it is "a poor education that misses the far away just as much as the close at hand" (p. 38). Equally we could say that it is a poor education that misses the close at hand just as much as the far away.

I have argued that Kitcher's (1988) mathematical naturalism, by focusing on rational transitions between mathematical practices, is simultaneously near and far away. As an enactment of mathematical naturalism, *Slow Maths* as outlined in this chapter and expanded in subsequent chapters captures the slowness,

diversity and excitement in the above quote. It brings together the close at hand and far away by emphasising mathematical, cultural and contextual connectedness. In the following three chapters I present detailed evidence from the discipline of mathematics itself, from the work of mathematicians, and from mathematics education research, that supports each of these dimensions of connectedness.

PART 2: CONNECTEDNESS

A narrative of connectedness: Handshakes

Early in my teaching career I was asked for my thoughts on the essence of mathematics. After some consideration I replied, "making connections". I tried to develop lessons that intentionally made connections between mathematical concepts, to the interests of students, to the real world, and where possible, to events in the history of mathematics. I developed one particular lesson that I then used as the first lesson with almost every new class, including preservice teacher education students. This lesson served as a way of getting to know students, engaging them in some mathematical thinking, and of introducing some important epistemological aspects of mathematics. I reproduce it here as a *lesson play* (Zazkis, Liljedahl, & Sinclair, 2009) The use of a lesson play personalises the situation, allowing the reader to feel part of the classroom. This is, of course, an idealised lesson play that sanitises the classroom dialogue, but it is authentic in that it describes the tasks, conversations and negotiation of meaning that I observed occurring each time I used the lesson.

<u>Posing the problem</u>

Steve: Welcome to Mr. Thornton's mathematics class. I need to tell you that mathematics is not a spectator sport, so let's start by doing some mathematics. Please draw a line approximately down the centre of a blank page of your book, so that you have divided the page into two roughly equal halves. Now draw a line across the centre of your page so that you have four roughly equal quarters, then two more lines line across the top half and bottom half of the page. So you should now have eight rectangles, roughly the same size.

A pause to ensure that all are ready.

Steve: I am going to ask you to draw some diagrams in each box. In the top left hand box could you please draw four straight lines that do not cross each other

no matter how long you draw them? I need to emphasise that the lines must be straight, and that lines continue indefinitely.

Steve draws an example on the board of two line segments that do not cross, but would cross if extended.

A pause while students work individually.

Girl 1[16] *(to Steve individually): Is this right?*

Steve (to Girl 1 individually): Do the lines cross? Would they cross if you made them longer?

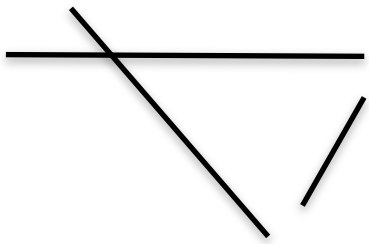
Similar conversations, including conversations between students, take place around the room. After a short time most students have drawn four parallel lines, however the answer is not revealed publicly.

Steve: So you have drawn four lines that do not intersect. Please label your diagram with the caption "0 intersections".

<u>Continuing the problem</u>

Steve: In the rectangle next to that can you please draw four straight lines, remembering that lines extend indefinitely, that intersect in exactly one point?

After a short time Steve draws an example on the board of three line segments, such as that shown below:



Steve: Is this one intersection?

[16] This is the first time I have met the class. At this stage of the lesson I do not know students' names.

Girl 2 (highlighting the intersection point): No. Two of the lines cross at one point, but if we made all the lines longer they would cross each other. So there would be 3 intersections.

Individual teacher-student and student-student conversations, similar to that described above, ensue. After a short time most students have drawn four concurrent lines, however this answer is not revealed publicly.

Steve: Please label this rectangle "1 intersection". You may have guessed the next question: in the third box please draw four straight lines that intersect in exactly two points, and label the rectangles "2 intersections".

Individual teacher-student and student-student conversations, similar to that described above, ensue. Many students become frustrated. No answers are revealed.

Steve: If you are having trouble you might like to move on to the next diagram, which of course is 3 intersections, then 4, 5, 6 and 7. It doesn't matter what order you do them in, and please talk about them with the person next to you. Mathematics is collaborative.

Individual teacher-student and student-student conversations continue. After several minutes:

Steve: By the way, I should warn you that sometimes in mathematics questions can be impossible.

Boy 1: Which ones?

Steve: That's for you to think about and talk about with other people. We will talk as a class later.

There is further frustration, but students make substantial progress in completing most diagrams. After students have diagrams in most rectangles:

<u>Sharing solutions</u>

Steve: I know that not all of you have finished all the diagrams, but that's ok. You can keep thinking about them while we talk about them together. Would someone like to come to the whiteboard and draw four straight lines with no intersection?

Several hands are raised, Steve points to Girl 1.

Steve: Yes, what's your name?

Girl 1: Jane

Steve: Thank you, Jane. Here's a whiteboard marker.

Jane draws four parallel lines in the appropriate space on the whiteboard.

Steve (to class): Who agrees with Jane that this is four straight lines with no intersections?

A large number of hands are raised. Steve shakes Jane's hand, much to her surprise.

Steve: Well done, Jane, congratulations[17].

Jane moves to sit down.

Steve: Wait a minute Jane. Would you mind staying here, please?

Steve: Can someone now please volunteer to draw four straight lines that intersect in exactly one point?

Several hands are raised. Steve points to Boy 1.

Steve: Yes, what's your name?

Boy 1: Jarrod

Steve: Thank you, Jarrod. Here's a whiteboard marker.

Jarrod draws four concurrent lines in the appropriate space on the whiteboard.

---

[17] This is not to provide positive reinforcement, but to accompany the handshake.

Steve (to class): Who agrees with Jarrod that this is four straight lines with exactly one intersection?

A large number of hands are raised. Steve shakes Jarrod's hand.

Steve: Congratulations, Jarrod. Jane, would you please shake Jarrod's hand and congratulate him?

Accompanied by laughter and puzzled looks, Jane shakes Jarrod's hand. Jarrod moves to sit down.

Steve: Wait a minute Jarrod. Would you mind staying here with Jane, please?

Steve: Can someone now please volunteer to draw four straight lines that intersect in exactly two points?

No hands are raised.

Steve: Does anyone have a solution? If not, would someone like to volunteer to write something appropriate in that rectangle?

One hand is raised by Girl 2.

Steve: Yes, what's your name?

Girl 2: Sarah.

Steve: Thank you, Sarah. Here's a whiteboard marker.

Sarah writes IMPOSSIBLE in the appropriate space on the whiteboard.

Steve (to class): Who agrees with Sarah that it is impossible to draw four straight lines with exactly two intersections?

A few hands are raised.

Steve: It looks as if quite a few of us think it is impossible. So congratulations, Sarah.

Steve shakes Sarah's hand.

Steve: Jane, would you please shake Sarah's hand and congratulate her? Jarrod, would you please shake Sarah's hand and congratulate her?

Accompanied by laughter and puzzled looks, Jane and Jarrod each shake Sarah's hand. Sarah moves to sit down.

Steve: Wait a minute Sarah. Would you mind staying here with Jane and Jarrod, please?

The process continues until eight students have provided solutions or asserted that a particular case is impossible. In each case the class has affirmed the solution. Each student has remained at the whiteboard and congratulated, by name and handshake, each of the other students.

Steve: Thank you for volunteering to write your solutions on the board. You can all go back to your seats.

<u>The follow-up problem</u>

After students have returned to their seats:

Steve: Of course, the next question is: How many handshakes were there?

A large number of groans follow.

Will (whose name is now known because he was one of the eight volunteers): 45

Steve: I don't want you to make your answer public yet. I would like you to take some time to think about it, and when you have an answer please write it down on your page.

After most students have an answer:

Steve: Would someone like to give me their answer? Will?

Will: 45

Steve writes 45 on the whiteboard.

Steve: Does anyone have a different answer?

Sarah: 36.

Steve writes 36 on the whiteboard.

Jane: 28.

Steve writes a list of suggested answers on the whiteboard. The list includes 45, 36, 35, 72, 64[18].

Steve: We have several suggested answers. Can you please have a look at them and decide which one you think is correct. It doesn't have to be the one you originally worked out. Now let's take a vote. Who thinks it is 45? 36? 35? 72? 64?

Students vote for the answer they think is correct and a count is tallied on the whiteboard.

<u>Sharing explanations</u>

Steve: Will, could you please explain why you think the answer is 45?

Will: Counting you there were 9 people at the front. The first person shook one hand, then 2, then 3 and so on. So I added 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9, and got 45.

Steve: Thank you, Will. Sarah, could you please explain how you got 36?

Sarah: I did the same as Will, but the last person only shook 8 hands, so I added 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8.

James (who suggested 35): I did the same as Sarah. I was wrong, because I added them incorrectly. I think it's 36 now.

Steve: Michelle, you said 72. Why is it 72?

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[18] These answers are almost always suggested by at least one student. 28 is also commonly suggested.

Michelle: I was the last person to come to the front. I shook 8 hands. There are 9 people, so 9 × 8 = 72[19].

Ian: I said 64 because there were 8 students and 8 × 8 = 64.

Steve: Now that we have explanations for all the answers, let's take another vote. You can change your mind if you want.

A second vote is taken. Many students are convinced by the explanations of others to change their answer.

Verifying the solution

Steve: So how can we find out which is correct?

Students in unison: Do it again.

The process is reenacted. Part way through a third vote is taken, as it becomes clear that the answer is not 64 or 72. The class agrees that there were 36 handshakes.

Debriefing the explanations

Steve: So let's talk about why some of the answers were wrong. We could give logical explanations for each of them, but obviously there was a mistake somewhere in most of them. Why was 45 wrong?

Will: Because I had the last person shaking 9 hands instead of 8. You don't shake your own hand. It would be 45 if there were 10 people.

Steve: Michelle, you said 72 because it was 9 × 8. That's a really interesting explanation, but why isn't it quite right?

Michelle: It would be correct if I halved it. I double counted because Will shaking hands with Jane and Jane shaking hands with Will is only one handshake.

---

[19] If students don't suggest 72, as the teacher I do, and give this explanation.

Steve: So we could have got 36 by saying that there were 9 people, each of whom shook 8 hands, then halving it to remove the double counting.

It is near the end of the lesson.

Making a connection

Steve: I have one final question for you to think about. I don't want an answer now, but please think about it overnight. What does crossing lines have to do with shaking hands?

Posing a third problem

The following day:

Steve: There is a (possibly apocryphal) story that a very lazy maths teacher once asked his class to add all the numbers from 1 to 100 while he read his morning paper. After less than 30 seconds young Carl[20] *approached the teacher with just one number (the correct answer) written on a piece of paper. My question is: what is the sum of all the numbers from 1 to 100, how did Carl do it and what is the connection between this problem, handshakes and intersecting lines?*

Although there are few connections with context in this particular lesson, the lesson play illustrates aspects of mathematical and cultural connectedness to which I will refer in Chapters 4 and 5, and of pedagogy to which I will refer in Chapter 7.

[20] The students are later informed that this was Carl Friedrich Gauss, who later became one of the most famous and productive mathematicians and physicists of all time.

CHAPTER 4: MATHEMATICAL CONNECTEDNESS—INSIGHTS FROM MATHEMATICS, MATHEMATICIANS AND MATHEMATICS EDUCATION RESEARCH

> Pure mathematics, on the other hand seems to me a rock on which all idealism founders: 317 is a prime, not because we think so, or because our minds are shaped in one way rather than another, but *because it is so*, because mathematical reality is built that way. (G. H. Hardy, 1940, p. 130, italics in original)

Synopsis

Insights from the discipline: Mathematics has long been regarded as the archetypal pure science, a field of knowledge in which the ultimate arbiter, at least in the context of pure mathematics, is the discipline itself.

Insights from practice: Professional mathematicians almost universally hold a view of mathematics as a coherent big picture. When writing mathematics, they allow the discipline to "speak for itself", and when researching mathematics striving for rigour is a prime motivation.

Insights from research: Effective teachers of mathematics are those who see the big picture of mathematics. They make connections between mathematical ideas and enable students to see how one idea builds on another.

Conclusion: There is compelling evidence that the connectedness of mathematical ideas is part of the essence of mathematics itself; adding to it is a goal of mathematicians; and enabling students to understand it is the work of teachers. This is slow work.

The preceding two chapters have made a case for reconsidering school mathematics, firstly as part of a humanising goal rather than framed within the

metaphors of education as production, cure or race, and secondly as encapsulating outside and inside views of mathematics as both a coherent body of knowledge and a human endeavour. Slowness, both temporally and as a philosophy of connectedness to culture and the world, provides a way of thinking that challenges dominant metaphors of education and values process and product equally. This chapter outlines the first of three aspects of connectedness that characterise what I term *Slow Maths*: mathematical connectedness. I argue that mathematical connectedness is implicit in the discipline of mathematics itself, in the work of professional mathematicians, and in the descriptions of effective teachers of mathematics and classroom practice in the mathematics education research literature.

At the outset it should be acknowledged that school mathematics is not academic mathematics. Despite the emphasis on developing mathematical proficiencies of understanding, fluency, reasoning and problem-solving articulated by the *Australian Curriculum: Mathematics*, school-aged students have neither the knowledge base nor the sophisticated analytical skills to be "mathematicians", in the accepted academic sense of the term. They do not generally solve original problems[21], publish in academic journals, or contribute to the community of mathematicians. Conversely, professional mathematicians do not complete exercises from textbooks or worksheets.

Nevertheless school mathematics is, for almost all students, the only image they possess of mathematics. This image—that is, the content and pedagogy of school mathematics—has the potential to engage students in, or alienate them from, mathematics. If the activity of school mathematics is to bear some resemblance to the activity of academic mathematics it is imperative that

[21] I use the term "original" in the sense of a problem whose solution is not known in the wider community. Of course, students solve many problems in school that are "original" to them such as those described in the Narrative of Connectedness introducing Part 2 of the thesis.

students in school solve problems that are unfamiliar to them, learn from others more experienced, read mathematics, communicate their thinking to an audience and make judgements about the relative worth of alternative mathematical arguments. That is, they need to engage with mathematics as a connected, internally consistent field of knowledge.

4.1 Insights from the discipline

Since the establishment in 1673 of a special Mathematical School for forty boys at Christ's Hospital in England, mathematics has been recognised as a central component in school curricula (Griffiths & Howson, 1974). It continues to occupy a position of high status and importance having, with English, one of the highest participation rates in senior secondary schooling (Ainley, Kos, & Nicholas, 2008). Along with literacy, numeracy is regarded as one of the essential capabilities that should be developed in all children and across the school curriculum (Australian Curriculum and Assessment Reporting Authority [ACARA], 2013). Despite evidence of decreasing enrolments in high levels of mathematics (Barrington, 2006) there is no reason to suggest that either the study of mathematics or the discipline of mathematics itself will lessen in value or importance.

4.1.1 Mathematics as a hard/pure field of knowledge

In his classic treatise *The Two Cultures and the Scientific Revolution*, C.P. Snow (1960) portrays the artistic culture and the scientific culture as diametrically opposed, making a plea for both sides to better understand each other. Snow's categorisation of knowledge was strongly influenced by his close friendship with the English mathematician G. H. Hardy, who might perhaps have identified with how the relationship between mathematics and other fields of knowledge is depicted in Figure 4.1.

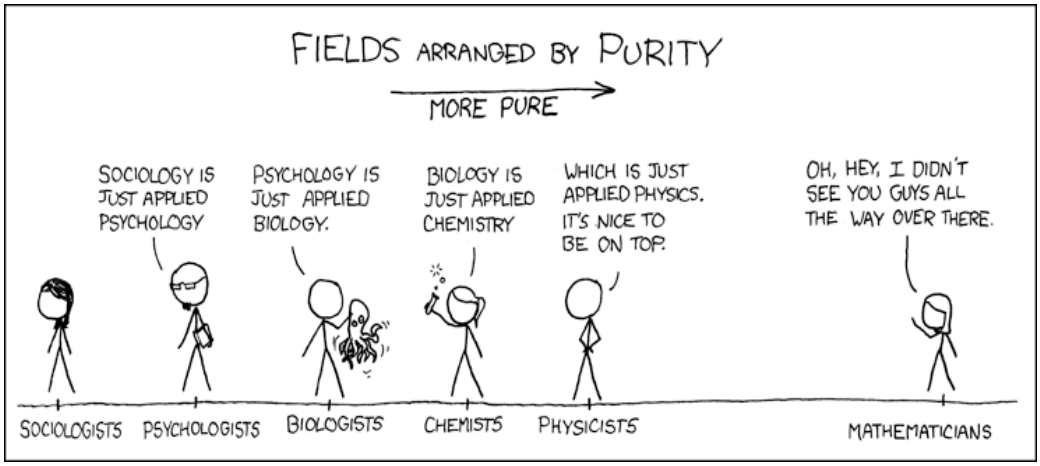

Figure 4.1: Fields arranged by purity (Munroe, 2008, reproduced under a Creative Commons licence)

A more subtle categorisation of fields of knowledge is given by Becher and Trowler (2001), who conducted an extensive survey of the nature of knowledge, career patterns, reputation and rewards, aspects of professional practice and costs and benefits of disciplinary membership within a number of fields in higher education. Their research was based on both reviews of literature within the fields and interviews with practising members of the field. It included thirteen mathematicians, as well as members of a variety of scientific, law, language and social science fields.

This research was partly a response to Snow's work, which the authors saw as a somewhat naïve, yet influential characterisation of the sciences and humanities. Becher and Trowler (2001) suggested that Snow's work offered a conceptually flawed polarisation between the worlds of the sciences and the humanities[22]. Instead they provided a two-dimensional analysis of knowledge as hard/soft and pure/applied, shown diagrammatically in Figure 4.2.

[22] In my view such an analysis also fails to take into account areas of knowledge that could be said to be neither scientific nor artistic, or even both artistic and scientific, such as teaching.

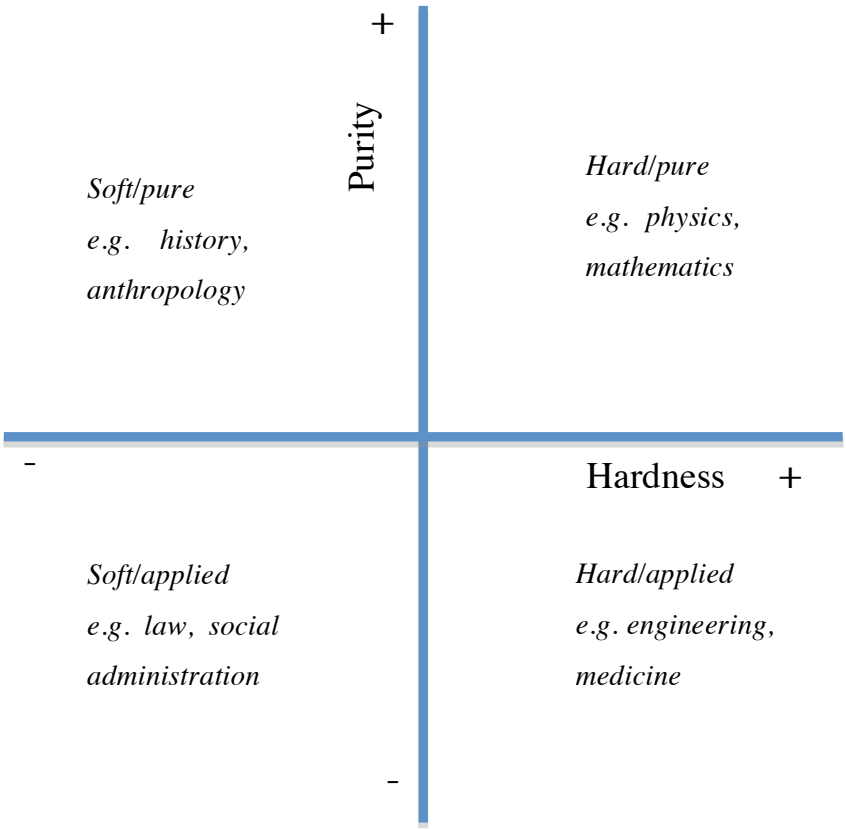

Figure 4.2: Academic territories (an interpretation of Becher & Trowler, 2001)

Table 4.1 provides an analysis of the nature of knowledge in each of the four quadrants. It shows the structure, values, approaches, criteria for verification and function of knowledge in each area.

Table 4.1 Knowledge and disciplinary groupings (Becher & Trowler, 2001, p. 36)

Disciplinary grouping	*Nature of knowledge*
Hard-pure	Cumulative; atomistic (crystalline/tree-like); concerned with universals, quantities, simplification; impersonal, value-free; clear criteria for knowledge verification and obsolescence; consensus over significant questions to address, now and in the future; results in discovery/explanation.

Soft-pure	Reiterative; holistic (organic/river-like); concerned with particulars, qualities, complication; personal, value-laden; dispute over criteria for knowledge verification and obsolescence; lack of consensus over significant questions to address; results in understanding/interpretation.
Hard-applied	Purposive; pragmatic (know-how via hard knowledge); concerned with mastery of physical environment; applies heuristic approaches; uses both qualitative and quantitative approaches; criteria for judgment are purposive, functional; results in products/techniques.
Soft-applied	Functional, utilitarian (know-how via soft knowledge); concerned with enhancement of [semi-] professional practice; uses case studies and case law to a large extent; results in protocols/procedures.

In this classification of knowledge, mathematics is clearly regarded as hard-pure, as knowledge is cumulative with considerable consensus over the criteria for knowledge verification and obsolescence. Through the processes of abstraction and symbolisation mathematics strives to capture the essence of a situation (Wittgenstein 1978) and through generalisation to describe universal truths. It is the language of the sciences, playing an essential role in the articulation of scientific concepts (Tiles, 1984), thus enabling explanation of real world phenomena[23].

23 Mathematics education on the other hand is primarily concerned with the enhancement of professional practice and can be regarded as an applied social science (soft-applied). Becher and Trowler's analysis of the academic territories associated with various fields of higher education sheds light on the practices within the various academic tribes. It shows how status and esteem is differentially conferred upon certain types of research and upon members of certain fields, with researchers in the hard area being more highly regarded than those in the soft and those in the pure area more highly regarded than those in the applied. The differential

While mathematics, theoretical physics and astronomy have always been intimately connected, the boundaries between mathematics and fields of study such as economics, the life sciences, linguistics and management are becoming less distinct (Davis & Hersh, 1986). However, I suggest that the characteristics of mathematics as a hard-pure field of knowledge detailed in Table 4.1 above have remained relatively unchanged. Quantification of phenomena, abstraction and the use of symbols to represent the essence of a situation, deductive logic that distinguishes a mathematical proof from a scientific theory, and generalisation to all such situations remain distinctive features of mathematical reasoning. Mathematics as a body of knowledge consists then not only of established results, concepts and theorems, but also of distinctive ways of reasoning. These ways of reasoning constitute a strong internal grammar (Bernstein, 1999), which in turn provides the basis for the evaluation of knowledge in mathematics using a knowledge legitimation code (Maton, 2000).

4.1.2 Mathematics as a field with strong internal grammar

Bernstein (1999) classifies mathematics as (1) *a vertical discourse* with (2) a *horizontal knowledge structure* characterised by (3) *a strong internal grammar*. He first distinguishes between discourses, or ways of knowing and being. Academic areas of study, legal or political argumentation, press editorials or religious expositions are all examples of vertical discourses. Typically the knowledge gained in a vertical discourse is systematic, coherent and principled, most often communicated or able to be communicated in written form. It is normally communicated by a more knowledgeable other, who recontextualises her knowledge for consumption by the learner. On the

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status of mathematics and education within universities goes some way towards explaining the long-standing battle for control of the school mathematics curriculum discussed in Chapter 1.

other hand the informal interactions and learning within a family or community, which are most often communicated orally and spontaneously, form horizontal discourses. Knowledge within a horizontal discourse is local, context-specific, and may be contradictory across contexts. Within a horizontal discourse individuals possess a repertoire of strategies learned from a communal reservoir, but these strategies are segmented, independent of one another. Table 4.2 summarises the distinction between vertical and horizontal discourses. Clearly mathematics is a vertical discourse as there are clear principles, recognised ways of reasoning and established rules of communication.

Table 4.2 Vertical and horizontal discourses (B. Bernstein, 1999, p. 162)

	Vertical discourse	**Horizontal discourse**
Practice	Official/Institutional	Local
Distributive principle	Recontextualisation	Segmentation
Social relation	Individual	Communalised
Acquisition	Graded Performance	Competence

At the second level, within these vertical discourses, Bernstein distinguishes between *hierarchical knowledge structures* such as Physics, which move towards ever greater generality, integration, and ultimately universal principles, and *horizontal knowledge structures* such as economics, linguistics or sociology that consist of a set of discrete languages for solving particular problems. Bernstein maintains that mathematics consists of a set of discrete languages such as geometry, number theory or discrete mathematics, each of which solves a range of language-specific problems. However, I suggest that these languages are less discrete than Bernstein maintains. Rather than the various branches of mathematics being distinct ways of solving particular problems, each could be considered a lens with which to view a problem from

a particular perspective. The isomorphism between the handshakes and the intersecting line problems in the lesson play at the start of Part 2 of the thesis illustrates this connection. There is a network of principles within and between these branches of mathematics, with knowledge development often taking place across boundaries. Furthermore there is a hierarchical structure within each branch of mathematics, with each having abstract principles that allow for the generalisation and integration of knowledge. However, regardless of whether mathematics is classified as a horizontal or vertical knowledge structure, it has a *strong internal grammar*.

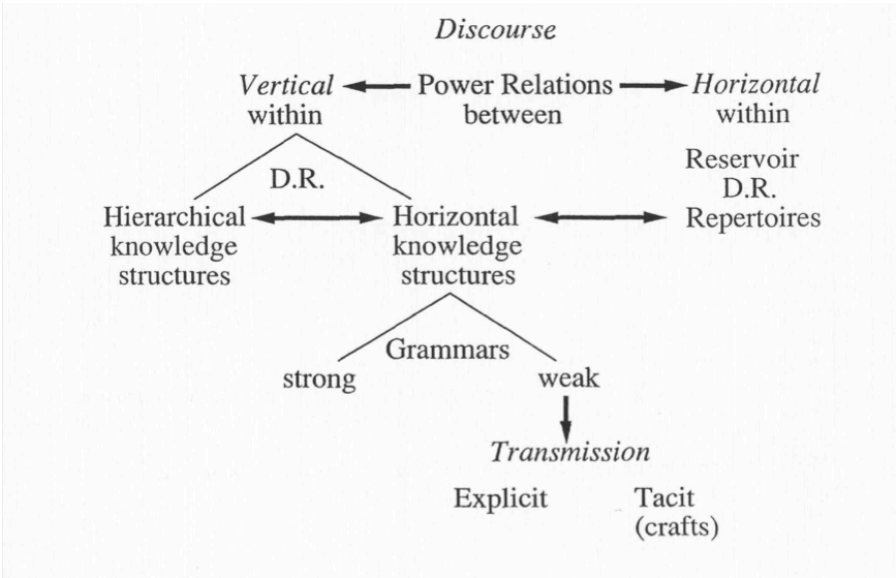
Bernstein terms the capacity to generate precise empirical descriptions and/or generate formal models within a field of knowledge the strength of the internal grammar. He first notes that all hierarchical knowledge structures such as physics have, by necessity, strong internal grammars. Within the wide variety of horizontal knowledge structures, he claims that economics, linguistics and parts of psychology have strong internal grammars, whereas sociology and cultural studies have much weaker internal grammars[24]. Of all fields of knowledge, with the possible exception of logic, mathematics has the strongest internal grammar, as it has accepted principles of logic, internal and external consistency and strives for a lack of gaps in reasoning.

Although not germane to the discussion of mathematics as a field of knowledge, Bernstein finally distinguishes between modes of transmission. Vertical knowledge structures, and horizontal knowledge structures with strong internal grammar, including mathematics, are necessarily *transmitted explicitly*

[24] I maintain that education as a field of study also has a weak internal grammar as there are few, if any, generalised theories capable of developing formal models or descriptions. However metaphors of education as production, cure or race, situated within the technological enframing, seek to establish such grammatical strength. As highlighted in the final Chapter I argue, along with Biesta (2009b) that attempts to establish education as a strong discipline ultimately remove the education from education.

either through formal teaching or in written forms such as journals. Horizontal knowledge structures with weaker internal grammars may be transmitted explicitly such as through formal teaching or exposition, or tacitly through modelling and copying as in the case of knowledge of crafts or trades.

Figure 4.3 illustrates Bernstein's taxonomy of knowledge structures, beginning at the level of discourse and moving to knowledge structure, grammar and transmission.



[D.R. means Distributive Rules]

Figure 4.3 Discourse and knowledge structures (Bernstein, 1999, p. 168)

If we are to take seriously the goals and purposes of school mathematics that I describe as *Slow Maths* students need to be empowered to move towards the generalisation and integration of knowledge that is typical of a knowledge structure with a strong internal grammar. However, coming to understand mathematics in this way is neither universal nor simple. For many students in school mathematics remains mysterious and invokes fear (Hembree, 1990), and consists of little more than a set of disconnected facts or skills each of which stands in isolation. These students possess a *repertoire* of strategies drawn

from the *reservoir* of strategies possessed by the community (see Figure 4.3), but with no way of either integrating these strategies into a coherent whole or using them to generate new strategies. I suggest that this is a major issue in mathematics education; for such students, mathematics has a *weak grammar*.

4.1.3 Mathematics as a field with a knowledge mode of legitimation

The analysis of discourses and knowledge structures discussed above is further developed by Maton (2000) who describes knowledge legitimation modes in various fields. He claims that these languages of legitimation are more than mere rhetoric; rather, they "represent the basis for competing claims to limited status and material resources" (p. 149). In particular he describes what he terms knowledge and knower legitimation modes, which are based on underlying principles concerning the epistemic and social relations. These principles structure both what can be legitimately claimed as knowledge within a given field, and who can legitimately claim or validate that knowledge.

Maton uses Bernstein's (1990) concepts of *classification*, that is the strength of boundaries between categories or contexts, and *framing*, that is the locus of control within a category, to discuss the nature of these principles. He describes how the epistemic and social relations that determine knowledge legitimation within a field vary according to the relative strength of the classification and framing on each dimension. A diagrammatic conception of Maton's knowledge legitimation modes is shown in Figure 4.4.

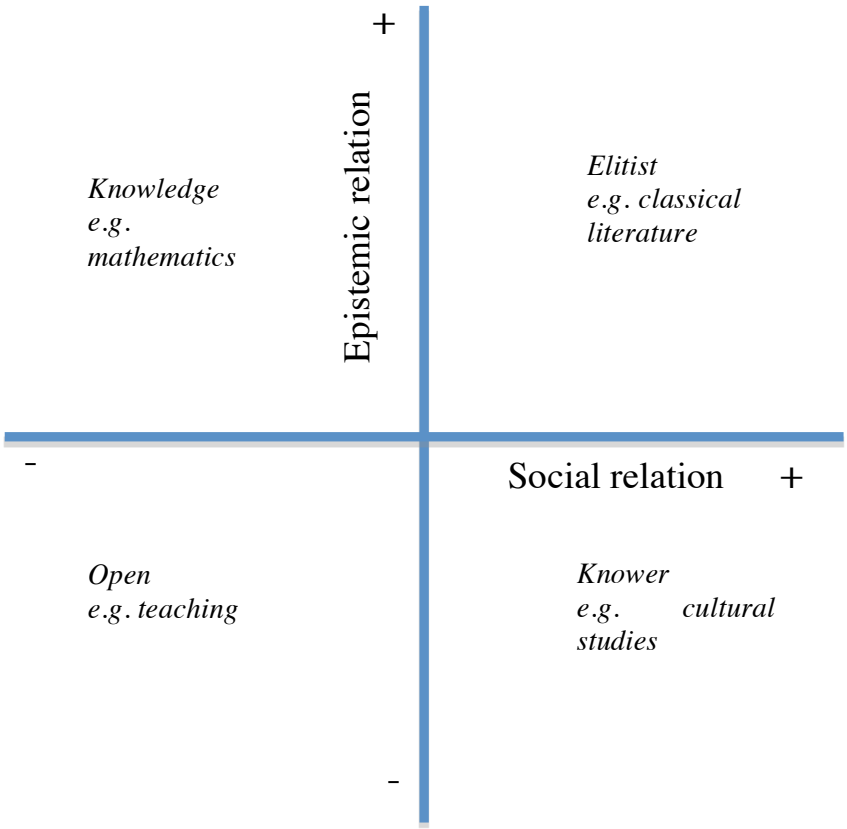


Figure 4.4: Modes of legitimation (adapted from Maton, 2000)

In the case of mathematics, the epistemic relation is both strongly classified and strongly framed. The strong internal grammar of mathematics discussed above makes it clear what *counts* as legitimate mathematics and exercises tight control over what is *accepted* as legitimate mathematics. On the other hand the social relation is relatively weakly classified and framed. Cultural differences and social disadvantage notwithstanding, in the end who develops mathematical knowledge is less important than the knowledge itself. For example when the Wolfskehl prize of 100 000 marks for a successful proof of Fermat's Last Theorem was announced, the University of Gottingen received a flood of entries. "Regardless of who had sent in a particular proof, every single one of them had to be scrupulously checked just in case *an unknown amateur* had stumbled upon the most sought after proof in mathematics" (Singh, 1998,

p. 143, italics added). Maton describes this as a knowledge mode of legitimation[25].

It is summed up by Corry (1989) in the (lengthy) quote below:

> On the one hand, more than in any other exact science or scholarly field, an isolated result in mathematics may be attained and corroborated without resorting to the authority of individuals or of written sources. Results may often be obtained through ingenuity and inspiration, and a thorough knowledge of sources and of the writings of the great masters is by no means a necessary condition for innovation (as it is in the humanities, for example). Proof remains the only validating procedure, and its standards are established through shared images of knowledge.
>
> In contrast, in fields where new results may invalidate previous ones, the authority of scientists may be jeopardized as research develops in new

---

[25] Maton contrasts a knowledge mode of legitimation with a knower mode, in which *who* holds the knowledge matters more than what it is. I argue that mathematics education has a knower mode of legitimation as it is weakly classified with respect to the epistemic dimension (within mathematics education research a variety of paradigms is acceptable (Lerman, 2000), and it is not always clear what counts as legitimate research), but strongly classified with respect to the social dimension (it matters who does the research within a particular social context). Journal and conference papers in the mathematics education research literature are reviewed according to relatively flexible criteria such as whether the paper builds on and interrogates published research, the open-endedness and thoughtfulness of the research questions, the clarity of description of methodology, the ethics of the research and the cohesion of the argument (Gordon, 2002). Like other knower legitimated fields of knowledge, mathematics education progresses via radical schisms, resulting in an insistence on the legitimacy of only current research.

I have argued elsewhere (Thornton, 2008) that these underlying differences in the way the two communities view knowledge are at the heart of much of the long-standing debate surrounding school mathematics curriculum, suggesting that "the debate over what counts in mathematics education and the school curriculum is, in effect, a battle for control of the epistemic device" (p. 524).

> directions. Consequently, these scientists may have good reasons (scientific, institutional, or political reasons) to refuse to accept new results. This kind of conflict of interests cannot arise within the body of knowledge of mathematics. There are no polemics, in principle, within the body of knowledge of mathematics. Proof, based on its accepted standards, has the last (and only) word. Authority of any kind is explicitly proscribed as a criterion for acceptance of any claim within the body of knowledge (p. 433).

Mathematics, then, is legitimated using a *knowledge* mode of legitimation. It was this knowledge mode of legitimation that I tried to establish in the lesson play described at the start of part 2 and that convinced me of the correctness of my result concerning RIFTWIBs in the problem described at the start of Part 1. As will be discussed later in this chapter, contemporary educational research into the characteristics of classrooms in which students develop deep and connected understandings of mathematics implicitly highlights this knowledge mode of legitimation.

4.2 Insights from practice

4.2.1 Linguistic features of the presentation of mathematics

If we accept that mathematics has a knowledge mode of legitimation, then the mathematics should "speak for itself". To ascertain the extent to which mathematicians share a community norm that mathematics speaks for itself, I analysed the length of abstracts in mathematics and mathematics education journals. The mean length of the abstracts of the 49 papers in the six issues of the *Journal of the Australian Mathematical Society* published in 2013 was 88.8 words. On the other hand abstracts of papers in four issues of the *Mathematics Education Research Journal* were almost twice as long, with a mean length of 161.5 words. This strongly suggests that mathematics educators feel the need to situate the research within a context, to convince the reader of its importance and to demonstrate that the research conducted was sufficiently rigorous to

address the issue. However, mathematicians have little need to make such a case. Rather than writing an apologetic that makes a case for the work, they allow the mathematics to speak for itself. This is, of course, not an indictment of mathematics education research as a discipline; rather it is indicative of the differing natures of the two disciplines.

Furthermore the writing style of mathematicians enlists the mathematical community as allies in the process of discovery, development and proof. A count of personal pronouns in the eight papers of the February 2012 edition of *Journal of the Australian Mathematical Society* reveals 557 uses of the personal pronoun *we*, and none of the pronoun *I*, despite two articles being individually authored. In almost all cases the pronoun *we* is used in an inclusive sense, such as "*we* now see that…" or "if *we* accept…" as in the first sentence of this section, rather than an exclusive sense referring to the authors alone. This contrasts sharply with the use of *we* or *I* in papers in mathematics education research journals, where, as most commonly in this thesis, the pronoun is used to refer specifically to the ideas or work of the author(s). I suggest that this use of the pronoun *we* in mathematics journals invites the reader to join the author(s) in the mathematical journey, rather than being used as a statement of ownership of the research.

The verbs following these uses of *we* also provide compelling evidence of the importance for the author(s) of knowledge legitimation. Table 4.2 shows the relative frequencies of the 20 most common verbs in the eight articles of the journal.

Table 4.2 Frequencies of 20 most common verbs in J Aust Ms 92(1)

Verb	Number of occurrences	Relative frequency
have	42	7.5%
assume	26	4.7%
use	26	4.7%
see	23	4.1%
obtain	18	3.2%
note	16	2.9%
consider	15	2.7%
define	15	2.7%
show	15	2.7%
apply	14	2.5%
prove	14	2.5%
deduce	12	2.2%
find	12	2.2%
write	11	2.0%
begin	10	1.8%
give	10	1.8%
observe	10	1.8%
derive	9	1.6%
know	9	1.6%

It is significant to note that, when used in a mathematical sense as in the journal articles, some 10% of the verbs are connected to justification of results[26] (*show*, *deduce*, *prove*, *derive*); 10% relate to setting up a problem (*assume*, *consider*, *define*); and a further 15% relate to results either during or at the conclusion of the process (*have*, *see*, *obtain*, *observe*). Few if any of the verbs relate to mechanical processes such as *calculating* or *simplifying*, verbs which abound in school mathematics texts.

Of course, one could argue that this is precisely the point of mathematics journal articles, and that the process leading up to the publication of results is inconsequential. That is, journal articles present the external rather than the internal view of mathematics. However, that merely serves to strengthen the case for mathematics as a discipline with a knowledge mode of legitimation. Regardless of how tortuous or tentative the process of developing a piece of mathematics may be, the ultimate goal of the mathematician is to produce a proof that stands up to rigorous scrutiny. I take as examples the proof of Fermat's Last Theorem, mentioned briefly in the previous section, and the four colour problem.

4.2.2 Wiles' proof of Fermat's Last Theorem

Fermat's Last Theorem states that for values of $n > 2$, there are no three positive integers a, b and c that satisfy the equation $a^n + b^n = c^n$. It is a problem of immense historical and cultural significance to mathematicians, arising from Pierre de Fermat's note in the margin of the page where he described the result that he had found "a truly marvellous demonstration of this proposition which this margin is too narrow to contain" (Singh, 1998, p. 66). What might be that proof? Could it really be so elegant yet so elusive? The rigour and elegance hinted at by Fermat, and the intellectual drive to find the proof, inspired

[26] This, of course, is in sharp contrast to verbs in the *Australian Curriculum: Mathematics*, where, as reported in Chapter 3, such verbs are almost absent.

mathematicians like Andrew Wiles for generations. It was not that Wiles had a particular need to prove the theorem to develop further results or to solve a pressing problem in the world. His sole reason for investing seven years of his research life was to see if he could do it: what drove him was to develop a proof with no gaps and unquestionable logic.

Wiles had been fascinated with the theorem since childhood; after all, it is simple to state and even relatively young children can easily compute specific cases. However, it was not enough for Wiles, and presumably the countless others who tried with more or less success to prove the theorem, simply to know of its existence and be satisfied that it appeared true. Empirical evidence, such as that sought by natural scientists, or a persuasive argument such as one sought by social scientists, was not enough. Rather he was driven by the need for legitimation by and within the discipline of mathematics.

Wiles' presentation of his proof is perhaps the ultimate demonstration of the aura with which mathematicians regard the inbuilt knowledge legitimation mode of mathematics. Singh's account of the proof and presentation of Fermat's Last Theorem reads like a mathematical mystery/adventure story.

> Fermat's Last Theorem, a problem that had captivated mathematicians for centuries, captured the imagination of the young Andrew Wiles. In Milton Road Library, ten-year-old Wiles stared at the most infamous problem in mathematics, undaunted by the knowledge that the most brilliant minds on the planet had failed to rediscover the proof. Young Wiles immediately set to work using all his textbook techniques to try to recreate the proof. Perhaps he could find something that everyone else, except Fermat, had overlooked. He dreamed he could shock the world.
>
> Thirty years later Andrew Wiles stood in the auditorium of the Isaac Newton Institute. He scribbled on the board and then, struggling to contain his glee, stared at his audience. The lecture was reaching its climax and the audience knew it. One or two of them had smuggled

> cameras into the lecture room and flashes peppered his concluding remarks
>
> With the chalk in his hand he fumed to the board for the last time. The final few lines of logic completed the proof. For the first time in over three centuries Fermat's challenge had been met. A few more cameras flashed to capture the historic moment. Wiles wrote up the statement of Fermat's Last Theorem, turned toward the audience, and said modestly: "I think I'll stop here."
>
> Two hundred mathematicians clapped and cheered in celebration. Even those who had anticipated the result grinned in disbelief (Singh, 1998, p. 35).

There was no need for words, no need to explain why the presentation was significant. The mathematics spoke for itself, as does all good mathematics. Of course, that was not the final chapter in the story, as rigorous checking of Wiles' proof revealed a logical gap. However, Singh's description shows the aura in which mathematics, as pure logic, is held by mathematicians.

4.2.3 Appel and Haken's proof of the four colour problem

The above discussion may give the impression that standards of rigour and proof in mathematics are immutable, fixed since Greek times. This is most certainly not the case, as Kleiner's (1991) examination of the history and evolution of mathematical proof clearly shows. Kleiner states:

> Looking back at 2500 years of the evolution of the notions of rigor and proof, we note that not only have the standards of rigor changed, but so have the mathematical tools used to establish rigor. Thus in ancient Greece, a theorem was not properly established until it was geometrized. In the Middle Ages and the Renaissance, geometry continued to be the final arbiter of mathematical rigor (even in algebra). Mathematicians' intuition of space appeared, presumably, more trustworthy than their insight into number—a continuing legacy of the consequences of the

> "crisis of incommensurability" in ancient Greece. The calculus of the 17th and especially the 18th century was no longer easily justifiable in geometric terms, and algebra became the major tool of justification (such as there was). There was a mix of the algebraic and geometric in Cauchy's work. With Weierstrass and Dedekind in the latter part of the 19th century, arithmetic rather than geometry or algebra had become the language of rigorous mathematics. To Plato, God ever geometrized, while to Jacobi, He ever arithmetized. The logical supremacy of arithmetic, however, was not lasting. In the 1880s Dedekind and Frege undertook a reconstruction of arithmetic based on ideas from set theory and logic (p. 301).

Kleiner traces the rise of the axiomatic method and the debate between formalists, logicists and intuitionists in the early twentieth century, the loss of certainty in axiomatic systems engendered by Gödel's Incompleteness Theorem in 1931, and the rethinking of the nature of proof precipitated by the computer-assisted proof of the four colour theorem in 1976 by Appel and Haken. The four colour theorem states that it is possible, using no more than four distinct colours, to colour any planar map in such a way that no two countries sharing a common border are identically coloured. Like Fermat's Last Theorem it is easy to state and easy to find examples of maps that require four colours. However, all attempts to construct a map that require five colours failed, and could quickly be reduced to four. Again the proof was elusive for at least 100 years.

Appel and Haken's proof used a computer to verify that each of some 1500 distinct configurations of map could be coloured using at most four distinct colours. Although the fundamental principles of the proof were publicly articulated, the proof contained thousands of pages of computer programs that were not published or open to scrutiny. Thus many mathematicians argued that it was incomplete. Others argued that computer hardware and software are subject to error and hence introduce a quasi-empirical element into the proof,

while others asserted that the proof failed to reveal the essence of the problem (Kleiner, 1991). Kleiner concludes that proof "is a social process and is based on the confidence of the mathematical community in the social systems that it has established for purposes of validation" (p. 311).

We thus appear to have two somewhat contradictory points of view. On the one hand Kleiner's analysis lends credence to mathematics as having a knower mode of legitimation in which proof is dependent on community confidence in social systems. On the other the rigorous checking of Wiles' proof of Fermat's Last theorem suggests that what matters above all else is knowledge. Perhaps, in the end, the distinctions are not as clear-cut as Maton (2000) would have us accept. As in Chapter 3 I suggest that these are perhaps better regarded as differences in inside/outside views of mathematics: looked at externally from afar, mathematical proof appears to rest on logic and human dimensions are hidden; looked at internally or near-at-hand, mathematical proof is a social process.

Nevertheless the drive for rigour remains one of the principal motivations of mathematicians. The logic of proof, however one conceives it, is the "profound intelligibility of mathematics" allowing us to discover "deep-lying reasons" and "common ideas" of theories (Bourbaki, 1950, p.223). It is what reveals the internal consistency of mathematics, and moves it beyond a collection of more-or-less isolated skills as so often presented to and perceived by students in school. It follows that a key goal of school mathematics is that of coming to understand mathematics as a connected sub-system of human thought. In the following section I discuss three pieces of influential research that show how the internal connectedness of mathematics has been a feature of research into teacher knowledge and classroom practice for more than 40 years.

4.3 Insights from research

4.3.1 Instrumental and relational understanding

The difference between mathematics as isolated skills and mathematics as a connected system is perhaps most clearly articulated in the mathematics education research literature by Skemp (1976). In his seminal article on the nature of understanding in mathematics, Skemp highlights the distinction between *instrumental* understanding, or knowing how, and *relational* understanding, knowing why, summarised in Table 4.3. This distinction profoundly influenced my own teaching (see Chapter 1 and the lesson play at the start of Part 2) in that it highlights the potential for a lack of shared beliefs in the classroom. As Skemp observes, it is not only the nature of understanding that is the subject of conflicting beliefs, but also the nature of mathematics itself. Using the example of finding oneself in an unfamiliar town, Skemp draws the analogy between an instrumental approach to mathematics and being given directions from one place to another. These directions may enable one to successfully reach the desired destination, but give no sense of how one might do so if there are obstacles or unexpected events. On the other hand he compares a relational approach to understanding with using a map or personally wandering the environment to discover how one landmark relates to another. This then enables one to successfully reach the destination from a number of different starting points or using a number of different routes, even if there are unexpected obstacles.

Skemp highlights the potential problems arising from mismatches between teachers' and students' views of mathematics and understanding. He suggests that where teachers attempt to teach relationally in an environment in which students wish to understand instrumentally, students will simply not want to know what the teacher is trying to explain. They will seek a set of rules that provides quick answers to questions they may encounter in a test of knowledge, and resist attempts to provide deeper understanding. "I don't want

to know why it works, just tell me how to do it to get the right answer", is a common cry of the student seeking instrumental understanding.

Of greater danger, according to Skemp, is the mismatch between teachers whose goal is to teach instrumentally and students who wish to understand relationally. Such students are likely to experience frustration at the teacher's insistence on a particular way of performing a procedure, when they have already used their understanding of underlying principles to develop their own approaches[27].

Table 4.3: A comparison of instrumental and relational understanding (adapted from Skemp, 1976)

Instrumental understanding	***Relational understanding***
Quick to learn, but easy to forget	Slow to learn, but long-term retention
Mathematics as isolated skills	Mathematics as a schema
Static, unable to generate new knowledge	Generative, builds on itself
Immediate rewards	Long-term rewards
Applicable in local contexts	Adaptable to new tasks
Requires external goal	A goal in itself

Just as the *Australian Curriculum: Mathematics* highlights both fluency and understanding as desirable mathematical proficiencies, so Skemp (1976) recognises that both instrumental and relational understanding are desirable goals of school mathematics. However, he concludes that "the two kinds of knowledge are so different that I think that there is a strong case for regarding them as different kinds of mathematics. If this distinction is accepted, then the word 'mathematics' is for many children indeed a false friend, as they find to

[27] This was the case with my own son who developed efficient mental strategies for subtracting two-digit numbers, but was told by his teacher that he must use a vertical algorithm.

their cost" (p. 26). He asserts that, despite situational pressures such as examination pressures, over-burdened syllabi, assessment difficulties and the psychological difficulty of restructuring existing schemas, for a teacher only relational understanding will suffice.

4.3.2 Connectionist orientations to teaching

Skemp's assertion is echoed in recent research by Askew, Brown, Rhodes, Wiliam and Johnson (1997) who identified connectionist teachers as more effective than those adopting a discovery or transmission orientation. Askew et al. gathered evidence relating to the beliefs, knowledge and actions of 90 teachers and related this to the achievement data of over 2000 pupils. From their sample they selected 18 case-study teachers who were shown to be effective in promoting student achievement gains. These 18 teachers were observed over two terms, and were interviewed with regard to their beliefs, practices, professional development activities and pedagogical and mathematical subject knowledge. The researchers concluded that what distinguishes highly effective teachers of numeracy from others is "a particular set of coherent beliefs and understandings which underpin their teaching of numeracy" (p. 1). In particular these beliefs included the need for teachers to have "a rich network of connections between different mathematical ideas" and "being able to select and use strategies which are both effective and efficient"[28] (p. 1).

Askew et al. found that teachers who gave priority to students acquiring a collection of standard arithmetical methods (perhaps the consequence of an

[28] Askew et al.'s use of the terms effectiveness and efficiency relate to the learning of specific concepts, rather than to the broader goals of schooling. Thus there is no inconsistency in valuing effective and efficient methods for solving mathematical problems while at the same time critiquing the drive for systemic effectiveness and efficiency that is characteristic of the technological enframing of mathematics discussed in Chapter 2.

extreme and naïve absolutist philosophy) or those who gave priority to the use of practical equipment (perhaps the consequence of an extreme relativist philosophy) produced lower gains in students than those who used teaching approaches that involved discussion of concepts and images to enable students to develop a rich network of mathematical concepts. They termed these three orientations to teaching *transmission*, *discovery* and *connectionist* respectively. Some of the differences between these orientations are shown in Table 4.4.

Table 4.4: Teachers' orientations to mathematics (adapted from Askew et al., 1997)

Orientation	***Transmission***	***Discovery***	***Connectionist***
Beliefs about solution methods	Teacher demonstrates a way that works, most often a standard algorithm	All methods equally acceptable, students' own strategies are the most important	Effectiveness and efficiency are equally important, particularly mental methods
Organisation of learning activities	Mathematical ideas presented in discrete packages	Students learn through a set of separate activities	Links between concepts and activities explicit
Application of mathematics	Best approached through decoding word problems to identify processes involved	Best approached through the use of practical apparatus	Best approached through challenges that require reasoning and justification
Students' capacity to learn	Ability determines success	Readiness determines success	Effective learning is the key

Beliefs about learning and teaching	Teaching over learning	Learning over teaching	Learning and teaching are mutually constructed through classroom dialogue
How students learn	Teacher reduces to simple questions, students follow instructions	Through individual activity based on actions on objects	Through complex and challenging problems and struggling to overcome difficulties

Similar conclusions were reached by Ma (1999) in her seminal research into the nature of teachers' understanding of mathematics in the USA and China. Ma describes Chinese teachers as having a "profound understanding of fundamental mathematics" (p. xxiii) and argues that to teach mathematics well teachers need to know the subject deeply. They need to know how one idea connects to another; they need to know how to approach mathematics and represent mathematical ideas in a multitude of ways; they need to know the fundamental recurring ideas of mathematics; and they need to know how mathematical knowledge builds on prior knowledge and leads to new knowledge. Ma's research was significant in that there was unusual agreement in the response of both mathematicians and mathematics education researchers, and generative in that it has led to a multitude of research projects relating to teachers' knowledge of mathematics.

Teachers' orientations to mathematics are thus critical in enabling students to appreciate the connectedness of mathematical ideas. Knowledge of skills or techniques, no matter how advanced, is not in itself enough. What matters is

the extent to which teachers, and hence students, see mathematics as a network of connected ideas, linked through logical reasoning. I suggest that these ideas change and develop primarily through the solution of problems that highlight mathematical connectedness. Such problems are important not because they are unsolved, but rather because they highlight key mathematical features such as generality, structure and proof.

4.3.3 Essentially mathematical tasks

Every field of knowledge has its own distinguishing features. One could argue that experimental evidence gathered through experiment is the lifeblood of school science, while creating and understanding literature is the lifeblood of school English. I suggest that engaging in mathematical tasks is the lifeblood of school mathematics. Watson and Mason (2007) describe such tasks in terms of their "Essential Mathematical-nesses", that is, the extent to which they promote in students the capacity to specialise and generalise, to conjecture and convince, and to classify and characterise. Such tasks allow students to move their attention between discerning details, recognising relationships, perceiving properties, and reasoning on the basis of those properties. Like the connectionist teachers in Askew et al.'s (1997) study or the Chinese teachers who had a profound understanding of fundamental mathematics in Ma's (1999) study, such tasks enable students to see structure and connectedness.

These types of tasks are not, however, typical of many school mathematics textbooks or classrooms. As described in Chapter 3, Vincent and Stacey (2008) found that only a very small proportion of tasks in textbooks written for year 8 students went beyond reproduction of standard techniques. Their study built on that conducted by Hollingsworth, Lokan and McCrae (2003), who analysed the tasks presented to students in 85 year 8 mathematics lessons in Australia, finding that fewer than one lesson in five contained tasks that focused on making connections. Even in those lessons that did contain such tasks, the

activity did not realise the potential of the tasks, as the teacher frequently presented the solution as a statement, rather than allowing students space to be hesitant and searching. In such a scenario, mathematics is seen as a set of disconnected results, validated by the teacher and textbook.

4.4 Conclusion

In this chapter I have argued the case for considering mathematical connectedness as one of the cornerstones of *Slow Maths*. The discipline of mathematics itself is internally consistent representing, at least from the outside, the archetypal field with a knowledge mode of legitimation. Despite the evolving views of what constitutes valid mathematical proof, striving for logical coherence and completeness remains a prime motivation for mathematicians. Classroom research similarly shows that effective teaching of mathematics relies on teachers having a deep and connected knowledge of mathematics that, in turn, is communicated to students.

Developing such an appreciation of mathematics necessarily takes time. It is slow work, requiring immersion into deep and sometimes difficult mathematics rather than superficial learning of a set of skills and techniques. It is not that these skills and techniques are not important: however I contend that their ultimate purpose is to give students the tools with which to engage in deeper mathematics. (Next sentence omitted)

I began the chapter with a quotation from G. H. Hardy's *A Mathematician's Apology*, in which he discusses the nature of mathematical ideas. He makes a powerful case for worthwhile problems being connected to significant mathematical ideas.

> The "seriousness" of a mathematical theorem lies, not in its practical consequences, which are usually negligible, but in the significance of the mathematical ideas which it connects. We may say, roughly, that a mathematical idea is "significant" if it can be connected, in a natural and

> illuminating way, with a large complex of other mathematical ideas. Thus a serious mathematical theorem, a theorem which connects significant ideas, is likely to lead to important advances in mathematics itself and even in other sciences. (G. H. Hardy, 1940, p. 16)

Hardy's book is both profound and moving. While his dogged insistence on the purity of mathematics and the absolute nature of proof is, perhaps, a product of its time and has been challenged by more recent thinking, he makes a cogent case for mathematics as a discipline worthy of study simply because it adds to human knowledge. He wrote at a time when he was nearing the end of his life, and when he considered that his days as a productive mathematician were long behind him. He concludes his book:

> The case for my life, then, or for that of anyone else who has been a mathematician in the same sense in which I have been one, is this: that I have added something to knowledge, and helped others to add more; and that these somethings have a value that differs in degree only, and not in kind, from that of the creations of the great mathematicians, or of any of the other artists, great or small, who have left some kind of memorial behind them. (G. H. Hardy, 1940, p. 151)

Hardy captures the essence of becoming a mathematician and what it means to add to human knowledge. I contend that *Slow Maths* is the classroom equivalent of Hardy's description of the case for his life. The mathematical knowledge acquired and developed by students in *Slow Maths* differs only in degree from those "somethings" that have been added to knowledge by mathematicians such as Hardy. Hardy also captures something of the richness and culture of mathematics and of the great mathematicians who have come before. I discuss this culture in Chapter 5, arguing that cultural connectedness is a second cornerstone of *Slow Maths*.

CHAPTER 5: CULTURAL CONNECTEDNESS—INSIGHTS FROM MATHEMATICS, MATHEMATICIANS AND MATHEMATICS EDUCATION RESEARCH

> How does it happen there are people who do not understand mathematics? If mathematics invokes only the rules of logic, such as are accepted by all normal minds; if its evidence is based on principles common to all men, and that none could deny without being mad, how does it come about that so many persons are here refractory? (Poincaré, 1908/2000, p. 85)

Synopsis

Insights from the discipline: The rapid growth in the number of sub-branches of mathematics shows that it is a dynamic, evolving discipline. Its historical development has both shaped and is shaped by culture.

Insights from practice: Generally speaking mathematicians do not work alone. They see themselves as contributing to the rich heritage of knowledge developed by mathematicians throughout history.

Insights from research: Contemporary research into effective classroom practice emphasises both social and cognitive aspects, highlighting the cultural norms in a community of practice.

Conclusion: Cultural connectedness locates mathematics as both a culture in its own right and as part of broader society.

The mathematician and philosopher Bertrand Russell once wrote that "[m]athematics, rightly viewed, possesses not only truth, but supreme beauty—a beauty cold and austere" (Russell, 1959, p. 60). While coldness and austerity may have an intrinsic beauty to mathematicians such as Russell, for many school students these qualities render mathematics inaccessible and remote.

> Math does make me think of a stainless steel wall—hard, cold, smooth, offering no handhold, all it does is glint back at me. Edge up to it, put

> your nose against it, it doesn't give anything back, you can't put a dent in it, it doesn't take your shape, it doesn't have any smell, all it does is make your nose cold. I like the shine of it—it does look smart, intelligent in an icy way. But I resent its cold impenetrability, its supercilious glare. (Buerk, 1982, p. 19)

For students such as the one whose journal contained this entry, mathematics is dehumanised, existing as part of a foreign and alien landscape.

Arguably textbooks do little to promote the human side of mathematics, other than what might be termed lip service to introduce a new topic. They give little sense of the history of mathematics or of the people who contributed to mathematics as we know it[29]. Perhaps unwittingly, then, they contribute to a view of mathematics as culture-free, disconnected from the lives and concerns of most members of society.

In this chapter I argue that cultural connectedness is a critical aspect of school mathematics. I take an inclusive view of culture, recognising that mathematics has its own distinctive culture, that mathematics has arisen in and contributes to different cultural contexts, and that school mathematics classrooms also represent a distinct culture. Connecting with the culture of mathematics and with mathematics as a cultural pursuit necessarily takes time and is central to what I term Sl*ow Maths.*

5.1 Insights from the discipline

Countless books and websites document the history of mathematics (e.g. Gullberg, 1997; Mankiewicz, 2000; O'Connor & Robertson, 2007), describe its contribution to the modern world (e.g. Kline, 1962, 1982), or highlight the

[29] Notable exceptions to this, written primarily for the non-mathematician and not used widely in schools, are *Mathematics: A Human Endeavour* (Jacobs, 1970) and *Mathematics From the Birth of Numbers* (Gullberg, 1997). Each of these locates mathematical concepts alongside their historical and cultural roots.

work of prominent mathematicians (e.g. E. T. Bell, 1965; Henrion, 1997). While it is beyond the scope of this thesis, or indeed beyond the capacity of any single work, to present a comprehensive historical review of the development of knowledge in mathematics, by any standards the growth in contemporary mathematics is staggering. "Intellectually, mathematics moves very quickly. Entire mathematical landscapes change and change again in amazing ways during a single career" (Thurston, 2006, p. 47).

5.1.1 The growth in contemporary mathematics

This changing landscape is highlighted by Ulam (1976) who estimates that some
200 000 new mathematical theorems are produced each year, and by the exponential growth in the number of mathematical journals. Table 5.1 shows the results of a search I conducted using the online database *Zentralblatt MATH* (2014) for journal articles produced during each 10-year period from 1810 to 2010, the first entry in the database being 1818. These results are shown graphically in Figure 5.1.

Table 5.1: Mathematics-related articles referenced during each decade in *Zentralblatt MATH*

Decade ending	**Articles**		**Decade ending**	**Articles**
1819	1		1919	29 380
1829	172		1929	38 667
1839	482		1939	96 149
1849	434		1949	43 284
1859	450		1959	87 706
1869	2080		1969	171 600

1879	12 533		1979	349 030
1889	17 964		1989	481 537
1899	24 091		1999	660 004
1909	31 073		2009	927 565

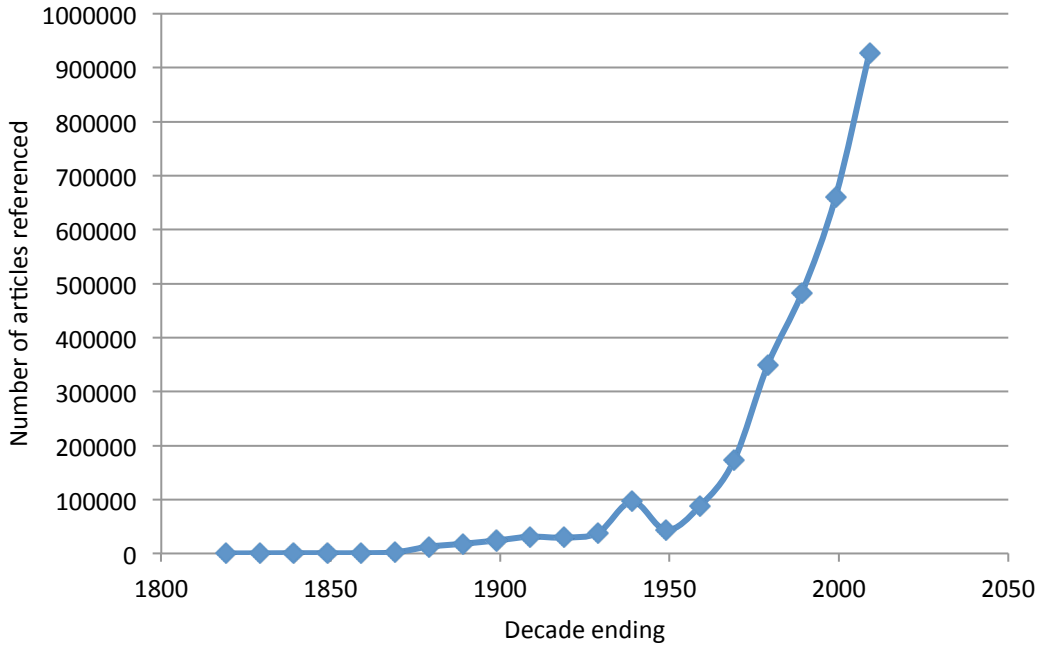


Figure 5.1: Graphical representation of number of mathematics-related articles referenced during each decade in *Zentralblatt MATH*

In the years 2010 to the time of writing, May 2014, 323 183 articles have been referenced.

The number of recognised branches and sub-branches of mathematics is increasing at a similar rate. In 1868 there were 12 subdivisions of the *Jahrbuch über die Fortschritte der Mathematik* with 38 subcategories; there are currently 63 subdivisions with approximately 3400 subcategories listed in the Mathematics Subject Classification (MSC) scheme developed by *Mathematical Reviews* and *Zentralblatt MATH* (Davis & Hersh, 1981). While it may seem

self-evident to say that mathematics is continually growing and evolving, it is far from the experience of students in school, whose view of mathematics is often one of a culture-free fixed body of knowledge.

This culture-free image of the development of mathematics has been characterised by Thurston, reflecting on his own career as a mathematician, as a definition-theorem-proof model.

> In caricature, the popular model holds that:
>
> D. mathematicians start from a few basic mathematical structures and a collection of axioms "given" about these structures, that
>
> T. there are various important questions to be answered about these structures that can be stated as formal mathematical propositions, and
>
> P. the task of the mathematician is to seek a deductive pathway from the axioms to the propositions or to their denials.
>
> We might call this the definition-theorem-proof (DTP) model of mathematics. (Thurston, 2006, p. 39)

Thurston argues that rather than striving to communicate logically defensible mathematical proofs in formal journals, mathematicians should spend more time in communicating mathematical ideas and ways of thinking, and in understanding different ways of thinking.

5.1.2 Ethnomathematics: challenging Eurocentric views of mathematics

Yet openness to diverse ways of thinking is not necessarily characteristic of school mathematics classrooms, and certainly not of school mathematics textbooks, where mathematics is presented almost exclusively as a product of Western culture. Such a view is challenged by recent work in ethnomathematics[30] (e.g. D'Ambrosio, 1997; Joseph, 2011) which provides

[30]The application of ideas from ethnomathematics in the context of a Western-world school is not without its critics (e.g. P. Dowling, 1998; Rowlands & Carson, 2002), who claim that many

numerous examples of how mathematical ideas arise in diverse cultures. While I do not argue that all ways of thinking mathematically are equal—the standards of rigour and proof expected in Western mathematics together with its advanced semiotic system give it a technical power far in excess of that of non-Western mathematics—I contend that ethnomathematics can add significantly to the humanising agenda advocated by Thurston and others.

At one level examples such as Japanese Temple Geometry Problems (Fukagawa & Rothman, 2008), the Kou-Ku Theorem (Swetz, 1977)[31], Egyptian fractions (Hurd, 1991) or the algebra of Al-Khwarizmi (Allaire & Bradley, 2001) provide history and culture-rich settings for the learning of traditional topics in the school mathematics curriculum. They show that mathematics was practised in diverse cultures, often using symbol systems that are dramatically different from those used today. They also show that the search for pattern and generality characteristic of mathematics, and the fascination with solving problems as an intellectual exercise, are universal.

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of the informal and context-bound mathematical ideas and ways of reasoning described in the ethnomathematics literature fail to meet the criteria of logic and rigour that are associated with academic mathematics. I agree with much of this critique, and hence do not advocate the radical abandonment of the content and reasoning typically found in school mathematics curricula. Rather the discussion in this section is intended to show how ideas from ethnomathematics can deepen and enrich "mainstream" school mathematics. Furthermore as my conception of *Slow Maths* is located in a Western cultural tradition I make no claims as to its universality.

[31] The Kou-Ku Theorem is a Chinese expression of what we now know as Pythagoras' Theorem, first communicated well before the time of Pythagoras. I have given examples of how Japanese temple geometry problems and the Kou-Ku Theorem might be used in a school mathematics classroom in a series of articles in the *Making of Mathematics* section of the *Australian Mathematics Teacher* in 2000 (Thornton, 2000).

At another level studying the mathematical ideas and modes of thinking of non-European cultures might provide new insights into how the mathematics curriculum is constructed in a mainstream context. One such example is the perception of position expressed by the Australian Indigenous people of Croker Island in the Northern Territory. Edmonds-Wathen (2011) describes how school mathematics curriculum documents reflect the sequence of acquisition of spatial language typical of English-speakers. "Children learn first the intrinsic frame of reference such as 'in front' and 'behind', then left and right, with north, south, east and west regarded as somewhat specialised and not part of everyday speech" (p. 220). On the other hand the language of cardinal directions is used from a very young age in many Indigenous languages such as Iwaidja, the most common language of Croker Island.

Edmonds-Wathen suggests that following the curriculum sequence derived from the language acquisition typical in English-speaking countries may not be the most appropriate for non-English speakers such as the children of Croker Island. I would go further and ask whether a curriculum sequence informed by other cultures and languages such as Iwaidja might not be beneficial for all students. Based on insights from the Mi'kmaw people of North America (Borden, 2013) where sharing is a more natural concept than acquisition and there are terms for division but not multiplication, a similar argument might be made for conceptualising multiplication as the inverse of division rather than vice versa as found in conventional school curriculum documents.

5.1.3 The culture of mathematics itself

Mathematics, then, is a universal pursuit. It is located within cultural contexts, evolving through history into what is now widely accepted as a universal language.

> In these days of conflict between ancient and modern studies, there must surely be something to be said for a study which did not begin with

> Pythagoras, and will not end with Einstein, but is the oldest and the youngest of all (G. H. Hardy, 1940, p. 9).

Mathematics, then, has its own culture, into which I argue it is the responsibility of teachers to begin to induct their students. In his speech to young researchers in mathematics at Cambridge University, renowned mathematician, Fields medallist and Abel Prize winner Sir Michael Atiyah looked back on 60 years of mathematics, describing "what views I have seen from the heights and what challenges lie ahead for the next generation" (Atiyah, 2010). He suggested there were many more mountain ranges to explore in mathematics, and stressed the cultural dimension of mathematics by projecting 28 slides, not one of which contained text. Rather each contained a photograph of a mathematician whom he had met or learned from. In effect he was saying, if not explicitly, "You are part of a long history and culture. Your work will add to the rich tapestry that constitutes the field of mathematics." This was also the implicit message I was giving to students in the lesson play introducing Part 2 of the thesis. By referring to Gauss initially as "young Carl" who, perhaps apocryphally, quickly summed the integers from 1 to 100, I was providing an opportunity for students to identify with someone who later became a prominent mathematician and physicist.

Atiyah concluded by giving seven pieces of advice to the young mathematicians: believe in yourself; beware of following bandwagons; aim to understand; be curious; explore the landscape; consider radical ideas; and (somewhat tongue in cheek) do not always follow advice. Although he did not use the term, Atiyah could equally well have said "slow down".

5.2 Insights from practice

Atiyah hinted at what motivates mathematicians: understanding, curiosity, exploration and ideas. In some cases it is the solution of an elusive problem; in others it is making a contribution to society by solving a socially significant

problem. Yet in almost all cases it appears that the overwhelming motivation for mathematicians to do mathematics is to add to knowledge; their motivation is "intellectual curiosity, desire to know the truth" (G. H. Hardy, 1940, p. 11). Of course, they care about whether the knowledge will make a difference, and they care about whether it adds to their reputation as scholars; but above all it is the value of knowledge itself that seems to matter[32].

5.2.1 Mathematics as a collaborative endeavour: the Polymath project

This is the culture of the discipline of mathematics—a collective intellectual curiosity, fuelled by collaboration within the community of mathematicians. Often this collaboration is formal through conferences and journals, often it is informal over afternoon tea, which is still described at Princeton as "the informal center of life in the department" (Department of Mathematics Princeton University, 2014, para. 2). As Thurston (2006) writes:

> We are inspired by other people, we seek appreciation by other people, and we like to help other people solve their mathematical problems. What we enjoy changes in response to other people. Social interaction occurs through face-to-face meetings. It also occurs through written and electronic correspondence, preprints, and journal articles…most mathematicians don't like to be lonely, and they have trouble staying excited about a subject, even if they are personally making progress, unless they have colleagues who share their excitement (p. 48).

[32] In an era when research funding is closely tied to so-called national priorities, it may well be time to reassert the value and importance of fundamental research such as that carried out in pure mathematics. Arguably the diminution of the value of pure research is one of the factors leading to the decline in the number of students studying the hard sciences in senior secondary schools and universities, and almost certainly a major factor in the well-documented decline in university mathematics departments (Gavin Brown, 2009).

In current times collaboration can also be stimulated via the Internet[33], such as the Polymath project (Gowers, 2009). In starting the project Gowers' goal was to promote online collaboration between mathematicians in order to share knowledge and ideas in solving a difficult problem. He posed an initial problem and in a little over a month announced a solution to a special case that could be generalised to prove the full theorem. Gowers claimed that the advantages of such a project are the speed with which results can be obtained and, perhaps more importantly, the record of the mathematical process, showing how ideas are formed, changed and discarded. The most recent Polymath project, Polymath 9 focusing on developing a proof or refutation of the famous P = NP problem, was proposed on 24 October 2013.

Polymath 8 is of particular interest as, like Fermat's Last Theorem, it concerns a problem understandable, although not solvable, by students at school. The problem is an aspect of the conjecture that there are infinitely many primes that differ by two, such as 11 and 13. Euclid proved some 2000 years ago that there are infinitely many primes, a proof which is delightfully elegant, clever, and absolutely understandable to school students.[34] However, although the largest known pair of twin primes has around 200 000 digits, and there are 800 trillion pairs of twin primes smaller than 10^{18}, to date no one has been able to show that there are infinitely many.

On 14 May 2013 Yitang Zhang at the University of New Hampshire announced a proof that there are indeed infinitely many prime pairs. However, these are not pairs that differ by two, but rather pairs that are no more than 70

[33] The use of technology to support such collaboration is in stark contrast to the technological enframing discussed in Chapter 2. Paradoxically technology as a democratic collaborative tool may well be the catalyst that causes society to reconsider the ways of thinking characteristic of the technological enframing.

[34] It is interesting to note that Euclid's proof is not mentioned in the *Australian Curriculum: Mathematics*, nor indeed is Euclid himself.

million apart. Zhang's result was the first time anyone had been able to put a finite gap on gaps between prime numbers and prompted a flurry of activity around the world. By 30 May Scott Morrison from the Australian National University announced that he had reduced the gap to 59 470 640, and on 4 June Terry Tao of the University of California launched a Polymath project. By 27 July the gap was down to 4680[35]. Within the space of three months arguably more progress was made on the twin prime conjecture than had been made in the previous 2000 years (Nielsen, 2014).

The research into bounded gaps between primes provides a stunning example of how mathematics evolves in a social and cultural context. In this case the context was provided by the collaborative technologies of the World Wide Web, which stimulated rapid progress on a previously somewhat intractable problem. Such collaboration is a far cry from what traditionally happens in school mathematics![36]

5.3 Insights from research

Although there are debates about whether knowledge growth in mathematics is incremental, occurs through paradigm shifts and knowledge revolutions or makes prior knowledge obsolete (Corry, 1989; Crowe, 1975; Mehrtens, 1976), the rapid growth in mathematical knowledge outlined above is at least partly a

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[35] Remarkably on 19 November, James Maynard, a postdoctoral researcher at the University of Montreal, reduced the gap to 600 using a totally different technique. Combining Maynard's technique with that of the earlier project has now resulted in a lower bound of 246.

[36] I distinguish collaboration in the sense of knowledge building, as in the Polymath project, from collaborative learning as it is often described in school classrooms. I suggest that the overwhelming majority of examples of collaborative learning in classrooms involve little more than students discussing how to solve a problem, rather than genuine knowledge building. Some moves towards collaborative learning in the sense of knowledge building in the school context may be evident in the *Metafora Project* (2014), in which students debate mathematical ideas and build knowledge in a specially constructed online environment.

result of the generative nature of mathematics. One mathematical idea suggests another, which in turn stimulates several more. The generative nature of mathematical ideas is, ideally, mirrored in the growth of mathematical knowledge within individual students and in the classroom community, where one concept lays the foundation for several others enabling students to progress in the volume and depth of knowledge possessed. That is, we could conceive of school mathematics as rational transitions between mathematical practices (Kitcher, 1988). This is not to suggest that this growth is linear or predictable—there are periods of relative stagnation often followed by sudden insights that can lead to rapid growth—but just as the cumulative body of knowledge known as mathematics grows, in an ideal world each individual student's mathematical knowledge grows.

Yet the quote from Poincaré at the start of this chapter asks: "How does it happen that there are people who do not understand mathematics?" This is, of course, the ultimate question of mathematics education research, asked not only 100 years ago by Poincaré, but in countless mathematics education journals and conferences since. While it is tempting to suggest that if the activities in mathematics classrooms were to mirror those undertaken by professional mathematicians students would learn mathematics better, this is only part of the picture. As my friend and colleague Ann Watson (2004) writes:

> ...mathematics might be seen as a practice whose standards are expressed through the nature of the tasks with which mathematicians engage. However this is not a very useful way to see school mathematics, since the tasks which engage professional mathematicians involve extended exploration, creation of new structures, conjecturing and convincing, argumentation, modelling, but certainly no practice exercises! So to understand school mathematics as it is requires rejection of professional practice as a model and something else to be described in its place (p. 104).

Unfortunately this "something else" is not easily described, nor, despite the positivist assumptions of the production, cure or race metaphors of education described in Chapter 2, may it be possible or even desirable to attempt to describe it universally. As William James (1925) said "You make a great, a very great mistake, if you think that psychology, being the science of the mind's laws, is something from which you can deduce definite programmes and schemes and methods of instruction for immediate schoolroom use" (chapter 1, para. 6). Here I describe what I term cultural connectedness within the school mathematics classroom in the hope that it may go some way towards describing school classrooms that promote the goals and purposes of school mathematics that I describe as *Slow Maths*.

Although cultural connectedness in the school classroom requires recognising and respecting the cultural background of students, it is much more than this. It is also about recognising and respecting the culture of mathematics and building a culture within the mathematics classroom that promotes the development of students as members of the mathematical community. It is both personal and communal, recognising the needs, values and aspirations of students as individuals within the classroom community. It is also both cognitive and social, seeking to develop both the knowledge base of students and their appreciation of mathematics as a cultural pursuit. It has deep connections to research such as the situated nature of learning (Lave & Wenger, 1991), ethnomathematics and its place in the school curriculum (Powell & Frankenstein, 1997), questions of equity and power relations (Burton, 2003), and beliefs and attitudes to mathematics (Leder et al., 2002). However, as the focus of this thesis is the school mathematics classroom I restrict the discussion to the culture of the mathematics classroom.

5.3.1 Sociomathematical norms

The lesson play on handshakes and intersecting lines I used to introduce Part 2 of the thesis highlights many aspects of mathematics that I felt were worth emphasising to students. These included:

(1) Mathematics is active, not passive.
(2) Many problems can be done in more than one way.
(3) Some problems are impossible.
(4) Problems that are superficially different may have the same underlying structure.
(5) Elegance and efficiency are to be sought in mathematics.

Without having read the relevant research at the time, as a relatively young teacher I was establishing what McClain and Cobb (2001) describe as sociomathematical norms. These are the beliefs and practices that regulate the interactions in a mathematics classroom. They include general social and intellectual norms such as the importance of explanation, listening and questioning, as well as specifically mathematical questions such as what counts as an acceptable explanation, what counts as a mathematically different explanation, or what makes a solution easy, elegant or efficient. These norms are part of what might be more generally described as the classroom culture. Although, as Watson highlights in the quote above, school mathematics can never mirror the activity of professional mathematicians, these sociomathematical norms come very close to describing the norms that regulate the discipline of mathematics.

McClain and Cobb (2001) describe classrooms in which the emphasis is on structure rather than procedures, and in which the goal is understanding and meaning rather than that of finding and writing solutions to problems. In such classrooms the mathematical agenda is guided towards the generalised and integrated thinking that is typical of mathematics as a connected body of

knowledge. The role of the teacher is not only someone who imparts knowledge or facilitates students' acquisition of knowledge, but more importantly one of representing the mathematical community and promoting a set of shared beliefs and practices about mathematics.

5.3.2 The dance of agency

The handshakes and intersecting lines lesson play also illustrates the importance I placed, at an early stage of my career, on problem-solving and investigation as central to the process of doing mathematics. This was, perhaps, a reflection of numerous professional development sessions and involvement in curriculum planning, however it was also an aspect of a more general orientation towards mathematics as a problem-solving endeavour. Yet even at that stage I realised that problem-solving and investigation alone do not establish a culture of mathematics in the classroom. Without having the language to describe it, I was pointing students to what Boaler (2003, after Pickering, 1995[37]) terms the *agency of the discipline*.

Boaler's (2003) study describes three teachers with very different approaches to mathematical activity. One of the teachers presented students with structured problem-solving related to real life, but by breaking the problems into small activities the teacher ensured that students engaged only in methods that were prescribed by her. Agency resided primarily with the teacher. A second teacher valued more open-ended problems, but refrained from providing structure. Agency resided with the students, but their solution methods and presentations were idiosyncratic and not necessarily well justified. The third teacher engaged

[37] To illustrate his notion of the agency of the discipline Pickering uses the example of the extension of complex numbers to higher spatial dimensions by Irish mathematician Sir William Rowan Hamilton in 1843. Hamilton's concept of a quaternion was rejected for some time by the mathematical community as quaternions failed to obey the commutative law. Ultimately, however, their properties were able to be reconciled with ideas from vector and matrix algebra.

students in mathematical practices, but also deflected agency to the discipline by asking students to try other cases or to consider connections with other areas of mathematics. Like the effective teachers of mathematics (Askew et al., 1997) I described in Chapter 4, these three teachers had very different views of school mathematics: one saw mathematics as a set of standard procedures to be acquired, another as an active process where all ideas are equally valuable, and the third as an internally coherent web of concepts authenticated by a knowledge mode of legitimation.

Boaler (2003) terms the constant switching between human agency and the agency of the discipline a *dance of agency*. Although intuition is crucial in creating initial thoughts, ultimately the solution to a problem must be judged by agreed conventions of rigour and justification. The solution must also be consistent with other mathematical ideas. I suggest that a crucial role of the teacher, then, is to establish sociomathematical norms and choreograph the dance of agency so as to create a culture of school mathematics that values structure and conceptual understanding, and values students as active participants. As Lerman (2001) says "learning school mathematics is nothing more than initiation into the practices of school mathematics" (p. 107). (Next sentence omitted)

5.3.3 Developing the language of mathematics.

In a similar vein to Lerman's statement about initiation into the practices of school mathematics, Pimm (1987) suggests that learning mathematics is little more than learning to "speak mathematics". As Johnston-Wilder and Lee (2008) write, "[B]ecoming able to articulate mathematical ideas, concepts and reasoning has a profound effect on the way pupils see themselves. The better a pupil can use the discourse that a mathematician may use, the more they become a mathematician" (p. 56). Just as the development of language is characteristic of becoming connected in human society, so I argue that the

development of the specialised language structures of mathematics is characteristic of becoming connected with the discipline of mathematics. It is thus a central aspect of the mathematical culture of the classroom. In this section I give an extremely brief overview of research relating to the language of mathematics, emphasising the central role of language in the development of mathematics in the individual and community.

Gillian Brown (1982) introduces the notion of listener-oriented versus message-oriented talk, claiming that developing more message-oriented speech is a key goal of education. Message-oriented speech is structured, has more syntactic marking between sequences and more logical connectors than listener-oriented speech, which tends to be broken into disconnections chunks, relies on physical context and uses non-specific denotations. Listener-oriented speech exhibits many of the features of Bernstein's (1959) restricted code, which restricts access to more formal mathematical norms and values. Brown concludes that simply encouraging more pupil "talk" is not sufficient, but that encouraging message-oriented talk is critical in enculturating students into the discipline of mathematics. In the handshakes and intersecting lines lesson play I intentionally sought explanations from students, asked others to evaluate those explanations and, if necessary, to modify their own thinking.

In a similar vein Chapman (1997) describes three components of classroom talk: interaction and theme, intertextuality, and transformational shifts. While talk is part of the social practice of all school classrooms, the text-connecting practices of school mathematics are transformative language practices in that they provide students with the means to think and speak like a mathematician. Chapman presents a two-dimensional model of transformational shifts in mathematics learning: modality and form, and describes the shift towards "more mathematical language" as one in which students' discourse moves from low to high modality and from metaphor to metonym.

However, this capacity to move from intuitive, metaphorical language to more mathematical, metonymic language may not be readily accessible for all students. In her ethnography of two classrooms, Zevenbergen (1998) shows that middle class students have greater access to the linguistic forms that are valued in classrooms. She concludes that the student's linguistic habitus has a substantial impact on their capacity to make sense of the discursive practices of the mathematics classroom and hence on their subsequent capacity to gain access to legitimate mathematical knowledge along with the power and status associated with that knowledge. Coming into the mathematical culture of the classroom is thus strongly associated with being able to access and use the linguistic structures of mathematics.

5.4 Conclusion

As Zevenbergen's (1998) research shows there is an important equity dimension to cultural connectedness in school mathematics. While it is well established that students from high socioeconomic backgrounds achieve significantly higher results than those from lower socioeconomic backgrounds (e.g. Sirin, 2005; Thomson, De Bortoli, Nicholas, Hillman, & Buckley, 2010), the research reviewed above suggests that one aspect of this is the degree to which students can become connected with the culture of mathematics. In what might be called the traditional mathematics classroom, access to the specialised language and ways of thinking that characterise the discipline of mathematics is differentially conferred on those who already possess the cultural and discursive resources valued in schools. Taking seriously the dimension of cultural connectedness thus challenges practices in which only those students perceived to be of high ability have access to the language structures and generalised thinking practised by the community of mathematicians.

I began the chapter with a quote from the mathematician Henri Poincaré, who might also lay claim to being the father of mathematics education research, posing the question of why so many people fail to understand mathematics. At

least part of the answer to Poincaré's question lies in the degree to which the classroom culture enables students to connect with the culture of mathematics itself. Where the classroom is restricted by an emphasis on learnification (Biesta, 2009a) rather than enculturation, the structure and logic of mathematics remains hidden, and students, except for the rare few who see beyond the confines of school mathematics, view mathematics as little more than a set of disconnected facts and procedures to be memorised.

In contrast Poincaré writes later in his essay on *Mathematical Creation* that his memory does not fail him because it is:

> guided by the general march of the reasoning. A mathematical demonstration is not a simple juxtaposition of syllogisms, it is syllogisms placed in a certain order, and the order in which these elements are placed is much more important than the elements themselves. If I have the feeling, the intuition, so to speak, of this order, so as to perceive at a glance the reasoning as a whole, I need no longer fear lest I forget one of the elements, for each of them will take its allotted place in the array, and that without any effort of memory on my part. (Poincaré, 1908/2000, p. 87)

The lessons from mathematics, mathematicians and mathematics education research described in this chapter help to inform an approach to teaching and learning mathematics that emphasises order, logic and the holistic nature of a mathematical argument. They also help to situate mathematics as a cultural pursuit, arising in a variety of contexts but with its own unique culture. I term this being *culturally connected*. As Poincaré suggests, when students perceive reasoning as a whole they need no longer fear forgetting.

Like mathematical connectedness, cultural connectedness is slow work. It takes time to establish the norms and practices of mathematics within the school mathematics classroom. It takes time to promote collaborative knowledge-building rather than individual acquisition of knowledge. It takes time to

develop the message-oriented talk typical of academic mathematics. However, I suggest that only when attention is paid to these aspects of the mathematics classroom will students come to appreciate mathematics as both having its own culture and being intimately connected to the broader culture of society.

CHAPTER 6: CONTEXTUAL CONNECTEDNESS—INSIGHTS FROM MATHEMATICS, MATHEMATICIANS AND MATHEMATICS EDUCATION RESEARCH

> Philosophy is written in that great book which ever is before our eyes—I mean the universe—but we cannot understand it if we do not first learn the language and grasp the symbols in which it is written. The book is written in mathematical language, and the symbols are triangles, circles and other geometrical figures, without whose help it is impossible to comprehend a single word of it; without which one wanders in vain through a dark labyrinth. (attributed to Galileo Galilei, cited in Burtt, 2012, p. 57)

Synopsis

Insights from the discipline: Mathematics has both an unreasonable effectiveness in explaining the world and grows in response to real world problems. Many modern fields of mathematics have developed simultaneously with aspects of fields such as biology, physics and economics.

Insights from practice: The interdisciplinary nature of much current research attests to the power of mathematics in modelling complex phenomena. The work of many mathematicians has been stimulated by their interest in other fields of knowledge.

Insights from research: Some of the most effective and innovative approaches to teaching and learning in school mathematics have deep connections between mathematics and the world. Enabling students to engage with the power of mathematics is a social justice as well as a pedagogical imperative.

Conclusion: The reciprocal relationship between mathematics and context makes mathematics both more relevant and more understandable.

A school principal whose name has long since vanished from memory once remarked to me in passing, words to the effect that "all we have to do to encourage children to learn mathematics better is to make it real." This is a not uncommon statement, yet at the same time it is one that is profoundly flawed. What we think is real for children may not be real or relevant in their minds—indeed, reality often resides in the imagination. This chapter therefore challenges simplistic notions of relevance and reality in school mathematics, and sets out to show that mathematics and context are deeply connected, with one depending on the other.

As discussed in Chapter 4, unlike many social sciences mathematics is not primarily concerned with the contextual details of a specific instance. It seeks the essential structure underlying a situation, looking for patterns and generalities that make one instance like another. The transformation of these generalities into abstract mathematical formulation then makes it possible to seek solutions that can then be reapplied to a specific case. This process of abstraction, generalisation and recontextualisation lies at the heart of mathematical modelling. It is responsible for the "unreasonable effectiveness of mathematics in the natural sciences" (Wigner, 1960). Simplistic attempts to teach all mathematics through context, which in the process marginalise abstract reasoning, thereby miss the essence of mathematics and relegate it to little more than functional skills for surviving in society.

The chapter commences with three examples from the field of mathematics itself that show that progress in mathematics is often stimulated by observations in the world and that the abstraction inherent in mathematics gives it its unreasonable effectiveness in the world. I then present case studies of two mathematicians whose work has been intimately connected to the world and instrumental in the development of information science and modern computing. Within the domain of school mathematics I discuss *Realistic Mathematics Education* from the Netherlands (Treffers & Goffree, 1985) as an

example of an approach to school mathematics that adopts a powerful, sophisticated and generative view of the connection between mathematics and context, and conclude with a discussion of the social justice implications of a low level approach to real world applications of mathematics.

6.1 Insights from the discipline

The rapid growth in the number of sub-disciplines of mathematics listed in the repository of mathematical knowledge *Zentralblatt Math* (2014) was highlighted in the previous chapter. Of the 63 sub-disciplines currently listed at least 19, such as fluid mechanics and information and communication are overtly interdisciplinary, while most others contain sub-categories that relate to applications of mathematics. For example, the classification Group Theory and Generalizations, whose researchers are normally located in departments of Pure Mathematics, explicitly mentions applications to physics and crystallography. Clearly while mathematics continues to be an important discipline in its own right, its scope has expanded as new problems have emerged in the natural and social sciences. While many examples could be discussed, three that are contemporary and accessible to students in school mathematics are chaos theory, knot theory and operations research.

6.1.1 Chaos Theory

Chaos theory was first suggested by Poincaré in the 1880s when studying the interaction of three bodies. Early work in chaos theory was continued by mathematicians such as Hadamard, stimulated by the desire to analyse the motion of particles. However, until the advent of the computer, the early ideas of Poincaré and Hadamard received relatively little attention (Ghys, 2012). In 1961 Edward Lorenz first noticed some anomalies in a computer simulation of weather patterns. When he repeated a simulation from the middle he noticed that the weather patterns the machine began to predict were completely different from those predicted before. The difference resulted from a very small

change in the input due to rounding a value accurate to six decimal places to three decimal places. Thus small differences in initial conditions led to wildly varying results.

> I started the computer again and went out for a cup of coffee. When I returned about an hour later, after the computer had generated about two months of data, I found that the new solution did not agree with the original one…I realized that if the real atmosphere behaved in the same manner as the model, long-range weather prediction would be impossible, since most real weather elements were certainly not measured accurately to three decimal places. (Lorenz in WMO Bulletin, 1996, p. 119)

This has since become known as the Butterfly Effect, immortalised in books and movies such as *Sliding Doors*, or indeed *The Butterfly Effect* itself.

However, the popular rhetoric that the world is inherently unpredictable is a misrepresentation of the mathematics (Ghys, 2012). Far from suggesting that we can never know what is likely to happen or that tiny changes in input are causal effects in catastrophic changes—the well-known suggestion that a butterfly flapping its wings in Paris causes a hurricane in Japan—chaos theory actually affirms that in the long run mathematics can help to understand and predict many real-world phenomena. The applications of chaos theory now extend across pursuits such as computer animation, economic forecasting and population dynamics. However, despite its rapid development, wealth of contemporary applications, the ready understandability of the logistic growth equation and the capacity to conduct rapid computer-generated simulations, chaos theory is not mentioned in the *Australian Curriculum: Mathematics*.

6.1.2 Knot Theory

Similarly, despite being a relatively simple and tangible example of topology, knot theory is absent from school curriculum documents in the Western world,

and if included at all in the school context is relegated to a mathematical recreation. Knot Theory developed in response to the need to categorise and make sense of Lord Kelvin's theory that atoms were knots in the ether, and attempted to categorise and classify different types of crossings. The demise of Kelvin's theory of matter led to a similar demise in interest in knot theory, but that was rekindled with the advent of modern topology (Adams, 2004).

This interest led to the development of mathematically defined methods for determining the equivalence of knots using Rademeister moves, and sophisticated notations to distinguish different knots such as the Alexander and Jones polynomials, the Dowker notation and Conway's tangles[38] (Adams, 2004). Knot theory thus became a respected field of study in its own right with the first issue of the *Journal of Knot Theory and its Applications* in 1992. The mathematics of knot theory has since become indispensable in studying the chirality of DNA and is increasingly significant in the construction of quantum computers.

6.1.3 Operations Research

One of the few topics from contemporary mathematics that has found its way into the school curriculum is Operations Research. Yet this is often located in non-specialist courses reserved for those students not bound for tertiary study in mathematics, such as the General Mathematics subject in the draft senior secondary *Australian Curriculum: Mathematics*. I argue that it is hence marginalised as somehow not quite worthy of serious study in schools. This is both a tragedy and an injustice.

[38] I vividly remember participating in a demonstration of the connection between tangles and fractions given by John Conway in 2005, and have since used the ideas with students in years 7, 8 and 9.

Operations Research as a field of study was stimulated by the need to make strategic decisions in the military. It resulted in both significant reductions in casualties among allied forces and significantly more effective strikes against opposition forces. The need for a systematic approach to operational problems led rapidly to the development of techniques such as critical path analysis, network optimisation and linear programming that have widespread application in business, logistics and government (Garfunkel & Malkevitch, 1994).

The above are but three examples of areas of contemporary mathematics that are intimately and deeply connected to the real world. Each arose from the need to make sense of or solve real world problems, each developed into a field of mathematics in its own right, and each has widespread applications in modern society. Yet each commands at most passing attention in a school mathematics curriculum dominated by skills and techniques several hundred years old. It is not that these skills and techniques are no longer important or are irrelevant, but if we are to reassert the connection between mathematics and the world I argue that we need to inject for all students more topics that are part of the fabric of contemporary mathematics.

Connections between school mathematics and the real world have even found their way into parliamentary debates. In the British House of Commons in 2003 the question of why quadratic equations should be included in the school curriculum was asked. This stimulated the publication of papers in the student mathematics magazine *Plus* (Budd & Sangwin, 2004) relating to potential applications of the quadratic equation[39]. The papers traced the history of the quadratic equation from the ancient Babylonians through Greek mathematics including the Golden Ratio and conic sections through to Copernicus, Galileo

[39] The record of this debate can be found in Hansard, United Kingdom House of Commons, 26 June 2003, Columns 1259-1269, 2003.

and Newton's laws of motion. They described how quadratic equations can be applied to the motion of a pendulum, airflow, population modelling and indirectly to the development of mobile phone technology. This merely scratches the surface of the potential applications of quadratic equations. Yet not one of these applications rates a mention in the *Australian Curriculum: Mathematics*, which requires students to "solve simple quadratic equations using a range of strategies". In the elaboration of this content description the Curriculum simply suggests "using a variety of techniques to solve quadratic *equations*, including grouping, completing the square, the quadratic formula and choosing two integers with the required product and sum" (ACARA, 2013, ACMNA241).

6.2 Insights from practice

G.H Hardy (1940) famously wrote in *A Mathematician's Apology*:

> There is one comforting conclusion which is easy for a real mathematician. Real mathematics has no effects on war. No one has yet discovered any warlike purpose to be served by the theory of numbers or relativity, and it seems very unlikely that anyone will do so for many years (p. 42).

Indeed, Hardy is reputed to have been totally unconcerned about any potential applications of the mathematics he was developing, although it is more likely that as a pacifist the above statement was an expression of his stance opposing the use of mathematics for military reasons. Little did he know, of course, that arguably the most pure branch of mathematics, number theory, would find a military use in the development of RSA codes in cryptography (Singh, 1999).

Of course, not all mathematicians are of a Hardyian pure mathematics persuasion. To illustrate this I relate the stories of two prominent mathematicians, one story well-known because of its popularisation through the telling of the history of the Enigma machine, and one less well-known but

perhaps more significant in pointing out the reciprocal relationship between mathematics and the real world. Both of these examples illustrate the power of mathematics in solving important real world problems, but both also show how consideration of real world problems contributes to the development of new areas of mathematics.

6.2.1 Alan Turing

The story of Alan Turing, frequently told in books and documentaries related to the breaking of the German Enigma code at Bletchley Park, is one such case. While Turing's story contains tragic social elements such as of homophobic persecution, depression and ultimately suicide,[40] it also highlights how a focus on the pressing real world issue of defeating the German forces in World War II stimulated the development of the modern-day computer. Turing's early work was related to Gödel's results on the limits of proof and computation, during which he developed the idea of the hypothetical device now known as the Turing machine. At this stage there was no clear link between Turing's mathematical work and the physical world—his focus was on computability within a mathematical system (Siegfried, 2012b).

At the same time Turing studied cryptology and began to build an electrical binary multiplier. This work was, of course, instrumental in enabling the construction of a cryptanalytical device that was used in conjunction with a captured German Enigma machine to allow the Allied forces to gain inside knowledge of enemy plans. The code-breaking work at Bletchley Park, in

[40] I suggest that if we are to assert cultural aspects of the development of mathematics, mathematicians need to be portrayed as human beings living in society. Thus these elements, tragic as they are, form an important part of the story and, perhaps tangentially, promote mathematics as a human endeavour.

which Turing's mathematical and statistical work was central, has been immortalised in print and film.[41]

> Science is infused with Turing's information-processing intuitions: computer science is not merely a branch of the scientific enterprise—it's at the heart of the enterprise. Modern science reflects Turing's vision. "He was," wrote Hodges [Turing's biographer] "the Galileo of a new science" (Siegfried, 2012a, p. 86).

6.2.2 Norbert Wiener

In his autobiography *I Am A Mathematician* Norbert Wiener (1956) describes how his career was stimulated by his capacity to cross the boundaries of mathematics and physics, leading to the development of new fields of knowledge, particularly cybernetics, a term coined by Wiener himself. On the occasion of his award of the National Medal of Science, the citation read by President Johnson stated "…for marvellously versatile contributions, profoundly original, ranging within pure and applied mathematics, and penetrating boldly into the engineering and biological sciences" (Rosenblith & Wiesner, 1966, p. 33). When Wiener arrived at the Massachusetts Institute of Technology the mathematics department was largely a service department to cater for the needs of engineering students. However, rather than being satisfied to deliver mathematics in isolation, Wiener took a keen interest in the problems of engineering. His work ranged from examining Brownian motion, spectral analysis and generalisations of Fourier series. Like Turing he was instrumental in the beginnings of the electronic computer, researching techniques for the separation of signal from noise. Like Turing, Wiener was

[41] I have a personal connection to Bletchley Park as my mother was employed during World War II to listen to and transcribe Morse code intercepted from German communications. She continues to take a keen interest in the history of Bletchley Park and the life of Alan Turing.

influential in the war effort, designing fire control apparatus for anti-aircraft guns.

Ultimately Wiener is credited with the development of the modern field of cybernetics, which has far-reaching implications for areas such as artificial intelligence and brain research. In his autobiography Wiener describes the interplay between the mathematics of analysis and the communication sciences and, despite his close association with Russell and Hardy, describes mathematicians who have no real contact with physics as having a "thin view of mathematics" (Wiener, 1956, p. 359).

In the epilogue to his autobiography Wiener (1956) also laments the restricted academic freedom of the current managerial era, dominated by what I described in Chapter 2 as education as production.

> I am lucky to have been born and grown up before the First World War, at a period when the vigor and elan of international scholarship had not yet been swamped by forty years of catastrophes. I am particularly lucky that it has not been necessary for me to remain for any considerable period a cog in a modern scientific factory, doing what I was told, accepting the problems given me by my superiors, and holding my own brain in commendam as a medieval vassal held his fiefs. If I had been born into this latter day feudal system of the intellect, it is my opinion that I would have amounted to little. From the bottom of my heart I pity the present generation of scientists, many of whom, whether they wish it or not, are doomed by the "spirit of the age" to be intellectual lackeys and clock punchers (p. 360).

6.3 Insights from research

In the introduction to this Chapter I related the story of a school principal who said, in my opinion naïvely, that teaching mathematics in context would solve all the problems that children experienced in understanding mathematics. As the above discussion highlights, the connections between mathematics and the

world are deep and generative. Creating pseudo-contexts (Beswick, 2011) in what might be described as typical textbook word problems does little to make mathematics relevant or to stimulate deeper mathematical understanding for students in the early years of high school. Even in senior secondary level courses that focus on applications of mathematics, textbooks (e.g. Bruce, Humphries, Haese, Haese, & Haese, 2012) are dominated by an approach in which students are presented with a real-life situation, given the mathematical tools or formulas to solve problems in the area, and then asked to do questions that are purely mathematical.

Responding to this perceived emphasis, in his blog "How should mathematics be taught to non-mathematicians?" Gowers (2012) suggests some 60 questions that could form the focus of mathematics courses with a genuine emphasis on mathematical modelling. Mathematical modelling has been described as "learning to do the analysis that guides understanding and sensible decisions" (Burkhardt, 2006, p. 180), as a way of connecting concepts, procedures and attitudes in the school mathematics curriculum (Garcia, Pérez, Higueras, & Casabó, 2006), as a way of seeing the world critically (Barbosa, 2006) and even as a "way of life" (Lamon, Parker, & Houston, 2003). Among the problems suggested by Gowers are how one might go about determining changes in average global temperature, how speed cameras work, and devising a method that guarantees finding your way out of a maze. Such problems are often ill-specified. They involve students in carefully analysing the real world situation, deciding what information matters and what can be ignored, and what assumptions or approximations must be made. They involve developing a mathematical formulation of the problem that can be investigated and solved using established tools and techniques, reapplying the solution back into the real world context and evaluating the extent to which the problem was solved.

6.3.1 Realistic Mathematics Education (RME)

The deep and generative connections within mathematics and between mathematics and the world in mathematical modelling are captured in Freudenthal's (1973) conception of *mathematisation*. Freudenthal distinguishes between *horizontal mathematisation*, which makes a problem from another field accessible to mathematical formulation such as in the mathematical modelling process, and *vertical mathematisation*, which refers to connections within mathematics itself, providing techniques for solving mathematical problems. These processes underpin the *Realistic Mathematics Education* (RME) approach to school mathematics in the Netherlands (Treffers & Goffree, 1985), illustrated in Figure 6.1.

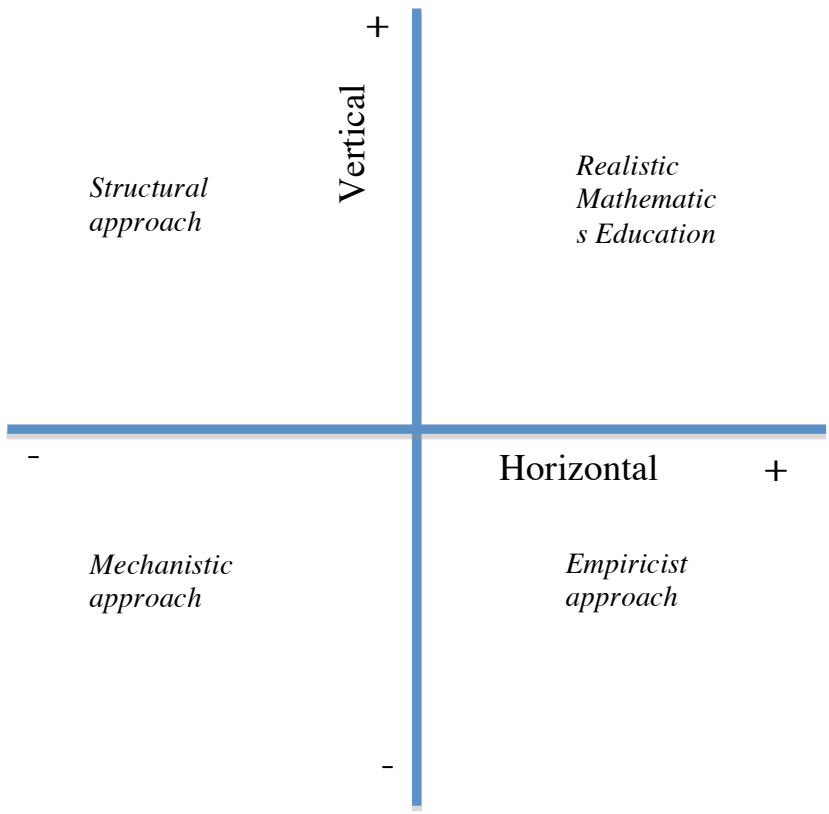


Figure 6.1: Vertical and horizontal mathematisation (Treffers & Goffree, 1985)

A *mechanistic* approach, typical of classrooms in which the teacher shows students a set of rules to follow with little or no explanation and in which students learn a set of techniques, is low in both horizontal and vertical mathematisation. It has little connection with the world beyond the school classroom, and there is little attempt to create a network of ideas of which students can make sense. It is typical of the pedagogy described in many of the TIMSS video study year 8 classrooms by McIntosh as "boring, artificial, low-level, irrelevant, mentally stifling" (2003, p. 106), and of the textbook problems studied by Vincent and Stacey (2008).

A *structural* approach highlights the structure within mathematics but does not attempt to make this structure applicable beyond the field of mathematics itself. Such an approach has strong vertical mathematisation, but weak horizontal mathematisation. It is typical of classrooms with a strong emphasis on relational understanding (Skemp, 1976), and of senior secondary courses designed for students intending to pursue tertiary studies in mathematics. While a structural approach has merit in emphasising understanding, I suggest that it is a somewhat elitist view of mathematics, giving higher status to pure knowledge than to applied knowledge (Becher & Trowler, 2001).

An *empiricist* approach develops mathematical ideas from empirical investigation. It emphasises pattern spotting and recognition in a real situation, but is unlikely to develop the reasoning that justifies the generalisations made. Noss, Healy and Hoyles (1997) warn of this:

> … school mathematics becomes constructed—by teachers and students alike—as a stereotypical data-driven 'pattern-spotting' activity in which it is acceptable to search for relationships by constructing tables of numeric data without appreciating any need to understand the structures underpinning them (p. 205)

Treffers and Goffree (1985) claim that a mechanistic approach does little to develop a sense of the structure or value of mathematics, while structural and

empirical approaches highlight only limited aspects of mathematics. They contrast this with the realistic approach adopted and developed in RME, which commences with a relatively ill-structured problem from the student's world and develops mathematical structure and meaning through mathematisation.

> It is more useful to know how to mathematize than to know a lot of mathematics. Teachers, in particular, would benefit by looking at their task in terms of teaching their students to mathematize rather than teaching them some mathematics (Wheeler, 1982, p. 45).

6.3.2 Domains of practice in school mathematics

Commencing from a very different starting point and with a very different agenda, Dowling's (1998) description of domains of practice in school mathematics (Figure 6.2) bears a remarkable similarity to the conception of mathematical practice described in RME. Building on Bernstein's (2000) notion of classification, and using the dimensions of content and expression, Dowling identifies four possible domains of practice:

- The *esoteric* domain is strongly classified with respect to both content and expression. That is, it consists of mathematical topics such as algebra, arithmetic and Euclidean geometry rather than real-life contexts, and within those topics the specialised language of mathematics is used. In the school mathematics classroom it could be described as "pure mathematics".
- The *public* domain is weakly classified with respect to both content and expression. It focuses on everyday contexts using everyday language. It is the mathematics that people do outside school, much of which is often not recognised as mathematics (Hogan & Morony, 2000).
- The *expressive* domain is strongly classified with respect to content, but weakly classified with respect to expression. That is, it focuses on content found within mathematics, such as algebra, but expresses it in

everyday language and symbols. The idea of a "function machine" (Australian Education Council, 1991, p. 197) is an example of recontextualising purely mathematical content, linear functions, into a more familiar setting using the everyday metaphor of a machine.

- The *descriptive* domain is strongly classified with respect to expression, but weakly classified with respect to content. That is, it recontextualises an everyday situation using mathematical terms and symbols. Solving word problems using algebraic equations is an example of the descriptive domain.

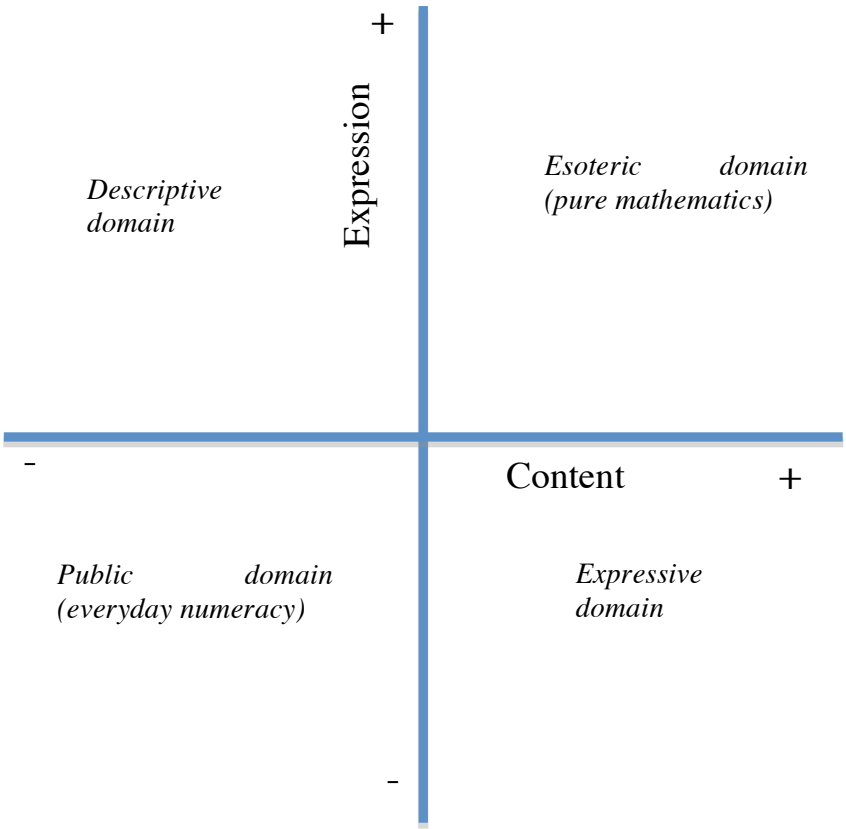


Figure 6.2: Domains of practice in school mathematics (adapted from Dowling, 1998, p. 135)

Dowling (1998) analyses how the language, problems, diagrams and contexts in school textbooks position students with respect to mathematics. He shows

that textbooks written for students who are classed as of lower ability, most commonly those from disadvantaged socioeconomic backgrounds, focus on the public domain, denying them access to generalised mathematical thinking and effectively positioning them as dependents in the classroom. However, textbooks written for higher achieving students, most commonly those from more advantaged socioeconomic backgrounds, focus more on the esoteric domain, giving students access to more challenging and abstract mathematics and positioning them as apprentices in the mathematics classroom.

Thus there is a strong social justice imperative in Dowling's analysis. While there is value in the types of mathematical activities that might be carried out within the other three domains, he makes a compelling case for all students to have access to the high levels of mathematical thinking in the esoteric domain. (Sentence omitted) The esoteric domain is both emancipatory and generative, providing students with access to the valued knowledge of the wider mathematical community.

6.4 Conclusion

The foregoing discussion is not a call for more context in school mathematics, and certainly not for the simplistic view expressed by the school principal that context can make mathematics understandable to all. Indeed there is considerable evidence that context can get in the way of students understanding the mathematics behind a problem, particularly for those students from disadvantaged backgrounds (Cooper & Dunne, 1999). Rather it is a call for a carefully considered and intellectually rigorous approach to making mathematics come alive through context and to using contextual stimuli to prompt mathematical learning.

As Wrigley, Lingard and Thomson (2012) write:

> A search for greater *relevance* is not enough, nor the proposal that learning become more *experiential*; both can mean an uncritical

> assimilation to the *status quo*. We prefer *connectedness* to relevance because it indicates both a respect for students' knowledges and interests and the need to scaffold learners into other knowledge forms, genres and media from which disadvantaged students should never be excluded (p. 99, italics in original).

Together, the preceding three chapters have outlined three dimensions of connectedness in school mathematics: mathematical, cultural and contextual. Little of what has been argued is new; rather, I have attempted to take familiar and well-accepted ideas and blend them in such a way that they challenge existing paradigms and practices in school mathematics. I argue that if we take seriously these three dimensions we cannot help but challenge dominant educational metaphors that are locked into what Heidegger (1977) terms the technological enframing (see Chapter 2), nor can we help but see mathematics from both the inside and outside (see Chapter 3).

I commenced this chapter with a quote from Galileo Galilei in which he asserted that mathematics is the language in which the physical world is written. Galileo's conclusions about the heliocentric nature of the universe brought him into direct conflict with some fellow astronomers and particularly the Holy Office of the Roman Catholic Church. He defended his views in his opus *Dialogue Concerning the Two Chief World Systems*, in which he wrote:

> In the long run my observations have convinced me that some men, reasoning preposterously, first establish some conclusion in their minds which, either because of its being their own or because of their having received it from some person who has their entire confidence, impresses them so deeply that one finds it impossible ever to get it out of their heads. Such arguments in support of their fixed idea as they hit upon themselves or hear set forth by others, no matter how simple and stupid these may be, gain their instant acceptance and applause. On the other hand whatever is brought forward against it, however ingenious and conclusive, they receive with disdain or with hot rage—if indeed it does

> not make them ill. Beside themselves with passion, some of them would not be backward even about scheming to suppress and silence their adversaries (Galilei, 1953, p. 322).

I am certainly not suggesting that what I have argued in this thesis is of comparable significance to the work of Galileo, nor necessarily that it will cause the mathematics or mathematics education communities to "scheme to suppress and silence" me! However, I suggest that there is a real danger that the arguments could be glossed over as familiar and already part of existing discourses and practices. Or worse, they may be subverted to serve the purposes of maintaining a technologically enframed agenda for school mathematics. That is, the arguments for efficiency or effectiveness located within the education as production, cure or race metaphors that have become so pervasive yet invisible in current educational discourse could recruit the three dimensions of connectedness for their own purposes.

In the final three chapters of the thesis I attempt to translate these ideas into the practical setting of school education. In Chapter 7 I examine curriculum, pedagogy and assessment using a *Slow Maths* lens and suggest a teaching/learning sequence that might enact the three dimensions of connectedness discussed in these chapters. In Chapter 8 I suggest a new dimension of knowledge for teaching that I term *cultural and contextual knowledge of mathematics* (omitted). In the final chapter I recap the key arguments of the thesis.

PART 3: INTO PRACTICE

The discussions of slowness and connectedness in Parts 1 and 2 suggest a number of potential areas of application and further research. I suggest that these include:

- Further research into how alternative metaphors for education might shape curriculum, pedagogy and assessment;
- Application of the principles of slowness beyond the context of school mathematics;
- Professional development programmes built on the principles of slowness rather than interventions that seek immediate rewards;
- Application of Kitcher's mathematical naturalism to the school context, reconceiving teaching and learning as rational transitions between mathematical practices;
- Analysis of curriculum and policy documents, classroom texts or online materials in terms of three dimensions of connectedness;
- Development of units of work and lesson materials that build on the three dimensions of connectedness; and
- Reconceiving aspects of teacher knowledge in terms of these dimensions of connectedness.

In Chapters 7 and 8 I concentrate on these last two possibilities.

CHAPTER 7: SLOW MATHS—TOWARDS A PEDAGOGY OF CONNECTEDNESS

> Most children in school fail…[They] fail in fact if not in name. They complete their schooling only because we have agreed to push them up through the grades and out of the schools, whether they know anything or not…they fail to develop more than a tiny part of the tremendous capacity for learning, understanding and creating with which they were born and of which they made full use during the first two or three years of their lives…
>
> They fail because they are afraid, bored, and confused.
>
> They are afraid, above all else, of failing, of disappointing or displeasing the many anxious adults around them, whose limitless hopes and expectations for them hang over their heads like a cloud.
>
> They are bored because the things they are given and told to do in school are so trivial, so dull, and make such limited demands on the wide spectrum of their intelligence, capabilities, and talents.
>
> They are confused because most of the torrent of words that pours over them in school makes little or no sense. It often flatly contradicts other things they have been told, and hardly ever has any relation to what they really know—to the rough model of reality they carry around in their heads. (J. Holt, 1964, pp. 9, 10)

Synopsis

Reflections on curriculum: Mathematics curriculum documents from around the world emphasise important goals for mathematics and highlight desirable proficiencies to be developed. However, these are all too often regarded as less important than the content to be covered.

Reflections on pedagogy: Even the most reform-oriented descriptions of classroom practice remain focused on mastering a predetermined set of skills.

Reflections on assessment: Ultimately what is assessed determines what is valued. The form and content of standardised testing has a significant impact on what happens in the mathematics classroom.

A thought experiment: The topic of simultaneous linear equations is used to create a sketch of a *slow* approach to curriculum, pedagogy and assessment.

In Chapters 4 to 6 I have given evidence that the dimensions of connectedness that underpin what I have termed *Slow Maths* have theoretical, lived and researched validity. I have argued that the discipline of mathematics itself, the work and life stories of mathematicians and influential pieces of mathematics education research all support the mathematical, cultural and contextual connectedness introduced in Chapter 3.

In this chapter I return to the school classroom context and the experiences I described in my personal journey in Chapter 1. I suggest that the quote from John Holt's (1964) classic study of school classrooms *How Children Fail* is no less relevant now than it was when he first wrote it in 1964. My observations of my own children and their friends tell me that most children still fail. They are still afraid, bored and confused.

And so I argue that we need a pedagogy of connectedness—one that can connect students to themselves, to their world and to mathematics. This is *slow* pedagogy. It is important to reiterate that slowness is a state of mind rather than a specific proposal. Indeed, arguing for one particular mode of implementation would contradict the principle of valuing variability discussed in Chapter 2. Hence this chapter is intended to be the opening of a discussion rather than the closing of an argument, and a reflection on how the principles of slowness

might be used to challenge conventional models of curriculum, pedagogy and assessment in mathematics.

7.1 Reflections on curriculum

It is noteworthy that of the 45 mathematicians mentioned by name in the preceding chapters, only one, Pythagoras, receives explicit mention in the *Australian Curriculum: Mathematics*. Few, if any of the mathematical topics such as knot theory or chaos theory receive any more than a passing mention. None of the concepts from mathematics education research is mentioned.

The writers and curriculum authorities may well argue that the curriculum is a statement of the content that should be taught and the proficiencies to be developed rather than a pedagogic prescription, and that teachers are free to use any methodologies or elaborations they feel are appropriate. In fact, this was the precise remit given to the *Australian Curriculum, Assessment and Reporting Authority*. Or they may argue that the document is dynamic, and that the current iteration represents a work in progress. However, I suggest that no curriculum is pedagogy-free. The absence of anything more than a passing reference to mathematicians, coupled with the almost complete absence of any mathematics developed more recently than the 18th century, illustrates the limited value placed on mathematics as a living, cultural pursuit. The absence of examples from contemporary society illustrates the limited value placed on the pivotal role played by mathematics in contemporary society. The absence of any reference to key concepts from mathematics education research shows the limited value placed on what we know about creating a classroom culture in which deep mathematical thinking is promoted and developed. Despite the avowed emphasis on mathematical proficiencies such as understanding, reasoning, fluency and problem-solving, I suggest that together these absences reinforce a view of mathematics as little more than a set of facts or skills to be learned, the teacher as a tame and complicit implementer of a fixed agenda,

and the student as a powerless pawn in the game of school mathematics. *Slow Maths* seeks to challenge this.

7.1.1 A comparison of curriculum documents

Mathematics curriculum documents around the world typically contain three distinct sections. The first is normally a preamble, stating why the study of mathematics is important and listing a small number of specific goals for school mathematics. For example, the *Australian Curriculum: Mathematics* (Australian Curriculum and Assessment Reporting Authority [ACARA], 2013, Rationale/Aims) lists three aims for school mathematics:

- To ensure that students are confident, creative users of mathematics, able to investigate, represent and interpret situations in their personal and work lives and as active citizens;
- To ensure that students develop an increasingly sophisticated understanding of mathematical concepts and fluency with processes, and are able to pose and solve problems and reason in Number and Algebra, Measurement and Geometry, and Statistics and Probability; and
- To ensure that students recognise connections between the areas of mathematics and other disciplines and appreciate mathematics as an accessible and enjoyable discipline to study.

Along with the stated aims of other international curriculum documents, these aims bear a remarkable similarity to the three dimensions of connectedness—mathematical, personal and contextual—articulated in Chapters 2 and 3[42].

The second component of all curriculum documents is a statement of the attributes that students are expected to develop through their study of

[42] In fact, a careful reading of curriculum documents was a significant influence in my formulation of the three aspects of connectedness described in this thesis.

mathematics. For example, the *Australian Curriculum: Mathematics* lists four so-called proficiency strands: understanding, fluency, reasoning and problem-solving, which are an adaptation of the five strands of conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition articulated by the National Research Council (2001).

The Singapore mathematics curriculum uses a framework that includes the development of positive attitudes, meta-cognitive skills and processes concerned with reasoning, communication and connections (Figure 7.1). Perhaps significantly this framework has remained unchanged for some 20 years (Fan & Zhu, 2007).

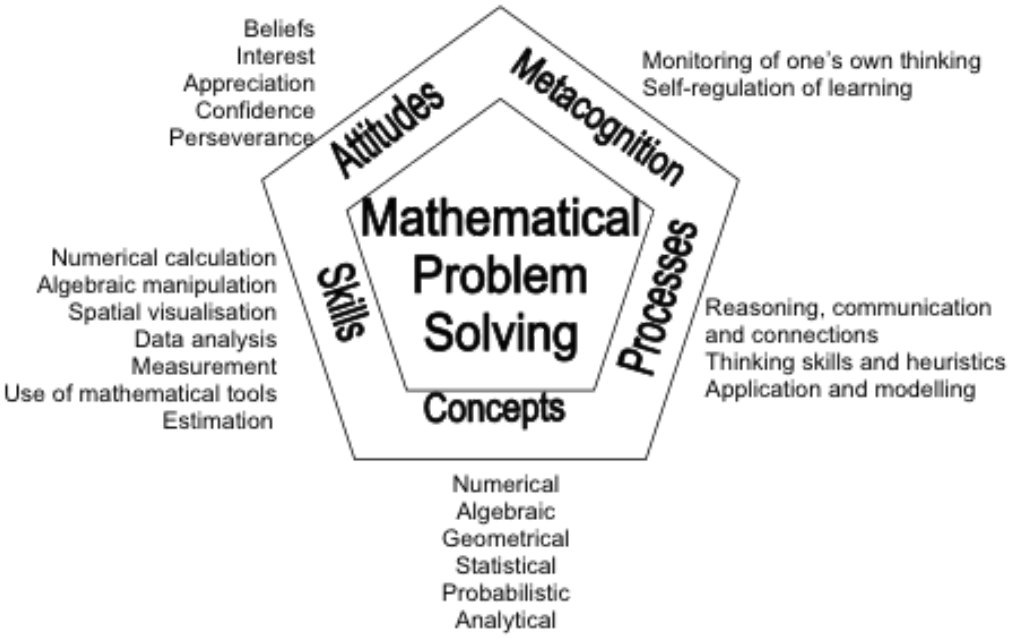


Figure 7.1: Mathematics framework (Ministry of Education Singapore, 2006, p. 6)

The *Common Core State Standards for Mathematics* (Common Core State Standards Initiative, 2010) from the USA lists eight standards for mathematical practice: make sense of problems and persevere in solving them; reason

abstractly and quantitatively; construct viable arguments and critique the reasoning of others; model with mathematics; use appropriate tools strategically; attend to precision; look for and make use of structure; look for and express regularity in repeated reasoning.

The Finland National Core Curriculum for Basic Education (2004) states:

> The task of instruction in mathematics is to offer opportunities for the development of mathematical thinking, and for the learning of mathematical concepts and the most widely used problem-solving methods. The instruction is to develop the pupil's creative and precise thinking, and guide the pupil in finding and formulating problems and in seeking solutions to them. The importance of mathematics has to be perceived broadly: it influences the pupil's intellectual growth and advances purposeful activity and social interaction on his or her part (p. 158)

Although there are relatively minor differences between the way each of these curriculum documents describes the goals of mathematics and the processes of thinking mathematically, the intent in each is clearly the same. Each explicitly emphasises the development of mathematical thinking, problem-solving and reasoning as central goals, each refers to the importance of mathematics in understanding the world, and each, perhaps implicitly, makes reference to the value of mathematics in one's personal and social life. Again, they bear a remarkable similarity to the three dimensions of mathematical, personal and contextual connectedness. These are undoubtedly worthy goals, yet I suggest that they are all too often subverted by what is usually the most detailed section of each document, a list of content laid out as a developmental sequence of learning.

In some cases the description of content may be relatively minimal in an attempt to emphasise depth over breadth (Common Core State Standards Initiative, 2010), yet in all cases a developmental progression is described in

which students are expected to follow a common sequence of learning. I suggest that the descriptions of content are based on several assumptions, each of which could be debated at length:

- Learning is hierarchical, progressing from foundational ideas to more complex knowledge;
- Learning is, if not totally predictable, at least typical relative to the age of the student;
- A common sequence of learning is appropriate for the majority of students in the majority of contexts;
- This sequence of learning is relatively stable across time and cultures;
- Curriculum development process is best done centrally; and
- The teacher is the implementer of curriculum, responsible for its interpretation in a given context, but not its formulation.

These assumptions almost invariably give rise to a formal curriculum that is compartmentalised into distinct areas of content and atomised according to perceived developmental levels. For example the *Australian Curriculum: Mathematics*, although initially designed to focus on a limited number of big ideas at each year level in three content domains of Number and Algebra, Statistics and Probability, and Measurement and Geometry, actually contains 278 content descriptions organised into 11 distinct year levels plus an additional level for students intending to proceed to higher levels of study (Table 7.1). Each content description then has one or more elaborations that expand on the skills and concepts described in the content description by providing examples of how they can be developed or applied. Although the document organises the content descriptions into sub-strands that flow through the curriculum, it is debatable whether or not such an extensive list of content focuses on a limited number of big ideas.

Table 7.1: Number of content descriptions in each content strand for each year level in the *Australian Curriculum: Mathematics*

Year level	Number and algebra	Measurement and geometry	Statistics and probability
F	**6**	**5**	**1**
1	**7**	**5**	**3**
2	**11**	**10**	**4**
3	**10**	**6**	**4**
4	**13**	**9**	**6**
5	**12**	**8**	**5**
6	**13**	**9**	**5**
7	**19**	**8**	**6**
8	**12**	**8**	**7**
9	**10**	**9**	**6**
10	**13**	**4**	**8**
10A	**7**	**6**	**3**

Such descriptions of content have a profound impact on teachers' enactment of a curriculum. It is hardly surprising that when confronted with a plethora of content their concerns focus almost exclusively on placement of content or resources[43]. Critiques of curriculum such as those in the MERGA publication *Engaging the Australian Curriculum Mathematics: Perspectives from the Field* (Atweh, Goos, Jorgensen, & Siemon, 2012) seem far removed from the consciousness of teachers. In our discussion of the overarching themes of the

[43] See Chapter 2 in which I discuss the focus of the contributions concerning the introduction of the *Australian Curriculum: Mathematics* made by teachers to the Australian Association of Mathematics Teachers email list.

Curriculum, Bill Atweh, Donna Miller and I (2012) conclude that it is both internally inconsistent in that the valued proficiencies articulated in the aims and rationale are not always evident in the content descriptions, and externally inconsistent in that it fails to enact more general goals of schooling such as active and engaged citizenship and building the capacity for lifelong learning. (Sentence omitted)

I suggest that this is due, at least in part, to the way curriculum documents are formulated. I suggest that they are developed with a mixture of *hindsight*, that is using a knowledge of the structure of the discipline of mathematics that only becomes apparent after significant higher-level study of mathematics, and *foresight*, that is using a knowledge of how the typical student progresses through levels of understanding. Both hindsight and foresight emphasise developmental progression, hierarchical structure, stability, and a research-development-dissemination model of curriculum (Begg, 2008) that positions teachers as implementers. While hindsight and foresight have a role to play in the formulation of curriculum, I argue that there is a much more important element—*insight*. Insight is context-specific. It is the special quality of expert teachers that enables them to see the connectedness of a particular piece of mathematics to other areas of mathematics, the world and the student.

Hence I suggest that we need a *spacious* curriculum (Angier & Povey, 1999); one that gives teachers room to explore with their students the culture and traditions of mathematics, room to use mathematics to understand and critique the world, and room to appreciate mathematics as part of themselves. This does not marginalise mathematical content in the quest for what might be seen as more holistic learning as critics of programs such as *New Basics* (Education Queensland, 2001) have argued, but just as in *slow food* the process of cooking generates the need to hone, develop and practice certain skills, so the process of engaging with authentic mathematics generates opportunities for students to practice and develop fluency and understanding of key skills and concepts.

> Deep conceptual learning or authentic learning takes time—this is 'slow learning' (in a sense akin to 'slow cooking') as opposed to the more superficial fast learning aimed at improving test scores. A more extended and flexible timeframe is needed to allow learners more control over activities. More open architectures of learning can involve, for example, creative projects leading to an open exhibition for parents in the tradition of Dewey; project method, based on an agreed problem or issue; the engagement of learners in resolving a real-world problem; or storyline, a form of thematic work based on the outline of a narrative. (Wrigley et al., 2012, p. 100)

Later in this chapter I give an example of how a unit on simultaneous linear equations might be developed to emphasise mathematical, cultural and contextual connectedness.

7.1.2 An aside: mathematics and numeracy

In order to maintain a focus on what is described in curriculum documents as *mathematics*, I have chosen not to discuss issues concerning the relationship between numeracy, or quantitative/mathematical literacy, and mathematics. The *Australian Curriculum* asserts that:

> [s]tudents become numerate as they develop the knowledge and skills to use mathematics confidently across all learning areas at school and in their lives more broadly. Numeracy involves students in recognising and understanding the role of mathematics in the world and having the dispositions and capacities to use mathematical knowledge and skills purposefully. (ACARA, 2013, introduction)

Steen (2001) uses the term quantitative literacy to describe "a habit of mind, an approach to problems that employs and enhances both statistics and mathematics" (p. 5). He goes on to present a variety of aspects of numeracy including elements such as confidence, cultural appreciation and making decisions with mathematics, expressions of quantitative literacy in citizenship,

culture, health or the professions, and skills in areas such as arithmetic, modelling, chance and reasoning.

The *Project for International Student Assessment* defines mathematical literacy as:

> [a]n individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded mathematical judgements and to engage in mathematics in ways that meet the needs of that individual's current and future life as a constructive, concerned and reflective citizen. (Organisation for Economic Cooperation and Development, 2004, p. 37)

Each of these descriptions highlights the capacity to see and interpret the world through mathematical eyes, which necessarily involves a set of dispositions and skills that include problem posing and solving, abstraction, generalisation, application and communication. Hence I suggest that *Slow Maths*, which foregrounds the culture and traditions of mathematics and its application to significant issues, makes debates about the relationship between mathematics and numeracy superfluous[44].

7.2 Reflections on pedagogy

The aspects of becoming mathematical described in the proficiencies of the *Australian Curriculum: Mathematics* or the standards for mathematical practice described in the *Common Core State Standards* develop slowly and over time. Engaging in activities related to one or more of these aspects requires one to work deeply, persistently and often slowly. Furthermore problem-solving and posing necessarily involve uncertainty and hesitancy, often as a "shared exploration where the destination is not known in advance"

[44] I would go further and suggest that the dominance of the term *numeracy* in Australian education has actually done a great disservice to the cause of mathematics. It deflects attention from the rigour and structure of mathematics, consigning much of the study of mathematics to a relatively simplistic application of basic mathematical skills.

(Smith, 2013, p. 379). The challenge for pedagogy, then, is the creation of an environment where time is allowed for the incubation of ideas, one that develops mathematical curiosity, and one in which mathematics is used to critique the world and take action.

Yet, with notable exceptions such as the three weeks allowed for students to solve up to six problems as part of the *Mathematics Challenge for Young Australians* (Australian Mathematics Trust, 2014), this time is rarely allowed in classrooms and even more rarely used by students. Studies by mathematics education researchers suggest that novice problem solvers rarely exercise meta-cognitive functions; that they dive quickly into explorations that may or may not prove fruitful; and that they seldom, if ever, take time to allow ideas to develop or to reflect on the appropriateness of a solution (Schoenfeld, 1985, 1992). While it is beyond the scope of this thesis to discuss in detail the research on higher order thinking and problem-solving, one message is clear: it takes time to think creatively and solve higher order problems.

In an effort to promote higher order thinking in mathematics, Sullivan (2011) highlights six characteristics of effective mathematics teaching:

(1) *Articulating goals*: Identify key ideas that underpin the concepts you are seeking to teach, communicate to students that these are the goals of the teaching, and explain to them how you hope they will learn (p. 25);
(2) *Making connections*: Build on what students know, mathematically and experientially, including creating and connecting students with stories that both contextualise and establish a rationale for the learning (p. 26);
(3) *Fostering engagement*: Engage students by utilising a variety of rich and challenging tasks that allow students time and opportunities to make decisions, and which use a variety of forms of representation (p. 26);
(4) *Differentiating challenges*: Interact with students while they engage in the experiences, encourage students to interact with each other, including asking and answering questions, and specifically plan to support students who need it and challenge those who are ready (p. 27);

(5) *Structuring lessons*: Adopt pedagogies that foster communication and both individual and group responsibilities, use students' reports to the class as learning opportunities, with teacher summaries of key mathematical ideas (p. 28); and

(6) *Promoting fluency and transfer*: Fluency is important, and it can be developed in two ways: by short everyday practice of mental processes; and by practice, reinforcement and prompting transfer of learnt skills (p. 29).

While there is merit in Sullivan's six principles, particularly in the connectedness implied in principle (2), I suggest that they are still based firmly on a view of mathematics as learning a set of skills and concepts, albeit in a progressive student-focused environment, rather than on a view of mathematics as being about becoming mathematical.

I have five major criticisms of the principles described by Sullivan:

(1) The skills and concepts articulated in the formal curriculum are taken as given, with the only question being how to teach them rather than whether or why they are worth learning.

(2) There is no mention of immersing students in the culture and traditions of mathematics; rather it appears to be up to the teacher to manufacture situations that will foster student engagement.

(3) The articulation of the goals of the "key ideas you are seeking to teach" and "how you hope they will learn" leaves little room for the flexibility and uncertainty inherent in creative problem-solving. In the handshakes and intersecting lines lesson play introducing Part 2 of this thesis the goal was intentionally hidden from students. This created a sense of mystery and curiosity among students.

(4) The lesson structure is described as "Launch; Explore; Summarise; Review" (p. 28), with a more detailed description of each phase in terms of the pedagogy embodied in Japanese lesson study. However, the danger in any such description is that teachers pay attention to the

form of the lesson rather than using it as the lens through which to view the relation between teaching and learning (Fernandez, Cannon, & Chokshi, 2003). In particular, a key aspect of Japanese lesson study is paying attention to students' thinking, anticipating a variety of responses, and hence valuing the variation in methods that might arise.

(5) Each lesson contains closure in the form of a teacher summary, with little or no room for a period of incubation that may well extend over several days. In contrast the handshakes and intersecting lines lesson posed a key unanswered question at the end: what do handshakes have to do with intersecting lines?

The principles espoused by Sullivan (2011) may well impact positively on the learning environment and enable students to gain deeper conceptual understanding, greater fluency and enhanced reasoning and problem-solving capacities, however I suggest that they continue to imply that what matters is the achievement of a predetermined endpoint. Thus even students who succeed may still have little appreciation of the culture and context of mathematics. *Slow Maths*, on the other hand, reconceives the relationship between teacher, students, knowledge and the world in terms of connectedness. Understanding, fluency, reasoning and problem-solving are essential components and outcomes of such pedagogy, but they are not all there is. The goal is to immerse students in the culture of mathematics and mathematical thinking, giving them time to become creative problem solvers and to identify as participants in a rich field of knowledge that explains and shapes the world. As Wrigley, Lingard and Thomson (2012, p. 99) write "pedagogy is bigger than methodology; it involves reflecting on society, values, history, environment and learning itself."

7.3 Reflections on assessment

Discussions of the value and impact of high-stakes performance-oriented approaches to assessment, such as those embodied in the *National Assessment*

Program in Literacy and Numeracy [NAPLAN] (ACARA, 2011), are extensive and well-rehearsed (Mansell, 2007; Polesel, Dulfer, & Turnbull, 2012). These criticisms include that they advantage or disadvantage certain groups of students such as Indigenous children (I. Hardy, 2013; Meaney & Evans, 2013); that they address a very limited range of desirable outcomes of schooling (Reid, 2010); that they have a backwash effect on teaching and learning by narrowing the curriculum to easily measured skills and concepts (Luke, 2010); and that they have a negative impact on students' sense of well-being and self-confidence (chu Ho, 2006).

A recent Senate inquiry into the effectiveness of NAPLAN (Senate Standing Committee on Education Employment and Workplace Relations, 2013) received 98 submissions addressing questions that included whether it was achieving its stated objectives, whether there were unintended consequences relating to its introduction, and its impact on teaching and learning strategies.[45] While there were diverse opinions expressed by respondents to the inquiry, many raised issues similar to those discussed above. In two submissions relating specifically to mathematics, Stacey (2013) noted that "NAPLAN inevitably sets a model for school mathematics, but this model neglects deep objectives for the curriculum, which are becoming increasingly important in a knowledge economy" (p. 1), while the Australian Association of Mathematics Teachers (2013) noted that "there is persistent advice that schools or teachers are 'preparing' students for NAPLAN and do this in lieu of their normal teaching and learning" (p. 2).

Whatever the merits or otherwise of high-stakes testing regimes, it is clear that if one adopts a metaphor of education as production, cure or race, some form of universal, easily administered form of assessment is indispensable. As

[45] Significantly, the question of whether the goals of NAPLAN were actually aligned with broader goals of schooling was not asked.

discussed in Chapter 2, I suggest that an over-emphasis on NAPLAN as the most visible and valued measure of the outcomes of schooling both supports and helps to construct a race metaphor for education, effectively silencing discussion of the goals of schooling. On the other hand a *slow* approach to assessment demands that students have opportunities to do hard mathematics, to work with others, to work with experts, to work on things over time, to debrief, review and reflect, and to identify and work on skills and concepts as and when required. Just as in food preparation, this requires coping with and developing strategies to deal with uncertainties or unexpected events as they emerge. It demands negotiation with students around their interests, what the authentic tasks in which they will engage might be, how they will manage their time and what the expected outcomes will be. Whether such tasks have an internal epistemic or external practical goal, what counts is the extent to which the end products achieve their purpose. Like slow food, there is both an intrinsic and extrinsic aspect to this judgment. The chef/mathematician herself gains a sense of satisfaction from the process, while the consumer/reviewer provides the validation of the end product.

7.4 A thought experiment in *Slow Maths*: simultaneous linear equations

To illustrate a possible approach to a traditional topic in school mathematics, I have chosen to present what is, in a sense, a thought experiment. I describe an approach to a unit of work focusing on the solution of simultaneous linear equations. The *Australian Curriculum: Mathematics* locates this as a content description in year 10, stating that students will "[s]olve linear simultaneous equations, using algebraic and graphical techniques including using digital technology". The content description has one elaboration: "associating the solution of simultaneous equations with the coordinates of the intersection of their corresponding graphs" (ACARA, 2013, ACMNA237). While there is some attention to mathematical connectedness in that the link between graphical and algebraic solutions is emphasised, there is no suggestion of

connection to context or culture. The emphasis on skills and knowledge is apparent.

The unit of work below may appear to be little more than a possibly creative and carefully planned unit that could be taught in any mathematics classroom. However, the end product obscures the motivation and thought processes in its conception. It starts with a carefully constructed classroom exchange that is designed to capture the essence of the topic through discussions that have mathematical and contextual connections, in a classroom culture that values the agency and discourse of the discipline of mathematics. It then moves into what may well be a somewhat traditional approach to teaching, but one that highlights key mathematical attitudes and appreciations that are often hidden in a typical textbook approach. It then explicitly makes mathematical, cultural and contextual connections. Finally it presents opportunities for students to extend their knowledge and understanding.

Phase 1: Orienting

This part of the unit provides three introductory problems that orient students to the topic.

> Problem 1: Heads and Legs
>
> There is an old children's puzzle called Heads and Legs.[46] A farmer walks into his paddock that contains a mixture of chickens and sheep. He counts 24 heads and 62 legs. How many chickens and sheep are there?
>
> *Solve the problem using any method that you like. Explain clearly your reasoning.*

[46] Of course, this problem is hardly new. Nor is it intended to be realistic. Rather it is presented as a puzzle to solve. It is called Heads and Legs in the Maths 300 bank of lessons (Williams, 2010), in which several possible solution methods are presented.

Problem 2: Whizzo Chocolate Factory

The Whizzo Chocolate Factory[47] manufactures two types of chocolates: Crunchy Frog, each costing 40 cents, and Cockroach Clusters, each costing 50 cents. I buy a mixture of 20 chocolates and paid a total of $8.70. How many of each do I buy?

Does your method work for this problem? If not, how do you need to modify it?

Problem 3: Truck tyres

A transport company runs a fleet of 8-wheeled trucks and 12-wheeled trucks. It has 38 trucks altogether and each time it orders a complete set of tyres for its fleet (excluding spares) it orders 364 tyres. How many 8 and 12-wheeled trucks does it have?

How would you modify your method to solve this problem?

Phase 2: Reviewing

This part of the unit is structured using a model from the *Mathematics Assessment Project* (Mathematics Assessment Resource Service, 2012) developed at the Shell Centre for Mathematics Education at the University of Nottingham and the University of California, Berkeley. It asks students to review and evaluate some hypothetical solutions to the problems and give feedback to the author. This promotes meta-cognitive strategies among the students in the class.

Here are three students' solutions to the problems. Give some feedback to each of the students. Start by writing down anything you do not understand or wish to have clarified. Then give some warm

[47] The idea for this question arose from a Monty Python sketch, entitled Crunchy Frog, concerning a chocolate shop proprietor being investigated by the Health Inspector. I used the problem to inject humour into mathematics lessons.

feedback that says what you like about the solution. Then ask some questions that you think might be hard to solve using the method they have suggested.

Student 1

Heads and legs

I guessed that there were 12 chickens and 12 sheep (half and half). I worked out that this would be 12 × 2 + 12 × 4 = 72 legs. This is not correct.

I tried 10 chickens and 14 sheep. This is 10 × 2 + 14 × 4 = 76 legs, which is worse than my first guess. So I obviously need fewer sheep and more chickens.

Next I guessed that there were 18 chickens and 6 sheep. I worked out that this would be 18 × 2 + 6 × 4 = 60 legs. This is almost right, but not enough legs. I probably need one more sheep.

So I guessed 17 chickens and 7 sheep. This would be 17 × 2 + 7 × 4 = 62 legs, which is correct.

Whizzo Chocolate Factory

The same method worked for this problem.

I started with half and half again, which was 10 of each chocolate. This cost 10 × 50 + 10 × 40 = 900 or $9. This was too much so I decided that I needed to reduce the cost by buying fewer Cockroach Clusters.

I tried 8 Cockroach Clusters and 12 Crunchy Frogs, which cost 8 × 50 + 12 × 40 = 880 or $8.80.

I decided that I needed to reduce the number of Cockroach Clusters by 1, so there were 7 Cockroach Clusters and 13 Crunchy Frogs. I checked this and it was correct.

Truck Tyres

The same method will work for Truck Tyres, but I decided to put the numbers in a table. I noticed a pattern that helped me to find the answer.

8-wheeled trucks	12-wheeled trucks	Tyres
19	19	380
20	18	376
21	17	372
22	16	368
23	15	364

Student 2

Heads and legs

I started by guessing that there might be 12 chickens and 12 sheep. I calculated that this would be $12 \times 2 + 12 \times 4 = 72$ legs. This is 10 legs too many.

Each time I swap a sheep for a chicken I lose 2 legs. So to lose 10 legs I need to swap 5 sheep for chicken. This gives me 17 chickens and 7 sheep.

Whizzo Chocolate Factory

The same method works. If I start with 10 of each, it costs $9. This is 30 cents too much, and since each Cockroach Cluster costs 10 cents more than a Crunchy Frog, I must have 3 Cockroach Clusters

too many. So there must be 7 Cockroach Clusters and 13 Crunchy Frogs.

Truck tyres

The same method works. In fact I can start with any guess, such as assuming that all the trucks have 8 wheels. This would be 38 × 8 = 304 tyres. This is 60 tyres too few. Swapping an 8-wheeled truck for a 12-wheeled truck gives me an extra 4 wheels per truck, so I need to swap 15 trucks. So there are 38 – 15 = 23 8-wheeled trucks and 15 12-wheeled trucks.

Student 3

Heads and legs

I assumed that there were x chickens. So there are $24 - x$ sheep. So the total number of legs is $2x + 4(24 - x)$. This must equal 62. So:

$$\begin{aligned} 2x + 4(24 - x) &= 62 \\ 2x + 96 - 4x &= 62 \\ 96 - 2x &= 62 \\ -2x &= -34 \\ x &= 17 \end{aligned}$$

So there are 17 chickens, which must mean there are 24 – 17 = 7 sheep.

Whizzo Chocolate Factory

The same method will work. If there are x Crunchy frogs, there are $20 - x$ Cockroach Clusters. So the total cost is $40x + 50(20 - x)$. This must equal \$9 or 900 cents. My equation is:

$$40x + 50(20 - x) = 870$$

$$40x + 1000 - 50x = 870$$
$$-10x = -130$$
$$x = 13$$

So I bought 13 Crunchy Frogs and 7 Cockroach Clusters.

Truck Tyres

This solution method will always work. In this case:

$$8x + 12(38 - x) = 364$$
$$8x + 456 - 12x = 364$$
$$-4x = -92$$
$$x = 23$$

So there are 23 8-wheeled trucks and 15 12-wheeled trucks.

Phase 3: Focusing

This part of the unit addresses the essence of the topic, developing the proficiencies of fluency and understanding described in the *Australian Curriculum: Mathematics*. It probably looks very similar to traditional teaching but has a somewhat different goal. Rather than focusing on the skills of algebraic, geometric or technological methods, it focuses on evaluating when each method might be helpful. It will involve the familiar exercise of answering textbook question but will also draw out the following key points, which are not always evident in a textbook approach.

1. The method of student 1, particularly the solution to *Truck Tyres*, is the basis of a solution method using spreadsheets.
2. The method of student 2 is essentially equivalent to an algebraic solution using elimination. By assuming that all are of one type (*Truck Tyres*) the algebraic manipulations used are exactly those used in solving by elimination.

3. The method of student 3 is an algebraic solution using substitution, with the first step reduced so that it is one equation in one variable.
4. Each word in the phrase *simultaneous linear equations* is significant in that the mathematical object is a set of equations, each member of which must be true, and each of which can be plotted as a straight line.
5. A single linear equation in two variables has no unique solution. However, there are three possible types of results when a second linear equation in the same two variables is introduced. Students should construct examples of pairs of simultaneous linear equations that have solutions that are unique, indeterminate or inconsistent. They might consider what will happen if a third equation is introduced.
6. A graphics and/or CAS calculator can be used to solve pairs of simultaneous linear equations. This may be particularly useful where the numbers involved are somewhat intractable.
7. The choice of solution method depends on the context, and students should consider when each might be most appropriate.
8. Although the examples used to introduce the topic are trivial and contrived, SLEs are frequently used as mathematical models of a real world situation. In context, the equations may be approximations, or more than two equations may be involved. In general, solving for n variables requires n equations or more. Often no solution will be possible.

Phase 4: Elaborating

This part of the unit emphasises the potential mathematical, cultural and contextual connections. It is designed to give students a choice, and to encourage all students to engage with genuine mathematical activity arsing within mathematics, culture or context.

Mathematical connectedness: extremely sensitive equations

Consider the pair of simultaneous linear equations:

$$2x + 3y = 7$$
$$x - y = 1$$

The solution is $x = 2$ and $y = 1$.

Now consider:

$$2x + 3y = 7$$
$$\mathrm{x} - \mathrm{y} = 1.1$$

The solution is x = 2.06 and y = 1.06.

Show that these two sets of solutions are correct.

Now consider the pair of simultaneous linear equations:

$$2x + 3y = 7$$
$$3x + 5y = 11$$

The solution is $x = 2$ and $y = 1$.

Now consider:

$$2x + 3y = 7$$
$$3x + 5y = 11.1$$

The solution is $x = 1.7$ and $y = 1.2$.

Show that these two sets of solutions are correct.

Notice that in the first two pairs of equations a small change in the initial values on the right hand side resulted in only a small change in the solution. However, in the second set a small change in initial values resulted in a surprisingly large change in the solution.

Find a pair of equations for which a small change in initial values results in an even larger change in the final solution. Find another pair for which it is even larger.

What is special about the coefficients used in the equations? Can you explain what is happening geometrically?

There are connections here to the Fibonacci sequence (the coefficients of the variables in extremely sensitive equations), the Golden Ratio (the gradients of the lines representing the two equations), and the idea of initial sensitivity (related to Chaos Theory). This requires a deep understanding of the geometric representation of simultaneous linear equations.

Cultural connectedness: Ancient Chinese mathematics[48]

Jiu Zhang Suan Shu (Nine Chapters on the Mathematical Art) is an ancient Chinese mathematical text written probably between 100 BCE and 100 CE. It is a collection of 246 mathematical problems on various topics, including equations. The text is probably even older: in 1984 a book dating from around 200 BCE was found in Hubei Province. This book was written on bamboo strips and contained very similar problems to Jiu Zhang Suan Shu.

<u>A problem from Jiu Zhang Suan Shu</u>

A certain number of persons purchase an article. If each contributes 8 dollars, the excess is 3 dollars: if each contributes 7 dollars, the deficiency is 4 dollars. How many persons are there and what is the cost of the article?

Rather than giving a solution to this particular problem, the text gives a general solution to problems of this type. It states, "excess and deficiency make up the discrepancy in the total contribution from all persons, while the difference between the two individual contributed amounts is the discrepancy in the contribution from one person. Dividing the former by the latter, we obtain the number of persons."

Explain why the numerical problem can be written as:

$$8x - y = 3$$

[48] Taken from Siu (1993).

$$7x - y = -4$$

Hence solve the system of equations.

Write a more general system of equations in which the numerical values are unspecified. Solve the general system and hence show that the general solution method in the Jiu Zhang Suan Shu is equivalent to the methods of modern algebra.

The above problem has obvious links to the history and culture of mathematics. While solving the equation and expressing a general solution algebraically are worthwhile exercises in themselves, of greater importance for students is to discuss why the symbol system of modern algebraic provides a more concise form of expression and leads to more generalisable methods of solution.

Contextual connectedness: Computer graphics

The website of Duke University in Durham, North Carolina[49] states: "The computer industry provides many lucrative jobs for math majors. Beyond mere proficiency in computer programming, math majors are trained to address the more fundamental issues involved in the creation of new algorithms. Furthermore, many sophisticated applications of computers such as creation of computer graphics and the compression of video and audio signals (to name a few examples) involve a great deal of deep mathematics, and as a result, many computer companies specifically hire math majors."

Here is a simplified computer graphics activity that requires simultaneous linear equations.

Programming computer graphics requires expressing each pixel on the screen to be written using its coordinates (x, y). To move an object the programmer then has to specify the mathematical operation

[49] Mathematics Department Duke University, n.d.

needed to translate and/or rotate each point to a new position. For example, a rotation through 60° anti-clockwise about the origin is specified by the operation:

$$x' = 0.5x + 0.866y$$
$$y' = -0.866x + 0.5y$$

where x' and y' are the new coordinates. (The coefficients can be found using trigonometry.)

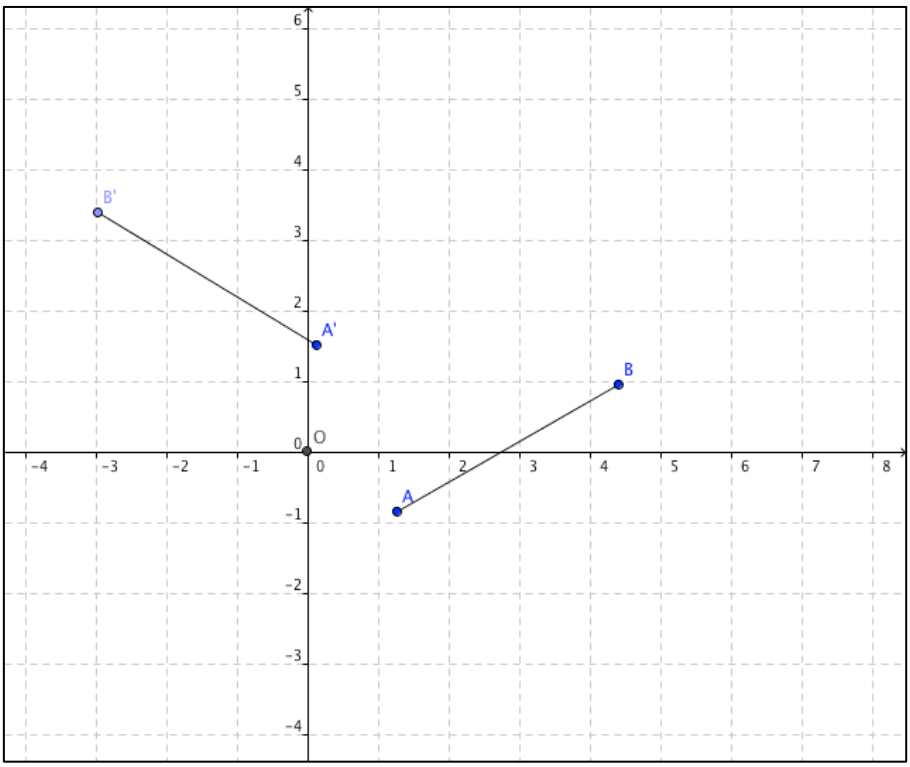
Draw a geometric shape such as a triangle on a pair of coordinate axes and calculate the new position of each vertex using the above pair of equations. Observe that the figure has been rotated through 60° anti-clockwise about the origin.

Often, of course, the programmer knows the starting position and where he or she would like the object to end up using a smooth motion. Hence he or she may need to work out the coefficients needed.

Draw a line segment somewhere on a blank piece of paper (representing a computer screen) and label the vertices A and B. Draw a congruent line segment somewhere else on your screen and label the vertices A′ and B′.

Assume that the segment A′ B′ is a rotation of AB through a certain point and angle.

Discuss how you would find the centre of rotation (hint: a circle must pass through both A′ and A, and a concentric circle through B′ and B), and draw a pair of coordinate axes using this point as the origin.



Example: Two line segments and the centre of rotation.

Write down as accurately as possible the coordinates of A, B, A' and B'. Call them (x_1, y_1), (x_2, y_2), (x_1', y_1') *and* (x_2', y_2') *respectively.*

Assume that the equations for the rotation of a point are given by:

$$x' = ax + by$$

$$y' = cx + dy$$

Use the coordinates of A and A', and B and B' to set up two pairs of equations and solve for a and b, and c and d.

Draw some other shapes on the axes and calculate the new position of each vertex using the equations you have just found. Check that these equations result in a rotation of these new shapes through the same angle as the rotation of AB onto A'B'.

The above example has obvious contextual connectedness to a contemporary situation. Other contexts, such as supply/demand graphs in economics or the costs of mobile phone plans could be used, but the advantages of this context are that it is visual, contemporary and allows results to be checked easily. It is, of course, grossly oversimplified; however, it highlights to students the application of mathematics in emerging fields and industries.

A note on assessment

Assessment is built into the unit through phases 2 (formative) and 4 (summative). The reviewing activities in phase 2 require students to give feedback on hypothetical samples of work and other students' work, which is presented as hypothetical but may be sourced from students in the class. Such peer evaluation is beneficial to both the giver and the receiver (Lundstrom & Baker, 2009), allowing both parties to refine their understanding. While some teachers may wish to assess students' knowledge and understanding in a relatively traditional way, the elaborating phase also provides comprehensive summative assessment. Students will need to explain why they focused on that particular elaboration and what they learned from it, both in terms of refining mathematical skills and their appreciation of mathematics. They will also discuss why they used a particular solution technique in that situation, and generalise to when a graphical, algebraic or technology-based technique may be most appropriate. I contend that being able to discuss these questions shows a far greater depth of understanding than the correct use of a specified technique to solve a given system of equations, as commonly required in more traditional test questions.

7.5 Conclusion

In this chapter I have outlined what a *slow* approach to curriculum, pedagogy and assessment might look like. I have also sketched a possible approach to a specific topic in a traditional high school mathematics curriculum. In some

ways the sketch may look only slightly different from a unit of work that good teachers have always designed. Indeed, a *slow* approach is not intended to revolutionise teaching and learning—food is still food regardless of how it is prepared and cooked. There may well be an argument for alternative types of schools or for "deschooling society" (Illich, 1973), but that is not the purpose of this thesis.

What I argue is different about a *slow* approach is that it has a different starting point. The rationale for teaching simultaneous linear equations is not that they are in the curriculum, but rather that they are an integral part of the tradition and culture of mathematics, and have deep connections to the world. This tradition dates back to the ancient mathematics of the Chinese and Babylonians and the applications include contemporary elements of the everyday world of students.

Other units designed using a *slow* approach may look totally different. They may be based in the workplace, with students identifying the mathematics that is used in a particular context. They may revolve around an authentic class activity such as planning an event. Or they may be an examination of an issue in the world where mathematics and statistics can be used to inform or obscure. Regardless of form, the primary goal of a *Slow Maths* unit is connectedness—to mathematics, to culture and to context.

I commenced the chapter with a quote from John Holt's study of classrooms *How Children Fail*, highlighting the fear, boredom and confusion felt by a large proportion of students in our schools. In his sequel *How Children Learn*, Holt describes school and classroom environments that offer children the space to follow their interests. He gives numerous examples from literacy, sport, art and particularly mathematics, of children exceeding all expectations when challenged and allowed to "mess about" with ideas. He explains the importance

of children seeing, "without hurry or pressure, how numbers change and grow and relate to each other" (J. Holt, 1967, p. 143). He writes:

> These stories show us a number of things about the ways in which children learn. They see the world as a whole, mysterious perhaps, but a whole none the less. They do not divide it up into airtight categories, as we adults tend to do. It is natural for them to jump from one thing to another, and to make the kinds of connections that are rarely made in formal classes and textbooks…Their learning does not box them in; it leads them out into life in many directions. Each new thing they learn makes them aware of other things to be learned. Their curiosity grows by what it feeds on. Our task is to keep it well supplied with food (p. 144).

Holt respects the reciprocal relationship between students and teachers, and respects the role of teachers as experts who "supply the food". Having the capacity to act as this expert requires teachers themselves to have expert knowledge. In Chapter 8 I review the literature relating to teacher knowledge and argue for a particular type of knowledge I term *cultural and contextual knowledge for teaching*.

CHAPTER 8: IMPLICATIONS FOR TEACHER KNOWLEDGE

> A musician wakes from a terrible nightmare. In his dream he finds himself in a society where music education has been made mandatory. "We are helping our students become more competitive in an increasingly sound-filled world." Educators, school systems, and the state are put in charge of this vital project. Studies are commissioned, committees are formed, and decisions are made— all without the advice or participation of a single working musician or composer. (Lockhart, 2009, p. 15)

Synopsis

Mathematical knowledge for teaching: Since Shulman's description of seven knowledge types for teaching there has been a massive research program investigating and describing the knowledge required by mathematics teachers.

A new form of teacher knowledge: The research into teacher knowledge has largely ignored what I term *cultural and contextual knowledge for teaching*, which I suggest is necessary for a pedagogy of connectedness.

Reflections on a trial: A preservice teacher education subject *Mathematising and Contextualising* was taught in 2013/2014 to a small number of students. The reflections of students are presented and discussed.

Conclusions: Cultural and contextual knowledge for teaching mathematics is generative in that it has the potential to engage teachers as mathematicians. It may even be more significant than well-researched forms of teacher knowledge such as pedagogical content knowledge.

It would be hard to imagine a music teacher who did not play music, or at the very least enjoy listening to music. It would be hard to imagine a physical

education teacher who did not play sport, or at the very least had not played sport at a younger age. I did once meet an English teacher who did not read books, but I imagine that this would be very rare. However, it is not at all hard to imagine a mathematics teacher who does not do any mathematics beyond the questions in the school textbook and who simply does not enjoy mathematics.[50] Seymour Papert (1993), the founder of LOGO and educational visionary, writes:

> The same effect [adult double talk] is produced when children are told school math is "fun" when they are pretty sure that teachers who say so spend their leisure hours on anything except this allegedly fun-filled activity…The children can see perfectly well that the teacher does not like math any more than they do and that the reason for doing it is simply that it has been inserted into the curriculum (p. 50).

In this chapter I argue that just as school mathematics needs to have mathematically, culturally and contextually connectedness, so we need teachers to also engage with mathematics as a cultural, historical and contextual pursuit. I commence with a review of the extensive literature on *mathematical knowledge for teaching* (MKT) (D. L. Ball & Bass, 2002) and argue that although the various components of MKT are important, they are not enough. I then propose a new form of teacher knowledge that I term *cultural and contextual knowledge for teaching* and argue that this is central to the connected approach to school mathematics described in preceding chapters. I describe the impact of a small-scale trial of a teacher education subject that looked specifically at cultural and contextual knowledge for teaching.

[50] I am not aware of any research that examines how teachers engage with the discipline beyond the classroom.

8.1 Mathematical knowledge for and in teaching

It has long been recognised that the mathematical knowledge needed by teachers is not the same as that needed by professional mathematicians or users of mathematics in various fields. Indeed, the mathematical knowledge needed for teachers has been referred to as a form of applied mathematics, where it is necessary to "understand sensitively the domain of application, the nature of its mathematical problems, and the forms of mathematical knowledge that are useful and usable in this domain" (Bass, 2004, p. 43).

8.1.1 Elementary Mathematics from an Advanced Standpoint

The earliest mention of the specific mathematical knowledge required by teachers was in Felix Klein's (1945) lectures on *Elementary Mathematics from an Advanced Standpoint*. It is noteworthy that Klein urged the cultivation of imagination and intuition along with mathematical rigour and an appreciation of the role of mathematics in making meaning of the world (Bass, 2004). One of the goals of his lectures was to show the "*mutual connections* between problems in the fields" (F. Klein, 1945, pp. 1-2, italics in original).

Klein's (1945) treatise on arithmetic, algebra and analysis argues that mathematics grows like a tree, both upwards and downwards simultaneously. At the same time as mathematical knowledge branches out in new directions, it also grows deeper and more rigorous. He argues that this must be the same with school mathematics: intuitive ideas and applications to the world promote students' growth of knowledge of mathematics, and at the same time this knowledge must be established on a firm basis. Although he advocates that teachers need to know and understand the fundamental field laws and properties of numbers, he does not argue for a purely formalist agenda. Rather he emphasises that intuition and sense-making must play a major role in the development of students' knowledge.

Commencing with a discussion of how teachers can help children to acquire knowledge of the counting numbers, Klein discusses how the formal laws of arithmetic underpin methods of calculation, and how they help to explain the need to extend the notion of number to negative numbers, fractions and irrational numbers. He gives a coherent and rigorous foundation for elementary ideas in number theory such as prime numbers, recurring decimals and continued fractions. What is significant about Klein's work is that it provides a coherent argument that teachers need a special kind of knowledge for teaching mathematics. While much of the underpinning mathematics, such as the field laws, may not be taught in schools, understanding that all the operations in number and algebra can be built from repeated application of the laws, is a fundamental part of teachers' knowledge.

8.1.2 Mathematical knowledge for teaching

Klein's work laid the foundation for more extended discussions of the knowledge required for teaching mathematics, of which I give only a brief overview here. Much of this work was stimulated by Shulman's (1986) seminal paper describing seven types of knowledge for teaching:

- General pedagogical knowledge;
- Knowledge of learners and their characteristics;
- Knowledge of educational contexts;
- Knowledge of educational ends, purposes, and values;
- Curriculum knowledge;
- Content knowledge related to a specific subject; and
- Pedagogical content knowledge related to a specific subject.

The first four are generic in that they apply equally to all areas of the curriculum, whereas the last three are specific to a given curriculum area. The notion of pedagogical content knowledge (PCK) has been particularly influential in mathematics education research.

Ball, Thames and Phelps (2008) describe what they term *Mathematical Knowledge for Teaching* (MKT), a model built on many years of empirical research in school mathematics classrooms. They highlight the distinction between the specialised knowledge of mathematics such as that required to diagnose student errors and choose suitable ways to intervene or to analyse thinking when students use unexpected methods to obtain answers, and the subject matter knowledge of mathematical skills and concepts. They argue that such analysis is something that many teachers perform with great facility, but that others, including mathematicians who may have advanced mathematical content knowledge, find very difficult. Their conceptualisation of mathematical knowledge for teaching has been used as the basis of research into the knowledge of preservice teachers (Callingham et al., 2011), practising teachers (D. L. Ball & Bass, 2002) and, to a lesser extent, mathematics teacher educators (Goos, 2009).

Ball et al. summarise the knowledge domains arising from their characterisation of teachers' knowledge using the now well-known "egg diagram" (Figure 8.1).

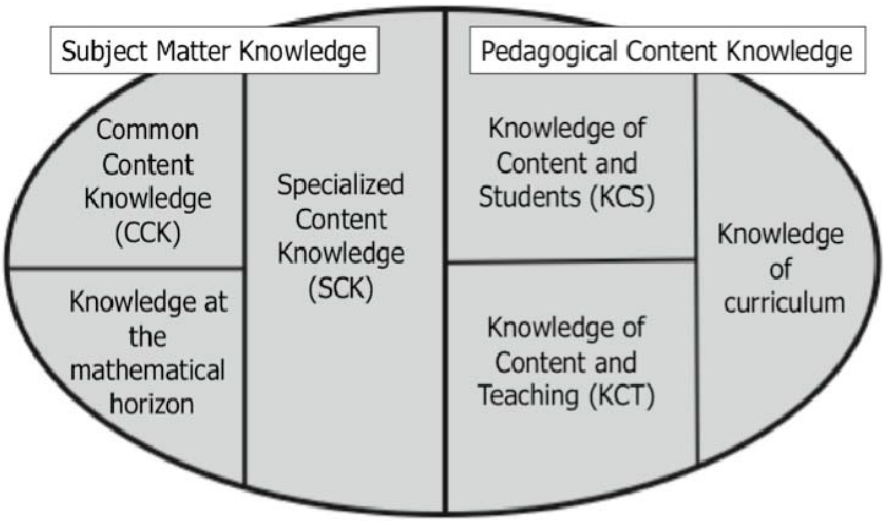


Figure 8.1 Domains of mathematical knowledge for teaching (D. L. Ball et al., 2008, p. 403)

The diagram expands on Shulman's categories of subject matter knowledge and pedagogical content knowledge by introducing several sub-categories.

Knowledge of curriculum includes knowledge of the formal curriculum, its domains, goals and assessment strategies, as well as a more general knowledge of which topics are typically taught at which grade levels. *Knowledge of content and students* refers to typical student actions such as error patterns, non-standard solution methods or points of difficulty; while *knowledge of content and teaching* refers to typical teacher knowledge such as the advantages of different representations or ways of sequencing new material. Together these three domains of knowledge form a slightly expanded notion of pedagogical content knowledge in mathematics.

Ball et al. (2008) also identify three distinct types of content knowledge. *Common content knowledge* refers to that held in common with others working in the mathematical field, and may include knowledge of common skills, concepts and algorithms, as well as ways of working mathematically. *Specialised content knowledge* refers to knowledge of mathematics that enables the teacher to do their work in the classroom. It is subtly different from pedagogical content knowledge in that it does not require knowledge of students or the practices of teaching. Rather it is knowledge about mathematics that teachers use, such as deciding whether a method will always work or evaluating the validity of an argument. Like Bass (2004) who termed mathematics teaching a kind of applied mathematics, Ball et al. argue that the work teachers do constitutes "a form of mathematical problem-solving used in the work of teaching" (p. 398).

More recently the term *knowledge at the mathematical horizon* (D. L. Ball & Bass, 2009) has been added to this conceptualisation of subject matter knowledge. This is described as knowledge that "supports a kind of awareness, sensibility, disposition that informs, orients and culturally frames instructional

practice" or "a kind of peripheral vision" (p. 5). Yet apart from one analysis in their paper of how a teacher was able to orchestrate a class discussion on the nature of odd and even numbers, there appears to have been little research on what constitutes knowledge at the mathematical horizon.

Ball and Bass (2009) describe four aspects of knowledge at the mathematical horizon:

> 1. A sense of the mathematical environment surrounding the current "location" in instruction;
> 2. Major disciplinary ideas and structures;
> 3. Key mathematical practices; and
> 4. Core mathematical values and sensibilities (p. 6)

They suggest that it is about "topics, practices and values and sheer mathematical honesty—that sense of the territory that helps to bring a sense of judgment and good taste to teachers' responsibilities toward their pupils" (p. 10). As they acknowledge, their conception of knowledge at the mathematical horizon is relatively undeveloped and unresearched, with little indication of the extent to which it contributes to more effective student learning. However, as I argue below, an expanded notion of knowledge at the mathematical horizon is essential if we are to enable students to build the mathematical, cultural and contextual connections described in previous chapters.

Failure to acknowledge the importance of knowledge at the mathematical horizon can, I suggest, lead to a reductionist view of the nature of teaching and learning. This is vividly illustrated in materials produced for school leaders in Western Australia (Swan, Woodley, & Marshall, 2012), which state that:

> In essence teaching mathematics involves three steps:
>
> 1. Following a clear sequence
> 2. Finding out where children are on that sequence

> 3. Choosing appropriate tasks and activities that will help all children move on from their current places on the sequence. Then repeat point 2 to see whether the children have moved on (p. 2).

This is clearly meant to enact the notion of pedagogical content knowledge, but I suggest that it is likely to lead to linear and pre-determined sequences of learning that do little to capture the connectedness of mathematics.

8.1.3 Situated conceptions of teacher knowledge

An alternative conception of teacher knowledge asserts that it can only be adequately observed in the practice of teaching. The *Knowledge Quartet* described in the Subject Knowledge in Mathematics project at the University of Cambridge (Rowland, Huckstep, & Thwaites, 2003) describes how theoretical knowledge is transformed into classroom action that ultimately takes account of unexpected events. It consists of four dimensions: foundation, transformation, connection and contingency.

> *Foundation* consists of trainees' knowledge, beliefs and understanding acquired in the academy, in preparation for their role in the classroom. Such knowledge and beliefs inform pedagogical choices and strategies in a fundamental way…*Transformation* concerns knowledge-in-action as demonstrated both in planning to teach and in the act of teaching itself …*Connection* binds together certain choices and decisions that are made for the more or less discrete parts of mathematical content…*Contingency* concerns classroom events that are almost impossible to plan for. (Rowland et al., 2003, pp. 97-98, italics added)

Rowland et al. claim that this conception of teachers' knowledge responds to criticisms of Ball et al.'s model that interactions between the various knowledge components are not described and that different teachers may develop different forms of PCK depending on their background and experience. The emphasis on contingency also recognises the situated nature of

PCK, in that how a teacher responds to a given situation may be very different depending on the context in which she is working. There is an obvious synergy between the recognition of the importance of connecting ideas and of the inherent unpredictability of the classroom described in the Knowledge Quartet and the dimensions of connectedness described in preceding chapters. Developing the dimension of contingency is clearly slow work.

A similar perspective is adopted by Watson and Barton (2011) who discuss teaching mathematics as "contextual application of mathematical modes of thinking." They argue that static models of teacher knowledge based on those proposed by Ball et al. miss out on the crucial aspect of teachers enacting mathematics. They describe how experienced teachers and mathematicians bring their mathematical knowledge to bear in a collaborative situation of designing tasks for students, and how mathematical ways of thinking enable teachers to make judgements and predictions, to react mathematically in the moment and to see what lies behind students' responses. They argue that teachers need to remain engaged in mathematical modes of thinking in order for the teacher to continue to be a mathematician in pedagogic situations.

There are obvious similarities and overlaps between the models of teacher knowledge discussed above. Each recognises the importance of foundational mathematical knowledge. Each also recognises the importance of specific knowledge required to teach mathematics effectively, which includes a deep understanding of elementary mathematics and knowledge of the connections, representations and contexts that make for more effective learning. Each builds on Klein's (1945) early discussion of elementary mathematics from an advanced standpoint. However none deals explicitly with what I term *cultural and contextual knowledge of mathematics*.

8.2 Cultural and contextual knowledge of mathematics: a new aspect of knowledge for teaching

I have argued in preceding chapters that teaching is not just about leading students through a pre-specified set of curriculum outcomes. It is also about inducting students into mathematical ways of understanding, seeing and acting in the world. I suggest that to do this, teachers themselves need to be able to understand and critique the nature of mathematics and its role in the world. Cultural and contextual knowledge of mathematics thus involves not only a deep knowledge of mathematical concepts, but also an understanding that mathematics arises in historical and cultural contexts, sometimes as an end in itself, and sometimes in response to the need to solve problems in the world. It involves understanding that mathematics continues to evolve and grow at an ever-increasing rate. It involves having a critical appreciation of how mathematics has been and continues to be used across all aspects of human society, sometimes to advance and inform, but sometimes to obscure or confuse.

I suggest that enriching teachers' cultural and contextual knowledge of mathematics may be at least as important as ensuring that they have strong content knowledge and pedagogical content knowledge. Having an appreciation of mathematics as a human endeavour, taking place throughout history and across all cultures, directly challenges notions of mathematics as linear and fixed. Having a rich cultural and contextual knowledge of mathematics moves teachers beyond being technical implementers of curriculum and enables them to become creative and critical problem solvers who can instil in students a love of mathematics and a deeper appreciation of its role in society.

The potential for teachers' cultural and contextual knowledge of mathematics to make mathematics come alive for students is beautifully captured by Barton

(Watson & Barton, 2011), reflecting on one of the "best" lessons he ever taught:

> The syllabus I am using requires five lessons on 2 × 2 matrices for my class of 14-year-olds. We have looked at arrays and gone through the operations +, −, ×, ÷ with other matrices and 1 × 2 vectors. The final section is on matrices as transformations of the unit square: reflections, stretches, shears and rotations. We do not quite finish, so I use a little of the next lesson in a tight syllabus. In response to an invitation to the students to give me a random matrix so we can look at its effect, I get a 3 × 3 matrix suggested. Smart kid. The class appears to have understood the 2-dimensional concept, so I extend, draw a unit cube and watch as they quickly pick up the idea and stretch and reflect it in a plane. No problem—until the same child, flushed with her success, asks about a 4 × 4 matrix with a smile, knowing that there are only three dimensions. I seize the moment to demonstrate the power of mathematics to go beyond our experience and soon hypercubes are being reflected through 3-D space using the patterns of 2 × 2 and 3 × 3 reflections. The keen students take home work on problems in 5 or 6 dimensions. But that lesson has been used up, and half the next one, and I am dreadfully behind my schedule. After the lesson, why did I not feel concerned? And why, 30 years later, do I remember that lesson as one of my best? (p. 65)

Barton knew about the connection between matrices and transformations; he knew about 3- and 4-dimensional space, and he had the confidence to allow students to experiment with ideas that were beyond the formal curriculum.

8.3 Reflections on a trial

As discussed above the development and value of teachers' cultural and contextual knowledge of mathematics is almost completely unresearched. I remember from my own experience being part of a group of teachers who were provided with six months sabbatical to "simply attend" classes in new areas of applied mathematics at the University of South Australia. The seven teachers

involved enjoyed the experience and gained a great deal of satisfaction from talking with each other and with the mathematicians in the university. The experience gave me a deeper understanding of areas of mathematics such as operations research and statistics and a much greater awareness of applications such as control systems and scheduling. I enjoyed the challenge of solving hard problems and of feeling part of a community of people engaged in similar pursuits. I was also prompted to read alternative descriptions of mathematics and its place in society such as those by Davis and Hersh (1981, 1986). While it is difficult to point to specific changes that resulted from the sabbatical, I am sure that it forms part of an accumulation of experiences that has shaped my views of mathematics and education.

A notable exception to the lack of attention to cultural and contextual knowledge in teacher education is the history of mathematics unit studied by preservice primary and secondary teachers at Sheffield Hallam University (Povey, 2013; Povey, Elliott, & Lingard, 2001). Preservice teachers were asked to write about the impact of the unit and in analysing the responses the researchers identified several themes. In particular preservice teachers commented on:

> the novelty of the learning; its impact on their enthusiasm for the subject; the experience of new relationships with the discipline; the linking of mathematics to humanity and human endeavour; connections made or not made with themselves as prospective teachers; and, tentatively, overt links with equity issues. (Povey et al., 2001, p. 11)

It was also suggested that preservice teachers were able to see mathematics from the inside, in the act of doing mathematics, rather than only from the outside as in traditional courses of mathematical content.

In a later study the teacher educators involved identified four benefits that they felt flowed from the course on the history of mathematics:

- to deepen mathematical understanding;
- to broaden and humanise mathematics;
- to develop critical thinking; and
- to provide motivation and fun for learners. (Povey, 2013, p. 148)

8.3.1 Mathematising and contextualising

Many of the same sentiments were also expressed by my own students in a unit at Charles Darwin University that we called *Mathematising and Contextualising*. The unit was developed as part of a new course for primary teachers, and was designed to be the third mathematics unit in the course. It was intended to follow a first year unit that aimed to develop mathematical content knowledge and a second year unit that aimed to develop pedagogical content knowledge and curriculum knowledge. It was specifically designed to engage students in using mathematics to analyse a current issue and in learning something about the history, culture or contexts of mathematics. It offered preservice teachers a choice of issues on which to write based on readings related to critical numeracy, history of mathematics, ethnomathematics and mathematical modelling. Preservice teachers completed an essay and maintained a blog reflecting on their reading and learning. I present below, with participants' permission, extracts from their blogs. The extracts show a deep engagement with cultural aspects of mathematics and a heightened appreciation of its human and historical origins.

Karen[51] has a relatively strong background in mathematics, having completed high level mathematics subjects in year 12. In her blog she wrote about three benefits for using the history of mathematics in the classroom: approaches to teaching, impact on student attitudes, and society and cultural factors. In discussing approaches to teaching she wrote: "traditional textbooks establish an

[51] Participants' names are pseudonyms.

approach to mathematics that, although logical with respect to the ideas being addressed, lacks the human element of endeavours in mathematics, which in their very nature occurred in a non-linear fashion." In discussing the impact of using historical perspectives on student attitudes she noted that students might be more motivated by historical dimensions and that it might help them to see that even mathematicians make mistakes. In discussing social aspects of mathematics, she explained the importance of seeing mathematics as socially constructed and continuously developing. She suggested of students: "Perhaps in their own way they, too, can add something of their own to the world of Mathematics?"

Karen also reflected on a lesson she had observed in a Steiner school, where the historical and cultural aspects of mathematics were emphasised. The lesson in question dealt with number systems through history, commencing with a discussion of an Aboriginal number system, and then dealing with Hindu-Arabic, Roman, Egyptian and Chinese-Japanese numbers systems. "It was at this point where the students were challenged to think about why we might be most comfortable with base ten." In reflecting on the lesson Karen wrote:

> I felt, watching this lesson and talking to the teacher afterwards, that this approach supported many of the ideas that I wrote about in my previous blogposts, including improving students' attitudes, humanizing the mathematics and looking at aspects of culture, all creating a natural, organic approach to teaching the subject. I'm certainly not advocating that every lesson should follow these sorts of approaches, but rather that they be integrated with conventional exercises, practical applications and ICT activities to provide greater meaning and human interest into the adolescent mathematics classroom. I've certainly managed to convince myself, and look forward to an opportunity to try some of these ideas for myself.

Michelle is studying to be an early childhood teacher, and wrote particularly about the use of meaningful contexts for young children. She argued that an early childhood approach would be beneficial if adopted throughout high school. She advocated meaningful choice for students in the mathematics they studied, following the argument in an article she had read by Garfunkel and Mumford (2011). She also recognised that not everything needs to be applied to real life, and agreed with some of the responses to the article that suggested that a number of things people do in life are simply to keep their minds and bodies "fresh and sharp". She concluded:

> In general I am unsure about where my beliefs in this topic now stand. Although I completely agree with the writer's position about making math in school a practical and purposeful connection to the lives of the students and future, I also agree that not everything we do is to progress our future career and must be relevant every day. Maybe a working combination of both can be used to progress the mathematical futures of all students both practically and cognitively. I hope to explore these ideas further, especially looking more into if teaching children how to learn really does benefit them in the long run.

Sarah has a major in politics and English and has never particularly enjoyed mathematics at school. She wrote:

> I feel that I make sense of the world in terms of its history and culture. Just by skimming the surface of the cultural roots of mathematics articles, all of a sudden mathematics has evolved from ancient hieroglyphs to a story, a history, a world I can relate to. I have never particularly enjoyed math. If I am not being forced to solve it in an academic forum, it can go unsolved in my books. I much prefer a good narrative to a mathematical brain teaser. Although, by beginning to delve into the cultural roots of mathematics, I find literature, art and intriguing stories.

Sarah chose to use technology to complete an interactive timeline showing the development of mathematical ideas from ancient times to the present. She intends to do this in the future as a research activity for primary school students in which they describe the cultural context, the significance of the mathematics developed and the people involved.

Tierney admitted that mathematics has never been her strong point, and suggested that this may have been because of how it was presented at school. She focused on cross-curriculum aspects of mathematics following a mathematical modelling perspective based on Galbraith (1998). She discussed two examples of how mathematical modelling can bring a cross-curriculum perspective to high school mathematics and wrote:

> Although Galbraith is addressing high school, the examples given do spark ideas of lessons for younger children and the idea of cross-disciplinary applications could easily be applied to primary education. It would perhaps be easier to achieve this effectively in a primary school class due to the fact that one teacher teaches their class a variety of subjects, compared to high school where teachers specialize in particular subjects.
>
> I do think this proposal on how mathematics can be applied in the curriculum would challenge, engage and help give a relevant context to the students work.

Each of the four students identified aspects of the history, culture or context of mathematics that they felt had influenced their view of mathematics and would influence their teaching. For some the focus on culture and context caused them to question the approach to learning mathematics they had experienced in school and to seek a more human face. Some identified the deeper mathematical knowledge that might come from learning about the history of mathematics and suggested some specific projects they would use in their teaching.

It is perhaps significant that the preservice teachers in the subject were studying entirely by distance. There were no face-to-face classes at which they could discuss ideas or where they could receive input from a lecturer. Their initial writing was thus entirely a response to what they had read and thought about in their own context. However, the social aspect of responding to what others had written on their blog also impacted on their thinking. They suggested new ideas, affirmed others' thoughts and provided encouragement to pursue ideas.

> I like how you have found the best way make this assignment relate to you. I guess that is what we as teachers should be doing for our students. Understanding the students' different learning styles and interests so we can relate the mathematical information in the way that will relate to them. (Tierney to Sarah)

For these preservice teachers, not only did they gain an appreciation of the social and cultural context in which mathematical knowledge develops, but they also participated, albeit at distance, in a social environment that stimulated their own knowledge development. As Petrou and Goulding (2011) note, most of the models of teacher knowledge discussed above adopt an individualistic perspective. They suggest that further research might focus on how mathematical knowledge in/for teaching is located in the collective. I suggest that this may be even more important if the focus is on the development of cultural and contextual knowledge of mathematics for teaching.

8.4 Conclusion

While the considerable research effort describing mathematical knowledge for teaching has undoubtedly led to a greater understanding of the importance and interaction of different types of knowledge, it may also unwittingly have led to a situation where the emphasis on pedagogical content knowledge has reinforced strict adherence to a predetermined curriculum and sequence of learning. In this chapter I have argued for a new aspect of teacher knowledge:

cultural and contextual knowledge of mathematics. This knowledge includes historical aspects of the development of mathematics, connections between mathematics and culture, an appreciation of the applications of mathematics in the world and critical numeracy. I have presented some preliminary research from a preservice teacher education subject that suggests such a focus may deepen mathematical knowledge, humanise mathematics, develop critical thinking, provide motivation and, perhaps most importantly, may challenge existing perceptions of mathematics.

I commenced the chapter with a quote from Paul Lockhart's simultaneously distressing examination of the current state of school mathematics and uplifting account of what it is to appreciate mathematics. In the conclusion to his book (Lockhart, 2009), he offers one piece of practical advice: "just play" (p. 139). He suggests that teachers "*especially* need to be playing around in Mathematical Reality" (p. 139). While I might not go as far as Lockhart and suggest throwing the textbooks and curriculum out of the window, I would certainly argue that without an appreciation of mathematics in context and culture, the curriculum becomes little more than a recipe to be followed. On the other hand appreciating the culture and context of mathematics, or as I have termed it *Slow Maths*, places the recipes in their proper perspective as ideas and guidance, but no more than that. I argue that the creative work of teaching and learning mathematics is only possible with a rich appreciation of mathematics and its culture.

CHAPTER 9: CONCLUSION

> [The] educational way [is] the slow way, the difficult way, the frustrating way, and, so we might say, the weak way, as the outcome of this process can neither be guaranteed nor secured…But in the long run it may well turn out to be the only *sustainable* way, since we all know that systems aimed at the total control of what human beings do and think eventually collapse under their own weight, if they have not already been cracked open from the inside before. (Biesta, 2014, pp. 3, 4, italics in original)

Synopsis

The argument revisited: I present a short restatement of the argument for *Slow Maths*.

Thesis and anti-thesis: I pose the final question of whether *Slow Maths* can be an evolution or needs to be a revolution.

9.1 The argument revisited

This thesis has presented an argument for rediscovering the essence of mathematics using the metaphor of slow food. In Part 1 of the thesis I established a case for slowness. Chapter 1 presented a personal journey, describing the joys of learning and teaching mathematics, coupled with the frustrations of holding views that did not seem to be in line with the dominant views of mathematics teaching. The personal tensions received a public face in the so-called *Math Wars*, which I argue were essentially a battle of philosophy and epistemology rather than pedagogy. I am certain that if I were teaching school mathematics in the current age of accountability and standardised testing these frustrations would be even more acute.

In Chapter 2 I named and critiqued the dominant metaphors for education: those of education as production and education as a cure. To these I added an increasingly prevalent metaphor: that of education as a race. I argued that each

of these is locked into a view of the world that is technological, with a means-end rationality that attempts to remove risk and variability. Yet risk and variability are at the heart of the educational endeavour, and should, I suggest, be valued and embraced rather than minimised. Hence I described a new and life-affirming metaphor for education that is based on the slow food movement—one that values traditions and culture, that mixes familiar ingredients in new ways and that has an ethical dimension that respects the educated human being as being. Since my principal concern is the field of mathematics I termed the approach *Slow Maths.*

Chapter 3 examined competing philosophies of mathematics, arguing that absolutist and relativist philosophies represent outside and inside views of mathematics respectively. As such they can be regarded as differences in representational grainsize rather than binary opposites. Holding this inside/outside view of mathematics allows one to see mathematics simultaneously as a coherent big picture and as a living, evolving body of knowledge in which uncertainty is central. I suggested that the current level of relative agreement between the mathematics community and the mathematics education community may be an uneasy peace, and that lasting agreement is more likely if absolutist and relativist philosophies are reconciled. I suggested that Kitcher's (1988) mathematical realism, which conceives of knowledge building in mathematics as rational transitions between mathematical practices, might help that reconciliation.

Part 2 of the thesis presented the philosophical basis of *Slow Maths* in terms of three dimensions of connectedness. It is important to recognise that I have not termed *Slow Maths* a *philosophy of mathematics education*, nor even a *philosophy* of school mathematics. Rather it is a metaphor for school mathematics that promotes a coherent and justifiable approach to curriculum, pedagogy and assessment. It does not purport to provide answers, but like all

metaphors it provides a lens through which to examine the practices of school mathematics teaching and learning.

Chapter 4 outlined what I termed *mathematical connectedness*. This dimension of connectedness is concerned with the internal grammar of mathematics, and its strong rules governing what counts as a legitimate mathematical argument. In the school curriculum it is about how one mathematical concept relates to another. In the school classroom it is about how students establish mathematical validity using the agency of the discipline. It is about understanding, explaining, establishing, verifying, defining and proving.

Chapter 5 outlined what I termed *cultural connectedness*. This dimension of connectedness is concerned with mathematics as a living body of knowledge, located and developing in cultural contexts and with its own distinctive culture. In the school curriculum it is about recognising the cultural and historical roots of mathematics, not as add-ons but as fundamental to developing deeper understanding. It is also about introducing contemporary areas of mathematics into the school curriculum. In the school classroom it is about students coming to see themselves as part of the rich culture of mathematics. It is about investigating, interpreting, developing, posing and exploring.

Chapter 6 outlined what I termed *contextual connectedness*. This dimension of connectedness is concerned with mathematics as a language to model and explain the world and as a body of knowledge that grows in response to problems in the world. In the school curriculum it is about articulating systematic and explicit use of contexts both as areas in which mathematics may be applied and as springboards for developing mathematical understanding. In the school classroom it is about students mathematising to model and solve real and significant problems. It is about modelling, predicting, solving, deciding, calculating, estimating and analysing.

In Part 3 of the thesis I gave examples of how *Slow Maths* might impact upon students and teachers respectively. In Chapter 7 I discussed the ways in which curriculum, pedagogy and assessment are currently described and enacted, arguing that no matter how progressive they appear they are locked into a technological way of seeing the world. Thus we are forever concerned with meeting standards and providing intervention for those that fail. I give a practical example of a unit of work that is explicitly built on the three dimensions of mathematical, cultural and contextual connectedness, but most importantly that accepts and embraces variability in input and outcome.

Chapter 8 presented a new aspect of mathematical knowledge for teaching arising from a metaphor of *Slow Maths*. I argued that cultural and contextual knowledge of mathematics might be at least as important as conventional descriptions of knowledge such as subject matter knowledge and pedagogical content knowledge. I gave an example of how a small number of preservice teachers developed a richer appreciation of mathematics through engaging in a subject related to history, culture and critical numeracy.

9.2 Thesis and anti-thesis

I have used many examples from mathematics, numerous stories of mathematicians and several key concepts from mathematics education research. Most are well known, at least to the professional mathematician and mathematics educator. This is intentional, as unless the mathematics education community can readily identify with the evidence underpinning *Slow Maths*, it is unlikely that they will identify with the concept itself.

The argument for creative, human and life-affirming approaches to education is not new. Gert Biesta (2014) has described *The Beautiful Risk of Education*, claiming that weakness lies at the very heart of education. He asserts that removing the risk from education may well remove education from education. My late MERGA colleague Jim Neyland (2010) has written passionately about

Rediscovering the Spirit of Education after Scientific Management, arguing that education should be enchanting, ecstatic, autotelic and comical. They and others like them have presented a case against an educational system obsessed with control much more eloquently than I can.

The argument for a mathematics curriculum and pedagogy that values history and culture, and that immerses students in creative problem-solving endeavours is not new. Paul Lockhart (2009) has written a deeply depressing but unnervingly accurate portrayal of a mathematics curriculum devoid of mathematics in *A Mathematician's Lament*. But he has described an alternative that puts the sheer joy of doing mathematics and being part of the rich tapestry of mathematical discovery at the centre of school mathematics. Magdalene Lampert (2001) has written a delightful account of *Teaching Problems and the Problems of Teaching* in her own classroom, one that is centred on children actively engaged in the process of developing mathematical knowledge through deep engagement in worthwhile problems. They and others like them have presented an image of what school mathematics might be.

So this thesis is very much in the spirit of a large body of work that rejects narrow and controlled approaches to school mathematics in favour of one that puts the creative act of problem-solving at the centre. What is new in this thesis is that these arguments are given a name and a metaphorical embodiment. I have called it *Slow Maths*, not simply because it takes time to learn mathematics deeply, but because slowness provides a fundamental challenge to managerial approaches to education.

I commenced this chapter with a quote from Gert Biesta's description of the real risks in education in his book *The Beautiful Risk of Education*. He argues that taking the risk out of education comes at a price, asking whether that price is worth paying. He asserts that "the call to make education strong, secure and predictable and risk-free is an expression of…impatience" (p. 3) and calls for

an education imbued with wisdom, values and democracy. He concludes his book with a call for what he terms a "pedagogy of the event", writing:

> …if we are genuinely interested in education as a process that has an interest in the coming into the world of free subjects…we need a pedagogy that is orientated positively toward the weakness of education. This is a pedagogy, in short, that is indeed willing to take the beautiful risk of education. (Biesta, 2014, p. 140)

In the preceding pages I have presented not only a thesis, an argument for *Slow Maths*, but also an anti-thesis, an argument against many of the current educational directions that stifle mathematical creativity in favour of results on standardised tests. The question that remains unanswered is whether it is actually possible to locate *Slow Maths* within the current educational system. Is rediscovering mathematics as a cultural and historical pursuit an evolutionary process? Or is the system already, as Biesta claims all systems obsessed with control eventually do, collapsing under its own weight? Is it so cracked from the inside that evolutionary change is simply not possible? The final question I pose, then, is does *Slow Maths* therefore require a revolution, a complete rejection of managerial approaches to education and the mathematics curriculum, pedagogy and assessment that resides within it?

AFTERWORD

I have chosen to present this thesis entirely as philosophical reflection, supported by literature. There is no new empirical data or analysis as the reader might expect in a traditional thesis in education or the social sciences. However, the thesis is not without data. The data is an amalgam of my own teaching experiences and the rich data that supports the research studies reviewed within the thesis. Hence the thesis could be termed a meta-reflection on the nature and purpose of school mathematics. Presenting first-hand data would, I believe, have detracted from the cohesion and power of the argument.

In many ways this thesis is a *slow* thesis. As I indicated in the Foreword it has taken a number of twists and turns, changes of topic, changes of location and changes of life. Perhaps most, if not all, PhD theses are *slow* in this sense. But I have also chosen to present it as a thesis that tells a *slow* story.

The thesis has a philosophical basis that I hope has become clear as it has progressed. I trust that my personal philosophy of mathematics and teaching as creative, human and intellectually rigorous endeavours has come through. I greatly value the culture and traditions with which I was brought up, which have become clearer as a teacher and which I hope I have been able to instil in at least a few of my students.

Throughout the thesis I have drawn upon established ideas to support what I hope is a fresh argument. One could argue that it introduces few new ideas, but I hope that it recruits existing ideas in fresh ways that will make sense to a range of audiences. Perhaps like all theses, the process is inherently uncertain, and the final product may bear little resemblance to the original intent. That is a process to be valued rather than lamented. Finally I trust that the thesis stands as something that is ethically defensible in that it takes a life-affirming view of mathematics and education. I pray that it is judged to be both good and worthwhile.

I certainly do not intend, nor do I wish, that *Slow Maths* become some kind of silver bullet or prescription for "fixing" mathematics education. I hope that no school ever proclaims itself as adopting or following a *Slow Maths* curriculum, or of introducing a *Slow Maths* intervention. My intent and hope is that thinking about school mathematics in a *slow* way might enable teachers and schools to ask deep questions about *what* is worth teaching and *why* students should learn it, rather than seeking ready-made fixes for perceived problems. This is an inherently variable and risky process with no predefined answers. It is also an intellectually rigorous process, and my hope is that at least some teachers might embrace the "beautiful risk" of *Slow Maths*.

It was once written about one of my educational heroes Lawrence Stenhouse, who was a founder of the teacher as researcher movement, that he had "an intellectual commitment to uncertainty". I hope that this thesis is, at least in some small respect, simultaneously intellectual, committed and uncertain. If so, and if it makes even a tiny fraction of the impact that Stenhouse's work once did, I shall be overjoyed.

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