USE OF THESES

This copy is supplied for purposes of private study and research only. Passages from the thesis may not be copied or closely paraphrased without the written consent of the author.
THE GENERAL THEORY OF
THE CONTRACTIONS OF LIE GROUPS,
LIE ALGEBRAS, AND THEIR REPRESENTATIONS

DAVID NATHANIEL ALEXANDER FORRESTER

DATE OF SUBMISSION: DECEMBER 1990

A thesis submitted for the degree of Doctor of Philosophy of the
Australian National University.
The contents of this thesis are entirely my own, original work, except where otherwise indicated.

D.N.A. Forrester

24/12/90

D.N.A. Forrester
ACKNOWLEDGEMENTS

The author wishes to thank his supervisor, Professor Derek W. Robinson for his continued keen interest, encouragement and support.

This thesis would not have come into being without the diligent co-supervision of Dr. Tony Dooley who gave freely of his time even to the point of domestic intrusion. The author here records his very great gratitude.

Special thanks are due to Professor Michael Cowling who gave freely of his time during 1990 and expressed a warm and friendly interest in my work.

Nevertheless, the greatest debt of thanks is due to

Beresford Clifford Waterman

who has truly been, and still is a major source of inspiration and energy in the author's life, as well as many others.

This acknowledgement would not be complete without the statement of my eternal thanks to Swami Satyananda Saraswati and Mahamandaleswar Sri Swami Sivanandaji Maharaj. The author is also most grateful for the support of Norma Edwards, Swami Bhaktimurti Saraswati, Swami Bhavanandana Saraswati, John Kesterton, and Dr. Harvey Turk.

Thanks and astonishment are due to Mayda Shahinian for the incredible feat of typing chapters 1, 2, 3 and 5. The author also wishes to thank Janette Neville for typing chapter 4.

Finally, the author wishes to thank Assoc. Professor Colin Sutherland for his early supervisory work during one of Professor Robinson's sabbatical leaves and Dr. Shaun Disney who has provided a lot of his time in answering questions of differential geometry and word processing.
While the author was writing this acknowledgement, he became aware of excited, little whispers gradually loudening to rumbles and thumps and reaching a muted clamour as he wrote the word “finally”.

The author therefore begs forgiveness for nearly omitting mention (very loud thumps and banging) of the inspiration supplied by the following Ones:

Sir Mujumup Wipsnibbles, Little Lord Botherington Wipsnibbles, Dame Grey-moor Staggers, Sir Njinud Smodge O.M.N.C., Icabod Snappums, Astrobel, Wampum Broot, Smoohah Bejoojit, those who have not told me their name as yet, and the transcendent One who lives (I use the word cautiously as I know there’s no beginning and end) in THE LOG.
THE LOG

Christmas Dinner

(No animal products used)

Grilled Funnelweb Fingers

Kookaburra Crunch

Magpie Muffins

Snake Snippets

Platypus Pumpkin Pie

Bluetongue Lizagne

Bull-ant Bavarois

Koala Jelly sprinkled with gumnuts for protein.
Motivated by the objective of explaining by group-theoretic means, why quark masses are different, this thesis generalises, globalises and clarifies the existing theory of the contractions of Lie algebras.

Whereas Lie group contractions were previously viewed in the domain of deformation theory, it has been found that Lie group contraction may be viewed as a homomorphism from a finite-dimensional Lie group to the contracted Lie group, or from an infinite-dimensional Lie group to the contracted Lie group. For the corresponding Lie algebra contraction, the Lie algebra of the domain of the contraction mapping may be a subalgebra of the Current Algebra as used by physicists, or a subalgebra of a factor algebra of Current Algebra. Intriguing connections are thereby made to elementary particle physics.

A very simple closed form for the necessary and sufficient conditions for the contraction of Lie algebras to take place has been found, as well as for the structure of the contracted algebra, and special cases of the contracted group.

Another result is that for any connected Lie group $G$, there is a sequence of nilpotent Lie groups of particularly simple structure, the limit of which recovers the structure of $G$.

For the contraction of representations we provide a pragmatic theorem which provides one, simple test which indicates whether all matrix elements or no matrix elements of a representation of the contracted Lie group will be the limit of matrix elements of a sequence of isomorphic copies of the original Lie group. That is, a simple yes/no test which provides an "all or nothing" answer for the question of whether there is one among the contraction of representations of the original Lie group that is equal to a given representation of the contracted Lie group.

Utilising the explicit form of the infinitesimal character of a unitary irreducible representation supplied by the method of orbits, it is shown how infinitesimal
characters may be meaningfully contracted. With assumptions on the contraction process and the orbits, a formula for contracting generalised characters is also given.
TABLE OF CONTENTS

1 Introduction and Notation
   1.0 Notation and Conventions.
   1.1 Finite Dimensional Lie Groups and Lie Algebras.
   1.2 Infinite Dimensional Lie Groups and Lie Algebras.
   1.3 Representations of Finite Dimensional Lie Groups and the Method of Orbits. Generalised Characters and Infinitesimal Characters.
   1.4 Differential Geometry and Differentiable Bundles.
   1.5 The Contractions of Lie Algebras and Lie Groups.
   The Contractions of Representations of Lie Groups.

2 Contractions of Lie Groups and Lie Algebras
   2.1 Motivations for this Work.
   2.2 Work Done by Other Authors and Work Done to the Present.
   2.3 Examples and Initial Definitions. Special Case of Embedding in a Larger Lie Group.
   2.4 Contraction of Lie Algebras. How Fully Generalised Saletan Contraction of Lie Algebras May be Viewed as a Homomorphism. Connection to the Current Algebras of Elementary Particle Physics.
   2.5 General Contractions as Homomorphisms from Banach-Lie Groups and as Homomorphisms from Finite-Dimensional Lie groups.
3 Applications of Contractions and Extensions of Contraction Definitions


3.2 The Rôle of Jet Bundles in Higher Dimensional General Contractions and Extension of Contraction of Lie Algebras to the Larger Domain of Laurent Polynomials, for Applications to Elementary Particle Physics.

4 Contractions of Representations of Lie Groups

4.1 Using the Method of Orbits. Contracting a Sequence of Forms. Contracting Orbits.

4.2 Contracting Polarizations and Stabilisers of Functionals.

4.3 Theorems Connecting the Method of Orbits on the Sequence of Isomorphic Lie Groups with the Method of Orbits on the Contracted Lie Group.

5 Contractions of Generalised and Infinitesimal Characters

5.1 Contractions of Generalised Characters of Irreducible Unitary Representations.

5.2 Contractions of Infinitesimal Characters.

6 Appendix

7 Bibliography

8 Index to Notation and Subjects
Chapter 1: Introduction and Notation

In Chapter 1, we will summarize all the results needed in Chapters 2, 3, 4 and 5 on Lie groups and Lie algebras in both finite and infinite dimensions, representations of Lie groups, differentiable bundles, contractions of Lie algebras and Lie groups and contractions of their representations. The objective of this chapter is to standardise notation and provide much-needed background material, as the thesis makes use of results in five different areas. We therefore make no claim to originality for Chapter 1.

Specifically, in §1.1 we define manifolds, Lie groups, tangent vector fields and the tangent bundle, leading to the exponential mapping and the Lie algebra of a Lie group and the Baker-Campbell-Hausdorff formula. Various theorems on homomorphisms of Lie groups and Lie subgroups associated to subalgebras are also given. A small section develops the adjoint and co-adjoint representations, co-adjoint orbits and invariant vector fields on homogeneous manifolds. In order to define integration on manifolds, exterior algebra bundles and differential forms are introduced. Definitions and properties of particular classes of Lie groups such as reductive, nilpotent, semisimple, cocompact nil-radical, semi-direct product and exponential Lie groups are given leading up to a brief structure theory for semisimple Lie groups and compact symmetric pairs. Section 1.1 concludes with the definition of tensor algebra, and universal enveloping algebra.

Section 1.2 is a re-development of theorems, definitions and propositions of §1.1 as they specifically apply to infinite-dimensional Banach-Lie groups, covering manifolds, Lie groups, tangent vector fields, the tangent bundle, Lie algebras and the exponential mapping and its properties. Section 1.2 concludes with propositions concerning homomorphisms, Lie subgroups and their corresponding subalgebras.
In §1.3 the concepts leading up to linear and unitary representations are given. An exposition proper of the method of orbits is then commenced, and is interspersed with comments and outlines to further illustrate and clarify the techniques of the procedure. In particular, we develop integral functionals, polarizations, the $\beta$-derivative and all necessary concepts of induced $\mathcal{G}$-modules of spaces of $C^\infty$ functions on $\mathcal{G}$ in order to give a $\mathcal{G}$-module, with $\mathcal{G}$ acting on a space of $C^\infty$ functions defined on $\mathcal{G}$, which is irreducible.

The next step is to elaborate on this irreducible $\mathcal{G}$-sub-module to provide an irreducible unitary representation of $\mathcal{G}$. This involves a thorough-going treatment of unitary induced representations as well as an "inducing in stages" theorem.

The next group of definitions and propositions of §1.3 deals with involutive Banach algebras of $L^1$ functions on $\mathcal{G}$ leading to the definition of a Generalised Character of a unitary representation and theorems giving its specific form for wide classes of Lie groups and several classes of their representations.

Finally, §1.3 closes with definitions of generalised functions on $\mathcal{G}$ with compact support, their algebra structure, the Gårding space of a continuous unitary representation of $\mathcal{G}$ and their relationship to the infinitesimal character of a unitary representation. The last theorem gives a specific, simple and geometrically appealing form for the infinitesimal character, which is used in Chapter 5.

In §1.4 we define the affine connection, covariant differentiation, and Riemannian manifolds. Geodesics, normal coordinates and the exponential map for manifolds are developed, and their relationship to canonical coordinates and the exponential map for Lie groups is given. Vector bundles are then introduced with the important (for Chapters 2, 3 and 4) example being given of the $q$'th cartesian cross-product bundle of $q$ tangent bundles over a Lie group $\mathcal{G}$.

We close §1.4 with the definition of fibre bundles over a manifold and the example of the $q$-jet bundle, to be used in §3.2.
Section 1.5 reproduces the Saletan and Inönü-Wigner theories of Lie algebra contraction, a Lemma on their domain of applicability and a definition and some properties of generalised Inönü-Wigner contraction. Global definitions and properties of Lie group contraction as given by Dooley (1983) (Global Inönü-Wigner contraction, with a mild reductivity condition), and Dooley and Ricci (1985) (contraction of a maximal compact subgroup inside a connected semisimple Lie group of real rank one) are exposited. Finally, the contraction of representations of a compact symmetric pair (Inönü-Wigner contraction) is then developed, as given by Dooley and Gaudry (1986), culminating in a theorem relating matrix elements of representations of the contracted group to limits (with respect to a contraction parameter) of matrix elements of the original Lie group.

§1.0 Notation and Conventions

• $G, G_1, G_2$ shall stand throughout to denote Lie groups, of finite or infinite dimension.

• The summation convention shall always apply except where otherwise indicated. Thus

$$\lambda^i \frac{\partial}{\partial x^i} \text{ stands for } \sum_{i=1}^{n} \lambda^i \frac{\partial}{\partial x^i}. $$

Generally where any two indices or power term indices in a polynomial occur together, they should be summed unless stated to the contrary. Thus,

$$\lambda^i \mu^j \left[ \frac{t^I X_i}{I!}, \frac{t^J X_j}{J!} \right] \text{ stands for, } \sum_{i,j=1}^{n,q} \lambda^i \mu^j \left[ \frac{t^I X_i}{I!}, \frac{t^J X_j}{J!} \right]. $$


§1.1 Finite-dimensional Lie Groups and Lie Algebras

In §1.1 we will mostly follow the elementary exposition of Cohn (1968), as Maissen (1962) in his treatise on Banach-Lie groups, translates Cohn’s scheme step by step to the infinite-dimensional setting and it is Maissen’s work which forms the bulk of §1.2. In places where Cohn’s approach is inappropriate, Sagle and Walde (1973), Kirillov (1976), and Helgason (1978) are quoted. In §1.2 we will point out what changes have to be made to §1.1 in order that the Definitions and Propositions for Lie groups hold in the infinite dimensional case.

Definition 1.1.1 (Cohn, §1.1)

A $C^\infty$ manifold, is a Hausdorff space $\mathcal{M}$ together with a family $\mathcal{F}$ of charts, each chart of which is a homeomorphism of an open subset of $\mathcal{M}$ into $\mathbb{R}^n$ such that $\mathcal{F}$ satisfies:

(a) At each point of $\mathcal{M}$ there is a chart which belongs to $\mathcal{F}$.
(b) Any two charts $\chi_1, \chi_2$ of $\mathcal{F}$ are $C^\infty$-related. That is, if $\chi_1, \chi_2$ have domains $\mathcal{N}_1, \mathcal{N}_2$ respectively $\chi_2 \circ \chi_1^{-1}$ is a $C^\infty$ function on $\chi_1(\mathcal{N}_1)$ and $\chi_1 \circ \chi_2^{-1}$ is a $C^\infty$ function on $\chi_2(\mathcal{N}_2)$.
(c) Any chart in $\mathcal{M}$ which is $C^\infty$-related to every chart of $\mathcal{F}$, itself belongs to $\mathcal{F}$.

A submanifold is a subset $\mathcal{M}_1$ of $\mathcal{M}$ such that:

(a) $\mathcal{M}_1$ is a manifold
(b) For any chart $(\chi, \mathcal{N})$ of $\mathcal{M}$, $\chi : \mathcal{N} \to \mathbb{R}^n$, the coordinate functions $\chi^i|_{\mathcal{M}_1} : \mathcal{M}_1 \to \mathbb{R}$ are $C^\infty$ functions on $\mathcal{M}_1$. That is, for any chart $(\chi_1, \mathcal{N}_1)$ of $\mathcal{M}_1$, $\chi_1 : \mathcal{N}_1 \to \mathbb{R}^{n_1}$, the functions $\chi^i \circ \chi_1^{-1} : \mathbb{R}^{n_1} \to \mathbb{R}$ are $C^\infty$.
(c) at each point $m_1$ of $\mathcal{N} \cap \mathcal{M}_1$, there is a subset $\left(\chi_{i1}|_{\mathcal{M}_1}, \cdots, \chi_{in_1}|_{\mathcal{M}_1}\right)$ which forms a chart of $\mathcal{M}_1$ at $m_1$. 

13
Definition 1.1.1(a) (Kirillov, §5.3)

Let \( \mathbb{R}^{n+} = \{(x_1, \ldots, x_n) \mid x_n > 0\} \).

A \( C^\infty \)-manifold with boundary is a Hausdorff space \( \mathcal{M} \) together with a family \( \mathcal{F} \) of charts, each chart of which is a homeomorphism of an open subset of \( \mathcal{M} \) onto an open subset of \( \mathbb{R}^n \) or onto a subset of \( \mathbb{R}^{n+} \) which is a neighbourhood of zero, relatively open in \( \mathbb{R}^{n+} \subset \mathbb{R}^n \), such that \( \mathcal{F} \) satisfies (1.1.1(a) (b) (c)).

Definition 1.1.2 (Cohn, §2.6)

A Lie group is a set \( \mathcal{G} \) such that

(a) \( \mathcal{G} \) is a group

(b) \( \mathcal{G} \) is a \( C^\infty \) manifold

(c) The mapping \( (g, h) \mapsto gh \) of the product manifold \( \mathcal{G} \times \mathcal{G} \) into \( \mathcal{G} \) is \( C^\infty \). That is, if \( (\chi_1, \mathcal{N}_1) \) is a chart at \( g \), \( (\chi_2, \mathcal{N}_2) \) a chart at \( h \) and \( (\chi_3, \mathcal{N}_3) \) a chart at \( gh \) and \( x_1, x_2 \) are coordinates in \( \chi_1(\mathcal{N}_1) \) and \( \chi_2(\mathcal{N}_2) \) respectively, then the function \( \chi_3(\chi_1^{-1}(x_1)\chi_2^{-1}(x_2)) \) is \( C^\infty \) in \( (x_1, x_2) \).

By virtue of the Implicit Function Theorem, we have that,

Proposition 1.1.3 (Cohn, 2.6.1)

In a Lie group \( \mathcal{G} \), the mapping \( x \mapsto x^{-1} \) is \( C^\infty \).

Remark 1.1.4

By a deep result of Montgomery, Gleason and Zippin (Montgomery and Zippin, 1955), a set \( \mathcal{G} \) which satisfies (1.1.2) with “\( C^\infty \)” replaced by “continuous” can be given a unique differentiable structure such that it is an analytic Lie group (replace “\( C^\infty \)” in (1.1.1) and (1.1.2) by “analytic”) and is hence a Lie group in the original sense of (1.1.2). In practice, it is often simplest to prove that differentiable structures are only twice continuously differentiable and hence \( C^\infty \).
Definition 1.1.5 (Cohn, §2.8)

A Lie subgroup \( \mathcal{H} \) of \( \mathcal{G} \) is a subset \( \mathcal{H} \) of \( \mathcal{G} \) such that

(a) \( \mathcal{H} \) is a subgroup of \( \mathcal{G} \)

(b) \( \mathcal{H} \) is a submanifold of \( \mathcal{G} \).

We would naturally wish a Lie subgroup of \( \mathcal{G} \) to have the following property:

Proposition 1.1.6 (Cohn, 2.8.1)

Any Lie subgroup of \( \mathcal{G} \) is a Lie group.

In order to discuss the important example of one-dimensional subgroups we introduce the concepts of dimension, \( C^\infty \) homomorphism and local \( C^\infty \) homomorphism. An alternative characterization of Lie subgroup to that of (1.1.5) is the following one, which is useful in the infinite dimensional case.

Proposition 1.1.7 (Cohn, 6.3.5)

\( \mathcal{H} \) is a Lie subgroup of \( \mathcal{G} \) if and only if

(a) \( \mathcal{H} \) is a subgroup of \( \mathcal{G} \)

(b) \( \mathcal{H} \) is a Lie group

(c) The identity mapping of \( \mathcal{H} \) into \( \mathcal{G} \) is continuous.

The following theorem is self-evidently a most useful one.

Theorem 1.1.8 (Cohn, 6.5.1)

Let \( \mathcal{H} \) be a closed subgroup of \( \mathcal{G} \). Then \( \mathcal{H} \) is a Lie subgroup of \( \mathcal{G} \).

Definition 1.1.9 (Cohn, §1.1)

The **dimension** of a \( C^\infty \) manifold \( \mathcal{M} \) at each point \( p \) in \( \mathcal{M} \) is defined to be the dimension of the image space of all possible charts at \( p \). This number is independent of the choice of chart at \( p \).

For Lie groups we have,
Lemma 1.1.10 (Cohn, 2.6.2)

The dimension of a Lie group is the same at all its points.

Definition 1.1.11 (Cohn, §2.6)

Let $\mathcal{G}_1, \mathcal{G}_2$ be Lie groups. A map $f: \mathcal{G}_1 \to \mathcal{G}_2$ is a $C^\infty$ homomorphism if

(a) $f$ is a homomorphism

(b) $f$ is $C^\infty$, that is, for any point $g_1 \in \mathcal{G}_1$ and for some chart $\chi_1$ at $g_1$ and $\chi_2$ at $f(g_1)$, $\chi_2 \circ f \circ \chi_1^{-1}$ is a $C^\infty$ map. A $C^\infty$ map $f: \mathcal{G}_1 \to \mathcal{G}_2$ whose inverse is $C^\infty$ and which is not necessarily a homomorphism, is called a diffeomorphism of $\mathcal{G}_1$ onto $\mathcal{G}_2$.

Definition 1.1.12 (Cohn, §2.2)

A neighbourhood of a point $g \in \mathcal{G}$ is an open subset containing $g$. A nucleus of $\mathcal{G}$ is a neighbourhood of the identity.

Definition 1.1.13 (Cohn, §1.7)

A local $C^\infty$ homomorphism $f$ of $\mathcal{G}_1$ into $\mathcal{G}_2$ is a $C^\infty$ map of a nucleus $\mathcal{N}_1$ of $\mathcal{G}_1$ into $\mathcal{G}_2$ such that $f(gh) = f(g)f(h)$ whenever $g, h, gh \in \mathcal{N}_1$. $f$ is a local $C^\infty$ isomorphism when its inverse $f^{-1}: f(\mathcal{N}_1) \to \mathcal{G}_2$ is $C^\infty$.

Proposition 1.1.14 (Cohn, 2.9.3)

Any connected one-dimensional Lie group is $C^\infty$ isomorphic either to the additive group of real numbers $\mathbb{R}$, or to $\mathbb{T}$, the additive group of real numbers mod 1. In particular, every one-dimensional Lie group is locally isomorphic to $\mathbb{R}$.

Before discussing Lie algebras, we need to define tangent vectors and derivatives of $C^\infty$ maps, and tangent vector fields.
Definition 1.1.15 (Cohn, 1.4.1)

Let $C^\infty_g$ denote the set of all real-valued functions defined and $C^\infty$ on a neighbourhood of $g$. A tangent vector at $g$ in $\mathcal{G}$ is a linear mapping $L : C^\infty_g \to \mathbb{R}$ which satisfies,

$$L(f_1f_2) = (Lf_1)f_2(g) + f_1(g)(Lf_2), \ \forall f_1, f_2 \in C^\infty_g.$$

Lemma 1.1.16 (Cohn, 1.4.2)

Let $(\chi_1, N_1)$ be a chart at $g_1$ in $\mathcal{G}$, with coordinates $(x^i)$. The set of tangent vectors at $g$ is an $N$-dimensional vector space with basis $\{L_i\}$,

$$L_i = \frac{\partial}{\partial x^i}|_g, \quad L_i(f) = \frac{\partial f(x^i)}{\partial x^i}|_{x^i=\chi_1(g)},$$

and is called the tangent space at $g$. It is denoted by $T_g(\mathcal{G})$ or $T_g$.

Definition 1.1.17

Let $\varphi : \mathcal{G}_1 \to \mathcal{G}_2$ be a $C^\infty$ map. The derivative of $\varphi$ at $g \in \mathcal{G}$, denoted $d\varphi|_g$ is the map,

$$d\varphi|_g : T_g \to T_{\varphi(g)}$$

given by $(d\varphi|_g L) : f \mapsto L(f \circ \varphi) \ \forall L \in T_g, \ \forall f \in C^\infty_{\varphi(g)}$.

The derivative of $\varphi$ at $e$ is simply written as $d\varphi$, and called the derivative of $\varphi$.

We now define tangent vector fields. They are an assignment of a vector in $T_g$ for each $g$ in $\mathcal{G}$, in a differentiable fashion, made precise by the concept of tangent bundle.

Definition 1.1.18 (Sagle and Walde §2.6)

The tangent bundle $T(\mathcal{G})$ to the Lie group $\mathcal{G}$ is the set, $T(\mathcal{G}) = \cup_{g \in \mathcal{G}} T_g(\mathcal{G})$ denoted by pairs $(g, X)$, $X \in T_g(\mathcal{G})$ which is an example of a fibre bundle to be defined in §1.4 and which has the properties that:
(a) $T(\mathcal{G})$ is a $C^\infty$ manifold.

(b) There is a $C^\infty$ surjection $\pi : T(\mathcal{G}) \to \mathcal{G}$ called the **projection map**, given by the map $\pi : (g, X) \mapsto g$.

(c) The set $\pi^{-1}(g)$, called a **fibre**, is the vector space $T_g(\mathcal{G})$.

(d) $T(\mathcal{G})$ is locally trivial. That is, there is a neighbourhood $\mathcal{N}_1$ of $g$ such that $\mathcal{N}_1 \times \mathbb{R}^n$ is diffeomorphic to $\pi^{-1}(\mathcal{N}_1)$. Letting $(\chi_1, \mathcal{N}_1)$ be a chart at $g$ with coordinates $(x^i)$, such a diffeomorphism is provided by the map

$$
\pi^{-1}(\mathcal{N}_1) \to \mathcal{N}_1 \times \mathbb{R}^n, \quad (\chi_1^{-1}(x^i), \lambda^i \frac{\partial}{\partial x^i}) \mapsto (\chi_1^{-1}(x^i), \lambda^i).
$$

It can be seen that this map gives $T(\mathcal{G})$ its $C^\infty$ structure.

**Definition 1.1.19** (Sagle and Walde, §2.7)

A **vector field** $X$ is a $C^\infty$ cross-section of $T(\mathcal{G})$. That is, $X$ is a $C^\infty$-map $X : \mathcal{G} \to T(\mathcal{G})$ such that $\pi \circ X(g) = g, \ \forall g \in \mathcal{G}$.

**Lemma 1.1.20** (Cohn, §1.7)

The set $\mathcal{L}$, of all vector fields on $\mathcal{G}$ is a vector space over $\mathbb{R}$. With respect to a chart $(\chi_1, \mathcal{N}_1)$ at $g$ with coordinates $(x^i)$, $X$ at $\chi_1^{-1}(x^i)$ is given by,

$$
X_{\chi_1^{-1}(x^i)} = \xi^i(x) \frac{\partial}{\partial x^i}, \text{ where the } \xi^i \text{ are } C^\infty \text{ functions on } \chi_1(\mathcal{N}_1).
$$

The vector fields $\mathcal{L}$ form a Lie algebra which is generally infinite dimensional. To see this we first need the definition of Lie algebra.

**Definition 1.1.21** (Cohn §3.1)

A **Lie algebra** is a vector space $\mathcal{L}$ together with a bilinear map $\mathcal{L} \times \mathcal{L} \to \mathcal{L}$, $(X, Y) \mapsto [X, Y]$, called the **bracket** which satisfies

(a) $[X, X] = 0$

(b) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$

It follows from the bilinearity of the bracket and (a) that $[X, Y] + [Y, X] = 0$.

A **Lie subalgebra** is a subspace $\mathcal{L}_1$ of $\mathcal{L}$ such that $[\mathcal{L}_1, \mathcal{L}_1] \subset \mathcal{L}_1$. 18
Lemma 1.1.22  (Sagle and Walde, §2.7)

Let $C^\infty(\mathcal{G})$ be the set of real valued, $C^\infty$ functions on $\mathcal{G}$. It follows from (1.1.15) and (1.1.19) that $X$ is a vector field if and only if $X$ is a linear map of $C^\infty(\mathcal{G})$ into itself which also satisfies,

$$X(f_1f_2) = (Xf_1)f_2 + f_1(Xf_2), \quad \forall f_1, f_2 \in C^\infty(\mathcal{G}).$$

Proposition 1.1.23  (Cohn, 3.1.2)

The set $\mathcal{L}$ of vector fields is a Lie algebra with bracket given by

$$[X, Y](f) = X \circ Y(f) - Y \circ X(f), \quad \forall f \in C^\infty(\mathcal{G}).$$

Lemma 1.1.24  (Cohn §3.1)

Let $X$ and $Y$ be a vector fields and $(\chi_1, \mathcal{N}_1)$ a chart at $g$ with coordinates $(x^i)$, then

$$[X, Y]_{\chi_1^{-1}(x^i)} = \left(\xi^i(x) \frac{\partial \eta^j}{\partial x^i}(x) - \eta^i(x) \frac{\partial \xi^j}{\partial x^i}\right) \frac{\partial}{\partial x^j}$$

where $X = \xi^i(x) \frac{\partial}{\partial x^i}$ and $Y = \eta^i(x) \frac{\partial}{\partial x^i}$.

Use will be made of the above form of the bracket to define a bracket of vector fields in the infinite dimensional setting.

Associated to any finite-dimensional Lie group $\mathcal{G}$, is a finite dimensional Lie algebra $\mathfrak{g}$. $\mathfrak{g}$ is the subalgebra of $\mathcal{L}$ of vector fields which are invariant under the operation of left multiplication by any element $h$ of $\mathcal{G}$. The concept of left-invariance is made precise by the following definition.

Definition 1.1.25  (Sagle and Walde, 5.1)

Let $\ell_h : \mathcal{G} \rightarrow \mathcal{G}$ be the $C^\infty$ map of left multiplication by $h \in \mathcal{G}$, $\ell_h : g \mapsto hg$ and let $d\ell_h$ be the derivative of $\ell_h$. Let $X \in \mathcal{L}$. Then $X$ is left invariant when

$$d\ell_h(X_e) = X_h.$$
Proposition 1.1.26  (Cohn, 3.2.3)

The set \( g \) of left invariant vector fields on \( G \) is a Lie subalgebra of \( \mathcal{L} \) and a Lie algebra of the same dimension as \( G \), which is hence vector space isomorphic to \( T_e \) via the map \((g \to T_e, \ X \mapsto X_e)\).

Let \((\chi, \mathcal{N})\) be a chart at \( e \).

The left-invariant vector fields uniquely determined by the coordinate form 
\[(1.1.20), \quad \frac{\partial}{\partial y^i} \bigg|_{x^{-1}(y)=e} = \frac{\partial}{\partial x^i}, \]
form a basis of \( g \), and are often called the infinitesimal transformations of the group (in the given chart).

Definition 1.1.27  (Cohn, §3.2)

Let \( \{X_i\} \) be a basis of \( g \). The structure constants of \( G \) with respect to the basis \( \{X_i\} \) are the \( n^3 \) real constants \( C^k_{ij} \) such that \( [X_i, X_j] = C^k_{ij}X_k \).

The structure constants determine the Lie algebra completely.

In order to discuss the connection of \( g \) to \( G \) via the exponential map, the following theorem is necessary.

Theorem 1.1.28  (Sagle and Walde, 5.9)

Let \( X \in g \). There exists exactly one analytic homomorphism \( \varphi : \mathbb{R} \to G \) such that \( d\varphi : \mathbb{R} \to T_e \) is the map \( d\varphi : r \mapsto rX_e, \ \forall r \in \mathbb{R} \)

Definition 1.1.29  (Sagle and Walde, 5.10)

The exponential mapping denoted \( \exp, \exp : g \to G \) is the map, \( \exp : X \mapsto \varphi(1) \) where \( X \) and \( \varphi \) are as given in Theorem (1.1.28).

Remark 1.1.30

Note that \( \exp \) is a \( C^\infty \) homomorphism of \( \text{span} \ \{X\} \) into \( G \) for any \( X \in g \).
**Proposition 1.1.31** (Sagle and Walde, 5.13)

There is a bounded connected neighbourhood $N_0$ of 0 in $g$ and a nucleus $N_e$ such that $\exp : N_0 \to N_e$ is an analytic diffeomorphism.

We shall frequently make use of a convenient chart at $e \in G$, called the canonical coordinate chart:

**Definition 1.1.32** (Cohn, §6.3)

Let $N_e$ be a nucleus of $G$ on which $\exp$ is a diffeomorphism, and let $\{X_i\}$ be a basis of $g$. The pair, $(\exp^{-1} : N_e \to g, N_e)$ regarding $g$ as an $n$-dimensional vector space, is a $C^\infty$ chart at $e$ in $G$ and the coordinates $(x^i)$ of the point $\exp(x^iX_i)$ are called canonical coordinates.

The most important theorem connecting $g$ with $G$ is the Baker-Campbell-Hausdorff formula.

**Theorem 1.1.33** (Sagle and Walde, 5.18ff)

There is a neighbourhood $N_0$ of 0 in $g$ such that for all $X, Y \in N_0$,

$$(\exp X)(\exp Y) = \exp F(X, Y)$$

where $F$ is analytic on $N_0 \times N_0$ and is given by

$$F(X, Y) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left[ \frac{\sum \tau(X^{p_1}, Y^{q_1}, X^{p_2}, Y^{q_2}, \ldots, X^{p_n}, Y^{q_n})}{C(p_1, q_1, \ldots, p_n, q_n)} \right]$$

where the second sum runs over the integers $p_i, q_i \geq 0$ with $p_i + q_i \geq 1$ for $i = 1, \ldots, n$ and

$$C(p_1, q_1, \ldots, p_n, q_n) = \sum_{i=1}^{n} (p_i + q_i)!p_1!q_1! \cdots p_n!q_n!$$

$$\tau(X^{p_1}, Y^{q_1}, \ldots, X^{p_n}, Y^{q_n}) = [[\cdots [[[X^{p_1}, Y^{q_1}], X^{p_2}], Y^{q_2}], \ldots X^{p_n}], Y^{q_n}]$$

21
where we use the notation

\[ [W^r, Z^s] = [[...[[W,W], W]...W], Z]...Z] \]

where \( r \) occurrences of \( W \) and \( s \) occurrences of \( Z \).

Corollary 1.1.34  (Sagle and Walde, §5.3)

The terms to order \( t^3 \) in \( F(tX, tY) \) of (1.1.33) are

\[
F(tX, tY) = tX + tY + \frac{1}{2}t^2[X,Y] + \frac{t^3}{12}([X,[X,Y]] + [Y,[Y,X]])
\]

We see from Theorem (1.1.33) that a nucleus of the Lie group \( \mathcal{G} \) is determined completely by the Lie algebra structure of \( \mathfrak{g} \).

We now state several, needed propositions on Lie subgroups and \( C^\infty \) homomorphisms and their connection to the Lie algebra \( \mathfrak{g} \) of \( \mathcal{G} \).

Proposition 1.1.35  (Helgason, 2.2.5)

Let \( \mathcal{H} \) be a Lie subgroup of \( \mathcal{G} \). Then the Lie algebra \( \mathfrak{h} \) of \( \mathcal{H} \) is a subalgebra of \( \mathfrak{g} \) and there is a neighbourhood \( \mathcal{N}_0 \) of 0 such that \( \exp(\mathcal{N}_0 \cap \mathfrak{h}) = \exp(\mathcal{N}_0) \cap \exp(\mathfrak{h}) \)
where \( \exp \) is a diffeomorphism on \( \mathcal{N}_0 \).

Proposition 1.1.36  (Cohn, 6.4.1)

Let \( \mathfrak{g}_1 \) be a Lie subalgebra of \( \mathfrak{g} \). The subgroup \( \mathcal{G}_1 \) generated by elements \( \exp X_1, X_1 \in \mathfrak{g}_1 \), is a connected, \( C^\infty \) Lie subgroup of \( \mathcal{G} \) whose Lie algebra is \( \mathfrak{g}_1 \).

Proposition 1.1.36(a)  (Cohn, 7.1.1)

Every connected Lie group (indeed, manifold) is path-connected.
Definition 1.1.36(b) (Cohn, §7.1)

Let $G_1$ be a subgroup of $G$ ($G_1$ need not be a Lie subgroup). The connected component of $G_1$ in $G$ is the subset,

$$\{g \mid g \in G, \exists \psi \in C_G[0,1] \ni \psi(0) \in G_1, \text{ and } \psi(1) = g\},$$

of $G$.

Proposition 1.1.36(c) (Cohn, §7.1)

The connected component of $G_1$ in $G$ is an open Lie subgroup of $G$.

Proposition 1.1.36(d) (Cohn, §7.1)

The relation of: $g_1 \sim g_2 \iff \exists \psi \in C_G[0,1] \ni \psi(0) = g_1$ and $\psi(1) = g_2$, is an equivalence relation of $G$, and $G$ is a disjoint union of maximal path-connected subsets, called connected components. If the number of connected components of finite, $G$ is said to be finitely connected; if countable, $G$ is countably connected.

Definition 1.1.37 (Cohn, §6.6)

An ideal $I$ of $g$ is a subalgebra of $g$ such that $[g, I] \subset I$. $g$ is semi-simple if it has no abelian ideals. The centre of $g$ is the set, $\{Y \mid Y \in g, [Y, X] = 0, \forall X \in g\}$.

Definition 1.1.38 (Cohn, §6.6)

A homomorphism of Lie algebras $g_1, g_2$ is a linear map $\phi : g_1 \to g_2$ such that

$$\phi[X, Y] = [\phi(X), \phi(Y)], \forall X, Y \in g_1.$$

Proposition 1.1.39 (Cohn, 6.6.1)

Let $g$ be any Lie algebra, and $I$ an ideal in $g$. Then $g/I$ is a Lie algebra and $g/I$ is a homomorphic image of $g$ under the natural homomorphism $X \mapsto X + I$ and the kernel of this homomorphism is $I$. 

23
Proposition 1.1.40 (Cohn, 6.6.4)

Let \( \mathcal{N} \) be a closed, normal subgroup of \( \mathcal{G} \), with Lie subalgebra \( \mathfrak{n} \). Then \( \mathfrak{n} \) is an ideal of \( \mathfrak{g} \) and the quotient group \( \mathcal{G}/\mathcal{N} \) is a Lie group with Lie algebra isomorphic to \( \mathfrak{g}/\mathfrak{n} \).

The following proposition provides useful information on quotient structures over \( \mathcal{G} \):

Proposition 1.1.40(a) (Cohn, 6.5.2)

Let \( \mathcal{G}_1 \) be a closed subgroup of \( \mathcal{G} \), and let \( \Phi_1 \) denote the natural mapping \( \Phi_1 : \mathcal{G} \to \mathcal{G}/\mathcal{G}_1 \). Then there is a \( C^\infty \) manifold structure for \( \mathcal{G}/\mathcal{G}_1 \) such that:

(a) \( \Phi_1 \) is \( C^\infty \)

(b) \( \Phi_1 \) has a local \( C^\infty \) cross-section at \( e \) and,

the analytic structure on \( \mathcal{G}/\mathcal{G}_1 \) is uniquely determined by conditions (a) and (b), and the manifold structure of \( \mathcal{G}/\mathcal{G}_1 \) is that induced by the manifold structure of \( \mathcal{G} \).

Proposition 1.1.41 (Cohn, 6.8.2)

Let \( \mathcal{H} \) be a connected Lie subgroup of a connected Lie group \( \mathcal{G} \), with Lie algebra \( \mathfrak{h} \). Then \( \mathfrak{h} \) is an ideal in \( \mathfrak{g} \) if and only if \( \mathcal{H} \) is normal in \( \mathcal{G} \).

Before stating the main theorem for homomorphisms of Lie algebras we quote the following Lemma which is often useful in practice.

Lemma 1.1.42 (Cohn, 6.3.2)

Any continuous homomorphism of \( \mathcal{G}_1 \) into \( \mathcal{G}_2 \) is analytic.

Remark 1.1.43 (Sagle and Walde, §5.1)

By (1.1.25), to every \( X \in T_e \) there is a unique left invariant vector field \( \tilde{X} \) such that \( \tilde{X}_e = X \). The assignment \( \tilde{X} \mapsto \tilde{X}_e \) is a vector space isomorphism of \( \mathfrak{g} \) with \( T_e \) as remarked in (1.1.26). This assignment becomes an isomorphism \( \mathfrak{g} \to T_e \) of Lie
algebras if we define the following bracket on $T_e : [X,Y] = [\tilde{X},\tilde{Y}]_e \ \forall X,Y \in T_e$.

With this bracket on $T_e$ we have the promised theorem.

**Theorem 1.1.44** (Cohn, 6.6.3)

Let $\Phi : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ be a continuous homomorphism. Then $d\Phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is a Lie algebra homomorphism, $\ker\Phi$ is a closed normal subgroup of $\mathcal{G}_1$ and $\ker d\Phi$ is its Lie algebra which is also an ideal in $\mathfrak{g}_1$. $\Phi(\mathcal{G}_1)$ is a Lie subgroup of $\mathcal{G}_2$ and has Lie algebra $d\Phi(\mathfrak{g}_1)$. $\Phi(\mathcal{G}_1)$ is $C^\infty$ isomorphic to $\frac{\mathcal{G}_1}{\ker\Phi}$ and has Lie algebra $\frac{\mathfrak{g}_1}{\ker d\Phi}$.

The following formula is very useful and will be called on in several places.

**Theorem 1.1.45** (Helgason, 2.1.12)

Let $\Phi : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ be a continuous homomorphism. Then

$$\Phi(\exp X) = \exp(d\Phi X), \ \forall X \in \mathfrak{g}_1$$

In dealing with the method of orbits in representation theory, we will need the concepts of adjoint and co-adjoint representations.

**Lemma 1.1.46** (Cohn, §6.8)

The operation of inner automorphism of $\mathcal{G}$ by an element $h$ of $\mathcal{G}$, $g \mapsto hgh^{-1}$ is $C^\infty$. Let $Ad(h)$ denote the derivative of this map. Then the map $Ad : \mathcal{G} \rightarrow GL(\mathfrak{g},\mathbb{R})$ is a $C^\infty$ homomorphism of Lie groups, called the adjoint representation of $\mathcal{G}$. The derivative of the map $Ad$, denoted $ad$, $ad : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g},\mathbb{R})$ is the map, $ad(X)(Y) = [X,Y]$.

**Definition 1.1.47** (Cohn, §1.5)

The dual of $\mathfrak{g}$, denoted $\mathfrak{g}^*$, called the functionals on $\mathfrak{g}$, is the vector space of linear maps $\mathfrak{g} \rightarrow \mathbb{R}$. The action of an element $\beta \in \mathfrak{g}^*$ on $\mathfrak{g}$ is denoted:

$$\beta : X \mapsto <\beta, X>, \ \forall X \in \mathfrak{g}.$$
Lemma 1.1.48 (Kirillov, §15.1)

The map $\text{Ad}^* : G \rightarrow GL(g^*, \mathbb{R})$ defined by

$$< \text{Ad}^*(g)\beta, X > = < \beta, \text{Ad}(g^{-1})X >$$

is a $C^\infty$ homomorphism of Lie groups, and is called the **co-adjoint representation of $G$**.

**Definition 1.1.49** (Kirillov, §15.1)

The **orbit of $G$ through $\beta$** denoted $O_\beta$ is defined to be the set $\text{Ad}^*(G)\beta \subset g^*$.

The following Lemma will show that the orbit through $\beta$ may be given the structure of a $C^\infty$-manifold.

**Lemma 1.1.51** (Helgason, 2.4.2)

Let $G_1$ be a closed subgroup of $G$. Then the left coset space $G_1 \backslash G$ is a $C^\infty$ manifold.

**Definition 1.1.52** (Kirillov, §15)

Let $G_1$ be a closed subgroup of $G$. An **invariant $C^\infty$ vector field** $\xi$ on $G_1 \backslash G$ is a $C^\infty$ vector field on $G_1 \backslash G$ such that

$$\xi_{r(g)(x)} = dr(g)\xi$$

where

$r(g) : G_1 \backslash G \rightarrow G_1 \backslash G$ is the $C^\infty$ action of $G$ by right multiplication on cosets.

**Lemma 1.1.53** (Kirillov, §15.1)

Let $G_\beta$ be the stabiliser of $\beta$ by the co-adjoint action of $G$. Then $G_\beta$ is a closed subgroup of $G$. The orbit of $G$ through $\beta$, $\text{Ad}^*(G)\beta$ is a $C^\infty$ manifold diffeomorphic to $G_\beta \backslash G$.

In §1.3, we shall need some of the concepts of differential forms. We will define differential forms and show that they form an algebra under the operation of exterior multiplication. Of course, the definition and properties associated to tangent vectors given so far for Lie groups also hold for $C^\infty$ manifolds.
Definition 1.1.54 (Cohn, §1.5, §1.6)

Let \( \mathcal{M} \) be a \( C^\infty \) manifold. The **dual tangent vector space** \( T^*_m \) to \( \mathcal{M} \) at \( m \), is the \( n \)-dimensional vector space of linear functionals \( T_m \to \mathbb{R}, X \to \langle \alpha, X \rangle, X \in T_m, \alpha \in T^*_m \).

Notation 1.1.55 (Cohn, §1.6)

Let \( \mathcal{M} \) be a \( C^\infty \) manifold. Let \( f \in C^\infty_m \) as per (1.1.15).

Define \( df \in T^*_m \) by

\[
\langle df, X \rangle = Xf.
\]

Let \((\chi, \mathcal{N})\) be a chart at \( m \) with coordinates \( x^i \). Then \( x^i \in C^\infty_m \), regarded as a map from \( \mathcal{G} \) to \( \mathbb{R} \), and the set \( \{dx^i\} \) forms a basis of \( T^*_m \), since \( \langle dx^i, \frac{\partial}{\partial x^j} \rangle = \delta^i_j \). Indeed \( \{dx^i\} \) is simultaneously a basis for \( T^*_m, \forall m \in \mathcal{N} \), as is \( \{\frac{\partial}{\partial x^i}\} \) a basis for \( T^*_m, \forall m \in \mathcal{N} \).

Differential forms will be defined as the cross-sections of a certain vector bundle, to be given below.

Proposition 1.1.56 (Cohn, 4.1.1)

Let \( V \) be a vector space. There is an associative algebra \( E(V) \) with unit element, generated by the basis elements of \( V \) with the defining relations for the product of: \( v \wedge v = 0, \forall v \in V, \) and \( E(V) \) is unique up to isomorphism. If \( V \) is of dimension \( n \), then \( E \) is of dimension \( 2^n \).

Definition 1.1.57 (Kirillov, §5.3, §5.4)

The **exterior algebra bundle** \( E(\mathcal{M}) \) to the \( C^\infty \) manifold \( \mathcal{M} \) is the set, \( E(\mathcal{M}) = \bigcup_{m \in \mathcal{M}} E(T^*_m) \) denoted by pairs \((m, \omega), \omega \in E(T^*_m)\), which is an example of a fibre bundle to be defined in §1.4 and which has the properties that:

(a) \( E(\mathcal{M}) \) is a \( C^\infty \) manifold.
(b) There is a $C^\infty$ surjection $\pi_E : E(\mathcal{M}) \to \mathcal{M}$ called the projection map, given by $\pi_E : (m, \omega) \mapsto m$.

(c) The set $\pi^{-1}_E(\mathcal{M})$, called a fibre is the vector space $E(T^*_m)$.

(d) $E(\mathcal{M})$ is locally trivial. That is, there is a neighbourhood $\mathcal{N}_1$ of $m$ such that $\mathcal{N}_1 \times \mathbb{R}^n$ is diffeomorphic to $\pi^{-1}_E(\mathcal{N}_1)$ where $n$ is the dimension of $\mathcal{M}$. Letting $(\chi_1, \mathcal{N}_1)$ be a chart at $m$ with coordinates $x^i$, such a diffeomorphism is provided by the map,

$$\pi^{-1}_E(\mathcal{N}_1) \to \mathcal{N}_1 \times \mathbb{R}^n,$$

$$(\chi_1^{-1}(x^i), \sum_{k=1}^{n} \lambda^{i_1, \ldots, i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \mapsto (\chi_1^{-1}(x^i), \lambda^{i_1, \ldots, i_\ell})$$

where $\ell = 1, \ldots, n$ and the sum on $i_\ell$ is implied.

Herewith the foreshadowed definition:

**Definition 1.1.58** (Kirillov, §5.3, §5.4)

A differential form $\omega$, is a $C^\infty$ cross-section of $E(\mathcal{M})$ (see 1.1.19).

**Definition 1.1.59** (Kirillov, §5.3, §5.4)

Let $\omega$ be a differential form on a $C^\infty$ manifold $\mathcal{M}$ of dimension $n$, such that if $(\chi, \mathcal{N})$ is a chart at $m \in \mathcal{M}$, with coordinates $x^i$, $\omega$ is the map,

$$\omega : \chi^{-1}(x^i) \mapsto \lambda^{i_1, \ldots, i_j}(x^i) dx^{i_1} \wedge \cdots \wedge dx^{i_j}, i_1 < i_2 < \cdots < i_j.$$

In this case, $\omega$ is called a (differential) $j$-form. An $n$-form $\omega$, necessarily has the simple form,

$$\omega : \chi^{-1}(x^i) \mapsto \lambda(x^i) dx^1 \wedge \cdots \wedge dx^n.$$

By analogy to (1.1.5), the form of $\omega$ in (1.1.59) provides an alternative characterization for $j$-forms.
Definition 1.1.60 (Cohn, §4.2)

Let \{X^i\} be any set of \(j\) vector fields on a \(C^\infty\) manifold \(\mathcal{M}\).

\(\omega\) is a \(j\)-form if and only if, for each \(m \in \mathcal{M}\), \(\omega_m\) is a multilinear anti-symmetric map,

\[
w_m : \prod_{i=1}^j T_m \to \mathbb{R}, \text{ such that the map,}\]

\[
\mathcal{M} \to \mathbb{R}, \quad m \mapsto \omega_m(X^1, \ldots, X^j) \quad \text{is } C^\infty.
\]

We will illustrate how an \(n\)-form may be used to define an integral for complex-valued functions on \(\mathcal{M}\), in an invariant way.

Proposition 1.1.61 (Kirillov, §5.3; Warner, 1983, §4.1)

Let \(\mathcal{M}\) be a \(C^\infty\)-manifold of dimension \(n\) which admits a nowhere-vanishing \(n\)-form (1.1.59). There is a family \(\mathcal{F}\) as per (1.1.1) of charts on \(\mathcal{M}\) such that an \(n\)-form \(\omega\), for any covering of \(\mathcal{M}\) with charts \((\chi_\alpha, \mathcal{N}_\alpha)\) and coordinates \(x^i\), uniquely defines the quantity,

\[
\int_{\mathcal{N}_\alpha} \omega = \int_{\chi_\alpha(\mathcal{N}_\alpha)} \lambda(x^1)dx^1 \cdots dx^n
\]

where \(\omega = \lambda(x^1)dx^1 \wedge \cdots \wedge dx^n\).

Further \(\omega\) is a sum of \(n\)-forms \(\omega_\alpha\), \(\omega = \sum_\alpha \omega_\alpha\) where \(\omega_\alpha(m) = 0\) for \(m\) outside of \(\mathcal{N}_\alpha\), and we define

\[
\int_{\mathcal{M}} \omega = \sum_\alpha \int_{\mathcal{N}_\alpha} \omega_\alpha,
\]

called the integral of the form \(\omega\) on the oriented manifold \(\mathcal{M}\).

Let \(\mathcal{G}\) be a closed subgroup of the Lie group \(\mathcal{G}\). If the \(C^\infty\) manifold \(\mathcal{G}_1 \setminus \mathcal{G}\) admits an invariant \(n_1\)-form \(\omega\) (where \(n_1\) is the dimension of \(\mathcal{G}_1 \setminus \mathcal{G}\)), then \(\omega\) is nowhere-vanishing and (1.1.61) applies to \(\mathcal{G}_1 \setminus \mathcal{G}\). Hence the following definition and lemma.
Definition 1.1.62 (Kirillov, §15.1)

Let $G_1$ be a closed subgroup of $G$. Let $\{X^i\}$ be any set of $j$ vector fields on the $C^\infty$ manifold $G_1 \backslash G$. In the notation of (1.1.52) $\omega$ is a $G$-invariant differential $j$-form on $G_1 \backslash G$ if it is a differential $j$-form on $G_1 \backslash G$ which satisfies,

$$\omega_m(X^1_m, \ldots, X^j_m) = \omega_r(g)(m)(dr(g)X^1_m, \ldots, dr(g)X^j_m),$$

for all $m \in G_1 \backslash G$.

Lemma 1.1.63 (Warner, 1983, §4.2)

Let $G_1$ be a closed subgroup of $G$. If $G_1 \backslash G$ admits a $G$-invariant differential $j$-form $\omega$, non-zero at some point $m \in G_1 \backslash G$, then $\omega$ is nowhere-vanishing.

The domain of applicability of the method of orbits which is outlined in §1.3, contains a wide range of classes of Lie groups. It is therefore necessary to define these classes in order to discuss their suitability to the orbits method. The following body of definitions and propositions achieves this.

Definition 1.1.64 (Sagle and Walde, §10.4)

Let $G_1, G_2$ be Lie subgroups of $G$. Let $(G_1, G_2)$ denote the subgroup of $G$ generated by all elements of the form $g_1 g_2 g_1^{-1} g_2^{-1}, \forall g_1 \in G_1, \forall g_2 \in G_2$.

Define the decreasing sequence of normal subgroups inductively:

$G^{(1)} = (G, G), G^{(k)} = (G^{(k-1)}, G^{(k-1)})$. If the sequence $G \supset G^{(1)} \supset G^{(2)} \supset \cdots$ terminates at $\{e\}$, then $G$ is solvable.

Proposition 1.1.65 (Sagle and Walde, 10.12)

There is a closed, maximal, solvable, normal, connected Lie subgroup of $G$, called the radical, $R$.

Definition 1.1.66 (Sagle and Walde, 10.14)

$G$ is semisimple if its radical is the set $\{e\}$. (The Lie algebra of $G$ is consequently semisimple (1.1.37))
Definition 1.1.67  (Sagle and Walde, §12.4)

\( G \) is a reductive Lie group, if for any subalgebra \( g_1 \) of \( g \), there is a vector space complement \( V \) of \( g_1 \) in \( g \), such that \([g_1, V] \subset V\).

Definition 1.1.67(a)  (Helgason, §4.3)

Let \( G \) be a compact, connected Lie group, \( K \) a closed subgroup, and \( \theta : G \to G \) a \( C^\infty \) involution. Let \( K_\theta \) be the set of fixed points of \( \theta \) and let \( (K_\theta)_0 \) be its identity component. If \( (K_\theta)_0 \subset K \subset K_\theta \). then \( (G, K) \) is called a compact symmetric pair, and \( \theta \) a Cartan involution of \( G \).

Definition 1.1.68  (Sagle and Walde, §11.1)

In the notation of (1.1.64), define the decreasing sequence of Lie subgroups of \( G \) inductively: \( G(1) = (G, G), G(k) = (G, G(k-1)) \). If the sequence \( G \supset G(1) \supset G(2) \supset \cdots \) terminates at \( \{e\} \), then \( G \) is nilpotent.

Proposition 1.1.69  (Sagle and Walde, §11.2)

A maximal, nilpotent, normal Lie subgroup of \( G \) exists, and is called the nilradical of \( G \).

Definition 1.1.70  (Lipsman, 1980)

A cocompact nilradical Lie group \( G \), is one whose nilradical \( N \), is simply connected and for which \( G/N \) is compact.

Proposition 1.1.71  (Lipsman, 1980)

A cocompact nilradical Lie group \( G \) is a semi-direct product of a compact Lie group \( G_1 \) and simply connected nilpotent Lie group, \( N \); where the semi-direct product of \( G_1 \) with \( N \), denoted \( G_1 \ltimes N \), is the set \( G_1 \times N \) with product defined via a homomorphism \( \Psi : G_1 \to \text{Aut}(N) \) such that the map \( G_1 \times N \to N \), \((g_1, n) \to \Psi(g_1)(n)\) is \( C^\infty \), and given by \((g_1, n_1)(g_2, n_2) = (g_1 g_2, n_1 \Psi(g_1)(n_2))\).
Definition 1.1.72 (Kirillov, §6.4)

An exponential Lie group is one which is covered by the canonical chart (1.1.32).

In order to discuss, in §1.3, the Principal Series representations for a non-compact semisimple Lie group $G$, will need some structure theory for $G$. The following group of definitions and lemmas provides for this.

Proposition 1.1.73 (Helgason, §2.6)

Define the Killing form $B$ to be the bilinear map $B : g \times g \to g$, $B(X, Y) = \text{Tr}(\text{ad}X \text{ad}Y)$. A connected Lie group $G$ is compact if and only if the Killing form on $g$ is strictly negative definite, in which case $g$ is also said to be compact.

Proposition 1.1.74 (Helgason, §3.7)

A semisimple Lie algebra $g$ has a direct sum decomposition $g = k + p$, where $k$ is a subalgebra, $[k, p] \subseteq p$ and $[p, p] \subseteq k$, which is called a Cartan decomposition. The Killing form, restricted to $k$ is strictly negative definite. If $G$ is a semisimple Lie group with Lie algebra $g$, the connected subgroup of $G$ with Lie algebra $\mathfrak{k}$ is compact (1.1.36).

The rank of a real, semisimple Lie algebra is given as follows.

Proposition 1.1.75 (Helgason, §3.7)

A complex, semisimple Lie algebra $g$ admits a maximal abelian subalgebra $\mathfrak{h}$, such that $\forall X \in \mathfrak{h}$, the linear map $\text{ad}X : g \to g$ is semisimple. $\mathfrak{h}$ is called a Cartan subalgebra. The dimension of $\mathfrak{h}$ is called the rank of $g$.

Definition 1.1.75(a) (Helgason, §3.7)

An involutive automorphism $\theta$ of a semisimple Lie algebra $g$ is a Cartan involution if the bilinear form (1.1.73), $B\theta(X, Y) = -B(X, \theta Y)$ is strictly positive definite.
Lemma 1.1.75(b) (Helgason, §9.1)

Let \( g \) be real, semisimple and \( d\theta \) a Cartan involution of \( g \). Then \( g \) has a Cartan decomposition \( g = k + p \) where \( k \) is the fixed point set of \( d\theta \).

**Definition 1.1.75(c)** (Helgason, §9.1)

Let \( g = k+p \) be as per (1.1.75(b)). Then the **real rank** of \( g \) is the dimension of any maximal abelian subspace of \( p \) (all such subspaces have the same dimension).

Roots can also be defined for real, semisimple Lie algebras, as well as complex ones.

**Definition 1.1.75(d)** (Helgason, 6.3.7ff)

Let \( g \) be real semisimple and let \( g = k + p \) be a Cartan decomposition of \( g \) as per (1.1.75(b)). Let \( h_p \) be a maximal abelian subalgebra of \( p \). For each functional \( \lambda : h_p \to \mathbb{R} \) define the subspaces \( g_\lambda \) of \( g \) to be

\[
 g_\lambda = \{ X \in g \mid [Y, X] = \lambda(Y)X, \quad \forall Y \in h_p \}. 
\]

\( \lambda \) is called a **root of** \( g \) **with respect to** \( h_p \) if \( \lambda \neq 0 \) and \( g_\lambda \neq 0 \). Let \( \Sigma \) denote this set of roots.

**Lemma 1.1.75(e)** (Helgason, 6.3.7ff)

In the notation of 1.1.75(d), for a real semisimple Lie algebra \( g \),

\[
 g = g_0 + \sum_{\lambda \in \Sigma} g_\lambda. 
\]

In order to distinguish positive roots we define Weyl chambers:

**Lemma 1.1.75(f)** (Helgason, §7.2)

In the notation of 1.1.75(d), let \( g \) be a real semisimple Lie algebra. Each root \( \lambda \in \Sigma \) defines the hyperplane, \( \{ Y \mid \lambda(Y) = 0, \ Y \in h_p \} \) in \( h_p \), the union of which divides the space \( h_p \) into finitely many connected components, called **Weyl chambers**. A Weyl chamber \( C \) is an open, convex subset of \( h_p \) with the property that \( Y \in C \Rightarrow \alpha Y \in C \), for \( \alpha \) strictly positive.
Definition 1.1.75(g) (Helgason, §6.3)

Let $g$ be a real, semisimple Lie algebra. Let $C^+$ be a fixed Weyl chamber as per (1.1.75(f)). The **positive roots with respect to** $C^+$ are the set,

$$\Sigma^+ = \{ \lambda \mid \lambda \in \Sigma, \lambda(Y) > 0, \ \forall Y \in C^+ \}.$$

We will need the following lemma in §1.5:

Lemma 1.1.75(h) (Helgason, §6.3)

Let $g$ be a real semisimple Lie algebra with $C^+$ as per (1.1.75(g)). Then there is a unique element $H$ of $C^+$ such that $B_{g}(H, H) = 1$ (see 1.1.75(a)).

Cartan subalgebras are used to define a system of roots for $g$ which are crucial to the structure theory:

Definition 1.1.76 (Helgason, §3.4)

Let $g$ be complex, semisimple and let $h$ be a Cartan subalgebra of $g$. Let $\alpha$ be any linear map $\alpha : h \to \mathbb{C}$ and define the subspaces $g^\alpha$ of $g$ to be

$$g^\alpha = \{ X \in g \mid \{Y, X\} = \alpha(Y)X, \ \forall Y \in h \}.$$

$\alpha$ is called a root if $g^\alpha \neq \{0\}$. Let $\Delta$ denote the set of all non-zero roots.

Proposition 1.1.77 (Helgason, 3.4.2)

Let $g$ be a complex, semisimple Lie algebra and $h$ a Cartan subalgebra with corresponding subspaces $g^\alpha$. Then,

(a) $g = h + \sum_{\alpha \in \Delta} g^\alpha$ (direct sum)

(b) $\dim g^\alpha = 1, \ \forall \alpha \in \Delta$

(c) $\forall \alpha, \beta \in \Delta \ \exists \ \alpha + \beta \neq 0, \ B(g^\alpha, g^\beta) = 0$

(d) $[g^\alpha, g^\beta] \subset g^{\alpha + \beta}$.

We choose a particular Cartan subalgebra and distinguish a subset of positive roots in $\Delta$: 34
Proposition 1.1.78 (Helgason, §6.3)

Let \( g \) be a real semisimple Lie algebra, and let \( g = k + p \) be a Cartan decomposition of \( g \). Let \( h_p \) denote any maximal abelian subspace of \( p \), and let \( h \) be any maximal abelian subalgebra of \( g \) containing \( h_p \). Then \( h \otimes \mathbb{C} \) is a Cartan subalgebra of \( g \otimes \mathbb{C} \). Let \( \Delta \) be the roots corresponding to \( h \otimes \mathbb{C} \) as per (1.1.76). Distinguish a particular element \( X_0 \) of \( h_p \) which is regular. That is, \( X_0 \) has centralizer in \( k \) of minimal dimension. Using \( X_0 \), define the set \( \Delta^+ \) of positive roots to be \( \Delta^+ = \{ \alpha | \alpha \in \Delta, \alpha(X_0) > 0 \} \).

Comments 1.1.78(a)

There are alternative ways of choosing a subset of positive roots in \( \Delta \). Lemma (1.1.75(f)) remains true for a complex semisimple Lie algebra with \( \Sigma \) replaced by \( \Delta \). The positive roots \( \Delta^+ \) may then be defined by (1.1.75(g)) again replacing \( \Sigma \) with \( \Delta \).

In order to give the structure theorem of Iwasawa decomposition of a real, semisimple Lie algebra, we cut the positive roots down to a still smaller set. This is done via a certain Lie automorphism of \( g \otimes \mathbb{C} \):

Lemma 1.1.79 (Helgason, §6.3)

Let \( g \) be a real, semisimple Lie algebra, and \( g = k + p \) a Cartan decomposition of \( g \). Let \( u \) be the real Lie algebra \( u = k + ip \).

Then \( u \otimes \mathbb{C} = g \otimes \mathbb{C} \). Define the maps \( \sigma : g \otimes \mathbb{C} \to g \otimes \mathbb{C} \), \( \tau : g \otimes \mathbb{C} \to g \otimes \mathbb{C} \) by,

\[
\sigma : X + iY \to X - iY, \quad \forall X, Y \in g
\]

\[
\tau : X + iY \to X - iY, \quad \forall X, Y \in u
\]

Then \( \theta = \sigma \tau \) is a Lie automorphism of \( g \otimes \mathbb{C} \).
**Proposition 1.1.80** (Helgason, 6.3.4)

Let $\mathfrak{g}$ be a real, semisimple Lie algebra and $\mathfrak{k}, \mathfrak{p}, \mathfrak{h}_p, \mathfrak{k}, \mathfrak{A}, \Delta, \Delta^+$ as per (1.1.78). The map $\alpha^\theta : \mathfrak{h} \rightarrow \mathbb{C}, \alpha^\theta : Y \mapsto \alpha(\theta Y)$ is a root, $\forall \alpha \in \Delta$. Define the subset $P^+$ of $\Delta^+$:

$$P^+ = \{ \alpha | \alpha \in \Delta^+, \alpha \neq \alpha^\theta \}.$$ 

Let $\mathfrak{n} = \sum_{\alpha \in P^+} \mathfrak{g}_\alpha$, \ $\mathfrak{n}_0 = \mathfrak{n} \cap \mathfrak{g}$, and $\mathfrak{s}_0 = \mathfrak{h}_p \oplus \mathfrak{n}_0$. Then $\mathfrak{n}$ and $\mathfrak{n}_0$ are nilpotent Lie algebras, $\mathfrak{s}_0$ is a solvable Lie algebra and $\mathfrak{g}$ is a direct sum of subalgebras, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{h}_p \oplus \mathfrak{n}_0$.

**Corollary 1.1.80(a)** (Helgason, 6.3.7)

Let $\mathfrak{g}$ be a real semisimple Lie algebra with $\mathfrak{g}$ and $\Sigma^+$ given by (1.1.75(d)) and (1.1.75(g)) respectively. In the notation of (1.1.80) we have,

$$\mathfrak{n}_0 = \sum_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda = \sum_{\lambda \in \Sigma^+} \mathfrak{g}_{-\lambda}$$

We have the following global version of the Iwasawa decomposition:

**Proposition 1.1.81** (Helgason, 6.5.1)

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{h}_p \oplus \mathfrak{n}_0$ be an Iwasawa decomposition of a semisimple real Lie algebra $\mathfrak{g}$. Let $G$ be any connected Lie group with Lie algebra $\mathfrak{g}$ and Lie subgroups $K, A_p, N$ with Lie algebra $\mathfrak{k}, \mathfrak{h}_p$ and $\mathfrak{n}_0$ respectively. Then the mapping $K \times A_p \times N \rightarrow G$, $(k, a, n) \mapsto kan$ is an analytic diffeomorphism. $A_p$ and $N$ are simply connected.

In §1.5 we will define certain representations of a compact, symmetric pair $(G, K)$ (see (1.1.67(a)). To this end we will need the definitions of roots and root subspaces which are analogous to those for semisimple Lie algebras.
Proposition 1.1.82 (Lassalle (1978), §1.1)

Let \( G \) be compact and \( g \) its Lie algebra. Let \( z \) be the centre (1.1.37) of \( g \), and let \( g' = [g, g] \). then,

\[ g = z \oplus g' \quad \text{and} \quad g' \text{ is semisimple and compact (see (1.1.66) and (1.1.73)).} \]

Lemma 1.1.83 (Lassalle (1978), §1.1)

Let \((G, K)\) be a compact symmetric pair and let \( g = k \oplus p \) be a Cartan decomposition as per (1.1.75(b)). In the notation of (1.1.82), let \( z_p = z \cap p, \quad p' = p \cap g', \quad k' = k \cap g' \). Let \( a \) be a maximal abelian subalgebra of \( p \); then \( a = z_p + a' \) where \( a' \) is a maximal abelian subalgebra of \( p' \). Let \( m' \) be an abelian subalgebra of \( p' \) commuting with \( a' \) and maximal with respect to these two properties; then \( t' \), given by \( t' = m' \oplus a' \), is a maximal abelian subalgebra of \( g' \) stable under \( \theta \). Further, \( t' \otimes C \) is a Cartan subalgebra of \( g' \otimes C \).

Comments 1.1.84

By (1.1.76), the root subspaces \( g'_\alpha \) and roots \( \alpha \) for \( t' \otimes C \) are now defined, together with the set of non-zero roots \( \Delta \). Applying (1.1.78) to \( g' \), we distinguish a regular element \( X_0 \) of \( g' \) by which to define the positive roots \( \Delta^+ \).

In §1.5, we shall need the following definition to describe the irreducible representations of a compact symmetric pair.

Definition 1.1.85 (Lassalle (1978), §1.1.)

Distinguish the subset \( P^+ \) of \( \Delta^+ \), in the notation of (1.1.83), as

\[ P^+ = \{ \alpha \mid \alpha \in \Delta^+, \alpha \mid_{g' \otimes C} \neq 0 \}, \]

where \( \Delta^+ \) is given by (1.1.84).

In order to give irreducible representations of a Cartan motion group in §1.5 we shall need a further definition.
Definition 1.1.86 (Lassalle (1978), §1.1)

Let \( \Sigma^+ = \{ \bar{\alpha} \mid \bar{\alpha} = \alpha \mid \mathfrak{g}' \otimes \mathbb{C}, \alpha \in P^+ \} \) with \( P^+ \) given by (1.1.85). We designate a Weyl chamber \( \mathfrak{g}'^+ \) associated to \( \Sigma^+ \) (see (1.1.75(f))). In the notation of (1.1.83), let

\[
\mathfrak{g}'^+ = \{ X \in \mathfrak{g}' \mid \alpha(iX) > 0, \ \forall \alpha \in \Sigma^+ \}
\]

and \( \mathfrak{g}^+ = \mathbb{Z}_p \oplus \mathfrak{g}'^+ \).

We will need the concept of universal enveloping algebra of a Lie algebra to discuss the contraction of infinitesimal characters in Chapter 4. The enveloping algebra is a factor algebra of the tensor algebra of \( \mathfrak{g} \):

Definition 1.1.87 (Helgason, §1.2)

Let \( \mathfrak{g} \) be a Lie algebra. The tensor algebra \( T(\mathfrak{g}) \) of \( \mathfrak{g} \) is the algebra of multilinear maps \( X, \) over \( \mathbb{R}, \ X : \prod_{i=1}^{n} \mathfrak{g}^* \to \mathbb{R}, \ n = 1, \ldots, \) with identity \( 1 \in \mathbb{R} \). The product \( X \otimes Y \) of two elements of \( T(\mathfrak{g}) \) is given by the product of two multilinear functions with respect to the product of \( \mathbb{R} \). Since \( (\mathfrak{g}^*)^* = \mathfrak{g} \), the linear functionals of \( \mathfrak{g}^* \) can be identified with \( \mathfrak{g} \) and hence \( \mathfrak{g} \subset T(\mathfrak{g}) \).

Herewith the definition of enveloping algebra:

Definition 1.1.88 (Helgason, §2.2)

Let \( T(\mathfrak{g}) \) be the tensor algebra of the Lie algebra \( \mathfrak{g} \) and let \( J \) be the two-sided ideal (see (1.3.39)) generated by elements of the form,

\( X \otimes Y - Y \otimes X - [X, Y], \ \forall X, Y \in \mathfrak{g} \). The universal enveloping algebra \( U(\mathfrak{g}) \) of \( \mathfrak{g} \) is the algebra \( U(\mathfrak{g}) = T(\mathfrak{g})/J \).

We shall need the following simplified version of the Paley-Wiener Theorem in Chapter 5:
Definition 1.1.89 (Rudin, 1973)

Let $\mathfrak{g}$ be a Lie algebra with inner product $(X,Y), X, Y \in \mathfrak{g}$, norm $\|X\| = \sqrt{\langle X, X \rangle}$, and Lebesgue measure $dX$, and $\phi: \mathfrak{g} \to \mathbb{C}$ a $C^\infty$ function with compact support contained in a neighbourhood of 0.

Let

$$f(X) = \int_{\mathfrak{g}} \phi(Y) e^{-i\langle X, Y \rangle} dY,$$

then, for $N = 0, 1, 2, \ldots$, there exists $\gamma_N$ positive, such that

$$|f(X)| \leq \gamma_N (1 + \|X\|)^{-N}.$$
§1.2 Infinite Dimensional Lie Groups and Lie Algebras

In Chapter 2, we shall need several results on Banach-Lie groups. The principal references for this subject are Maissen (1962) and Lang (1962). Most definitions and propositions of §1.1 still hold true for Banach-Lie groups with "\( \mathbb{R}^n \)" uniformly replaced by "\( \mathcal{B} \)" where \( \mathcal{B} \) is a Banach space (defined below), and with "differentiable" replaced by "Fréchet-differentiable", while other propositions hold true but with further qualifications. We will define Fréchet differentiability below, then restate all definitions and propositions from §1.1 which apply to Banach-Lie groups and which will be needed in Chapter 2.

**Definition 1.2.1** (Lang, §1.2)

A **Banach space** \( \mathcal{B} \), is a complete, normed, linear space. Let the norm be denoted, \( \| \xi \|, \; \xi \in \mathcal{B} \).

**Definition 1.2.2** (Maissen, §1.b)

A map \( f : \mathcal{B} \rightarrow \mathcal{B} \) is **Fréchet-differentiable** at \( x \) in \( \mathcal{B} \), whenever there is, for each \( x \), a bounded, linear map \( L(x) \) such that

\[
\lim_{\|h\| \to 0} \frac{\|f(x + h) - f(x) - L(x)h\|}{\|h\|} = 0, \quad h \in \mathcal{B}.
\]

When such an \( L \) exists, it is denoted \( df(x) \) or \( df\big|_x \). \( df(x) \) is the **derivative** of \( f \) at \( x \). It is sometimes written without an argument as \( df \).

We define the \( k' \)th Fréchet derivative of \( f \) inductively:

**Definition 1.2.3** (Lang, §1.3; Hildebrandt and Graves (1927))

If \( f : \mathcal{B} \rightarrow \mathcal{B} \) is \( (k - 1) \) times Fréchet-differentiable at \( x \), then \( f \) is \( k \) times Fréchet-differentiable at \( x \), whenever there exists a continuous multilinear map...
\[ d^k f(x) : \prod_{i=1}^{k} \mathcal{B} \to \mathcal{B}, \quad d^k f : (x, h^1, \ldots, h^k) \mapsto d^k f(x)(h^1, \ldots, h^k), \text{ with norm on } \prod_{i=1}^{k} \mathcal{B} \text{ given by} \]

\[
\| (h^1, \ldots, h^k) \|_k = \sum_{i=1}^{k} \| h^i \|, \text{ such that,}
\]

\[
\frac{\| d^{k-1} f(x + h^k)(h^1, \ldots, h^{k-1}) - d^{k-1} f(x)(h^1, \ldots, h^{k-1}) - d^k f(x)(h^1, \ldots, h^k) \|}{\| (h^1, \ldots, h^k) \|_k} = 0
\]

in the limit, \( \lim_{\| (h^1, \ldots, h^k) \|_k \to 0} \).

**Definition 1.2.4** (Lang, §1.3; Hildebrandt and Graves (1927))

A map \( f : \mathcal{B} \mapsto \mathcal{B} \) is Fréchet-\( C^k \) if \( f \) is \( k \)-times Fréchet differentiable and

\[ d^k f : (x, h^1, \ldots, h^k) \mapsto d^k f(x)(h^1, \ldots, h^k) \]

is a continuous map of \( \prod_{i=1}^{k+1} \mathcal{B} \) into \( \mathcal{B} \). \( f \) is Fréchet-\( C^\infty \) if \( f \) is Fréchet-\( C^k \) for any \( k \).

**Definition 1.2.5** (Maissen, §2.1)

A Fréchet-\( C^\infty \) Manifold is given by Definition (1.1.1) with “\( C^\infty \)” replaced by “Fréchet-\( C^\infty \)” and “\( \mathbb{R}^n \)” replaced by “\( \mathcal{B} \)”.

**Definition 1.2.6** (Maissen, §3.1)

A Banach-Lie Group is given by Definition (1.1.2) with “\( C^\infty \)” replaced by “Fréchet-\( C^\infty \)”.

**Proposition 1.2.7** (Maissen, §3.1)

In a Banach Lie group \( \mathcal{G} \), the mapping \( x \mapsto x^{-1} \) is Fréchet-\( C^\infty \).
In infinite dimensions there are two types of differentiable subgroup, namely differentiable subgroup, (generalising (1.1.5) (Maissen, Definition 12.1), and Banach-Lie subgroup (generalising (1.1.7)) to be defined below.

If \( \mathcal{H} \) is a differentiable subgroup of a Banach-Lie group \( \mathcal{G} \), then \( \mathcal{H} \) is a Banach-Lie subgroup of \( \mathcal{G} \). The converse of this statement however, is not true and depends on whether the Banach space is complementable or not, broadly speaking (Maissen, Theorem 12.1). For most applications, the weaker definition of Banach Lie subgroup suffices:

**Definition 1.2.8** (Maissen, Definition 3.12.2)

A subset \( \mathcal{H} \) of a Banach Lie group \( \mathcal{G} \) is a **Banach-Lie subgroup** of \( \mathcal{G} \) when:

(a) \( \mathcal{H} \) is a subgroup of \( \mathcal{G} \)

(b) \( \mathcal{H} \) is a Lie group

(c) The identity mapping of \( \mathcal{H} \) into \( \mathcal{G} \) is continuous.

**Definition 1.2.9** (Maissen, §3.10)

Let \( \mathcal{G}_1, \mathcal{G}_2 \) be Banach-Lie groups. A map \( f : \mathcal{G}_1 \to \mathcal{G}_2 \) is a **Fréchet-\( C^\infty \)** Homomorphism or a **Fréchet diffeomorphism** according to Definition (1.1.11) with “\( C^\infty \)” replaced by “Fréchet-\( C^\infty \)”.

Lemma (1.1.41) also holds in the infinite dimensional case and is very useful:

**Lemma 1.2.10** (Maissen, Theorem 3.10.1)

Let \( \mathcal{G}_1, \mathcal{G}_2 \) be Banach-Lie groups. Any continuous homomorphism of \( \mathcal{G}_1 \) into \( \mathcal{G}_2 \) is at least twice Fréchet-differentiable.

The definitions of tangent vector and tangent vector field for Banach-Lie groups represent a significant departure from their finite-dimensional analogues. This is because, for example, it is awkward to characterize the space of linear functionals defined on the space of Fréchet-\( C^\infty \) functions on \( \mathcal{B} \). The definition of
tangent vector given for Banach-Lie groups is equivalent to (1.1.15) when $B$ is finite dimensional.

**Definition 1.2.11** (Lang, 2.2)

A **tangent vector** at $g$, to the Banach-Lie group $G$ is an equivalence class $\{(\chi, N, X)\}_g$ of triples $(\chi, N, X)$ where $(\chi, N)$ is a chart at $g$, and $X \in B$. The equivalence relation $\sim$ on the set of all such triples is,

$$ (N, \chi, X) \sim (N_1, \chi_1, X_1) \iff d(\chi_1 \circ \chi^{-1})X = X_1 $$

where the derivative $d(\chi_1 \circ \chi^{-1})$ of $\chi_1 \circ \chi^{-1}$ is the linear map of $B$ into $B$ defined by (1.2.3), and $(\chi, N), (\chi_1, N_1)$ are charts at $g$.

**Lemma 1.2.12** (Lang, 2.2)

Let $(\chi, N)$ be a fixed chart at $g$ in a Banach-Lie group $G$, and let the set of all tangent vectors at $g$ be denoted by $T_g(G)$ or $T_g$ and be called the **tangent space to $G$ at $g$**. The map $X \mapsto \{(\chi, N, X)\}_g, B \rightarrow T_g$, transports a Banach space structure to $T_g$ such that $B$ and $T_g$ are isometric.

**Definition 1.2.13** (Lang, §2.2)

Let $f : G_1 \rightarrow G_2$ be a Fréchet-$C^\infty$ map of the Banach-Lie groups $G_1, G_2$. The **derivative of $f$ at $g$**, denoted $df(g)$ or $df|_g$ is the linear map $df(g) : T_g \rightarrow T_{f(g)}$, given, in the notation of Definition (1.2.11), by

$$ df(g) : \{(N_1, \chi_1, X_1)\}_g \mapsto \{(N_2, \chi_2, d(\chi_2 \circ f \circ \chi_1^{-1})|_{\chi_1(g)}X_1)\}_{f(g)} \} $$

We write $df$ for $df(e)$ and call $df$ the **derivative of $f$**.

The definition (1.2.11) of a tangent vector, makes it unnecessary to take recourse to tangent bundles solely to define vector fields:
Definition 1.2.13(a) (Maissen, §2.4)

A vector field $X$ on a Banach-Lie group $\mathcal{G}$, is a map, $X : \mathcal{G} \rightarrow \bigcup_{g \in \mathcal{G}} T_g(\mathcal{G})$

$$X : g \mapsto \{(\mathcal{N}, \chi, \xi(g))\}_g,$$

denoted $X : g \mapsto X_g$

where $(\mathcal{N}, \chi)$ is any chart at $g$ and $\xi : \mathcal{G} \rightarrow \mathcal{B}$ is a Fréchet-$C^\infty$ map of $\mathcal{G}$ into $\mathcal{B}$. That is, for any point $g_1 \in \mathcal{G}$, and chart $(\chi_1, \mathcal{N}_1)$ at $g_1$, $\xi \circ \chi_1^{-1} : \mathcal{N}_1 \rightarrow \mathcal{B}$ is Fréchet-$C^\infty$.

Lemma 1.2.14 (Maissen, §2.4)

The set $\mathcal{L}$ of all vector fields on the Banach-Lie group $\mathcal{G}$ is a vector space, isomorphic with the vector space $C^\infty_{/\mathcal{B}}(\mathcal{G})$ of Fréchet-$C^\infty$ maps $\mathcal{G} \rightarrow \mathcal{B}$ with isomorphism given, in the notation of (1.2.13) by

$$X \mapsto (g \mapsto \xi(g)).$$

The form of the bracket of finite-dimensional vector fields (1.1.24) is the inspiration for defining a bracket for the space of vector fields on a Banach-Lie group $\mathcal{G}$:

Lemma 1.2.15 (Maissen, §2.4)

Let $X, Y \in \mathcal{L}$ be vector fields on a Banach-Lie group $\mathcal{G}$, given, in the notation of (1.2.13) by,

$$X : g \mapsto \{(\mathcal{N}, \chi, \xi(g))\}_g$$

$$Y : g \mapsto \{(\mathcal{N}, \chi, \eta(g))\}_g,$$

then $\mathcal{L}$ is a Lie algebra, as defined by (1.1.21), with bracket given by,

$$[X, Y] : g \mapsto \{(\mathcal{N}, \chi, d\eta(g)\xi(g) - d\xi(g)\eta(g))\}_g.$$

The characterization of the left-invariant vector fields for a Banach-Lie group is a straightforward generalisation of (1.1.25):
Definition 1.2.16 (Maissen, §3.3)

Let $\mathcal{G}$ be a Banach-Lie group. Then a vector field $X \in \mathcal{L}$ is **left-invariant** when it satisfies Definition (1.1.25) with "$C^\infty$" replaced by "Fréchet-$C^\infty$".

Lemma 1.2.17 (Maissen, Theorem 3.3.1)

The set $\mathfrak{g}$ of left-invariant vector fields on a Banach-Lie group $\mathcal{G}$ is a Lie subalgebra of $\mathcal{L}$ and hence a Lie algebra called the **Lie algebra $\mathfrak{g}$ of $\mathcal{G}$**. $\mathfrak{g}$ is vector space isomorphic to $T_e$ via the map $(\mathfrak{g} \to T_e, X \mapsto X_e)$ which transports a Banach space structure to $\mathfrak{g}$.

The exponential mapping is defined analogously to the finite-dimensional case of (1.1.28) and (1.1.29):

Lemma 1.2.18 (Maissen, Theorems 3.6.1, 3.6.2)

Let $X \in \mathfrak{g}$, the Lie algebra of the Banach-Lie group $\mathcal{G}$. There exists exactly one homomorphism $\varphi : \mathbb{R} \to \mathcal{G}$ such that $d\varphi : \mathbb{R} \to T_e$ is the map,

\[ d\varphi : r \mapsto rX_e, \forall r \in \mathbb{R}. \]

Definition 1.2.19 (Maissen, §3.6)

Let $\mathcal{G}$ be a Banach-Lie group. The **exponential mapping** for $\mathcal{G}$, denoted $\exp$, $\exp : \mathfrak{g} \to \mathcal{G}$ is the map,

\[ \exp : X \mapsto \varphi(1) \] where $X$ and $\varphi$

are as given in Theorem (1.2.18).

Remark 1.2.20

$\text{Exp}$ is a Fréchet-$C^\infty$ homomorphism of the one-dimensional, abelian Lie group $\text{span} \{X\}$ into $\mathcal{G}$, for any $X \in \mathfrak{g}$.
**Lemma 1.2.21** (Maissen, Theorem 3.6.1)

Let $\mathcal{G}$ be a Banach-Lie group with Lie algebra $\mathfrak{g}$ having Banach space structure as per (1.2.17). There is an open ball $B_0$, centre 0, with respect to the norm on $\mathfrak{g}$ such that $\exp : B_0 \to \exp(B_0)$ is a Fréchet diffeomorphism.

**Remark 1.2.22** (Maissen, Theorem 3.3.1)

A Lie algebra structure may be defined on $T_e$ exactly as described in (1.1.43). This makes the derivative of a homomorphism $\Phi$, into a Lie algebra homomorphism.

**Theorem 1.2.23** (Maissen, Theorem 3.10.2)

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be Banach Lie groups, and $\Phi$ a Fréchet-$C^\infty$ homomorphism $\Phi : \mathcal{G}_1 \to \mathcal{G}_2$. Then $d\Phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a continuous homomorphism of Lie algebras as given by Definition (1.1.37).

The relationship between $\Phi$ and $d\Phi$ is further elucidated by the following theorem.

**Theorem 1.2.24** (Maissen, §3.10)

Let $\Phi : \mathcal{G}_1 \to \mathcal{G}_2$ be a Fréchet-$C^\infty$ homomorphism of Banach-Lie groups $\mathcal{G}_1$ and $\mathcal{G}_2$, then

$$\Phi(\exp X) = \exp(d\Phi X) \quad \text{for all} \quad X \in \text{an open ball, centre 0, of } \mathfrak{g}_1.$$

**Remarks 1.2.25**

We have quoted only the analogous propositions and Lemmas to those of §1.1 which will be needed in the sequel. The Baker-Campbell-Hausdorff formula holds for Banach-Lie groups (Maissen, Theorem 3.9.1) in a form identical to that given by Sagle and Walde, Theorem 5.18, and the closed subgroup theorem (1.1.8) can be generalised (Maissen, Theorem 3.12.5) to the Banach-Lie case to show that the
identity component of $\mathcal{H}$ is a Lie subgroup of $\mathcal{G}$ provided there is a corresponding Lie algebra $\mathfrak{h}$ for $\mathcal{H}$, which is closed in $\mathfrak{g}$.

Naturally, all Lie algebraic theorems and definitions of §1.1 hold in the infinite-dimensional case except those that make specific reference to $\mathfrak{g}$ being finite dimensional.

The analogous result to (1.1.44) is given by Maissen, Theorem 3.12.7, where he shows that if $\Phi : \mathcal{G}_1 \to \mathcal{G}_2$ is a Fréchet-$C^\infty$ homomorphism, the identity component of the kernel of $\Phi$ is a Lie subgroup of $\mathcal{G}_1$, the identity component of $\Phi(\mathcal{G}_1)$ is a Lie subgroup of $\mathcal{G}_2$, the Lie algebra of $\ker \Phi$ is $\ker d\Phi$ and the Lie algebra of $\Phi(\mathcal{G}_1)$ is $d\Phi(\mathfrak{g}_1)$. 
§1.3 Representations of Finite Dimensional Lie Groups and the Method of Orbits. Generalised Characters and Infinitesimal Characters

In Chapter 3, we will give theorems relating the unitary representations of a sequence of isomorphic Lie groups \( \{ G_n \} \) to those of a Lie group \( G_0 \), called the contraction of \( G \) (The concept of contraction of Lie groups and Lie algebras is developed in §1.5 and §2.3 ff). The Method of Orbits, which is a technique for constructing the representations of a Lie group, has a structure which lends itself well to the programme of Chapter 3 which neatly exploits the concepts of the Orbits Procedure. We give a summary in this section on the Orbits Method. We also give the definitions of Generalised Character and Infinitesimal Character for infinite-dimensional representations, as the theory for the contractions of Generalised and Infinitesimal Characters, which are two applications of the contraction of representations, will be exposited in Chapter 5. The principal reference for this section (§1.3) will be Kirillov (1976).

Induced representations form part of the underpinnings of the method of orbits and we commence with a summary of unitary representations and induced \( G \)-modules.

**Lemma 1.3.1** (Kirillov, §4.1)

Let \( \mathcal{H} \) be a complex vector space with an inner product \( \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C} \) satisfying

\[
(\alpha X_1 + \beta X_2, Y) = \alpha (X_1, Y) + \beta (X_2, Y)
\]

\( \forall \alpha, \beta \in \mathbb{C}, \forall X_1, X_2, Y \in \mathcal{H} \)

\[
(X, Y) = (Y, X), \forall X, Y \in \mathcal{H}.
\]

Then \( \mathcal{H} \) is a Banach space (1.2.1) with norm \( \|X\| = \sqrt{(X, X)} \), and is called a **Hilbert space**. All Hilbert spaces considered herein will be assumed separable. Consequently an orthonormal basis \( \{ e_i \} \) of \( \mathcal{H} \) will always exist.
Definition 1.3.2 (Kirillov, §13.2)

A unitary operator \( U : \mathcal{H} \rightarrow \mathcal{H} \) on a Hilbert space \( \mathcal{H} \) is a linear map such that

\[
(UX, UY) = (X, Y), \quad \forall X, Y \in \mathcal{H}.
\]

A unitary representation \( \mathcal{R} \) of a Lie group \( \mathcal{G} \) is a homomorphism \( \mathcal{R} : \mathcal{G} \rightarrow \mathcal{U}(\mathcal{H}) \) where \( \mathcal{U}(\mathcal{H}) \) is the group of unitary operators on \( \mathcal{H} \).

Definition 1.3.3 (Kirillov, §13.2)

Let \( V \) be a complex vector space. A linear representation \( \mathcal{R}_1 \) of \( \mathcal{G} \), is a homomorphism \( \mathcal{R}_1 : \mathcal{G} \rightarrow \mathcal{L}(V) \) where \( \mathcal{L}(V) \) is the group of linear operators on \( V \). Two representations \( \mathcal{R}_1 : \mathcal{G} \rightarrow \mathcal{L}(V_1) \) and \( \mathcal{R}_2 : \mathcal{G} \rightarrow \mathcal{L}(V_2) \) are equivalent if there is a vector space isomorphism \( \Psi : V_1 \rightarrow V_2 \) such that \( \Psi^{-1} \mathcal{R}_2(g)\Psi = \mathcal{R}_1(g) \), \( \forall g \in \mathcal{G} \) (equivalent representations are isomorphic objects in the category of representations of \( \mathcal{G} \) in complex linear spaces (Kirillov, §7.1)). The pair \( (\mathcal{R}_1, V_1) \) is called a \( \mathcal{G} \)-module. Two \( \mathcal{G} \)-modules are isomorphic if they are equivalent as representations.

The method of orbits makes use of the method of induced representations, principally developed by Mackey in the 1950's (see Mackey (1976)) for locally compact, topological groups. We will employ the concept of induced \( \mathcal{G} \)-module, borrowed from Induced Representation Theory, but the form of the inner product, making our \( \mathcal{G} \)-module into a unitary representation, will be somewhat different. The inner product will be defined toward the end of §1.3.

We now give the definition induced \( \mathcal{G} \)-module:

Definition 1.3.4 (Kirillov, §13.4)

Let \( \mathcal{G}_1 \) be a closed, hence Lie subgroup of \( \mathcal{G} \), and let \( \mathcal{R}_1 : \mathcal{G}_1 \rightarrow \mathcal{U}(\mathcal{H}_1) \) be a finite-dimensional unitary representation of \( \mathcal{G}_1 \) on a Hilbert space \( \mathcal{H}_1 \). Let
$C^\infty(\mathcal{G}, \mathcal{G}_1, \mathcal{R}_1)$ be the set of $C^\infty$ maps $f$, $f : \mathcal{G} \to \mathcal{H}_1$

\[ f(g_1 g) = \mathcal{R}_1(g_1) f(g), \quad \forall g_1 \in \mathcal{G}_1, \forall g \in \mathcal{G}. \]

The induced $\mathcal{G}$-module by $\mathcal{R}_1$, denoted $\text{Ind}(\mathcal{G}, \mathcal{G}_1, \mathcal{R}_1)$, is the linear representation $\mathcal{R} \uparrow_{\mathcal{G}_1}^{\mathcal{G}}$ of $\mathcal{G}$,

\[ \mathcal{R} \uparrow_{\mathcal{G}_1}^{\mathcal{G}} : \mathcal{G} \to \mathcal{L}(C^\infty(\mathcal{G}, \mathcal{G}_1, \mathcal{R}_1)) \]

where $\mathcal{R} \uparrow_{\mathcal{G}_1}^{\mathcal{G}}$ is the homomorphism of $\mathcal{G}$ into the linear operators on the vector space $C^\infty(\mathcal{G}, \mathcal{G}_1, \mathcal{R}_1)$, given by,

\[
\mathcal{R} \uparrow_{\mathcal{G}_1}^{\mathcal{G}} (g) : f \mapsto f_g \text{ where } \\
\quad f_g(g_2) = f(g_2 g), \quad \forall g, g_2 \in \mathcal{G}.
\]

Comments and Outline of Procedure ..... (1.3.5)

The next group of definitions and propositions describes how to construct a unitary representation of $\mathcal{G}$ via the method of orbits. The method is to induce a $\mathcal{G}$-module from a 1-dimensional unitary representation of a certain Lie subgroup of $\mathcal{G}$, then to restrict to an invariant subspace of the induced $\mathcal{G}$-module. An invariant subspace of a linear representation $\mathcal{R} : \mathcal{G} \to \mathcal{L}(V)$ is a subspace $V_1$ of $V$ such that $\mathcal{R}(\mathcal{G})V_1 \subset V_1$. The idea is that the linear representation $\mathcal{R}|_{V_1} : \mathcal{G} \to \mathcal{L}(V_1)$, called a subrepresentation, will be irreducible. The representation $\mathcal{R}|_{V_1}$ is irreducible when there are no subspaces $V_2$ of $V_1$ invariant under $\mathcal{R}|_{V_1}$ other than \{0\} and $V_1$ itself. It is a principal aim of the method of orbits to find all the irreducible unitary representations of an arbitrary Lie group $\mathcal{G}$.

The final stage of the Orbit Method is to put a Hilbert space structure on this invariant subspace. This procedure will form the final part of the exposition of the orbits process.
Following the procedure described in (1.3.5), the first step in the method of orbits is to find a subgroup of \( G \) and a one-dimensional representation of this subgroup from which to induce the required \( G \)-module. The source of these representations is supplied by the forms \( \beta \in g^* \) (see 1.1.47):

**Proposition 1.3.6** (Kirillov, §15.3)

Let \( G_\beta \) denote the stabiliser of \( \beta \) by the co-adjoint action (1.1.48) of \( G \) on \( g^* : \quad G_\beta = \{ g \mid g \in G, \langle \operatorname{Ad}^* g(\beta), X \rangle = \langle \beta, X \rangle, \forall X \in g \}. \) The Lie algebra \( g_\beta \) of \( G_\beta \) is

\[
\{ X \mid X \in g, [\beta, [X, Y]] = 0, \forall Y \in g \}
\]

and \( \beta \), when restricted to \( g_\beta \) is a Lie-algebra homomorphism \( g_\beta \rightarrow \mathbb{R} \).

The homomorphism \( \beta : g_\beta \rightarrow \mathbb{R} \), provides us with a local homomorphism \( G_\beta \rightarrow T \) (Sagle & Walde, Theorem 6.8) where \( T = \{ z \mid z \in \mathbb{C}, |z| = 1 \} \). Thus, Proposition (1.3.6) points to the source of subgroups and one-dimensional representations:

**Definition 1.3.7**

A functional \( \beta \in g^* \) is integral if there exists a continuous homomorphism \( \rho : G_\beta \rightarrow T \) with derivative \( d\rho = \beta |_{g_\beta} \). Here, \( T = \{ z \mid z \in \mathbb{C}, |z| = 1 \} \); and \( \rho \) is called a character of \( G_\beta \).

**Comments 1.3.8**

Equipped with the closed subgroup \( G_\beta \) of \( G \) and the representation \( \rho : G_\beta \rightarrow T \), the induced \( G \)-module \( \text{Ind}(G, G_\beta, \rho) \) can be constructed. The next step is to find a subspace of \( C^\infty(G, G_\beta, \rho) \) on which the restricted representation will be irreducible. This is done by defining a complex subalgebra \( h(\beta) \) of \( g \otimes \mathbb{C} \) called a polarization, and a generalised Lie derivative (Helgason, 1978, 1.B.1) by every element \( z \) of \( h(\beta) \) such that the required subspace is the set of all functions \( f : G \rightarrow \mathbb{C} \) of
$C^\infty(\mathcal{G}, \mathcal{G}_\beta, \rho)$ such that the generalised Lie derivative of $f$ by every element $Z$ of $\mathcal{h}(\beta)$ is zero.

We remark that an alternative condition exists for $\beta$ being integral. This involves a cohomological condition (Kirillov, §15.3) on the manifold structure of the orbit of $\mathcal{G}$ through $\beta$, $\text{Ad}^*(\mathcal{G})\beta$.

We now define a polarization for a functional $\beta \in \mathfrak{g}^*$.

**Definition 1.3.9** (Ausländer and Kostant, 1971, 1.4.1 and §1.6)

Let $\beta \in \mathfrak{g}^*$. A **polarization for $\mathfrak{g}$ at $\beta$** is a complex Lie subalgebra $\mathcal{h}(\beta)$ of $\mathfrak{g} \otimes \mathbb{C}$ such that:

(i) $\mathcal{h}(\beta) \supset \mathfrak{g}_\beta$ and $\mathcal{h}(\beta)$ is stable under $\text{Ad}(\mathcal{G}_\beta)$

(ii) $\langle \beta, [\mathcal{h}(\beta), \mathcal{h}(\beta)] \rangle = 0$ and $\mathcal{h}(\beta)$ is a maximal vector subspace of $\mathfrak{g} \otimes \mathbb{C}$ with respect to this property.

(iii) $\mathcal{h}(\beta) + \overline{\mathcal{h}(\beta)}$ is a Lie subalgebra of $\mathfrak{g} \otimes \mathbb{C}$.

**Lemma 1.3.10** (Ausländer and Kostant, 1971, 1.4.1 & §1.6)

The maximal isotropy condition (1.3.9 (ii)) is equivalent to the statement:

$$\dim_{\mathbb{C}} \frac{\mathfrak{g} \otimes \mathbb{C}}{\mathcal{h}} = \frac{1}{2} \dim_{\mathbb{R}} \frac{\mathfrak{g}}{\mathfrak{g}_\beta}.$$

We now define the generalised derivative on elements of $C^\infty(\mathcal{G}, \mathcal{G}_\beta, \rho)$ alluded to in (1.3.8), and define the $\mathcal{G}$-module which will be irreducible.

**Definition 1.3.11** (Ausländer and Kostant, 1971, §1.5)

Let $f \in C^\infty(\mathcal{G}, \mathcal{G}_\beta, \rho)$. The **$\beta$-derivative** of $f$ in the complex direction $Z = Z_1 + iZ_2 \in \mathcal{h}(\beta)$ is

$$\tilde{\nabla}_Z f(g) = \frac{d}{ds} f((\exp sZ_1)g)|_{s=0} + \frac{d}{ds} f((\exp sZ_2)g)|_{s=0} - 2\pi i \langle \beta, Z \rangle f(g)$$
Proposition 1.3.12 (Auslander and Kostant, 1971, §1.5)

The $\beta$-derivative $\tilde{\nabla}_Z$ is a linear map of $C^\infty(G, G_\beta, \rho)$ into itself for any $Z \in g \otimes \mathbb{C}$.

The subspace of $C^\infty(G, G_\beta, \rho)$ given by

$$C^\infty(G, G_\beta, \rho; h(\beta)) = \{ f \mid f \in C^\infty(G, G_\beta, \rho), \tilde{\nabla}_Z f = 0, \forall Z \in h(\beta) \}$$

is an invariant subspace of the $G$-module $\text{Ind}(G, G_\beta, \rho)$ given by (1.3.4), and this $G$-module is denoted $\text{Ind}(G, G_\beta, \rho; h(\beta))$.

We now illustrate an equivalent construction of $C^\infty(G, G_\beta, \rho; h(\beta))$ in order to show how to build a Hilbert space structure from $C^\infty(G, G_\beta, \rho; h(\beta))$. To do this, the inducing in stages theorem is outlined which states that given closed subgroups $G_1 \subset G_2$ in $G$, the $G$-module $\text{Ind}(G, G_1, \rho)$, $\rho$ a character of $G_1 \rho : G_1 \to T$, with induced representation $\mathcal{R} \uparrow_{G_1}^{G_2}$, is equivalent to $\text{Ind}(G, G_2, \mathcal{R} \uparrow_{G_1}^{G_2})$ where $\text{Ind}(G, G_2, \mathcal{R} \uparrow_{G_1}^{G_2})$ is the representation formed by first inducing from $G_1$ and $\rho$ to $\text{Ind}(G_2, G_1, \rho)$ with representation $\mathcal{R} \uparrow_{G_1}^{G_2}$, and then forming the $G$-module $\text{Ind}(G, G_2, \mathcal{R} \uparrow_{G_1}^{G_2})$.

We first construct $G_1$ and $G_2$ from the polarization $h(\beta)$:

Definition 1.3.13 (Kirillov, §15.3)

Let $g_1 = h(\beta) \cap g$, so $g_1$ is a real subalgebra of $g$. Let $G_1^0$ be the connected subgroup of $G$ corresponding to $g_1$ (see 1.1.36) and let $G_1$ be the subgroup of $G$ generated by $G_1^0$ and $G_\beta$.

Definition 1.3.14 (Kirillov, §15.3)

Let $g_2 = (h(\beta) + \overline{h}(\beta)) \cap g$, so $g_2$ is a real subalgebra of $g$. Let $G_2^0$ be the connected subgroup of $G$ corresponding to $g_2$ and let $G_2$ be the subgroup of $G$ generated by $G_2^0$ and $G_\beta$. 

53
The following proposition establishes the conditions under which a suitable character of $G_1$ will exist.

**Proposition 1.3.15** (Auslander and Kostant, 1971, 1.5.1)

(*The generalised Pukansky Condition*)

If the orbit $\text{Ad}^*(G_2)\beta$ is closed in $G^*$, then the character $\rho$ of $G_\beta$ extends to a character $\bar{\rho}$ of $G_1$.

**Proposition 1.3.16** (Gaal, 1973, §6.9)

Suppose $G_1$ and $G_2$ are closed in $G$. Let $\mathcal{R} \uparrow_{G_1}^{G_2}$ be the representation of $G_2$ in the definition (1.3.4) of $\text{Ind}(G_2, G_1, \bar{\rho})$.

Form the induced representation $\text{Ind}(G, G_2, \mathcal{R} \uparrow_{G_1}^{G_2})$ in the following way:

Let $C^\infty(G, G_2, \mathcal{R} \uparrow_{G_1}^{G_2})$ be the vector space,

$$C^\infty(G, G_2, \mathcal{R} \uparrow_{G_1}^{G_2}) = \{ f \mid f \in C^\infty(G \times G_2), f : G \times G_2 \to \mathbb{C},$$

$$f(g, g_1g_2) = \tilde{x}(g_1)f(g, g_2)$$

$$f(g_2g, \cdot) = \mathcal{R} \uparrow_{G_1}^{G_2}(g_2)f(g, \cdot)$$

$$= f(g, \cdot g_2), \forall g_1 \in G_1, \forall g_2 \in G_2, \forall g, \in G \}. $$

The representation $\mathcal{R} \uparrow_{G_2}^{G}$ is given by, $\mathcal{R} \uparrow_{G_2}^{G}(h)f(g, \cdot) = f(gh, \cdot)$ The map $\Psi$

$$\Psi : C^\infty(G, G_1, \bar{\rho}) \to C^\infty(G, G_2, \mathcal{R} \uparrow_{G_1}^{G_2})$$

$$\Psi(f)(g, g_2) = f(g_2g), \quad \Psi(f)(g, \cdot) = f(\cdot g), \quad \forall g \in G, \forall g_2 \in G_2,$$

is a vector space isomorphism, and

$$\Psi^{-1} \mathcal{R} \uparrow_{G_2}^{G}(g)\Psi = \mathcal{R} \uparrow_{G_1}^{G}(g), \quad \forall g \in G,$$

hence the representations $\mathcal{R} \uparrow_{G_2}^{G}, \mathcal{R} \uparrow_{G_1}^{G}$ are equivalent.

For a theorem on the continuity of $\Psi$, see Proposition (1.3.38(a)).
Before applying the inducing in stages theorem (1.3.16) to find an alternative form for $C^\infty(\mathcal{G}, \mathcal{G}_\beta, \rho; \mathcal{H}(\beta))$, the next two propositions will make this alternative form more evident:

**Proposition 1.3.17** (Auslander and Kostant, 1971, §1.4, §1.5)

Let $\mathcal{G}$ be as per (1.3.13), and suppose that $\rho$ extends to a character $\tilde{\rho}$ of $\mathcal{G}_1$, and $\mathcal{G}_1$ is closed in $\mathcal{G}$. $\mathcal{G}_1 \otimes \mathbb{C} = \mathcal{H}(\beta) \cap \overline{\mathcal{H}(\beta)}$, and we have the following equality

$$\{ f \mid f \in C^\infty(\mathcal{G}, \mathcal{G}_\beta, \rho), \tilde{\nabla}_Z f = 0, \forall Z \in \mathcal{H}(\beta) \cap \overline{\mathcal{H}(\beta)} \} = C^\infty(\mathcal{G}, \mathcal{G}_1, \tilde{\rho}).$$

of $\mathcal{G}$-modules, $C^\infty(\mathcal{G}, \mathcal{G}_\beta, \rho; (\mathcal{H} \cap \overline{\mathcal{H}})(\beta))$ and $C^\infty(\mathcal{G}, \mathcal{G}_1, \tilde{\rho})$.

**Proposition 1.3.18** (Auslander and Kostant, 1971, §1.4, §1.5)

Let $\mathcal{G}_1, \mathcal{G}_2$ be as per (1.3.13) and (1.3.14) respectively, and suppose they are closed subgroups of $\mathcal{G}$, with $\rho$ extending to a character $\tilde{\rho}$ of $\mathcal{G}_1$. Let $\gamma = \beta|_{\mathcal{G}_2}$. Then $(\mathcal{G}_2)_\gamma = \mathcal{G}_1$ and $\mathcal{H}(\beta) \subset \mathcal{G}_2 \otimes \mathbb{C}$ is a polarization (1.3.9) for $\mathcal{G}_2$ at $\gamma$.

Write $\mathcal{H}_2(\gamma)$ for $\mathcal{H}(\beta)$. Let $f \in C^\infty(\mathcal{G}, \mathcal{G}_\beta, \rho; \mathcal{H}(\beta))$ then by (1.3.17) $f \in C^\infty(\mathcal{G}, \mathcal{G}_1, \tilde{\rho})$. Define the functions, $f_g \in C^\infty(\mathcal{G}_2, \mathcal{G}_1, \tilde{\rho})$ by

$$f_g(g_2) = f(g_2g), \quad \forall g \in \mathcal{G}, \forall g_2 \in \mathcal{G}_2,$$

then $f_g \in C^\infty(\mathcal{G}_2, \mathcal{G}_1, \tilde{\rho}; \mathcal{H}_2(\gamma))$.

The above proposition prompts us to make the definition of a $\mathcal{G}$-module induced from $\text{Ind}(\mathcal{G}_2, \mathcal{G}_1, \tilde{\rho}, \mathcal{H}_2(\gamma))$.

**Definition 1.3.19** (Auslander and Kostant, 1971, §1.5)

Let $\mathcal{R}_2 \uparrow_{\mathcal{G}_1}^{\mathcal{G}_2}$ denote the representation $\mathcal{R} \uparrow_{\mathcal{G}_1}^{\mathcal{G}_2}$ (1.3.4) acting on $C^\infty(\mathcal{G}_2, \mathcal{G}_1, \tilde{\rho}; \mathcal{H}_2(\gamma))$.

Let $C^\infty(\mathcal{G}, \mathcal{G}_2, \mathcal{R}_2 \uparrow_{\mathcal{G}_1}^{\mathcal{G}_2})$ denote the vector space

$$C^\infty(\mathcal{G}, \mathcal{G}_2, \mathcal{R}_2 \uparrow_{\mathcal{G}_1}^{\mathcal{G}_2}) = \left\{ f \mid f \in C^\infty(\mathcal{G} \times \mathcal{G}_2), f : \mathcal{G} \times \mathcal{G}_2 \to \mathbb{C}, \right. \left. f(g, \cdot) \in C^\infty(\mathcal{G}_2, \mathcal{G}_1, \tilde{\rho}; \mathcal{H}_2(\gamma)) \right\}$$

$$f(g_2g, \cdot) = \mathcal{R}_2 \uparrow_{\mathcal{G}_1}^{\mathcal{G}_2} f(g, \cdot), \quad \forall g \in \mathcal{G}, \forall g_2 \in \mathcal{G}_2.$$
and let \( \text{Ind}(\mathcal{G}, \mathcal{G}_2, \mathcal{R}_2 \uparrow_{\mathcal{G}_1}^{\mathcal{G}_2}) \) be the corresponding \( \mathcal{G} \)-module with representation \( \mathcal{R} \uparrow_{\mathcal{G}_1}^{\mathcal{G}_2} \) acting as in (1.3.4).

Herewith the promised theorem giving the alternative form for \( \text{Ind}(\mathcal{G}, \mathcal{G}_\beta, \rho; \hbar(\beta)) \).

**Theorem 1.3.20** (Auslander and Kostant, 1971, §1.5)

Suppose that (1.3.13) and (1.3.14) are closed in \( \mathcal{G} \), and that \( \rho \) extends to a character \( \tilde{\rho} \) and \( \mathcal{G}_1 \) (c.f. (1.3.15):

In the notation of (1.3.18) and (1.3.19), the \( \mathcal{G} \)-module \( \text{Ind}(\mathcal{G}, \mathcal{G}_2, \mathcal{R}_2 \uparrow_{\mathcal{G}_1}^{\mathcal{G}_2}) \) is isomorphic to the \( \mathcal{G} \)-module \( \text{Ind}(\mathcal{G}, \mathcal{G}_\beta, \rho; \hbar(\beta)) \) via the vector space isomorphism,

\[
\Psi : C^\infty(\mathcal{G}, \mathcal{G}_\beta, \rho; \hbar(\beta)) \to C^\infty(\mathcal{G}, \mathcal{G}_2, \mathcal{R}_2 \uparrow_{\mathcal{G}_1}^{\mathcal{G}_2})
\]

\[
\Psi(f)(g, g_2) = f(g_2 g)
\]

\[
\Psi(f)(g, \cdot) = f(\cdot g), \quad \forall g \in \mathcal{G}, \forall g_2 \in \mathcal{G}_2.
\]

To facilitate extending \( C^\infty(\mathcal{G}, \mathcal{G}_\beta, \rho; \hbar(\beta)) \) to a Hilbert space we develop the concepts of unitary induced representations. In order to create the Hilbert space structure, we need some measure-theoretic results:

**Proposition 1.3.21** (Kirillov, Theorem 9.1.1)

If \( \mathcal{G} \) is a Lie group, there is a left-invariant Borel measure on \( \mathcal{G} \) denoted \( d_\ell g \) and called left Haar measure and a right-invariant Borel measure on \( \mathcal{G} \) denoted \( d_r g \) and called right Haar measure. Both left- and right-invariant measures are unique up to a numerical factor.

**Definition 1.3.22** (Kirillov, §4.5)

Let \( \mathcal{G} \) be a Lie group, \( \mathcal{H} \) a Hilbert space and \( \{e_i\} \) an orthonormal basis of \( \mathcal{H} \). With respect to a measure on \( \mathcal{G} \), a map \( f : \mathcal{G} \to \mathcal{H} \) is measurable if each map \( \mathcal{G} \to C, \ g \mapsto (f(g), e_i) \) is measurable for each \( e_i \) in \( \{e_i\} \).
Lemma 1.3.23 (Kirillov §9.1)

There is a continuous homomorphism $\Delta_g : G \to \mathbb{R}^+$ of $G$ into the multiplicative group of positive real numbers which satisfies,

$$d_r(g_1g) = \Delta_g(g_1)d_r g, \quad \forall g, g_1 \in G.$$

Proposition 1.3.24 (Gaal, 1973, 5.3.4)

Let $G$ be a Lie group and $G_c$ a closed subgroup of $G$. There is a continuous function $\delta$,

$$\delta : G \to (0, \infty) \quad \text{such that}$$

$$\delta(hg) = \frac{\Delta_g(h)}{\Delta_{g_c}(h)} \delta(g), \quad \forall g \in G, \forall h \in G_c.$$

We now give the Hilbert space structure for an induced, unitary representation, which is essentially due to Mackey. This vector space generalises the induced $G$-module of (1.3.4).

Proposition 1.3.25 (Kirillov, §13.2)

Let $G_c$ be a closed subgroup of $G$, and $R_c : G_c \to U(H_c)$ a unitary representation of $G_c$ on a Hilbert space $H_c$ with inner product $(\cdot, \cdot)_c$. By analogy to (1.3.4) let $L^2(G, G_c, R_c)$ denote the set of measurable (1.3.22) maps $f : G \to H_c$ which satisfy,

$$f(hg) = R_c(h)f(g), \quad \forall h \in G_c, \forall g \in G$$

and

$$\int_G (f(g), f(g))_c \delta(g)d_r g < \infty;$$

then $L^2(G, G_c, R_c)$ is a Hilbert space with inner product,

$$(f_1, f_2) = \int_G (f_1(g), f_2(g))_c \delta(g)d_r g.$$
Proposition 1.3.26 (Kirillov, §13.2)

Let \( G_c \) be a closed subgroup of \( G \) and \( R_c : G_c \to \mathcal{U}(\mathcal{H}_c) \) a unitary representation.

Define the function \( r \),

\[
r(g_1, g) = \frac{\delta(g_1 g)}{\delta(g_1)}, \quad \forall g, g_1 \in G.
\]

For each \( g \in G \), the map

\[
R \uparrow_{G_c}^G : L^2(G, G_c, R_c) \to L^2(G, G_c, R_c),
\]

\[
R \uparrow_{G_c}^G (g)(f)(g_1) = f(g_1)\sqrt{r(g_1, g)}
\]

where \( f(g_1) = f(g_1 g) \) (in analogy to (1.3.4)), is a homomorphism,

denoted \( R \uparrow_{G_c}^G : G \to \mathcal{U}(L^2(G, G_c, R_c)) \), and is called the unitary representation of \( G \) induced by \( R_c : G_c \to \mathcal{U}(\mathcal{H}_c) \), denoted \( \text{Ind}L^2(G, G_c, R_c) \).

The answer to the natural question of continuity of an induced representation is supplied by the following definition and lemma.

Definition 1.3.27 (Kirillov, §13.2)

A unitary representation \( R : G \to \mathcal{U}(\mathcal{H}) \) is continuous when the map,

\( G \times \mathcal{H} \to \mathcal{H}, \ (g, f) \mapsto R(g)f \) is continuous in both variables.

Lemma 1.3.28 (Kirillov, §13.2)

The unitary representation \( \text{Ind}L^2(G, G_c, R_c) \) is continuous if \( R_c \) is continuous.

The measure \( \delta(g)d_{r}g \) generates a quasi-invariant measure \( d\mu \), on \( G_c \setminus G \) and there is another form for the inner product of (1.3.25):
Definition 1.3.29 (Kirillov, §9.1)

Let $G_c$ be a closed subgroup of $G$. A measure $d\mu$ on $G_c \setminus G$ is **quasi-invariant** if $d\mu(Xg) = m_g d\mu(X)$, $\forall g \in G$, $\forall X \in G_c \setminus G$, where $m_g : G_c \setminus G \to \mathbb{C}$ is a Borel-measurable function. $d\mu$ is **invariant** if $d\mu(Xg) = d\mu(X)$.

Lemma 1.3.30 (Kirillov, §13.2)

Let $s : G_c \setminus G \to G$ be a cross-section map which is Borel measurable. Then there is a quasi-invariant measure $d\mu$ on $G_c \setminus G$ such that

$$\int_G (f_1(g), f_2(g))_c d\mu(g) = \int_{G_c \setminus G} (f_1(s(x)), f_2(s(x)))_c d\mu(X)$$

When $G_c \setminus G$ admits an invariant measure, the induced representation of $G$ by $G_c$ is simpler:

**Proposition 1.3.31** (Gaal, 1973, §6.2)

Let $G_c$ be a closed subgroup of $G$, and $\mathcal{R}_c : G_c \to \mathcal{U}(\mathcal{H}_c)$ a unitary representation of $G_c$. Suppose that $G_c \setminus G$ admits an invariant Borel measure $\mu$. The vector space $L^2(G, G_c, \mathcal{R}_c)$ of measurable (1.3.22) maps $f : G \to \mathcal{H}_c$ which satisfy,

$$f(hg) = \mathcal{R}_c(h) f(g), \quad \forall h \in G_c, \forall g \in G$$

and

$$\int_{G_c \setminus G} (f(s(x)), f(s(x)))_c d\mu(X) < \infty$$

where $s$ is any fixed, Borel-measurable cross-section $s : G_c \setminus G \to G$, is a Hilbert space with inner product,

$$(f_1, f_2) = \int_{G_c \setminus G} (f_1(s(x)), f_2(s(x)))_c d\mu(X).$$

**Proposition 1.3.32** (Gaal, 1973, §6.2)

Let $G_c$ be a closed subgroup of $G$, $\mathcal{R}_c : G_c \to \mathcal{U}(\mathcal{H}_c)$ a unitary representation of $G_c$, and $L^2(G, G_c, \mathcal{R}_c)$ be given by (1.3.31).
Let $g \in \mathcal{G}$. The map,

$$\mathcal{R} \uparrow_{\mathcal{G}_c}^\mathcal{G} (g) : \mathcal{L}^2(\mathcal{G}, \mathcal{G}_c, \mathcal{R}_c) \to \mathcal{L}^2(\mathcal{G}, \mathcal{G}_c, \mathcal{R}_c),$$

$$\mathcal{R} \uparrow_{\mathcal{G}_c}^\mathcal{G} (g)(f) = f_g, \quad \text{where} \quad f_g(g_1) = f(g_1 g), \quad \forall g_1 \in \mathcal{G}$$

is the unitary representation $\mathcal{R} \uparrow_{\mathcal{G}_c}^\mathcal{G} : \mathcal{G} \to \mathcal{U}(\mathcal{L}^2(\mathcal{G}, \mathcal{G}_c, \mathcal{R}_c))$ induced by $\mathcal{R}_c$, denoted $\text{Ind}\mathcal{L}^2(\mathcal{G}, \mathcal{G}_c, \mathcal{R}_c)$.

The first step in providing a Hilbert space structure for $\text{Ind}(\mathcal{G}, \mathcal{G}_2, \mathcal{R}_2 \uparrow_{\mathcal{G}_c}^\mathcal{G}) \simeq \text{Ind}(\mathcal{G}, \mathcal{G}_2, \mathcal{R}_2; \mathcal{h}(\beta))$ (see (1.3.20)) is to find a Hilbert space structure for $C^\infty(\mathcal{G}_2, \mathcal{G}_1, \mathcal{R}_1; \mathcal{h}(\gamma))$. Fortunately, the form $\gamma$ (1.3.18) provides a $\mathcal{G}_2$-invariant measure $\mu$ on $\mathcal{G}_1 \setminus \mathcal{G}_2$ enabling a simpler definition of the Hilbert space.

**Lemma 1.3.33** (Kirillov, §15.1)

Let $\mathcal{G}_2, \mathcal{G}_1$ be closed subgroups of $\mathcal{G}$, as per (1.3.13) and (1.3.14). From (1.3.18), $\mathcal{G}_1 = (\mathcal{G}_2)_\gamma$. Identifying $T_\gamma(\mathcal{G}_1 \setminus \mathcal{G}_2)$ with $\mathcal{G}_1 \setminus \mathcal{G}_2$ (where $\gamma$ is the coset $\mathcal{G}_1$), let $X, Y \in \mathcal{G}_2$, and let $\xi(X), \xi(Y)$ be invariant vector fields (1.1.52), such that

$$\xi_\gamma(X) = X + \mathcal{G}_1, \quad \xi_\gamma(Y) = Y + \mathcal{G}_1.$$

Define the map $\omega_\alpha : T_\alpha(\mathcal{G}_1 \setminus \mathcal{G}_2) \times T_\alpha(\mathcal{G}_1 \setminus \mathcal{G}_2) \to \mathbb{R}$,

$$\omega_\alpha(\xi_\alpha(X), \xi_\alpha(Y)) = \langle \alpha, [X,Y] \rangle$$

where $\alpha \in O_\gamma$, and $O_\gamma$ is identified with $\mathcal{G}_1 \setminus \mathcal{G}_2$. Then the map $\alpha \mapsto \omega_\alpha$ is a $\mathcal{G}_\alpha$-invariant 2-form on $\mathcal{G}_1 \setminus \mathcal{G}_2$.

The $\mathcal{G}_2$-invariant 2-form $\omega$ provides the $\mathcal{G}_2$-invariant measure $\mu$ on $\mathcal{G}_1 \setminus \mathcal{G}_2$ as heralded:
Proposition 1.3.34  (Auslander and Kostant, 1971, §1.5)

Since \( g_1 \) is the radical of the anti-symmetric bilinear map: \( g_2 \times g_2 \rightarrow \mathbb{R}, \) \((X,Y) \mapsto \langle \gamma, [X,Y] \rangle\), \( g_2/g_1 \) is even dimensional whence \( G_1 \backslash G_2 \) is even dimensional, of dimension \( 2n_1 \) say. The \( 2n_1 \)-form \( \omega \wedge \cdots \wedge \omega \) (product \( n_1 \) times as per (1.1.56) and (1.1.58)) is invariant (1.1.62) under the action (1.1.52) of \( G_2 \). By (1.1.63) and (1.1.61) a measure \( \mu \) is determined on \( G_1 \backslash G_2 \) which is \( G_2 \)-invariant.

Corollary 1.3.35  (Kirillov, §15.6)

The anti-symmetric bilinear form \((X,Y) \mapsto \langle \beta, [X,Y] \rangle\) defines a two-form \( \omega \) on \( G_\beta \backslash G \), a manifold of dimension \( 2n_2 \), say.

The \( 2n_2 \)-form, \( \frac{1}{n_2!} \omega \wedge \cdots \wedge \omega \) (product \( n_2 \) times) defines a measure \( \mu \) on the orbit \( O_\beta = G_\beta \backslash G \).

Herewith step one in providing a Hilbert space structure for
\[ \text{Ind}(G, G_2, R_2 \uparrow_{G_1}^G) : \]

Lemma 1.3.36  (Auslander and Kostant, 1971, §1.5)

Let \( G_1, G_2 \) be closed in \( G \), and as per (1.3.13) and (1.3.14). Suppose that \( \rho \) extends to a character \( \tilde{\rho} \) of \( G_1 \) (c.f. (1.3.15)). Let \( \mu \) be the \( G_2 \)-invariant measure on \( G_1 \backslash G_2 \) of (1.3.34). Form the induced representation \( \text{Ind}L^2(G_2, G_1, \tilde{\rho}) \) of (1.3.32), completing \( L^2(G_2, G, \tilde{\rho}) \) if \( \mu \) is not Borel measurable. The subspace \( S^\infty = C^\infty(G_2, G_1, \tilde{\rho}; \mathcal{H}_2(\gamma)) \cap L^2(G_2, G_1, \tilde{\rho}) \) forms a \( G_2 \)-submodule of \( \text{Ind}L^2(G_2, G_1, \tilde{\rho}) \).

The closure of this subspace, denoted \( L^2(G_2, G_1, \tilde{\rho}; \mathcal{H}_2(\gamma)) \), provides a unitary representation of \( G_2 \), denoted \( \text{Ind}L^2(G_2, G_1, \tilde{\rho}; \mathcal{H}_2(\gamma)) \).

Now the final step to yield a Hilbert space structure for
\[ \text{Ind}(G, G_2, R_2 \uparrow_{G_1}^G) \simeq \text{Ind}(G, G_\beta, \rho; \mathcal{H}(\beta)). \]
Lemma 1.3.37 (Auslander and Kostant, 1971, §1.5)

Let $\mathcal{G}_1, \mathcal{G}_2$ be closed in $\mathcal{G}$, and as in (1.3.13) and (1.3.14). Suppose that $\rho$ extends to a character $\tilde{\rho}$ of $\mathcal{G}_1$ (c.f. (1.3.15)). Let $\mathcal{R}_2 \uparrow_{\mathcal{G}_1}^{\mathcal{G}_2}$ be the representation of $\mathcal{G}_2$ acting on $\mathcal{L}^2(\mathcal{G}_2, \mathcal{G}_1, \tilde{\rho}; \mathcal{H}_2(\gamma))$. The desired unitary representation of $\mathcal{G}$ is the induced representation,

$$\text{Ind} \mathcal{L}^2(\mathcal{G}, \mathcal{G}_2, \mathcal{R}_2 \uparrow_{\mathcal{G}_1}^{\mathcal{G}_2})$$
as in (1.3.26).

Comments 1.3.38 (Kirillov, §15.3)

The relationship of $\text{Ind}(\mathcal{G}, \mathcal{G}_2, \mathcal{R}_2 \uparrow_{\mathcal{G}_1}^{\mathcal{G}_2})$ of (1.3.20) to $\text{Ind}\mathcal{L}^2(\mathcal{G}, \mathcal{G}_2, \mathcal{R}_2 \uparrow_{\mathcal{G}_1}^{\mathcal{G}_2})$ is that $\mathcal{L}^2(\mathcal{G}, \mathcal{G}_2, \mathcal{R}_2 \uparrow_{\mathcal{G}_1}^{\mathcal{G}_2})$ is "generated" by a subspace of $C^\infty(\mathcal{G}, \mathcal{G}_2, \mathcal{R}_2 \uparrow_{\mathcal{G}_1}^{\mathcal{G}_2})$ of (1.3.19). Specifically, this subspace is $C^\infty(\mathcal{G}, \mathcal{G}_2, \mathcal{R}_2 \uparrow_{\mathcal{G}_1}^{\mathcal{G}_2})$ where $\mathcal{R}_2 \uparrow_{\mathcal{G}_1}^{\mathcal{G}_2}$ is the restriction of the representation $\mathcal{R}_2 \uparrow_{\mathcal{G}_1}^{\mathcal{G}_2}$ to the subspace $S^\infty$ of (1.3.36). $\text{Ind}\mathcal{L}^2(\mathcal{G}, \mathcal{G}_2, \mathcal{R}_2 \uparrow_{\mathcal{G}_1}^{\mathcal{G}_2})$ is the unitary representation of $\mathcal{G}$ induced from the unitary representation $\mathcal{R}_2 \uparrow_{\mathcal{G}_1}^{\mathcal{G}_2}$ of $\mathcal{G}_2$. The method of orbits as outlined here provides all the irreducible representations of a very wide class of Lie groups including compact, solvable and co-compact nilradical groups as well as the Principal Series of representations of non-compact, semisimple Lie groups to be discussed below (see Dooley 1983). These irreducible representations are given by $\text{Ind}\mathcal{L}^2(\mathcal{G}, \mathcal{G}_2, \mathcal{R}_2 \uparrow_{\mathcal{G}_1}^{\mathcal{G}_2})$.

If the representation $\text{Ind}\mathcal{L}^2(\mathcal{G}, \mathcal{G}_2, \mathcal{R}_2 \uparrow_{\mathcal{G}_1}^{\mathcal{G}_2})$ is constructed using any other functional $\alpha$ of $O_\beta$, the representations will be equivalent. Thus the representation constructed by the method of orbits depends only on the orbit $O_\beta$ of $\mathcal{G}$ through $\beta$. It is appropriate at this point to note the following proposition which will be useful in Chapter 3. It is a generalisation of (1.3.16):
Proposition 1.3.38(a) (Gaal, §6.9) (Inducing in Stages Theorem)

Let $\text{Ind}(G_2, G_1, \hat{\rho})$ be the induced representation $\mathcal{R} \uparrow_{G_1}^{G_2}$ (1.3.32) of $G_2$ with respect to the $G_2$-invariant measure $\mu$ on $G_1 \backslash G_2$ as per (1.3.34). Now form the induced representation of $G$, $\text{Ind}^2(G, G_2, \mathcal{R} \uparrow_{G_1}^{G_2})$ as per (1.3.26).

On the other hand form the induced representation of $G$, $\text{Ind}^2(G, G_1, \hat{\rho})$ as per (1.3.26). Then, $\text{Ind}^2(G, G_2, \mathcal{R} \uparrow_{G_1}^{G_2})$ and $\text{Ind}^2(G, G_1, \hat{\rho})$ are equivalent (1.3.3) and the isomorphism map $\Psi_I$ is an isometry of Hilbert spaces.

We will now address the necessary concepts leading up to the definition of Generalised Character and Infinitesimal characters and their relationship to the method of orbits. In Chapter 5 we will study the contraction of generalised characters.

A generalised character of a representation $\mathcal{R}$ of a Lie group $G$ is a functional on an ideal in $L^1(G, d_\gamma g)$. We give the definitions of $L^1(G, d_\gamma g)$ and the ideal, then the definition of a generalised character for $\mathcal{R}$ and $G$ illustrating that it determines the representation $\mathcal{R}$ up to equivalence. Further we illustrate an explicit formula for the generalised character of a representation which makes the study of generalised characters particularly attractive. This explicit form makes the study of the contractions of generalised characters a possible tool for further conjecture on the domain of applicability of the formula.

Definition 1.3.39 (Gaal, 1973, §1.1)

A Banach algebra $\mathcal{A}$ with involution is an associative algebra which is a Banach space (1.2.1) over $\mathbb{C}$ such that $\|xy\| \leq \|x\| \cdot \|y\|$, $\forall x, y \in \mathcal{A}$ and which has a continuous map $\mathcal{A} \to \mathcal{A}$, $x \mapsto x^*$ such that $(\lambda x + \mu y)^* = \overline{\lambda} x^* + \overline{\mu} y^*$, $(xy)^* = y^* x^*$, $(x^*)^* = x$, $\forall x, y \in \mathcal{A}$, $\lambda, \mu \in \mathbb{C}$. An ideal $\mathcal{I}$ of $\mathcal{A}$ is a vector subspace of $\mathcal{A}$ such that for any $w \in \mathcal{I}$, $xw \in \mathcal{I}$ and $wx \in \mathcal{I}$, $\forall x \in \mathcal{A}$.
Lemma 1.3.40 (Kirillov, §10.2)

The set $L^1(G, dtg) = \{ f \mid f : G \to C, \int_G |f(g)|dtg < \infty \}$ is a Banach algebra with product given by the convolution of two functions,

$$(f_1 * f_2)(g) = \int_G f_1(h)f_2(h^{-1}g)dh$$

and involution, $f^*(g) = \Delta_G(g)f(g^{-1})$ with $\Delta_G$ given by (1.3.23).

Lemma 1.3.41 (Gaal, 1973, §1.8)

Let $H$ be a Hilbert space, and let $L(H)$ be the algebra of continuous linear maps $A : H \to H$ with norm given by the operator norm,

$$\|A\| = \sup_{\|x\|=1} \|Ax\|.$$ 

The Adjoint of $A$ is the unique linear map $A^* : H \to H$ defined by $(x, Ay) = (A^*x, y), \forall x, y \in H$. $L(H)$ is a Banach algebra with involution $A \mapsto A^*$ which in this case is an isometry.

The following homomorphism of algebras will be used to define the generalised character.

Proposition 1.3.42 (Gaal, 1973, 5.5.4)

Let $\mathcal{R} : G \to U(H)$ be a continuous unitary representation of $G$ and $f \in L^1(G, dtg)$ (1.3.40) with norm $\| \cdot \|_1$. The linear operator $\sigma_{\mathcal{R}}(f) : H \to H$ defined by

$$(y, \sigma_{\mathcal{R}}(f)x) = \int_G f(g)(y, \mathcal{R}(g)x)dtg$$

is bounded as an operator on $H$. Furthermore, the linear map $\sigma_{\mathcal{R}} : L^1(G, dtg) \to L(H)$ is a bounded homomorphism of algebras. $\sigma_{\mathcal{R}}(f)$ is formally denoted,

$$\sigma_{\mathcal{R}}(f) = \int_G f(g)\mathcal{R}(g)dtg.$$ 

Herewith the definition of generalised character:
Proposition 1.3.43 (Gaal, 1973, §7.5)

Let \( \mathcal{R} : \mathcal{G} \to \mathcal{U}(\mathcal{H}) \) be a continuous unitary representation of \( \mathcal{G} \) and let \( \{e_i\} \) be an orthonormal basis of \( \mathcal{H} \). Let \( \mathcal{I}_\mathcal{R} \) be the set of all \( f \in L^1(\mathcal{G}, d\ell g) \) such that \( \sigma_\mathcal{R}(f) \) is of trace class, that is, \( \text{tr}(\sigma_\mathcal{R}(f)) = \sum_{i=1}^{\infty}(e_i, \sigma_\mathcal{R}(f)e_i) < \infty \). The trace class operators on \( \mathcal{H} \) form an ideal of \( \mathcal{L}(\mathcal{H}) \) and consequently \( \mathcal{I}_\mathcal{R} \) is an ideal of \( L^1(\mathcal{G}, d\ell g) \). Define the functional \( \chi_\mathcal{R} : \mathcal{I}_\mathcal{R} \to \mathbb{C}, \ chi_\mathcal{R} : f \mapsto \text{tr}(\sigma_\mathcal{R}(f)) = \hat{f}(\mathcal{R}) \).

The functional \( \chi_\mathcal{R} \) is called the generalised character of \( \mathcal{R} \). The representation \( \mathcal{R} \) is said to have a generalised character if \( \mathcal{I}_\mathcal{R} \) is non-zero.

The importance of generalised characters is underlined by the following proposition:

Proposition 1.3.44 (Gaal, 1973, 7.5.14)

If the irreducible representation \( \mathcal{R} : \mathcal{G} \to \mathcal{U}(\mathcal{H}) \) has a character, then \( \mathcal{R} \) is determined to within equivalence by the restriction of \( \chi_\mathcal{R} \) to any ideal on which it is defined and does not vanish identically.

There is a wide class of Lie groups and irreducible representations for which \( \chi_\mathcal{R} \) and \( \mathcal{I}_\mathcal{R} \) are non-zero.

For Lie groups, \( \mathcal{I}_\mathcal{R} \) is always non-zero:

Theorem 1.3.45 (Kirillov, §11.2)

Let \( \mathcal{G} \) be a Lie group and let \( C^\infty_0(\mathcal{G}) \) be the space of \( C^\infty \) functions \( \mathcal{G} \to \mathbb{C} \) of compact support. A \( C^\infty \) function \( f : \mathcal{G} \to \mathbb{C} \) is said to have compact support if the set \( \{ g \mid g \in \mathcal{G}, f(g) \neq 0 \} \) is contained in a compact subset of \( \mathcal{G} \). The set \( C^\infty_0(\mathcal{G}) \) is an ideal in \( L^1(\mathcal{G}, d\ell g) \), and for any irreducible representation \( \mathcal{R} : \mathcal{G} \to \mathcal{U}(\mathcal{H}) \), \( C^\infty_0(\mathcal{G}) \subseteq \mathcal{I}_\mathcal{R} \).

Kirillov (§15.6) has conjectured a universal formula for \( \chi_\mathcal{R} \) for all representations of all Lie groups:
**Theorem 1.3.46** (Kirillov, §15.6)

Let $\mathcal{R}$ be any unitary representation of a compact, simply connected Lie group or of an exponential Lie group, $G$. Then, if $\Omega$ is the orbit corresponding to $\mathcal{R}$, (see comments (1.3.38))

$$
\chi_\pi(f) = \hat{f}(\mathcal{R}) = \int_{\Omega} \int_{g} e^{i(\alpha, X)} f(X)(P_\Omega(X))^{-1} dX d\mu(\alpha)
$$

where $f \in C_0^\infty(g(0))$ and $g(0)$ is a neighbourhood of 0 in $g$ on which $\exp$ is a diffeomorphism, $\mu$ is the measure (1.3.35) on the orbit $\Omega$, and $P_\Omega \in C^\infty(g)$ is invariant under inner automorphisms.

This formula is truly universal for a large number of cases:

**Corollary 1.3.47** (Kirillov, §15.6)

In the notation of (1.3.46), if the representation $\mathcal{R}$ corresponds to an orbit $\Omega$ of maximal dimension then the function $P_\mathcal{U} \in C^\infty(g)$, given by

$$P_\mathcal{U}(X) = \det^{\frac{1}{2}} \left( \frac{\sinh \text{ad}(X)}{\text{ad}(X)} \right)$$

satisfies $P_\mathcal{U}(X) = P_\Omega(X)$ for any representation with corresponding orbit $\Omega$.

The universal character formula also holds for the Principal and Discrete Series of representations of a non-compact semisimple Lie group. We define these series below:

**Proposition 1.3.48** (Barut and Rackza, 1986, §19.1)

Let $G$ be a non-compact, semisimple Lie group, and let $G = KAN$ be an Iwasawa decomposition of $G$ as in (1.1.81). Let $M$ be the centralizer of $A$ in $K$. The subgroup $P = MAN$ is called the **minimal parabolic subgroup** of $G$. All finite-dimensional unitary irreducible representations of $P$ are of the form $\mathcal{R}(man) = \chi(a)\mathcal{R}_1(m)$, $m \in M, a \in A, n \in N$ where $\chi$ is a character of $A$ and $\mathcal{R}_1$
is a continuous, irreducible finite-dimensional representation of $M$. A **Principal Series representation** of $G$ is the induced unitary representation $\text{Ind}L^2(G, P, \mathcal{R})$ as in (1.3.26).

**Proposition 1.3.49** (Hurt, 1983, §14.1)

Let $\mathcal{R}$ be a continuous irreducible unitary representation of $G$ in a Hilbert space $\mathcal{H}$ which has an orthonormal basis $\{e_i\}$. $\mathcal{R}$ is a **discrete series representation** if every function $g \mapsto (\mathcal{R}(g)e_i, e_j)$ is square integrable on $G$ with respect to Haar measure (1.3.21). Let $K$ be a maximal compact subgroup of $G$. A discrete series representation exists for $G$ if and only if $\text{rank}(G) = \text{rank}(K)$ (see 1.1.75).

Herewith the promised theorem for the universal generalised character formula for the principal and discrete series representations of a semisimple Lie group:

**Theorem 1.3.50** (Rossmann, 1978)

Let $G$ be a non-compact semisimple Lie group such that:

(a) $G$ has only finitely many connected components

(b) $\text{Ad}(G) \subset \text{Int}(g \otimes \mathbb{C})$ where $\text{Int}(g \otimes \mathbb{C})$ denotes the analytic subgroup of $G\mathcal{L}(g \otimes \mathbb{C})$ whose Lie algebra is the homomorphic image $\text{ad}(g \otimes \mathbb{C})$ of $g \otimes \mathbb{C}$.

(c) The centre of the connected component of the identity of $G$ is finite.

Then if $\mathcal{R}$ is a principal or discrete series representation of $G$, $\chi_{\mathcal{R}}$ has the form (1.3.46) with $P_H = Pu$ as in (1.3.47).

We now give the definitions and propositions leading up to the presentation of infinitesimal characters. We illustrate how a generalised character can be found as a solution of a generalised differential equation involving the infinitesimal character, leading to the classification of a wide class of representations of complex semisimple Lie groups. The techniques of the method of orbits provide an explicit
form for the infinitesimal character for any Lie group and any irreducible representation. This form is particularly amenable to the application of the contraction process, studied in Chapter 5.

The infinitesimal character is a homomorphism of the centre $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ into the field of complex numbers. The enveloping algebra is isomorphic to an algebra of generalised functions on $\mathcal{G}$:

**Definition 1.3.51** (Kirillov, §10.4)

Let $\mathcal{G}$ be a Lie group with Lie algebra $\mathfrak{g}$. $f$ is a generalised function on $\mathcal{G}$ with compact support $K$ if $f$ is a linear function $f : C^\infty(\mathcal{G}) \to \mathbb{R}$ such that there exists a smallest compact subset $K$ of $\mathcal{G}$ for which $f(\phi) = 0$, for all $\phi \in C^\infty(\mathcal{G})$ which vanish together with their derivatives on $K$. Let $\Phi(\mathcal{G})$ denote the set of all generalised functions $\phi$ on $\mathcal{G}$ with compact support, and let $\Phi(\mathcal{G}, K)$ be the subset of $\Phi(\mathcal{G})$ of those $\phi$ with compact support $K$.

$\Phi(\mathcal{G}, K)$ can be given an algebra structure, which for $K = \{e\}$ is isomorphic to the Universal Enveloping algebra of $\mathfrak{g}$:

**Lemma 1.3.52** (Kirillov, §10.4)

Let $\mathcal{G}$ be a Lie group with Lie algebra $\mathfrak{g}$, and $\Phi(\mathcal{G})$ as per (1.3.51). Let $L_g : C^\infty(\mathcal{G}) \to C^\infty(\mathcal{G})$ be the maps,

$$L_g(\phi)(g_1) = \phi(g_1^{-1} g_1), \quad \forall g \in \mathcal{G}, \forall \phi \in C^\infty(\mathcal{G}).$$

Given $f \in \Phi(\mathcal{G})$ define the map $L^f : C^\infty(\mathcal{G}) \to C^\infty(\mathcal{G})$ by $L^f(\phi)(g) = f(L_g^{-1}(\phi))$.

Then the convolution of two elements $f_1$ and $f_2$ of $\Phi(\mathcal{G})$ is given by,

$$f_1 \ast f_2(\phi) = f_1(L^{f_2}(\phi)) \quad \text{and}$$

$\Phi(\mathcal{G})$ is an algebra over $\mathbb{R}$. $\Phi(\mathcal{G}, K)$ is a subalgebra of $\Phi(\mathcal{G})$. 68
Theorem 1.3.53 (Kirillov, 10.4.1)

Let $G$ be a Lie group with Lie algebra $g$. Then,

$$\Phi(G, \{e\})$$

is isomorphic to $U(g)$ given by (1.1.88).

Any continuous unitary representation $\mathcal{R} : G \to U(H)$ of a Lie group $G$ has a corresponding representation of $\Phi(G)$ in a dense subspace of $H$ called the Gårding space. The restriction of this representation to $\Phi(G, \{e\})$ gives a representation of $U(g)$, used to define the infinitesimal character of $\mathcal{R}$:

Definition 1.3.54 (Kirillov, §10.5)

Let $\mathcal{R} : G \to U(H)$ be a continuous unitary representation of $G$. The Gårding space $\mathcal{H}_\infty$, is the set of all $\xi \in H$ such that the map $H \to C$, $g \mapsto (\eta, \mathcal{R}(g)\xi)$ is $C^\infty$ for each $\eta \in H$.

Theorem 1.3.55 (Kirillov, 10.5.1)

For a continuous unitary representation $\mathcal{R} : G \to U(H)$, the Gårding space $\mathcal{H}_\infty$ is dense in $H$.

Proposition 1.3.56 (Kirillov, §10.5)

Let $\mathcal{R} : G \to U(H)$ be a continuous unitary representation of $G$. Let $r_{\eta, \xi}$ be the $C^\infty$ map $r_{\eta, \xi} : G \to C$, $g \mapsto (\eta, \mathcal{R}(g)\xi)$ where $\eta, \xi \in \mathcal{H}_\infty$ and $(\cdot, \cdot)$ is the inner product on $H$. The representation $\mathcal{R}$ generates a representation $\mathcal{R}_\infty : \Phi(G) \to \mathcal{L}(\mathcal{H}_\infty)$ by the formula,

$$(\eta, \mathcal{R}_\infty(f)\xi) = f(r_{\eta, \xi}), \quad \forall f \in \Phi(G).$$

$\mathcal{L}(\mathcal{H}_\infty)$ is the algebra of linear operators $\mathcal{H}_\infty \to \mathcal{H}_\infty$ defined on all of $\mathcal{H}_\infty$.

The restriction of $\mathcal{R}_\infty$ to $\Phi(G, \{e\})$ is a representation of $U(g)$ by (1.3.53).

The following theorem enables as to use $\mathcal{R}_\infty$ to define the infinitesimal character.
Theorem 1.3.57 (Kirillov, 11.3.1)

Let $\mathcal{R} : \mathcal{G} \to \mathcal{U}(\mathcal{H})$ be any irreducible unitary representation of a Lie group $\mathcal{G}$. The restriction of $\mathcal{R}^{\infty}$ (as per (1.3.56)) to the centre $Z(\mathcal{g})$ of $U(\mathcal{g}) \simeq \Phi(\mathcal{G}, \{e\})$ is the subalgebra of $\mathcal{L}(\mathcal{H}^{\infty})$ of scalar operators on $\mathcal{H}^{\infty}$.

Finally, we can define the infinitesimal character of an irreducible representation:

Definition 1.3.58 (Kirillov, §11.3)

Let $\mathcal{R} : \mathcal{G} \to \mathcal{U}(\mathcal{G})$ be an irreducible unitary representation of $\mathcal{G}$. By (1.3.57), there is a homomorphism $\lambda_{\mathcal{R}} : Z(\mathcal{g}) \to \mathbb{C}$ given by

$$\mathcal{R}^{\infty}(f) = \lambda_{\mathcal{R}}(f) \cdot 1, \quad \forall f \in Z(\mathcal{g})$$

where $1 : \mathcal{H}^{\infty} \to \mathcal{H}^{\infty}$ is the identity map. The map $\lambda_{\mathcal{R}}$ is called the infinitesimal character of the representation $\mathcal{R}$.

The importance of the infinitesimal character is underlined by the following lemma:

Lemma 1.3.59 (Kirillov, §11.3)

Let $\mathcal{R} : \mathcal{G} \to \mathcal{U}(\mathcal{H})$ be an irreducible unitary representation of $\mathcal{G}$ which has a generalised character $\chi_{\mathcal{R}}$ (see (1.3.43)). $\chi_{\mathcal{R}}$ has domain $C_{0}^{\infty}(\mathcal{G})$ (1.3.45). Let $\lambda_{\mathcal{R}}$ be the infinitesimal character of $\mathcal{R}$ as per (1.3.58). The convolution (1.3.52) is defined for functionals $C_{0}^{\infty}(\mathcal{G}) \to \mathbb{R}$ and,

$$z \ast \chi_{\mathcal{R}}(\phi) = \lambda_{\mathcal{R}}(z)\chi_{\mathcal{R}}(\phi), \quad (1.3.60)$$

$$\forall z \in Z(\mathcal{g}), \forall \phi \in C_{0}^{\infty}(\mathcal{G}).$$

Comments 1.3.61

For complex semisimple Lie groups, generalised characters of irreducible unitary representations have been found as solutions of equations (1.3.60) in the class
of linear functionals on $C_0^\infty(\mathcal{G})$ that are invariant under inner automorphisms of $\mathcal{G}$. This method leads to a complete classification of a wide class of so-called simple representations containing all unitary representations and representations in Banach spaces. A simple representation $\mathcal{R} : \mathcal{G} \to \mathcal{U}(\mathcal{H})$ is one for which $\mathcal{R}^\infty$ restricted to $Z(\mathfrak{g})$ has range contained in the scalar linear operators on $\mathcal{H}^\infty$.

We now construct a homomorphism $\lambda_\Omega : Z(\mathfrak{g}) \to \mathbb{C}$ corresponding to an orbit $\Omega$ associated to an irreducible representation $\mathcal{R}$ in the sense of (1.3.38). The form of $\lambda_\Omega$ has a direct geometric appeal and if $\mathcal{R}$ has generalised character $\chi_\mathcal{R}$ given by (1.3.46) then $\lambda_\Omega$ coincides with $\lambda_\mathcal{R}$, the infinitesimal character of $\mathcal{R}$. We will use the following theorem to construct $\lambda_\Omega$:

**Theorem 1.3.62** (Kirillov, §15.7; Duflo (1971))

$Z(\mathfrak{g})$ is algebra isomorphic to the algebra $\mathcal{P}_T^n(\mathfrak{g}^*)$ of polynomials on $\mathfrak{g}^*$ which are invariant under the co-adjoint action of $\mathcal{G}$.

**Proposition 1.3.63** (Kirillov, §15.7)

Let $\mathcal{R} : \mathcal{G} \to \mathcal{U}(\mathcal{H})$ be a unitary irreducible representation of $\mathcal{G}$ which has a generalised character $\chi_\mathcal{R}$ given by (1.3.46) and (1.3.50), and let $\Omega$ be the orbit associated to $\mathcal{R}$ in the sense of (1.3.38). Elements of $\mathcal{P}_T^n(\mathfrak{g}^*)$ are constant on orbits.

Let $\beta$ be any point on the orbit $\Omega$ and let us identify $Z(\mathfrak{g})$ with $\mathcal{P}_T^n(\mathfrak{g}^*)$ as per (1.3.62). Define the map $\lambda_\Omega$,

$$\lambda_\Omega : Z(\mathfrak{g}) \to \mathbb{C}$$

$$\lambda_\Omega : p \mapsto p(\beta), \ p \in \mathcal{P}_T^n(\mathfrak{g}^*).$$

Then $\lambda_\Omega$ coincides with $\lambda_\mathcal{R}$ as given by (1.3.58).

This geometric form of $\lambda_\mathcal{R}$ will be seen to lend itself easily to the contraction process in Chapter 5.
§1.4 Differential Geometry and Differentiable Fibre Bundles

In this section we will summarize a few, needed results from differential geometry as well as topics from the theory of fibre bundles in order to give the definitions of the \( q \)th product tangent bundle and \( q \)-jet bundles. Extensive use of the \( q \)th product tangent bundle will be made throughout Chapters 2, 3 and 4. \( q \)-jet bundles will find an important application in §3.2 on more general contraction definitions. The principal references for this section will be Helgason (1978) and Golubitsky and Guillemin (1974).

In order to define a derivative of curves on \( \mathcal{G} \) (called the covariant derivative) in an invariant way we need the concept of a connection:

**Definition 1.4.1** (Helgason, §1.4)

Let \( \mathcal{L}(\mathcal{M}) \) be the vector space of \( C^\infty \) vector fields on the manifold \( \mathcal{M} \) (see 1.1.19) and let \( \mathcal{L}(\mathcal{L}(\mathcal{M}), \mathcal{L}(\mathcal{M})) \) be the vector space of linear mappings of \( \mathcal{L}(\mathcal{M}) \) into itself. An affine connection \( \nabla \) is a linear map \( \nabla : \mathcal{L}(\mathcal{M}) \to \mathcal{L}(\mathcal{L}(\mathcal{M}), \mathcal{L}(\mathcal{M})) \), \( X \mapsto \nabla_X \) which satisfies:

(i) \( \nabla_{f_1 X + f_2 Y} = f_1 \nabla_X + f_2 \nabla_Y \), \( \forall X, Y \in \mathcal{L}(\mathcal{M}), \forall f_1, f_2 \in C^\infty(\mathcal{M}) \)

(ii) \( \nabla_X (f Y) = f \nabla_X (Y) + (X f) Y \), \( \forall X, Y \in \mathcal{L}(\mathcal{M}), \forall f \in C^\infty(\mathcal{M}) \)

(see 1.1.22).

**Notation 1.4.2** (Helgason, §1.4)

Let \( (\chi, \mathcal{N}) \) be a chart at a point \( m \) of \( \mathcal{M} \), with coordinates \( x^i \). The \( C^\infty \) functions \( \Gamma^k_{ij} \) are defined by

\[
\nabla_{\frac{\partial}{\partial x^i}} \left( \frac{\partial}{\partial x^j} \right) = \Gamma^k_{ij}(x) \frac{\partial}{\partial x^k} \quad \text{(see (1.1.16)).}
\]

We will use the connection to define the covariant derivative, relating this derivative to conventional differentiation with respect to the coordinates \( x^i \), which will involve the \( \Gamma^k_{ij} \).
Definition 1.4.3 (Helgason, §1.7)

Let $X \in \mathcal{L}(\mathcal{M})$ and let $\varphi$ be a $C^\infty$ map of an open interval $I$ of $\mathbb{R}$ into $\mathcal{M}$. The curve $t \rightarrow \varphi(t)$ is called an integral curve of $X$ when

$$X_{\varphi(t)} = d\varphi|_t \left( \frac{d}{dt} \right)$$

$(d\varphi|_t \left( \frac{d}{dt} \right))$ is the tangent vector to the curve $t \mapsto \varphi(t)$. See (1.1.15)–(1.1.17).

Lemma 1.4.4 (Helgason, 1.7.1)

Let $\nabla$ be an affine connection on a $C^\infty$ manifold $\mathcal{M}$ and let $X, Y \in \mathcal{L}(\mathcal{M})$. Let $t \mapsto \varphi(t)$, $I \rightarrow \mathcal{M}$ be an integral curve of $X$ as in (1.4.3). With respect to the chart (1.4.2) at $\varphi(t)$, let $Y$ be denoted $Y_{\chi^{-1}(x^i)} = Y^k(x) \frac{\partial}{\partial x^k}$, in the notation of (1.1.20). Abbreviate $Y^k(\chi \circ \varphi(t))$ to $Y^k(t)$, $(\chi \circ \varphi)^i(t)$ to $x^i(t)$ and $\Gamma^k_{ij}(\chi \circ \varphi(t))$ to $\Gamma^k_{ij}(t)$. Then,

$$\left( \nabla_X(Y) \right)_{\varphi(t)} = \left( \frac{dY^k}{dt}(t) + \Gamma^k_{ij}(t) \frac{dx^i}{dt}(t) Y^j(t) \right) \frac{\partial}{\partial x^k} \big|_{\varphi(t)}$$

and $\left( \nabla_X(Y) \right)_{\varphi(t)}$ is called the covariant derivative of $Y$ with respect to $X$.

Every Lie group admits a Riemannian connection and in a manifold with a Riemannian connection, it is always possible to choose a chart so that the covariant derivative appears the same as the usual derivative:

Lemma 1.4.5 (Bishop and Crittenden, 1964, 7.1.1, Theorem 2; Gaal, 1973, 8.5.3)

Every Lie group $\mathcal{G}$ is a Riemannian manifold. That is, $\forall g \in \mathcal{G}$ there is a positive definite symmetric bilinear form $T_g \times T_g \rightarrow \mathbb{R}$, $(X, Y) \mapsto \langle X, Y \rangle$ which is $C^\infty$ in the sense that for any chart $(\chi, \mathcal{N})$ at $g$ with coordinates $x^i$, the functions $g_{ij}(x) = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle$ are $C^\infty$ functions of $x^i$ on $\chi(\mathcal{N})$. 

73
Every Riemannian manifold has a metric space structure. The metric space function is called the **Riemannian metric**.

**Lemma 1.4.5(a)** (Bishop and Crittenden, 7.1.1, Lemma 1)

Let $\mathcal{M}$ be a $C^\infty$ manifold. Let $\varphi : [0, 1] \to \mathcal{M}$ be $C^\infty$ and define the arc length of $\varphi$ to be $|\varphi| = \int_0^1 (X_{\varphi(t)}X_{\varphi(t)})dt$ where $X_{\varphi(t)}$ is the tangent vector field to $\varphi[0, 1]$. The definition is the same if $\varphi$ is continuous and piecewise $C^\infty$ (called **broken-$C^\infty$**). Let $m_1, m_2 \in \mathcal{M}$, and let

$$\Gamma(m_1, m_2) = \{ \varphi \mid \varphi \text{ is broken-$C^\infty$}, \varphi(0) = m_1, \varphi(1) = m_2 \},$$

then the map, $\mu : \mathcal{M} \times \mathcal{M} \to \mathbb{R} \cup \{ +\infty \}$,

$$\mu(m_1, m_2) = \inf_{\varphi \in \Gamma(m_1, m_2)} |\varphi|$$

is a metric on $\mathcal{M}$.

**Lemma 1.4.6** (Bishop and Crittenden, 1964, §7.3, §6.4.4)

At any point $m$ of a Riemannian manifold $\mathcal{M}$, it is always possible to choose coordinates $(N, \chi)$ called **normal coordinates** such that the covariant derivative of $Y$ with respect to $X$ at $\varphi(t)$ in the notation of (1.4.4) is,

$$\left( \nabla_X(Y) \right)_{\varphi(t)} = \left( \frac{dY^k}{dt} \right)(t) \left( \frac{\partial}{\partial x^k} \right)_{\varphi(t)}.$$

In particular, canonical coordinates at $e$ in a Lie group $\mathcal{G}$ are normal coordinates.

**Corollary 1.4.7**

Let $\varphi : [0, 1] \to \mathcal{G}$ be a $C^\infty$ curve, and let $X_{\varphi(t)}$ be the tangent vector field to $\varphi[0, 1]$. Then the $q'$th covariant derivative of $X$ with respect to $X$ in normal coordinates is

$$\left( \left( \nabla_X \right)^q(X) \right)_{\varphi(t)} = \left( \frac{d^qX^k}{dt^q} \right)(t) \left( \frac{\partial}{\partial x^k} \right)_{\varphi(t)}$$

74
in the notation of (1.4.4). We denote \( X_{\varphi(t)} \) as \( \nabla \varphi(t) \) and \( (\nabla X)^{g^{-1}}(X)_{\varphi(t)} \) as \( \nabla^g \varphi(t) \). If we wish to write the \((q + 1)\)-tuple of the point \( \varphi(t) \), together with the vectors \( \nabla \varphi(t), \ldots, \nabla^g \varphi(t) \), we shall often put this as \((\nabla^0, \nabla, \ldots, \nabla^g) \varphi(t) \).

Analogously to canonical coordinates at \( e \) in a Lie group \( G \), there are also normal coordinates defined by the exponential mapping at each point \( m \) of a manifold \( \mathcal{M} \). We shall define the exponential mapping and give its properties in the next group of definitions and lemmas:

We will need the concept of geodesic to define the exponential map:

**Definition 1.4.8** (Helgason, §1.5)

Let \( I \subset \mathbb{R} \) be an open interval containing 0. Let \( \varphi : I \to \mathcal{M} \) be a \( C^\infty \) map into the \( C^\infty \) manifold \( \mathcal{M} \). \( \varphi \) is called a **geodesic** if

\[
\nabla \nabla \varphi(t) (\nabla \varphi(t))_{\varphi(t)} = 0.
\]

**Lemma 1.4.8(a)** (Bishop and Crittenden, §8.1, Theorem 2)

Let \( \mathcal{M} \) be a \( C^\infty \) manifold and \( m_1, m_2 \in \mathcal{M} \). Then the geodesic joining \( m_1 \) to \( m_2 \) minimises the arc length over all broken-\( C^\infty \) curves joining \( m_1 \) to \( m_2 \).

**Lemma 1.4.9** (Helgason, 1.5.3, §1.6)

Let \( \mathcal{M} \) be a \( C^\infty \) manifold with affine connection. Let \( m \in \mathcal{M} \) and let \( X \in T_m, X \neq 0 \).

There exists a unique geodesic \( \varphi_X : I \to \mathcal{M} \) for some \( I \) as in (1.4.8) such that,

\[
\varphi_X(0) = m, \quad \dot{\varphi}_X(0) = X.
\]

We define the **exponential map** at \( \mathcal{M} \), \( \exp_m : T_m \to \mathcal{M} \) by, \( \exp_m : X \mapsto \varphi_X(1) \).
Proposition 1.4.10 (Helgason, 1.6.1)

Let \( \mathcal{M} \) be a \( C^\infty \) manifold with affine connection. Let \( m \in \mathcal{M} \). There exists an open neighbourhood \( \mathcal{N}_0 \) of 0 in \( T_m \) and an open neighbourhood \( \mathcal{N}_m \) of \( m \) in \( \mathcal{M} \) such that \( \exp_m : \mathcal{N}_0 \to \mathcal{N}_m \) is a diffeomorphism.

Lemma 1.4.11 (Helgason, §1.6)

In the notation of (1.4.10) let \( \chi_m = \exp_m \) so that \( (\chi_m, \mathcal{N}_m) \) is a chart at a point \( m \) of a Riemannian manifold \( \mathcal{M} \). Then \( (\chi_m, \mathcal{N}_m) \) is a normal coordinate chart at \( m \) in the sense of (1.4.6).

The following lemma will be useful in Chapter 2:

Lemma 1.4.11(a) (Helgason, §2.1)

Let \( G \) be a Lie group. The exponential mapping for Lie groups and \( \exp_e \) as given by (1.4.10) coincide. Further, for any \( g \in G \) and \( X \in T_e \),

\[
\exp_g (d\ell_g X) = g \exp(X) \quad \text{(see (1.1.25))}.
\]

We now introduce the concept of vector bundles giving as examples the tangent bundle and the \( q'\)th product tangent bundle.

Definition 1.4.12 (Golubitsky, §1.5)

A \( C^\infty \) vector bundle is a pair of \( C^\infty \) manifolds \( E \) and \( \mathcal{M} \) together with a \( C^\infty \) surjection \( \pi : E \to \mathcal{M} \) which satisfy:

(a) \( \pi^{-1}(m) \), denoted \( E_m \), \( \forall m \in \mathcal{M} \) is a vector space with addition and scalar multiplication continuous with respect to the topology on \( E_m \) induced from \( E \).

(b) \( \forall m \in \mathcal{M} \), there is a neighbourhood \( \mathcal{N}_m \) of \( m \) in \( \mathcal{M} \) such that \( \pi^{-1}(\mathcal{N}_m) \) is diffeomorphic to \( \mathcal{N}_m \times \mathbb{R}^n \) for some positive integer \( n \).
Definition 1.4.13 (Golubitsky, 1.5.13)

Let $\pi_1 : E^1 \to M^1$ and $\pi_2 : E^2 \to M^2$ be $C^\infty$ vector bundles. A diffeomorphism $\phi : E^1 \to E^2$ is a vector bundle isomorphism if:

(a) There exists a diffeomorphism $\psi : M^1 \to M^2$ such that $\psi \circ \pi_1 = \pi_2 \circ \phi$

(b) $\forall m \in M_1, \phi : E^1_m \to E^2_{\psi(m)}$ is linear.

Remark 1.4.14

The tangent bundle (1.1.18) to a $C^\infty$ manifold $M$ is an example of a $C^\infty$ vector bundle.

Hereewith the definition of the $q$th product tangent bundle.

Lemma 1.4.15 (Golubitsky, §1.5)

Let $G$ be a Lie group and let $T^q(G)$ be the set,

$$T^q(G) = \bigcup_{g \in G} \left( g, \prod_{i=1}^q T_g \right),$$

with map $\pi : T^q(G) \to G$,

$$\pi : (g, X_1, \ldots, X_q) \mapsto g.$$

Let $(\chi, \mathcal{N})$ be a chart at $g \in G$ with coordinates $x^i$. By analogy to (1.1.18), define the maps,

$$C_g : \pi^{-1}(\mathcal{N}) \to \mathcal{N} \times \mathbb{R}^n$$

by

$$C_g : (\chi^{-1}(x^i), X_1^{i_1} \frac{\partial}{\partial x^{i_1}}, \ldots, X_q^{i_q} \frac{\partial}{\partial x^{i_q}}) \mapsto (\chi^{-1}(x^i), X_1^{i_1}, \ldots, X_q^{i_q}).$$

With topology generated by the maps $C_g$, $\pi : T^q(G) \to G$ is a $C^\infty$ vector bundle.

Jet bundles, which will be used in §2.8 are not in general vector bundles but rather are examples of fibre bundles:
Definition 1.4.16 (Golubitsky, 2.2.8)

Let $E, M$ and $F$ be $C^\infty$ manifolds and $\pi : E \to M$ a $C^\infty$ surjection. $E$ is a fibre bundle over $M$ with fibre $F$ and projection $\pi$ if

(i) for each $m \in M$ there is a neighbourhood $N_m$ of $m$ such that $\pi^{-1}(N_m)$ is diffeomorphic to $N_m \times F$

(ii) Let $\pi_m : N_m \times F \to N_m$ be the canonical projection and let $\phi_m : \pi^{-1}(N_m) \to N_m \times F$ be the diffeomorphism of (i), then

$$\pi = \pi_m \circ \phi_m.$$ 

Jet bundles are derived from the set of $C^\infty$ maps between two manifolds $M_1$ and $M_2$. Firstly we need an equivalence relation for $C^\infty$ maps from $M_1$ to $M_2$:

Lemma 1.4.17 (Golubitsky, §2.2)

Let $M_1, M_2$ be $C^\infty$ manifolds, let $f, g : M_1 \to M_2$ be $C^\infty$ maps with $f(m_1) = g(m_1) = m_2$ and let $q$ be a non-negative integer. The relation defined by: $(f \sim_{q,m_1} g) \iff (d^k f(m_1) = d^k g(m_1), k = 0, 1, \ldots, q)$ is an equivalence relation (see (1.2.3)).

Proposition 1.4.18 (Golubitsky, §2.2)

Let $M_1, M_2$ be $C^\infty$ manifolds of dimensions $n_1$ and $n_2$ respectively and let $J^q(M_1, M_2)_{m_1,m_2}$ denote the set of equivalence classes by $\sim_{q,m_1}$ as per (1.4.17) of $C^\infty$ maps $f : M_1 \to M_2$ with $f(m_1) = m_2$. Let

$$J^q(M_1, M_2) = \bigcup_{(m_1,m_2) \in M_1 \times M_2} J^q(M_1, M_2)_{m_1,m_2} \text{ (disjoint).}$$

Let $V^q_{n_1}$ be the vector space of polynomials in $n_1$ variables of degree at most $q$ and let $V^q_{n_1,n_2} = \bigoplus_{i=1}^{n_2} V^q_{n_1}$. Then $J^q(M_1, M_2)$, called the $q$-jet bundle, is a $C^\infty$ fibre bundle as per (1.4.16) with fibre the Euclidean space $V^q_{n_1,n_2}$ and projection map given as follows: For each $\rho \in J^q(M_1, M_2)$ $\exists m_1, m_2$ for which $\rho \in J^q(M_1, M_2)_{m_1,m_2}$; the map $\pi : J^q(M_1, M_2) \to M_1 \times M_2$ given by $\pi : \rho \to (m_1,m_2)$ is a $C^\infty$ surjection. The differentiable structure is exhibited in
the following way: Let $N_1 \times N_2$ be a neighbourhood of $(m_1, m_2)$ in $M_1 \times M_2$, $(x_1, N_1)$ and $(x_2, N_2)$ the respective charts and let $\rho \in \pi^{-1}(N_1 \times N_2)$ with $f: M_1 \to M_2$ a representative of the equivalence class $\rho$. Put $f_{12} = x_2 \circ f \circ x_1^{-1}$ and let $P_q f_{12}$ be the polynomial of degree $q$ given by the first $q$ terms of the Taylor series (that is, all the homogeneous polynomials $P^i(x_1)$ in the variable $x_1$ of the chart $(x_1, N_1)$ which by definition satisfy

$$P^i(\lambda x_1) = (\lambda)^j P^i(x_1), \quad j = 1, \ldots, q,$$

of which there are $q$ in number) of the coordinate function $f^i$ of $f$ at $x_i(m_1)$ after the constant term. The diffeomorphism $\phi_{(m_1, m_2)}$ of (1.4.16) is given by,

$$\phi_{(m_1, m_2)}: \pi^{-1}(N_1 \times N_2) \to N_1 \times N_2 \times V^q_{n_1, n_2},$$

$$\phi_{(m_1, m_2)}: \rho \mapsto (\pi(\rho), P_q f_{12}^{1}, \ldots, P_q f_{12}^{n_2})$$

**Comments 1.4.19**

Observe in the notation of (1.4.18), that an element of $J^q(M_1, M_2)(m_1, m_2)$ is uniquely determined by the polynomial which is all the terms to order $q$ of the Taylor series of $x_2 \circ f \circ x_1^{-1}$, at $x_i(m_1)$ for some $C^\infty$ map $f: M_1 \to M_2$. A $q$-jet bundle is just an invariant way of describing this.

Associated to any $C^\infty$ map $f: M_1 \to M_2$ is the $q$-jet of $f$:

**Lemma 1.4.20** (Golubitsky, 2.2.1)

Define the map $j^q: C^\infty(M_1, M_2) \to C^\infty(M_1, J^q(M_1, M_2))$

$$j^q f(m_1) = \{\text{equivalence class of } f \text{ in } J^q(M_1, M_2)_{m_1, f(m_1)}\}.$$

The map $j^q f: M_1 \to J^q(M_1, M_2)$ is $C^\infty$ and is called the $q$-jet of $f$.
§1.5 The Contractions of Lie Algebras and Lie Groups

The Contractions of Representations of Lie Groups

In this section, we will summarise the basic ideas and results of contractions of Lie algebras and Lie groups, and contractions of representations for a compact, symmetric pair. The main references will be Saletan (1961), Dooley and Ricci (1985) and Dooley and Gaudry (1986).

Proposition 1.5.1 (Saletan, §A.2)

Let \( g \) be a Lie algebra with underlying vector space \( V \) and let \( U : [0,1] \times g \to g \) be a \( C^\infty \) map linear in the second argument with \( U(t) \) invertible for \( t \in (0,1] \) and singular for \( t = 0 \). If the limits,

\[
\lim_{t \to 0} U^{-1}(t)[U(t)X, U(t)Y], \quad \forall X, Y \in g,
\]

then a new Lie algebra \( g_0 \) is defined, given by \( g_0 = V \), with bracket,

\[
[X, Y]_0 = \lim_{t \to 0} U^{-1}(t)[U(t)X, U(t)Y], \quad \forall X, Y \in V.
\]

\( g_0 \) is called the \textbf{contraction of} \( g \) \textit{by} \( U \) or the \textbf{contracted Lie algebra}. This condition is necessary and sufficient for \( g_0 \) to be a Lie algebra.

Remark 1.5.2

In order to obtain specific results, Saletan considered the special case of \( U(t) = u + tI \) where \( I \) is the identity map and \( u : g \to g \) linear. This restriction on \( U(t) \) is minor:

Lemma 1.5.3 (Lévy-Nahas (1966), §B)

Let \( W : [0,1] \times g \to g \) be any \( C^\infty \) map linear in the second argument, such that \( W(t) \) is invertible \( \forall t \in [0,1] \). The contraction of \( g \) by \( U_1 \), \( U_1(t) = u + tI \), is isomorphic to the contraction of \( g \) by \( U_2 \), \( U_2(t) = u + tW(t) \).
**Notation 1.5.4**

Let $q$ be the smallest integer for which $u^q+1 g = u^q g$. Letting $V$ be the underlying vector space of $V$, put $V_1 = u^q V$ and $V_2 = \ker u^q$. Then $V = V_1 \oplus V_2$. Denote the restriction of the bracket of $g$ to $V_i$ by $[\cdot, \cdot]_i$.

With the above restriction in place, Saletan obtained the following result:

**Theorem 1.5.5** (Saletan, §B.5)

Let $g$ be a Lie algebra with underlying vector space $V$, and $U : [0, 1] \times g \to g$ of the form $U(t) = u + tI$. In the notation of (1.5.4) and (1.5.1) the contraction $g_0$ of $g$ by $U$ is defined and is a Lie algebra if and only if,

$$u^2[X,Y]_2 - u[uX,Y]_2 - u[X,uY]_2 + [uX,uY]_2 = 0, \quad \forall X,Y \in g.$$

The bracket on $g_0$ is given by,

$$[X,Y]_0 = u^{-1}[uX,uY]_1 - u[X,Y]_2 + [uX,Y].$$

Let $g_0^{(1)} = g_0$, and $g_0^{(i+1)}$ denote the contraction of $g_0^{(i)}$ by $U$. Then $g_0^{(i+j)}$ is the contraction of $g_0^{(i)}$ by $W : [0, 1] \times g \to g$, $W : t \mapsto u^j + tI$. $g_0^{(q)}$ is isomorphic to $g_0^{(q+k)}$ $k = 0, 1, 2, \ldots$ $u^j V$ is a subalgebra of $g$ and $g_0^{(i)}$, $i,j = 1, \ldots, q$ and $\ker u^j$ is a solvable ideal of $g_0^{(i+j)}$, $j = 1, \ldots, q$, $i = 0, \ldots, q$; $j + i \leq q$. No algebra $g_0^{(i)}$ is semi-simple. In this case, $g_0$ is called the **Saletan contraction of $g$**.

Special cases of Saletan contraction corresponding to particular choices of $u$ are Inönü-Wigner contraction (hereinafter called $I-W$ contraction) and generalised $I-W$ contraction which permit a more group theoretic statement of Theorem (1.5.5):
Lemma 1.5.6 (Kupczynski, 1969)

Let $U : [0, 1] \times g \to g$ be of the form $U = u_0 + u_1 t + \cdots + u_q t^q$
where $u_i : g \to g$ are linear maps, which satisfy

$$\text{Im} u_i \cap \text{Im} u_j = \{0\}, \quad i \neq j$$

$$\bigoplus_{i=0}^{q} \text{Im} u_i = g, \quad \text{Im} u_i + \ker u_i = g.$$  

Then the contraction $g_0$, of $g$ by $U$ is defined and is a Lie algebra if and only if

$$[\text{Im} u_i, \text{Im} u_j] \subseteq \sum_{j=0}^{i+j} \text{Im} u_j, \quad \forall i, j \geq 0 \quad \text{such that } (i + j) \leq q.$$  

$g_0$ is the generalised I–W contraction of $g$. For $q = 1$, $g_0$ is defined and
is a Lie algebra if and only if $\text{Im} u_0$ is a subalgebra of $g$, and is called the I–W
contraction of $g$.

Corollary 1.5.7 (Kupczynski, 1969)

In the notation of (1.5.6), for the contraction of $g$ with $q = 1$, $g_0$ is defined
and is a Lie algebra if and only if $\text{Im} u_0$ is a subalgebra of $g$, and is called the I–W
contraction of $g$. Observe that $\text{Im} u_0$ is a subalgebra of $g$ also for generalised $I$–$W$ contraction ($q \geq 1$).

A simple, global version of $I$–$W$ contraction is due to Dooley (1983):

Proposition 1.5.8 (Dooley, 1983)

Let $G$ be any Lie group, reductive with respect to a Lie subgroup $G_1$.
That is, $g = g_1 + V_1$ and $\text{ad}(G_1)V_1 \subseteq V_1$. Let $V$ be the underlying vector space of $g$. Then the inner product $G_1 \ltimes V_1$ (see 1.1.71) with respect to the adjoint action
of $G_1$ on $V_1$ has Lie algebra which is the $I$–$W$ contraction of $g$ by the derivative
$U(t) = d\phi(t) : V \to V$ of the $C^\infty$ maps

$$\phi(t) : G_1 \ltimes V_1 \to \mathcal{G}, \quad \phi(t) : (g_1, v) \mapsto (\exp_{G} tv)g_1.$$
A global version of Saletan contraction for the special case of contraction inside a connected semisimple Lie group of real rank one has been given by Dooley and Ricci (1985).

We need the following proposition before defining a family of maps $\phi(t)$ analogously to (1.5.8).

**Proposition 1.5.9** (Dooley and Ricci (1985), §2)

Let $G$ be a connected semisimple Lie group of real rank one (see 1.1.75(c)) and let $G = KAN$ be an Iwasawa decomposition of $G$. For each $g \in G$, let $k(g)$ denote the $K$ component of $g$ in the Iwasawa decomposition. Let $\overline{N} = \theta(N)$ where $\theta$ is a Cartan involution (1.1.67(a)) of $G$, and $M$ the centralizer of $A$ in $K$. The map $k : \overline{N}M \rightarrow K$ is a $C^\infty$ diffeomorphism of $\overline{N}M$ onto a dense submanifold of $K$.

**Proposition 1.5.10** (Dooley and Ricci (1985), §2.5)

Let $G$ be a connected, semisimple Lie group of real rank one and let $H$ be the unique element of $g$ as per (1.1.75(h)) with respect to the Cartan decomposition $g = k + p$ where $k$ is the Lie algebra of $K$ in the Iwasawa decomposition above. The contraction maps $\phi(t) : \overline{N}M \rightarrow K$ are defined for $t > 0$ as

$$
\phi(t)(\overline{m}) = k(\exp(tH)\overline{m}(\exp - tH))
$$

where the map $k$ is given by (1.5.9), and is a $C^\infty$ diffeomorphism on to a dense submanifold of $K$. The derivative $d\phi(t)$ of $\phi(t)$ is,

$$
d\phi(t) : \overline{k} + m \rightarrow k,
$$

$$
d\phi(t) : X \rightarrow \text{Ad} (\exp tH)X + \theta (\text{Ad} (\exp tH)X)
$$

where $\theta$ is the Cartan involution (1.1.75(b)) with fixed point set $k$ and,

$$
\forall X, Y \in \overline{k} + m, 
\lim_{t \to +\infty} d\phi^{-1}(t)[d\phi(t)X, d\phi(t)Y] = [X, Y].
$$
The last formula of Proposition (1.5.10) illustrates, on comparison with (1.5.1) and (1.5.8), that the \( d\phi(t) \) define a contraction at the Lie algebra level. A similar result holds at the global level:

**Proposition 1.5.11** (Dooley and Ricci (1985), §2.7)

Let \( G \) be a connected semisimple Lie group of real rank 1. Let the notation and the maps \( \phi(t) : N M \to K \) be given by (1.5.10). Then for all \( n_1, n_2 \in N \), and \( m_1, m_2 \in M \),

\[
\lim_{t \to +\infty} \phi_t^{-1}(\phi(t)(n_1 m_1)\phi(t)(n_2 m_2)) = n_1 m_1 n_2 m_2.
\]

**The contractions of Representations of Lie Groups**

We now give the main ideas and results for the contractions of representations of Lie groups in the case covered by Dooley and Gaudry (1986), as mentioned earlier.

Using Dooley's (1983) contraction process of (1.5.8) for a compact symmetric pair \((G, K)\) (see (1.1.67(a))), Dooley and Gaudry (1986) show how any element in the class of generic irreducible representations of the contraction (1.5.8) of \( G \) with respect to \( K \) is the limit of a family of representations of \( G \).

In order to relate the representations of \( G \) and \( G_0 \) we need the representation theory of \( G \). A class of representations for \( G \) or for \( G_0 \) is determined by a class of finite-dimensional representations of a subgroup of \( G_0 \):

**Lemma 1.5.12** (Dooley and Gaudry, §2.1)

Let \((G, K)\) be a compact symmetric pair (1.1.67(a)) and \( K \ltimes V \) the contraction (1.5.8) of \( G \), with Lie algebra \( k + V \). Let \( \mathfrak{a} \) be a maximal abelian subalgebra of \( V \), \( A = \exp \mathfrak{a} \), and let \( M \) be the centralizer of \( A \), in \( K \). An irreducible representation of \( MA \) has the form,

\[
\eta \otimes e^{i\phi} : m \exp X \to e^{i\phi(X)} \eta(m)
\]
where $X \in \mathfrak{g}$, $\phi \in \mathfrak{g}^*$ and $\eta \in \hat{M}$ (the irreducible unitary representations of $M$).

Further for $m = \exp Y \in M \cap A$, $e^{i\phi(Y)}\eta(m)$ is the identity map on the Hilbert space $\mathcal{H}_\eta$ of the representation $\eta$ of $M$.

$\mathcal{H}_\eta$ is a finite-dimensional subspace of $C^\infty(M, \mathbb{C})$, the $C^\infty$-functions from $M$ to $\mathbb{C}$.

The representation theory for $G$ is precisely that given by the method of orbits outlined in §1.3. A more specific form is made possible by the structure theory of compact Lie groups and holomorphic induction (which is equivalent to the method of orbits (1.3.38)).

**Proposition 1.5.13** (Dooley and Gaudry, §2.1)

Let $(G, K)$ be a compact symmetric pair. Let $\mathfrak{u}_+$ be the subalgebra of $\mathfrak{g}' \otimes \mathbb{C}$ of (1.1.83) given by,

$$\mathfrak{u}_+ = \sum_{\alpha \in P_+} \mathfrak{g}'_\alpha$$

where $\mathfrak{g}'_\alpha$ is specified in (1.1.84) and $P_+$ in (1.1.85).

Let $X \in \mathfrak{g} \otimes \mathbb{C}$ act as a left-invariant vector field in the sense of (1.1.22). In the notation of (1.5.12), define the map $\nabla_X : C^\infty(G, \mathcal{H}_\eta) \to C^\infty(G, \mathcal{H}_\eta)$ by

$$\nabla_X F(g)(m) = [\text{Ad}(m)X] \cdot F(g)(m),$$

for $F : G \to \mathcal{H}_\eta$, $\forall g \in G$, $\forall m \in M_0$, the connected component of $M$.

Define the subspace $C^\infty(G, MA, \eta \otimes e^{i\phi}; \mathfrak{u}_+)$ (which is also given by (1.3.12)) of $C^\infty(G, \mathcal{H}_\eta)$, to be

$$C^\infty(G, MA, \eta \otimes e^{i\phi}; \mathfrak{u}_+) = \{ F | F \text{ is } C^\infty, F : G \to \mathcal{H}_\eta, \eta \otimes e^{i\phi}(ma)F(ma) = F(g), \forall g \in G, m \in M_0, a \in A \}$$

$$\nabla_X F = 0, \forall X \in \mathfrak{u}_+ \}$$

85
For each representation $\eta \otimes e^{i\phi}$ (1.5.12) of $MA$, there is an irreducible representation $\sigma_{\phi,\eta}$ of $G$:

$$
\sigma_{\phi,\eta} : G \to \mathcal{L}(C^\infty(G, MA, \eta \otimes e^{i\phi}; \mathbb{D}_+))
$$

$$
\sigma_{\phi,\eta}(g) : F \mapsto F_g
$$

where $F_g(h) = F(g^{-1}h) \forall g, h \in G$.

We now give the generic irreducible unitary representations of $K \ltimes V$. These representations are given by (1.3.37).

**Proposition 1.5.14** (Dooley and Gaudry, §2.1)

Let $(G, K)$ be a compact symmetric pair, and $K \ltimes V$ the contraction (1.5.8) of $G$. In the notation of (1.5.12), let $\eta$ be an irreducible unitary representation of $M$ on the finite-dimensional Hilbert space $\mathcal{H}_\eta$, and let $\phi \in (\mathfrak{g}^+)^*$ in the spirit of (1.1.86). Let the Hilbert space $L^2(K \ltimes V, M, \eta)$ be given by (1.3.31) (with left cosets replaced uniformly by right cosets). Then to each representation $\eta \otimes e^{i\phi}$ (1.5.12) of $MA$, there is a unitary irreducible representation $\rho_{\phi,\eta}$ of $K \ltimes V$:

$$
\rho_{\phi,\eta} : K \ltimes V \to \mathcal{U}(L^2(K \ltimes V, M, \eta))
$$

$$
\rho_{\phi,\eta}(k, v)f(k_0) = e^{i\phi(Ad(k_0^{-1})v)}f(k^{-1}k_0),
$$

$\forall f \in L^2(K \ltimes V, M, \eta), \ v \in V, \ k, k_0 \in K$.

In order to state a theorem which shows how a given representation of $G_0 = K \ltimes V$ is approximated by a family of representations of $G$, we need a Lemma relating the space $C^\infty(G, MA, \eta \otimes e^{i\phi}; \mathbb{D}_+)$ to the Hilbert space $L^2(K \ltimes V, M, \eta)$.

**Lemma 1.5.15** (Dooley and Gaudry, §2.2)

Let $(G, K)$ be a compact symmetric pair, and $K \ltimes V$ the contraction (1.5.8) of $G$. Let $\mathcal{H}_{\phi,\eta} = C^\infty(G, MA, \eta \otimes e^{i\phi}; \mathbb{D}_+)$, and $\mathcal{H}^\eta = L^2(K \ltimes V, M, \eta)$. The mapping

$$
\mathcal{R}_{\phi,\eta} : \mathcal{H}_{\phi,\eta} \to \mathcal{H}^\eta \text{ is an injection.}
$$
Further, if $\phi$ is any $K$-class 1-weight of $G$, and $s$ is a $K$-fixed vector in $\mathcal{H}_{\phi,1}$, with $s(k) = 1, \forall k \in K$ (by (1.6.2) $s$ is complex-valued), then

$$\mathcal{R}_{\psi + \phi, \eta} s \cdot f = \mathcal{R}_{\psi, \eta} f.$$ 

The approximation theorem may now be neatly stated with the following notation:

**Notation 1.5.16**

Fix an inner product $(X, Y) = -B_\theta(X, Y)$ on $g$ where $B_\theta$ is the killing form (1.1.75(a)) and identify $g^*$ with $g$, with respect to this inner product. Let $\|X\|$ be the corresponding norm.

**Theorem 1.5.17** (Dooley and Gaudry, §2.2)

Let $(G, K)$ be a compact symmetric pair and $G_0 = K \ltimes V$ the contraction (1.5.8) of $G$. With $\eta \otimes e^{i\phi}$ given by (1.5.12), let $\rho_{\phi, \eta}$ be a corresponding representation of $K \ltimes V$. For a family of representations $\sigma(\psi_0 + \gamma, \eta)$ of $G$ given by (1.5.13) with $\gamma$ being any $K$-class 1-weight of $G$; for any $f \in \mathcal{R}_{\psi_0, \eta} \mathcal{H}_{\psi_0, \eta}$ and any $M_1, M_2, M_3 > 0$; there exists a constant $c$ determined solely by $f$ and the $M_i$, such that whenever $\phi \in (g^+)^*$ (see (1.1.86)), and

$$\|v\| \leq M_1, \|(\psi_0 + \gamma)/\lambda\| \leq M_2, \|\psi_0 + \gamma - \lambda \phi\| \leq M_3$$

we have

$$\|\rho_{\phi, \eta}(k, v)f - \mathcal{R}_{\psi_0 + \gamma, \eta} \sigma_{\psi_0 + \gamma, \eta}(\phi(t)(k, v))\mathcal{R}_{\psi_0 + \gamma, \eta}^{-1} f\|_{\mathcal{H}_\eta} \leq \frac{c}{\lambda}$$

where $\|\cdot\|_{\mathcal{H}_\eta}$ is the norm on $\mathcal{H}_\eta$ of (1.5.15) and $\phi(t)$ is given by (1.5.8).

For the contraction theory of (1.5.9) to (1.5.11) inside a connected semisimple Lie group of real rank one, there is a corresponding theorem for the approximation of a representation of $\overline{N}M$ by a family of representations of $K$ analogous to (1.5.17). See Dooley and Ricci (1985).
Chapter 2: Contractions of Lie Groups and Lie algebras

§2.1 Motivation for this Work

The two motivations for this study are physical and mathematical. The first purpose was to take the initial steps in answering the question of Elementary Particle physics: "Why are quark masses different?" It had been postulated (see Hegerfeldt and Henning, 1968) that an explanation of this fact would be obtained by combining the Poincaré and internal symmetry group $SU(n)$ as subgroups of a larger group whose irreducible representations would contain one spanned by the quark wave-functions each with different eigenvalues of the mass-squared operator, as opposed to the same eigenvalues of the mass-squared operator for all representations of $SU(n)$.

Theorems soon appeared showing that in an irreducible representation of the larger group all particles will have the same mass (see Barut and Rączka, 1968). These were rigorous theorems which made the assumption that the space-time symmetry group was the Poincaré group, taking no account of cosmological effects. However, cosmological effects which might appear to have a vanishingly small effect on interactions at high energies can determine the existence or non-existence of certain particles (Gürsey, 1964). Further, the idea of combining dynamic and internal symmetries to explain quark masses should not be totally abandoned in view of the partial success of Greenberg (1964) who showed that an abelian subgroup can be added to $SU(3)$ to give a reasonable mass formula for a baryon octet. Hence the main thrust should have been to combine the cosmological symmetry group with $SU(n)$.

In the early evolution of the Universe, (if we are to assume the "Big Bang" model), very near to its time of creation, the density of matter would have been so great that the gravitational force would have been comparable with the strong force, and cosmological effects at this time could then have determined the exis-
tence of particles of significantly large rest-mass. If we make the assumption that
the distribution of matter at this time was homogeneous, then a de Sitter-type
model of the Cosmos might be expected to apply. This model is distinguished by
its simple symmetry group.

It has been shown by M. Lévy-Nahas (1967) that the only algebras that con­
tract to the Poincaré algebra are those of $SO(4,1)$ and $SO(3,2)$. Hence, if there
is a cosmological group symmetry which becomes the Poincaré group $\mathcal{P}$ in the
limit of vanishing space-time curvature, as the Universe expands, it must be either
$SO(4,1)$ or $SO(3,2)$, the de Sitter groups. The idea which motivated this thesis
was to investigate whether $SU(n)$ and $SO(4,1)$ or $SO(3,2)$ could be combined
non-trivially in the process of contraction, yielding a mass-squared operator lead­
ing to different quark masses and limiting to the mass-squared operator for the
Poincaré group as the space-time curvature tends to zero. The naïve approach
of seeking a larger Lie group containing $SU(n)$ and $SO(4,1)$ (or $SO(3,2)$) is un­
likely to work, as it is known that any semi-direct product of $SU(n)$ with $SO(4,1)$
(respectively $SO(3,2)$) is somorphic to their direct product. The hope is that a
correct formulation of Lie group contraction will show a way of combining these
symmetries somewhat in the spirit of supersymmetry theories (see Müller-Kirsten
and Wiedemann (1987), p151). The success of this type of enquiry would depend
then, on a deep understanding of the global theory of Lie group contractions and
of contractions of representations of the group (see comments at the end of this
section).

All early treatments of Lie group contraction, however, were only at the Lie
algebra level except for Mickelsson and Niederle (1972), but no satisfactory global
treatment existed until Dooley (1983) who treated only İnönü-Wigner contraction
for the special case where the group was reductive with respect to the preserved
subgroup (see also Dooley and Rice (1983), Dooley and Gaudry (1984, 1986) and
Dooley and Rice (1985)), and Dooley and Ricci (1985) which contained general results for a contraction of the maximal compact subgroup inside semi-simple Lie groups. (Subsequent to discovering general, global contractions of Lie groups the author became aware of the above-mentioned works of Dooley et al. The results of this thesis however, generalise their definitions of Lie group contractions very substantially but do not achieve results such as (1.5.17) and Dooley and Ricci (1985) Theorem 8.1, because of the level of generality attempted. Thus, there were no global theories of Lie group contraction without restrictions on the group concerned.

Consequently, the secondary motivation was to provide a mathematically sound, global approach which was as physically motivated as possible. This approach succeeded, and it is encouraging that it uncovered a totally unexpected connection to current algebras (Dolan 1984) establishing the desired link back to Elementary Particle physics; as well as providing the neat group-theoretical structure postulated to somehow exist by Saletan (1961, p12). The question of the existence of this group-theoretical structure however, was not subsequently investigated by Saletan or any following author until recently (see references cited above).

It now appears possible that the formalism of Lie group contraction will allow internal and dynamical symmetry to be combined somewhat along the lines of supersymmetry theories because of the demonstration that the contraction process for Lie algebras can be viewed as a homomorphism $dX$ from a subalgebra of the current algebra of $su(n) \times so(4,1)$ or $su(n) \times so(3,2)$ to the contracted algebra.

For I-W contraction, there is a subspace $\mathcal{T}$ of this algebra which has as image by the contraction homomorphism, the generator of the time-translation subgroup, which may be loosely interpreted as a mass-operator. The elements of $\mathcal{T}$, do not commute with $su(n)$ and it is hoped that in a correct physical theory, appropriate
elements of an algebra containing $T$ will be singled out to supply a canonical mass-squared operator which has physical significance, supplying the correct quark masses for representation spaces spanned by the quark wave-functions, and which contracts by the homomorphism $dX$ to the mass-squared operator for flat space-time.
The seminal work for Lie group contractions was done only on Lie algebra contractions by Segal (1951). İnönü and Wigner (1953) were able to obtain many more results than Segal by restricting to a special case, and Saletan (1961) proved several theorems in a much more general setting than İnönü-Wigner. Lévy-Nahas (1967, p1218), proved that Saletan’s definitions were the most general possible within his framework, except for the “more singular contractions”, a special case of which she treated in that paper. Theorems for the general case of “more singular” Saletan contractions were given by Kupczynski (1969). A list of references to further early work along much the same lines as the above-mentioned authors, can be found in the book of Gilmore (1974). Similarly, other early work dealing with contractions and expansions of Lie algebras (and specific Lie groups) and their representations was done by Hermann (1966), Bacry and Lévy-Leblond (1968), Mukunda (1969), Doebner and Melsheimer (1967), Lykhmus (1969), Ström (1965, 1970), Evans (1967), Wolf (1971), and Rawnsley (1972). In all these cases contractions of representations in general were either very specific or treated in a limited way, and all general treatments were at the Lie algebra level.

Mickelsson and Niederle (1972) were the first to give a general, global definition of Lie group contraction, but it had the disadvantage of needing pre-knowledge of what the contracted group was to be, rather than having the facility of calculating it from the original Lie group, and given family of mappings. Further, their treatment was still essentially local, not global. Celeghini and Tarlini (1981 and 1982), addressed the problem of contraction of group representations by defining the contraction in the representation of the group, thus giving a class of contractions of the original group which depended on the particular representation. At about the same time Dooley (1983) outlined a global approach to the İnönü-Wigner style of contraction of Lie groups and their representations, and pushed
this through for the case when the group, and the preserved subgroup of the contractions formed a Riemannian symmetric pair (see also Dooley and Rice (1983) and Dooley and Gaudry (1984, 1986)). Dooley and Ricci (1985) then addressed the case of contraction inside semi-simple Lie groups of real rank 1. See also Ricci (1986). Vide Cazzaniga (1985) who duplicated independently of the author, a few of the results of this thesis only for the special case of a connected, compact semi-simple Lie group where the contracted group is abelian. Applications of Lie groups to gauge theories in elementary particle physics can be found in the works of Celeghini, Magnollay, Tarlini and Vitiello (1985), Celeghini, Tarlini and Vitiello (1984), and Tataru-Mihai and Vitiello (1982).

The results of this thesis fill in some of the gaps in the theory of contractions of Lie groups and their representations. It is shown herein that very general, global contractions can be carried out on any Lie group and these contraction processes can be described within the framework of a very natural and simple geometric and group theoretic structure. Considerable insight is thereby gained, of both the global and Lie algebraic structure of Inönü-Wigner and Saletan contractions (especially in Section 3.1) including being able to "state in a simple group-theoretical way the content of these (Saletan's (1961), p12 and Eqs (17)) equations." the probable existence of which Saletan and many subsequent authors had postulated. The physical motivation of the definition of Lie group contraction given here bears fruit in exhibiting unexpected connections with infinite dimensional Lie groups and Current Algebras which are both in use by physicists in their investigations of gauge theories. The connections described in section 2.4 point strongly to the unexpected role of Lie group contractions in finding non-perturbative solutions of the free-field Quantum Chromodynamic field equations. In Chapter 3, the cumbersome formulae and methods of Saletan are replaced by very simple ones and the algebraic structure of the contracted algebra is almost self-evident in strong
contrast to Saletan’s structure theorems which are somewhat surprising in the context of his theory.

Finally, in an area that is considered to be notoriously complex and technically difficult (see the remarks of Celeghini and Tarlini (1982, p180)), the principal strength of the approach of this thesis is that of the very great simplification it provides in the methods and understanding of the process of Lie group contraction (particularly by re-casting all contraction processes as homomorphisms and by providing simple closed forms of the equations for generalised Saletan contraction).
§2.3 Initial Definition and Examples.

Special Case of Embedding in Larger Lie group

Example of the Poincaré Group Contraction

In what follows, we make precise for the Poincaré group the undefined method used by physicists for contracting the example of the one dimensional Lorentz group which is:

"To take the limit \( c \to \infty \) in

\[
G = \left\{ \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \begin{pmatrix} 1 & -v \\ -\frac{v}{c} & 1 \end{pmatrix} : v \in (-c, c) \right\}
\]

This will be done by making a physical analysis of the process of taking "the limit \( c \to \infty \)" and applying it to the Poincaré group. We will then generalise this process to obtain the definition of contraction.

The process of "taking the limit \( c \to \infty \)" in a fixed cosmological model of the universe is not a physical one, as the speed of light should appear constant for any model of the Cosmos. Thus, as \( c \) varies with some parameter \( t \), it is natural to consider a distinct, disjoint model of the universe for each value of \( c(t) \); a limit universe being obtained for \( c \to \infty \). To implement this, simplistically take each universe \( U_t \) as the subset \( \{t\} \times \mathbb{R}^4 \) of \( \mathbb{R}^5 \) letting \( c(t) \to \infty \) monotonically as \( t \to 0 \). As the speed of light in each universe is \( c(t) \), then the equation of the expanding light sphere is \( x_1^2 + x_2^2 + x_3^2 - c^2(t)x_4^2 = 0 \), so the metric is given by \( s^2(t) = x_1^2 + x_2^2 + x_3^2 - c^2(t)x_4^2 \).

Let \( c(1) \) be the conventional value of the speed of light, and \( \mathcal{L} \) the full Lorentz group of invariance of \( s^2(t) \), for the universe \( U_1 \). Define the maps \( \varphi_t : U_1 \to U_t \) to be:

\[
\varphi_t(1, x_1, x_2, x_3, x_4) \mapsto (t, x_1, x_2, x_3, \frac{c(1)}{c(t)}x_4).
\]
Representing $\varphi_t$ as a $4 \times 4$ matrix, $\varphi_t L \varphi_t^{-1}$ will be the group of invariance of $s^2(t)$ in $U_t$. In order to obtain a new group for the universe $U_0$, one might be tempted to take the limits (in $M(4 \times 4)$, the $4 \times 4$ real matrices), $\lim \varphi_t L \varphi_t^{-1}$ for each $L$ in $L$ such that the limit exists. Somewhat surprisingly these limits together yield only the 3-dimensional rotation group with space and time reflections, not the full Gallilei group as desired. The solution to this minor dilemma is obtained from a closer physical and mathematical examination of the structures so far defined:

In the case of the 1-dimensional Lorentz group $L^1$, acting in the $x_1 - t$ plane, if $L^1_t$ is the 1-dimensional group of invariance of $s^2(t)$ and $L^1 \in L^1$, $L^1_t \in L^1_t$ and $v$ the velocity parameter of $L^1$, then

$$\varphi_t L^1(v) \varphi_t^{-1} = L^1_t \left( \frac{vc(t)}{c(1)} \right)$$  \hspace{1cm} (2.1)

Thus, as we take the limit $t \to 0$ in $\varphi_t L^1(v) \varphi_t^{-1}$, not only is the speed of light (massless particles) approaching infinity but the speed of all non-zero rest mass particles is approaching infinity as well because of the term $\frac{vc(t)}{c(1)}$. Thus in the limit universe, all velocities are infinite, and hence it is natural to obtain only the rotation group and reflections as the limit group. From equation (2.1), we see that if we take $v$ or $L^1(v)$ as a function of $t$ we might obtain a limit universe with finite velocities. Hence we are led to consider taking limits of the form, $\lim_{t \to 0} \varphi_t L^1(t) \varphi_t^{-1}$ where $L^1 \in C^1_{L^1}[0,1]$ say (the once-continuously differentiable functions from $[0,1]$ to $L^1$).

Alternatively, from the mathematical viewpoint, observe that the maps $\varphi_t$ may be used to define a map $\Phi$,

$$\Phi : (0,1] \times L \rightarrow (0,1] \times M(4 \times 4)$$

$$\Phi : (t,L) \mapsto (t, \varphi_t L \varphi_t^{-1})$$

96
and that the set \( \Phi((0,1] \times \mathcal{L}) \) will be a fibre bundle if we induce its topology by \( \Phi \) from \((0,1] \times \mathcal{L})\). In this setting, taking the limit \( t \to 0 \) of \( \varphi_t L \varphi_t^{-1} \) is the same as running along a section as \( t \to 0 \) of the fibre bundle \( \Phi((0,1] \times \mathcal{L}) \). When we obtain the limit points of these curves, we are tacking on a “limit fibre” \( \{0\} \times \mathcal{L}_0 \) to the bundle. The sections of \( \Phi((0,1] \times \mathcal{L}) \) are more general than this so we are thereby enjoined to take limits of the form \( \lim_{t \to 0} \varphi_t L(t) \varphi_t^{-1} \), possibly solving our dilemma by allowing \( L \) to vary with \( t \).

We are now in a position to make precise the example of the contraction of the Poincaré group, which will serve as a prelude to the general definitions:

Let \( \mathcal{P} \) be the full Poincaré group and let \( \Phi \) be the map,

\[
\begin{align*}
\Phi &: (0,1] \times \mathcal{P} \to (0,1] \times M(5 \times 5) \\
\Phi &: (t, P) \mapsto (t, \varphi_t P \varphi_t^{-1})
\end{align*}
\]

where \( \varphi_t : \mathbb{R}^5 \to \mathbb{R}^5 \) is the map,

\[
\varphi_t = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & t
\end{pmatrix}.
\]

Now \( \mathcal{P} = \left\{ \begin{pmatrix} L & \ell \\ 0 & 1 \end{pmatrix} \mid L \in \mathcal{L}, \ell \in \mathbb{R}^4 \right\} \),

and we wish to compute all the limits,

\[
\lim_{t \to 0} \begin{pmatrix} L & \ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}, \quad (*)
\]

for every \( P \in C^1_{\mathcal{P}}[0,1] \), such that the limit exists. Define the set \( \mathcal{D}^1 \subset C^1_{\mathcal{P}}[0,1] \) to be

\[
\mathcal{D}^1 = \{ P \mid P \in C^1_{\mathcal{P}}[0,1] \text{ and the limit } (*) \text{ exists} \}.
\]
and define the contraction $\mathcal{P}_0$ of $\mathcal{P}$ to be

$$\mathcal{P}_0 = \{ P_0 \mid P_0 = \lim_{t \to 0} (t, \varphi_t P(t) \varphi_t^{-1}), \ P \in \mathcal{D}_1 \}$$

There is a natural mapping $\mathcal{X}$ from $\mathcal{D}_1$ to $\mathcal{P}_0$,

$$\mathcal{X}(P) = \lim_{t \to 0} (0, \varphi_t P(t) \varphi_t^{-1}).$$

For $\mathcal{D}_1$ we obtain,

$$\mathcal{D}_1 = \{ P \mid P \in C^1_\mathcal{D}[0,1], \ P(0) \in O(3) \times (T \times T) \}$$

where $T$ is the discrete, time reversal subgroup; $T$ is the group of time translations and $T$ acts on $T$ in the natural way. Then if we compute all the limits as $t \to 0$ for

$$\varphi_t P(t) \varphi_t^{-1} = \begin{pmatrix} L_1 & \frac{t_1}{t} & \frac{t_2}{t} \\ \frac{t_2}{t} & L_2 \\ tL_3 & L_2 & \frac{t_4}{t} \end{pmatrix}$$

with $P \in \mathcal{D}_1$, where the $L_i$ are submatrices of $L(t) \in \mathcal{L}$, $\ell(t) = (\ell_1, \ell_2, \ell_3, \ell_4)^T(t)$, and $L_1$ is a $3 \times 3$ matrix, we obtain

$$\mathcal{X}(\mathcal{D}_1) = \{ \mathcal{X}(P) \mid P \in \mathcal{D}_1 \} = \left\{ \begin{pmatrix} R & v_1 & a_1 \\ v_2 & a_2 \\ v_3 & a_3 \\ 0 & \pm 1 & a_4 \end{pmatrix} \mid R \in O(3), \ v_i \in \mathbb{R}, \ a_i \in \mathbb{R} \right\}.$$

We recognise the group $\mathcal{P}_0 = \mathcal{X}(\mathcal{D}_1)$ as the full Gallilei group with time translations. Notice that $\mathcal{D}_1$ is a group and the map $\mathcal{X} : \mathcal{D}_1 \to \mathcal{P}_0$ is in fact a homomor-
For if $P, Q \in \mathcal{D}^1$, 

$$\mathfrak{X}(PQ) = \lim_{t \to 0} (0, \varphi_t P(t)Q(t)\varphi_t^{-1}) = \lim_{t \to 0} (0, \varphi_t P(t)\varphi_t^{-1}\varphi_t Q(t)\varphi_t^{-1}) = (0, (\lim_{t \to 0} \varphi_t P(t)\varphi_t^{-1})(\lim_{t \to 0} \varphi_t Q(t)\varphi_t^{-1})) = \mathfrak{X}(P)\mathfrak{X}(Q).$$

We shall see in section 2.5 that $\mathcal{D}^1$ is a Banach-Lie group and $\mathfrak{X}$ is a $C^\infty$-Fréchet homomorphism.

We have succeeded in making precise definitions of contraction for the Poincaré group obtaining the full Gallilei group as the contraction, and uncovering the unexpected group structure of $\mathfrak{X} : \mathcal{D}^1 \to \mathcal{P}_0$. The general definitions of contraction will now be given.

We now want to re-cast the above example in the most general possible setting:

**Initial Definitions**

Let $\mathcal{G}$ be the Lie group (of dimension $n$) to be contracted and $E$ a finite-dimensional Euclidean space. Let $\sigma$ be a positive integer and let $[0, 1]^\sigma$ denote $\bigotimes_{i=1}^\sigma [0, 1]$.

**Definition 2.3.2**

Let $\phi : \mathcal{G} \times [0, 1]^\sigma \to E$ be a $C^\infty$ map such that $\phi(\cdot, t)$ is a $C^\infty$-embedding (i.e. $\phi(\cdot, t)$ is $1-1$ and $d\phi(\cdot, t)$ is $1-1$ at every point) of $\mathcal{G}$ into $E$ for all $t$ in $(0, 1]^\sigma$, which is zero on the identity. For given $i_j$ such that $\sum_{j=1}^\sigma i_j = -q$, with $q$ a positive integer, define the map $\Psi$, 

$$\Psi(g, t) = t_1^{i_1} \cdots t_\sigma^{i_\sigma} \phi(g, t).$$

99
We call $\Psi(\cdot, t)$ for each $t$, a **Laurent map**, as $\Psi$ has a Laurent-type series in $t$ when $\phi$ is jointly analytic in its arguments.

Let the map $\Phi$ be the graph of $\Psi$, so that,

$$\Phi(g, t) = (\Psi(g, t), t).$$

**Examples of Laurent Maps and the Basic Idea of General Contraction**

When $G$ is countably connected (1.1.36(d)), $G$ can be embedded in a Euclidean space of dimension $2n$ or less (the topology of a countably connected Lie group has a countable base and a theorem of Grauert (1958) shows that $G$ can be regularly, analytically embedded in $\mathbb{R}^{2n}$). Let $\xi : E \times [0, 1]^n \to E$ be a $C^\infty$-map such that $\xi(\cdot, t)$ is 1—1 and $d\xi(\cdot, t)$ is 1—1 at every point. Regarding $G$ as a submanifold of $E$, $\phi$ is generated by restricting $\xi$ to $G \times [0, 1]^n$; and hence there is an abundance of Laurent maps, $\Psi$.

Let $\sigma = 1$. The idea of contracting $G$ is to induce an isomorphic product on $\Psi(G, t)$ using the maps $\Psi(\cdot, t)$, and to compute the "limit" of these isomorphic groups $\Psi(G, t)$ by calculating $\lim_{t \to 0} \Psi(g(t), t)$ for all $g \in C^\sigma_G[0, 1]$ (the set of all $q$-times continuously differentiable maps from $[0, 1]$ to $G$) such that the limit exists; the set of all these limit points is to be the contraction $G_0$ of $G$. The following example of the contraction of the circle $T$ to the straight line illustrates how this works:

Identify $\mathbb{R}^2$ with the complex plane and consider the circle $S^1 = \{z \mid |z + 1| = 1\}$. Define the map $\Psi$ to be

$$\Psi(z, t) = \frac{z}{t}.$$ 

By analogy to the case of the Poincaré group, $D^1$ will be the set of all $C^1_{S^1}[0, 1]$ curves $z(t)$ such that the limit $\lim_{t \to 0} \Psi(z(t), t)$ exists.

Therefore $D^1 = \{z \mid z \in C^1_{S^1}[0, 1], \ z(0) = 0\}$. 
Let \( z, w \in S^1 \). The product on the circle is given by

\[
\pi(z, w) = (z + 1)(w + 1) - 1
\]

\[
= zw + z + w,
\]

and from this formula it follows that \( D^1 \) is a group.

Let \( z_t, w_t \in \Psi(S^1, t) \). Then the isomorphic product on \( \Psi(S^1, t) \) is given by

\[
\pi_t(z_t, w_t) = \Psi(\pi^{-1}(z_t, t), \Psi^{-1}(w_t, t))
\]

The contraction of \( S^1 \) will be \( G_0 = \{ z \mid z = \lim_{t \to 0} \frac{w(t)}{t}, w \in D^1 \} \)

\[
= \{ z \mid z = \lim_{t \to 0} \left( \frac{-1 + \cos \theta(t)}{t} + \frac{i \sin \theta(t)}{t} \right), \theta(0) = 0 \}
\]

\[
= \{ z \mid z = (0 + i \theta(0)) \} = \{ z \mid z \in iR \}
\]

which is the imaginary axis.

The product on \( G_0 \) should be given by computing all limits

\[
\lim_{t \to 0} \pi_t(\Psi(z(t), t), \Psi(w(t), t)) \text{ with } z, w \in D^1 :
\]

Then,

\[
\lim_{t \to 0} \pi_t(\Psi(z(t), t), \Psi(w(t), t)) = \lim_{t \to 0} \frac{1}{t}(z(t)w(t) + z(t) + w(t))
\]

\[
= z(0)w(0) + w(0)z(0) + z(0) + w(0)
\]

\[
= z(0) + w(0)
\]

from the definitions of \( \pi, \pi_t, \Psi \) and \( D^1 \).

Since \( G_0 = \{ z \mid z = \hat{w}(0), w \in D^1 \} \), \( G_0 \) must be an abelian Lie group.

Observe now, again by analogy to the Poincaré case, that the natural mapping defined by

\[ \mathcal{X} : D^1 \to G_0 \]
\[ \mathcal{X}(z) = \lim_{t \to 0} \Psi(z(t), t) \]

is a homomorphism:

\[
\mathcal{X}(zw) = \lim_{t \to 0} \frac{1}{t} (z(t)w(t) + z(t) + w(t)) = \dot{z}(0) + \dot{w}(0)
\]

\[
= \lim_{t \to 0} \frac{z(t)}{t} + \lim_{t \to 0} \frac{w(t)}{t}
\]

\[
= \mathcal{X}(z) + \mathcal{X}(w).
\]

We now wish to consider the geometric picture of the contraction of \( S^1 \) to \( i\mathbb{R} \).

As \( t \) approaches zero, \( \Psi(S^1, t) \) is a circle of ever-increasing radius \( \frac{1}{t} \), with centre \( \frac{1}{t} \), which in the limit becomes the straight line \( i\mathbb{R} \). The collection of circles for \( t \in (0, 1] \) forms the half-plane with imaginary axis and the interior of the unit circle, centre \(-1 + i0\), removed.

A more natural geometric object is formed by considering the graph of \( \Psi \).

Then the graph of the set \( \bigcup_{t \in (0, 1]} (\Psi(S^1, t), t) \cup \mathcal{G}_0 \) gives a definite geometric shape of an inverted vase, the \( C \)-plane-parallel sections of which are the sets \( (\Psi(S^1, t), t) \) and all the "lip" of the vase is \((i\mathbb{R}, 0)\). A covering by coordinate patches is given by the maps:

\[
\chi_1^{-1} : \begin{cases} 
(\mu, t) \mapsto \left( \frac{e^{i\pi \tanh \mu t} - 1}{t}, t \right) & \text{for } t \in (0, 1], \mu \in (-\infty, \infty) \\
(\mu, 0) \mapsto (i\mu\pi, 0), & \mu \in (-\infty, \infty)
\end{cases}
\]

\[
\chi_2^{-1} : \begin{cases} 
(\mu, t) \mapsto \left( \frac{e^{i\pi (\tanh \mu t + 1)} - 1}{t}, t \right) & \text{for } t \in (0, 1], \mu \in (-\infty, \infty)
\end{cases}
\]

The mapping \( \chi_1 \) is:

\[
\chi_1 : (w, \tau) \mapsto \left( \frac{1}{\tau} \tanh^{-1} \left( \frac{1}{i\pi} \log(\tau w + 1) \right), \tau \right)
\]
where the cut in the C-plane is the negative real axis.

Overlap of the charts occurs for $\mu t < 0$ and $\mu t > 0$. For $\mu t < 0$,

$$\chi_1 \circ \chi_2^{-1} (\mu, t) = \frac{1}{t} \tanh^{-1} (\tanh \mu t + 1)$$

and for $\mu t > 0$,

$$\chi_1 \circ \chi_2^{-1} (\mu, t) = \frac{1}{t} \tanh^{-1} (\tanh \mu t - 1),$$

so the charts are $C^\infty$-related. It is readily verified that the projection map,

$$\left( \frac{z}{t}, t \right) \mapsto t$$

$$(i\mu, 0) \mapsto 0$$

is $C^\infty$.

Hence the set $\bigcup_{t \in (0,1]} (\Psi(S^1, t), t) \cup \mathcal{G}_0$ is a $C^\infty$-fibre bundle. The topology induced from the charts on $\bigcup_{t \in (0,1]} (\Psi(S^1, t), t)$ is the same topology as that induced from $T \times (0,1]$ via the maps $\Psi$.

With the example of the contraction of the circle to the real line in hand, the motivation for the following definitions is clear.

**N. B. Except for section 3.2 we will henceforth take $\sigma = 1$.**

In the following definitions, the map $\Phi$ is fixed.

**Definition 2.3.3**

Let $\mathcal{G}_{(0,1]} \subset E \times (0,1]$ be the set,

$$\mathcal{G}_{(0,1]} = \{ \Phi(\mathcal{G}, t) \mid t \in (0,1] \}$$

with the topology of the trivial bundle induced by the mapping $\Phi$ from $\mathcal{G} \times (0,1]$.

$\mathcal{G}_{(0,1]}$ is a $C^\infty$-manifold with projection map $\pi : \Phi(g, t) \mapsto t$. 103
Definition 2.3.4

Let \( S^q \) be the set of all continuous sections \( s \) of \( \mathcal{G}_{[0,1]} \) which correspond to a \( C^q \)-section \( g \) of \( \mathcal{G} \times [0,1] \) such that \( s(t) = \Phi(g(t), t) \), for \( t \in (0,1] \) and the limit \( t \to 0^+ \) of \( s(t) \) exists.

Definition 2.3.5

Let \( \mathcal{G}_0 \) be the set of endpoints \( (t = 0) \) of all curves which are elements of \( S^q \).

We call \( \mathcal{G}_0 \) the contraction of \( \mathcal{G} \). Define \( \mathcal{G}_{[0,1]} = \mathcal{G}_{[0,1]} \cup \mathcal{G}_0 \).

Definition 2.3.6

Let \( D^q \) be the set of all \( C^q \)-sections of \( \mathcal{G} \times [0,1] \) corresponding uniquely as in Definition (2.3.4) to a continuous section of \( \mathcal{G}_{[0,1]} \) in \( S^q \).

Definition 2.3.7

Define the map

\[
\mathbf{X} : D^q \to \mathcal{G}_0
\]

\[
\mathbf{X} : g \mapsto \lim_{t \to 0^+} \Phi(g(t), t).
\]

\( \mathbf{X} \) is called the contraction mapping and \( D^q \) the domain of the contraction mapping.

Example of Definitions

In the example of the Poincaré group contraction, the group structure of \( GL(5, \mathbb{R}) \) has to be exploited to show that the map \( \mathbf{X} \) is a homomorphism. We will now contract the Poincaré group another way, using only the geometry of \( \mathbb{R}^{25} \), and the group structure of \( \mathcal{P} \), to give an example of the above definitions.

Take the matrix representation of the Poincaré group as before, but regard this representation, as an embedding of \( \mathcal{P} \) in \( E = \mathbb{R}^{25} \).
Define the map $\Psi$ to be:

$$
\Psi : \begin{pmatrix}
L_1 & L_4 & \ell_1 \\
L_2 & L_3 & \ell_2 \\
0 & 0 & 1
\end{pmatrix} \mapsto \begin{pmatrix}
L_1 & L_4 & \ell_1 \\
L_2 & L_3 & \ell_2 \\
0 & 0 & 1
\end{pmatrix}.
$$

It is readily verified that $\Psi(\cdot, t)$ is not a homomorphism in contrast to the previous treatment of the contraction of $\mathcal{P}$. Now $\mathcal{D}^1$ can also be calculated as the set of all $C^1_p[0, 1]$ curves $g$, such that the limit, $\lim_{t \to 0} \Psi(g(t), t)$ exists.

Then

$$
\mathcal{D}^1 = \left\{ g \in C^1_p[0, 1] \mid g(0) = \begin{pmatrix} R & 0 & 0 \\ 0 & \pm 1 & \ell_4 \\ 0 & 0 & 1 \end{pmatrix}, \quad R \in O(3), \quad \ell_4 \in \mathbb{R} \right\}
$$

$\mathcal{D}^1$ is a group with respect to point by point multiplication, because the endpoints of elements of $\mathcal{D}^1$ form a group in this case. As in the example of the Laurent maps, the idea is that the product on $\mathcal{P}_0$, the contraction of $\mathcal{P}$, will be given by

$$
\pi_0(\mathcal{X}(g)\mathcal{X}(h)) = \lim_{t \to 0} \Psi(g(t)h(t), t)
$$

Letting

$$
g(t) = \begin{pmatrix} L_1(t) & L_4(t) & \ell_1(t) \\ L_2(t) & L_3(t) & \ell_2(t) \\ 0 & 0 & 1 \end{pmatrix},
$$

we have

$$
\mathcal{X}(g) = \begin{pmatrix} L_1(0) & \dot{L}_4(0) & \ell(0) \\ 0 & \pm 1 & \ell_4(0) \\ 0 & 0 & 1 \end{pmatrix}
$$

$L_1(0) \in \mathbb{R}^3$, $\dot{L}_4(0) \in \mathbb{R}^3$, $\ell(0) \in \mathbb{R}^3$, $L_4(0) \in \mathbb{R}$, $\ell_4(0) \in \mathbb{R}$, $L_1(0) \in O(3)$. 

105
Thus $\mathcal{P}_0$ will clearly be the full Gallilei group if the conventional product agrees with the product $\pi_0$. Now, the conventional product on $\mathcal{P}_0$ is

$$\mathcal{X}(g)\mathcal{X}(g') = \begin{pmatrix} L_1(0)L'_1(0) & L_1(0)L'_4(0) + \hat{L}_4(0) & L_1(0)\ell'(0) + \hat{L}_4(0)\ell'_4(0) + \ell'(0) \\ 0 & \pm 1 & \pm \ell'_4(0) + \ell(0) \\
0 & 0 & 1 \end{pmatrix}$$

and

$$\mathcal{X}(gg') = \lim_{t \to 0} \Psi(g(t)g'(t),t)$$

$$= \lim_{t \to 0} \begin{pmatrix} L_1L'_1 + L_4L'_4 & (L_1L'_4 + L_4L'_2)/t & (L_1\ell' + L_4\ell'_4 + \ell)/t \\
0 & 0 & L_3\ell' + L_2\ell'_4 + \ell_4 \end{pmatrix}$$

$$= \begin{pmatrix} L_1(0)L'_1(0) & L_1(0)L'_4(0) + \hat{L}_4(0) & L_1(0)\ell'(0) + \hat{L}_4(0)\ell'_4(0) + \ell'(0) \\ 0 & 0 & \pm \ell'_4(0) + \ell(0) \\
0 & 0 & 1 \end{pmatrix}$$

where $L_2(0) = \pm 1$, $L'_2(0) = \mp 1$; so the products are the same, and $\mathcal{P}_0$ is indeed the full Gallilei group. We have also verified that the contraction mapping $\mathcal{X}$ is a homomorphism.

**Special Case of Embedding in a Larger Lie Group**

In the case of Saletan contraction of Lie algebras (1.5.5), we saw that necessary and sufficient conditions were required to hold on $g$ in order that $g_0$ be a Lie algebra, but that these conditions were quite obscure. There is however, an important class of special cases of Definition (2.3.2), which we now give, for which it can be proved that $\mathcal{D}'$ and $\mathcal{G}_0$ are always both Lie groups and $\mathcal{X}$ a Lie homomorphism with no conditions on $\mathcal{G}$ at all.

**Case 2.3.8**

- Each Laurent map $\Psi(\cdot,t)$ is a $C^\infty$ monomorphism of $\mathcal{G}$ into a Lie group $\mathcal{G}_L$ which is embedded in $E$, via a $C^\infty$ diffeomorphism into.
- For all $g$ in $\mathcal{D}'$, the limits in $E$, $\lim_{t \to 0} \Psi(g(t),t)$, are contained in $\mathcal{G}_L$. 

106
Example of Case 2.3.8

Let $G$ be any analytic subgroup of $GL(n, R)$ and let $E = R^n$. Let $\varphi(t)$ be a $C^\infty$ curve in $M(n \times n, R)$ for $t \in [0,1]$, where $\det \varphi(t)$ is a polynomial with term of lowest order $q$ in $t$, and $\varphi(0)$ is singular (In (3.1.7) we will see that if $\varphi(0)$ is merely singular the same contraction will result as if $\det \phi(t)$ were polynomial to an appropriate order). This $\varphi(t)$ simply generalises the $\varphi_t$ of the Poincaré example.

Define $\Psi$ to be

$$\Psi(g, t) = \varphi(t)g\varphi^{-1}(t)$$

$$= (\det \varphi(t))^{-1}\varphi(t)g(\text{adj } \varphi(t))$$

(Adj is the classical matrix adjoint)

and this $\Psi$ is of the form (2.3.2). Let $g \in D^q$. Then

$$\det \Psi(g(t), t) = \det(\varphi(t)g(t)\varphi^{-1}(t))$$

$$= \det(g(t)), t \in (0,1]$$

therefore

$$\lim_{t \to 0} \det \Psi(g(t), t) = \det(g(0)) \neq 0$$

and hence $\Psi$ satisfies (2.3.8).

We will now prove two assertions concerning $D^q$, $X$ and $G_0$:

**Proposition 2.3.9(a)**

Observe that $C^q_G[0,1]$ is a group with respect to pointwise multiplication. Let the Laurent maps $\Psi(\cdot, t)$ be monomorphisms of $G$ into $G_L$ as in (2.3.8). Then,

$D^q$ is a group (a subgroup of $C^q_G[0,1]$),

$G_0$ a group, and

$X$ a homomorphism.

**Corollary 2.3.9(b)**

With $\Psi$ given by Proposition (2.3.9(a)), $D^q$ and $G_0$ are both Lie groups and $X$ a Banach-Lie homomorphism (see 1.2).
Proof of 2.3.9(b):

Theorem (2.5.30) which holds without any restrictions on the Laurent map, and Proposition (2.3.9(a)) together assert that $D^q$ is a Lie group and $G_0$ is a Lie group with product given by

$$X(g)X(h) = \lim_{t \to 0} \Psi(g(t)h(t), t) \text{ (in $E$)}$$

$$= (\lim_{t \to 0} \Psi(g(t), t))(\lim_{t \to 0} \Psi(h(t), t) \text{ (in $G_L$)}).$$

So the product function on $G_0$ thus given, is the same as that of $G_L$. Theorem (2.5.30) also asserts that $X$ is a Banach-Lie homomorphism. ■

Proof of 2.3.9(a):

The set of $C^q$ sections of $G \times [0,1]$ form a group under point by point multiplication, so we only need to show closure in $D^q$ and inclusion of the identity. Let $g$ and $h$ be in $D^q$. Then by definition $\lim_{t \to 0} E \Psi(g(t), t)$ and $\lim_{t \to 0} E \Psi(h(t), t)$ both exist.

Now,

$$\left(\lim_{t \to 0} E \Psi(g(t), t)\right)\left(\lim_{t \to 0} E \Psi(h(t), t)\right)^{-1} = \lim_{t \to 0} \Psi(g(t), t)(\Psi(h(t), t)^{-1}$$

$$= \lim_{t \to 0} \Psi(g(t)h^{-1}(t), t)$$

$$= \lim_{t \to 0} E \Psi(g(t)h^{-1}(t), t)$$

and hence the limit exists, because

$$\lim_{t \to 0} E \Psi(g(t), t) = \lim_{t \to 0} E \Psi(g(t), t)$$ for any $g \in D^q$,

$\Psi$ is a monomorphism, the product and inverse functions on $G_L$ are continuous, and $G_L$ has the relative topology in $E$. Therefore $gh^{-1} \in D^q$, so $D^q$ is a group.
Now, for the same reasons,

\[ X(gh^{-1}) = \lim_{t \to 0} \Psi(g(t)h^{-1}(t), t) \]

\[ = \lim_{t \to 0} \left( \Psi(g(t), t) \Psi(h(t), t) \right)^{-1} \]

\[ = (\lim_{t \to 0} \Psi(g(t), t)) \left( \lim_{t \to 0} \Psi(h(t), t) \right)^{-1} \]

\[ = X(g)(X(h))^{-1} \]

Further,

\[ X(e) = \lim_{t \to 0} \Psi(e, t) = \lim_{t \to 0} e = e, \]

hence \( X \) is a homomorphism. \( G_0 \) will now be a group because it is equal to \( X(D^\theta) \) by definition.

**Examples 2.3.10**

The example of Case (2.3.8) generalises the example of the contraction of the Poincaré group. A more specific example of this general case is the contraction of the De Sitter group to the Poincaré group.

Let the De Sitter group be \( SO(3,2,\mathbb{R}) \); each element leaving invariant the metric \( M = (1, +1, +1, -1, -1) \); and let \( \Psi \) be given by \( \Psi(g, t) = \varphi(t) g \varphi^{-1}(t) \),

\[ g \in SO(3,2,\mathbb{R}) \text{ with } \varphi(t) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 & t \end{pmatrix}. \]

In a similar way to Saletan contraction, the subgroup with respect to which the contraction is taken, is not chosen in advance, but is dynamically determined by \( G \) and the choice of \( \Psi \). In this case it is \( SO(3,1) \), and for this and other Inönü-Wigner-type contractions, the contraction subgroup is isomorphic to the set of all constant curves which are
elements of $D^1$ (see §3.1). For another choice of $\Psi$, the constant curves in $D^1$ may only consist of the identity element, resulting in the contraction of $G$ to $\mathbb{R}^n$ (again, see §3.1).
§2.4 Contraction of Lie Algebras. How Generalised Saletan Contraction of Lie Algebras May be Viewed as a Homomorphism

Introduction

In order to show how Saletan contraction is encompassed by the general, global definitions, Definitions (2.3.3) to (2.3.7) will be differentiated to define the contraction of Lie algebras. Within this context, Generalised Saletan contraction (1.5.1) will be demonstrated to be a homomorphism of an infinite-dimensional Lie algebra $\mathfrak{d}^q$ to the contracted Lie algebra $\mathfrak{g}_0$. It is remarkable that for Lie algebra contractions as general as (1.5.1), a very simple Lie-algebraic necessary and sufficient condition can be given (2.4.8) for $\mathfrak{g}_0$ to be a Lie algebra in terms of $\mathfrak{d}^q$. The infinite-dimensional Lie algebra $\mathfrak{d}^q$ is effectively a subalgebra of Current Algebra (Dolan, 1984), which appears in elementary particle physics. It is then natural to ask whether there is an intermediate Lie algebra of finite dimension such that the contraction mapping may be effectively viewed as a homomorphism of this algebra to $\mathfrak{g}_0$. Such an algebra exists (denoted $dP(\mathfrak{d}^q)$) and is a subalgebra of a factor algebra of the Current Algebra. This section will then be concluded with two very general theorems giving among other things several, equivalent, necessary and sufficient conditions for $\mathfrak{g}_0$ to be a Lie algebra and the contraction map to be a homomorphism. One of these conditions is a generalisation of the most general Saletan conditions (1.5.1) for contractions. We point out that the results (2.4.7) and (2.4.8) hold a fortiori for the very general class of Lie algebra contractions given by (1.5.1) which includes the special case (1.5.2) originally considered by Saletan (1961). An immediate corollary of (2.4.8) is that $\mathfrak{g}_0$ is a Lie algebra if and only if $dP(\mathfrak{d}^q)$ is a Lie algebra for the case of Saletan contraction (1.5.1). As $dP(\mathfrak{d}^q)$ is finite-dimensional, this is a very desirable result indeed. We conclude this section with a discussion of the relationship of the algebras $\mathfrak{d}^q$ and $dP(\mathfrak{d}^q)$ to current algebra, and the implications for Elementary Particle Physics.
Definition 2.4.1

Let \( g_{(0,1)} \subset E \times (0,1) \) be the set

\[
g_{(0,1)} = \{ d\Phi(g, t) \mid t \in (0,1) \}
\]

with the topology of the trivial bundle induced by the mapping \( d\Phi \) from \( g \times (0,1) \).

Definition 2.4.2

Let \( s^q \) be the set of all continuous sections \( S \) of \( g_{(0,1)} \) which correspond to a \( C^q \)-section \( X \) of \( g \times [0,1] \) such that

\[
S(t) = d\Phi(X(t), t), \quad \text{for } t \in (0,1)
\]

and the limit \( t \to 0 \) of \( S(t) \) exists in \( E \).

Definition 2.4.3

Let \( g_0 \) be the set of endpoints of all curves which are elements of \( s^q \). We call \( g_0 \) the contraction of \( g \).

Definition 2.4.4

Let \( d^q \) be the set of all \( C^q \)-sections of \( g \times [0,1] \) corresponding uniquely as in Definition (2.4.2) to a continuous section of \( g_{(0,1)} \) in \( s^q \).

Definition 2.4.5

Define the map \( d\mathcal{X} : d^q \to g_0 \)

\[
d\mathcal{X} : X \mapsto \lim_{t \to 0} d\Phi(X(t), t)
\]

\( d\mathcal{X} \) is called the derivative of the contraction mapping and \( d^q \) the domain of the contraction mapping.

Example of Generalised Saletan Constructions

The important and easy applications of the above definitions are the generalised Saletan contractions foreshadowed in the introduction to this section.
Definition 2.4.6

Saletan contraction (Saletan, 1961) can be globally characterised by the following two conditions:

(i) The image of the tangent space at $e$ in $G$ by the derivative $d\Psi(\cdot, t)$ of the Laurent maps, is the same for all $t$ in $(0,1]$.

Letting the image space of the $d\Psi(\cdot, t)$'s as in (i) be $V \subset E$

(ii) The family of inverse maps $d\Psi^{-1}(\cdot, t) : V \to g$ can be extended to $t \in [0,1]$ such that $d\Psi^{-1}(v, \cdot) \in C^g_{\mathbb{R}}[0,1] \forall v \in V$.

Note: Inducing via the $\Psi(\cdot, t)$ from $G$ an isomorphic Lie group structure on $\Psi(G, t)$, and identifying $G$ with $\Psi(G, 1)$, we have a sequence of isomorphic Lie groups $\mathcal{U}(t) : G \to \mathcal{U}(t)G$ where $\mathcal{U}(t) = \Psi(\cdot, t) \circ \Psi^{-1}(\cdot, 1)$ with each Lie algebra sharing the same vector space $V$. We may then identify $\Psi$ with $\mathcal{U}$. Let the derivative of $\mathcal{U}(t)$ be $u(t)$.

The following two propositions hold independently of (2.4.6) for maps $U(t) = u^{-1}(t)$ given by (1.5.1):

Proposition 2.4.7

For the case of Saletan contraction, if Saletan’s necessary and sufficient conditions (1.5.1) hold then, $\mathfrak{d}$ is a Lie algebra (viz., a subalgebra of the Lie algebra $C^g_{\mathbb{R}}[0,1]$ with bracket taken pointwise) $\mathfrak{d}_0$ has the Saletan Lie algebra structure and $d\mathfrak{X}$ is a homomorphism.

Proof:

With the above identifications $\mathfrak{d}_0$ is given by

$$\mathfrak{d}_0 = \{X \mid X \in C^g_{\mathbb{R}}[0,1], \lim_{t \to 0^+} u(t)X(t) \text{ exists}\}$$

(Here "\(\lim\)" means that the limit is taken in the Euclidean space $E$ of (2.3.2).) Let $X, Y$ be in $\mathfrak{d}_0$. Then the curves $u(t)X(t)$ and $u(t)Y(t)$ are in $C^g_{\mathbb{R}}[0,1]$. 

113
Now if Saletan’s necessary and sufficient conditions hold, namely that
\[ \lim_{t \to 0} u(t)[u^{-1}(t)X', u^{-1}(t)Y'] \exists \text{ for all } X' \text{ and } Y' \text{ in } g, \]
then by viewing these conditions in terms of the structure constants on \( g \) the limit
\[ \lim_{t \to 0} u(t)[u^{-1}(t)W(t), u^{-1}(t)Z(t)] \exists \text{ for all continuous curves } W(t), Z(t). \]
Thus the limit will exist for the curves \( u(t)X(t) \) and \( u(t)Y(t) \) and hence the limit,
\[ \lim_{t \to 0} u(t)[X(t), Y(t)] \exists. \]
Hence \( d^g \) is a Lie algebra. The Lie algebra structure of \( d_0 \) is given by,
\[ [X, Y]_0 = \lim_{t \to 0} u(t)[u^{-1}(t)X, u^{-1}(t)Y]. \]
Let \( W(t), Z(t) \) be continuous curves in \( g \),
then,
\[ \lim_{t \to 0} u(t)[u^{-1}(t)W(t), u^{-1}(t)Z(t)] = [W(0), Z(0)]_0, \]
and putting \( W(t) = u(t)X(t), Z(t) = u(t)Y(t) \) with \( X, Y \in d^g \) we have,
\[ \lim_{t \to 0} u(t)[X(t), Y(t)] = [\lim_{t \to 0} u(t)X(t), \lim_{t \to 0} u(t)Y(t)]. \]
That is, the map \( d\mathcal{X} \) which is defined by
\[ d\mathcal{X} : X \mapsto \lim_{t \to 0} u(t)X(t), \]
is a homomorphism.

Corollary 2.4.8

For the case of Saletan contraction, Saletan’s necessary and sufficient conditions hold if and only if \( d^g \) is a Lie algebra.

Proof:

If \( d^g \) is a Lie algebra then since curves \( u^{-1}(t)X \) and \( u^{-1}(t)Y \) are in \( d^g \) for all \( X, Y \) in \( g \), the limit \( \lim_{t \to 0} u(t)[u^{-1}(t)X, u^{-1}(t)Y] \) exists. The converse follows from (2.4.7).
Before demonstrating the connection of $g^q$ to Current algebra, we will prove the two theorems mentioned in the introduction to §2.4 giving necessary and sufficient conditions for contraction.

It is immediate from (2.4.5) that the domain of the contraction mapping $d\mathcal{X}$ is in general an infinite dimensional vector space. It is natural to seek some finite dimensional space and a projection map $dP$ such that $d\mathcal{X}$ is a composition of maps,

$$d\mathcal{X} : d^q \xrightarrow{dP} dP(d^q) \xrightarrow{dX} g_0.$$ 

The following definition and lemmas provide this finite-dimensional space, which plays a greatly clarifying role in the necessary and sufficient conditions for contraction to take place.

**Definition 2.4.9**

Let $\mathcal{P}^q(g)$ be the Lie algebra of all polynomials in $t$ with coefficients from $g$, factored by the ideal of polynomials of degree strictly greater than $q$. From the vector space viewpoint $\mathcal{P}^q(g)$ is the space of polynomials of order $q$ in $t$, on $g$. Let $\{X\}$ be a coset of $X$ by the ideal.

**Remark**

We will see later that $\mathcal{P}^q(g)$ is the Lie algebra of the bundle $\Pi_{i=1}^q T(G)$ where $T(G)$ is the tangent bundle to $G$ equipped with a suitable product. Later, in section 3.2 we will define a similar algebra that will be related to Jet bundles over $G$.

**Lemma 2.4.10**

The map $dP$,

$$dP : C^q_0[0,1] \rightarrow \mathcal{P}^q(g)$$

$$dP : X \mapsto \left\{ \sum_{I=0}^{q} \frac{X^I(0)t^I}{I!} \right\}$$

is a homomorphism.
Proof:

Note that $dP$ effectively just maps $X$ to the sum of the first $q + 1$ terms of its Taylor series.

Now $dP(X) = \sum_{i=0}^{q} \left\{ \frac{X^{(i)}(0)}{i!} \right\}$ and

$$dP([X,Y]) = \sum_{i=0}^{q} \left\{ \frac{[X,Y]^{(i)}(0)}{i!} \right\}$$

$$= \sum_{i=0}^{q} \left\{ \sum_{j=0}^{i} \left[ \frac{X^{(j)}(0)}{j!}, \frac{Y^{(i-j)}(0)}{(i-j)!} \right] t^i \right\} \quad (2.4.11)$$

Computing $[dPX,dPY]$, and dropping terms of order greater than $q$ in $t$,

$$[dPX,dPY] = \sum_{\ell=0}^{q} \sum_{m=0}^{q-\ell} \left\{ \frac{X^{(\ell)}(0)}{\ell!}, \frac{Y^{(m)}(0)}{m!} \right\} t^\ell m$$

and putting $\ell + m = i$ and $\ell = j$,

$$[dPX,dPY] = \sum_{j=0}^{q} \sum_{i=j}^{q} \left\{ \left[ \frac{X^{(j)}(0)}{j!}, \frac{Y^{(i-j)}(0)}{(i-j)!} \right] t^i \right\}.$$

This is equal to the R.H.S. of (2.4.11) upon reversing the sums. Therefore $dP[X,Y] = [dPX,dPY]$, and as $dP$ is linear, $dP$ is a homomorphism.

Lemma 2.4.11

There is a map $d\mathcal{X}_P : dP(\mathbb{A}^q) \to \mathcal{G}_0$ such that $d\mathcal{X}$ is the lifting of $d\mathcal{X}_P$ to $\mathbb{A}^q$.

Viz., $d\mathcal{X} = d\mathcal{X}_P \circ dP$.

Proof:

Define $d\mathcal{X}_P$ to be the map,

$$d\mathcal{X}_P : \{dP(X)\} \mapsto d\mathcal{X}(X), \quad X \in \mathbb{A}^q.$$

116
To show that $dX_P$ is well-defined, we show that if $\{dP(X)\} = \{dP(Y)\}$ with $X, Y \in d^q$, then $d\mathcal{X}(X) = d\mathcal{X}(Y)$. Suppose $\{dP(x)\} = \{dP(Y)\}$, then

$$\sum_{i=0}^{q} \frac{X^{(i)}(0)}{i!} t^i = \sum_{i=0}^{q} \frac{Y^{(i)}(0)}{i!} t^i + r(t)$$

where $r(t)$ is a polynomial of order $t^{q+1}$.

Writing,

$$X(t) = \left( \sum_{i=0}^{q} \frac{X^{(i)}(0)}{i!} t^i \right) + \left( X(t) - \sum_{i=0}^{q} \frac{X^{(i)}(0)}{i!} t^i \right)$$

and similarly for $Y(t)$, we get

$$X(t) - Y(t) = r(t) + \left( X(t) - \sum_{i=0}^{q} \frac{X^{(i)}(0)}{i!} t^i \right) - \left( Y(t) - \sum_{i=0}^{q} \frac{Y^{(i)}(0)}{i!} t^i \right).$$

Observing that $d\Psi(t)$ has the form

$$d\Psi(t) = t^{-q}d\phi(t)$$

where $\phi(t)$ is a $C^\infty$-map of $\mathcal{G} \times [0,1]$ into $E$, and using L'Hôpital’s rule and the Leibnitz rule,

$$d\mathcal{X}(X - Y) = \lim_{t \to 0} d\Psi(t)(X(t) - Y(t))$$

$$= \frac{d^q}{dt^q}(d\phi(t)(r(t))) \bigg|_{t=0} + \frac{d^q}{dt^q}(d\phi(t)(X(t) - \sum_{i=0}^{q} \frac{X^{(i)}(0)}{i!} t^i)) \bigg|_{t=0}$$

$$+ \frac{d^q}{dt^q}(d\phi(t)(Y(t) - \sum_{i=0}^{q} \frac{Y^{(i)}(0)}{i!} t^i)) \bigg|_{t=0}$$

$$= 0.$$ 

Thus $d\mathcal{X}(X) = d\mathcal{X}(Y)$ as we wished to show, and so $d\mathcal{X}_P$ is well-defined. It clearly satisfies $d\mathcal{X} = d\mathcal{X}_P \circ dP$. 

\[\square\]
Note: As a simplification, we will henceforth write $\Phi$ and $\Psi$ with their first arguments only, the second argument being implied, where it won't cause confusion. This slight abuse of notation, which occurs throughout this work, is amply compensated for by the gain in clarity.

Definition 2.4.12

The contracting map (2.3.2) $d\Phi$ induces a map,

$$d\Phi_s : g^q \rightarrow g^q$$

as follows:

$$d\Phi_s(X)(t) = \begin{cases} 
  d\Phi(X(t)), & t \in (0, 1] \\
  \lim_{t \rightarrow 0} d\Phi(X(t)), & t = 0
\end{cases}$$

where $X \in g^q$.

Definition 2.4.13

A bracket can be defined for pairs of elements $S, U$ in $g^q$ where it exists:

$$[S, U](t) = \begin{cases} 
  d\Phi[d\Phi^{-1}S(t), d\Phi^{-1}U(t)], & t \in (0, 1] \\
  \lim_{t \rightarrow 0} d\Phi[d\Phi^{-1}S(t), d\Phi^{-1}U(t)], & t = 0
\end{cases}$$

Note 2.4.14

Before proceeding to the two theorems, we note from Definitions (2.4.2) and (2.4.3) that it is immediate that $g^q_0$ is a vector space and has a basis; but the following example illustrates that the dimension of $g^q_0$ may be less than the dimension of $g$. Consequently we can only give generalised Saletan-like conditions for contraction when the dimensions are the same:

Example 2.4.15

We may contract the Poincaré group $G$, which has dimension 10, to the full Gallilei group without time translations, which has dimension 9, by taking $G_L =$
$GL(5, \mathbb{R})$, $E = \mathbb{R}^{25}$ and

$$
\varphi_t = \begin{pmatrix}
1 & 1 & 1 \\
1 & t & 1
\end{pmatrix}
$$

as in example (2.3.10). Performing the contraction at the Lie algebra level, the derivative of the embedding map is

$$
d\Psi(t) : X \mapsto \varphi_t X \varphi_t^{-1}, \quad X \in \mathfrak{g} \subset M(5 \times 5, \mathbb{R})
$$

and

$$
\mathfrak{d}^1 = \left\{ X \left| \begin{array}{l}
X \in C^1_t[0, 1], \\
R \in O(3) \\
\tau_i \in \mathbb{R}
\end{array} \right\}
\right. \quad X(0) = \begin{pmatrix} R & 0 & \tau_1 \\
0 & 0 & \tau_2 \\
0 & 0 & \tau_3 \\
0 & 0 & 0 \end{pmatrix}
$$

and hence

$$
\mathfrak{g}_0 = \left\{ X \left| X = \begin{pmatrix}
R & \nu_1 & \tau_1 \\
\nu_2 & \nu_2 & \tau_2 \\
\nu_3 & \nu_3 & \tau_3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \\
\nu_i \in \mathbb{R} \\
\tau_i \in \mathbb{R}
\right\},
$$

which is the Lie algebra of the Gallilei group without time translations, which has dimension 9.

**Note 2.4.16**

Further to Note (2.4.14), we have to point out a further, expected limitation to the two theorems. In Corollary (2.4.8) we proved a very simple, group-theoretic necessary and sufficient condition for contraction to take place. This was because we could choose the image of each Laurent map to be the same space for each $t \in (0, 1]$. Now, it is not possible to globalise this Lie-algebraic Saletan contraction (in the hope of finding similar, simple conditions) to the extent that $\Psi(\mathcal{G}, t)$ is
the same set for all \( t \in (0,1] \), due to the fact that \( \mathcal{G} \) may be compact and the form of the embedding map in (2.3.2) implies that \( \Psi(\mathcal{G}, t) \) will be unbounded for \( t \in (0,1] \) (the set \( \{\Psi(\mathcal{G}, t) \mid t \in (0,1]\} \) will in general be unbounded in \( E \)). As a consequence, the conditions of the theorem for contraction to take place in full generality will be stronger than those of Corollary (2.4.8).

Following Saletan (1961, (17)) the theorem below gives several, equivalent necessary and sufficient conditions for contraction to take place. What is remarkable, is that conditions (1) and (2) are entirely group-theoretic in nature, while the contraction process is very general indeed.

**Theorem 2.4.17**

The following statements are equivalent

1. \( dP(d\mathfrak{q}) \) is a subalgebra of \( \mathcal{P}(g) \) and \( \ker d\mathfrak{X}_P \) is an ideal of \( dP(d\mathfrak{q}) \).
2. \( d\mathfrak{q} \) is a Lie algebra and \( \ker d\mathfrak{X} \) is an ideal of \( d\mathfrak{q} \).
3. \( \mathfrak{q} \) is a Lie algebra with bracket as in Definition (2.4.13) and the set 
   \( \mathfrak{q}_{\text{ker}} = \{ S \mid S \in \mathfrak{q}, S(0) = 0 \} \) is an ideal of \( \mathfrak{q} \).
4. \( \mathfrak{g}_0 \) is a Lie algebra with bracket induced unambiguously as in Definition (2.4.13) from any two curves in \( \mathfrak{q} \) with endpoints on two elements of \( \mathfrak{g}_0 \).

In cases when \( \dim \mathfrak{g} = \dim \mathfrak{g}_0 \), the following condition, which is a generalised Saletan condition, is equivalent to each of the above four statements:

5. There is a set of \( n \) sections \( S_i \in \mathfrak{q} \) which for \( t \in [0, \epsilon) \) (for some \( 0 < \epsilon \leq 1 \)) form a basis of the fibres of \( \mathfrak{g}_{[0,1]} \) over each \( t \), and of \( \mathfrak{g}_0 \); and if \( d\Psi(t) \) has matrix \( u(t) \) with respect to these bases and a basis \( \{X_i\} \) of \( \mathfrak{g} \), then

\[
\lim_{t \to 0} u(t)^j_{ik} C_{mp}^{-1}(t)^m_{ij} u^{-1}(t)_k^p \text{ exists.}
\]

**Remark**

For a substantially improved version of this theorem, which uses results found later in the thesis, see Appendix A.2. Note Corollary (2.4.8) where restrictions on
the Laurent maps lead to simpler necessary and sufficient conditions. See also §3.1
where mild restrictions on the Laurent maps fulfill (1) completely for the case of
Inönü-Wigner and generalised Inönü-Wigner contraction.

Proof:

We will prove (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (1) for the general case, and then
(4) ⇒ (5) ⇒ (1) for the case, dim \( g = \dim g_0 \).

Proof of (1) ⇒ (2):

Let \( Z \in \mathfrak{d}^q \). For any \( Z \in C^q_0[0,1] \), \( Z \) is in \( \mathfrak{d}^q \) if and only if

\[
\lim_{t \to 0} t^{-q}d\phi(t)Z(t)
\]

exists.

Let

\[
X(t) = \sum_{i=0}^{q} \frac{t^i Z^{(i)}(0)}{i!}, \quad Y(t) = Z(t) - \sum_{i=0}^{q} \frac{t^i Z^{(i)}(0)}{i!}.
\]

Then \( Z(t) = X(t) + Y(t) \). Observe that \( Y(t) \) has all derivatives up to the \( q \)’th
order at \( t = 0 \) zero, hence \( Y \in \ker d\mathfrak{X} \) and \( Y \in \mathfrak{d}^q \). Note that, more generally,
\( \ker dP \subset \ker d\mathfrak{X} \). Then the limit,

\[
\lim_{t \to 0} t^{-q}d\phi(t)X(t) = \lim_{t \to 0} t^{-q}d\phi(t)Z(t) - \lim_{t \to 0} t^{-q}d\phi(t)Y(t)
\]

exists, and hence \( X \in \mathfrak{d}^q \). Conversely, if \( X \in \mathfrak{d}^q \), then \( Z \in \mathfrak{d}^q \), as the limit,

\[
\lim_{t \to 0} t^{-q}d\phi(t)Z(t) = \lim_{t \to 0} t^{-q}d\phi(t)Y(t) - \lim_{t \to 0} t^{-q}d\phi(t)X(t)
\]

exists.

Thus \( Z \in \mathfrak{d}^q \) if and only if \( X \in \mathfrak{d}^q \).

Now let \( Z_1, Z_2 \in \mathfrak{d}^q \), then \( Z_i \) decomposes as above to \( Z_i = X_i + Y_i \).

Therefore

\[
\alpha Z_1 + \beta Z_2 = (\alpha X_1 + \beta X_2) + (\alpha Y_1 + \beta Y_2).
\]

Since \( dP(\mathfrak{d}^q) \) is a vector space, \( \{\alpha X_1 + \beta X_2\} \in dP(\mathfrak{d}^q) \),

121
we have $\alpha X_1 + \beta X_2 \in d^q$ because $d^q + \ker dP \subset d^q$ as we saw above.

Therefore $\alpha Z_1 + \beta Z_2 \in d^q$ since $\alpha Y_1 + \beta Y_2 \in \ker dP$.

Now, $[Z_1, Z_2] = [X_1, X_2] + [X_1, Y_2] + [Y_1, X_2] + [Y_1, Y_2]$ and from re-writing the bracket in terms of structure constants we see that the last three terms have all derivatives up to the $q'$th, zero at $t = 0$. Further, because $dP(d^q)$ is a Lie algebra and $\ker dP \subset \ker d\mathcal{X}$, we have $[X_1, X_2] \in d^q$.

Therefore $[Z_1, Z_2] \in d^q$ and hence $d^q$ is a Lie algebra.

Now suppose $\ker d\mathcal{X}_P$ is an ideal of $dP(d^q)$. Let $Z_1, Z_2 \in \ker d\mathcal{X}$ with $Z_i = X_i + Y_i$ as above. $\alpha Z_1 + \beta Z_2 = (\alpha X_1 + \beta X_2) + (\alpha Y_1 + \beta Y_2)$ and $\alpha Z_1 + \beta Z_2 \in \ker d\mathcal{X}$ if and only if $\alpha X_1 + \beta X_2 \in \ker d\mathcal{X}$. Now $\{X_1\} \in \ker d\mathcal{X}_P$ and $\{X_2\} \in \ker d\mathcal{X}_P$, therefore $\{\alpha X_1 + \beta X_2\} \in \ker d\mathcal{X}_P$ since $\ker d\mathcal{X}_P$ is a subspace of $dP(d^q)$. But as $\ker d\mathcal{X} + \ker dP \subset \ker d\mathcal{X}$, we have $\alpha X_1 + \beta X_2 \in \ker d\mathcal{X}$. Hence $\ker d\mathcal{X}$ is a subspace of $d^q$.

Let $Z_1 \in \ker d\mathcal{X}$ and $Z_2 \in d^q$, then

$$[Z_1, Z_2] = [X_1, X_2] + [X_1, Y_2] + [Y_1, X_2] + [Y_1, Y_2]$$

as above. Again, the last three terms have all derivatives up to the $q'$th, zero at $t = 0$. Then $[Z_1, Z_2] \in \ker d\mathcal{X}$ if and only if $[X_1, X_2] \in \ker d\mathcal{X}$. But $\{\{X_1\}, \{X_2\}\} = \{[X_1, X_2]\} \in \ker d\mathcal{X}_P$. Hence $[X_1, X_2] \in \ker d\mathcal{X}$ as $\ker d\mathcal{X} + \ker dP \subset \ker d\mathcal{X}$. Therefore $[Z_1, Z_2] \in \ker d\mathcal{X}$, thus proving (1) $\Rightarrow$ (2).

Proof of (2) $\Rightarrow$ (3):

Let $U, V \in d^q$ and let $X = d\Phi_s^{-1}U$, $Y = d\Phi_s^{-1}V$. Now $\alpha X + \beta Y \in d^q$ and since $d\Phi_s$ is linear

$$d\Phi_s(\alpha X + \beta Y) = \alpha d\Phi_s X + \beta d\Phi_s Y = \alpha U + \beta V$$

122
and so $\alpha U + \beta V \in \mathfrak{g}^q$.

Now $[X, Y] \in \mathfrak{g}^q$ and hence $d\Phi_s([X, Y]) \in \mathfrak{g}^q$. But

$$d\Phi_s[X, Y](t) = \begin{cases} d\Phi[d\Phi^{-1}U(t), d\Phi^{-1}V(t)] & \text{for } t \in (0, 1] \\ \lim_{t \to 0} d\Phi[d\Phi^{-1}U(t), d\Phi^{-1}V(t)] & \text{for } t = 0 \end{cases}$$

and with the product on $\mathfrak{g}^q$ as given in (2.4.13), this establishes that $[U, V] \in \mathfrak{g}^q$. Thus $\mathfrak{g}^q$ is a Lie algebra. Note that the set $\{S \mid S \in \mathfrak{g}^q, S(0) = 0\}$ is precisely $d\Phi_s(ker d\mathfrak{X})$.

Now let $U \in \mathfrak{g}^q$, $V \in d\Phi_s(ker d\mathfrak{X})$ with $X$ and $Y$ as above. Then $[X, Y] \in ker d\mathfrak{X}$

and so $d\Phi_s[X, Y] = [U, V] \in d\Phi_s(ker d\mathfrak{X})$.

Therefore $d\Phi_s(ker d\mathfrak{X})$ is an ideal by linearity of $d\Phi_s$. This proves $(2) \Rightarrow (3)$.

**Proof of $(3) \Rightarrow (4)$:**

Let $X_0, Y_0 \in \mathfrak{g}_0$. There must be $U, V \in \mathfrak{g}^q$ with $U(0) = X_0, V(0) = Y_0$. Now $\alpha U + \beta V \in \mathfrak{g}^q$, whence $\alpha X_0 + \beta Y_0 \in \mathfrak{g}_0$ and $\mathfrak{g}_0$ is a vector subspace of $E$. Let the bracket on $\mathfrak{g}_0$ be, $[X_0, Y_0]_0 = [U, V](0)$. We must show that this is well defined. Suppose that $U_2, V_2 \in \mathfrak{g}^q$ and $U_2(0) = X_0, V_2(0) = Y_0$. Then $U = U_2 + U_3$, $V = V_2 + V_3$ where $U_3, V_3$ are in $\{S \mid S \in \mathfrak{g}^q, S(0) = 0\}$.

Now $[U, V] = [U_2, V_2] + [U_2, V_3] + [U_3, V_2] + [U_3, V_3]$ and since $\mathfrak{g}_{ker}$ is assumed to be an ideal of $\mathfrak{g}^q$, we have $([U_2, V_3] + [U_3, V_2] + [U_3, V_3]) \in \mathfrak{g}_{ker}$ and therefore $[U, V](0) = [U_2, V_2](0)$ whence the bracket on $\mathfrak{g}_0$ is well-defined. Clearly, the bracket is anti-symmetric and satisfies the Jacobi identity because $[U, [V, W]](t) + [W, [U, V]](t) + [V, [W, U]](t) = 0$ for $t \in (0, 1]$ and the left hand side of this equation converges to $[X_0, [Y_0, Z_0]] + [Z_0, [X_0, Y_0]] + [Y_0, [Z_0, X_0]]$ (where $W(0) = Z_0$) by the continuity of addition on $E$. This completes the proof of $(3) \Rightarrow (4)$.
Proof of (4) \Rightarrow (1):

Let \( X, Y \) be arbitrary elements of \( dq \); then \( dP(X), dP(Y) \) will be arbitrary elements of \( dP(dq) \). Now \( \alpha dX(X) + \beta dX(Y) \in g_0 \) by assumption. i.e. \( dX(\alpha X + \beta Y) \in g_0 \). Then \( \alpha X + \beta Y \in dq \) and so \( \alpha dP(X) + \beta dP(Y) \in dP(dq) \) by the linearity of \( dP \). Hence \( dP(dq) \) is a subspace of \( P^d(g) \). Now \( [dX(X), dX(Y)] \in g_0 \) by assumption. But

\[
[dX(X), dX(Y)] = \lim_{t \to 0} d\Phi[X(t), Y(t)] = dX[X, Y],
\]

therefore \( [X, Y] \in dq \) and since \( dP \) is a homomorphism, \( [dP(X), dP(Y)] \in dP(dq) \).

Hence \( dP(dq) \) is a subalgebra of \( P^d(g) \).

Let \( Z \in \ker dX \). Then \( dP(Z) \in \ker dX_P \). Now, by assumption, \( [dX(Z), dX(X)] = 0 \) for any \( X \in dq \). But by definition, \( 0 = [dX(Z), dX(X)] = dX[Z, X] \), so \( [Z, X] \in \ker dX \). Since \( dP \) is a homomorphism, \( \ker dX_P \) is an ideal of \( dP(dq) \) if \( \ker dX_P \) is a subspace of \( dP(dq) \). But this is immediate because \( dX[\alpha Z_1 + \beta Z_2, X] = 0 \), \( \forall Z_1, Z_2 \in \ker dX, \forall X \in dq \). This proves (4) \Rightarrow (1).

Proof of (4) \Rightarrow (5):

Let indices with respect to a basis of \( E \) be given by Greek letters and indices with respect to those of \( g \), by Roman letters. Let \( \{X^a_0\} \) be a basis of \( g_0 \), and extend this basis to a basis \( \{X_a\} \) of \( E \). There must be \( n \) sections \( S_i \in g^q \) with \( S_i(0) = X^i_0 \), and \( S_i(0) = \delta^a_i X_a \).

Let \( S_i(t) = \gamma^a_i(t)X_a \) where \( \gamma(t) \in g_\ell(E, R), \gamma^a_\beta(0) = 0 \) for \( \beta > n \) and \( \gamma^a_\beta(0) = \delta^a_i \). Let \( \det_n \) be a function defined on \( g_\ell(E, R) \) which gives the determinant of the top, left-hand \( n \times n \) submatrix. Clearly, \( \det_n \) is a continuous function on \( g_\ell(E, R) \), and as \( \det_n(\gamma(t)) \neq 0 \) for \( t = 0 \) there must exist \( 0 < \epsilon \leq 1 \) such that \( \det_n(\gamma(t)) \neq 0 \) for \( t \in [0, \epsilon) \). Thus, for \( t \in [0, \epsilon) \), \( \gamma(t) \) has top left-hand \( n \times n \) submatrix with rank \( n \), and so \( \gamma(t) \) must have rank \( n \). Therefore \( \{S_i(t)\} \) is a linearly independent
set for \( t \in [0, \varepsilon) \). Let \( \{Y_i\} \) be a basis of \( g \). Now,

\[
d\Psi[d\Psi^{-1}S_i(t), d\Psi^{-1}S_j(t)] = d\Psi[u^{-1}(t)^kY_k, u^{-1}(t)^jY_j]
\]

\[
= d\Psi Y_m C_{kt}^m u^{-1}(t)^k u^{-1}(t)^j
\]

\[
= S_p(t)u(t)^p C_{kt}^m u^{-1}(t)^k u^{-1}(t)^j
\]

\[
= X_\alpha \gamma_\alpha^\alpha(t)u(t)^p C_{kt}^m u^{-1}(t)^k u^{-1}(t)^j
\]

Therefore the limit \( t \to 0 \) exists for \( \gamma_\alpha^\alpha(t)u(t)^p C_{kt}^m u^{-1}(t)^j \) for all \( \alpha \), and in particular for \( \alpha = 1, 2, \ldots, n \). But the top left-hand \( n \times n \) submatrix of \( \gamma(t) \) is invertible and hence,

\[
\lim_{i \to 0} u(t)^p C_{kt}^m u^{-1}(t)^k u^{-1}(t)^j \text{ exists,}
\]

thus establishing \( (4) \Rightarrow (5) \).

**Proof of \( (5) \Rightarrow (1) \):**

We will actually prove \( (5) \Rightarrow (2) \) then \( (2) \Rightarrow (1) \), using the fact that \( dP \) is a homomorphism.

Let \( X, Y \in d^g \), then \( [X(t), Y(t)] \) is defined for \( t \in [0, 1] \) and \( [X, Y] \) is in \( C_f^g[0, 1] \). Thus if

\[
\lim_{t \to 0} d\Phi[X(t), Y(t)] \exists, \text{ then } [X, Y] \text{ is in } d^g.
\]

Now let \( U = d\Phi_sX, V = d\Phi_sY \). Then we have,

\[
d\Phi[X(t), Y(t)] = d\Phi[d\Phi^{-1}U(t), d\Phi^{-1}V(t)]
\]

\[
= d\Phi[d\Phi^{-1}S_i(t), d\Phi^{-1}S_j(t)]\lambda_i(t)\mu_j(t)
\]

where \( U(t) = \lambda_i(t)S_i(t), V(t) = \mu_j(t)S_j(t) \).

Therefore

\[
d\Phi[X(t), Y(t)] = S_p(t)u(t)^p C_{kt}^m u^{-1}(t)^k u^{-1}(t)^j \lambda_i(t)u_j(t).
\]
Now \( \lim_{t \to 0} S_p(t) \) exists and \( \lim_{t \to 0} \lambda_i(t) \mu_j(t) \) exists because \( U \) and \( V \) are in \( \mathfrak{g}^q \), so by the assumption of (5), \( \lim_{t \to 0} d\Phi[X(t),Y(t)] \) exists. Therefore \( [X,Y] \in \mathfrak{g}^q \). Let \( \alpha, \beta \in \mathbb{R} \).

Since \( d\Phi(\alpha X(t) + \beta Y(t)) = \alpha d\Phi X(t) + \beta d\Phi Y(t) \) the limit \( \lim_{t \to 0} d\Phi(\alpha X + \beta Y)(t) \) will exist, hence \( \mathfrak{g}^q \) is a Lie algebra.

Now ker \( d\mathfrak{X} \) is a subspace of \( \mathfrak{g}^q \) since \( d\mathfrak{X} \) is linear. Let \( X \in \text{ker} \ d\mathfrak{X}, \ Y \in \mathfrak{g}^q \).

Then we must show that \( \lim_{x \to 0} d\Phi[X(t),Y(t)] = 0 \).

Now, \( d\Phi X(t) = \lambda_i(t) S_i(t) \) and \( d\Phi Y(t) = \mu_j(t) S_j(t) \), and

\[
\lim_{t \to 0} d\Phi X(t) = \lambda_i(0) S_i(0) = 0
\]

whence \( \lambda_i(0) = 0 \).

Again, let \( U = d\Phi_s X, \ V = d\Phi_s Y \), then

\[
d\Phi[X(t),Y(t)] = d\Phi[d\Phi^{-1}U(t),d\Phi^{-1}V(t)] = d\Phi[d\Phi^{-1}S_i(t),d\Phi^{-1}S_j(t)]\lambda_i(t)\mu_j(t)
\]

\[
= \left( S_p(t) U(t)^p_m C_{lk}^{\mu} U^{-1}(t)_{l}^k U^{-1}(t)_{j}^f \right) (\lambda_i(t)\mu_j(t)) .
\]

Now the limit, as \( t \to 0 \) of the first bracket exists, while the second bracket is zero for \( t = 0 \)

we have \( \lim_{t \to 0} d\Phi[X(t),Y(t)] = 0 \)

and \( \text{ker} \ d\mathfrak{X} \) is an ideal of \( \mathfrak{g}^q \).

Since \( dP \) is a homomorphism, \( dP(\mathfrak{g}) \) is a subalgebra of \( \mathcal{P}^q(g) \), and since \( dP(\text{ker} \ d\mathfrak{X}) = \text{ker} \ d\mathfrak{X}_P \), then \( \text{ker} \ d\mathfrak{X}_P \) is an ideal of \( dP(\mathfrak{g}) \). Thus establishing (5) \( \Rightarrow \) (1).

Before stating the second theorem, which completes the homomorphism-structural picture of Lie algebra contraction presaged by Theorem (2.4.17), we need a map which takes an element of \( \mathfrak{g}^q \) to its endpoint in \( g_0 \).
Definition 2.4.18

Let $d\epsilon$ be the map,

$$d\epsilon : \mathfrak{g}^q \to \mathfrak{g}_0, \quad d\epsilon : S \to S(0).$$

Theorem 2.4.19

When any of the conditions of Theorem (2.4.17) holds, the following diagram commutes:

$$\begin{array}{c}
\mathfrak{g}^q \\
\downarrow d\Phi \\
\mathfrak{g}^q \\
\downarrow d\mathfrak{X} \\
\downarrow dP \\
\mathfrak{g}_0 \\
\downarrow d\Phi \\
dP(d^q)
\end{array}$$

where $d\mathfrak{X}$, $d\mathfrak{X}_P$, and $d\epsilon$ are homomorphisms and $d\Phi$ is an isomorphism.

Proof:

It is immediate from the definitions that

$$d\epsilon \circ d\Phi = d\mathfrak{X} = d\mathfrak{X}_P \circ dP,$$

and that $d\Phi$, $d\epsilon$ and $d\mathfrak{X}_P$ are linear maps. By (4) of Theorem (2.4.17), with $X, Y \in \mathfrak{g}^q$, $U = d\Phi X$, $V = d\Phi Y$,

$$d\mathfrak{X}[X, Y] = \lim_{t \to 0} d\Phi[d\Phi^{-1}U(t), d\Phi^{-1}V(t)]$$

$$= [U(0), V(0)]$$

$$= [d\mathfrak{X}(X), d\mathfrak{X}(Y)].$$
Further,

\[ d\mathfrak{X}_P\{X, \{Y\}\} = d\mathfrak{X}_P\{X, Y\} \] by (2.4.17(1))

\[ = d\mathfrak{X}[X, Y] \]

\[ = [d\mathfrak{X}(X), d\mathfrak{X}(Y)] \]

\[ = [d\mathfrak{X}_P\{X\}, d\mathfrak{X}_P\{Y\}] \]

From (4) of the Theorem (2.4.17), it is immediate that \( de \) is a homomorphism, and

\[
d\Phi_s[X, Y] = \begin{cases} 
  d\Phi[X(t), Y(t)], & t \in (0, 1] \\
  \lim_{t \to 0} d\Phi[X(t), Y(t)], & \text{for } t = 0
\end{cases}
\]

\[ = \begin{cases} 
  d\Phi[d\Phi^{-1}U(t), d\Phi^{-1}V(t)], & t \in (0, 1] \\
  \lim_{t \to 0} d\Phi[d\Phi^{-1}U(t), d\Phi^{-1}V(t)], & t = 0
\end{cases} \]

\[ = [U, V] \]

\[ = [d\Phi_s(X), d\Phi_s(Y)] \]

and clearly \( d\Phi_s \) is 1–1.

Thus the diagram above commutes and all the maps are homomorphisms as asserted. \( \square \)

Remarks 2.4.20

Theorem (2.4.17(1)) enables us to make a group theoretic statement of the Generalised Saletan conditions (2.4.17(5)) in terms of the finite dimensional Lie algebra, \( dP(d^q) \), or in terms of the infinite dimensional Lie algebra, \( d^q \). This result is entirely new.

Theorem (2.4.19) allows us to express Lie algebra contraction as a homomorphism from a finite dimensional Lie algebra \( dP(d^q) \) to the contracted algebra \( g_0 \), or as a homomorphism from an infinite dimensional Lie algebra \( d^q \), to the contracted
This broadens the picture of Lie algebra contractions to more than just deformations of Lie algebras (Dooley, 1983).

The group-theoretic structure of Lie algebra contraction has thus been shown to be surprisingly simple. This simple structure may be promising for Elementary Particle Physics because of the additional connection to spontaneous symmetry breaking and current algebras:

**Connection of the Algebras** \( \mathfrak{g}^q \) **and** \( dP(\mathfrak{g}^q) \) **to Current Algebra**

Let \( \mathcal{P}(\mathfrak{g}) \) be the Lie algebra of all polynomials of positive degree in \( t \in [0,1] \).

The complexification of \( \mathcal{P}(\mathfrak{g}) \) is precisely the algebra that arises when one computes the Lie algebra of all moments of the currents of a Quantum Field Theory (QFT) with dynamical symmetry \( \mathcal{G} \) (Dolan, 1984). This is the algebra which physicists call Current Algebra.

**Definition 2.4.20**

Let \( \mathfrak{g}^q(\mathcal{P}) = \mathfrak{g}^q \cap \mathcal{P}(\mathfrak{g}) \).

The central theorems in the contraction of Lie algebras are (2.4.17) and (2.4.19). We have the following corollary:

**Corollary 2.4.21**

Theorem (2.4.17) and Theorem (2.4.19) still hold true when their statements are modified by the uniform substitution of \( \mathfrak{g}^q(\mathcal{P}) \) for \( \mathfrak{g}^q \).

**Proof:** This follows from the observation that \( d\mathcal{X}(\mathfrak{g}^q(\mathcal{P})) = d\mathcal{X}(\mathfrak{g}^q \cap \mathcal{P}(\mathfrak{g})) \).

**Remarks 2.4.22**

Corollary (2.4.21) illustrates that the restricted domain of the contraction mapping, \( \mathfrak{g}^q(\mathcal{P}) \), quite naturally occurs as a subalgebra of current algebra. It is only when we are taking global considerations into account, that \( \mathfrak{g}^q \) and \( D^q \) are more convenient; especially in proving that \( D^q \) is a Banach-Lie group.
Now $\mathcal{P}(g) \otimes \mathbb{C}$ is known to be an infinite-dimensional symmetry of the self-dual, free field, Yang-Mills equations of Quantum Chromodynamics (QCD) (Dolan, 1984). On the other hand, Hongoh, Matsumoto, and Umezawa (1978) and de Concini and Vitiello (1976) have shown that in gauge theories such as QCD, which exhibit spontaneous breakdown of symmetry, the symmetry group that appears in observations proves to be a contraction of the dynamical invariance group.

The entirely new connection of contractions to Current algebra presented here, raises the intriguing possibility that contractions of Lie groups have a non-trivial rôle to play in obtaining non-perturbative solutions of the QCD field equations.

Even more intriguing is that is is conjectured (Dolan, 1984) that the full symmetry group of the unrestricted, free-field Yang-Mills equations of QCD will have Lie algebra $\mathcal{P}_L(g) \otimes \mathbb{C}$ (where $\mathcal{P}_L(g)$ is the Lie algebra of Laurent polynomials in a non-zero real variable), while contraction of Lie algebras is straightforwardly generalised to take $\mathcal{P}_L(g)$ as its domain. This will be done in section 3.2 where it will be seen that there is no intermediate Lie algebra of finite dimension to play the role of $dP(d^q)$, but nevertheless theorems analogous to (2.4.17) and (2.4.19) can be proven.

Note that the Lie algebra $dP(d^q)$ is a subalgebra of the quotient of Current algebra by the ideal of polynomials of degree greater than $q$. 

130


\section*{2.5 General Contractions as Homomorphisms from Banach-Lie Groups and as Homomorphisms from Finite-Dimensional Lie Groups}

The objective of this section is to give theorems analogous to Theorem (2.4.17) and (2.4.19), but for global, as opposed to local, contractions. Before producing these theorems however, we will prove several propositions and introduce the necessary structures.

We first give the global analogue of $P^q(g)$ as given by Definition (2.4.9), which will enable us to produce an analogous statement to (2.4.17(1)). We will see below, that the group so defined, whose Lie algebra could have an important role to play in finding non-perturbative QCD solutions may also have an important role to play in the representation theory of semi-simple Lie groups.

\textbf{Definition 2.5.1}

Let $T(G)$ be the tangent bundle of $G$, and let $T^q(G)$ be the bundle formed by taking the cross product of $q$ copies of the tangent space $T_g$ at each point $g$ of $G$ (the topology and manifold structure of $T^q(G)$ is induced from trivialising charts of $T(G)$ in a straightforward manner. See Lemma (1.4.15)).

In Proposition (2.5.8) we will see that $T^q(G)$ has Lie algebra $P^q(g)$, once a Lie group structure has been given for $T^q(G)$ in Propositions (2.5.3) and (2.5.4).

\textbf{Remark 2.5.2}

In the following proposition we will define a product on $T^q(G)$ using differentiogeometric methods. This product can be given explicitly in terms of the Lie structure of $G$, and this will be demonstrated in Proposition (2.5.4). The definition given in Proposition (2.5.3) however, is conceptually important, as it provides a basis for the means of proving that $D^q$ is a Banach-Lie group; a fact which would be difficult to prove directly, as the criterion for a $C^q_G[0,1]$ curve to be in $D^q$
involves highly non-linear conditions on its first \( q \) derivatives at \( t = 0 \).

**Proposition 2.5.3**

\( T^q(\mathcal{G}) \) is a Lie group with product defined in the following way:

Let \( u, v \in T^q(\mathcal{G}) \). Then there are curves \( g, h \in C^q_0[0,1] \) such that,

\[
\left( \nabla, \ldots, \nabla^q \right) g(t)|_{t=0} = u, \quad g(0) = \pi(u)
\]
\[
\left( \nabla, \ldots, \nabla^q \right) h(t)|_{t=0} = v, \quad h(0) = \pi(v)
\]

and the product given by,

\[
uv = \left( \nabla, \ldots, \nabla^q \right)(g(t)h(t))|_{t=0},
\]

is well-defined. Here \( \nabla \) is covariant differentiation along a curve and \( \nabla^i \) the \( i \)th covariant derivative, while \( \pi \) is the projection map \( \pi : T^q(\mathcal{G}) \to \mathcal{G} \), all as defined in Section 1.4.

**Proof:**

By Definition (2.3.2), \( \mathcal{G} \) can be imbedded in \( E \) and hence it must have a Riemannian structure; the Riemannian connection corresponding to this structure is symmetric, hence \( \mathcal{G} \) can be covered by normal coordinate charts (see (1.4.6)).

**Definition of Special chart at \( u \in T^q(\mathcal{G}) \):**

Let a normal coordinate chart \( (x^i) \) at \( g(0) \) be given. Then covariant differentiation becomes ordinary differentiation with respect to this chart (1.4.7). This normal chart induces a trivialising chart (1.4.12(b)) at any \( u \) with \( \pi(u) = g(0) \) by choosing the same basis \( \left\{ \frac{\partial}{\partial x^i} \right\} \) for each of \( q \) copies of the tangent space \( T_g \) for \( g \) in a neighbourhood of \( g(0) \) in \( \mathcal{G} \) (see (1.4.15)).

Suppose that \( u \in T^q(\mathcal{G}) \) has coordinates \( (u^0, u^1, \ldots, u^q) \), \( u^i \in \mathbb{R}^n \), with respect to this chart. Then defining via the chart \( x^i \), a curve \( g(t), g \in C^q_0[0,1] \) by,

\[
x(t) = u^0 + u^1 t + \frac{u^2 t^2}{2!} + \cdots + \frac{u^q t^q}{q!}
\]
we get \((\nabla, \ldots, \nabla^q)g(t)|_{t=0} = u\), with \(g(0) = \pi(u)\). We now show that the product is well-defined. Let \(g, h\) be given as above, and suppose \(g', h' \in C^0_\partial [0,1]\) also satisfy
\[
(\nabla, \ldots, \nabla^q)g'(t)|_{t=0} = u, \quad g'(0) = \pi(u)
\]
\[
(\nabla, \ldots, \nabla^q)h'(t)|_{t=0} = v, \quad h'(0) = \pi(v);
\]
then \(\nabla^i g(t)|_{t=0} = \nabla^i g'(t)|_{t=0}, \quad \nabla^i h(t)|_{t=0} = \nabla^i h'(t)|_{t=0}, \quad i = 0, \ldots, q.

Let normal charts \(x^i, y^i\) at \(g(0)\) and \(h(0)\) respectively, be given, so that the curves \(g, h, g', h'\) are denoted by \(x(t), y(t), x'(t), y'(t)\). Let the product on \(\mathcal{G}\) with respect to \(x^i\) and \(y^i\) be given by \(\rho(x^i, y^i)\). Then \((\nabla, \ldots, \nabla^q)g(t)h(t)|_{t=0}\) and \((\nabla, \ldots, \nabla^q)g'(t)h'(t)|_{t=0}\) are given in special coordinates as above by
\[
\rho(x(0), y(0)) + \frac{d}{dt} \rho(x(t), y(t))|_{t=0} + \cdots + \frac{d^q}{dt^q} \rho(x(t), y(t))
\]
and
\[
\rho(x'(0), y'(0)) + \frac{d}{dt} \rho(x'(t), y'(t))|_{t=0} + \cdots + \frac{d^q}{dt^q} \rho(x'(t), y'(t))
\]
respectively. But these expressions are identical as
\[
\frac{d^i x(t)}{dt^i}|_{t=0} = \frac{d^i x'(t)}{dt^i}|_{t=0}, \quad i = 0, \ldots, q
\]
and
\[
\frac{d^i y(t)}{dt^i}|_{t=0} = \frac{d^i y'(t)}{dt^i}|_{t=0}, \quad i = 0, \ldots, q
\]
Thus, \(uv\) is well-defined.

Now \(T^q(\mathcal{G})\) is an analytic manifold, so we only need show that \(T^q(\mathcal{G})\) is a group whose product is analytic.

Let \(u, v, w \in T^q(\mathcal{G})\) have associated curves \(g, h, k\). Then the product is associative because
\[
(\nabla, \ldots, \nabla^q)(ghk)(t)|_{t=0} = u(vw).
\]
Let \(e(t)\) be the curve in \(\mathcal{G}\) such that \(e(t) = e\) for \(t \in [0,1]\), and define the identity in \(T^q(\mathcal{G})\) to be 1 = \((\nabla, \ldots, \nabla^q)e(t)|_{t=0}\); then the inverse of \(u\) is given by \((\nabla, \ldots, \nabla^q)g^{-1}(t)|_{t=0}\). Hence \(T^q(\mathcal{G})\) is a group.
Let \( u \) and \( v \) be given in special coordinates by \((u^0, u^1, \ldots, u^q)\) and \((v^0, v^1, \ldots, v^q)\). Observe that
\[
\frac{d^i}{dt^i} \rho(x(t), y(t))\big|_{t=0}
\]
is analytic in the variables \( x(0), y(0) \) and is a multinomial in the variables \( x^{(i)}(0), y^{(i)}(0), \)
\( i = 1, \ldots, q \). Let us write \( \sigma^i(u^j, v^k) = \frac{d^i}{dt^i} \rho(x(t), y(t))\big|_{t=0} \). Then the expression for
\( uv \) in coordinates is
\[
(\sigma^0(u^j, v^k), \sigma^1(u^j, v^k), \ldots, \sigma^q(u^j, v^k))
\]
and this function is analytic in \((u^j, v^k)\) since the functions \( \sigma^i(u^j, v^k) \) are analytic. Therefore \( T^q(G) \) is a Lie group.

The following proposition elucidates the structure of \( T^q(G) \) and is central to describing the structure of \( G_0 \), as we will see explicitly in section \( §3.1 \).

**Proposition 2.5.4**

There is a \( C^\infty \)-isomorphism \( \psi_q \) from \( T^q(G) \) to the Lie group represented by
\((q + 1)\)-tuples \((g, X_1, \ldots, X_q)\), \( g \in G \), \( X_i \in g \) with product given by,
\[
(g_1, X_1, X_2, \ldots, X_q) \cdot (g_2, Y_1, Y_2, \ldots, Y_q)
= (g_1g_2, F_1(\text{Ad}g_2(X), Y), \ldots, F_q(\text{Ad}g_2(X), Y)) \tag{2.5.5}
\]
where \( F_i(X_j, Y_k) \) is the coefficient of \( \frac{t^i}{i!} \) in the expression
\[
F\left(\sum_{j=1}^{q} \frac{t^j X_j}{j!}, \sum_{k=1}^{q} \frac{t^k Y_k}{k!}\right)
\]
where \( F \) is the function given by the Baker-Campbell-Hausdorff formula (1.1.33, 1.1.34),
\[
\exp F(X, Y) = \exp X \cdot \exp Y.
\]
Proof:

The set of \((q+1)\)-tuples with the natural product (2.5.5) and simple manifold structure is clearly a Lie group. Let \(u \in T^q(\mathcal{G})\). There is always a curve \(g \in C^\infty_0[0,1]\), such that \((\nabla^0, \nabla, \ldots, \nabla^q)g(t)|_{t=0} = u\) with

\[
g(t) = g_1 \exp(tX_1 + \frac{t^2X_2}{2!} + \cdots + \frac{t^qX_q}{q!})
\]

(2.5.6)

where \(\pi(u) = g_1\) with \(\pi\) as in (1.4.15). The isomorphism is given by

\[
\psi_q : u \mapsto (g_1, X_1, \ldots, X_q)
\]

which is 1-1 and well-defined since \((\nabla^0, \nabla, \ldots, \nabla^q)g(0)\) is uniquely determined by \((g_1, X_1, \ldots, X_q)\). \(\psi_q\) is clearly \(C^\infty\).

To see the homomorphism property, let

\[
h(t) = g_2 \exp(tY_1 + \frac{t^2Y_2}{2!} + \cdots + \frac{t^qY_q}{q!})
\]

where \((\nabla^0, \nabla, \ldots, \nabla^q)h(t)|_{t=0} = v\), for some \(v \in T^q(\mathcal{G})\) and observe that,

\[
g(t)h(t) = g_1g_2(g_2^{-1} \exp(\sum_j \frac{t^jX_j}{j!})g_2) \exp(\sum_k \frac{t^kY_k}{k!})
\]

\[
= g_1g_2 \exp F(\sum_j \frac{t^j}{j!} \text{Ad}g_2(X_j), \sum_k \frac{t^kY_k}{k!})
\]

where \(F\) is given by the BCH formula (1.1.33, 1.1.34) \((t\) can be taken as small as necessary for the BCH formula to apply, as we are only concerned with what happens for \(t = 0\)). Expressing \(g(t)h(t)\) as above in the form (2.5.6) and using (2.5.5) establishes the homomorphism property.
Corollary 2.5.7

For \( q = 1 \), the product (2.5.5) is:

\[
(g_1, X_1) \cdot (g_2, Y_1) = (g_1 g_2, \text{Ad}_{g_2}(X_1) + Y_1),
\]

which is the semidirect product of the Lie algebra \( g \) by the group \( G \), with respect to the adjoint action.

For \( q=2 \):

\[
(g_1, X_1, X_2) \cdot (g_2, Y_1, Y_2) = (g_1 g_2, \text{Ad}_{g_2}(X_1) + Y_1, \text{Ad}_{g_2}(X_2) + Y_2 + [\text{Ad}_{g_2}(X_1), Y_1]).
\]

For \( q=3 \):

\[
(g_1, X_1, X_2, X_3) \cdot (g_2, Y_1, Y_2, Y_3) = (g_1 g_2, \text{Ad}_{g_2}(X_1) + Y_1, \text{Ad}_{g_2}(X_2) + Y_2 + [\text{Ad}_{g_2}(X_1), Y_1],
\]

\[
\text{Ad}_{g_2}(X_3) + Y_3 + \frac{3}{2} [\text{Ad}_{g_2}(X_1), Y_2] + \frac{3}{2} [\text{Ad}_{g_2}(X_2), Y_1]
\]

\[
+ \frac{1}{2} [\text{Ad}_{g_2}(X_1), [\text{Ad}_{g_2}(X_1), Y_1]] + \frac{1}{2} [Y_1, [Y_1, \text{Ad}_{g_2}(X_1))]].
\]

**Proof:** Use (1.1.34).

In view of Proposition (2.5.4) and earlier remarks, the Lie algebra structure of \( T^q(G) \) comes as no surprise:

**Proposition 2.5.8**

\( T^q(G) \) has Lie algebra isomorphic to \( P^q(g) \), with exponential map

\[
\text{Exp} : \left. \frac{t^j X_J}{J!} \right|_t = \exp_X \exp \left( \frac{t^J X_J}{J!} \right), \ldots, \nabla^q \exp \left( \frac{t^J X_J}{J!} \right) \bigg|_{t=0} \quad (\text{sum convention on } J),
\]

where \( \exp \) is the exponential mapping for \( G \).

**Proof:**

Let coordinates for \( T^q(G) \) be labelled \( \alpha^I, \beta^J \) where \( I, J = 0, \ldots, q \) and \( i, j = 1, \ldots, n \) and \( n \) is the dimension of \( G \).
Choose special charts at $e$ in $G$ and $1$ in $T^q(G)$, by taking a normal chart $(x^i)$ at $E$ and taking $(x^i, u^{Jj})$ as coordinates for points $(x^i, u^{Jj} \frac{\partial}{\partial x^j})$ in a neighbourhood of $1$. We will proceed by calculating the left-invariant infinitesimal transformations (1.1.26) and then calculating their commutators for a chosen basis of the Lie algebra $g$. 

Let $\eta^{Ii}$ be the product function of $T^q(G)$ with respect to the special chart above. We wish to compute the transformation function (1.1.26)

$$\Psi^{Ij}_j(\alpha) = \frac{\partial \eta^{Ii}}{\partial \beta^{Jj}}(\alpha, \beta)|_{\beta=0}.$$ 

With respect to the special chart above $\alpha, \beta$ may be represented by polynomial curves $\tilde{\beta}^J t^J, \tilde{\alpha}^I t^I$ (sum convention on $I, J$; tildes indicate vectors in $\mathbb{R}^n$).

Let $\eta^i$ be the product function on $G$, and $\psi^i_j$ the corresponding transformation function. Then $\eta^{Ii}$ is given by the coefficients of $t^I$, up to order $q$, in the Taylor expansion of

$$\eta^i(\alpha^K t^K, \tilde{\beta}^J t^J).$$

We now assert that

$$\psi^{Ii}_j(j) = \begin{cases} 
\text{Coefficient of } t^{I-J} \text{ in} \\
\text{Taylor expansion of } \psi^i_j(\alpha^K t^K) \\
0 
\end{cases} 
\text{ for } J \leq I, 
\text{ for } J > I. \quad (2.5.9)$$
To see this,

\[ \psi_{j}^{I_{j}} = \frac{\partial \eta_{j}^{I_{i}}}{\partial \beta_{j}}(\alpha, \beta)|_{\beta = 0} \]

\[ = \frac{1}{I!} \frac{\partial}{\partial \beta_{j}} \left( \frac{\partial I}{\partial \beta_{j} \partial \beta_{j}} \eta^{i}(\alpha^{K} t^{K}, \beta_{M} t^{M}) \right)|_{t = 0} \]

\[ = \frac{1}{I!} \frac{\partial}{\partial t^{I}} \left( \frac{\partial I}{\partial \beta_{j} \partial \beta_{j}} \eta^{i}(\alpha^{K} t^{K}, \beta_{M} t^{M}) \right)|_{t = 0} \]

\[ = \frac{1}{I!} \frac{\partial}{\partial t^{I}} \left( \frac{\partial I}{\partial \beta_{j} \partial \beta_{j}} \eta^{i}(\alpha^{K} t^{K}, 0) t^{I} \right)|_{t = 0} \]

\[ = \left\{ \begin{array}{ll} \text{Coefficient of } t^{I-J} \text{ in } \psi_{j}^{i}(\alpha^{K} t^{K}) \quad & \text{for } J \leq I \\ 0 & \text{for } J > I, \end{array} \right. \]

as we asserted.

Take as basis for \( T^{q}(G) \),

\[ X_{j} = \psi_{j}^{I_{i}} \frac{\partial}{\partial \alpha^{I_{i}}} \]  

(2.5.10)

If we calculate \( \frac{\partial}{\partial \alpha^{I_{i}}} \psi_{L \ell}^{K}(\alpha) \), it is a similar calculation to that of (2.5.9), and we have

\[ \frac{\partial \psi_{L \ell}^{K_{k}}(\alpha)}{\partial \alpha^{I_{i}}} = \left\{ \begin{array}{ll} \text{coefficient of } t^{K-L-I} \text{ in } \frac{\partial \psi_{L \ell}^{K_{k}}(\alpha M_{t}^{M})}{\partial x_{i}} \quad & \text{for } I \leq (K - L), \\
0 & \text{for } I > (K - L). \end{array} \right. \]

Then calculating the commutators,

\[ [X_{j}, X_{L \ell}] \psi_{j}^{I_{i}} \frac{\partial}{\partial \alpha^{I_{i}}} \psi_{L \ell}^{K_{k}} \frac{\partial}{\partial \alpha^{K_{k}}} - \psi_{L \ell}^{K_{k}} \frac{\partial}{\partial \alpha^{K_{k}}} \psi_{j}^{I_{i}} \frac{\partial}{\partial \alpha^{I_{i}}} \]
\[
\left\{ \frac{1}{(I-J)!} \frac{d^{I-J}}{dt^{I-J}} \psi_j^i(\alpha^M t^M) \right\} \left\{ \frac{1}{(K-L-I)!} \frac{d^{K-L-I}}{dt^{K-L-I}} \frac{\partial \psi_j^k}{\partial x^i}(\alpha^M t^M) \right\} \bigg|_{t=0} \frac{\partial}{\partial \alpha^{Kk}}
\]

\[
= \left\{ \frac{1}{(I-L)!} \frac{d^{I-L}}{dt^{I-L}} \psi_k^i(\alpha^M t^M) \right\} \left\{ \frac{1}{(K-J-I)!} \frac{d^{K-J-I}}{dt^{K-J-I}} \frac{\partial \psi_j^k}{\partial x^i}(\alpha^M t^M) \right\} \bigg|_{t=0} \frac{\partial}{\partial \alpha^{Kk}}.
\]

\[(2.5.11)\]

In (2.5.11) the sum convention applies; for the first bracket, the sum on \(I\) is from \(J\) to \(q\) and the sum on \(K\) is from \(L\) to \(q\). For the second bracket the sum on \(I\) is from \(L\) to \(q\) and the sum on \(K\) from \(J\) to \(q\). In the first and second brackets the constraints \(I < K - L\) and \(I < K - J\) apply respectively.

Now we have,

\[
\sum_{I=J}^q \sum_{K=L+J}^{q-K-L} = \sum_{K=L+J}^{q} \sum_{I=J}^{K-L} ,
\]

and making the change of variables \(K = K, N = I - J\) the sum becomes,

\[
\sum_{K=L+J}^{q} \sum_{N=0}^{K-L-J} ,
\]

so that we may apply the Leibniz rule to the first bracketed term of (2.5.11) to obtain,

\[
\frac{1}{(K-L-J)!} \frac{d^{K-L-J}}{dt^{K-L-J}} \left\{ \psi_j^i(\alpha^M t^M) \frac{\partial \psi_j^k}{\partial x^i}(\alpha^M t^M) \right\} \bigg|_{t=0} \frac{\partial}{\partial \alpha^{Kk}}
\]

(sum on \(K\) from \(L + J\) to \(q\)).

Similarly, for the second bracketed term of (2.5.11) we obtain,

\[
\frac{-1}{(K-L-J)!} \frac{d^{K-L-J}}{dt^{K-L-J}} \left\{ \psi_k^i(\alpha^M t^M) \frac{\partial \psi_j^k}{\partial x^i}(\alpha^M t^M) \right\} \bigg|_{t=0} \frac{\partial}{\partial \alpha^{Kk}}.
\]

Adding these two terms together, and using the identity,

\[
\psi_j^i(x) \frac{\partial \psi_j^k(x)}{\partial x^i} - \psi_k^i(x) \frac{\partial \psi_j^k(x)}{\partial x^i} = C_m^p \psi_m^k(x),
\]

139
\[ [X_{J_l}, X_{L_l}] = \frac{1}{(K - L - J)!} \frac{d^{(K-L-J)}}{dt^{(K-L-J)}} \left( C_{t_j}^m \psi^k_m (\alpha^M t^M) \right) \bigg|_{t=0} \frac{\partial}{\partial \alpha^{K_k}} \]

\[= \sum_{K=J+L} \sum_{m} C_{t_j}^m \psi^k_{(L+J)m} (\alpha) \frac{\partial}{\partial \alpha^{K_k}} \]

\[= \begin{cases} C_{t_j}^m X_{(L+J)m} & \text{for } L + J \leq q \\ 0 & \text{for } L + J > q \end{cases} \quad (2.5.11(a)) \]

confirming that the Lie algebra of \( T^q(\mathcal{G}) \) is precisely \( \mathcal{P}^q(\mathfrak{g}) \), with isomorphism \( X_{L_m} \mapsto t^L X_{0_m} \), taking \( \{X_{0_m}\} \) as a basis for \( \mathfrak{g} \).

To prove that \( \text{Exp} \) is the required map, we use (1.1.28) and (1.1.29). Clearly \( \text{Exp} \) is an analytic map of \( \mathcal{P}^q(\mathfrak{g}) \) into \( T^q(\mathcal{G}) \). It is also a homomorphism of the one dimensional subspace through \( \frac{t^J X_J}{J!} \) since

\[ \exp \left( (s_1 + s_2) \frac{t^J X_J}{J!} \right) = \exp(s_1 \frac{t^J X_J}{J!}) \exp(s_2 \frac{t^J X_J}{J!}) \]

and

\[ \text{Exp} \left( (s_1 + s_2) \frac{t^J X_J}{J!} \right) = (\nabla^0, \nabla, \ldots, \nabla^q) \left( \exp(s_1 \frac{t^J X_J}{J!}) \exp(s_2 \frac{t^J X_J}{J!}) \right) \bigg|_{t=0} \]

\[= (\nabla^0, \nabla, \ldots, \nabla^q) \exp(s_1 \frac{t^J X_J}{J!}) \bigg|_{t=0} \cdot (\nabla^0, \nabla, \ldots, \nabla^q) \exp(s_2 \frac{t^J X_J}{J!}) \bigg|_{t=0} \]

\[= \text{Exp}(s_1 \frac{t^J X_J}{J!}) \text{Exp}(s_2 \frac{t^J X_J}{J!}). \]

With respect to the special chart the derivative of \( \text{Exp}(s \frac{t^J X_J}{J!}) \) at \( s = 0 \) is indeed

\[ \frac{t^J X_J}{J!}. \]

Hence \( \text{Exp} \) is the exponential mapping for \( T^q(\mathcal{G}) \)

Comments 2.5.11(b)

Proposition (2.5.4) leads us now to an important and interesting result, namely that for any Lie group \( \mathcal{G} \), there is a sequence of nilpotent Lie groups \( \mathcal{G}_n \) which have

140
as limit the group of all analytic curves in $\mathcal{G}$ with endpoint the identity. Further, these groups have a very simple manifold structure and their product is given in terms of the Lie algebra bracket of $g$. Since a neighbourhood of the identity of $\mathcal{G}$ can be re-constructed from this group of curves, we have a sequence of nilpotent Lie groups which recovers in the limit, a neighbourhood of the identity of $\mathcal{G}$. As all the representations are known for nilpotent Lie groups and are relatively simple in structure, this raises the compelling question of whether new, concrete representations of say, a semi-simple Lie group $\mathcal{G}$ could be obtained as limits of the representations of the $\mathcal{G}_n$. Indeed, it is possible that several, unexpected series of representations of totally arbitrary Lie groups may be found in this way. We hope to address this question in future publications.

Herewith the heralded proposition:

**Proposition 2.5.12**

The sets $\mathcal{G}^J = \Pi_{j=1}^J g_j$, with product given by

$$X \cdot Y = (F_1(X,Y), \ldots, F_J(X,Y)), \quad X, Y \in \mathcal{G}^J, \quad (2.5.13)$$

(in the notation of (2.5.4)) are nilpotent Lie groups. The set $\mathcal{G}^{\infty}$,

$$\mathcal{G}^{\infty} = \{X \mid X \in \prod_{j=0}^{\infty} g_j, \sum_{j=1}^{\infty} \frac{t^j X_j}{j!} < \infty, \forall t \in [0,1]\}$$

can be equipped with a product, $X \cdot Y = (F_1(X,Y), F_2(X,Y), F_3(X,Y), \ldots)$ which is well-defined, and $\mathcal{G}^{\infty}$ is a group with respect to this product. With metric $\rho$ given by

$$\rho(X,Y) = \sup_{t \in [0,1]} \left| \sum_{j=1}^{\infty} \frac{t^j X_j}{j!} - \sum_{j=1}^{\infty} \frac{t^j Y_j}{j!} \right|,$$

$\mathcal{G}^{\infty}$ is the limit of the $\mathcal{G}^J$'s in the sense that for any $Y \in \mathcal{G}^{\infty} \exists \{X_J\}$, $X_J \in \mathcal{G}^J$ such that $\lim_{J \to \infty} \rho(X_J, Y) = 0$. 141
There is a neighbourhood $N_1$ of 0 in $\mathcal{G}$ for which the set

$$\mathcal{G}^{\infty}(N_1) = \{X \mid X \in \mathcal{G}^{\infty}, \, X[0,1] \subset N_1, \text{ where } X(t) = \sum_{j=1}^{\infty} \frac{t^j X_j}{j!}\}$$

is a local topological group. Further, there is a topology on the set $A^{\infty}(N_1)$ of all analytic maps $g : [0,1] \to \mathcal{G}$ with $g(0) = e$, and $g[0,1] \subset \exp(N_1)$ with pointwise product, such that the map,

$$\mathcal{G}^{\infty}(N_1) \to A^{\infty}(N_1),$$

$$X \mapsto \exp \left( \sum_{i=1}^{\infty} \frac{X_i t^i}{i!} \right)$$

is a bi-continuous isomorphism of local topological groups.

**Proof:**

By taking $g_1 = g_2 = e$ in (2.5.5), we see that $\mathcal{G}^J$ is a group. Topologically $\mathcal{G}^J$ is just $(\mathcal{G})^J \simeq \mathbb{R}^{nJ}$. The product, being given in terms of the functions $F_j$ which are linear combinations of brackets of the variables $(X_1, \ldots, X_J, Y_1, \ldots, Y_J)$ to a level of nesting up to the $j$'th order, must be analytic on $(\mathcal{G})^J$. Hence the $\mathcal{G}^J$ are Lie groups.

By (2.5.4), $\mathcal{G}^J$ is $C^{\infty}$ isomorphic to the analytic subgroup $(T_e)^J$ of $T^q(\mathcal{G})$. The Lie algebra of $(T_e)^J$ is spanned by the polynomials in $t$, to order $t^J$ in $g$ which vanish at $t = 0$, as can be seen from Proposition (2.5.8), and this Lie algebra is clearly nilpotent. Since $\mathcal{G}^J$ is connected, it is nilpotent as claimed. Let $X, Y \in \mathcal{G}^{\infty}$. Since $F$ is analytic on $g \times g$ at $(0,0)$, $F\left( \sum_{j=1}^{\infty} \frac{t^j X_j}{j!}, \sum_{k=1}^{\infty} \frac{t^k Y_k}{k!} \right)$ is analytic in $t$ for $t$ small enough, and $F_j(X,Y)$ exists for all $j$. Hence the product is well-defined on
Let
\[ g(t) = \exp(tX_1 + \frac{t^2X_2}{2!} + \cdots) \]
\[ h(t) = \exp(tY_1 + \frac{t^2X_2}{2!} + \cdots), \]
where,
\[ \sum_{j=1}^{\infty} \frac{(X \cdot Y)^j t^j}{j!} = \log(g(t)h(t)) \]

for \( t \) small enough.

Exploiting (2.5.16), \( G^\infty \) is closed under products and inverses, and is associative and hence a group. To see that \( G^J \to G^\infty \), let \( Y \in G^\infty \) and define \( X_j = (Y_1, Y_2, \ldots, Y_j, 0, 0, \ldots) \). Then \( \lim_{j \to \infty} \rho(X_j, Y) = 0 \), immediately.

\( G^\infty(N_1) \) is a local topological group:

Let \( N_0 \) be as in (1.1.33). Choose a connected neighbourhood \( N_1 \) of 0 and a compact subset \( B_0 \) containing 0 such that
\[ N_1 \subset B_0 \subset N_0. \]

For \( X(t), Y(t) \) analytic in \( t \in [0, 1] \), let \( N_\delta(X) \) denote the open subset of the metric space, \( G^\infty \)
\[ N_\delta(X) = \{ Y \mid \rho(Y, X) < \delta \}. \]

Given \( N_\varepsilon(X \cdot Y) \) we must find \( N_{\delta_1}(X), N_{\delta_2}(Y) \) such that
\[ N_{\delta_1}(X) \cap G^\infty(N_1) \cdot N_{\delta_2}(Y) \cap G^\infty(N_1) \subset N_\varepsilon(X \cdot Y) \cap G^\infty(N_1). \]

Let \( W \in N_{\delta_1}(X) \cap G^\infty(N_1) \), and \( Z \in N_{\delta_2}(Y) \cap G^\infty(N_1) \), then
\[ \sup_{t \in [0, 1]} |F(W(t), Z(t)) - F(X(t), Y(t))| < \varepsilon \]
for \( \delta_1, \delta_2 \) small enough. \( F \) is uniformly continuous on \( B_0 \), hence \( \delta_1, \delta_2 \) can be chosen so that
\[ |F(W(t), Z(t)) - F(X(t), Y(t))| < \varepsilon \quad \forall t \in [0, 1] \]
establishing the desired condition.

Secondly, we must have \( W.Z[0,1] \subset \mathcal{N}_1 \). The continuity of the product on \( \mathcal{G} \), and (2.5.16) provide that \( \delta_1, \delta_2 \) can be chosen small enough to ensure this.

To show that

\[
(N_{\delta}(X) \cap \mathcal{G}^\infty(\mathcal{N}_1))^{-1} \subset N_{\epsilon}(X^{-1}) \cap \mathcal{G}^\infty(\mathcal{N}_1),
\]

the proof involves the same steps as above. Hence \( \mathcal{G}^\infty(\mathcal{N}_1) \) is a local topological group.

\( \mathcal{A}^\infty(\mathcal{N}_1) \) is a local topological group:

Let \( \mu \) be the Riemannian metric on \( \mathcal{G} \) as in 1.4.5(a). Define a distance function \( \rho_1 \) for two analytic curves \( h, g : [0,1] \rightarrow \mathcal{G} \)

\[
\rho_1(g,h) = \sup_{t \in [0,1]} \mu(g(t),h(t)).
\]

For \( g[0,1], h[0,1] \subset \exp(N_0) \), \( \rho_1 \) is just \( \rho \) when expressed in exponential coordinates (1.4.11(a)). The proof that \( \mathcal{A}^\infty(\mathcal{N}_1) \) is a local topological group is formally similar to that for \( \mathcal{G}^\infty(\mathcal{N}_1) \), exploiting the uniform continuity of the product on \( \exp(B_0) \).

Because \( \rho_1 \) expressed in coordinates is just \( \rho \) (the minimum distance (geodesic) between two points in \( \mathcal{G} \) is just the straight line), the map

\[
\mathcal{G}^\infty(\mathcal{N}_1) \rightarrow \mathcal{A}^\infty(\mathcal{N}_1), \ X \mapsto \exp \left( \sum_{i=1}^{\infty} \frac{X_i t^i}{i!} \right)
\]

is an isometry. The homomorphism property follows from (2.5.16).

We are now in a position to state the global version of Theorem (2.4.17) modulo three preliminary definitions and two lemmas. This theorem gives us global, necessary and sufficient conditions for contraction to take place in a group-theoretic context.
framework, thus achieving Saletan's programme, now at a global level, of finding these neat and simple necessary and sufficient conditions which he postulated should exist. Definition (2.3.2) and examples such as (2.3.8) and §3.1 show that these conditions hold with a surprising amount of generality. In fact, it has not been at all clear until the present work, what form these global conditions should take (except for the results of Cazzaniga (1985) for the special case described in the introduction to §2.2).

The global version of Definition (2.4.12) is as follows:

Definition 2.5.17

Recall the convention, adopted at Definition (2.4.12) of writing \( \Phi \) and \( \Psi \) with their first arguments only, the second argument being implied, where this won't cause confusion.

The contracting map \( \Phi \) induces a map

\[
\Phi_s : \mathcal{D}^q \mapsto S^q
\]
given by

\[
\Phi_s(g)(t) = \begin{cases} 
\Phi(g(t)), & t \in (0, 1] \\
\lim_{t \to 0} \Phi(g(t)), & t = 0
\end{cases}
\]

where the limit is taken in \( E \times [0, 1] \).

The following definition is the global analogue of Definition (2.4.13).

Definition 2.5.18

Define the product, where it exists, of any two elements \( s_1, s_2 \) of \( S^q \) to be,

\[
s_1(t)s_2(t) = \Phi(\Phi^{-1}(s_1(t))\Phi^{-1}(s_2(t))), \quad t \in (0, 1]
\]

\[
s_1(0)s_2(0) = \lim_{t \to 0} \text{E}_{\times [0, 1]}(\Phi(\Phi^{-1}(s_1(t)))\Phi^{-1}(s_2(t))), \quad t = 0
\]
In Lemma (2.4.10) we defined the “projection” map \( dP : C_g^q[0,1] \to \mathcal{P}^q(g) \). We now want to define an integral of this map, in order to state the global version of (2.4.17(1)).

**Definition 2.5.19**

Let \( P \) be the map,

\[
P : C_g^q[0,1] \to T^q(G)
\]

\[
P : g \mapsto (\nabla^0, \nabla, \ldots, \nabla^q)g(0)
\]

**Remark 2.5.20**

The map \( P \) is a natural one in view of the Lie structure on \( T^q(G) \) given in Proposition (2.5.3). In canonical coordinates at \( e \), for an analytic curve \( g \), \( P \) is the map

\[
\sum_{i=0}^{\infty} \frac{X_i t^i}{i!} \mapsto (X_0, X_1, \ldots, X_q),
\]

so that it is intuitively clear that \( dP \) is the derivative of the Banach-Lie homomorphism (1.2.9), \( P : C_g^q[0,1] \to T^q(G) \). This will be proved below, at Proposition (2.5.65), after a Banach-Lie structure for \( C_g^q[0,1] \) has been defined. We now need the global analogue of Lemma (2.4.10).

**Lemma 2.5.21**

The map

\[
P : C_g^q[0,1] \to T^q(G)
\]

\[
P : g \mapsto (\nabla^0, \nabla, \ldots, \nabla^q)g(0)
\]

is a homomorphism.

**Proof:** Immediate from the proof of (2.5.3).

We have the following global analogue of Lemma 2.4.10, which is necessary to re-render Theorem (2.4.17).
Lemma 2.5.22

There is a map \( \mathcal{X}_P : P(\mathcal{D}^q) \to \mathcal{G}_0 \) such that \( \mathcal{X} = \mathcal{X}_P \circ P \)

Proof:

Define \( \mathcal{X}_P \) by,

\[
\mathcal{X}_P(P(g)) = \mathcal{X}(g)
\]

(2.5.23)

We need only show that \( \mathcal{X}_P \) is well-defined. This will follow if we can show that \( P(g) = P(h) \) implies \( \mathcal{X}(g) = \mathcal{X}(h) \) for all \( g, h \in \mathcal{D}^q \):

Now \( P(g) = P(h) \) implies \( g(0) = h(0) \), so letting \( \{\chi, x^i\} \) be a normal chart at \( g(0) \),

\[
\Phi \circ \chi^{-1}(x(t)) = (t^q \phi(\chi^{-1}(x(t)), t), t).
\]

Therefore

\[
\lim_{t\to 0} \Phi \circ \chi^{-1}(x(t)) = \left( \frac{1}{q!} \frac{d^q}{dt^q} \phi(\chi^{-1}(x(t)), t) \bigg|_{t=0}, 0 \right),
\]

for \( g \in \mathcal{D}^q \) with \( \chi(g(t)) = x(t) \).

And so \( \mathcal{X}(g) \) depends only \( (\nabla, \ldots, \nabla^q)g(0) \) and \( g(0) \).

Since \( P(g) = P(h) \) implies \( (\nabla, \ldots, \nabla^q)g(0) = (\nabla, \ldots, \nabla^q)h(0) \) this gives \( \mathcal{X}(g) = \mathcal{X}(h) \) as required.

A comparison with Theorem (2.4.17) readily shows that the following theorem is indeed its global analogue.

Theorem 2.5.24

The following statements are equivalent:

1. \( P(\mathcal{D}^q) \) is a subgroup of \( T^q(\mathcal{G}) \) and \( \ker \mathcal{X}_P \) is a normal subgroup of \( P(\mathcal{D}^q) \).
2. \( \mathcal{D}^q \) is a subgroup of the group \( C^q_{[0,1]} \) and \( \ker \mathcal{X} \) is a normal subgroup of \( \mathcal{D}^q \).
3. \( \mathcal{S}^q \) is a group and the set, \( \mathcal{S}_{\ker} = \{ s \mid s \in \mathcal{S}^q, s(0) = e_s(0) \} \) is a normal subgroup of \( \mathcal{S}^q \) (where the curve \( e_s(t) \) is the identity of \( \mathcal{S}^q \)).
(4) $G_0$ is a group, with product induced per (2.5.17) unambiguously as in Definition (2.5.18) from any two curves in $S^q$ and with inverse induced per (2.5.17) from the inverse of any curve in $S^q$.

Remark 2.5.25

See also Appendix A.2 for a neater version of this theorem using results which appear later in the thesis. Note that $u \in \ker X_P$ if and only if $X_P(u) = X_P(1)$. Similarly, $g \in \ker X$ if and only if $X(g) = X(e)$. We will see in §3.1 that in the case of Inönü-Wigner and generalised Inönü-Wigner contractions, mild conditions on $\Phi$ fulfil (1) completely. We have already seen a large class of similar examples in Proposition (2.3.9(a)).

Proof of Theorem (2.5.24): We will prove $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$

Proof of $(1) \Rightarrow (2)$:

Now $P$ is homomorphism of $C^q_0[0,1]$ onto $T^q(G)$. Therefore $P^{-1}(P(D^q))$ is a subgroup of $C^q_0[0,1]$. We must show $P^{-1}((D^q)) = D^q$. Now, certainly $D^q \subset P^{-1}(P(D^q))$, so we must show $D^q \supset P^{-1}(P(D^q))$. Let $g \in P^{-1}(P(D^q))$ whereby $\exists h \in D^q$ such that $P(g) = P(h)$. Since $P$ is a homomorphism $\exists k \in \ker P$ such that $g = hk$. We now show that for $h \in D^q$, $k \in \ker P$, $hk \in D^q$. Choose two normal coordinate charts $(\chi_1, x), (\chi_2, y)$ at $h(0)$ and $k(0)$ respectively. Now $h \in D^q$ if and only if

$$\frac{dJ}{dt} \phi(\chi_1^{-1}(x(t), t))|_{t=0} = 0, \quad J = 0, \ldots, q-1. \quad (2.5.26)$$

Let the product on $G$ in coordinates be given by $\zeta$. So we must show

$$\frac{d^J}{dt^J} \phi(\zeta(x(t), y(t)), t)|_{t=0} = 0, \quad (2.5.27)$$

(where $\chi$ is a chart at $h(0)k(0)$).

The above expression is a multinomial in the variables $\frac{d^Kx(0)}{dt^K}, \frac{d^Ly(0)}{dt^L}$ with
coefficients being $C^\infty$ functions of the variables $x(0), y(0)$. As $k \in \ker P$, \( \frac{d^Jy(0)}{dt^J} = 0, \ J = 0, \ldots, q \)

Therefore

\[
\frac{d^J}{dt^J} \phi(e^{-1}(\zeta(x(t), y(t))), t) \big|_{t=0} = \frac{d^J}{dt^J} \phi(e^{-1}(\zeta(x(t), y(0))), t) \big|_{t=0} = \frac{d^J}{dt^J} \phi(e^{-1}(x(t))) \big|_{t=0} = 0 \quad \text{by (2.5.26)};
\]

hence $g = hk \in D^q$ and $D^q$ is a subgroup of $C_0^q[0, 1]$.

Now suppose that $\ker \mathcal{X}_P$ is a normal subgroup of $P(D^q)$. $P^{-1}(\ker \mathcal{X}_P)$ is a normal subgroup of $D^q$ and $\ker \mathcal{X} \subset P^{-1}(\ker \mathcal{X}_P)$ and we are left with showing that $\ker \mathcal{X} \supset P^{-1}(\ker \mathcal{X}_P)$. As before, we only need show that for $k \in \ker \mathcal{X}$ and $p \in \ker P$, $kp \in \ker \mathcal{X}$. Take two normal coordinate charts \{\( \chi_1, x \)\}, \{\( \chi_2, y \)\} at $k(0)$ and $p(0)$ respectively.

We must show that

\[
\frac{d^q}{dt^q} \phi(e^{-1}(\zeta(x(t), y(t))), t) \big|_{t=0} = 0
\]

(where $\mathcal{X}$ is a chart at $k(0)p(0)$) given that \( \frac{d^J}{dt^J}y(0) = 0, \ J = 0, \ldots, q \)

and

\[
\frac{d^q}{dt^q} \phi(e^{-1}(x(t)), t) \big|_{t=0} = 0.
\]

This is exactly analogous to the proof of (2.5.27).

**Proof of (2) $\Rightarrow$ (3):**

Let $s_1, s_2 \in S^q$ with $\Phi_s^{-1}(s_1) = g_1$ and $\Phi_s^{-1}(s_2) = g_2$. By (2.5.18),

\[
\Phi(g_1(t)g_2(t)) = s_1(t)s_2(t) \quad \text{for } t \in (0, 1],
\]

149
and since \( g_1g_2 \in \mathcal{D}' \), \( s_1(0)s_2(0) \) is defined and agrees with (2.5.18). Hence the product is defined on \( \mathcal{S}' \). The identity is \( e_s(t) = \Phi_s(e)(t) \), and if \( s = \Phi_s(g) \), the inverse of \( s \) is \( \Phi_s(g^{-1}) \). The product is associative, because we can take the limit \( t \to 0 \) in the equality

\[
(s_1(t)s_2(t))s_3(t) = s_1(t)(s_2(t)s_3(t)), \; t \in (0,1].
\]

Therefore \( \mathcal{S}' \) is a group.

We now show that \( \mathcal{S}_{\text{ker}} \) is a normal subgroup of \( \mathcal{D}' \). Let \( s \in \mathcal{S}' \) and \( s_1 \in \mathcal{S}_{\text{ker}} \). We must show that \( (ss_1s^{-1})(0) = e_s(0) \). Now for \( t \in (0,1] \)

\[
(ss_1s^{-1})(t) = \Phi((gg_1g^{-1})(t)) \text{ where } \Phi_s(g) = s \text{ and } \Phi_s(g_1) = s_1. \text{ Since } s_1(0) = e_s(0) = \mathcal{X}(e) = \mathcal{X}(g_1) \text{ then } g_1 \in \text{ker}\mathcal{X}. \text{ By assumption, there is a } g_2 \in \text{ker}\mathcal{X} \text{ such that } gg_1g^{-1} = g_2. \text{ Therefore } \Phi_s(gg_1g^{-1})(0) = \Phi_s(g_2)(0) = e_s(0) \text{ and so } ss_1s^{-1} \in \mathcal{S}_{\text{ker}} \text{ and } \mathcal{S}_{\text{ker}} \text{ is a normal subgroup.}
\]

**Proof of (3) \( \Rightarrow \) (4):**

We must first show that there is a well-defined product on \( \mathcal{G}_0 \). Let \( g_1, g_2 \in \mathcal{G}_0 \).

There must be \( s_1, s_2 \in \mathcal{S}' \) with \( s_1(0) = g_1, s_2(0) = g_2 \). Define

\[
g_1g_2 = s_1(0)s_2(0) \quad (2.5.28)
\]

We must show that if \( s_3(0) = g_1, s_4(0) = g_2 \) then \( s_1(0)s_2(0) = s_3(0)s_4(0) \), where \( s_1(0)s_2(0) \) is defined in (2.5.18).

Now if \( s_1(0) = s_3(0) \), and \( s_2(0) = s_4(0) \), \( \exists s_5, s_6 \in \mathcal{S}_{\text{ker}} \) such that

\[
s_1 = s_3s_5, \; s_2 = s_4s_6
\]

Therefore

\[
s_1s_2 = s_3s_5s_4s_6
\]

\[
= s_3s_4(s_4^{-1}s_5s_4)s_6
\]

\[
= s_3s_4(s_7s_6), \; s_7 \in \mathcal{S}_{\text{ker}}
\]

whence \( s_1(0)s_2(0) = s_3(0)s_4(0) \) because \( s_7s_6 \in \mathcal{S}_{\text{ker}} \).
So the product is well-defined. Define the identity on $G_0$ to be $e_s(0)$. If $g_0 = s(0), s \in S^q$, define $g_0^{-1} = (s^{-1}(t))_{t=0}$. Clearly $g_0 g_0^{-1} = g_0^{-1} g_0 = e_s(0)$ and the product is associative because the product on $S^q$ is associative. Hence $G_0$ is an abstract group.

**Proof of (4) $\implies$ (1):**

We will show that $D^q$ is a subgroup of $C_0^q[0,1]$ and $\ker \Phi$ is a normal subgroup of $D^q$, then use the fact that $P$ is a homomorphism.

Let $g, h \in D^q$. In order to show that $gh \in D^q$ it is sufficient to show that

$$\lim_{t \to 0} \Phi(g(t)h(t))$$

exists.

Let $s_1 = \Phi_s(g), s_2 = \Phi_s(h)$; then $s_1(0)s_2(0)$ is defined and

$$s_1(0)s_2(0) = \lim_{t \to 0} \Phi(\Phi^{-1}(s_1(t))\Phi^{-1}(s_2(t))) \text{ by (2.5.18)},$$

$$= \lim_{t \to 0} \Phi(g(t)h(t)).$$

Hence the limit exists and $gh \in D^q$. Independently of whether $G_0$ is a group or not, $D^q$ always has an identity by (2.3.2).

Let $g \in D^q$. Then $X(g) = g_0 \in G_0$ and $g_0^{-1} = s^{-1}(0)$, where $s(0) = g_0$ and $s^{-1} \in S^q$. By (2.5.18), $s^{-1}(t) = \Phi_s(g^{-1})(t)$ whence $g^{-1} \in D^q$. Then $D^q$ is a subgroup of $C_0^q[0,1]$. Let $k \in \ker X$ and $g \in D^q$ and $s_1 = \Phi_s(k), s_2 = \Phi_s(g)$.

Then

$$\Phi_s(g^{-1}kg)(0) = s_2^{-1}(0)s_1(0)s_2(0) = s_1(0) = e_0$$

giving $X(g^{-1}kg) = e_0$ whence $g^{-1}kg \in \ker X$ and $\ker X$ is a normal subgroup of $D^q$. (1) now follows by using the fact that $P$ is a homomorphism.

With one further definition, we are now able to state the global generalisation of Theorem (2.4.19)
Definition 2.5.29

Let \( \epsilon \) be the map

\[
\epsilon : S^q \to G_0, \\
\epsilon : s \mapsto s(0).
\]

Theorem 2.5.30 (see §1.2 for definitions)

If any of the conditions of Theorem (2.5.24) hold then:

- \( D^q \) is a Banach-Lie group and a Banach-Lie subgroup (1.2.8) of the Banach-Lie group \( C^q_0[0,1] \).
- \( P(D^q) \) is a closed, analytic subgroup of \( T^q(G) \);
- \( G_0 \) is a Lie group with manifold structure inherited from \( P(D^q) \) by \( \mathfrak{X}_P \);
- \( S^q \) is a Banach-Lie group;
- \( \mathfrak{X}, \Phi_s, P \) and \( \epsilon \) are \( C^\infty \)-Fréchet homomorphisms;
- \( \mathfrak{X}_P \) is an analytic homomorphism;
- and the following diagram commutes:

\[
\begin{array}{ccc}
S^q & \xrightarrow{\epsilon} & G_0 \\
\Phi_s \downarrow & & \downarrow \mathfrak{X}_P \\
D^q & \xrightarrow{\mathfrak{X}} & P(D^q) \\
P \downarrow & & \downarrow \mathfrak{X}_P \\
\end{array}
\]

About the Proof of theorem (2.5.30)

We will prove Theorem (2.5.30) by proving a number of lemmas and propositions first, followed by a proof which draws all these results together.

Our first objective is to prove that \( C^q_0[0,1] \) is a Banach-Lie group and \( D^q \) is a Banach-Lie subgroup under the conditions of Theorem (2.5.24). The product on \( D^q \) as given in (2.3.9(a)) and (2.5.24) is straightforward but the Banach-Lie
manifold structure for $D^q$ appears obscure. This is because the conditions for a curve $g(t)$ to be in $D^q$ are complicated and highly non-linear. Letting $(x^i)$ be a chart at $g(0)$ and the embedding map $\phi$ of Definition (2.3.2) be written in coordinates as $\phi(x^i, t)$, the necessary and sufficient condition for $g$ to be in $D^q$ is

$$\frac{d^J}{dt^J} \phi(x^i(t), t)|_{t=0} = 0, \quad J = 0, \ldots, q - 1.$$  

(2.5.31)

These conditions are $C^\infty$-functions of $x^i(0)$, and multinomials in the $(x^i(t))^{(J)}|_{t=0}$ variables.

We overcome this problem by finding an isomorphism $I^q$ of $C^q_0[0,1]$ into a new group of curves on $T^q(G)$. By contrast to $D^q$, $I^q(D^q)$ has a simple structure and we show it to be a Banach-Lie group. We also show that $C^q_0[0,1]$ and $I^q(C^q_0[0,1])$ are Banach-Lie groups and $I^q$ a Banach-Lie isomorphism, which allows us to establish that $D^q$ is a Banach-Lie group.

We remark at this point that all applications of Lie group contractions as developed in this work can be pursued without ever using the facts that $D^q$ is a Banach-Lie group and $X$ a Banach-Lie homomorphism. The connection of $q^q$ to Current Algebras however (see (2.4.22)) and the importance of the Banach-Lie groups whose Lie algebras are Current Algebras, to Quantum Field Theory (Dolan, 1984), impel us to verify by necessarily lengthy means that these properties are indeed true.

We will require a slightly more general map than $I^q$ in order to prove that $C^q_0[0,1]$ is also a Banach-Lie group.

**Definition 2.5.32**

Let $r$ be any integer, $r \geq q$ and let $I^{r-q}$ be the map

$$I^{r-q} : C^r_0[0,1] \to C^{r-q}_{T^q(G)}[0,1]$$

$$I^{r-q} : g(t) \mapsto (\nabla^0, \nabla^1, \ldots, \nabla^q)g(t).$$

153
Thus $I^{r,q}(g)$ is the curve which is the $q$-tuple of vectors $(V^0, V, \ldots, V^q)g(t)$ at $g(t)$ for each $t \in [0,1]$.

The following Lemma illustrates the simple structure of $I^{r,q}(D^q)$. The product for the groups of curves $C^r_g[0,1]$ and $C^{r-q}_{T^q(g)}$ is the pointwise product.

**Lemma 2.5.33**

The map $I^{r,q} : C^r_g[0,1] \rightarrow C^{r-q}_{T^q(g)}$ given by (2.5.32) is an injective homomorphism, and the set $I^{r,q}(D^q)$ is given by

$$I^{r,q}(D^q) = \left\{ t \mapsto (V^0, V, \ldots, V^q)g(t) \middle| g \in C^r_g[0,1], (\nabla^0, \nabla, \ldots, \nabla^q)g(0) \in P(D^q) \right\}$$

(2.5.34)

**Proof:**

Let $g, h \in C^r_g[0,1]$. We wish to show that

$$I^{r,q}(g)(t)I^{r,q}(h)(t) = (V^0, V, \ldots, V^q)(g(t)h(t))$$

$$= I^{r,q}(gh)(t).$$

To see this, choose curves $k_t, \ell_t \in C^r_g[0,1-t]$ for each $t$ in $[0,1]$ with $k_t(s) = g(s + t)$, $\ell_t(s) = h(s + t)$. Clearly each $k_t, \ell_t$ can be extended to a $C^r_g[0,1]$ curve and letting $\nabla_s$ be covariant differentiation along a curve with parameter $s$, we have, by definition of the product on $T^q(G)$, for each $t \in [0,1]$,

$$I^{r,q}(g)(t)I^{r,q}(h)(t) = (\nabla^0_s, \nabla_s, \ldots, \nabla^q_s)(k_t(s)\ell_t(s))\big|_{s=0}$$

$$= (\nabla^0_s, \nabla_s, \ldots, \nabla^q_s)(g(s + t)h(s + t))\big|_{s=0}$$

$$= (\nabla^0_t, \nabla_t, \ldots, \nabla^q_t)(g(t)h(t))$$

$$= I^{r,q}(gh)(t).$$

Hence $I^{r,q}$ is a homomorphism. $I^{r,q}$ is 1-1 since $I^{r,q}(g) = I^{r,q}(h)$ means $(V^0, V, \ldots, V^q)g(t) = (V^0, V, \ldots, V^q)h(t)$ for $t \in [0,1]$ which implies $g(t) = h(t)$.
To prove (2.5.34), let \( g \in \mathcal{D}^q \), then \( P(g) \in P(\mathcal{D}^q) \), that is \( (\nabla^0, \nabla, \ldots, \nabla^q)g(0) \in P(\mathcal{D}^q) \).

Conversely, the supposition \( g \in C^q_{\mathcal{D}^q}[0,1] \) with \( (\nabla^0, \nabla, \ldots, \nabla^q)g(0) \in P(\mathcal{D}^q) \) implies \( g \in \mathcal{D}^q \) since \( \ker P \subset \mathcal{D}^q \). This establishes (2.5.34) and completes the proof. \[\square\]

We are now in a position to prove a theorem which will enable us to show that \( C^q_{\mathcal{D}^q}[0,1], T^q:q(C^q_{\mathcal{D}^q}[0,1]) \text{ and } T^q:q(\mathcal{D}^q) \) are Banach-Lie groups. We first establish some notation and a pivotal lemma.

**Notation 2.5.35**

Let \( \Sigma \) be any analytic subgroup of \( T^q(G) \), and let \( C^r_{\mathcal{T};\Sigma} \) denote the set,

\[
C^r_{\mathcal{T};\Sigma} = \{ u \mid u \in T^r(q(C^q_{\mathcal{D}^q}[0,1])), u(0) \in \Sigma \}
\]

**Remark 2.5.36**

\( \Sigma \) will play the rôle of \( P(\mathcal{D}^q) \) as given in (2.5.34) in establishing that \( T^q:q(\mathcal{D}^q) \) is a Banach-Lie group.

We are now going to show that \( C^r_{\mathcal{T};\Sigma} \) is a Banach-Lie group, but first we will need a Banach-Lie algebra (see 1.2.17)) as Lie algebra of \( C^r_{\mathcal{T};\Sigma} \).

**Notation 2.5.37**

From Proposition (2.5.8), all the subspaces \( \xi_J = \text{span}\{X_J \mid j = 1,\ldots,n\} \) are vector space isomorphic to \( \mathfrak{g} \) via the maps \( \xi_J : X_J \mapsto X_0J \). Choose a norm \( |\cdot|^' \) on the Lie algebra of \( T^q(G) \) such that \( |X|^' = |\xi_J(X)|' \). This choice will make the norm defined below on \( C^r_{\mathcal{D};\mathcal{T};\Sigma} \) a standard one. We identify \( \xi_0 \) with \( \mathfrak{g} \).
Lemma 2.5.38

Let $t^q(g)$ be the Lie algebra of $T^q(G)$ and $\mathfrak{g}$ be the Lie algebra of $\Sigma$. Let $V_J$ denote the projection of $V \in t^q(g)$ onto the subspace $t_J$. The vector space given by

$$C^{r,q}_{d^I,\mathfrak{g}} = \left\{ V \left| \begin{array}{l}
V \in C^{r,q}_{t^q(g)}[0,1] \\
V_0 \in C^r_\mathfrak{g}[0,1] \\
V_J(t) = \frac{1}{J!} \frac{d^J}{dt^J} V_0(t) \\
V(0) \in \mathfrak{g}
\end{array} \right. \right\}$$

with norm

$$\|V\| = \sum_{J=0}^{r} \sup_{t \in [0,1]} \left| \frac{d^J}{dt^J} V_0(t) \right|^J$$

(2.5.39)

is a Banach space and a Lie algebra with the bracket taken pointwise. The bracket is continuous with respect to the Banach space norm.

Proof:

The function (2.5.39) is a standard norm. Let $\{V_n\}$ be a Cauchy sequence. Then by the properties of this norm, there is a $V_0 \in C^r_\mathfrak{g}[0,1]$ such that,

$$\lim_{n \to \infty} \sum_{J=0}^{r} \sup_{t \in [0,1]} \left| \frac{d^J}{dt^J} V_n(t) - \frac{d^J}{dt^J} V_0(t) \right|^J = 0.$$ 

(2.5.40)

Letting $V_J(t) = \frac{1}{J!} \frac{d^J}{dt^J} V_0(t)$ we will have found the required limit $V$ of the sequence $\{V_n\}$ if $V(0) \in \mathfrak{g}$. This follows immediately from (2.5.40). Hence the normed vector space is complete.

Let $V, W \in C^{r,q}_{d^I,\mathfrak{g}}$. We have $V = \sum_{J=0}^{q} V_J$, $W = \sum_{K=0}^{q} W_K$. 

156
Now

\[ [V(t), W(t)]_I = \left[ \sum_J V_J(t), \sum_K W_K(t) \right]_I \text{ (projection onto } \mathfrak{t}_I) \]

\[ = \sum_{K=0}^{I} [V(I-K)(t), W_K(t)] \]

\[ = \sum_{K=0}^{I} \frac{1}{(I-K)!} \frac{1}{K!} \left[ \frac{d^{(I-K)}}{dt^{(I-K)}} V_0(t), \frac{d^K}{dt^K} W_0(t) \right] \]

\[ = \frac{1}{I!} \frac{d^I}{dt^I} [V_0(t), W_0(t)] \text{ on using (2.5.11(a))} \]

Since \([V(0), W(0)] \in \mathfrak{z}, \ [V, W] \in C^{r,q}_{dI;\mathfrak{z}}\) and hence \(C^{r,q}_{dI;\mathfrak{z}}\) is a Lie algebra. On \(C^{r,q}_{dI;\mathfrak{z}} \times C^{r,q}_{dI;\mathfrak{z}}\) take the natural norm,

\[ ||(V, W)||_1 = ||V|| + ||W|| \quad (2.5.41) \]

Since the bracket \([U, V]\) is bilinear; by Lang (1962) it will be continuous if it is bounded. The boundedness follows on writing

\[ ||[V, W]||_1 = \sum_{J=0}^{r} \sup_{t \in [0,1]} \left| \sum_{I=0}^{J} \binom{J}{I} \frac{d^I V_0(t)}{dt^I}, \frac{d^{(J-I)} W_0(t)}{dt^{(J-I)}} \right|' \]

\[ = \sum_{J=0}^{r} \sup_{t \in [0,1]} \left| \sum_{I=0}^{J} \binom{J}{I} C_{ij}^{k} \frac{d^I V_{ij}(t)}{dt^I}, \frac{d^{(J-I)} W_{ij}(t)}{dt^{(J-I)}} X_{ik} \right|' \]

and using standard inequalities to obtain \( ||[V, W]||_1 \leq M(||V|| + ||W||) \) for some \(M \geq 0\). This completes the proof.

Proposition 2.5.42

Using the criteria of (1.2.6), \(C^{r,q}_{dI;\Sigma}\) is a Banach-Lie group with Lie algebra \(C^{r,q}_{dI;\mathfrak{z}}\), and topology induced from the exponential map,

\[ \exp_q(V)(t) = \text{Exp}(V(t)) \quad (2.5.43) \]
where \( \text{Exp} \) is the exponential mapping for \( T^q(G) \).

**Proof:**

We first show that \( C^{r,q}_{T;\Sigma} \) is a group.

From Lemma (2.5.33) \( T^q(C^r_{G}[0,1]) \) is a group. Since \( \Sigma \) is a group, curves which are elements of \( T^q(C^r_{G}[0,1]) \) with endpoints of \( \Sigma \) will form a subgroup. By (2.5.35) \( C^{r,q}_{T;\Sigma} \) is a group.

To verify that \( \text{exp}_q \) is a map of \( C^{r,q}_{dT;\Sigma} \) into \( C^{r,q}_{T;\Sigma} \), evaluate \( \text{Exp} \) with respect to the special chart of Proposition (2.5.8) based on an exponential (1.1.32) chart at \( e \). In these coordinates \( \nabla^K_s \exp \left( \frac{d^j}{dt^j} V_0(t) \right) |_{s=0} \) has the same coordinates as \( \nabla \exp V_0(t) \). Then \( \text{exp}_q \) may be re-expressed,

\[
\text{exp}_q(V)(t) = (\nabla^0, \nabla, \ldots, \nabla^q) \exp V_0(t). \tag{2.5.44}
\]

From (2.5.43) we see that \( \text{exp}_q(V)(0) \in \Sigma \) whence \( \text{exp}_q \) is indeed a map into \( C^{r,q}_{T;\Sigma} \).

Let \( 1 \) be the identity of \( C^{r,q}_{T;\Sigma} \). We will use \( \text{exp}_q \) to provide a coordinate map at \( 1 \).

To construct a chart at \( 1 \), let \( N_0 \) be a neighbourhood of zero in \( t^q(G) \) such that \( (N_0)(N_0)^{-1} \) is contained in the open set where \( \text{Exp} \) is a diffeomorphism and such that \( N_0(N_0)^{-1} \) is contained in the open set on which the Baker-Campbell-Hausdorff formula holds (1.1.33). Define \( N_1 \), the domain of the coordinate map \( \chi_1 \) at \( 1 \) to be

\[
N_1 = \text{exp}_q \{ V \mid V \in C^{r,q}_{dT;\Sigma}, V[0,1] \subset N_0 \}
\]

and define \( \chi_1 \) to be

\[
\chi_1 : N_1 \rightarrow C^{r,q}_{dT;\Sigma},
\]

\[
\chi_1 : u \mapsto \text{exp}_q^{-1}(u)
\]

where

\[
\text{exp}_q^{-1}(u)(t) = \text{Exp}^{-1}(u(t)). \tag{2.5.45}
\]
We now construct charts at arbitrary points $u$ of $C^I_{T;\triangle}$. Let $\ell_u$ be the operation of left translation by an element $u$ in $C^I_{T;\triangle}$. The natural choice for a coordinate neighbourhood $N_u$ at $u$ and coordinate mapping $\chi_u$ is

$$N_u = \ell_u(N_1)$$

$$\chi_u : N_u \to C^I_{d,T;\triangle}$$

$$\chi_u = \exp_q^{-1} \circ \ell_u^{-1}$$

(2.5.46)

Note that the maps $\chi_1$ and $\chi_u$ are $1 - 1$ by (2.5.43) and the group property of $C^I_{T;\triangle}$. To see $\chi_1(N_1) = \chi_u(N_u)$ is open in $C^I_{d,T;\triangle}$, let $V \in \chi_1(N_1) = \chi_u(N_u)$ and note that as $N_0$ is open in $t^q(\triangle)$ and $[0,1]$ is compact, there is an $\epsilon > 0$ such that the open ball in $t^q(\triangle)$ of radius $\epsilon$ at each point $V(t)$ is contained in $N_0$. By (2.5.39) the open ball in $C^I_{d,T;\triangle}$ of radius $\epsilon$ is contained in $\chi_1(N_1) = \chi_u(N_u)$ which is hence open.

We now show that overlapping charts are $C^\infty$-Fréchet related to complete the manifold structure of $C^I_{T;\triangle}$.

Suppose that $N_u$ and $N_v$ overlap. Let $V \in \chi_v^{-1}(N_v)$.

Therefore

$$\chi_u \circ \chi_v^{-1}(V) = \exp_q^{-1} \circ \ell_u^{-1} \circ \ell_v \circ \exp_q(V)$$

$$= \exp_q^{-1} \circ \ell_u^{-1} \circ \exp_q(V).$$

Now $\exists x \ni x \in uN_0$ and $x \in vN_0$;

$\therefore \exists n_1, n_2 \in N_0 \ni un_1 = x = vn_2$

and so $u^{-1}v = n_1n_2^{-1} \in N_0(N_0)^{-1}$ which is contained in the neighbourhood of 1 on which Exp is a diffeomorphism. Then $\exists W_0 \in C^I_{\triangle}[0,1]$ such that

$$\chi_u \circ \chi_v^{-1}(V)(t) = \exp_q^{-1}((\nabla^0,\ldots,\nabla^q) \exp W_0(t)(\nabla^0,\ldots,\nabla^q) \exp V_0(t))$$

$$= \exp_q^{-1}((\nabla^0,\ldots,\nabla^q) \exp W_0(t) \exp V_0(t))$$

$$= (1, \frac{d}{dt}, \frac{1}{2} \frac{d^2}{dt^2}, \ldots, \frac{1}{q!} \frac{d^q}{dt^q}) \exp^{-1}(\exp W_0(t) \exp V_0(t))$$

$$= (1, \frac{d}{dt}, \ldots, \frac{1}{q!} \frac{d^q}{dt^q}) F(W_0(t), V_0(t))$$

(2.5.47)
where $F$ is given by the BCH formula.

The formula (2.5.47) is closely related to the formula for the product with respect to coordinates, and this is useful as the proof that $\chi_{uv}(\chi_u^{-1}(U)\chi_v^{-1}(V))$ is a Fréchet-$C^\infty$ function of $(U, V)$ is very similar to the proof that $\chi_u \circ \chi_v^{-1}(V)$ is a Fréchet-$C^\infty$ function of $V$. Thus it is only necessary to give an outline of the proof of the former assertion. We have,

$$
\chi_{uv}(\chi_u^{-1}(U)\chi_v^{-1}(V))(t) = \exp^{-1}(q(U(t))\exp(V(t)))
$$

where $v_0 \in C^r_{\mathcal{G}}[0,1]$ and $(\nabla^0, \ldots, \nabla^q)v_0(t) = v(t)$

$$
= \exp^{-1}((\nabla^0, \ldots, \nabla^q)\exp(\operatorname{Ad}v_0(t)(U_0(t)))\exp V_0(t))
$$

$$
= (1, \frac{d}{dt}, \ldots, \frac{1}{q!} \frac{d^q}{dt^q})F(\operatorname{Ad}(v_0(t))U_0(t), V_0(t))
$$

(2.5.48)

which is very similar in form to (2.5.47). To show that $\chi_u \circ \chi_v^{-1}$ is $C^\infty$-Fréchet we first show that it is once Fréchet differentiable then proceed by induction.

Let $Q = \chi_u \circ \chi_v^{-1}$ and

$$
Q_j(V)(t) = \frac{1}{J!} \frac{d^J}{dt^J} F(W_0(t), V_0(t))
$$

(2.5.49)

We must show that there is a bounded, linear map $L$ on $C_{\mathcal{G}_{dJ,q}}^{r+q}$ such that

$$
\lim_{\|H\| \to 0} \frac{\|Q(V + H) - Q(V) - L(H)\|}{\|H\|} = 0
$$

(2.5.50)

Writing $D^J \equiv \frac{1}{J!} \frac{d^J}{dt^J}$ and $D \equiv (1, \frac{d}{dt}, \frac{1}{2} \frac{d^2}{dt^2}, \ldots, \frac{1}{q!} \frac{d^q}{dt^q})$ we assert that if

$$
|Q_j(V + H)(t) - Q_j(V)(t) - L_j(H)(t)|' \leq |K_j(DH_0(t))|
$$

(2.5.51)
where \( K_j \) is a multinomial function of lowest order 2 in any of the \( n(q + 1) \) components of \( \mathcal{D}H_0(t) \), then (2.5.50) holds.

To see this, write

\[
\frac{\|Q(V + H) - Q(V) - L(H)\|}{\|H\|} = \frac{\sum_{J=0}^{r} \sup_{t \in [0,1]} |QJ(V + H)(t) - QJ(V) - L_J(H)|'}{\sum_{M=0}^{r} \sup_{t \in [0,1]} |H_M(t)|'}
\]

Now \( K_j \) can be written

\[
K_j(\mathcal{D}H_0(t)) = K_{ji}^j(\mathcal{D}H_0(t))D^JH_0(t)
\]

where the \( K_{ji}^j \) are multinomial functions of \( \mathcal{D}H_0(t) \) of lowest order 1, and the sum on \( I_i \) is implied, and where \( H(t) = H_{Mi}(t)X_{Mi} \), extending Notation (2.5.37). Then,

\[
\frac{\|Q(V + H) - Q(V) - L(H)\|}{\|H\|} \leq \sum_{J=0}^{r} \sup_{t \in [0,1]} |K_{ji}^j(\mathcal{D}H_0(t))| \cdot |D^JH_0(t)|
\]

where the sum on \( I_i \) is taken over all three pairs \( (I_i) \) together

\[
\leq \sum_{J=0}^{r} \sup_{t \in [0,1]} |K_{ji}^j(\mathcal{D}H_0(t))| \sup_{t \in [0,1]} |D^JH_0(t)|
\]

which has limit zero as \( \|H\| \) goes to zero as we wished to show.
We will now establish (2.5.51). We take $L$ to be the linear map

$$L : C^{r,q}_{dI,g} \to C^{r,q}_{dI,g}$$

$$L_J(H)(t) = D^J \left( \frac{\partial F(W_0(t), V_0(t))}{\partial V_{0i}(t)} H_{0i}(t) \right)$$

(2.5.52)

To see that $L$ is bounded take

$$L_J(H)(t) = D^J \left( \frac{\partial F(W_0(t), V_0(t))}{\partial V_{0i}(t)} H_{0i}(t) \right)$$

$$= \sum_{M=0}^{r} \binom{J}{M} \frac{1}{J!} \frac{d^{(J-M)}}{dt^{(J-M)}} \left( \frac{\partial F(W_0(t), V_0(t))}{\partial V_{0i}(t)} \right) H_{M_i}(t)$$

and use the fact that the coefficients of $H_{M_i}(t)$ have a maximum on $[0,1]$. Let $N$ be the largest of the maxima, then

$$|L_J(H)(t)|' \leq \sum_{M_i} |N||H_{M_i}(t)|'$$

$$\leq 2|N| \sum_{M} |H_{M}(t)|'.$$

Therefore $\|L(H)\| \leq 2|N||H||$, hence $L$ is bounded.

(2.5.51) is proven by using Taylor's Theorem with the integral form of the remainder:

Now, writing $F(V_0(t)) = F(W_0(t), V_0(t))$

$$Q_J(V + H)(t) - Q_J(V(t)) - L_J(H)(t)$$

$$= D^J(F(V_0(t) + H_0(t)) - F(V_0(t)) - \frac{\partial F}{\partial V_{0i}(t)} H_{0i}(t))$$

$$= \frac{D^J}{2} \int_0^1 (1 - \lambda^2) \frac{\partial^2 F}{\partial \lambda^2}(V_0(t) + \lambda H_0(t))d\lambda$$

$$= \frac{D^J}{2} \int_0^1 (1 - \lambda^2) \frac{\partial^2 F}{\partial V_i \partial V_j}(V_0(t) + \lambda H_0(t)) H_{0i}(t) H_{0j}(t)d\lambda$$

$$= \frac{1}{2} \sum_{M=0}^{J} \sum_{I=0}^{M} \int_0^1 (1 - \lambda^2) D^{J-M} \frac{\partial^2 F}{\partial V_i \partial V_j}(V_0(t) + \lambda H_0(t)) d\lambda D^{M-I} H_{0i}(t) \cdot D^I H_{0j}(t).$$

162
Let

\[ R^{JM}_{ij} = \frac{(1 - \lambda^2)}{2} D^{JM} \frac{\partial^2 \mathcal{F}(V_0(t) + \lambda H_0(t))}{\partial V^i \partial V^j}; \]

\( R^{JM}_{ij}(V_0(t), \lambda, H_0(t), D^K V_0(t), D^L H_0(t)) \) is a multinomial in the variables \( D^I V_0(t), \) 
\( D^J H_0(t) \) with coefficients analytic in the variables \( (V_0(t), \lambda, H_0(t)). \)

For our purposes we view \( R^{JM}_{ij} \) as an analytic function \( Z^{JM}_{ij} \) of \((2 + (g + 1)n)\) variables:

\[
Z^{JM}_{ij}(t, \lambda, H_0(t), D^K H_0(t)) = R^{JM}_{ij}(V_0(t), \lambda, H_0(t), D^K V_0(t), D^L H_0(t))
\]

As we will be taking the limit \(||H|| \to 0\), we can restrict \( H \) to be in a closed ball of radius \( r \) and hence \( |D^I H_0(t)|' \leq r \). Thus \( Z^{JM}_{ij} \) is an analytic function on 
\([0, 1] \times [0, 1] \times \prod_{i=0}^q B_r\) where \( B_r \) is the closed ball at 0 in \( g \) of radius \( r \) and has a maximum on this closed set, say \( W^{JM}_{ij} \). Hence

\[
|Q_J(V + H)(t) - Q_J(V(t)) - L_J(H)(t)|' \leq \sum_{M=0}^I \sum_{I=0}^M |W^{JM}_{ij}|'|D^{M-I} H_0(t)| \cdot |D^I H_{0j}(t)|
\]

proving (2.5.51). Thus, overlapping charts are once Fréchet differentiably related.

To show that \( \chi_u \circ \chi_v^{-1} \) is Fréchet-\( C^\infty \), we proceed by induction. Assume that \( \chi_u \circ \chi_v^{-1}(V) \) is \( k \)-times Fréchet differentiable. For each \( V \) the \( k' \)th derivative of \( \chi_u \circ \chi_v^{-1} \) is the continuous, multilinear map

\[
L_k : \prod_{i=1}^k C^{r_i q}_{dT} \to C^{r_i q}_{dT}
\]
\[ L_k(H^1, \ldots, H^k)(t) = D^J \left( \frac{\partial^k F(W_0(t), V_0(t))}{\partial V_{0i_1}(t) \cdots \partial V_{0i_k}(t)} H_{0i_1}^1(t) \cdots H_{0i_k}^k(t) \right) \tag{2.5.53} \]

with the norm on \( \prod_{i=1}^k C_{d; \mathcal{D}, \mathcal{S}}^{r, q} \) given by \( \| (H^1, \ldots, H^k) \|_k = \sum_{i=1}^k \| H^i \| \) and we must show, letting

\[ \mathcal{F}_{i_1 \ldots i_k}^k (V_0(t)) = \frac{\partial^k F(W_0(t), V_0(t))}{\partial V_{0i_1}(t) \cdots \partial V_{0i_k}(t)} \]

that,

\[ \left| D^J \left( \mathcal{F}_{i_1 \ldots i_k}^k (V_0(t) + H_0^{k+1}(t)) H_{0i_1}^1(t) \cdots H_{0i_k}^k(t) - \mathcal{F}_{i_1 \ldots i_k}^k (V_0(t)) H_{0i_1}^1(t) \cdots H_{0i_k}^k(t) \right) \right| \]

\[ -L_{k+1}(H^1, \ldots, H^{k+1})(t) \leq |K_j(D H_0^{k+1}(t), D H_0^1(t), \ldots, D H_0^k(t))| \tag{2.5.54} \]

where \( K_j \) as in (2.5.51) is a multinomial function of the \((k + 1)n(q + 1)\) variables 
\( D H_0^{k+1}(t), D H_0^1(t), \ldots, D H_0^k(t) \) and where no term has order less than 2 in the \(n(q + 1)\) components of \( D H_0^{k+1}(t) \), and that (2.5.54) holds for all \( H^1, \ldots, H^k \in C_{d; \mathcal{D}, \mathcal{S}}^{r, q} \).

The proof that \( L_{k+1} \) is a bounded multinomial function of \((H^1, \ldots, H^{k+1})\) is very analogous to the proof that \( L \) is bounded with the addition of the fact that multinomial maps from \( \prod_{i=1}^k \mathcal{S} \) to \( \mathcal{S} \) are bounded.

The proof of (2.5.54) is again standard and entirely analogous to the proof of (2.5.51), resting on the fact that \( \mathcal{F} \) is a \( C^\infty \) function on \( \mathcal{S} \). Hence \( \chi_u \circ \chi_v^{-1} \) is Fréchet-\( C^\infty \).

We now show that the product is Fréchet-\( C^\infty \).

In view of the remarks following equation (2.5.47), the proof that 
\( \chi_{uv}(\chi_u^{-1}(U)\chi_v^{-1}(V))(t) \) as given by (2.5.48) is a Fréchet-\( C^\infty \) function of \((U, V)\), is once more very similar to the proof of (2.5.50) except that \( V \) is replaced by \((U, V)\), \( H \) is replaced by \((H, K)\) and \( \| H \| \) is replaced by \( \|(H, K)\|_2 \) as given by (2.5.53)ff. The modifications required, of the proof of (2.5.50), are formal only and
we omit them for brevity. Hence the product is Fréchet-$C^\infty$. With the topology on $C_{r,T;\Sigma}^{r,q}$ induced by the coordinate charts, $C_{r,T;\Sigma}^{r,q}$ is a Banach-Lie group by (1.2.6) and (1.2.7).

**We now calculate the Lie algebra of $C_{r,T;\Sigma}^{r,q}$.**

By (1.2.17) the basis of the Lie algebra is the set of left-invariant vector fields of $C_{r,T;\Sigma}^{r,q}$. By analogy to the finite-dimensional left-invariant vector fields and on account of (2.5.52), we assert that these vector fields are represented locally by the maps $\xi^H$, with domain $C_{d,T;\Sigma}^{r,q}$, defined by

$$\xi^H_t(U)(t) = D^i\left(\frac{\partial F(U_0(t), V_0(t))}{\partial V_0(t)}\right)_{V_0(t)=0} H_0(t) = D^i\left(\frac{\partial F(U_0(t), V_0(t))}{\partial V_0(t)}\right)_{V_0(t)=0}$$

where $H_0 \in C_{d,T;\Sigma}^{r,q}$, and $D H_0(0) \in \mathfrak{g}$. (2.5.55)

The $\xi^H_t(U)(t)$ play a rôle analogous to the transformation functions in the finite-dimensional case. The $\xi^H$ have $C_{d,T;\Sigma}^{r,q}$ as image space because of the form of (2.5.55) and because the linear map,

$$H_0(t) \mapsto \left.\frac{\partial F(U_0(t), V_0(t))}{\partial V_0(t)}\right|_{V_0(t)=0} H_0(t), \text{ for each } t,$$  

is the derivative of left translation in $T^g(G)$ by the element $\exp(U_0(t))$, whence $(\nabla^0, \ldots, \nabla^n)\exp(U_0(t))|_{t=0}$ is in $\Sigma$, and $\xi(U)(0)$ is in $\mathfrak{g}$.

Let $T_u^C$ be the tangent space to $C_{r,T;\Sigma}^{r,q}$ at $u$; then in the notation of §1.1, (1.2.15) and (1.2.16), we postulate the left-invariant vector fields to be

$$\{\xi^H\}_u = \{(\mathcal{N}_1, \chi_1, \xi^H(\chi_1(u)))\}_u$$  

(2.5.57)

where $(\mathcal{N}_1, \chi_1)$ are given by (2.5.45), and $\xi^H$ by (2.5.55).

Note that

$$\{\xi^H \mid H \in C_{d,T;\Sigma}^{r,q}\} = C_{d,T;\Sigma}^{r,q}$$

since the linear map (2.5.56) is invertible for all $t$ in $[0,1]$.
We now show that the \( \{\xi^H\} \) are left-invariant vector fields.

Now \( \{\xi^H\} \) will be left-invariant if

\[
\{\xi^H\}_v = d\ell_{v^{-1}}\{\xi^H\}_u
\]  
(2.5.58)

Let \( \chi_1(u) = U, \chi_1(v) = V, \chi_1(w) = W, H \in T_u^G \). With respect to coordinates \((N_1, \chi_1)\), the operation of left translation by \( w \) at \( T_u^G \) is given by the map

\[
(d\ell_{\chi_1^{-1}(w)}(H))_j = D^J\left( \frac{\partial F(W_0(t), U_0(t))}{\partial U_0(t)} H_{0j}(t) \right)
\]  
(2.5.59)

which follows from (2.5.48) and the proof of (2.5.51). (2.5.58) will follow if we can show that

\[
\xi^H(V) = d\ell_{\chi_1^{-1}(v)\chi_1^{-1}(U^{-1})}^H(u).
\]  
(2.5.60)

Let \( W = \chi_1(\chi_1^{-1}(V)\chi_1^{-1}(U^{-1})) \), then by (2.5.59)

\[
(d\ell_{\chi_1^{-1}(w)}^H(U))_j(t) = 
D^J\left( \frac{\partial F(W_0(t), U_0(t))}{\partial U_0(t)} \frac{\partial F^j(U_0(t), Z_0(t))}{\partial Z_0(t)} \right)_{Z_0(t) = 0} H_{0j}(t).
\]

Note that,

\[
\text{Exp} Z(t) = (\nabla^0, \ldots, \nabla^q) \exp Z_0(t)
\]

\[
W_0(t) = \exp^{-1} \left( \exp V_0(t)(\exp U_0(t))^{-1} \right)
\]

and by the definition of left-invariant vector fields on \( G \),

\[
\frac{\partial F(W_0(t), U_0(t))}{\partial U_0(t)} \frac{\partial F^i(U_0(t), Z_0(t))}{\partial Z_0(t)} \bigg|_{Z_0(t) = 0} = \frac{\partial F(V_0(t), Z_0(t))}{\partial Z_0(t)} \bigg|_{Z_0(t) = 0},
\]

whence

\[
(d\ell_{\chi_1^{-1}(w)}^H(U))_j(t) = \xi^H_j(V)
\]

thus establishing (2.5.60).
We now calculate the Lie algebra of $C_{\mathcal{I};\Sigma}^{r,q}$.

Let $\{\xi^H\}_u, \{\xi^K\}_u$ be two left-invariant vector fields, given by (2.5.57). Then, from §1.1, (1.2.15) and (1.2.16),

$$\left[\{\xi^H\}_u, \{\xi^K\}_u\right] = \left\{d\xi^H(\xi^K)(u) - d\xi^K(\xi^H)(u)\right\}_u$$  \hspace{1cm} (2.5.61)

where $d\xi^H$ is the Fréchet derivative of $\xi^H$. With the form of $d\xi^H$ clear from the example of (2.5.52), we have

$$(d\xi^H(\xi^K)(u) - d\xi^K(\xi^H)(u))_j(t)$$

$$= D^J\left(\frac{\partial^2 F^j(U_0(t), V_0(t))}{\partial U_0(t)\partial V_0(t)}\right)_{|V_0(t)=0}K_{0j}(t)$$

$$- \frac{\partial^2 F^j(U_0(t), V_0(t))}{\partial U_0(t)\partial V_0(t)}_{|V_0(t)=0}H_{0j}(t)$$

$$= D^J\left(C^{t}_{ik}\frac{\partial F^i(U_0(t), V_0(t))}{\partial V_0(t)}\right)_{|V_0(t)=0}K_{0j}(t)$$

by the expression for the bracket of left-invariant vector fields on $\mathcal{G}$,

$$= \xi^{[H,K]}_j(U)(t)$$

where

$$[H, K]_j(t) = D^J\left(C^{t}_{ik}H_{0i}(t)K_{0k}(t)\right)$$

$$= D^J([H_0(t), K_0(t)]).$$

Thus the Lie algebra of $C_{\mathcal{I};\Sigma}^{r,q}$ is isomorphic with the Lie algebra $C_{d\mathcal{I};\mathcal{G}}^{r,q}$ of (2.5.38) by the isomorphism $H \mapsto \{\xi^H\}$.

**We show that** $\exp_q$, given by (2.5.43) **is the exponential mapping for** $C_{\mathcal{I};\Sigma}^{r,q}$ **by showing that for any** $V \in C_{d\mathcal{I};\mathcal{G}}^{r,q}$, $\exp$ **is the unique Fréchet-$C^\infty$ homomorphism of** $\text{span}\{V\}$ **into** $C_{\mathcal{I};\Sigma}^{r,q}$, **with derivative at 1 being** $V$. **Now**

$$\exp_q((s_1 + s_2)V)(t) = \text{Exp}((s_1 + s_2)V(t))$$

$$= \text{Exp}(s_1 V(t))\text{Exp}(s_2 V(t))$$

$$= \exp_q(s_1 V)(t)\exp_q(s_2 V)(t)$$

167
and $\exp_q$ is the required homomorphism.

By definition of the chart $(\chi_1, N_1)$ of (2.5.45) at 1, $\exp_q$ is trivially Fréchet-$C^\infty$, with derivative $V$ at 1. This completes the proof of Proposition (2.5.42) ■

The promised fruits of Proposition (2.5.42) are the following:

**Corollary 2.5.62**

$C_g^q[0,1]$ is a Banach-Lie group.

**Proof:** Put $q = 0$ in (2.5.42), then set $r = q$. ■

**Proposition 2.5.63**

The map (2.5.32) $T^q: C_g^q[0,1] \rightarrow C_{\hat{T}^q(\mathcal{G})}^{q,q}$ is a Fréchet-$C^\infty$ isomorphism of Banach Lie groups.

**Proof:**

By (2.5.33) and (2.5.35) $T^q$ is an isomorphism.

Let $(\chi_g, N_g)$ be a chart (2.5.46) at $g \in C_g^q[0,1]$ and let $(\chi_{\hat{T}^q(\mathcal{G})}, N_g)$ be a chart (2.5.46) at $T^q(g) \in C_{\hat{T}^q(\mathcal{G})}^{q,q}$. Then by the definition of these charts and the homomorphism property of $T^q$,

$$\chi_{\hat{T}^q(\mathcal{G})} \circ T^q \circ \chi_g^{-1}(X)(t) = \text{Exp}^{-1}((\nabla^0, \ldots, \nabla^q) \exp X(t))$$

where Exp is given by (2.5.8). Observe that Exp is the map,

$$\text{Exp} : s^J D^J X(t) \mapsto (\nabla^0_s, \ldots, \nabla^q_s) \exp(s^J D^J X(t))\big|_{s=0}$$

where $D^J$ is given at (2.5.51), and

$$(\nabla^0_s, \ldots, \nabla^q_s) \exp(s^J D^J X(t))\big|_{s=0} = (\nabla^0, \ldots, \nabla^q) \exp X(t)$$

This is because, in exponential coordinates at $e$ in $\mathcal{G}$,

$$\nabla^I_s \exp(s^J D^J X(t))\big|_{s=0} = \nabla^I \exp X(t) \ (\text{no sum on } I).$$
\[ x(t) = \chi_g^{-1} \circ \chi_{q(t)} \circ y(t) \]

By (2.5.39) and (2.5.62), \( X \) is a linear isometry of the Banach spaces \( C_{dT_g}^q \) and \( C_{dt_g}^q \), and is Frléchet-\( C^\infty \), a fortiori, together with its inverse. ■

In order to show that \( T^q(\mathcal{D}) \) is a Banach-Lie subgroup of \( C_{dT_g}^q \) in the sense of (1.2.8), we prove the following result.

**Proposition 2.5.64**

\( C_{dT_g}^q \) is a Banach-Lie subgroup of \( C_{dT_g}^q \) in the sense of (1.2.8) for any analytic subgroup \( \Sigma \) of \( T^q(\mathcal{G}) \).

**Proof:**

We only need show that the injection \( j : C_{dT_g}^q \to C_{dT_g}^q \) which is the identity on \( C_{dT_g}^q \) is Frléchet-\( C^\infty \).

Following the proof of Proposition (2.5.63) the map, \( \chi_{j(u)} \circ j \circ \chi_u^{-1} \) is a linear, injective isometry of the Banach space \( C_{dT_g}^q \) into the Banach space \( C_{dT_g}^q \), and must be Frléchet-\( C^\infty \). ■

We now fulfill the promise made at (2.5.20) to show that \( P \) is a Frléchet-\( C^\infty \) homomorphism, and \( dP \) is its derivative. We also need this proposition to complete the proof of Theorem (2.5.30).

**Proposition 2.5.65**

The map \( P \), defined by (2.5.19) is a Frléchet-\( C^\infty \) homomorphism of \( C_{\mathcal{G}}^q \) into \( T^q(\mathcal{G}) \) and the map \( dP \) defined by (2.4.10) which is a continuous homomorphism of \( C_{\mathcal{G}}^q \) into \( T^q(\mathcal{G}) \), is the Frléchet derivative of \( P \) at \( e \).

**Proof:**

We have already shown at (2.5.21) that \( P \) is a homomorphism. To see that \( P \)
is Fréchet-$C^\infty$, let \( g \in C^q_0[0,1] \) with chart \((\chi_g, \mathcal{N}_g)\) given by (2.5.46). By analogy to (2.5.46) let the chart \((\chi_{P(g)}, \mathcal{N}_{P(g)})\) at \( P(g) \) in \( T^q(G) \) be given by

\[
\chi_{P(g)}(V) = \text{Exp}^{-1} \circ \ell_{P(g^{-1})} \\
\mathcal{N}_{P(g)} = \ell_{P(g)} B_1,
\]

where \( B_1 \) is a neighbourhood of 1 in \( T^q(G) \) on which \( \text{Exp} \) is a diffeomorphism.

Then

\[
\chi_{P(g)} \circ P \circ \chi_g^{-1} = \text{Exp}^{-1} \circ \ell_{P(g^{-1})} \circ P \circ \ell_g \circ \exp_q(V) \\
= \text{Exp}^{-1}((\nabla^0, \ldots, \nabla^q) \exp V(t)|_{t=0}) \\
\text{by the homomorphism property} \\
= \frac{dV(t)}{dt}|_{t=0}
\]

(2.5.66)

from the proof of (2.5.63)

This map is linear, and will be Fréchet-$C^\infty$ if it is bounded.

Boundedness follows from the inequalities,

\[
|\chi_{P(g)} \circ P \circ \chi_g^{-1}(V)|' = |\frac{dV(t)}{dt}|_{t=0}' \\
\leq \sum_{j=0}^{q} |\frac{d^jV(t)}{dt^j}|_{t=0}' \\
\leq \sum_{j=0}^{q} \sup_{t \in [0,1]} |\frac{d^jV(t)}{dt^j}'|' \\
= \|V\|.
\]

Since \( P \) is a linear map with respect to charts \((\chi_1, \mathcal{N}_1)\) at 1 \( \in T^q(G) \), and \((\chi_e, \mathcal{N}_e)\) at \( e \in C^q_0[0,1] \), its derivative is the same map. From (2.5.66), the derivative of \( P \) is given by the map,

\[
V \mapsto \frac{dV(t)}{dt}|_{t=0}
\]

170
which agrees with the definition of $dP$ at (2.4.10) on utilising (2.5.37) and the isomorphism $X_{Lm} \mapsto t^L X_m$ of (2.5.8). $dP$ is continuous because it is linear and bounded. ■

We will now prove Theorem (2.5.30) by pulling together all the Definitions, Propositions and Lemmas from Definition (2.5.32) to Proposition (2.5.65).

**Proof of Theorem (2.5.30):**

We show that $P(D^q)$ is a closed, hence analytic subgroup of $T^q(G)$:

$P(D^q)$ is a subgroup of $T^q(G)$ by assumption and we show that it is closed. Let $\{u_m\}$ be a sequence in $P(D^q)$ converging to a point $u$ in $T^q(G)$. Let $u$ be given by (2.5.4), by $(g, X_1, \ldots, X_q)$. From (2.3.2) $u$ is in $P(D^q)$ if and only if

$$
\frac{d^J}{dt^J} \phi(g \exp\left(\frac{t^L X_I}{I!}\right), t)|_{t=0} = 0, \quad J = 0, \ldots, q - 1. \quad (2.5.67)
$$

(2.5.67) determines $q$ analytic functions $f^J$ of $u$, which are multinomial in the $X_{I_i}$. Write (2.5.67) as,

$$
f^J(g, X_1, \ldots, X_q) = 0, \quad J = 0, \ldots, q - 1. \quad (2.5.68)
$$

Hence $u_m \to u$ implies $f^J(u_m) \to f(u) = 0$ whence $u$ is in $P(D^q)$ and $P(D^q)$ is closed.

We now show that $D^q$ is a Banach-Lie group:

By Corollary (2.5.62) and Proposition (2.5.63) the map,

$$
T^q: C^q[0,1] \to C^q_{T:T^q}(G) \text{ is a Fréchet-}C^\infty \text{ isomorphism. By (2.5.34), } T^q(D^q) = C^q_{T:P(D^q)}.
$$

By Proposition (2.5.64) with $\Sigma = P(D^q)$, $T^q(D^q)$ is a Lie subgroup of $C^q_{T:T^q}(G) = T^q(C^q[0,1])$. Since $T^q$ is a Fréchet-$C^\infty$ isomorphism $D^q$ is a Lie subgroup of $C^q[0,1]$ and is hence a Lie group.

We define a topology for $G_0$:
By assumption, \( \ker \mathfrak{X}_P \) is a normal subgroup of \( P(D^q) \). We show that it is a closed, and hence analytic subgroup of \( P(D^q) \). Let \( \{U_m\} \) be a sequence in \( \ker \mathfrak{X}_P \) converging to a point \( U \) in \( P(D^q) \). Proceeding as for (2.5.67) and (2.5.68), there is an analytic function \( f \), such that \( U \in \ker \mathfrak{X}_P \iff f(g, X_1, \ldots, X_q) = 0 \). Then \( U_m \to U \) implies \( f(U_m) \to f(U) = 0 \) whence \( U \in \ker \mathfrak{X}_P \) and \( \ker \mathfrak{X}_P \) is closed in \( P(D^q) \).

We may now give \( \mathcal{G}_0 \) the analytic structure of the quotient group, \( P(D^q)/\ker \mathfrak{X}_P \) (That \( \mathcal{G}_0 \) is finite-dimensional follows immediately from the fact that \( P(D^q) \) is Lie subgroup of the finite-dimensional Lie group \( T^q(\mathcal{G}) \)). To see that \( \mathfrak{X} \) and \( \mathfrak{X}_P \) are both homomorphisms, let \( g, h \in D^q \), with \( s_1 = \Phi_s(g), s_2 = \Phi_s(h) \),

\[
\text{therefore } \mathfrak{X}(gh) = \lim_{t \to 0} \Phi(\Phi^{-1}(s_1(t))\Phi^{-1}(s_2(t)))
\]

\[
= s_1(0)s_2(0) \quad \text{by (2.5.18)}
\]

\[
= \mathfrak{X}(g)\mathfrak{X}(h) \quad \text{by (2.5.24)}
\]

whence it also follows that \( \mathfrak{X}(e) = e_0 \) where \( e_0 \) is the identity of \( \mathcal{G}_0 \). Then by definition of \( \mathfrak{X}_P \), \( \mathfrak{X}_P(P(g)P(h)) = \mathfrak{X}_P(P(g))\mathfrak{X}_P(P(h)) \), \( \mathfrak{X}_P(1) = e_0 \). By the choice of topology for \( \mathcal{G}_0 \), \( \mathfrak{X}_P \) is an analytic homomorphism.

\( S^q \) is a Banach-Lie Group and \( \Phi_s, \mathfrak{X}, P, \) and \( \epsilon \) are \( C^\infty \)-Fréchet:

We have already seen in (2.5.65) that \( P \) is always Fréchet-\( C^\infty \). Induce the topology and manifold structure of \( S^q \) from \( D^q \) via \( \Phi_s \). Inducing the group structure for \( S^q \) via \( \Phi_s \) as in (2.5.17); (2.5.18) makes \( S^q \) and \( D^q \) Fréchet-\( C^\infty \) isomorphic Lie groups, and \( \Phi_s \) a \( C^\infty \)-Fréchet diffeomorphism. Both \( \mathfrak{X} \) and \( \epsilon \) are Fréchet-\( C^\infty \) because \( \mathfrak{X} = \mathfrak{X}_P \circ P \) and \( \epsilon = \mathfrak{X} \circ \Phi_s^{-1} \).

It is appropriate at this point to include a theorem which connects the global contraction theory of this section, with the Lie algebra contraction theory of §2.4.
Theorem 2.5.69

Let Exp_q, Exp and exp_0 be the exponential mappings for \( D^q \), \( P(D^q) \) and \( \mathcal{G}_0 \) respectively. Then the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{G}_0 & \xrightarrow{\exp} & P(D^q) \\
\downarrow \text{Exp} & & \downarrow \text{Exp} \\
\mathcal{D}^q & \xrightarrow{dP} & dP(D^q) \\
\downarrow \text{Exp}_q & & \downarrow \text{Exp}_q \\
D^q & \xrightarrow{\text{dP}(dq)} & dP(D^q)
\end{array}
\]

Proof:

We first show that \( dP(dq) \) is the Lie algebra of \( P(D^q) \).

Let \( u \) be in \( T^q(\mathcal{G}) \) with \( u = (g, X_1, \ldots, X_q) \) as per (2.5.4). \( u \) is in \( P(D^q) \) if and only if,

\[
\frac{d^J}{dt^J} \phi(\chi^{-1}(g \exp(\frac{t^I X_I}{I!})), t)|_{t=0} = 0, \quad J = 0, \ldots, q - 1,
\]  
(2.5.70)

(from the proof of (2.5.30)). (2.5.70) generates a \( C^\infty \) function \( f \) on \( T^q(\mathcal{G}) \),

\[
f : T^q(\mathcal{G}) \to \mathbb{R}^N
\]

(where \( N \) is the dimension of the vector space \( E \))

\[
f^J(g, X_1, \ldots, X_q) = \frac{d^J}{dt^J} \phi(\chi^{-1}(g \exp(\frac{t^I X_I}{I!})), t)|_{t=0}
\]  
(2.5.71)

The condition (2.5.70) makes \( f \) a trivial homomorphism of \( P(D^q) \) into \( \mathbb{R}^N \) as it satisfies,

\[
f(1) = 0, \quad \text{by (2.3.2)}
\]

\[
f(uv) = 0 = f(u) + f(v).
\]

Let \( V \in \mathfrak{g}^q(g) \) be in a sufficiently small neighbourhood of 0, so that

\[
f \circ \text{Exp}(V) = \exp(dfV) = dfV, (\exp \text{ is exponential mapping for } \mathbb{R}^N)
\]
since \( \exp \) is the identity on \( \mathbb{R}^N \). By (2.5.71) \( V \) is in the Lie algebra of \( P(D^q) \) if and only if

\[
df V = 0
\]  

(2.5.72)

We calculate the derivative of \( f \) at 1:

Since \( u \) is near 1 \( f \) becomes

\[
f^J(X_0, X_1, \ldots, X_q) = \frac{dJ}{dt} \phi(\chi^{-1}(\exp(X_0 + \frac{t^J X_I}{I!})), t) \big|_{t=0}.
\]

Write \( \frac{t^J X_I}{I!} \) to mean \( \sum_{I=0}^q \frac{t^J X_I}{I!} \) and let \( (x^i) \) be coordinates at \( e \) with respect to \( \chi \).

Then

\[
\frac{\partial f^J(X_I)}{\partial x_K} = \frac{dJ}{dt} \frac{\partial}{\partial x_K} \phi(\chi^{-1}(\exp(\frac{t^J X_I}{I!})), t) \big|_{t=0}
\]

\[
= \frac{dJ}{dt} \frac{\partial \phi}{\partial x_k}(\chi^{-1}(\exp(\frac{t^J X_I}{I!})), t) \frac{t^K}{K!} \big|_{t=0}
\]

\[
= (J) \frac{d^{J-K}}{dt^{J-K}} \frac{\partial \phi}{\partial x_k}(\chi^{-1}(\exp(\frac{t^J X_I}{I!})), t) \big|_{t=0}
\]

(no sum on \( J, K \))

Therefore

\[
\left. \frac{\partial f^J}{\partial x_K} \right|_{u=1} = (J) \frac{d^{J-K}}{dt^{J-K}} \frac{\partial \phi}{\partial x_k}(\chi^{-1}(0), t) \big|_{t=0} \tag{2.5.73}
\]

Then \( V \) is in the Lie algebra of \( P(D^q) \) if and only if

\[
\left. \frac{\partial f^J}{\partial x_K} \right|_{u=1} V_{Kk} = 0 \quad (\text{sum on } Kk) \tag{2.5.74}
\]

with \( \left. \frac{\partial f^J}{\partial x_K} \right|_{u=1} \) given by (2.5.73).

Now from §2.4, \( \frac{dJ}{dt} \left( \frac{\partial \phi(t) V_{Kk} t^K}{K!} \right) \big|_{t=0} = 0. \) That is,

\[
\left. \frac{dJ}{dt} \left( \frac{\partial \phi^{-1}(0), t) V_{Kk} t^K}{K!} \right) \right|_{t=0} = 0
\]

174
which becomes

\[
\left( J \right) \frac{d^{J-K}}{dt^{J-K}} \left( \frac{\partial \phi^{-1}(0, t)}{\partial x^k} \right) \bigg|_{t=0} V_k = 0 \quad \text{(sum } Kk\text{)}
\]

which is exactly (2.5.74). Hence the Lie algebra of \( P(D^q) \) is \( dP(d^q) \).

We now show that \( d^q \) is the Lie algebra of \( D^q \).

\( I^{q,q} \) is a map from \( C^q_0[0,1] \) to \( C^q_{T^q}(q) \). \( C^q_0[0,1] \) has Lie algebra \( C^q_{dT^q} \equiv C^q_0[0,1] \) and \( C^q_{T^q}(q) \) has Lie algebra \( C^q_{dT^q}(q) \) by Proposition (2.5.42). Hence the derivative homomorphism \( dI^{q,q} \) of the Lie algebras is a map

\[
dI^{q,q} : C^q_0[0,1] \to C^q_{dT^q}(q)
\]

and is a Lie algebra isomorphism. We know from (2.5.42) that the Lie algebra of \( I^{q,q}(D^q) \) is \( C^q_{dT:dP(d^q)} \). Then \( d^q \) will be the Lie algebra of \( D^q \) if we can show that,

\[
dI^{q,q}(d^q) = C^q_{dT:dP(d^q)}.
\]

First we calculate the derivative of \( I^{q,q} \). Choose charts (2.5.45) \((\mathcal{U}_e, \chi_e)\) and \((\mathcal{V}_1, \chi_1)\) at \( e \in C^q_0[0,1] \) and \( 1 \) in \( C^q_{T^q}(q) \) respectively.

Then

\[
\chi_1 \circ I^{q,q} \circ \chi_e^{-1}(X)(t) = \text{Exp}^{-1}((\nabla^0, \ldots, \nabla^q) \exp X(t))
\]

\[
= DX(t) \quad \text{from the proof of (2.5.63)}.
\]

As \( \chi_1 \circ I^{q,q} \circ \chi_e^{-1} \) is a linear map, its derivative is the same. Hence

\[
dI^{q,q}(X)(t) = DX(t). \quad \text{(2.5.75)}
\]

Let \( X \in d^q \), then

\[
DX(0) = dP(X) \in P(d^q)
\]

and from (2.5.75) and (2.5.38), we have

\[
dI^{q,q}(d^q) = C^q_{dT:dP(d^q)}
\]

whence \( d^q \) is the Lie algebra of \( D^q \).
To show that \( g_0 \) is the Lie algebra of \( G_0 \) we only need show that \( dX_P \) is the derivative homomorphism of \( X_P \). This will show that \( \ker dX_P \) is the Lie algebra of \( \ker X_P \). And as \( G_0 = P(Dq)/\ker X_P \), \( G_0 \) must then have Lie algebra \( \frac{dP(d^2)}{\ker dX_P} \), which is precisely \( g_0 \).

Now \( X_P \) is the map,

\[
X_P : (g, X_1, \ldots, X_q) \mapsto \frac{d^q}{dt^q} \phi(g \exp(t^I X_I), t) \bigg|_{t=0}
\]

(2.5.76)

and \( dX_P \) is the map,

\[
dX_P : (X_0, X_1, \ldots, X_q) \mapsto \frac{d^q}{dt^q} (d\phi(t)(\frac{t^I X_I}{I!}), t) \bigg|_{t=0} \quad \text{(sum on } I) \]

\[= \left( \begin{array}{c} q \\ I \end{array} \right) \frac{d^{q-I}}{dt^{q-I}} \frac{\partial \phi(\chi^{-1}(0), t)}{\partial x^i} \bigg|_{t=0} X_I_i. \]

(2.5.77)

If we now calculate the derivative of \( X_P \) at 1 from (2.5.76), we will arrive at (2.5.77) in exactly the same way as for (2.5.73).

Hence \( dX_P \) is the derivative of \( X_P \) and \( g_0 \) is the Lie algebra of \( G_0 \).

By Proposition (2.5.65) \( dP \) is the derivative of \( P \) and so the diagram commutes by Theorem (1.2.24). This completes the proof of (2.5.69)

\[\square\]

Remark 2.5.78

At this point, we make the observation that \( \text{Exp}_q \), the exponential mapping for \( Dq \) is given by (2.5.43) as,

\[
\text{Exp}_q : Dq \rightarrow Dq
\]

\[
\text{Exp}_q(X)(t) = \exp(X(t)).
\]

The following lemma is a useful corollary of (2.5.69):

176
Lemma 2.5.79

There is a neighbourhood $\mathcal{N}_0$ of 0 in $\mathfrak{g}$ such that, $\lim_{t \to 0} \exp_{\Phi(g,t)} d\Phi(X(t)) = \exp_0 d\mathfrak{X}(X)$, $\forall X \in \mathfrak{g}$, with $X(0) \in \mathcal{N}_0$.

Proof: Now $\Phi$ is a homomorphism, and by (1.1.45),

$$\exp_{\Phi(g,t)} d\Phi(X(t)) = \Phi(\exp X(t)).$$

Therefore

$$\lim_{t \to 0} \exp_{\Phi(g,t)} d\Phi(X(t)) = \lim_{t \to 0} \Phi(\exp X(t))$$

$$= \mathfrak{X}(\exp X)$$

$$= \exp_0 d\mathfrak{X}(X) \text{ by (2.5.69)} \quad \square$$

In this chapter, we show how the theory of §2.4 and §2.5 gives a simple and elegant description of global IW and global generalised IW contractions (see (1.5.6)) elucidating the structures involved. The advantage of these descriptions is that they by-pass completely the necessity of finding Laurent maps $\Psi(\cdot, t)$ and of solving non-linear equations to find $D^q$ and $P(D^q)$. Further, for the case of generalised Saletan contraction of Lie algebras, given by (2.4.6), (2.4.7) and (2.4.8) (which is the full generalisation of the original Saletan contraction given by (1.5.5) and of the contractions of Kupczynski (1969) and Lévy-Nahas (1967)) we show the remarkable fact that explicit, simple, compact formulae exist to describe $d^q$ and $\ker d\mathcal{X}$ greatly improving on the standard descriptions of

$$
\mathcal{D}^q = \left\{ X \mid X \in C^q_{\mathbb{R}}[0, 1], \left. \frac{d^i}{dt^i}(d\phi(t)X(t)) \right|_{t=0} = 0, \; i = 0, \ldots, q - 1 \right\}
$$

and

$$
\ker d\mathcal{X} = \left\{ X \mid X \in C^q_{\mathbb{R}}[0, 1], \left. \frac{d^q(d\phi(t)X(t))}{dt^q} \right|_{t=0} = 0 \right\}
$$

which have the disadvantage of giving a rather too general formula for the $X^{(i)}(0), i = 0, \ldots, q$, and only as solutions yet to be found of a linear system of equations. By contrast, for the case of generalised Saletan contraction, $d^q$ and $\ker d\mathcal{X}$ can be given in closed form. It is this closed form which then enables us to give a global version of Saletan contraction, relatively rich in structure, in the same way as IW and generalised IW contractions are treated above.

All the results of this section are entirely new and unexpected. It appears likely that this global formulation of Saletan contraction (with structural details)
in particular, will enable an exhaustive and thorough treatment of the generalised Saletan contractions (2.4.6) of representations of Lie groups, going much further with the contraction of the representation theory than is achieved in Chapter 4, and fulfilling Saletan's (Saletan, 1961) original programme.

**Definition 3.1.1**

The group $G_0$ of (2.3.5) with $\sigma = q = 1$ in (2.3.2) is called the **global Inönü-Wigner (IW) contraction** of $G$.

A proposition giving the structure of global IW contractions in this still very general setting can be given with only mild restrictions on the Laurent maps (2.3.2):

**Proposition 3.1.2**

Let $\phi$ be given by (2.3.2) and let $G_0$ be the global IW contraction of $G$. Suppose that

(a) the set, $G_1 = \{g \mid g \in G, \phi(g, 0) = 0\}$ is a subgroup of $G$, and

(b) $g \in \ker \xi \Rightarrow g(0) = e$ (see (2.5.25)).

Then, $G_0$ and $P(D^1)$ are Lie groups, $\ker \xi_P$ a normal Lie subgroup of $P(D^1)$ and $\xi_P$ a $C^\infty$ homomorphism. Further, $G_0$ is the semi-direct product $G_{1,0} \ltimes T$ of Lie subgroups $G_{1,0}$ and $T$ of $G_0$, where $G_{1,0}$ is $C^\infty$-isomorphic to $G_1$ and $T$ is a connected, abelian and non-compact normal subgroup of $G_0$.

**Proof:**

We show that condition (2.5.24(1)) holds:

Now

$$P(D^1) = \{(g_1, X) \mid g_1 \in G_1, X \in g_1\}$$

(3.1.3)

in the notation of (2.5.4). By (2.5.4) $P(D^1)$ is a subgroup of $T(G) \iff G_1$ is a subgroup of $G$.

Let $(\chi, \eta)$ be a chart at $e$. 179
Now by condition (b), \( g \in \ker X \) if and only if

\[
\left. \left( \frac{\partial \phi(x^{-1}(x(t)), t) \, dx^i(t)}{\partial x^i} \right) \right|_{t=0} = 0
\]

(where \( \chi(g(t)) = x(t) \)). Whence,

\[
\ker X_P = \{(e, X) \mid X \in g_1\}
\]  \hspace{1cm} (3.1.4)

By (2.5.7), \( \ker X_P \) is a normal subgroup of \( P(D^1) \). By (2.5.30) or by observation, \( g_0 \) and \( P(D^1) \) are Lie groups and \( X_P \) a \( C^\infty \)-homomorphism, and \( \ker X_P \) is a closed, normal Lie subgroup of \( P(D^1) \). By (3.1.3) and (3.1.4), \( g_{1,0} = g_1 \) and \( T = \{(e, X) \mid x \in g\}/\{(e, X) \mid X \in g_1\} \) and the action of \( g_{1,0} \) on \( T \) is given by the action of \( g_1 \) on cosets of \( \{(e, X) \mid X \in g\} \) by \( \{(e, X) \mid X \in g_1\} \).

**Comments 3.1.5**

Given a Lie group \( G \) with Lie subgroup \( G_1 \), one may define the contraction \( g_0 \) of \( G \) directly by putting,

\[
G_0 = \{(g_1, X) \mid g_1 \in g_1, X \in g\}/\{(e, X) \mid X \in g_1\}
\]

which is self-evidently a Lie group by the structure of \( T(G) \) (2.5.7). The beauty of giving \( g_0 \) directly as a factor group of a Lie subgroup of \( T(G) \) by a normal Lie subgroup, is that the need to search for appropriate Laurent maps is obviated. This approach allows us to globalize generalised IW contraction in the following proposition:

**Proposition 3.1.6**

Let \( G \) be a Lie group with Lie algebra \( g \) and let \( g_0 \) be the generalised contraction of \( g \) by \( U \) as per (1.5.6) with \( S_i = \text{Im} u_i \). Let \( g_{S_0} \) be the connected Lie subgroup of \( G \) with the algebra \( S_0 \). Then, in the notation of Proposition (2.5.4), the Lie group given by the factor group \( G_0 \), of the Lie subgroup

\[
G_{dP(g^e)} = \{(g_0, X_1, \ldots, X_q) \mid g_0 \in g_{S_0}, X_i \in S_0 + S_1 + \cdots + S_i\}
\]
of \( T^q(G) \), by its normal Lie subgroup,

\[
\mathcal{G}_{ker\, d\mathfrak{X}_p} = \{(e, Y_1, \ldots, Y_q) \mid Y_i \in S_0 + \cdots + S_{i-1}\},
\]

has Lie algebra \( \mathfrak{g}_0 \).

**Remark 3.1.7**

Here \( \mathcal{G}_{dP(d^q)} \) plays the rôle of \( P(D^q) \), \( \mathcal{G}_{ker\, d\mathfrak{X}_p} \) the rôle of \( ker\, \mathfrak{X}_P \), and \( \mathfrak{X}_P \) may be seen as the natural homomorphism

\[
\mathfrak{X}_P : \mathcal{G}_{dP(d^q)}/\mathcal{G}_{ker\, d\mathfrak{X}_p} \rightarrow \mathfrak{g}_0.
\]

**Proof of (3.1.6):**

To see that \( \mathcal{G}_{dP(d^q)} \) is a group, let

\[
(k_0, Z_1, \ldots, Z_q) = (g_0, X_1, \ldots, X_q). (h_0, Y_1, \ldots, Y_q),
\]

\( Z_i \) is given by (2.5.4) as a multilinear function of \( (\text{Ad}(h_0)X_j, Y_\ell) \). Since \( [S_0, S_0 + \cdots + S_j] \subset S_0 + \cdots + S_j \) by (1.5.6) and \( \mathcal{G}_{S_0} \) is connected, \( \text{Ad}(\mathcal{G}_{S_0})(S_0 + \cdots + S_j) \subset S_0 + \cdots + S_j \). Then \( Z_i \) is a linear combination of terms \( \text{Ad}(h_0)X_i + Y_\ell \) and \( [W_{j_1}, [Y_\ell, [\cdots, [W_{j_\ell}, Y_\ell] \cdots]] \) with \( W_{j_\ell} \in S_0 + \cdots + S_\ell \), where

\[
\sum_{\ell=1}^{k+1} j_\ell = i.
\]

Because of the condition \( \sum_{\ell=1}^{k+1} j_\ell = i \) and the fact that \( [S_k, S_\ell] \subset S_{k+\ell} \)

\[
\text{(and hence}
\]

\( [S_0 + \cdots + S_k, S_\ell + \cdots + S_\ell] \subset S_0 + \cdots + S_{k+\ell} \), we have \( Z_i \in S_0 + \cdots + S_i \), whence \( \mathcal{G}_{dP(d^q)} \) is closed with respect to products. Since the inverse of \( (g_0, X_1, \ldots, X_q) \)

is \( (g_0^{-1}, -\text{Ad}g_0^{-1}X_1, \ldots, -\text{Ad}g_0^{-1}X_q) \), it is contained in \( \mathcal{G}_{dP(d^q)} \). The differentiable structure of \( \mathcal{G}_{dP(d^q)} \) is immediate from the definition of \( \mathcal{G}_{dP(d^q)} \). Hence \( \mathcal{G}_{dP(d^q)} \) is \( C^\infty \)-isomorphic to a Lie subgroup of \( T^q(G) \). To show that \( \mathcal{G}_{ker\, d\mathfrak{X}_p} \) is a group, we may reproduce the argument for \( \mathcal{G}_{dP(d^q)} \), this time showing that all elements of the form \( Q_i = [W_{j_1}, [Y_\ell, [\cdots, [W_{j_\ell}, Y_\ell] \cdots]] \) where \( \sum_{\ell=1}^{k+1} j_\ell = i \), are in \( S_0 + \cdots + S_{i-1} \)
as opposed to $S_0 + \cdots + S_i$. This follows a fortiori since $W_{j_k} \in S_0 + \cdots + S_{j_k-1}$ and if there were $m$ arguments in the above bracket expression, $Q_i$ would be in $S_0 + \cdots + S_{i-m}$. The differentiable structure of $G_{ker dX_P}$ is also self-evident and hence it is a Lie subgroup of $G_{dP(d^t)}$. To see that $G_{ker dX_P}$ is a normal subgroup of $G_{dP(d^t)}$, we use the formula for $\text{Exp} : P^q(g) \rightarrow T^q(G)$ given by (2.5.8) which shows that the $C^\infty$-isomorphic image of $G_{ker dX_P}$ by $\psi^{-1}_q$ of (2.5.4) is the image by $\text{Exp}$ of the ideal

$$G_{ker dX_P} = \{ tY_1 + t^2Y_2 + \cdots + t^qY_q \mid Y_i \in S_0 + \cdots + S_{i-1} \}$$

(by abuse of notation we identify a polynomial of order $t^q$ with its coset by the map $dP$) of the Lie algebra $G_{dP(d^t)} = \{ X_0 + tX_1 + \cdots + t^qX_q \mid X_i \in S_0 + \cdots + S_i \}$ and this image is all of $\psi^{-1}_q(G_{dP(d^t)})$. $G_{dP(d^t)}$ is a Lie algebra because for the spanning elements $\{ t^iX_i \}$ (no sum), $X_i \in S_0 + \cdots + S_i$ as an element of $G_{dP(d^t)}$, $[t^iX_i, t^jX_j] = t^{i+j}[X_i, X_j]$ and $[X_i, X_j] \in S_0 + \cdots + S_{i+j}$ by (1.5.6). For the same reasons $G_{ker dX_P}$ is a Lie subalgebra of $G_{dP(d^t)}$, and is an ideal because for the spanning elements $\{ t^iY_j \}$ (no sum), $Y_j \in S_0 + \cdots + S_{j-1}$ as an element of $G_{ker dX_P}$,

$$[t^iX_i, t^jY_j] = t^{i+j}[X_i, Y_j] \quad \text{and} \quad [X_i, Y_j] \in S_0 + \cdots + S_{i+j-1}.$$

Then $G_{ker dX_P}$ is a normal Lie subgroup of $G_{dP(d^t)}$. To see that $G_0$ has Lie algebra $G_0$, we find the Lie algebra $dP(d^q)$ and ker $dX_P$ for generalised IW contraction and show that they are the Lie algebras of $G_{dP(d^t)}$ and $G_{ker dX_P}$. To compute $dP(d^q)$ we may restrict $d^q$ to be in $P(g)$ (2.4.22). By the definition of $U(t)$, if $X \in P(g)$, and $\Pi_{S_j}$ denotes the canonical projection $\Pi_{S_j} : g \rightarrow S_j$,

$$X \in d^q \iff \Pi_{S_j}(X^{(i)}(0)) = 0, \quad j = 1, \ldots, q; \ i = 0, \ldots, j - 1.$$

Then

$$d^q = \{ t^{i} \Pi_{S_0} X_0 + t^{i+1} \Pi_{S_1} X_1 + \cdots + t^{q} \Pi_{S_q} X_q + t^{q+1}Y \mid X_i, Y \in P(g) \}.$$
\[ \{ Z_0 + t Z_1 + \cdots + t^q Z_q + t^{q+1} X \mid Z_i \in S_0 + \cdots + S_i, X \in \mathcal{P}(g) \}, \]

hence

\[ dP(\mathfrak{g}^q) = \{ Z_0 + t Z_1 + \cdots + t^q Z_q \mid Z_i \in S_0 + \cdots + S_i \} \]

which is the Lie algebra of \( \mathcal{G}_{dP(\mathfrak{g}^q)} \) by the definition of \( \mathcal{G}_{dP(\mathfrak{g}^q)} \) and the definition of \( \text{Exp} : \mathcal{P}(g) \to T^q(\mathcal{G}) \) of (2.5.8). We now compute \( \text{ker} \, d\mathfrak{X}_P \). Using \( \text{ker} \, d\mathfrak{X}_P = dP(\text{ker} \, d\mathfrak{X}) \), \( X \in \text{ker} \, d\mathfrak{X} \subset \mathfrak{g}^q \subset \mathcal{P}(\mathfrak{g}) \) if and only if

\[ \Pi_{S_j}(X^{(i)}(0)) = 0, \quad j = 0, \ldots, q; \quad i = 0, \ldots, j; \]

whence \( \text{ker} \, d\mathfrak{X} = \{ t \Pi_{S_0} X_0 + t^2 \Pi_{S_1} X_1 + \cdots + t^q \Pi_{S_{q-1}} X_{q-1} + t^{q+1} Y \mid X_i, Y \in \mathcal{P}(\mathfrak{g}) \} \)

\[ = \{ t W_1 + \cdots + t^q W_q + t^{q+1} Y \mid W_i \in S_0 + \cdots + S_{i-1}, Y \in \mathcal{P}(\mathfrak{g}) \}. \]

Therefore \( \text{ker} \, d\mathfrak{X}_P = dP(\text{ker} \, d\mathfrak{X}) \)

\[ = \{ t W_1 + \cdots + t^q W_q \mid W_i \in S_0 + \cdots + S_{i-1} \}, \]

which by the definitions of \( \mathcal{G}_{\text{ker} \, d\mathfrak{X}_P} \) and \( \text{Exp} \) (see (2.5.8)) is the Lie algebra of \( \mathcal{G}_{\text{ker} \, d\mathfrak{X}_P} \). Hence \( \mathcal{G}_0 \) has Lie-algebra \( \mathfrak{g}_0 \) as asserted. \( \blacksquare \)

Generalised Saletan Contraction of Lie Algebras

Discussion 3.1.7

In this sub-section, we give the most general possible version of Saletan contraction (2.4.6) of Lie algebras, completely free of the necessity of being derived from global contraction in the sense that all the definitions (2.4.1) to (2.4.5) are a priori derived from Definition (2.3.2).

Let \( u : [0,1] \times \mathfrak{g} \to \mathfrak{g} \) be \( C^\infty \) map, linear in its second argument, with \( u(t) \) invertible for \( t \in (0,1] \) and singular for \( t = 0 \). In definitions (2.4.1) to (2.4.5), to obtain Saletan contraction, we take \( E = \mathfrak{g} \) and \( d\Phi(X,t) = (u^{-1}(t)X,t), \quad X \in \mathfrak{g} \).
We must first show that \( u^{-1}(t) = \frac{d\phi(t)}{t^q} \) for some \( C^\infty \) map \( d\phi : [0,1] \times \mathbb{R} \rightarrow \mathbb{R} \) linear on \( \mathbb{R} \), and some \( q \).

Let \( \varphi(t) = \det u(t) \). Then \( \varphi(0) = 0 \). Let \( q \) be the integer such that \( \varphi^{(i)}(0) = 0 \), \( i = 0, \ldots, q-1 \) and \( \varphi^{(q)}(0) \neq 0 \). Write \( u^{-1}(t) \) as,

\[
 u^{-1}(t) = \frac{1}{\varphi(t)} \text{adj}(u(t)) \quad \text{(classical matrix adjoint)}
\]

\[
 = \left( \frac{t^q}{\varphi(t)} \right) \text{adj}(u(t)) \frac{1}{t^q}.
\]

If we show that \( \frac{t^q}{\varphi(t)} \) is \( C^\infty \), then we may take \( d\phi(t) = \left( \frac{t^q}{\varphi(t)} \right) \text{adj}(u(t)) \), and our definition of \( d\Phi(X,t) = (u^{-1}(t)X,t), X \in \mathbb{R} \) will fit Definitions (2.4.1) to (2.4.5).

This choice is confirmed by the following lemma:

**Lemma 3.1.8**

Let \( \varphi : [0,1] \rightarrow \mathbb{R} \) be a \( C^\infty \) map such that \( \varphi(t) \neq 0 \), \( t \in [0,1] \), \( \varphi^{(i)}(0) = 0 \), \( i = 0, \ldots, q-1 \), and \( \varphi^{(q)}(0) \neq 0 \) for some positive integer \( q \). Then the function, \( \varphi_q \) given by

\[
 \varphi_q(t) = \begin{cases} 
 \frac{\varphi(t)}{t^q}, & t \in (0,1] \\
 \lim_{t \to 0} \frac{\varphi(t)}{t^q}, & t = 0 
\end{cases}
\]

and the function \( (\varphi_q(t))^{-1} \) are \( C^\infty \) on \( [0,1] \).

**Proof:**

To see that \( \varphi_q \) is \( C^\infty \) on all of \( [0,1] \), proceed by induction. Take the Taylor series for \( \varphi(t) \) to order \( t^q \) with the Lagrange form of the remainder \( \mathcal{L}(t^{q+1}) \), which has order \( t^{q+1} \). Then

\[
 \lim_{t \to 0} \frac{\varphi(t)}{t^q} = \frac{\varphi^{(q)}(0)}{q!},
\]

and taking the Taylor series of \( \varphi(t) \) to order \( t^{q+1} \),

\[
 \lim_{t \to 0} \frac{\varphi_q(t) - \varphi_q(0)}{t} = \frac{\varphi^{(q+1)}(0)}{(q + 1)!},
\]
and \( \varphi_q^{(1)}(t) = \frac{1}{t^{q+1}} (t\varphi^{(1)}(t) - q\varphi(t)) \) which has Taylor series,

\[
\varphi_q^{(1)}(t) = \frac{\varphi^{(q+1)}(0)}{(q+1)!} + \frac{t\varphi^{(q+2)}(0)}{(q+2)!} + \frac{0(t^{q+3})}{t^{q+1}}.
\]

Proceeding to higher derivatives of \( \varphi_q(t) \) in this way,

\[
\varphi_q^{(i)}(t) = \frac{\varphi^{(q+i)}(0)}{(q+i)!} + \frac{t\varphi^{(q+i+1)}(0)}{(q+i+1)!} + 0(t^2)
\]

and \( \varphi_q \) is \( C^\infty \) on all of \([0,1]\). To see that \( (\varphi_q(t))^{-1} \) is \( C^\infty \) on \([0,1]\), observe that

\[
\frac{d}{dt}(\varphi_q(t))^{-1} = (-1)(\varphi_q(t))^{-2}\varphi_q^{(1)}(t),
\]

\[
\frac{d^2}{dt^2}(\varphi_q(t))^{-1} = (2)(\varphi_q(t))^{-3}\varphi_q^{(1)}(t)\varphi_q^{(1)}(t) - (\varphi_q(t))^{-2}\varphi_q^{(2)}(t)
\]

and the expression for \( \frac{d^i}{dt^i}(\varphi_q(t))^{-1} \) will be a multinomial in the variables

\( (\varphi_q(t))^{-2}, (\varphi_q(t))^{-3}, \ldots, (\varphi_q(t))^{-(i+1)}, \varphi_q^{(1)}(t), \varphi_q^{(2)}(t), \ldots, \varphi_q^{(i)}(t) \), with integer coefficients. Since \( \varphi_q(t) \neq 0, t \in [0,1] \), \( (\varphi_q(t))^{-1} \) is \( C^\infty \) on \([0,1]\). ■

Since the more specialized Saletan contraction given by (1.5.5) represents a large class of contraction examples (1.5.3) we illustrate how it fits in with the current scheme:

**Example 3.1.9**

Let \( U(t) = u + tI \) and notation be given by (1.5.2) to (1.5.5). Then the contraction of \( g \) with respect to

\[
d\Phi : (X,t) \to (U^{-1}(t)X,t) \quad \text{per (2.4.1)ff}
\]

has \( U^{-1}(t) = \frac{d\phi(t)}{t^q} \) where \( q \) is the smallest integer such that \( u^{q+1} = u^q \), and

\[
d\phi(t)_{|V_1} = t^q(u + tI)^{-1}
\]

\[
d\phi(t)_{|V_2} = \sum_{j=1}^{q} (-u)^{j-1} t^{q-j}
\]

185
is a $C^\infty$ map, $d\phi: [0,1] \times g \to g$, linear on $g$; and $g_0$ of (2.4.3) is isomorphic with $g_0$ of (1.5.5).

**Proof:**

By (1.5.4), $g = V_1 \oplus V_2$. Take the power series of $t^g U^{-1}(t) = t^g(u + tI)^{-1}$ on $V_2$, observing that $u|_{V_2}$ is nilpotent of order $q$. The proof that contraction (2.4.3) is precisely contraction (1.5.5) is given by Theorem (2.4.17(5)).

We now give the characterization of $d^q$ and $\ker dX$, but first we need a technical Lemma:

**Lemma 3.1.10** ($C^\infty$ analogue of Corollary (2.4.21))

Let $d^q_\infty$ denote the set $[0,1]$ with $d^q$ given by (2.4.4). Theorem (2.4.17) and Theorem (2.4.19) still hold true when their statements are modified by the uniform substitution of $d^q_\infty$ for $d^q$.

**Proof:**

Follows from the observation that $dX(d^q) = dX(d^q \cap C^\infty_0[0,1])$.

**Theorem 3.1.11**

Let $d^q$, $g_0$, and $\ker dX$ be given by Definitions (2.4.1) to (2.4.5), but with $d\phi(t) = t^g u^{-1}(t)$, $u(t)$ being specified by (3.1.7). Let

$$uC^\infty_g[0,1] = \{Y \mid Y(t) = u(t)X(t), X \in C^\infty_g[0,1]\}$$

and

$$uC^\infty_g(0)[0,1] = \{Y \mid Y(t) = u(t)X(t), X \in C^\infty_g[0,1], X(0) = 0\}.$$

Then

$$d^q_\infty = uC^\infty_g[0,1]$$

and

$$\ker dX = uC^\infty_g(0)[0,1].$$

**Proof:**

It is immediate that $uC^\infty_g[0,1] \subset d^q_\infty$. Let $X \in d^q_\infty$. Then $\lim_{t \to 0} u^{-1}(t)X(t)$ exists.
Let
\[ Y(t) = \begin{cases} u^{-1}(t)X(t), & t \in [0,1] \\ \lim_{t \to 0} u^{-1}(t)X(t), & t = 0 \end{cases} \] (3.1.12)

In the notation of (3.1.7),
\[ u^{-1}(t) = \left( \frac{t^q}{\varphi(t)} \right) \frac{\text{adj}(u(t))}{t^q}, \quad t \in (0,1]. \]

By (3.1.8),
\[ u^{-1}(t) = (\varphi_q(t))^{-1} \frac{\text{adj}(u(t))}{t^q} \]
where \((\varphi_q)^{-1} \in C^\infty_R[0,1].\) Then the function \( t \mapsto (\varphi_q(t))^{-1} \text{adj}(u(t))X(t) \) is \( C^\infty \) on \([0,1],\) and the limit, \( \lim_{t \to 0} (\varphi_q(t))^{-1} \text{adj}(u(t))X(t) \) exists. Putting \( Z(t) = (\varphi_q(t))^{-1} \text{adj}(u(t))X(t) \) we have \( Z \in C^\infty\mathcal{g}[0,1] \) and \( Z^{(i)}(0) = 0, \ i = 0, \ldots, q - 1.\)

Applying Lemma (3.1.8) to each coordinate function of \( Z,\) it follows that \( Y \in C^\infty\mathcal{g}[0,1]. \) Then \( X(t) = u(t)Y(t) \) and hence \( X \in uC^\infty\mathcal{g}[0,1], \) proving that \( d^q = uC^\infty\mathcal{g}[0,1]. \) Clearly \( uC^\infty\mathcal{g}(0)[0,1] \subset \ker d\mathcal{X}. \) Let \( X \in \ker d\mathcal{X}. \) Then \( X \in uC^\infty\mathcal{g}[0,1]. \) From (3.1.12) \( X(t) = u(t)Y(t), \ t \in [0,1], \) \( Y \in C^\infty\mathcal{g}[0,1] \) and \( Y(0) = 0, \) showing that \( \ker d\mathcal{X} \subset uC^\infty\mathcal{g}(0)[0,1]. \) This establishes the theorem. ■

**Corollary 3.1.12(a)**

For generalised Saletan contraction (3.1.7) of a Lie algebra \( \mathcal{g} \) (where the conditions of (2.5.1) are assumed to hold), the set \( u(0)\mathcal{g} \) is a Lie subalgebra of \( \mathcal{g}. \)

**Proof:**

Since \( uC^\infty\mathcal{g}[0,1] \) is a Lie subalgebra of \( C^\infty\mathcal{g}[0,1], \) endpoints of elements of this set must be a Lie subalgebra of \( \mathcal{g}, \) hence \( u(0)\mathcal{g} \) is a Lie subalgebra of \( \mathcal{g}. \) ■

**Example 3.1.13**

The structures of \( dP(q^q) \) and \( \ker d\mathcal{X}_p \) are easily computed using (3.1.11), and are particularly simple since the structures are the same whether \( u \) is \( C^\infty \) or is a polynomial of degree \( q \) in \( t. \)
Let \( \{X\}_q \) denote the element in \( \mathcal{P}^q(g) \) which is the coset of \( X \in C^\infty_2[0,1] \) by \( \ker dP \). Elements of \( \mathcal{P}^q(g) \) are uniquely determined by polynomials \( X \) of order \( q \). Without loss of generality, \( u(t) \) may be assumed to be a polynomial of degree \( q \) in \( t \).

Then,

\[
dP(d^q) = \{ \{Y\}_q \mid Y(t) = u(t)X(t), X \in \mathcal{P}(g), X \text{ of degree } q. \}
\]

and

\[
\ker dX_P = \{ \{Y\}_q \mid Y(t) = tu(t)X(t), X \in \mathcal{P}(g), X \text{ of degree } q. \}.
\]

For the special case of Saletan contraction in the form (1.5.5), the expression for \( dP(d^q) \) with \( d^q \) given at the beginning of §3.1, in terms of \( q \)'th order polynomials representing \( \mathcal{P}^q(g) \) has the elegant form,

\[
dP(d^q) = \{ \{X\}_q \mid X(t) = X_0 + X_1 t + \cdots + \frac{X_q t^q}{q!}, \sum_{j=0}^{q-1} (-u)^j X_j = 0 \}
\]

and similarly

\[
\ker dX_P = \{ \{X\}_q \mid X(t) = X_0 + X_1 t + \cdots + \frac{X_q t^q}{q!}, \sum_{j=0}^{q-1} (-u)^j X_j = 0. \}
\]

While the corresponding closed forms are immediately seen to be solutions to the above equations:

\[
dP(d^q) = \{ \{X\}_q \mid X(t) = X_0 + X_1 t + \cdots + \frac{X_q t^q}{q!},
\]

\[
X_0 = uY_0, \ X_j = uY_j + Y_{j-1}, \ j = 1,\ldots,q, \ Y_j \in g\}
\]

and

\[
\ker dX_P = \{ \{X\}_q \mid X(t) = X_0 + X_1 t + \cdots + \frac{X_q t^q}{q!},
\]

\[
X_0 = 0, \ X_1 = uY_0, \ X_j = uY_{j-1} + Y_{j-2}, \ j = 2,\ldots,q, \ Y_j \in g\}
\]
With the explicit description of $d_q^\infty$ and $\ker dX$ for generalised Saletan contraction in hand, it is not surprising that we can construct a contraction $G_0$ of $G$ which has Lie algebra $g_0$ given by the contraction of $g$ with respect to the fully generalised Saletan contraction procedure (3.1.7), directly and explicitly, without the use of Laurent maps:

**Definition 3.1.14**

Let $u_q : P^q(g) \to P^q(g)$ be the map

$$u_q : \{X\}_q \mapsto \{uX\}_q, \; X \in P(g)$$

where $uX$ is the element of $C^\infty[0,1]$ given by $uX : t \mapsto u(t)X(t)$. (In practical computations only the first $q + 1$ terms of the Taylor series of $u$ at $t = 0$ need be used in place of $u(t)$.)

Let $U_q : T^q(G) \to T^q(G)$ denote the $C^\infty$ map (in the notation of (2.5.4)),

$$U_q : (e, X_1, \ldots, X_q) \mapsto (e, Y_1, \ldots, Y_q)$$

where

$$\{tY_1 + \cdots + \frac{t^q Y_q}{q!}\}_q = u_q\{tX_1 + \cdots + \frac{t^q X_q}{q!}\}_q$$

**Theorem 3.1.15**

Let $G$ be a Lie group with lie algebra $g$, and $g_0$ the generalised Saletan contraction of $g$ as per (3.1.7) where the conditions of (1.5.1) are assumed to hold. Let $G_{dP(d^q)}$ be the connected Lie subgroup of $T^q(G)$ with Lie algebra $dP(d^\infty) = dP(d_q)$ given by (3.1.11). Let $U_q : T^q(G) \to T^q(G)$ be the $C^\infty$ map given by (3.1.14). Then $U_q(T^q(G))$ is a closed, connected, normal, nilpotent subgroup of $G_{dP(d^q)}$ and the factor Lie group, $G_0 = G_{dP(d^q)}/U_q(T^q(G))$ has Lie algebra $g_0$.

**Proof:**

$U_q(T^q(G))$ is clearly closed and connected in $T^q(G)$. To see that it is normal in $G_{dP(d^q)}$ observe that the Lie algebra of $U_q(T^q(G))$ is $S = \{V \mid V \in u_qP^q(g), \; V = \ldots \}$.
$u_q\{X\}_q, X(0) = 0\}$ and as $u_q\mathcal{P}(g)$ is a Lie algebra, $S$ is an ideal of $u_q\mathcal{P}(g) = dP(d^q_\theta)$. Hence $U_q(T^q_\theta(G))$ is normal in $G_{dP(d^q_\theta)}$.

$S$ is nilpotent because representative polynomials of elements of $S$ have highest order $q$ and lowest order 1, in $t$. Hence $S$ is $q$-step nilpotent. That the Lie group $G_0$ has Lie algebra $g_0$ follows from (3.1.11) and (3.1.13).
§3.2 More General Contractions

In this section, we investigate more general contractions than those of (2.3.5) by taking $\sigma > 1$ in (2.3.2) and by generalising $\mathfrak{d}^q$ to include the Laurent polynomials on $g$ as promised at the end of §2.4. The generalisation of (2.5.24(1)), the necessary and sufficient conditions for contraction, and thereby (2.5.30), will be seen to involve the jet bundle $J^q([0,1]^\sigma, G)$ of (1.4.18). The advantage of considering this type of contraction is that it broadens the possibilities of contraction "inside a larger dimensional Lie group" as given by (2.3.8), (2.3.9(a)) and (2.3.9(b)) which has the appealing property of having no necessary and sufficient conditions required to hold for contraction. A further filip for its investigation is the hope of understanding more of the structure of $\mathfrak{d}^q$ (2.4.4) by studying contractions for $\sigma > 1$, but with $i_1 = i_2 = \cdots = i_\sigma = 1$ in (2.3.2), possibly elucidating contractions with $\sigma = 1$ and $q > 1$.

Even more relevant yet, is the fact that, in view of the conclusions in Remark (2.4.22) that contractions of Lie groups may have a non-trivial rôle to play in obtaining non-perturbative solutions of the QCD field equations (the perturbative approach does not work for particle interactions at the energies in which particle theorists are most interested), more general contractions ($\sigma > 1$) involve jet bundles which form the background scenario for the definition of Bäcklund maps, themselves most useful for solving non-linear partial differential equations not unlike these field equations of QCD.

The final part of this section will be to show how $\mathfrak{d}^q$ can be generalised as a subalgebra of the Laurent polynomials, giving the revised versions of propositions and definitions leading up to a new Theorem (2.4.17) and (2.4.19).

We proceed by generalising in turn, the propositions and definitions of §2.3, §2.4 and §2.5, putting $\sigma > 1$, in so far as they have bearing on the above-mentioned aims, omitting infinite-dimensional theorems such as (2.5.42).
Definition 3.2.1 (Definitions of $\mathcal{G}_{(0,1]^\sigma}, \mathcal{S}^q, \mathcal{G}_0, \mathcal{D}^q$ and $\mathcal{X}$.)

In Definitions (2.3.3) to (2.3.7) uniformly replace $(0,1]$ by $(0,1]^\sigma$ (the cross-product of $(0,1], \sigma$-times) and $[0,1]$ by $[0,1]^\sigma$.

We now give an example of contraction, which shows that different contractions for $\sigma > 1$ will result depending on how $t$ approaches zero.

Example 3.2.2(a)

We contract the matrix form of $SO(4,1)$ in the spirit of Example (2.3.8) with Laurent map (3.2.1), $\Psi(g, s, t) = \varphi(s, t)g\varphi^{-1}(s, t)$

where

$$\varphi(s, t) = \begin{pmatrix} 1 & O \\ 1 & s \\ O & t \end{pmatrix}.$$  

Let $g \in SO(4,1)$, and $G_1$ and $2 \times 2$ submatrix of the $4 \times 4$ matrix

$$g = \begin{pmatrix} G_1 & G_2 & G_3 \\ G_4 & G_5 & G_6 \\ G_7 & G_8 & G_9 \end{pmatrix}.$$  

Then

$$\Psi(g, s, t) = \begin{pmatrix} G_1 & G_2/s & G_3/t \\ sG_4 & G_5 & s/tG_6 \\ tG_7 & t/sG_8 & G_9 \end{pmatrix}.$$  

Computing $G_0$, if we take the limit $(s, t) \to (0,0)$ such that $s/t$ approaches 1, we obtain

$$G_0 = \left\{ \begin{pmatrix} R & \tau_1 & \tau_2 \\ 0 & L \end{pmatrix} \middle| \begin{array}{l} R \in SO(2) \\ \tau_i \in \mathbb{R}^2 \\ L \in SO(1,1) \end{array} \right\}.$$  

On the other hand computing $G_0$ so that $s/t \to \alpha$ where $\alpha \neq 1$ and $\alpha > 0$, we get

$$G_0 = \left\{ \begin{pmatrix} R & \tau_1 & \tau_2 \\ 0 & ALA^{-1} \end{pmatrix} \middle| \begin{array}{l} R \in SO(2), \tau_i \in \mathbb{R}^2 \\ L \in SO(1,1), \\ A = (\begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & 1/\sqrt{\alpha} \end{pmatrix} \right\},$$  

192
which is distinct from, but isomorphic to the previous contraction.

Finally, computing $G_0$ by taking $\frac{s}{t} \to 0$ gives us

$$G_0 = \left\{ \begin{pmatrix} R & \tau_1 & \tau_2 \\ 0 & \iota_1 & 0 \\ 0 & r & \iota_2 \end{pmatrix} \middle| R \in SO(2), \tau_i \in \mathbb{R}^2 \right\}.$$

These last two contractions could not have been obtained from the Example of Case (2.3.8) with $\sigma = 1$ and $\varphi(t)$ in simple diagonal form. Consequently, it is worthwhile generalising (2.3.9):

**Convention 3.2.2(b)**

In view of 3.2.2(a) we will choose, for all subsequent contractions in §2.8, to take the limit $t \to 0$ in such a way that, in the notation of (2.3.2),

$$\frac{t_1^{k_1} \ldots t_\sigma^{k_\sigma}}{t_1^{l_1} \ldots t_\sigma^{l_\sigma}}$$

approaches a finite number if $\sum_{i=1}^{\sigma} k_i = q$, and diverges if $\sum_{i=1}^{\sigma} k_i \leq q$. This would be the case for the example of $t_i = (t)^{n(i)}$, $i = 1, \ldots, \sigma$, where $n(i)$ is a positive integer function of $i$ and $t$ is univariate. The objective of this convention is to make the generalisation of the theorems of Chapter 2 as straightforward as possible while still embracing significantly different examples such as (3.2.2(a)).

**Proposition 3.2.3** (Generalises the case of when the images of the Laurent maps are contained inside a Lie group.)

Let the Laurent maps $\Psi(\cdot, t)$ of (3.2.1) satisfy Case (2.3.8) with $t \in (0,1]^\sigma$. Then, in the notation of (3.2.1), $\mathcal{D}^q$ is a subgroup of the group $C_q^\sigma[0,1]^\sigma$, the group of $q$-times differentiable maps from $[0,1]^\sigma$ into $\mathcal{G}$ with respect to pointwise multiplication, and further $G_0$ is a group and $\mathcal{X}$ a homomorphism.

**Proof:**
The proof of (2.3.9(a)) holds true with [0, 1] and (0, 1] uniformly replaced by \([0, 1]_\sigma\) and \((0, 1]\_\sigma\) respectively and \(\mathcal{G}_0, \mathcal{D}_\sigma\) and \(\mathcal{X}\) given by (3.2.1). ■

Comments 3.2.4

In order to keep the notation simple, \(\mathcal{D}_\sigma\) has been chosen in (3.2.1) as a subset of \(\mathcal{C}^\sigma_\sigma[0, 1]\). Instead of \(\mathcal{C}^\sigma_\sigma[0, 1]\), we could have chosen the more streamlined set of maps \([0, 1]^\sigma \to \mathcal{G}\) that are \(i_j\)-times differentiable in the \(j\)th component of \(t\), for \(j = 1, \ldots, \sigma\) (in the notation of (2.3.2)). The superfluity in the use of \(\mathcal{C}^\sigma_\sigma[0, 1]\) is exactly analogous to using \(\mathcal{C}^\sigma_\sigma+r[0, 1]\) in the definition of \(\mathcal{D}_\sigma\) for \(\sigma = 1\), and no information is lost.

We will offset the statement of the analogue of (2.3.9(b)), by the treatment of contractions of Lie algebras for the case \(\sigma > 1\) (which will show that the definitions and propositions (including proofs) can be straightforwardly adapted to the case of \(\sigma > 1\)), as this will motivate the constructions necessary for the generalisation to a finite dimensional equivalent of (2.3.9(b)).

Definition 3.2.5 (Definitions of \(g(0,1]_\sigma, \mathcal{A}_\sigma, \mathcal{A}_\mathcal{G}_0, \mathcal{A}_\mathcal{D}_\sigma\) and \(d\mathcal{X}\).)

In Definitions (2.4.1) to (2.4.5), uniformly replace \((0,1]\_\sigma\) and \([0,1]\_\sigma\) by \([0,1]_\sigma\). Saletan contraction (1.5.1) can be generalised trivially to the case \(\sigma > 1\) and will be used in Propositions (3.2.7) and (3.2.8).

Proposition 3.2.6 (Saletan Contraction for \(\sigma > 1\))

Proposition (1.5.1) remains true when \([0,1]_\sigma\) and \((0,1]_\sigma\) are uniformly substituted for \([0,1]\) and \((0,1]\) respectively.

Proof:

\(\mathcal{G}_0\) is a vector space by definition.

Now \[ [X, Y]_0 = \lim_{t\to 0} U^{-1}(t)[U(t)X, U(t)Y], \]

194
and the bracket is antisymmetric and bi-linear. The Jacobi identity for $g_0$ follows from taking the limit $t \to 0$ in the identity,

$$\sum_{\text{Cyclicperm} (X,Y,Z)} U^{-1}(t)[[U(t)X, U(t)Y], U(t)Z] = 0.$$ 

The generalised versions of (2.4.7) and (2.4.8) still hold true:

**Proposition 3.2.7**

Let generalised Saletan contraction be given by (2.4.6), with $[0,1]$ and $(0,1]$ uniformly replaced by $[0,1]^{\sigma}$ and $(0,1]^{\sigma}$ respectively. If Saletan's condition of (3.2.6) holds, then $d^\sigma$ is a subalgebra of the Lie algebra $C^\sigma_\mathfrak{g}[0,1]^{\sigma}$ with bracket taken pointwise, $g_0$ as given by (3.2.5) has the Saletan Lie algebra structure of (3.2.6), and $d\mathfrak{x}$ is a homomorphism.

**Proof:**

Uniformly substitute $[0,1]^{\sigma}$ for $[0,1]$ in the proof of (2.4.7).

**Proposition 3.2.8**

Let generalised Saletan contraction be given by (2.4.6) with $[0,1]$ and $(0,1]$ uniformly replaced by $[0,1]^{\sigma}$ and $(0,1]^{\sigma}$ respectively. Then Saletan's condition of (2.8.6) holds if and only if $d^\sigma$ is a Lie algebra.

**Proof:**

See the proof of (2.4.8) which still holds for $\sigma > 1$.

Our aim is to prove theorems analogous to (2.4.17) and (2.4.19). To this end, we generalise $dP$ and $d\mathfrak{x}_P$. The generalisation of $dP$ as given by (2.4.10) is less obvious than for the preceding results.

**Lemma 3.2.9**

Let $\mathcal{P}^\sigma_\mathfrak{g}(\mathfrak{g})$ be the Lie algebra of all polynomials in $\sigma$ variables, $t \in [0,1]^{\sigma}$, factored by the ideal of all polynomials in $t$ of total degree in the $\sigma$ variables.
greater than \( q \). Then the map,

\[
dP : C^q_g[0,1] \to P^q_\sigma(g)
\]

\[
dP : X \mapsto \left\{ \sum_{N=0}^{q} \sum_{i=1}^{\sigma} \sum_{K_i=N}^{K_i} \frac{t^K}{K!} X^K(0) \right\}
\]

where \( K! = K_1!K_2! \cdots K_\sigma! \), \( t^K = t_1^{K_1}t_2^{K_2} \cdots t_\sigma^{K_\sigma} \),

\[
X^K(0) = \left. \frac{\partial^N X(t)}{\partial t_1^{K_1}\partial t_2^{K_2} \cdots \partial t_\sigma^{K_\sigma}} \right|_{t=0},
\]

(and where \( \{X\} \), for \( X \) a polynomial in \( t \), denotes the coset containing \( X \) by the above mentioned ideal) is a Lie algebra homomorphism.

**Proof:**

Let \( X, Y \) be polynomials on \( g \) in \( t \) of order \( K_1, K_2 \) respectively. Let \( X \mapsto \overline{X} \) denote the operation of dropping off terms in \( X \) of order above \( q \) in \( t \) (that is, terms \( t_1^{k_1} \cdots t_\sigma^{k_\sigma} \) are dropped if \( \sum_{i=1}^{\sigma} k_i > q \)). To prove that \( dP[X,Y] = [dPX,dPY] \), the problem is reduced to showing that,

\[
[X,Y] = \overline{[X,Y]}.
\]

Now \( X = \overline{X} + 0(t^{q+1}) \) and \( Y = \overline{Y} + 0(t^{q+1}) \) where \( 0(t^{q+1}) \) denotes terms \( t_1^{K_1}t_2^{K_2} \cdots t_\sigma^{K_\sigma} \), with \( \sum_{I} K_I > q \).

Therefore

\[
[X,Y] = \overline{[X + 0(t^{q+1}), Y + 0(t^{q+1})]}
\]

\[
= \overline{[X,Y]} + 0(t^{q+1})
\]

\[
= \overline{[X,Y]} \text{ establishing the result.}
\]

We generalise the map \( dX_P \):
Lemma 3.2.10

The map $dX_p : dP(d^q) \mapsto g_0$ given by

$$dX_p : \{dP(X)\} \mapsto dX_0(X),$$

in the notation of (2.8.9), is well-defined and satisfies $dX = dX_p \circ dP$.

Proof:

The proof of (2.4.11) remains valid when we replace the first $q + 1$ terms of the Taylor series of $X$ and $Y \in C^q_\mathbb{E}[0,1]$ by the expressions for $dP(X)$ and $dP(Y)$ given by (3.2.9) making $t \in [0,1]^\sigma$. $d\Psi(t)$ has the more general form,

$$d\Psi(t) = t_1^{i_1} \cdots t_\sigma^{i_\sigma} d\phi(t),$$

$$\sum_{j=1}^{\sigma} i_j = -q \text{ (taking } i_j < 0 \text{ without loss of generality) and the proof holds when we replace } \frac{d^q}{d^t} \text{ by a linear combination of all derivatives, } \frac{\partial^q}{\partial t_1^{k_1} \cdots \partial t_\sigma^{k_\sigma}} \text{ on account of Convention (3.2.2(b)).}$$

We can now generalise Theorem (2.4.17):

Theorem 3.2.11 (Necessary and Sufficient Conditions for Contractions)

Theorem (2.4.17) holds with the following uniform substitutions: $\mathcal{P}^q_g(\varrho)$ for $\mathcal{P}^q_g(\varrho)$, $[0,\epsilon)^\sigma$ for $[0, \epsilon)$ and $(0,1)^\sigma$ for $(0,1]$; where $(0,1)^\sigma$ and $[0,1)^\sigma$ have been substituted for $(0,1]$ and $[0,1]$ respectively in Definition (2.4.13).

Proof:

We give the necessary uniform substitutions in the proof of (2.4.17). All changes include $[0,1)^\sigma$ for $[0,1]$, $(0,1)^\sigma$ for $(0,1]$ and $t$ multivariate for $t$ univariate.

Proof of (1) $\Rightarrow$ (2): Replace $\sum_{i=0}^{q} t^i Z^{(i)}(0)$ by the expression for $dP(Z)$ in (3.2.9); replace “derivatives up to $q$’th order” by “derivatives $\frac{\partial^r Y(t)}{\partial t_1^{i_1} \cdots t_\sigma^{i_\sigma}}$ where $r \leq q$”; replace $t^{-q}$ by $t_1^{i_1} \cdots t_\sigma^{i_\sigma}$ $(i_j < 0)$. 197
Proof of (2) ⇒ (3): As per the changes specified above.

Proof of (3) ⇒ (4): Changes as given above.

Proof of (4) ⇒ (1): As given above.

Proof of (4) ⇒ (5): As given above; and [0, ε) replaced by [0, ε)\(^\sigma\).

Proof of (5) ⇒ (1): As per changes above. ■

Theorem 2.4.19 holds with very little modification:

**Theorem 3.2.12 (Differentiable Properties of Contraction)**

Theorem (2.4.19) is true for \(a > 1\).

**Proof:**

Modify the proof of (2.4.19) by replacing \((0, 1] \text{ and } [0, 1]\) by \((0, 1]^\sigma \text{ and } [0, 1]^\sigma\) respectively. ■

**Comments 3.2.12(a)**

In (3.2.12), the domain of \(dX_P\) is contained in the Lie algebra of multinomials on \(g\) in \(\sigma\) variables. In view of Comments (1.4.19) we expect that the domain of the generalisation \(X_P\) of \(dX_P\) will be contained in \(J^q([0, 1]^\sigma, \mathcal{G})\). The generalisation of \(T^q(\mathcal{G})\) of (2.5.1) is \(\bigcup_{g \in \mathcal{G}} J^q([0, 1]^\sigma, \mathcal{G})_{0,g}\), denoted \(J^q([0, 1]^\sigma, \mathcal{G})_{0,g}\) (as given by (1.4.18)), which is a subset of the \(q\)-jet bundle \(J^q([0, 1]^\sigma, \mathcal{G})\). We will now show that \(J^q([0, 1]^\sigma, \mathcal{G})_{0,g}\) is a Lie group, then suitably generalise the map \(P\) of (2.5.21), then prove a condensed version of (2.5.30), our principal aim being to show the role of jet bundles for the case \(\sigma > 1\).

**Proposition 3.2.13**

Let \(J^q([0, 1]^\sigma, \mathcal{G})_{0,g} = \bigcup_{g \in \mathcal{G}} J^q([0, 1]^\sigma, \mathcal{G})_{0,g}\) in the notation of (1.4.18), with elements \(\{f_1\}, \{f_2\}, f_i \in C_\mathcal{G}^\infty[0, 1]^\sigma\), and product \(\{f_1\} \cdot \{f_2\} = \{f_1 f_2\}\) where \(f_1 f_2\) is the map \(f_1 f_2 : [0, 1]^\sigma \to \mathcal{G}, t \mapsto f_1(t) f_2(t)\). (\(\{f_i\}\) is the equivalence class of \(f_i\) by the equivalence relation (1.4.17))

Then \(J^q([0, 1]^\sigma, \mathcal{G})_{0,g}\) is a Lie group.
Proof:

By formal analogy to the proof (2.5.3) that $T^2(G)$ is a group it follows that $J^q([0, 1]^\sigma, G)_{0,g}$ is a group with respect to the above product, with identity $\{e\} \in J^q([0, 1]^\sigma, G)_{0,e}$, where $e$ is the map $e : [0, 1]^\sigma \to G$ which is constantly the identity. To obtain the manifold structure of $J^q([0, 1]^\sigma, G)_{0,g}$, let $\{f\}$ be an arbitrary element of $J^q([0, 1]^\sigma, G)_{0,g}$, and $(\chi, N)$ a chart at $g$ in $G$. Let $f_\chi = \chi \circ f$ and $f_\chi^i$ be its coordinate functions. For convenience, identify $V_{q,n}^q$ (see (1.4.18)) as a coordinate space by suitable choice of basis polynomials. Define the map

$$\pi : J^q([0, 1]^\sigma, G)_{0,g} \to G$$

$$\pi : \{f\} \in J^q([0, 1]^\sigma, G)_{0,g} \mapsto g,$$

then the coordinate maps are given by

$$\chi^\pi : \pi^{-1}(N) \to \mathbb{R}^n \times V_{q,n}^q$$

$$\chi^\pi : \{f\} \in J^q([0, 1]^\sigma, G)_{0,g} \mapsto (\chi(g), P_q f_1^1, \ldots, P_q f_n^1) \quad \text{(see (1.4.18)).}$$

The $\chi^\pi$ are surjective maps by (1.4.18), which induce a topology on $J^q([0, 1]^\sigma, G)_{0,g}$. Overlapping charts $(\chi_1^\pi, \pi^{-1}(N_1), x_1, y_1^1, \ldots, y_1^n)$ and $(\chi_2^\pi, \pi^{-1}(N_2), x_2, y_2^1, \ldots, y_2^n)$ at $\{f_1\}$ and $\{f_2\}$ respectively, correspond to overlapping charts $(\chi_1, N_1)$ and $(\chi_2, N_2)$ in $G$. Writing $\chi_2^\pi \circ (\chi_1^\pi)^{-1}$ in coordinate form,

$$(x_2, y_2^1, \ldots, y_2^n) = (x_2(x_1), y_2^1(y_1), \ldots, y_2^n(y_1)).$$

$x_2$ is a $C^\infty$ function of $x_1$ and the $y_2^i$ are linear functions of $y_1$, induced by a change of variables in the Taylor series of $f_\chi^i$. Hence overlapping charts are $C^\infty$-related. To see that the product is $C^\infty$, let $(\chi_3^\pi, \pi^{-1}(N_3), x_3, y_3^1, \ldots, y_3^n)$ be a chart at $\{f_1\}\{f_2\}$. Then,

$$\chi_3^\pi((\chi_1^\pi)^{-1}(x_1, y_1)(\chi_2^\pi)^{-1}(x_2, y_2))$$

$$= (\chi_3(\chi_1^{-1}(x_1)\chi_3^{-1}(x_2)), P_q((\chi_1^{-1} \circ f_{x_1}(\chi_2^{-1} \circ f_{x_2}))^1_{x_3}, \ldots, P_q((\chi_1^{-1} \circ f_{x_1}(\chi_2^{-1} \circ f_{x_2}))^n_{x_3})$$

199
(where \( f_{x_i} = \chi_i \circ f_i \)) and the \( y_j \) will be multinomial functions of \( y_1 \) and \( y_2 \) and \( C^\infty \)-functions of \( x_1 \) and \( x_2 \), while \( x_3 \) will be a \( C^\infty \)-function of \( x_1 \) and \( x_2 \) only.

Hence \( J^q([0,1]^\sigma, \mathcal{G})_{0,\sigma} \) is a Lie group.

Regarding elements of \( T^q(\mathcal{G}) \) as equivalence classes of \( C^\infty \) maps \([0,1] \rightarrow \mathcal{G} \), we have the obvious corollary:

**Corollary 3.2.14** (Case \( \sigma = 1 \))

\[ J^q([0,1], \mathcal{G})_{0,\sigma} \text{ is precisely } T^q(\mathcal{G}). \]

The generalisation of \( P \) (2.5.19) uses \( q \)-jets:

**Definition 3.2.15**

Let \( f : [0,1]^\sigma \rightarrow \mathcal{G} \) be \( C^\infty \) and let \( j^q f \) be the \( q \)-jet of \( f \) given by (1.4.20), with respect to \( J^q([0,1]^\sigma, \mathcal{G}) \).

Let \( P \) be the map,

\[ P : C^\infty_\mathcal{G}[0,1]^\sigma \rightarrow J^q([0,1]^\sigma, \mathcal{G})_{0,\sigma} \]

\[ P : f \mapsto j^q f(0). \]

We employ \( P \) in the following, compact generalisation of (2.5.30).

**Theorem 3.2.16**

Define the map \( \mathfrak{X}_P : P(D^q \cap C^\infty_\mathcal{G}[0,1]^\sigma) \rightarrow \mathcal{G}_0 \) by (2.5.23) with \( P \) as in (3.2.15) and \( D^q \) given by (3.2.1). \( P \) is a homomorphism and \( \mathfrak{X}_P \) is well defined. Let \( D^q_\infty = D^q \cap C^\infty_\mathcal{G}[0,1]^\sigma \). If either,

(a) \( D^q_\infty \) is a subgroup of \( C^\infty_\mathcal{G}[0,1]^\sigma \) and \( \ker \mathfrak{X} \) is a normal subgroup of \( D^q_\infty \), or

(b) \( P(D^q_\infty) \) is a subgroup of \( J^q([0,1]^\sigma, \mathcal{G})_{0,\sigma} \), and \( \ker \mathfrak{X}_P \) a normal subgroup, then \( \mathcal{G}_0 \) and \( P(D^q_\infty) \) are Lie groups and \( \mathfrak{X}_P \) a \( C^\infty \) homomorphism. Further, (a) and (b) are equivalent.
Proof:

To show that $X_P$ is well-defined, the proof of (2.5.22) may be modified by replacing $\frac{d^q \phi}{dt^q}$ by a linear combination of all derivatives of the form $\frac{\partial^q}{\partial t_1^{j_1} \cdots \partial t_\sigma^{j_\sigma}}$, "of order $q$" and with the observation that (in view of (1.4.19)) if $g, h \in C_\sigma^\infty[0,1]^\sigma$ and $P(g) = P(h)$ then all derivatives of $\chi \circ g$ to order $q$ will agree with those of $\chi \circ h$. In view of the expression for $X(g)$ in coordinate form, $X(g) = X(h)$ as required, and $X_P$ is well-defined.

$P$ is a homomorphism: Let $f_1, f_2 \in C_\sigma^\infty[0,1]^\sigma$.

Then $P(f_1f_2) = \{\text{equivalence class of } f_1f_2 \text{ in } J^q([0,1]^\sigma, G)_{0,f_1f_2(0)}\}$

$= \{f_1\}\{f_2\} = P(f_1)P(f_2)$

in the notation of (3.2.13); and $P(e) = \{e\}$. We now show that (a) and (b) are equivalent. That (a) $\Rightarrow$ (b) follows from $P$ being a homomorphism and the definition of $X_P$. To see (b) $\Rightarrow$ (a), we follow the proof and notation of (2.5.24), (1) $\Rightarrow$ (2): Now $P^{-1}(P(D^q_\infty))$ is a subgroup of $C_\sigma^\infty[0,1]^\sigma$, and to see that $P^{-1}(P(D^q_\infty))$ equals $D^q_\infty$ we only need show $D^q_\infty \supseteq P^{-1}(P(D^q_\infty))$. This is accomplished by establishing (2.5.27) with $\frac{d^J \phi}{dt^J}$ replaced by all derivatives of the form $\frac{\partial^J}{\partial t_1^{j_1} \cdots \partial t_\sigma^{j_\sigma}}$.

Now if $k \in \text{ker } P$, then $\frac{\partial^J y(0)}{\partial t_1^{j_1} \cdots \partial t_\sigma^{j_\sigma}} = 0; J = 0, \ldots, q; \forall j_i \geq 0 \ni \sum_{i=1}^\sigma j_i = J$. Now

$$\frac{\partial^J}{\partial t_1^{j_1} \cdots \partial t_\sigma^{j_\sigma}}(\chi^{-1}(\zeta(x(t), y(t))), t)|_{t=0}$$

will be multinomial in the terms $\frac{\partial^K x(0)}{\partial t_1^{k_1} \cdots \partial t_\sigma^{k_\sigma}}$ and $\frac{\partial^K y(0)}{\partial t_1^{k_1} \cdots \partial t_\sigma^{k_\sigma}}$ whence all terms
including any one of the latter variables will vanish. Hence

\[
\frac{\partial^J}{\partial t_{i_1}^{j_1} \cdots \partial t_{i_\sigma}^{j_\sigma}} \phi(\zeta(x(t), y(t)), t) \bigg|_{t=0} = \frac{\partial^J}{\partial t_{i_1}^{j_1} \cdots \partial t_{i_\sigma}^{j_\sigma}} \phi(x^{-1}(\zeta(x(t), y(0))), t) \bigg|_{t=0} = \frac{\partial^J}{\partial t_{i_1}^{j_1} \cdots \partial t_{i_\sigma}^{j_\sigma}} \phi(x_1^{-1}(x(t)), t) \bigg|_{t=0} = 0,
\]

by (3.2.2(b)) as we wished to show, and \( \mathcal{D}_{\infty}^g \) is a subgroup of \( C^\infty_0[0,1]^q \). To show that \( \ker \mathcal{X} \) is a normal subgroup of \( \mathcal{D}_{\infty}^g \), we again follow (2.5.24), but must show that

\[
\lim_{t \to 0} \frac{1}{t_{i_1}^{j_1} \cdots t_{i_\sigma}^{j_\sigma}} \phi(x^{-1}(\zeta(x(t), y(t))), t) \bigg|_{t=0} = 0 \tag{3.2.17}
\]

given that \( \frac{\partial^J y(0)}{\partial t_{j_1}^{j_1} \cdots \partial t_{j_\sigma}^{j_\sigma}} = 0; \ J = 0, \ldots, q \ \forall j_i > 0 \ \exists \sum_{i=1}^\sigma j_i = J, \)

and that

\[
\lim_{t \to 0} \frac{1}{t_{i_1}^{j_1} \cdots t_{i_\sigma}^{j_\sigma}} \phi(x_1^{-1}(x(t)), t) \bigg|_{t=0} = 0.
\]

By expanding the LHS of (3.2.17) in terms of partial derivatives of \( x(t) \) and \( y(t) \) and \( t = 0 \) as before, we get,

\[
\lim_{t \to 0} \frac{1}{t_{i_1}^{j_1} \cdots t_{i_\sigma}^{j_\sigma}} \phi(x^{-1}(\zeta(x(t), y(t))), t) \bigg|_{t=0} = \lim_{t \to 0} \frac{1}{t_{i_1}^{j_1} \cdots t_{i_\sigma}^{j_\sigma}} \phi(x^{-1}(\zeta(x(t), y(0))), t) \bigg|_{t=0} = \lim_{t \to 0} \frac{1}{t_{i_1}^{j_1} \cdots t_{i_\sigma}^{j_\sigma}} \phi(x_1^{-1}(x(t)), t) \bigg|_{t=0} = 0
\]

and \( \ker \mathcal{X} \) is a normal subgroup of \( \mathcal{D}_{\infty}^g \).

If we show that \( P(\mathcal{D}_{\infty}^g) \) is closed in \( J^q([0,1]^q, G)_0, g \) then \( P(\mathcal{D}_{\infty}^g) \) will be a Lie subgroup (1.1.8).
Let \( \{f\} \in P(D_\infty^g) \) with \( \{f\} \in J^q([0,1]^\sigma, G)_0,\sigma \) and \((\chi, N, x^i)\) a chart at \( g \). \( \{f\} \) is in \( P(D_\infty^g) \) if and only if

\[
\frac{\partial^J}{\partial t_1^{j_1} \cdots \partial t_\sigma^{j_\sigma}} \phi(\chi^{-1}(\chi \circ f(t)), t)|_{t=0} = 0
\]

\( J = 0, \ldots, q - 1, \sum_{i=1}^\sigma j_i = J, j_i \geq 0. \)

In terms of the chart \( \chi^\sigma \) of (3.2.13), the above conditions determine \( C^\infty \)-functions \( F_{j_1,\ldots,j_\sigma}^J \) of the \((x, y)\) satisfying, \( F_{j_1,\ldots,j_\sigma}^J(x, y) = 0 \) in the same way as for (2.5.68). Then any sequence in \( P(D_\infty^g) \) converging to \( \{f\} \) in \( J^q([0,1]^\sigma, G)_0,\sigma \) will converge to a point of \( P(D_\infty^g) \) and hence \( P(D_\infty^g) \) is closed by the same argument as in the proof of (2.5.30) at (2.5.68). Similarly, \( \ker \chi_P \) is closed in \( P(D_\infty^g) \) and is a Lie subgroup of \( P(D_\infty^g) \). With the topology and manifold structure of \( P(D_\infty^g)/\ker \chi_P \), \( G_0 \) is a Lie group and \( \chi_P \) a \( C^\infty \)-homomorphism.

Equipped with the necessary theorems we conclude this section by giving the generalisation of (2.3.9(b)) promised in (3.2.4).

**Corollary 3.2.18**

Let each Laurent map \( \Psi(\cdot, t) \) be a \( C^\infty \)-monomorphism of \( G \) into a Lie group \( G_L \) as in (2.3.8) with \((0,1] \) replaced by \((0,1]^\sigma \). Then \( G_0 \) is always a Lie group and the contraction mapping is given by \( \chi = \chi_P \circ P \) where \( \chi_P \) is a \( C^\infty \)-homomorphism of the Lie groups, \( P(D_\infty^g) \) onto \( G_0 \).

**Proof:** Proposition (3.2.3) and Theorem (3.2.16).

**The Rôle of Laurent Polynomials in Contractions of Lie Algebras.**

The objective of this section is to fulfil our undertaking in (2.4.22) by showing how \( \mathfrak{g}^q \) may be generalised to include the Laurent polynomials on \( \mathfrak{g} \) in a non-trivial way. The idea being to highlight the possible physical import of the applications of this contraction theory. We first give the following class of examples:
Example 3.2.19

Let $\phi$ be given by (2.3.2) and suppose that $d\phi(t)$ is polynomial in $t$. Let $K$ be the highest positive order of $t$ in the Laurent polynomial $\frac{d\phi(t)}{t^q}$. In evaluating $g_0$ of (2.4.5), if we allow $X : (0,1] \to \mathbb{R}$ to be a Laurent polynomial in $t$, then the limit $\lim_{t \to 0} \frac{d\phi(t)}{t^q} X(t)$ can only exist for Laurent polynomials $X(t)$ of minimum degree $-K$ in $t$. Then $d\mathcal{X}(X)$ becomes,

$$d\mathcal{X}(X) = \lim_{t \to 0} \frac{d\phi(t)}{t^{q+K}(X(t))}$$

where $t^K X(t)$ is polynomial in $t$. In view of Corollary (2.4.21) it appears that in this case, an expansion of the domain of $d\mathcal{X}$ to include the Laurent polynomials leads to a contraction closely related to a contraction with respect to $\frac{d\phi(t)}{t^{q+K}}$ with domain $\mathcal{L}^{q+K}(\mathcal{P})$ (2.4.20). We now give the general theory which will enable us to prove this in precise form.

Definition 3.2.20 (Re definition of $g_{(0,1)}$, $\mathbb{L}^q$, $g_0$, $\mathbb{L}^q$ and $d\mathcal{X}$)

Let $g_{(0,1)}$ be given by (2.4.1); $\mathbb{L}^q$ by (2.4.2) but with $g \times [0,1]$ replaced by $g \times (0,1]$ and “$C^q$-section” by “Laurent polynomial section”; $g_0$ by (2.4.3); $\mathbb{L}^q$ by (2.4.4) with $g \times [0,1]$ replaced by $g \times (0,1]$ and with “$C^q$-section” replaced by “Laurent polynomial section”; and $d\mathcal{X}$ given by (2.4.5).

Let $\mathcal{P}_L(g)$ denote the Lie algebra of Laurent polynomials on $\mathcal{G}$, in a single variable $t \in (0,1]$.

Comments 3.2.21

We will now generalise Theorem (2.4.17) for this case of an expanded domain $\mathcal{D}^q$ of the contraction mapping. Let $\mathcal{P}_L^q(g)$ denote the subalgebra (it is not an ideal) of $\mathcal{P}_L(g)$ of polynomials of degree greater than $q$. Note that if we generalised $dP$ as a map $dP : \mathcal{P}_L(g) \to \mathcal{P}_L(g)/\mathcal{P}_L^q(g)$, the range of $dP$ is now not finite.
dimensional. As the initial motivation for defining $dP$ was that its range should be finite-dimensional, we omit the use of $dP$ in the following theorem.

**Theorem 3.2.22**

Let $g_{(0,1)}$, $g$, $d_g$ and $dX$ be given by (3.2.20). Then with respect to these definitions, statements (2), (3), (4) and (5) of Theorem (2.4.17) are equivalent.

**Proof:**

The only modifications to the proof of (2.4.17) that are necessary are for arguments that require an element $X$ of $d_g$ to be a map $X : [0,1] \rightarrow g$ and not just a map $X : (0,1] \rightarrow g$. Hence the proofs of (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) hold with the new definitions mutatis mutandis. We give the proof of (4) $\Rightarrow$ (2).

**Proof of (4) $\Rightarrow$ (2):**

Let $X, Y \in d_g$. By assumption,

$$\alpha dX(X) + \beta dX(Y) \in g_0, \quad \forall \alpha, \beta \in \mathbb{R}$$

i.e. $dX(\alpha X + \beta Y) \in g_0$, whence $\alpha X + \beta Y \in d_g$ and $d_g$ is a subspace of $P_L(g)$.

Further, $[dX(X), dX(Y)] \in g_0$ by assumption. Now,

$$[dX(X), dX(Y)] = \lim_{t \rightarrow 0} d\Phi(t)[X(t), Y(t)]$$

by definition of the product on $g_0$,

$$= dX([X, Y])$$

Then $[X, Y] \in d_g$ which is hence a subalgebra of $P_L(g)$.

Let $Z \in \ker dX$. Then $[dX(Z), dX(X)] = 0$, $\forall X \in d_g$.

Therefore $0 = [dX(Z), dX(X)] = dX([Z, X])$ and so $[Z, X] \in \ker dX$. Ker $dX$ is clearly a subspace of $d_g$ and so is an ideal, establishing (4) $\Rightarrow$ (2).

The proof of (4) $\Rightarrow$ (5) of (2.4.17) holds with the new definitions (3.2.2) without change.
We give the proof of (5) \(\Rightarrow\) (2):

Let \(X, Y \in \mathcal{J}^q\); then \([X(t), Y(t)]\) is defined for \(t \in (0, 1]\) and \([X, Y] \in \mathcal{P}_L(g)\). Thus if \(\lim_{\mathcal{E} \times [0,1]} d\Phi[X(t), Y(t)]\) exists, then \([X, Y]\) will be in \(\mathcal{J}^q\). To complete this proof, the proof from this stage on of (5) \(\Rightarrow\) (2) in (2.4.17) holds with the new definitions (3.2.20).

This establishes (5) \(\Rightarrow\) (2) (done in the proof of (5) \(\Rightarrow\) (1)) and completes the proof of the theorem. ■

We now present the proposition promised in example (3.2.19):

Observe that, in the following lines, \(\mathcal{J}^{q+K}\) is in fact \(\mathcal{J}^{q+K}(\mathcal{P})\). In view of (2.4.21) however, we economise on notation by writing simply \(\mathcal{J}^{q+K}\).

**Proposition 3.2.23**

Let \(d\Phi(\cdot, t)\) and \(d\phi(\cdot, t)\) denote the derivatives of the maps \(\Phi, \phi\) defined in (2.3.2), or be freshly defined linear maps with the properties of derivatives of \(\Phi\) and \(\phi\) as given by (2.3.2).

Suppose that \(d\phi(t)\) is polynomial in \(t\), with highest positive order of \(t\) being \(K\). Let \(g_0, L, d\mathcal{X}_L\) denote respectively the re-naming of the sets \(g_0, \mathcal{J}^q\) and \(d\mathcal{X}\) given by (3.2.20), corresponding to the Laurent map derivatives, \(d\mathcal{T}(\cdot, t) = \cdot\) of (2.3.2).

Let \(g_0, d^{q+K}, d\mathcal{X}\) be given by (2.4.3), (2.4.4) and (2.4.5) respectively, relative to the Laurent map derivatives, \(d\Psi(\cdot, t) = \frac{d\phi(\cdot, t)}{t^q}\) of (2.3.2).

Then, defining \(t^Kd^q_L = \{X|X(t) = t^KY(t), Y \in d^q_L\}\), and similarly defining \(t^K\ker d\mathcal{X}_L\), we have, \(t^Kd^q_L = d^{q+K}, t^K\ker d\mathcal{X}_L = \ker d\mathcal{X}, d\mathcal{X}(d^{q+K}) = d\mathcal{X}_L(d^q_L)\) and \(d^q_L\) is a Lie algebra if and only if

\[ [d^{q+K}, d^{q+K}] \subset d^{q+2K} \]
where $\mathfrak{d}_{q+2K}$ is the set given by (2.4.4) with Laurent map derivative, $d\Psi(\cdot, t) = \frac{d\phi(\cdot, t)}{t^{q+2K}}$.

$\ker d\mathfrak{X}_L$ is an ideal of $\mathfrak{d}_L^q$ if and only if

$$[\mathfrak{d}_{q+K}, \ker d\mathfrak{X}] \subseteq \ker d\mathfrak{X}_{(q+2K)}$$

where $d\mathfrak{X}_{(q+2K)}$ is the contraction map for $\mathfrak{d}_{q+2K}$.

**Remark 3.2.24.**

Because of negative powers of $t$ in elements of $\mathfrak{d}_L^q$, we have that $\mathfrak{d}_L^q$ and $\mathfrak{d}_{q+K}$ are not isomorphic, and hence $d\mathfrak{X}(\mathfrak{d}_{q+K})$ and $d\mathfrak{X}_L(\mathfrak{d}_L^q)$ are not isomorphic, even though they are set-theoretically the same.

**Proof of (3.2.23):**

$$t^K \mathfrak{d}_L^q = t^K \{ \{ Y | Y \in \mathcal{P}_L(g) , \lim_{t \to 0} \frac{d\phi(t)Y(t)}{t^q} \text{ exists} \}$$

$$= t^K \{ \{ Y | Y \in \mathcal{P}_L(g), \lim_{t \to 0} \frac{d\phi(t)Y(t)}{t^q} \text{ exists and } Y \text{ is of lowest order } t^{-K} \}$$

$$= t^K \{ \{ Y | Y \in \mathcal{P}_L(g), Y(t) = \frac{1}{t^K} X(t), X \in \mathcal{P}(g) \text{ and } \lim_{t \to 0} \frac{d\phi(t)X(t)}{t^{q+K}} \text{ exists} \}$$

$$= \mathfrak{d}_{q+K}.$$

Similarly, $t^K \ker d\mathfrak{X}_L = \ker d\mathfrak{X}$.

That $t^K \mathfrak{d}_L^q = \mathfrak{d}_{q+K}$ follows immediately from the definitions of $\mathfrak{d}_L^q$ and $\mathfrak{d}_{q+K}$ and that $d\mathfrak{X}(\mathfrak{d}_{q+K}) = d\mathfrak{X}_L(\mathfrak{d}_L^q)$ also follows directly from the definitions of $d\mathfrak{X}$ and $d\mathfrak{X}_L$. Suppose that $\mathfrak{d}_L^q$ is a Lie algebra. Then for $X, Y \in \mathfrak{d}_{q+K}$,

$$[\frac{X(t)}{t^K}, \frac{Y(t)}{t^K}] = \frac{Z(t)}{t^K} \text{ with } Z \in \mathfrak{d}_{q+K};$$

therefore $[X(t), Y(t)] = t^K Z(t)$ whence

$[X, Y] \in \mathfrak{d}_{q+K}$. Further,

$$\lim_{t \to 0} \frac{d\phi(t)[X(t), Y(t)]}{t^{q+2K}} = \lim_{t \to 0} \frac{d\phi(t)Z(t)}{t^{q+K}}.$$
yielding $[X, Y] \in \mathfrak{d}^{q+2K}$.

Suppose that $\mathfrak{d}^{q+K}$ is a Lie algebra, and $[\mathfrak{d}^{q+K}, \mathfrak{d}^{q+K}] \subseteq \mathfrak{d}^{q+2K}$.

Let $X, Y \in \mathfrak{d}^{q+K}$. Then

$$\left[ \frac{X(t)}{t^K}, \frac{Y(t)}{t^K} \right] = \frac{Z(t)}{t^{2K}} \quad \text{with} \quad Z \in \mathfrak{d}^{q+2K}.$$ 

Consequently

$$\lim_{t \to 0} \frac{d\phi(t) \left[ \frac{X(t)}{t^K}, \frac{Y(t)}{t^K} \right]}{t^q} = \lim_{t \to 0} \frac{d\phi(t)Z(t)}{t^{q+2K}}$$

and the limit exists, showing $\mathfrak{d}_L^q$ to be a Lie algebra.

The proof of the conditions for $\Ker d\mathfrak{X}_L$ to be an ideal of $\mathfrak{d}_L^q$ is similar. ■
Chapter 4: Contractions of Representations of Lie Groups.

§4.1 Using the method of orbits. Contracting a sequence of forms.

Contracting orbits.

In this chapter we endeavour to extend theorems of the type of (1.6.6) and Dooley and Ricci (1985), Theorem 8.1 to the fully general case of showing how the representations of the sequence of Lie groups $\Phi(\mathcal{G}, t)$ for arbitrary $\mathcal{G}$ and general $\Phi$ will in the limit become representations of $\mathcal{G}_0$, the contracted Lie group. It would be too much to expect exactly analogous results at this level of generality, although substantial progress has been made in this direction. These results will be modulo a mild connectivity condition on $\mathcal{G}$, and the requirement that $\dim \mathcal{G} = \dim \mathcal{G}_0$, which is totally standard for all treatments of Lie group contraction in existence as given by other authors (note that the second condition does not apply to Theorem (4.3.13)).

We give a pragmatic theorem which provides one, simple test which indicates whether all matrix elements or no matrix elements of a representation of the contracted Lie group will be the limit of matrix elements of a sequence of isomorphic copies of the original Lie group. That is, a simple yes/no test which provides an "all or nothing" answer for the question of whether there is one among the contraction of representations of the original Lie group that is equal to a given representation of the contracted Lie group.

We assume throughout Chapter 3 that the method of orbits (see §1.3) "works" for $\mathcal{G}_0$ and "sufficiently many" groups $\Phi(\mathcal{G}, t)$ in the sense that there is a sequence $\{t_n\} \subset [0, 1], t_n \to 0$ such that for each functional $\beta_{t_n} \in (d\Phi(g_{t_n})))^*$, with $\beta_0 = \lim_{t_n \to 0} \beta_{t_n}$, the particular representations of $\Phi(\mathcal{G}, t_n)$ and $\mathcal{G}_0$ constructed by the method of orbits for each $\beta_{t_n}$ and $\beta_0$ will be irreducible.

209
The underlying ideas of Chapter 2 and their subsequent development provide a very natural and quick framework for approaching the contraction of representations. The structure of the theory naturally encompasses the awkward fact that the stabiliser of $\beta_0$ will in general be of higher dimension than the stabiliser of $\beta_t$ for $t \in (0,1]$. Similarly the possibility of the dimension of the polarization for $\beta_0$ being of higher dimension than that of the polarisation for $\beta_t, t \in (0,1]$ is easily allowed for, and the criteria for deciding whether a suitable polarization for $\beta_0$ exists or not can be given in terms of conditions on either $d^q \otimes \mathbb{C}$ or $dP(d^q) \otimes \mathbb{C}$.

Finally, suppose that the Hilbert spaces $\{\mathcal{H}_{tn}, \mathcal{H}_0\}$ each carry an irreducible representation of each of the Lie groups $\{\Phi(\mathcal{G}, t_n), \mathcal{G}_0\}$ respectively, constructed via the method of orbits as alluded to above. The approach to contractions of Lie groups developed in this work facilitates a remarkably simple proof of a powerful "Bootstrap" theorem for the contraction of representations which states that if we can find just one sequence of elements $\{f_{tn}, f_0\}$ with $f_{tn} \in \mathcal{H}_{tn}$, $f_0 \in \mathcal{H}_0$ such that $f_0$ is non-zero and $\{f_{tn}, f_0\}$ satisfies an appropriate "continuity" condition as $n \to \infty$, then any operator $\mathcal{R}_0$ in the representation of $\mathcal{G}_0$ on $\mathcal{H}_0$ is the limit (in a precise sense to be described below) of a sequence of operators $\mathcal{R}_{tn}$ from the representation of $\Phi(\mathcal{G}, t_n)$ on $\mathcal{H}_{tn}$. This is just the sort of result which Saletan (1961) postulated should exist. The Bootstrap Theorem applies to a very wide class of Lie groups and to arbitrary representations, and was totally unexpected at this level of generality (the only requirement on the Lie group is a connectivity condition). A further advantage of this theorem is that, even though its statement is within the context of the method of orbits, it is largely independent of the method of orbits, as we will see in §4.3.

The balance of section 4.3 is concerned with proving the existence of the sequence $\{f_{tn}, f_0\}$ as described above.
Applying the Contraction Process and its Philosophy to the Method of Orbits.

In §1.3 we saw how co-adjoint orbits in the space of functionals $g^*$ formed the underpinnings and thread along which the method of orbits was developed. It behooves us then, to study the contraction of functionals in $g^*$ in detail.

Following this, the contractions of orbits and the contractions of polarizations (done in section 4.2) as defined in §1.3 will be drawn together in §4.3.

In Chapter 2, we saw how $g$ was contracted to $g_0$ by defining a map from $C^\infty_{[0,1]}$ onto $g_0$. A similar approach works for the contraction of $g^*$ to $g_0^*$. First we must dualise $d\Phi$:

**Definition 4.1.1**

Let the dual $(d\Phi^{-1})^*(-, t)$ of the maps $d\Phi^{-1}(-, t): d\Phi(g, t) \rightarrow g$, be given by the maps,

$$(d\Phi^{-1})^*(-, t): g^* \rightarrow d\Phi(g, t)^*,$$

$$\langle (d\Phi^{-1})^* (\beta, t), X_t \rangle = \langle \beta, d\Phi^{-1}(X_t, t) \rangle$$

where $\beta \in g^*$ and $X_t \in d\Phi(g, t)$.

**Remark (Notation)**

As we did for $d\Phi(-, t)$ at (2.4.12), we write $(d\Phi^{-1})^*(-)$ for $(d\Phi^{-1})^*(-, t)$ where it will not cause confusion.

From Chapter 2, we expect that the maps $(d\Phi^{-1})^*(-, t)$ will provide an analogous map to $dX$, whose domain will be a space of curves on $g^*$ and whose range will be $g_0^*$. The following example confirms this.
Example 4.1.2

Let $G$ be a Lie subgroup of $GL(n, \mathbb{R})$. Let $\beta \in \mathfrak{g}^*$, be represented by the $n \times n$ matrix $\beta_{ij}$, such that $\beta$ is the map,

$$\beta : \mathfrak{g} \to \mathbb{R}$$

$$\beta : X \mapsto X_{ij} \beta_{ij} \quad \text{(sum convention)},$$

where $X \in \mathfrak{g}$ is an $n \times n$ matrix.

We want to define a contraction of $\beta$ associated to a given Lie group contraction:

Let the Laurent maps $\Psi(\cdot, t)$ for a contraction be given by the example of Case (2.3.8).

Then the functional $(d\Phi^{-1})^* \beta$ of $d\Phi(\mathfrak{g}, t)^*$ given by

$$((d\Phi^{-1})^* \beta, X_t) = X_t^{ij}(\varphi^{-1}_{ki}(t)\beta^{kl}\varphi_{jl}(t)),$$

$X_t \in d\Phi(\mathfrak{g}, t)$, \hspace{1cm} (4.1.3)

and

$$(d\Phi^{-1})^* \beta = (\varphi^T)^{-1}(t)\beta \varphi^T(t),$$

gives a sequence of functionals which in the limit may provide a functional $\beta_0$ on $\mathfrak{g}_0^*$.

If we naively try to "contract" $\beta$ by computing the limit,

$$\beta_0 = \lim_{t \to 0} (\varphi^T)^{-1}(t)\beta \varphi^T(t) \quad \text{(the limit is taken in } M(n \times n, \mathbb{R})) \hspace{1cm} (4.1.4)$$

for arbitrary $\beta$, this limit won't in general exist, and for all the limits that do exist, not all of $\mathfrak{g}_0^*$ will be obtained.

The obvious solution is to take $\beta \in C^2_{\mathfrak{g}}[0, 1]$ in the philosophy of Chapter 2, and then (4.1.4) will exist provided $\beta$ satisfies a system of homogeneous linear equations in the variables $\frac{d^i \beta(0)}{dt^i}, \ i = 0, \cdots, q$. One might hope that by this process, all elements of $\mathfrak{g}_0^*$ will be obtained. That this is not the case, is illustrated by the following example:
Example 4.1.5

In a special case of Example (4.1.2), namely the contraction of the de Sitter group (outlined in (2.3.10)) to the Poincaré group \( \mathcal{P} \), we will see that for any \( \beta \in C^1_0[0,1] \) such that (4.1.4) exists, the functional \( \beta_0 \) will always be trivial on the translation subalgebra of the Poincaré algebra, and hence the representation of the Poincaré group constructed by the method of orbits from \( \beta_0 \) will be trivial on the translation subgroup. This is a consequence of the fact that if we attempt to define \( \beta_0 \) on \( g_0 \) in the following way:

\[
\langle \beta_0, d\mathcal{X}(X) \rangle = \lim_{t \to 0} ((d\Phi^{-1})^*(\beta(t), t), d\Phi(X(t), t)) \forall X \in \mathfrak{g}^1, \tag{4.1.6}
\]

we then have,

\[
\langle \beta_0, d\mathcal{X}(X) \rangle = \lim_{t \to 0} (\beta(t), X(t)) \\
= (\beta(0), X(0)).
\]

That is, only the endpoint of \( \beta \) contributes to the value of \( \beta_0 \) on \( g_0 \). As \( \beta(t) \) acts by inner product on \( g \), we may take \( \beta(t) \in g \).

Therefore

\[
(\varphi^T)^{-1}(t)\beta(t)\varphi^T(t) = \begin{pmatrix} B(t) & t\Gamma(t) \\ \Delta(t)/t & E(t) \end{pmatrix} \tag{4.1.7}
\]

where \( \beta(t) = \begin{pmatrix} B(t) \\ \Delta(t) \\ E(t) \end{pmatrix} \) and \( B(t) \) is a \( 4 \times 4 \) matrix. For \( \beta(t) \) to be in \( g \), we must have,

\[
\beta(t) = \begin{pmatrix} B(t) \\ -\Gamma^T(t)M \end{pmatrix} \Gamma(t), \quad \mathcal{M}B^T(t) = -\mathcal{M}B(t), \quad \mathcal{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Computing the limit as \( t \to 0 \) in (4.1.7) per (4.1.4) we get

\[
\beta_0 = \begin{pmatrix} B(0) \\ -\Gamma^T(0)\mathcal{M} \end{pmatrix}, \quad \text{where } \Gamma^T(0)\mathcal{M} = 0.
\]

213
By (4.1.3) $\beta_0$ is defined on $g_0$ by

$$\langle \beta_0, d\mathcal{X}(X) \rangle = \lim_{t \to 0} (d\Phi X)_{ij}((d\Phi^{-1})^\ast \beta(t))_{ij} = (d\mathcal{X}(X))_{ij}(\beta_0)_{ij}, \quad \forall X \in d^1$$

and is trivial on the translation subalgebra of $\mathcal{P}$.

The translation subgroup $T$ is contained in the stabiliser of $\beta_0$:

To see this take $g_0 \in \mathcal{P}$,

$$g_0 = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}, \quad \tau \in \mathbb{R}^4;$$

the co-adjoint action of $\mathcal{P}$ on $\beta_0$ will limit to:

$$\beta_0 \mapsto (g_0^{-1})^T \beta_0 g_0^T,$$

therefore

$$\begin{pmatrix} B(0) \\ -\tau^T B(0) - \Gamma^T(0) \mathcal{M} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and so

$$(d\mathcal{X}(X))_{ij}((g_0^{-1})^T \beta_0 g_0^T)_{ij} = (d\mathcal{X}(X))_{ij}(\beta_0)_{ij}$$

since the lower, left-hand submatrix of $d\mathcal{X}(X)$ is zero, thus proving the assertion.

Since the translations are isomorphic to $\mathbb{R}^4$, and $\beta_0$ integrates to a character $\rho_0$ of the stabiliser of $\beta_0$, $\rho_0$ will be trivial on the translations.

Let $f_0$ be an element of the Hilbert space of the unitary representation of $\mathcal{P}$ constructed from $\beta_0$ by the method of orbits (1.3.37) for some polarization $h_0(\beta_0)$. Then,

$$f_0(h_0 g_0) = \rho_0(h_0)f_0(g_0), \quad \forall g_0 \in \mathcal{P}, \quad \forall h_0 \in \mathcal{P}_{\beta_0}.$$
Thus the representation of $\mathcal{P}$ is trivial on the translation subgroup $T$, as asserted.

**Contracting a Sequence of Functionals and Contracting Co-adjoint Orbits.**

The problem raised by Example (4.1.5) is overcome by an appeal to the philosophy of the methods of Chapter 2.

Suppose we take

$$\beta(t) = \frac{\gamma(t)}{t}, \quad \gamma \in C^1_{\mathcal{F}}[0, 1].$$  \hspace{1cm} (4.1.8)

In this case, when we compute the limit (4.1.6) we obtain instead,

$$\langle \beta_0, d\mathcal{X}(X) \rangle = \langle \dot{\gamma}(0), X(0) \rangle + \langle \gamma(0), \dot{X}(0) \rangle,$$

provided that the condition: $\langle \gamma(0), X(0) \rangle = 0$ holds.

In this case, the condition means that the Lorentz group is contained in the Kernel of $\beta_0$.

This restriction on $\beta(t)$ ensures that $\beta_0$ is well-defined on $\mathcal{L}_0$ by (4.1.9).

Taking $\beta(t) = \frac{1}{t} \begin{pmatrix} B(t) & \Gamma(t) \\ -\Gamma^T(t)\mathcal{M} & 0 \end{pmatrix}$,
we have $\beta_0 = \begin{pmatrix} \dot{B}(0) & \Gamma(0) \\ -\dot{\Gamma}^T(0)\mathcal{M} & 0 \end{pmatrix}$,
where $B(0) = 0$, $\Gamma^T(0)\mathcal{M} = \dot{\Gamma}^T(0)\mathcal{M} = 0$ and $\beta_0$ is no longer trivial on the translations.

We generalise (4.1.8) by the following lemma:

**Lemma 4.1.10**

For a contraction defined by (2.3.2), and for any functional $\beta_0 \in \mathcal{L}_0^*$, there is a curve $\gamma \in C^\infty_{\mathcal{F}}[0, 1]$ such that

$$\lim_{t \to 0} \frac{\gamma(t)}{t} \cdot X(t) = \langle \beta_0, d\mathcal{X}(X) \rangle.$$
Convention 4.1.11

In the note at Definition (2.4.6), we assumed without loss of generality, that \( G \) was embedded in \( E \). We adopt this convention here. Hence all functionals \( \beta \) on \( g \) can be represented with respect to the inner product on \( E \), denoted \( (\cdot, \cdot) \).

Proof of (4.1.10):

Let \( \beta_0 \in g^0 \) be given. \( \exists Y_0 \in g_0 \) such that \( (\beta_0, X_0) = (Y_0, X_0), \forall X_0 \in g_0. \) Let \( Y \in g^q \cap C^\infty_0[0,1] \ni d\tilde{\alpha}(Y) = Y_0. \) Then,

\[
\lim_{t \to 0} (d\Phi(Y(t), t), d\Phi(X(t), t)) = (Y_0, X_0) = (\beta_0, X_0), \forall X \in g^q \ni d\tilde{\alpha}(X) = X_0.
\]

(4.1.12)

Let \( d\Phi^T(\cdot, t) \) denote the "transpose" of the map \( d\Phi(\cdot, t) \) (regarded as the restriction to \( g \) of a linear map \( E \to E \)) with respect to the inner product on \( E \). Then

\[
(d\Phi(Y(t)), d\Phi(X(t))) = (d\Phi^T d\Phi(Y(t)), X(t)).
\]

(4.1.13)

By Lemma (4.1.10), the curve \( Z(t) \) defined by

\[
Z(t) = d\Phi(Y(t), t), \quad t \in (0,1]
\]

\[
Z(0) = \lim_{t \to 0} d\Phi(Y(t), t)
\]

is in \( C^\infty_E[0,1] \) and hence \( d\Phi^T d\Phi(Y(t)) = d\Phi^T(Z(t)) \) is of the form \( \frac{W(t)}{t^q} \) where \( W \in C^\infty_E[0,1] \). Defining \( (\gamma(t), X) = (W(t), X), \forall X \in g \), the proof is complete. ■

By direct analogy to Definition (2.4.4), and in view of Lemma (4.1.10), we define a domain for the contraction of \( g^* \) to \( g^*_0 \):

Definition 4.1.14

Let \( d^*_q \) denote the set of all curves of functionals \( \beta(t) \in g^* \) of the form

\[
\beta(t) = \frac{\gamma(t)}{t^q}, \quad \gamma \in C^q_0[0,1] \text{ such that:}
\]

\[
\gamma^2(0) < \gamma(1)
\]

216
(a) \( \lim_{t \to 0} (\beta(t), X(t)) \) exists, \( \forall X \in \mathfrak{g}^q \).

(b) There exists \( Y \in \mathfrak{g}^q \) for which

\[
\lim_{t \to 0} (d\Phi(Y(t)), d\Phi(X(t))) = \lim_{t \to 0} (\beta(t), X(t)), \forall X \in \mathfrak{g}^q.
\]

**Comments 4.1.15**

To complete the analogy with Lie algebra contraction, one would expect \( \mathfrak{g}_0^* \) to be the image of some linear map with domain \( (\mathfrak{g}^q)^* \). The domain of the desired map is \( \mathfrak{g}_q^* \) which is in 1-1 correspondence with a subspace of \( (\mathfrak{g}^q)^* \), the correspondence being,

\[
d_q^* \rightarrow (\mathfrak{g}^q)^* \\
\{ \beta \in \mathfrak{g}_q^* \} \rightarrow \{ X \mapsto \lim_{t \to 0} (\beta(t), X(t)), X \in \mathfrak{g}^q \}
\]

Herewith the foreshadowed definition:

**Definition 4.1.16**

Let \( d\mathfrak{X}^* \) denote the linear map,

\[
d\mathfrak{X}^* : \mathfrak{g}_q^* \rightarrow \mathfrak{g}_0^*, \\
d\mathfrak{X}^* : \beta \mapsto (d\mathfrak{X}(Y), \cdot),
\]

where \( Y \in \mathfrak{g}^q \) corresponds to \( \beta \) per Definition (4.1.14). Denote \( d\mathfrak{X}^*(\beta) \) also as \( \beta_0 \).

We need to know that all elements of \( \mathfrak{g}_0^* \) can be obtained via \( d\mathfrak{X}^* \):

**Lemma 4.1.17**

The map \( d\mathfrak{X}^* : \mathfrak{g}_q^* \rightarrow \mathfrak{g}_0^* \) is onto.
Proof: Follows from the proof of (4.1.10).

In §1.3, we saw that a representation was determined by a co-adjoint orbit of \( G \) through an element \( \beta \) of \( g^* \), hence we wish to define the contraction of orbits.

As expected from (4.1.16) and Chapter 2, the domain of the contraction mapping will be a co-adjoint orbit of \( D^q \) through an element \( \beta \) of \( d_q^* \). Before seeing this, we need a lemma and the definition of this co-adjoint action.

Lemma 4.1.18

Let \( g \in D^q \) and \( X \in d^q \). Then \( \text{Ad}(g)X \in d^q \), and
\[
\text{d}X(\text{Ad}(g)X) = \text{Ad}(X(g))\text{d}X(X).
\]

Proof:

Let \( \iota_{g_1} \) denote inner automorphism by a group element \( g_1 \) of \( G \), and let \( g, h \in D^q \). Now,
\[
\mathcal{X}_P \circ \iota_{P(g)}(P(h)) = \mathcal{X}_P(P(g)^{-1}P(h)P(g))
\]
\[
= \iota_{\mathcal{X}_P(P(g))}(P(h)),
\]
by the homomorphism property of \( \mathcal{X}_P \).

Differentiating both sides,
\[
\text{d} \mathcal{X}_P \circ \text{Ad}(P(g))U = \text{Ad}(\mathcal{X}_P(P(g))) \circ \text{d} \mathcal{X}_P U, \quad \forall U \in dP(d^q).
\]
i.e. \( \text{d} \mathcal{X}_P \circ dP(\text{Ad}g)X = \text{Ad}(\mathcal{X}_P(P(g))) \circ \text{d} \mathcal{X}_P \circ dP(X) \)

by Proposition (2.5.65), where \( U = dP(X) \).

Therefore \( \text{d} \mathcal{X}(\text{Ad}(g)X) = \text{Ad}(X(g))\text{d} \mathcal{X}(X), \) and \( \text{Ad}(g)X \in d^q \).
Lemma 4.1.19

Let $\text{Ad}^*$ be the map

$$\text{Ad}^*: \mathcal{D}^q \rightarrow \mathcal{L}(\mathcal{C}_g(0,1))$$

$$(\text{Ad}^*(g)\beta)(t) = \text{Ad}^*(g(t))\beta(t),$$

$g \in \mathcal{D}^q, \beta \in \mathcal{C}_g(0,1), t \in (0,1].$

Then $\text{Ad}^*(\mathcal{D}^q)(\mathcal{d}_q^*) \subset \mathcal{d}_q^*.$

Proof:

Since $\text{Ad}^*(g)\beta$ is clearly of the form $\frac{\gamma(t)}{t^q}, \gamma \in \mathcal{C}_g(0,1], g \in \mathcal{D}^q$, we only need to show that there is a $Y \in \mathcal{d}_q^*$ such that,

$$\lim_{t \to 0} (d\varphi(Y(t)), d\varphi(X(t)) = \lim_{t \to 0} ((\text{Ad}^*(g)\beta)(t), X(t)), \forall X \in \mathcal{d}_q^*. \quad (4.1.19(a))$$

Let $Z$ correspond to $\beta$ as in (4.1.14).

The limit of the R.H.S. of (4.1.19(a)) exists and is equal to

$$\langle \beta_0, \text{Ad}\mathcal{X}(g)^{-1}d\mathcal{X}(X) \rangle = (d\mathcal{X}(Z), \text{Ad}\mathcal{X}(g)^{-1}d\mathcal{X}(X))$$

Since $(\text{Ad}\mathcal{X}(g)^{-1})^T d\mathcal{X}(Z)$ is in $\mathcal{d}_0$, we simply choose $Y \in \mathcal{d}_q^*$ such that

$$d\mathcal{X}(Y) = (\text{Ad}\mathcal{X}(g)^{-1})^T d\mathcal{X}(Z).$$

Lemma 4.1.20

Let $\beta \in \mathcal{d}_q^*$. The co-adjoint orbit of $\mathcal{G}_0$ through $d\mathcal{X}^*(\beta)$ is given by,

$$d\mathcal{X}^*(\text{Ad}^*(\mathcal{D}^q)\beta) = \text{Ad}(\mathcal{G}_0)d\mathcal{X}^*(\beta).$$
Proof:

Now \( d\mathfrak{X}^*(\text{Ad}^*(g)\beta, \, d\mathfrak{X}(X)) \)

\[
= \lim_{t \to 0} (\beta(t), \, \text{Ad}g^{-1}(t)X(t)), \quad \forall \beta \in d^*_q, \quad \forall g \in D^q, \quad \forall X \in d^q,
\]

\[
= \lim_{t \to 0} (d\Phi(Y(t)), \, d\Phi(\text{Ad}g^{-1}(t)X(t))) \text{ by (4.1.14)}.
\]

By Lemma (4.1.18), \( d\mathfrak{X}(\text{Ad}gX) = \text{Ad}\mathfrak{X}(g)d\mathfrak{X}(X) \). Hence

\[
(d\mathfrak{X}^*(\text{Ad}^*(g)\beta), \, d\mathfrak{X}(X)) = (d\mathfrak{X}(Y), \, \text{Ad}\mathfrak{X}(g^{-1})d\mathfrak{X}(X))
\]

\[
= (d\mathfrak{X}^*(\beta), \, \text{Ad}(\mathfrak{X}(g))^{-1}d\mathfrak{X}(X))
\]

\[
= (\text{Ad}^*\mathfrak{X}(g)d\mathfrak{X}^*(\beta), \, d\mathfrak{X}(X)),
\]

establishing the lemma. \( \blacksquare \)
§4.2 Contracting Polarizations and Stabilisers of Functionals

As outlined in §4.1, our starting point in contracting a sequence of representations of $\Phi(G,t)$ to that of $G_0$ is to let an element $\beta \in \mathcal{U}_q$ first be chosen. This curve $\beta(t)$ provides the “thread” whereby we link up the method of orbits on each $\Phi(G,t)$ with the method of orbits on $G_0$. Obviously, we would like the method of orbits to work on each $\Phi(G,t)$ for $t \in (0,1]$, as well as on $G_0$. For this to happen, we at least must have that each functional $\beta(t)$, $t \in (0,1]$ and functional $d\mathcal{X}^*(\beta)$ is integral. It is not always possible however, to choose a curve of forms $\beta(t)$ such that $\beta(t)$ is integral for each $t \in (0,1]$ and $\beta_0 = d\mathcal{X}^*(\beta)$ is integral. The following example illustrates this:

Example 4.2.1

We consider the contraction of the matrix group $SO(3)$ to $M(2)$. The integral orbits of $SO(3)$ in $so(3)^*$ are the spheres of radius $k = 0, 1, 2, 3, \ldots$. The integral orbits of $M(2)$ in $m(2)^*$ are given with respect to a basis $\{e_1, e_2, e_3\}$ of $m(2)^*$, with $\beta \in g^*$ acting on $g$ by the dot product defined by $\{e_1, e_2, e_3\}$ as an orthonormal basis of $g$: They are the points $m e_1$, $m \in \mathbb{Z}$, and the cylinders with axis the $e_1$-axis, of all non-zero radii, where

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

For this contraction, $E = \mathbb{R}^9 = M(3 \times 3, \mathbb{R})$. The Laurent maps are

$$\Psi(\cdot, t) : g \mapsto \varphi(t)g\varphi^{-1}(t) \text{ with}$$

$$\varphi(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t \end{pmatrix}.$$

Now $\mathcal{U}_q$ is given by curves $\beta$, $\beta(t) = \frac{\gamma(t)}{t}$, $\gamma \in C^1_{\mathbb{R}}[0,1]$. For $\beta(t)$ to be integral for each $t \in (0,1]$, we must have $\beta(t)$ on a sphere of integer radius. By continuity of
$\beta$, $\beta(0,1]$ must be contained in one sphere only. Each $\beta(t)$ will therefore correspond to the same representation, depending on the choice of polarization.

To be assured of having $\beta(t)$ correspond to inequivalent representations for different $t$’s we are forced to consider $\beta$ only on sequences $\{t_n\}$, $t_n \to 0$. Our ultimate aim is to prove theorems relating representations of $G_0$ (by the method of orbits) to representations of $\Phi(G, t_n)$ (by the method of orbits), for some sequence $t_n \to 0$. In order to study this limiting process we will need the machinery for the contraction of stabilisers and polarizations, the building blocks ((1.3.7), (1.3.9)) of the method of orbits.

**Contracting Stabilisers**

We will now define the stabiliser of $\beta \in d^*_q$ by the action of $D^q$ on $\beta$ via $\text{Ad}^*$, and show that its contraction is $(G_0)_{\beta_0}$ (see (1.3.6)).

**Definition 4.2.2**

Let $\beta \in d^*_q$, define the stabiliser of $\beta$ to be the set $(D^q)_\beta$ given by:

$$(D^q)_\beta = \{g|g \in D^q, \exists X \in d^q, \lim_{t \to 0}((\text{Ad}^*g(t)\beta(t), X(t)) - \langle \beta(t), X(t) \rangle) = 0\}$$

$$= \{g|g \in D^q, d\bar{\mathcal{X}}^*(\text{Ad}^*g)\beta = d\bar{\mathcal{X}}^*(\beta)\}$$

**Proposition 4.2.3**

$\bar{\mathcal{X}}((D^q)_\beta) = (G_0)_{\beta_0}$ and hence the diagram commutes,

$$
\begin{array}{ccc}
(D^q)_\beta & \xrightarrow{\bar{\mathcal{X}}} & (G)_{\beta_0} \\
\downarrow{P} & & \downarrow{X_\beta} \\
P((D^q)_\beta) & \xleftarrow{\bar{\mathcal{X}}^*} & \\
\end{array}
$$

**Proof:**

Let $\beta_0 \in d^*_0$. By Lemma (4.1.17), there exists a $\beta \in d^*_q$ such that $d\bar{\mathcal{X}}^*(\beta) = \beta_0$. 

222
Let $g \in (D^q)_\beta$. By Definition (4.2.2),

$$\langle \beta_0, d\mathcal{X}(X) \rangle = \lim_{t \to 0} \langle \text{Ad}^* g(t) \beta(t), X(t) \rangle$$

$$= \lim_{t \to 0} \langle \beta(t), \text{Ad} g^{-1}(t) X(t) \rangle$$

$$= \lim_{t \to 0} \langle d\Phi(Y(t)), d\Phi(\text{Ad} g^{-1}(t) X(t)) \rangle$$

for all $X \in \mathfrak{g}^q$, and $Y$ corresponding to $\beta$ as in Definition (4.1.14).

By Lemma (4.1.18),

$$\langle \beta_0, d\mathcal{X}(X) \rangle = \langle \beta_0, \text{Ad}(\mathcal{X}(g))^{-1} d\mathcal{X}(X) \rangle$$

$$= \langle \text{Ad}^*(\mathcal{X}(g)) \beta_0, d\mathcal{X}(X) \rangle$$

whence $\mathcal{X}(g) \in (G_0)_{\beta_0}$. Conversely if $g_0 \in (G_0)_{\beta_0}$, $\exists g \in D^q$ such that $\mathcal{X}(g) = g_0$ and hence

$$\langle \beta_0, d\mathcal{X}(X) \rangle = \langle \text{Ad}^*(\mathcal{X}(g)) \beta_0, d\mathcal{X}(X) \rangle$$

i.e. $\langle d\mathcal{X}^*(\beta), d\mathcal{X}(X) \rangle = \lim_{t \to 0} \langle \text{Ad}^* g(t) \beta(t), X(t) \rangle$

giving $g_0 \in \mathcal{X}((D^q)_\beta)$ and $\mathcal{X}((D^q)_\beta) = (G_0)_{\beta_0}$ and the diagram commutes.

It is natural to ask whether $(D^q)_\beta$ and $P((D^q)_\beta)$ are both Lie groups, and whether $P((D^q)_\beta)$ is the stabiliser of some element of $(dP(d^q))^*$. To answer this question we need the following definition, derived from (4.1.16).

**Definition 4.2.4**

Let $dP^*$ be the linear map

$$dP^* : \mathfrak{g}_q^* \rightarrow (dP(d^q))^*$$

$$\langle dP^*(\beta), U \rangle = \sum_{J=0}^{q} \frac{q!}{(q-J)!} \langle \gamma^{(q-J)}(0), U^J \rangle$$

$$= \langle d\mathcal{X}^*(\beta), d\mathcal{X}_P(U) \rangle,$$
where $\beta(t) = \frac{\gamma(t)}{t}$, by Definition (4.1.14), and $\{U_J\}$ are the coordinates of $U \in dP(d^q)$ with respect to the basis $\{X_{Lm}\}$ of Proposition (2.5.8).

**Proposition 4.2.5**

$(D^q)_\beta$ is a Banach-Lie subgroup (1.2.8) of $D^q$, $P((D^q)_\beta)$ is a (closed) Lie subgroup of $P(D^q)$, and $P((D^q)_\beta) = (P(D^q))_{dP^\bullet(\beta)}$.

**Proof:**

By Proposition (2.5.63), $P^q : C^q_0[0,1] \rightarrow C^q_{T^q;T^q(g)}$ is an isomorphism. If $P((D^q)_\beta)$ is already a Lie subgroup of $P(D^q)$ we have immediately by Proposition (2.4.64) and Proposition (2.5.63) that $(D^q)_\beta$ is a Banach-Lie group. To show that $(D^q)_\beta$ is a Lie subgroup of $D^q$, it is sufficient by (1.2.8) to show that the injection,

$$j : C^q_{T^q;P((D^q)_\beta)} \rightarrow C^q_{T^q;P(D^q)}$$

is $C^\infty$.

The proof that $j$ is an injection is entirely analogous to the proof of Proposition (2.5.64); hence it is only necessary to show that $P((D^q)_\beta)$ is a closed (hence Lie) subgroup of $P(D^q)$.

Let $Y$ be the element of $d^q$ corresponding to $\beta$ as in definition (4.1.14), and let $dP(Y) = V$.

By Definition (4.2.2),

$$P((D^q)_\beta) = \{ P(g) \mid g \in D^q, \; dX^*(Ad^*(g)\beta) = dX^*(\beta) \}$$

$$= \{ P(g) \mid g \in D^q, \lim_{t \to 0} (\beta(t), Adg^{-1}(t)X(t)) = (dX^*(\beta), dX(X)), \; \forall X \in d^q \}$$

$$= \{ u \mid u \in P(D^q), \; (dX^*(\beta), dX_P(Ad(u^{-1})U)) = (dX^*(\beta), dX_P(U)),$$

$$\; \forall U \in dP(d^q) \}$$

$$= \{ u \mid u \in P(D^q), \; Ad^*(u)dP^\bullet(\beta) = dP^\bullet(\beta) \}$$

224
Since \( P((D^q)_\beta) \) is the stabiliser in \( P(D^q) \) of \( dP^*(\beta) \), it is closed and is hence a Lie subgroup of \( P(D^q) \), establishing the proposition.

As we have defined a "dual" for the map \( dP \), it is appropriate at this point to define a "dual" for \( dX_P \) and show the mutual relationship of \( dP^*, dX^* \) and \( dX_P^* \);

**Lemma 4.2.5(a)**

Let \( \alpha \in dP^*(d_q^*) \) with \( \beta \in \mathfrak{g}_q^* \) such that \( dP^*(\beta) = \alpha \). Let \( dX_P^* \) denote the map,

\[
dX_P^*: dP^*(d_q^*) \rightarrow \mathfrak{g}_q^\ast
\]
\[
dX_P^*: \alpha \mapsto (dX^*(\beta), \cdot) ;
\]

then the following diagram commutes:

\[
\begin{array}{ccc}
d_q^* & \xrightarrow{dX} & \mathfrak{g}_q^* \\
\downarrow{dP^*} & & \downarrow{dX_P^*} \\
dP^*(d_q^*) & \xrightarrow{dX_P} & \mathfrak{g}_q^\ast 
\end{array}
\]

(note that \( dP^*(d_q^*) \subseteq (dP(d_q^*))^\ast \).

**Proof:** Immediate from (4.1.14), (4.2.4) and (4.2.5).

**Comments 4.2.6**

As the objective is to relate the method of orbits for each pair \( \{G(t), (D^q(t)) \} \) to the same on \( \{G_0, \beta_0 \} \), we observe that for any \( g \in D^q \) with \( g(t) \in G_\beta(t) \), \( t \in (0,1] \), we have \( g \in (D^q)_\beta \) by (4.2.2).

To clarify the relationship of \( G_\beta(t) \), \( t \in (0,1] \) to \( (D^q)_\beta \) and \( G_\beta_0 \), we give the following illustration:
Example 4.2.7

For the contraction example of (4.2.1), let

\[
\begin{align*}
    w_1 &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
    w_2 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \\
    w_3 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}
\end{align*}
\]

so that \( \{w_1, w_2, w_3\} \) is an orthonormal basis of \( \mathfrak{so}(3) \) with inner product \( \langle \cdot, \cdot \rangle \).

Let \( \beta(t) = \frac{r w_3}{t}, \ r \in \mathbb{R}, \) acting on \( \mathfrak{so}(3) \) with respect to this inner product.

Let \( Y \in \mathfrak{d}^1 \) corresponding to \( \beta \) by (4.1.14) be \( Y(t) = r t w_3 \).

By (4.1.16) \( (dX^*(\beta), X_0) = (d\mathcal{X}(Y), X_0), \forall X_0 \in \mathfrak{m}(2) \) and \( d\mathcal{X}(Y) = r e_3 \). Then

\[
\mathcal{G}_{\beta(t)} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & R \end{pmatrix} \bigg| R \in SO(2) \right\},
\]

\[
\mathcal{G}_{\beta_0} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \tau \\ 0 & 0 & 1 \end{pmatrix} \bigg| \tau \in \mathbb{R} \right\},
\]

\[
\mathcal{D}^1 = \{ g | g \in C^1_{SO(3)}[0, 1], \ g(0) = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}, \ R \in SO(2) \},
\]

\[
\mathfrak{d}^1 = \{ X | X \in C^1_{\mathfrak{so}(3)}[0, 1], \ X(0) = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}, \ S \in \mathfrak{so}(2) \},
\]

\[
\mathcal{G}_0 = \left\{ \begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix} \bigg| R \in SO(2), \ T \in \mathbb{R}^2 \right\},
\]

\[
\mathfrak{g}_0 = \left\{ \begin{pmatrix} S & \tau \\ 0 & 0 \end{pmatrix} \bigg| S \in \mathfrak{so}(2), \ \tau \in \mathbb{R}^2 \right\}.
\]

Now,

\[
\langle \beta_0, d\mathcal{X}(X) \rangle = \lim_{t \to 0} \langle \beta(t), X(t) \rangle = \langle r w_3, \dot{X}(0) \rangle,
\]

provided \( \langle w_3, X(0) \rangle = 0 \), which is assured by \( X \) being in \( \mathfrak{d}^1 \). (If this is not the case, we can choose \( \text{Ad}^* g(t) \beta(t) \) in place of \( \beta(t) \), for any \( g \in \mathcal{D}^1 \).
Then using (2.5.4) to compute $(D^1)_\beta$,

$$(D^1)_\beta = \left\{ g \left| g \in C^1_{\Omega}[0,1], \, g(0) = e, \, \nabla g(t)\bigg|_{t=0} \in \text{Span} \{w_1, w_3\} \right. \right\},$$

in the notation of (1.4.7).

And the subgroup $D^1 \cap \{ g|g(t) \in \mathcal{G}_\beta(t), t \in (0,1]\}$ equals

$$\left\{ g|g \in C^1_{\Omega}[0,1], \, g(0) = e, \, \nabla g(t)\bigg|_{t=0} \in \text{Span} \{w_3\} \right\}. \quad \text{The relationship between these two groups is an illustration of the general case that for } (g, X_1, \cdots, X_q) \in P((D^1)_\beta) \text{ and } (h, Y_1, \cdots, Y_q) \in P(D^1 \cap \{ g|g(t) \in \mathcal{G}_\beta(t), t \in (0,1]\}), \text{ the difference between the two elements is frequently only in their nilpotent factors, } (e, X_1, \cdots, X_q) \text{ and } (e, Y_1, \cdots, Y_q). \text{ This is made more precise by the following lemma:}

**Lemma 4.2.8**

Let $\beta(t) = \frac{\gamma(t)}{t}$ as in (4.1.14); then $(D^q)_\beta \subseteq \{ g \left| g \in D^q, \, g(0) \in \mathcal{G}_{\gamma(0)} \right. \}$.

**Proof:**

Now $g \in (D^q)_\beta$ if and only if

$$\lim_{t \to 0} \frac{1}{t^q} \langle g(t), \text{Ad}_{g^{-1}(t)} X(t) \rangle = \lim_{t \to 0} \frac{1}{t^q} \langle g(t), X(t) \rangle, \quad \forall X \in \mathfrak{g}^q.$$ 

That is, if and only if,

$$\sum_{i=0}^{q} \binom{q}{i} \langle \gamma^{(q-i)}(0), \frac{d^i}{dt^i} (\text{Ad}_{g^{-1}(t)} X(t))\bigg|_{t=0} \rangle$$ 

$$= \sum_{i=0}^{q} \binom{q}{i} \langle \gamma^{(q-i)}(0), X^{(i)}(0) \rangle, \quad \forall X \in \mathfrak{g}^q. \quad (4.2.9)$$

Let $X \in C^q_{\Omega}[0,1]$ with $X^{(i)}(0) = 0, \, i = 0, \cdots, q - 1$; then by (2.4.4), $X \in \mathfrak{g}^q$.

Then (4.2.9) reduces to

$$\langle \gamma(0), \text{Ad}_{g^{-1}(0)} X^{(q)}(0) \rangle = \langle \gamma(0), X^{(q)}(0) \rangle$$
and the result follows because \(X^q(0)\) is arbitrary.

In order to discuss the contraction of polarizations, we need to define the Lie algebra of \((\mathcal{D}^q)_\beta\), by analogy to definition (4.2.2).

**Definition 4.2.10**

Let \(\beta \in \mathfrak{d}^q\); define the stabiliser of \(\beta\) in \(\mathfrak{d}^q\) to be \((\mathfrak{d}^q)_\beta\) where

\[
(\mathfrak{d}^q)_\beta = \{ X \mid X \in \mathfrak{d}^q, \langle d\mathfrak{X}^*(\beta), d\mathfrak{X}([X,Y]) \rangle = 0, \forall Y \in \mathfrak{d}^q \}.
\]

**Lemma 4.2.11**

\((\mathfrak{d}^q)_\beta\) is the Lie algebra of \((\mathcal{D}^q)_\beta\).

**Proof:**

By Proposition (2.5.42), \(\mathcal{I}^q:(\mathcal{D}^q)_\beta = C^q_{\mathcal{I}}(\mathcal{D}^q)_\beta\) which has Lie algebra \(C^q_{\mathcal{I}}\) where \(\mathfrak{g}\) is the Lie algebra of \(P((\mathcal{D}^q)_\beta)\). If we show that \(\mathfrak{g} = dP(\mathfrak{d}^q)_\beta\), then employing (by (2.5.42)) the fact that,

\[
d\mathcal{I}^q: C^q_{\mathcal{I}}[0,1] \rightarrow C^q_{\mathcal{I}}(\mathfrak{d}^q)[0,1]
\]

\[
d\mathcal{I}^q(V)(t) = \sum_{J=0}^{\infty} \frac{1}{J!} \frac{d^JV(t)}{dt^J} t^J
\]

is a Lie algebra homomorphism, this will establish that \((\mathfrak{d}^q)_\beta\) is the Lie algebra of \((\mathcal{D}^q)_\beta\).

This follows immediately:

From (4.2.5), \(P((\mathcal{D}^q)_\beta) = (P(\mathcal{D}^q))_{dP^*}(\beta)\), and from (4.2.10),

\[
\langle d\mathfrak{X}^*(\beta), d\mathfrak{X}([X,Y]) \rangle = 0 \text{ if and only if } \langle d\mathfrak{X}^*(\beta), d\mathfrak{X}[P(X), P(Y)] \rangle = 0.
\]

Then by (4.2.4) and (4.2.10), \(\mathfrak{g} = dP(\mathfrak{d}^q)_\beta\), and this completes the proof.

**Corollary 4.2.12**

\(dP(\mathfrak{d}^q)_\beta\) is the Lie algebra of \(P((\mathcal{D}^q)_\beta)\).
The following lemma is useful for determining from \( \beta \), whether \( dX^*(\beta) \) will be integral or not (in the sense of (1.3.7)).

**Lemma 4.2.13**

Let \( \beta \in d_q^* \). \( \beta_0 = dX^*(\beta) \) is integrable to a character \( \rho_0 \) of \( G_{\beta_0} \) if and only if:

(a) \( dP^*(\beta) \) is integrable to a character \( \rho_P \) of \( (P(D^q))_{dP^*(\beta)} \), and

(b) \( \text{Ker} X_P \cap (P(D^q))_{dP^*(\beta)} \subseteq \text{Ker} \rho_P \).

**Proof:**

Follows immediately from the fact that

\[ G_0 = P(D^q)/\text{Ker} X_P. \]

**Contracting Polarizations**

In the method of orbits (§1.3), an irreducible unitary representation is determined by the co-adjoint orbit of \( G \) through \( \beta \in \mathfrak{q}^* \) and a polarization for \( \beta \). We need to understand how polarizations contract in order to relate representations of \( \Phi(G, t) \) to those of \( G_0 \).

Let \( \beta \in \mathfrak{g}^* \). We will define an infinite dimensional polarization for \( \beta \), as a subalgebra of \( \mathfrak{d}^q(\otimes \mathbb{C}) \), and then show that the contraction of this polarization gives a polarization for \( \beta_0 = dX^*(\beta) \).

**Definition 4.2.14**

A subalgebra \( \mathfrak{h}(\beta) \subseteq \mathfrak{d}^q \otimes \mathbb{C} \) will be called a polarization for \( \beta \), if it satisfies the following conditions:

(i) \( \lim_{t \to 0} (\beta(t), [W(t), Z(t)]) = 0 \), \( \forall W, Z \in \mathfrak{h}(\beta) \)

(ii) \( \mathfrak{h}(\beta) \) is stable under \( \text{Ad} ((D^q)_\beta) \), and \( \mathfrak{h}(\beta) \supset (\mathfrak{d}^q)_\beta \)

(iii) \( \mathfrak{h}(\beta) + \overline{\mathfrak{h}(\beta)} \) is a subalgebra of \( \mathfrak{d}^q \otimes \mathbb{C} \)

(iv) \( \mathfrak{h}(\beta) \) is maximal with respect to property (i).
Remark 4.2.15

The inner product on \( E \) is extended by linearity to \( E \otimes \mathbb{C} \), and \( \text{Ad} \) is similarly extended to \( \mathfrak{g} \otimes \mathbb{C} \).

Definition 4.2.16

Let \( \mathfrak{h}(\beta_0) \) denote the subalgebra of \( \mathfrak{g}_0 \otimes \mathbb{C} : \mathfrak{h}(\beta_0) = d\mathfrak{X}(\mathfrak{h}(\beta)) \)

Proposition 4.2.17

\( d\mathfrak{X}(\mathfrak{h}(\beta)) = \mathfrak{h}(\beta_0) \) is a polarization for \( \beta_0 \) and \( dP(\mathfrak{h}(\beta)) \) is a polarization for \( dP^*(\beta) \), thus implying that the following diagram commutes,

\[
\begin{array}{ccc}
\mathfrak{h}(\beta) & \xrightarrow{d\mathfrak{X}} & \mathfrak{h}(\beta_0) \\
\downarrow dP & & \downarrow d\mathfrak{X}_P \\
\mathfrak{dP}(\mathfrak{h}(\beta)) & & \\
\end{array}
\]

Proof:

(i) Let \( W_0, Z_0 \in \mathfrak{h}(\beta_0) \). Then \( \exists W, Z \in \mathfrak{h}(\beta) \ni d\mathfrak{X}(W) = W_0, \ d\mathfrak{X}(Z) = Z_0 \) by (4.2.16), and \( \langle \beta_0, [W_0, Z_0] \rangle = \lim_{t \to 0} \langle \beta(t), [W(t), Z(t)] \rangle = 0 \) by (4.2.14)(i).

(ii) \( \text{Ad} (D^q_\beta \mathfrak{h}(\beta) \subseteq \mathfrak{h}(\beta) \) by (4.2.14)(ii).

Therefore \( \text{Ad}(\mathfrak{X}(D^q_\beta))d\mathfrak{X}(\mathfrak{h}(\beta)) \subseteq d\mathfrak{X}(\mathfrak{h}(\beta)) \) by (4.1.18)

i.e. \( \text{Ad}(\mathfrak{g}_0)\mathfrak{h}(\beta_0) \subseteq \mathfrak{h}(\beta_0) \). By (4.2.14)(ii) \( (D^q_\beta) \beta \subseteq \mathfrak{h}(\beta) \)

therefore \( \mathfrak{g}_{\beta_0} = d\mathfrak{X}(D^q_\beta) \subseteq d\mathfrak{X}(\mathfrak{h}(\beta)) = \mathfrak{h}(\beta_0) \).

(iii) Now \( d\mathfrak{X}(\mathfrak{h}(\beta) + \mathfrak{h}(\beta)) = \mathfrak{h}(\beta_0) + \mathfrak{h}(\beta_0) \).

Since \( d\mathfrak{X} \) is a homomorphism, \( \mathfrak{h}(\beta_0) + \mathfrak{h}(\beta_0) \) is a subalgebra of \( \mathfrak{g}_0 \otimes \mathbb{C} \) by (4.2.14) (iii)

(iv) Let \( \mathfrak{h}'(\beta_0) \) be another algebra with \( \mathfrak{h}'(\beta_0) \supseteq \mathfrak{h}(\beta_0) \), and satisfying,

\( \langle \beta_0, [W_0', Z_0'] \rangle = 0, \ \forall W_0', Z_0' \in \mathfrak{h}'(\beta_0) \).

230
Then \( d\mathcal{X}^{-1}(h'(\beta_0)) \supset h(\beta) \) and for any \( W, Z \) in \( d\mathcal{X}^{-1}(h'(\beta_0)) \),

\[
0 = \langle \beta_0, [d\mathcal{X}(W), d\mathcal{X}(Z)] \rangle = \lim_{t \to 0} \langle \beta(t), [W(t), Z(t)] \rangle.
\]

Since \( h(\beta) \) is maximal with respect to (4.2.14)(i),

\[
d\mathcal{X}^{-1}(h'(\beta_0)) = h(\beta)
\]

whence \( h'(\beta_0) = h(\beta_0) \)

and \( h(\beta_0) \) is maximally isotropic. Hence \( h(\beta_0) \) is a polarization for \( \beta_0 \).

Using the notation of Definition (4.1.14) we have that

\[
\langle dP^*(\beta), U \rangle = \langle d\mathcal{X}_P(dP_Y), d\mathcal{X}_P(U) \rangle,
\]

and

\[
\lim_{t \to 0} \langle \beta(t), [W(t), Z(t)] \rangle = \langle d\mathcal{X}_P(dP_Y), d\mathcal{X}_P[dP_W, dP_Z] \rangle = 0, \quad \forall W, Z \in h(\beta)
\]

Hence \( dP(h(\beta)) \) is isotropic with respect to \( dP^*(\beta) \).

It follows directly from Definition (4.2.14)(ii)(iii) and the fact that \( dP \) is a homomorphism that \( dP(h(\beta)) \) is stable under \( \text{Ad}(P(D^q)_\beta) \) and that \( dP(h(\beta)) \supset dP(d^q)_\beta \) where by Corollary (4.2.12) and Lemma (4.2.5(a)), \( dP((d^q)_\beta) \) is the Lie algebra of the stabiliser of \( dP^*(\beta) \). Also \( dP(h(\beta)) + dP(h(\beta)) \) is a subalgebra of \( dP(d^q) \otimes \mathbb{C} \).

Suppose that \( h'(dP^*(\beta)) \supset dP(h(\beta)) \) and that \( \langle dP^*(\beta), [U, V] \rangle = 0, \forall U, V \in h'(dP^*(\beta)) \). Let \( h'(\beta) = dP^{-1}(h'(dP^*(\beta))) \). Then for all \( W, Z \) in \( h'(\beta) \),

\[
0 = \langle dP^*(\beta), [dPW, dPZ] \rangle = (d\mathcal{X}_P(dP_Y), d\mathcal{X}_P[dPW, dPZ]) = \lim_{t \to 0} \langle \beta(t), [W(t), Z(t)] \rangle.
\]

231
Since by assumption \( h'(\beta) \supset h(\beta) \) we must have \( dP(h'(\beta)) = dP(h(\beta)) \)
i.e. \( h'(dP^*(\beta)) = dP(h(\beta)) \), so that \( dP(h(\beta)) \) is maximally isotropic with respect
to \( dP^*(\beta) \), making \( dP(h(\beta)) \) a polarization for \( dP^*(\beta) \).

**Remark 4.2.18**

Let \( \beta \in g_q^* \), with \( \beta(t) = \frac{2t}{t} \). Somewhat analogously to Lemma (4.2.8), the
endpoints of elements of \( h(\beta) \) are on an isotropic subspace for \( \gamma(0) \). Letting \( W, Z \in h(\beta) \), this follows from (4.2.14(i)) because it is necessary that \( \langle \gamma(0), [W(0), z(0)] \rangle = 0 \) in order for the limit in (4.2.14) (i)) to exist.

**Remark 4.2.19**

From \( h(\beta) \), a polarization \( h(\beta_0) \) can be constructed. The pair \( \{\beta(t), h(\beta(t))\} \)
determines a representation of \( \Phi(G, t) \) for each \( t \), while \( \{\beta_0, h(\beta_0)\} \) determines a
representation of \( G_0 \). This representation of \( G_0 \) is associated directly with \( h(\beta) \)
and \( \beta \), but there is no immediate connection with the \( h(\beta(t)) \)’s. This leads us to
make the following condition on \( h(\beta) \), in view of the analogy provided by (4.2.6):

**Condition 4.2.20**

We require \( h(\beta) \) of Definition (4.2.14) to satisfy the additional condition:

\[
h(\beta) \supset \{Z | Z \in d_q \otimes C, \ Z(t) \in h(\beta(t))\}
\]

**Summary 4.2.21**

We have seen how, given a sequence of representations “labelled” by \( \beta(t), t \in (0, 1] \) we have the associated, “contracted” functional \( \beta_0 \in g_q^* \). The curve \( \beta(t) \)
has a stabiliser \( (D^q)_\beta \) with respect to the co-adjoint action of \( D^q \) on \( \beta \), and the
contraction of \( (D^q)_\beta \) is \( G_{\beta_0} \), the stabiliser of \( \beta_0 \). \( G_{\beta_0} \) is also the image by \( X_P \)
of the finite-dimensional group, \( P((D^q)_\beta) \), the stabiliser of \( dP^*(\beta) \). Associated
with the curve \( \beta(t) \) and its stabiliser \( (D^q)_\beta \), is a polarization \( h(\beta) \) which is a
subalgebra of $\mathfrak{g} \otimes C$; the contraction of $h(\beta)$ (extended to complex algebras) is a polarization $h(\beta_0)$ for $\beta_0$ and $G_{\beta_0}$. The image by $dX_P$ of $dP(h(\beta))$ (a finite-dimensional polarization for $dP^*(\beta)$ and $P((\mathcal{D})_{\beta})$, is also equal to $h(\beta_0)$. The question of whether $\beta_0$ is integral or not can be determined entirely by conditions on $P((\mathcal{D})_{\beta})$.

Thus the sequence of representations “labelled” by $\{\beta(t), h(\beta(t)), t \in (0, 1]\}$ determines, by the method of orbits, a representation “labelled” by $\{\beta_0, h(\beta_0)\}$ which we assume (see §4.1) to be irreducible. We are now interested in seeing how operators of representations for $\beta(t)$ approach operators of the representation for $\beta_0$. 

233
§4.3 Theorems Connecting the Method of Orbits on the Sequence of Isomorphic Lie Groups with the Method of Orbits on the Contracted Lie Group

Let $\beta \in d^*$ and let $\{t_n\} \subseteq [0,1]$ be a sequence $t_n \to 0$ such that $\beta(t_n)$ is integral for each $t_n$. Assume that the condition of Lemma (4.2.13) holds, and $\beta_0$ is integral. In §3.2 we saw how a sequence of representations "labelled" by $\{\beta(t), h(\beta(t))\}$ determine a representation "labelled" by $\{\beta_0, h(\beta_0)\}$. In this section our aim is to find the conditions under which a given operator $R_0$ of a representation of $G_0$ labelled by $\{\beta_0, h(\beta_0)\}$ is the limit of a sequence of operators $G_n$ of the representation of $\Phi(G,t_n)$ labelled by $\{\beta(t_n), h(\beta(t_n))\}$.

In the example of section 2.3 we also saw how $G_{[0,1]}$ was a $C^\infty$-bundle over $[0,1]$. This is true in full generality, when $\dim G = \dim G_0$, and this fact will be useful in what follows.

**Proposition 4.3.1**

Suppose $\dim G_0 = \dim G$. Then,

(a) $G_{[0,1]}$ is a $C^\infty$-manifold with boundary (1.1.1(a)) with $C^\infty$ projection map 
$$\tau_{[0,1]} : G_{[0,1]} \to [0,1],$$
and,

(b) $G_0$ and the $\Phi(G,t)$ are closed submanifolds of $G_{[0,1]}$.

**Proof:**

Let $(N_0, \chi_0, \lambda^i)$ be a canonical (1.1.32) chart at $e_0$ in $G_0$ where $N_0$ is the open subset of $\mathbb{R}^n$ which is the image of the domain of the coordinate chart $\chi_0$ at $e_0$. Let $\lambda^i$ denote the coordinates of a point with respect to this chart. Similarly, let $(N, \chi, x^i)$ be a canonical chart at $e$ in $G$.

We will proceed by first defining a chart in $G_{[0,1]}$ at $e_0$. From this chart, a family of charts of $G_{[0,1]}$ will be constructed, which cover all of $G_0$. The rest of $G_{[0,1]}$ will be covered by mapping a coordinate covering of $G$ onto $G_{[0,1]}$ via the map $\Phi$.  

234
Restrict the chart $\chi_0$ so that its domain coincides with the domain of the local cross-section map $\theta : \mathcal{G}_0 = P(D^q) / \text{Ker } \mathcal{X}_P \rightarrow P(D^q)$ of (1.1.40(a)). Choose a canonical chart at $1 \in P(D^q)$. With respect to the two coordinate charts the mapping $\theta$ appears as its derivative, $d\theta$. We write $d\theta$ in matrix form as $\theta^{j,i}$. 

Let $\theta(t)$ be the $C^\infty$ curve of linear maps $\theta(t) : \mathbb{R}^n \rightarrow \mathbb{R}^n$, 

$$\theta^j(t)(\lambda) = \sum_{J=0}^{q} \frac{t^J \theta^{j,i} \lambda^i}{J!} \lambda^i.$$ 

If there is no $t$ in $(0,1]$ such that $\theta(t)$ is invertible, add a term $\frac{t^{q+1} \theta^{j,i} \lambda^i}{(q+1)!}$, so that it is.

Since $\det (\theta(t))$ is polynomial in $t$, and non-zero for some $t$, it must be non-zero for all $t$ in $(0,\epsilon)$, for some $\epsilon > 0$. We write $\theta(t)$ as an implied sum: $\theta(t)(\lambda) = \frac{t^J \theta^{j,i} \lambda^i}{J!}$. Viewing the image of $\theta(t)$ as in $g$, define an $n$-parameter family of $C^\infty$ curves by 

$$g(\lambda^i, t) = \chi^{-1} \circ \theta(t)(\lambda^i), \ t \in [0,\epsilon).$$

(By the definition of canonical coordinates in $\mathcal{G}$ and $T^q(\mathcal{G})$, the domain of $\chi_0$ can be further restricted so that $\theta(t)N_0 \subseteq N$ for $t$ small enough.)

Then, with respect to these coordinates, 

$$\chi \circ g(\lambda^i, t) = \theta(t)(\lambda^i)$$

and therefore, 

$$\nabla^J \chi^j \circ g(\lambda^i, t))|_{t=0} = \theta^{j,i} \lambda^i.$$ 

By the definition of the exponential mapping, $\text{Exp}$, of $T^q(\mathcal{G})$ given in (2.5.8), the coordinates of $P(g(\lambda^i))$ are $\theta^{j,i} \lambda^i$ and hence $P(g(\lambda^i)) \in P(D^q)$ whence $g(\lambda^i) \in D^q$.

Note that for each $t \in (0,\epsilon)$, the maps $g(\cdot, t)$ provide a chart on $\mathcal{G}$:
Define this chart to be \( \chi_1 = (\theta(t))^{-1} \circ \chi \cdot \chi_1 \) is a local homeomorphism and is \( C^\infty \)-related to \( \chi \) because \( \chi_1 \circ \chi^{-1}(\chi^i) = (\theta(t))^{-1}(\chi^i) \) and \( (\theta(t))^{-1}(\chi^i) \) has Jacobian a rank \( n \).

The family of curves \( \{ g(\lambda^i, \cdot) \} \) now provides a chart at \( e_0 \) on \( G_{[0,1]} \) given by the map,

\[
\frac{1}{t^q} \phi(g(\lambda^i, t), t), t \mapsto (\lambda^i, t), t \in (0, \epsilon), \lambda \in \mathcal{N}_0 \\
\left( \lim_{t \to 0} \frac{\phi(g(\lambda^i, t), t)}{t^q}, 0 \right) \mapsto (\lambda^i, 0), \lambda \in \mathcal{N}_0. \tag{4.3.2}
\]

For \( t = 0, (4.3.2) \) gives a chart on \( G_0 \). For each \( t \in (0, \epsilon) \) it gives a chart on \( \Phi(G, t) \).

The map (4.3.2) is clearly 1-1. We will generate a family of these charts so that \( G_0 \) is covered. The topology of \( G_{[0,1]} \) will be generated by these charts (and the charts for \( G_{(0,1)} \) to be defined below), and it will be seen that this topology is compatible with the topology already defined for \( G_{(0,1)} \).

In order to construct a chart of \( G_{[0,1]} \) at \( g_0 \in G_0 \), let \( g_1 \in D^q \) be a \( C^\infty \) curve such that \( \mathcal{X}(g_1) = g_0 \). Now consider, as before, the \( n \)-parameter family of \( C^\infty \) curves, \( g_1(t)g(\lambda^i, t), t \in [0, \epsilon) \). As before, the maps \( g_1(t)g(\cdot, t) \) provide a chart on \( G \) for each \( t \in (0, \epsilon) \) by the properties of the product on \( G \), and \( \mathcal{X}(g_1g(\mathcal{N}_0)) \) is a neighbourhood of \( g_0 \) in \( G_0 \).

The family of curves \( \{ g_1(\cdot)g(\lambda^i, \cdot) \} \) again provides a chart of \( G_{[0,1]} \) at \( g_0 \) given by the map,

\[
\frac{1}{s^q} \phi(g_1(s)g(\lambda^i, s), s), s \mapsto (\lambda^i, s), s \in (0, \epsilon), \lambda \in \mathcal{N}_0 \\
\left( \lim_{s \to 0} \frac{\phi(g_1(s)g(\lambda^i, s), s)}{s^q}, 0 \right) \mapsto (\lambda^i, 0), \lambda \in \mathcal{N}_0. \tag{4.3.3}
\]

The composition of overlapping charts for the points specified by \( s = t = 0 \) is \( C^\infty \) by the \( C^\infty \)-differentiability of left translation by \( g_0 \). For \( s = t \neq 0 \), the composition
of charts is the map,

\[(\lambda, t) \mapsto (\theta(t))^{-1} \circ \chi(g_1^{-1}(t)g(\lambda, t)), \text{ which is } C^\infty.\]

(See Appendix A.1 for a clearer proof using results which appear later in the thesis.)

To cover the balance of \(\mathcal{G}_{[0,1]}\) with \(C^\infty\)-related charts, we provide a covering of \(\mathcal{G}_{(0,1]}\) with charts. To do this, take a covering of \(\mathcal{G}\) with charts \((\mathcal{N}_2, \chi_2, z_i)\). Each chart generates a chart for \(\mathcal{G}_{(0,1]}\) covering the set,

\[\left\{ \left( \frac{1}{t^q} \phi(g, t), t \right) \mid g \in \mathcal{N}_2, t \in (0,1] \right\}\]

and the chart is given by

\[\left( \frac{1}{t^q} \phi(g, t), t \right) \mapsto (\chi_2(g), t). \quad (4.3.4)\]

these charts are \(C^\infty\)-related to each other and to the charts (4.3.2) and (4.3.3).

To see that \(\mathcal{G}_0\) and \(\Phi(\mathcal{G}, r)\) are submanifolds of \(\mathcal{G}_{[0,1]}\) merely take \(t = 0\) or \(t = r\) in (4.3.2), (4.3.3) or (4.3.4). They are clearly closed. The projection map

\[\pi_{[0,1]} : \mathcal{G}_{[0,1]} \rightarrow [0,1] \text{ is } \pi_{[0,1]} : (g, t) \mapsto t\]

and is \(C^\infty\).

\[\blacksquare\]

Remark 4.3.5

Our aim is to show that for any operator \(\mathcal{R}_0\) in the image of the representation of \(\mathcal{G}_0\) constructed by the method of orbits from \(\{\beta_0, \hbar(\beta_0)\}\), \(\mathcal{R}_0\) is the limit in some sense of operators \(\mathcal{R}_n\) in the representation of \(\Phi(\mathcal{G}, t_n)\) constructed by the method of orbits from \(\{\beta(t_n), \hbar(\beta(t_n))\}\). A major step on the way to fulfilling this aim,
is the proof of the "Bootstrap" proposition described in §4.1. But first we need some notation and definitions.

**Notation 4.3.6**

In view of (4.2.1), we must restrict ourselves to considering representations of \( \Phi(\mathcal{G}, t_n) \) for a sequence \( \{t_n\} \), to subsume all possible examples (there are cases for which this restriction is unnecessary, such as the contraction of simply connected, nilpotent Lie groups). It will simplify notation if, in the sequel, \( t \) takes values in the set \( \{t_n, 0\} \) only. With this convention \( \{\mathcal{G}_t\} \) will denote the Lie groups \( \{\Phi(\mathcal{G}, t_n), \mathcal{G}_0\} \); \( C^\infty_t \) will denote the \( C^\infty \) maps from \( \mathcal{G}_t \) to \( \mathbb{C} \); and \( \{\mathcal{H}_t\} \) will denote the Hilbert spaces \( \{\mathcal{H}_{t_n}, \mathcal{H}_0\} \).

Given a sequence of elements \( f_t \) of \( \mathcal{H}_t \), where \( \mathcal{H}_t \) is constructed by the method of orbits ((1.3.27) and (1.3.26)), we need a meaningful concept for the convergence of \( \{f_t\} \) to \( f_0 \). To achieve this, we must regard every element \( f_t \) of \( \mathcal{H}_t \) as a map from \( \mathcal{G}_t \) to \( \mathbb{C} \):

**Lemma 4.3.7**

The unitary representation of \( \mathcal{G} \), \( \text{Ind} \mathcal{L}^2(\mathcal{G}, \mathcal{G}_2, \mathcal{R}_2 \uparrow_{\mathcal{G}_1}^{\mathcal{G}_2}) \) constructed by the method of orbits (1.3.37) is equivalent to a subrepresentation of \( \text{Ind} \mathcal{L}^2(\mathcal{G}, \mathcal{G}_1, \rho) \) as given by (1.3.26). Further, the isomorphism is an isometry of Hilbert spaces.

**Proof:**

By (1.3.36) and (1.3.37), \( \text{Ind} \mathcal{L}^2(\mathcal{G}, \mathcal{G}_2, \mathcal{R}_2 \uparrow_{\mathcal{G}_1}^{\mathcal{G}_2}) \) is a closed, invariant subspace of \( \mathcal{L}^2(\mathcal{G}, \mathcal{G}_2, \mathcal{R} \uparrow_{\mathcal{G}_1}^{\mathcal{G}_2}) \) given by (1.3.38(a)). The result now follows from (1.3.38(a)).

**Convention 4.3.8**

In view of (4.3.7), if \( \mathcal{R} : \mathcal{G} \to \mathcal{U}(\mathcal{H}) \) is a representation of \( \mathcal{G} \) constructed by the method of orbits, we will now regard all elements \( f \) of \( \mathcal{H} \) as functions \( f : \mathcal{G} \to \mathbb{C} \).
The following definition of convergence for sequences \( \{f_t\} \) is the intuitively acceptable one.

**Definition 4.3.9**

Let \( R_t : \mathcal{G}_t \to \mathcal{U}(\mathcal{H}_t) \) be unitary representations of \( \mathcal{G}_t \), constructed by the method of orbits ((1.3.36), (1.3.37) and (4.3.7)). Let \( \{f_t\} \) be a sequence of elements \( f_t \in \mathcal{H}_t \). We say that \( \{f_t\} \) is \( \mathcal{D}^q \)-continuous if

\[
\lim_{t \to 0} f_t(\Phi(g(t), t)) = f_0(\Phi(g)),
\]

for all \( g \) in \( \mathcal{D}^q \).

Before stating the "Bootstrap" proposition promised in §4.1, we give the "mild connectivity condition" on \( \mathcal{G} \) also mentioned therein.

**Definition 4.3.10**

Let \( \mathcal{G} \) be a Lie group, and suppose that one of the conditions of Theorem (2.5.24) holds, with respect to a fixed Laurent map \( \Psi \) (2.3.2). Let \( \mathcal{E}(\mathcal{D}^q) \) denote the subgroup of \( \mathcal{G} \) given by \( \mathcal{E}(\mathcal{D}^q) = \{g(0) | g \in \mathcal{D}^q\} \). Then we say that \( \mathcal{G} \) is \( \mathcal{D}^q \)-connected if,

\[
\forall g \in \mathcal{G}, \exists \phi : [0, 1] \to \mathcal{G},
\]

continuous, such that \( \phi(0) \in \mathcal{E}(\mathcal{D}^q) \) and \( \phi(1) = g \).

**Example 4.3.11**

If \( \mathcal{G} \) is connected, then \( \mathcal{G} \) is \( \mathcal{D}^q \)-connected, since every connected manifold is path-connected (1.1.36 (a)). Suppose that \( \mathcal{G} \) is finitely connected (1.1.36(d)). If the intersection of \( \mathcal{E}(\mathcal{D}^q) \) with each connected component (1.1.36(d)) is non-empty, then \( \mathcal{G} \) is \( \mathcal{D}^q \)-connected.

The maximal \( \mathcal{D}^q \)-connected subset of \( \mathcal{G} \), is the connected component (1.1.36 (b)) of \( \mathcal{E}(\mathcal{D}^q) \), which is open in \( \mathcal{G} \). The following lemma shows that \( \mathcal{D}^q \)-connectivity is a minor restriction on \( \mathcal{G} \) as far as contractions of Lie groups are concerned:
Lemma 4.3.12

For a fixed Laurent map \( \Psi (2.3.2) \): Let \( D^q \) and \( X \) correspond by (2.3.6) and (2.3.7) to the Lie group \( G \), satisfying any condition of Theorem (2.5.24).

Let \( G_1 \) be the connected component (see (1.1.36(c))) of \( E(D^q) \) in \( G \), with \( D_1^q \) corresponding by (2.3.6) and (2.3.7) to the Lie group \( G_1 \).

Then, \( D_1^q = D^q \) and hence \( X(D_1^q) = X(D^q) \).

Proof:

Let \( g \in D_1^q \). Then \( \lim_{t \to 0} \Phi(g(t), t) \) exists, where \( g \in C^q_{G_1} [0, 1] \). Hence \( g \in D^q \), and \( D_1^q \subseteq D^q \). Letting \( g \in D^q \), then \( \lim_{t \to 0} \Phi(g(t), t) \) exists and \( g(0) \in E(D^q) \), whence \( g[0, 1] \) is contained in the connected component of \( E(D^q) \) in \( G \), viz. \( G_1 \).

Hence \( g \in D_1^q \), \( D_1^q = D^q \) and \( X(D_1^q) = X(D^q) \). \[ \square \]

We now state the "Bootstrap" theorem announced in §4.1 on limiting matrix elements of representations of \( G \) to matrix elements of representations of \( G_0 \):

Theorem 4.3.13 ("Bootstrap" Theorem)

Suppose that \( G \) is \( D^q \)-connected, and let \( \{ G_t \} \) be the sequence of Lie groups (4.3.6). Let \( \mathcal{R}_t : G_t \to \mathcal{U}(\mathcal{H}_t) \) be a unitary representation, constructed by the method of orbits ((1.3.36), (1.3.37), (4.3.7)), for each \( G_t \).

Then, for any \( f_0 \in \mathcal{H}_0 \), there exists a \( D^q \)-continuous sequence \( \{ f_t \}, f_t \in \mathcal{H}_t \), such that

\[
\mathcal{R}_0(X(g))f_0(X(h)) = \lim_{t \to 0} \mathcal{R}_t(\Phi(g(t)))f_t(\Phi(h(t)))
\]

for all \( g, h \in D^q \),

if and only if

there exists a sequence of \( D^q \)-continuous elements \( \{ \tilde{f}_t \}, \tilde{f}_t \in \mathcal{H}_t \) with \( \tilde{f}_0 \neq 0 \).

240
Proof:

Consider the unitary, irreducible representation $\mathcal{R}_\nu : \mathcal{G}_\nu \to \mathcal{U}(\mathcal{H}_\nu)$ and the subset $\mathcal{H}_\nu^{(D^q)}$ of $\mathcal{H}_\nu$ given by,

$$\mathcal{H}_\nu^{(D^q)} = \{ f \mid f \in \mathcal{H}_\nu \ni \exists \{ f_t \}, \ f_t \in \mathcal{H}_t, \ \forall t$$

with $\{ f_t \} D^q$-continuous, and $f_\nu = f$.

(That is, each element of $\mathcal{H}_\nu^{(D^q)}$ is the restriction of a $D^q$-continuous sequence.)

We assert that $\mathcal{H}_\nu^{(D^q)}$ is an invariant subspace of $\mathcal{H}_\nu$:

Let $f, \tilde{f} \in \mathcal{H}_\nu^{(D^q)}$ with corresponding $D^q$-continuous sequences $\{ f_t \}, \ \{ \tilde{f}_t \}$.

Then $\alpha f + \beta \tilde{f} \in \mathcal{H}_\nu^{(D^q)}$, $\forall \alpha, \beta \in \mathbb{C}$ since the sequence $\{ \alpha f_t + \beta \tilde{f}_t \}$ has $\alpha f_t + \beta \tilde{f}_t = \alpha f + b\tilde{f}$. Hence $\mathcal{H}_\nu^{(D^q)}$ is a subspace of $\mathcal{H}_\nu$.

To show that $\mathcal{H}_\nu^{(D^q)}$ is an invariant subspace of $\mathcal{H}_\nu$, let $f \in \mathcal{H}_\nu^{(D^q)}$. By (4.3.7) $\mathcal{G}_\nu$ acts on $f$ by the right regular representation. Then we must show that the map

$$f(\cdot, g_\nu) : \mathcal{G}_\nu \to \mathbb{C} \text{ is in } \mathcal{H}_\nu^{(D^q)}$$

for each $g_\nu \in \mathcal{G}_\nu$. To do this, we need only find a $D^q$-continuous sequence $\{ \tilde{f}_t \}, \ \tilde{f}_t \in \mathcal{H}_t$, with $\tilde{f}_\nu(\cdot) = f(\cdot, g_\nu)$:

Let $\{ f_t \}, \ f_t \in \mathcal{H}_t$ be the $D^q$-continuous sequence such that $f_\nu = f$. The desired sequence $\{ \tilde{f}_t \}$ will be given by

$$\tilde{f}_t(\cdot) = f_t(\cdot, \tilde{g}_t), \ \forall t$$

where

$$\tilde{g}_t = \Phi(\bar{\tilde{g}}(t), t), \ t \neq 0$$

$$\tilde{g}_0 = \lim_{t \to 0} \Phi(\bar{\tilde{g}}(t), t) = \mathcal{X}(\bar{\tilde{g}})$$

for some $\tilde{g} \in D^q$. We now find this element of $D^q$. If $t' = 0$, choose $\bar{g} \in D^q$ with $\mathcal{X}(\bar{g}) = g_0$. If $t' \neq 0$, consider $\Phi^{-1}(g_\nu, t') \in \mathcal{G}$. Since $\mathcal{G}$ is $D^q$-connected, there
exists a $\psi \in C_{[0,1]}$ with $\psi(0) \in \mathcal{E}(\mathcal{D}^q)$ and $\psi(1) = \Phi^{-1}(g', t')$. Let $\bar{g} \in \mathcal{D}^q$ with $\bar{g}(0) = \psi(0)$. Then the points $\bar{g}[0,1]$ and $\Phi^{-1}(g', t')$ are in the same connected component (1.1.36 (d)). Since the requirements for $\bar{g}$ to be in $\mathcal{D}^q$ involve only $\bar{g}(0)$ and the first $q$ derivatives of $\bar{g}$ at $t = 0$, it is possible to choose a $\bar{g} \in \mathcal{D}^q$ with $\bar{g}(0) \in \mathcal{E}(\mathcal{D}^q)$ and $\bar{g}(t') = \Phi^{-1}(g', t')$.

With $\bar{g}$ now chosen, it is clear that the sequence $\{\bar{f}_t\}$ has $\bar{f}_t \in \mathcal{H}_t$, because $\mathcal{G}_t$ acts on $f_t \in \mathcal{H}_t$ invariantly by the right regular representation (4.3.7). That $\{\bar{f}_t\}$ is $\mathcal{D}^q$-continuous, follows from the $\mathcal{D}^q$-continuity of $\{f_t\}$, the group structure of $\mathcal{D}^q$, and $\bar{g}$ being an element of $\mathcal{D}^q$.

Since $\bar{f}_t(\cdot) = f'(g)$, this shows that $\mathcal{H}_t^{(\mathcal{D}^q)}$ is an invariant subspace of $\mathcal{H}_t$.

Now let $\{\bar{f}_t\}$ be the $\mathcal{D}^q$-continuous sequence of the statement of the theorem, with $\bar{f}_0 \neq 0$. By the definition of $\mathcal{D}^q$-continuity there must be a $\tau$ such that $\bar{f}_t \neq 0$, $\forall t \leq \tau$.

As we are interested only in what happens as $t \to 0$, we can assume without loss of generality that $\bar{f}_t \neq 0$, $\forall t$.

Since $\bar{f}_t \neq 0$, $\forall t$, then $\mathcal{H}_t^{(\mathcal{D}^q)}$ is non-trivial for each $t$. By the irreducibility of $\mathcal{R}_t : \mathcal{G}_t \to \mathcal{U}(\mathcal{H}_t)$ and the invariance of the subspace $\mathcal{H}_t^{(\mathcal{D}^q)}$, we must have $\mathcal{H}_t^{(\mathcal{D}^q)} = \mathcal{H}_t$, $\forall t$.

Let $f_0 \in \mathcal{H}_0$, then there must exist for $f_0$, a $\mathcal{D}^q$-continuous sequence $\{f_t\}$, $f_t \in \mathcal{H}_t$.

For any $g, h \in \mathcal{D}^q$ we have

$$
\mathcal{R}_0(\mathcal{I}(g))f_0(\mathcal{I}(h)) = \lim_{t \to 0} f_t(\Phi(h(t))g(t))
$$

by $\mathcal{D}^q$-continuity and the definition of $\mathcal{R}_0$,

$$
= \lim_{t \to 0} \mathcal{R}_t(\Phi(g(t)))f_t(\Phi(h(t)))
$$

by the definition of $\mathcal{R}_t$;
as asserted in the statement of the theorem. The converse of the statement is immediate.

Comments 4.3.14

Observe that the Bootstrap Theorem (4.3.13) holds largely independently of the method of orbits. Only the following aspects of the method of orbits are used:
(a) Elements $f_t \in \mathcal{H}_t$ are functions on $\mathcal{G}_t$ into $\mathbb{C}$ (or any fixed vector space, $\forall t$),
(b) $\mathcal{R}_t$ acts by the right regular representation on elements of $\mathcal{H}_t$,
(c) $\mathcal{R}_t : \mathcal{G}_t \rightarrow \mathcal{U}(\mathcal{H}_t)$ is irreducible.

Thus, Theorem (4.3.13) would hold for representations $\mathcal{R}_t : \mathcal{G}_t \rightarrow \mathcal{U}(\mathcal{H}_t)$ satisfying (a)(b)(c) only, with no recourse to the method of orbits. The method of orbits however, does provide a coherent procedure for choosing the sequence of irreducible representations $\mathcal{R}_t : \mathcal{G}_t \rightarrow \mathcal{U}(\mathcal{H}_t)$, $\forall t$, based on a choice of $\beta \in \mathcal{G}_t^*$ only (modulo some freedom in the choice of polarization $\mathcal{h}(\beta)$). We naturally wish to fulfill the conditions of the Bootstrap Theorem (4.3.13) and find a $D^q$-continuous sequence $\{f_t\}, f_t \in \mathcal{H}_t$ with $f_0 \neq 0$. One approach, is to use the theory developed so far for the contraction of $\mathcal{G}_t$ and polarizations to seek a $D^q$-continuous sequence $\{f_t\}$ determined by a $C^\infty$ function on $\mathcal{G}_{[0,1]}$. We can in fact (this is somewhat more fundamental), give conditions on a $C^\infty$ curve of $C^\infty$ functions on $\mathcal{G}$ to supply an appropriate sequence. Our task then, is to find what restrictions must be placed on a $C^\infty$ function $F : \mathcal{G} \times (0,1] \rightarrow \mathbb{C}$ in order to provide the $D^q$-continuous sequence $\{f_t\}, f_t \in \mathcal{H}_t, \forall t$, with $f_0 \neq 0$.

We call $F$ a "seed function" if it satisfies these restrictions:

Definition 4.3.14

A $C^\infty$ map $F : \mathcal{G} \times (0,1] \rightarrow \mathbb{C}$ is called a seed function if it satisfies the following conditions:
(i) \( \lim_{t \to 0} (X \Phi(g(t)))^T F \circ \Phi^{-1}(\Phi(g(t), t)) \) exists, for any \( C^\infty \) vector field \( X \) on \( G_{[0,1]} \), for any positive integer \( J \), and for all \( g \in D^q \).

(ii) \( F(g^{-1}g_1, t) = \rho_t(g)F(g_1, t), t \neq 0 \ \forall g_1 \in G_{\beta(t),} \ \forall g \in G \) where \( \rho_t : G_{\beta(t)} \to T \) is a character of \( G_{\beta(t)} \) with derivative \( \beta(t) \) (c.f. the method of orbits).

(iii) \( \nabla_Z F(g, t) = 0, \forall Z \in \mathfrak{h}(\beta(t)), t \neq 0, \forall g \in G \) (see (1.3.11)).

(iv) \( \lim_{t \to 0} F(g(t), t) = \rho_0(X(g)), \forall g \in (D^q)_\beta, \) where \( \rho_0 \) is a character of \( G_{\beta_0} \) with derivative \( \beta_0 \).

(v) \( \lim_{t \to 0} F(g(t)g_1(t)) = \left( \lim_{t \to 0} F(g(t), t) \right) \left( \lim_{t \to 0} F(g_1(t), t) \right) \ \forall g \in (D^q)_\beta, \forall g_1 \in D^q \)

(iv) \( \lim_{t \to 0} \nabla_{d\Phi(Z(t))} F \circ \Phi^{-1}(\Phi(g(t), t)) = 0, \forall Z \in \mathfrak{h}(\beta), \forall g \in D^q \) (see (1.3.11)).

While these conditions in \( F \) are certainly cumbersome, nonetheless they are almost necessary and sufficient in ensuring the existence of a \( D^q \)-continuous sequence of \( C^\infty \) functions \( \{f_t\} \):

**Proposition 4.3.15**

If a seed function \( F \) exists, then a sequence of \( D^q \)-continuous maps \( \{f_t\}, f_t \in H_t \) with \( f_0 \neq 0 \) exists, and each \( f_t \) is \( C^\infty \).

If a sequence of \( D^q \)-continuous maps \( \{f_t\}, f_t \in H_t \) exists, with \( f_0 \neq 0 \), which extends to a \( C^\infty \) function \( \tilde{F} \) on \( G_{[0,1]} \), then \( \tilde{F} \circ \Phi \) is a seed function.

**Proof:**

Let \( F \) be a seed function. Taking \( J = 0 \) in (4.3.14 (i)), and using the form of the coordinate charts (4.3.2) on \( G_{[0,1]} \), \( F \circ \Phi^{-1} \) has a unique extension to a continuous map \( \tilde{F} \) on all of \( G_{[0,1]} \) by standard metric space theory (e.g. Copson (1968, §65)). For any \( g_0 \in G_0 \), let \( (\lambda, t) \) be the coordinates of a chart at \( g_0 \) in \( G_{[0,1]} \) as given in (4.3.1). Take \( J = 1 \), and choose the vector fields \( X_i = \frac{\partial}{\partial \lambda^i} \). By (4.3.14(i)) we can extend \( \frac{\partial F \circ \Phi^{-1}}{\partial \lambda^i} \) to a function continuous on all of the chart. By

244
taking partial integrals of $\frac{\partial \tilde{F}}{\partial \lambda^i}$ and using the uniqueness of the extension of $F \circ \Phi^{-1}$ to $\tilde{F}$, we conclude that $\tilde{F}$ is differentiable on $G_{[0,1]}$. Proceeding by induction, $\tilde{F}$ is $C^\infty$ on $G_{[0,1]}$. Now define a sequence $\{f_t\}$ by $f_t(\cdot) = \tilde{F}(\cdot, t)$. By (4.3.14(ii) & (iii)) $f_t \in \mathcal{H}_t$, $t \neq 0$. By (4.3.14 (v)),

$$f_0(\mathcal{X}(g)\mathcal{X}(g_1)) = f_0(\mathcal{X}(g))f_0(\mathcal{X}(g_1)), \forall g \in (\mathcal{D}^g)_\beta, \forall g_1 \in \mathcal{D}^g$$

$$= \rho_0(\mathcal{X}(g))f_0(\mathcal{X}(g_1)) \text{ by (4.3.14(iv)).}$$

Therefore $f_0(h_0g_0) = \rho_0(h_0)f_0(g_0)$, $\forall h_0 \in G_{\beta_0}, \forall g_0 \in G_0$ by (4.2.3).

By (4.3.14(i) & (vi)), (2.5.69) and the group properties of $\mathcal{D}^g$,

$$\nabla_{\mathcal{D}^g(z)} f_0(\mathcal{X}(g)) = 0, \forall z \in \mathcal{H}(\beta), \forall g \in \mathcal{D}^g,$$

whence by (4.2.17), $\nabla_{Z_0} f_0(g_0) = 0$, $\forall Z_0 \in \mathcal{H}(\beta_0)$, $\forall g_0 \in G_0$. Therefore $f_0 \in \mathcal{H}_0.\{f_t\}$ is $\mathcal{D}^g$-continuous because it is the restriction of a $C^\infty$ map $\tilde{F}$ on $G_{[0,1]}$. $f_0$ is non-zero by (4.3.14 (iv)).

Conversely, let $\{f_t\}$, $f_t \in \mathcal{H}_t$ be a $\mathcal{D}^g$-continuous sequence of maps with $f_0 \neq 0$ which extends to a $C^\infty$ map $\tilde{F}$ on $G_{[0,1]}$. Then (4.3.14 (i)(ii)(iii)(iv)) are immediately satisfied. Let $g \in (\mathcal{D}^g)_\beta$, $g_1 \in \mathcal{D}^g$, then

$$\lim_{t \to 0} \tilde{F} \circ \Phi(g(t)g_1(t)) = F(\mathcal{X}(g)\mathcal{X}(g_1))$$

$$= f_0(\mathcal{X}(g)\mathcal{X}(g_1))$$

$$= \rho_0(\mathcal{X}(g))f_0(\mathcal{X}(g_1))$$

$$= f_0(\mathcal{X}(g))f_0(\mathcal{X}(g_1))$$

$$= \tilde{F}(\mathcal{X}(g))\tilde{F}(\mathcal{X}(g_1))$$

$$= \lim_{t \to 0} \tilde{F} \circ \Phi(g(t)) \cdot \lim_{t \to 0} \tilde{F} \circ \Phi(g_1(t)),$$

so that (4.3.14 (v)) is fulfilled.

245
By the definition of the $\beta$-derivative (1.3.11), and (2.5.69), and by the group property of $\mathcal{D}^q$,

$$\lim_{t \to 0} \nabla_d \Phi(Z(t)) \tilde{F} \circ \Phi(g(t)) \text{ exists, } \forall Z \in \mathbb{H} (\beta), \forall g \in \mathcal{D}^q,$$

and is equal to $\nabla_d \tilde{F}(\mathcal{X}(g)) = \nabla_d \tilde{F}(\mathcal{X}(g)) = \nabla_d \tilde{F}(\mathcal{X}(g))$, which is zero, satisfying (4.3.14 (vi)).

Consequently $\tilde{F} \circ \Phi$ is a seed function as asserted.
§5.1 Contractions of Generalised Characters of Irreducible Unitary Representations

In this section, we use the explicit, universal formula ((1.3.46), (1.3.47)) of the generalised character $\chi_\mathcal{R}$ (1.3.43) of a representation $\mathcal{R}$ to show that, if $\mathcal{G}_t (\forall t)$ has an irreducible unitary representation $\mathcal{R}_t$ and a generalised character $\chi_{\mathcal{R}_t}$ given by the formulas (1.3.46) and (1.3.47), then in the notation of (1.3.46), for any appropriate sequence of functions $f_t \in C_0^\infty(\mathcal{G}_t(0))$ we have

$$\lim_{t \to 0} \chi_{\mathcal{R}_t}(f_t) = \chi_{\mathcal{R}_0}(f_0).$$

(5.1.1)

We prove the above result for a slightly more general case than $\mathcal{G}$ compact and connected or exponential (1.1.72), with some rather special conditions on the co-adjoint orbits $\Omega_t$. It appears very likely that (5.1.1) can be proved for a much wider class of Lie groups, and one shall endeavour to solve this problem in future publications.

The first step in proving (5.1.1) is to show how the inner integral in (1.3.46) may be meaningfully contracted. This will clarify the process of contracting both integrals together, as the procedure for contracting the second integral is a local application of the method of contracting the inner integral.

In order to state the required proposition neatly, we need the following corollary of (4.3.1):

**Corollary 5.1.2**

Let $q_{[0,1]} = \bigcup_{t \in (0,1]} d\Phi(q,t) \cup q_0$ and suppose that $\dim q = \dim q_0$.

Then, $q_{[0,1]}$ is a $C^\infty$-vector bundle (1.4.12) which is vector bundle isomorphic (1.4.13) to $q \times [0,1]$. 

247
Proof:

Using the notation of (4.3.1), let \([0, \epsilon)\) be such that \(\det(\theta(t)) \neq 0, \forall t \in [0, \epsilon)\).

Define \(g_{[0,\epsilon)}\) to be

\[
g_{[0,\epsilon)} = \bigcup_{t \in (0,\epsilon)} d\Phi(g,t) \cup g_0.
\]

By the proof of (4.3.1) we have the map

\[
g_{[0,\epsilon)} \to g \times [0, \epsilon)
\]

\[
\left(\frac{d\phi(t)}{dt}(\theta(t)\lambda), t\right) \mapsto (\lambda, t), \quad t \in (0,\epsilon), \lambda \in \mathbb{R}^n,
\]

whence, with topology on \(g_{[0,\epsilon)}\) induced by this map, \(g_{[0,\epsilon)}\) is vector bundle isomorphic to \(g \times [0, \epsilon)\), and hence \(g_{[0,1]}\) is vector bundle isomorphic to \(g \times [0, 1]\).

\[\Box\]

The following notation will be useful:

**Notation 5.1.3**

\(g_{[0,1]}(0)\) will denote an open subset of \(g_{[0,1]}\) such that \(g_t \cap g_{[0,1]}(0)\) is a neighbourhood of 0 on which the exponential mapping for \(g_t, \forall t\), is a diffeomorphism.

Such a neighbourhood always exists by (5.1.2) and the relationship, \(\exp_{g_t}(d\Phi(t)X) = \Phi(\exp_g(X), t), \forall X \in g\) (see (1.1.45)).

Herewith the proposition for contracting the inner integral in the formula (1.3.46) for the generalised character.

**Proposition 5.1.4**

Let \(f\) be an element of \(C^\infty_0(g_{[0,1]}(0))\) with \(Pu\) given for arbitrary Lie algebras by (1.3.47). Then,

\[
\lim_{t \to 0} \int_{g_t} e^{i(\Lambda^*(\Phi(g(t)))d\Phi^*\beta(t),X_t)} f(X_t)(Pu(X_t))^{-1}dX_t
\]

248
for all \( \beta \in \mathcal{D}_q^* \) and \( g \in \mathcal{D}_q^* \).

**Proof:**

The proof consists of rewriting the integral on the left hand side of (5.1.5) so that the variable and domain of integration are independent of \( t \) enabling the limit \( t \to 0 \) to be taken. This is achieved by making a certain, \( t \)-dependent change of variables in the integral.

Let \( \theta(t) \) be given by the proof of (4.3.1). Then we make the \( t \)-dependent change of parameters in (5.1.5) by,

\[
\begin{align*}
\varphi(t) & \rightarrow \mathbb{R}^n \\
X_t = d\Phi(t)\theta(t)\lambda, & \quad t \in [0, \epsilon), \lambda \in \mathbb{R}^n,
\end{align*}
\]

and the determinant of the Jacobian is

\[
\left| \frac{\partial d\Phi(t)\theta(t)\lambda}{\partial \lambda} \right| = |d\Phi(t)\theta(t)|.
\]

Then we have, for \( t \in [0, \epsilon) \),

\[
\mathcal{J}^*\mathcal{D}_q^* \mathcal{D}_q^* \mathcal{D}_q^*
\]

\[
since e^{i(\Lambda d^* \Phi(\theta(t)) d\Phi^* \beta(t), X_t)} f(X_t)(P\mu(X_t))^{-1} dX_t
\]

\[
= \int_{\mathbb{R}^n} e^{i(\Lambda d^* \Phi(\theta(t)) d\Phi^* \beta(t), \Lambda \theta(t)\lambda)} f(d\Phi(t)\theta(t)\lambda)(P\mu(d\Phi(t)\theta(t)\lambda))^{-1} \left| \frac{\partial d\Phi(t)\theta(t)\lambda}{\partial \lambda} \right| d\lambda
\]

As the integration is Lebesgue and the region of integration compact, the Lebesgue dominated convergence theorem applies and we may take the limit of both sides of the above equation, taking the limit inside the integral on the RHS:

\[
\lim_{t \to 0} d\Phi(t)\theta(t)\lambda = \lambda,
\]

\[
\lim_{t \to 0} \left| \frac{\partial d\Phi(t)\theta(t)\lambda}{\partial \lambda} \right| = 1.
\]
\[ \lim_{t \to 0} \langle \text{Ad}^* \Phi(g(t)) d\Phi^* \beta(t), d\Phi(t) \theta(t) \lambda \rangle = \langle \text{Ad}^* g_0 \beta_0, \lambda \rangle \]

whence,

\[
\begin{align*}
\lim_{t \to 0} & \int \mathcal{G} \, e^{i(\text{Ad}^* \Phi(g(t)) d\Phi^* \beta(t), X_t) f(X_t)(P_U(X_t))^{-1} dX_t} \\
&= \int_{\mathcal{G}_0} e^{i(\text{Ad}^* g_0 \beta_0, \lambda) f(\lambda)(P_U(\lambda))^{-1} d\lambda} \\
&= \int_{\mathcal{G}_0} e^{i(\text{Ad}^* g_0 \beta_0, X_0) f(X_0)(P_U(X_0))^{-1} dX_0},
\end{align*}
\]

establishing the proposition. \(\blacksquare\)

**Comments 5.1.5(a)**

In order to establish (5.1.1) we will need to prove some technical propositions. In Proposition (4.3.1) we showed that \( \mathcal{G}_{[0,1]} \) was a \( C^\infty \)-manifold with boundary. We can give a similar result for the co-adjoint orbits of the \( \mathcal{G}_t \), with reasonable conditions connecting the co-adjoint orbits for each \( t \):

**Notation 5.1.5(b)**

Let \( \beta \in \mathcal{G}_t^* \) with

\[
\beta_t = \begin{cases} 
(d\Phi^{-1})^* \beta(t), & t \in (0,1] \\
\mathcal{X}^*(\beta), & t = 0
\end{cases}
\]

\[
\mathcal{G}_t = \begin{cases} 
\Phi(\mathcal{G}, t), & t \in (0,1] \\
\mathcal{X}(\mathcal{D}), & t = 0
\end{cases}
\]

\( \Omega_t = \text{Ad}^* \mathcal{G}_t(\beta_t) \), and

\[
\Omega_{[0,1]} = \bigcup_{t \in [0,1]} \Omega_t.
\]

**Proposition 5.1.6**

Suppose that the following conditions hold, connecting the \( \Omega_t \)'s:

(i) \( \dim \mathcal{G} = \dim \mathcal{G}_0 \)
(ii) \( \dim g_\beta(t) = \dim g_{\beta_0} \)

(iii) \( g_{\beta_0} = d\mathcal{X}\{X \mid X \in d^q, X(t) \in g_\beta(t), t \in (0, 1)\}. \)

Then,

(a) \( \Omega_{[0,1]} \) is a \( C^\infty \)-manifold with boundary, and has a \( C^\infty \) projection map \( \pi_{[0,1]} : \Omega_{[0,1]} \to [0, 1] \)

(b) \( \Omega_0 \) and \( \Omega_t \) are closed submanifolds of \( \Omega_{[0,1]} \).

**Proof:**

We will choose special bases of \( g_t \) and define a chart on \( \Omega_{[0,1]} \) with respect to these, using the exponential map on \( \mathcal{G}_t \). The proof is somewhat similar to the proof of (4.3.1).

By (iii) \( \exists Y^i \in d^q \), with \( Y^i(t) \in g_\beta(t) \) such that \( d\mathcal{X}(Y^i) \) is a basis of \( g_{\beta_0} \), and the vectors \( \{d\Phi(Y^i(t))\} \) are linearly independent for all \( t \) in \( [0, \epsilon] \) for some \( \epsilon > 0 \), by the proof of (2.4.17(5)).

By (ii), \( \{Y^i(t)\} \) must be a basis of \( g_\beta(t) \), for \( t \in [0, \epsilon] \).

For convenience, we may take \( \epsilon = 1 \) without loss of generality.

Again, using the argument of the proof of (2.4.17(5)), the basis \( \{Y^i(t)\} \) may be extended so that \( d\mathcal{X}(Y^i) \) is a basis of \( g_0 \) and \( \{Y^i(t)\} \) is a basis of \( g \) for \( t \in (0, 1) \). The \( Y^i \) can always be chosen as \( C^\infty \) functions of \( t \).

We adopt the convention that \( Y^\alpha(t) \) (with a greek index) refers to any element of this extended basis, while \( \{Y^i(t)\} \) refers to \( m \) elements in the complement of the set of those vectors which form a basis of \( g_\beta(t) \) for \( t \in (0, 1) \]. The basis \( \{Y^\alpha(t)\} \) is ordered with the sub-basis \( \{Y^i(t)\} \) first.

We now define a \( C^\infty \) sequence of linear maps \( \theta(t) : g_0 \to g \) that enables us to view \( \Omega_{[0,1]} \) as a subset of \( g_0^* \): Define the maps,

\[
\theta(t) : g_0 \to g \\
\theta(t) : d\mathcal{X}(Y^\alpha) \to Y^\alpha(t), \ t \in [0, 1].
\] (5.1.7)
With respect to a fixed basis \{X_\alpha\} of \( g \) and basis \{d\mathcal{X}(Y_\alpha)\} of \( \mathfrak{g}_0 \), if \( Y_\alpha(t) = \theta^\alpha(t)X_\eta \), we can choose the \{Y_\alpha\} as \( C^\infty \)-functions on \([0,1]\), so that \( \theta^\alpha(t) \) is \( C^\infty \) on \([0,1]\) and is invertible for each \( t \in (0,1] \), and is the matrix of the linear map \( \theta(t) \) with respect to these bases. By choosing special bases
\[
\{d\Phi(t)Y_\alpha(t)\} \text{ of } g_t \text{ and } \{d\mathcal{X}(Y_\alpha)\} \text{ of } \mathfrak{g}_0,
\]
we can obtain a convenient, concrete realisation of \( \Omega_{[0,1]} \):

Now,
\[
\lim_{t \to 0} \langle \text{Ad}_t^* g(t)\beta(t), Y_\alpha(t) \rangle = \lim_{t \to 0} \langle \text{Ad}_t^* g(t)\beta(t), \theta(t)d\mathcal{X}(Y_\alpha) \rangle = \lim_{t \to 0} \langle \theta^*(t)\text{Ad}_t^* g(t)\beta(t), d\mathcal{X}(Y_\alpha) \rangle,
\]
and we define \( \Omega'_t \) to be:
\[
\Omega'_t = (\theta^*(t)\text{Ad}_t^*(\mathcal{G})\beta(t), t), \quad t \in (0,1]
\]
\[
\Omega'_0 = \left\{ \left( \lim_{t \to 0} \theta^*(t)\text{Ad}_t^*(g(t))\beta(t), 0 \right) \mid g \in \mathcal{D}' \right\},
\]
with \( \Omega'_{[0,1]} = \bigcup_{t \in [0,1]} \Omega'_t \). Define the \( 1-1 \) and onto map,
\[
\Theta : \Omega'_{[0,1]} \to \Omega_{[0,1]}
\]
\[
\Theta : \left\{ \begin{array}{c}
\gamma'_t \mapsto (d\Phi^{-1}(t))^* \circ (\theta^*)^{-1}(t)(\gamma'_t) \\
\gamma'_0 \mapsto \gamma'_0
\end{array} \right. \quad \text{5.1.7(a)}
\]
We will show that \( \Omega'_{[0,1]} \) is a \( C^\infty \) manifold with boundary and then induce the topology of \( \Omega_{[0,1]} \) via the map \( \Theta \).

We now give a chart on \( \Omega'_{[0,1]} \) at \( \beta_0 \):

Let \( P : \mathbb{R}^m \to \mathbb{R}^n \) be the natural projection onto the copy of \( \mathbb{R}^m \) given by \( \{(x_1, x_2, \ldots, x_m, 0, \ldots, 0) \mid x_i \in \mathbb{R} \} \), and let \( \mathcal{N}_0 \) be a neighbourhood of \( 0 \) in \( \mathbb{R}^n \) such that the set \( \{\lambda^\alpha Y^\alpha(t) \mid t \in [0,1], \lambda \in \mathcal{N}_0 \} \), is contained in the domain on
which exp is a diffeomorphism, and such that the set \( \{ \lambda^\alpha Y_0^\alpha \mid \lambda \in \mathcal{N}_0 \} \) bears the same relationship to \( \exp_0 \). Recalling our index convention, the chart \( \chi \) is given by,

\[
(\theta^*(t) \text{Ad}^*(\exp(\lambda^i Y^i(t)))) \beta(t), t \mapsto (\lambda^i, t), \ t \in (0, 1], (\lambda^i, 0) \in \mathcal{N}_0,
\]

\[
\left( \lim_{t \to 0} \theta^*(t) \text{Ad}^*(\exp(\lambda^i Y^i(t))) \beta(t), 0 \mapsto (\lambda^i, 0), (\lambda^i, 0) \in \mathcal{N}_0 \right) \quad (5.1.7(b))
\]

This chart is a \( 1-1 \) map onto \( [0, 1] \times \mathcal{P}(\mathcal{N}_0) \) by the choice of the \( \{ Y^i \} \) and by the fact that

\[
\lim_{t \to 0} \theta^*(t) \text{Ad}^*(\exp(\lambda^i Y^i(t))) \beta(t) = \text{Ad}^*(\exp_0(\lambda^i \mathcal{X}(Y^i))) \beta_0.
\]

To obtain a covering of \( \Omega'_{[0,1]} \) by charts, let \( \gamma_0 \in \Omega_0 \) with \( \text{Ad}^*(g_0) \beta_0 = \gamma_0 \) and \( \mathcal{X}(g) = g_0 \) with \( g \in \mathcal{D}^\eta \), and \( g \in \mathcal{C}^\infty \).

A chart \( \chi_1 \) at \( \gamma_0 \) is given by,

\[
(\theta^*(s) \text{Ad}^*(g(s) \exp(\lambda^i Y^i(s)))) \beta(s), s \mapsto (\lambda^i, s), \ s \in (0, 1], (\lambda^i, 0) \in \mathcal{N}_0,
\]

\[
\left( \lim_{s \to 0} \theta^*(s) \text{Ad}^*(g(s) \exp(\lambda^i Y^i(s))) \beta(s), 0 \mapsto (\lambda^i, 0), (\lambda^i, 0) \in \mathcal{N}_0 \right) \quad (5.1.7(c))
\]

A chart \( \chi_2 \) in the neighbourhood of \( \gamma_t \in \Omega'_t \) is simply given by,

\[
(\theta^*(s) \text{Ad}^*(g \exp(\lambda^i Y^i(s)))) \beta(s), s \mapsto (\lambda^i, s),
\]

for \( s \in (t - \delta, t + \delta), (\lambda^i, 0) \in \mathcal{N}_0 \), with \( \delta \) a fixed, positive number in \( (0, t) \), where \( \text{Ad}^*(\Phi(g, t)) \beta_t = \gamma_t \), for \( g \in \mathcal{G} \).

Let \( \chi_3 \) be a chart at another point of \( \Omega_0 \) corresponding to the element \( g_1 \) of \( \mathcal{D}^\eta \) which is \( \mathcal{C}^\infty \) as for \( \chi_1 \). Define the maps \( \tilde{\theta}(t) \),

\[
\tilde{\theta}(t) : g \to g
\]

\[
\tilde{\theta}(t) : X^\alpha \to \theta^\alpha \eta(t) X^\eta.
\]

253
Then,

$$\chi_i \circ \chi^{-1}_a(\lambda^i, t) = \tilde{\theta}^{-1}(t) \circ \exp^{-1}(g^{-1}(t)g_1(t)\exp(\lambda^iY^i(t))).$$  \hspace{1cm} (5.1.8)$$

Now each of the curves $g^{-1}(t), g_1(t), \exp(\lambda^iY^i(t))$ is in $D^q$ so

$$g^{-1}(t)g_1(t)\exp(\lambda^iY^i(t)) \text{ is in } D^q, \text{ and is in the domain of the canonical chart.}$$

Then,

$$g^{-1}(t)g_1(t)\exp(\lambda^iY^i(t)) = \exp X_\lambda(t)$$

for some $X_\lambda \in D^q$, where $X_\lambda(t)$ is a $C^\infty$ function of $(\lambda, t)$.

Re-writing (5.1.8),

$$\chi_i \circ \chi^{-1}_a(\lambda^i, t) = \tilde{\theta}^{-1}(t)X_\lambda(t)$$

$$= (d\Phi(t)\tilde{\theta}(t))^{-1}(d\Phi(t)X_\lambda(t))$$

(5.1.8(a))

By Lemma (3.1.8) and the fact that $\lim_{t \to 0} d\Phi(t)\tilde{\theta}(t) = I$, $\chi_i \circ \chi^{-1}_a$ is a $C^\infty$ function of $(\lambda^i, t)$ (see also the proof of (5.1.9(a))). The proofs that other pairs of charts are $C^\infty$-related, are similar. Hence $\Omega_{[0,1]}$, with topology induced by its charts, is a $C^\infty$-manifold with boundary, whence $\Omega_{[0,1]}$ is a $C^\infty$-manifold with boundary.

Define the projection map,

$$\pi_{[0,1]} : \Omega_{[0,1]} \to [0, 1]$$

$$\pi_{[0,1]} : (\gamma_t, t) \mapsto t$$

which is the map $(\lambda^i, t) \mapsto t$ with respect to any of the above coordinate charts, and is clearly $C^\infty$.

That $\Omega_0$ and $\Omega_t$ are closed submanifolds of $\Omega_{[0,1]}$ follows immediately by fixing $t$ or $s$ in any of the charts $\chi, \chi_1, \chi_2$; thus establishing the proposition.

At the beginning of this section, we remarked that to prove (5.1.1) involves contracting the inner and outer integrals in the formula (1.3.46). To give a theorem limiting integrals on $\Omega_t$, we will need a technical lemma on the existence of a chart which is effectively global on $\Omega_{[0,1]}$. 

254
Lemma 5.1.9

Suppose that conditions (5.1.6(i)(ii)(iii)) are fulfilled, so that $\Omega_{[0,1]}$ is a $C^\infty$-manifold with boundary, and that (iv) $\mathcal{G}$ is compact and connected, exponential (1.1.72), or has a canonical chart covering a dense subset of $\mathcal{G}$.

Then,

there is a dense, open subset $\mathcal{U}$ of $\Omega_{[0,1]}$ on which a $C^\infty$ chart is globally defined.

Proof:

We first find a covering of $\Omega_0$ and then extend this to $\Omega_{[0,1]}$. We extend the range of the coordinate chart (5.1.7(b)):

Let

$$\tilde{\theta}_M(t) : \mathbb{R}^n \to \mathbb{R}^n \quad \text{be the map}$$

$$\tilde{\theta}_M(t) : x^\alpha \mapsto \theta^\alpha_n(t)x^n$$

with $\theta^\alpha_n(t)$ given in the proof of (5.1.6).

Now, as we will see in Lemma (5.1.9(a)), $\tilde{\theta}_M^{-1}(t)$ defines a generalised Saletan contraction of $g$ whose domain is the same as that of $d\mathcal{X}$.

Let $\mathcal{N}$ be a maximal neighbourhood of 0 in $g$ on which $\exp$ is a diffeomorphism. Then $\exp(\mathcal{N})$ is dense in $\mathcal{G}$. With respect to the basis $\{X_\alpha\}$ of $g$, let $\mathcal{N}_2 = \{ x \mid x \in \mathbb{R}^n, x^\alpha X_\alpha \in \mathcal{N} \}$. Then, for

$$(\lambda^i, 0) \in \tilde{\theta}_M^{-1}(t)\mathcal{N}_2,$$ we have $\lambda^i Y^i(t) \in \mathcal{N}$, $t \in (0,1]$;

and defining

$$\tilde{\theta}_M^{-1}(0)\mathcal{N}_2 = \{ \lim_{t \to 0} \tilde{\theta}_M^{-1}(t)x(t) \mid X \in g, X(t) = x^\alpha(t)X_\alpha, x[0,1] \subset \mathcal{N}_2 \},$$

the modified version of (5.1.7(b)) is the map

$$\chi : (\theta^*(t)\text{Ad}^*(\exp(\lambda^i Y^i(t))))\beta(t), t \mapsto (\lambda^i, t), t \in (0,1], (\lambda, 0) \in \tilde{\theta}_M^{-1}(t)\mathcal{N}_2$$

$$\chi : (\lim_{t \to 0} \theta^*(t)\text{Ad}^*(\exp(\lambda^i Y^i(t))))\beta(t), 0 \mapsto (\lambda^i, 0), (\lambda, 0) \in \tilde{\theta}_M^{-1}(0)\mathcal{N}_2$$

255
The notation "$\hat{\theta}^{-1}_M(0)N_2$" is intended to convey the fact that the neighbourhoods $\hat{\theta}^{-1}_M(t)N_2$ geometrically deform to $\hat{\theta}^{-1}_M(0)N_2$ as $t \to 0$.

Let $P$ be given by the proof of (5.1.6). Now

$$\lim_{t \to 0} \theta^* (t) A d^* (\exp(\lambda^i Y_i(t))) \beta(t) = A d^* (\exp(\lambda^i d\mathcal{X}(Y_i))) \beta_0,$$

and $\chi$ will be $1 - 1$ and onto the set $\bigcup_{t \in [0,1]} (P(\hat{\theta}^{-1}_M(t)N_2), t)$, and a candidate for a chart, if $\exp_0$ is a diffeomorphism on the domain

$$\{\lambda_0^i d\mathcal{X}(Y_i^\alpha) | \lambda_0 \in \hat{\theta}^{-1}_M(0)N_2\} = \{d\mathcal{X}(X) | X \in G^q, X[0,1] \subset N\}$$

and $\hat{\theta}^{-1}_M(0)N_2$ is open in $\mathbb{R}^n$. That $\hat{\theta}^{-1}_M(0)N_2$ is open in $\mathbb{R}^n$, will follow (by Lemma (5.1.9(b)) if $\bar{N}_0 = \{d\mathcal{X}(X) | X \in G^q, X[0,1] \subset N\}$ is open in $g_0$.

Now (2.5.46), with the map $\chi_1$ and neighbourhood $N_1$ as given there, shows that $\chi_1(N_1)$ is open in $C^{rq}_{d\mathcal{X},\mathcal{X}}$. Using the fact that $\mathcal{I}^{rq}$ and $\mathcal{X}$ are continuous open maps onto their images (see (2.5.63) and (2.5.30), and using $G_0 = D^q/\ker \mathcal{X}$), this establishes that $\bar{N}_0$ is open in $g_0$.

It follows from (2.5.69) with $\exp_0$ given by $\exp_0 = \mathcal{X}_P \circ \exp \circ d\mathcal{X}^{-1}_P$ and the form of $\exp$ given by (2.5.8), that $\exp_0$ is a diffeomorphism on all of $\bar{N}_0$.

The proof that $\chi$ is $C^\infty$-related to the other charts of (5.1.6) amounts to re-verifying that the map given by (5.1.8(a)) is $C^\infty$ on the set $N_0 \times [0,1]$, with $N_0$ in this case defined in the proof of (5.1.6).

Let $\mathcal{U} = \chi^{-1}\left(\bigcup_{t \in [0,1]} (P(\hat{\theta}^{-1}_M(t)N_2), t)\right)$. Certainly $\chi^{-1}(P(\hat{\theta}^{-1}_M(t)N_2), t))$ is dense in $\Omega_t$ for $t \neq 0$; as using (5.1.7(a)) and identifying $\Omega'_{[0,1]}$ with $\Omega_{[0,1]}$,

$$\chi^{-1}(P(\hat{\theta}^{-1}_M(t)N_2), t)) = A d^*(\exp_t \circ d\Phi(t)(\mathcal{N})) \beta_t$$

and $\exp_t \circ d\Phi(t)(\mathcal{N})$ is dense in $\mathcal{G}_t$ because $\exp(\mathcal{N})$ is dense in $\mathcal{G}$. By the results of
Lemma (5.1.9(b)),

\[ \chi^{-1}((P(\tilde{\theta}_M^{-1}(0)N_2),0)) = \{ \text{Ad}^*(\exp_0(\lambda^i d\mathfrak{X}(Y^i)))\beta_0 \mid (\lambda^i,0) \in \tilde{\theta}_M^{-1}(0)N_2 \} \]

\[ = \text{Ad}^*(\exp_0(\tilde{N}_0))\beta_0. \]

Let \{X_j\} be the basis of \( \mathfrak{g}'(G) \) given at (2.5.11(a)).

Then defining,

\[ \mathcal{N}_q = \{X_j\lambda^{ij} \mid X_{0j}\lambda^{0j} \in \exp^{-1}(N), \lambda^{ji} \in \mathbb{R}, I > 0 \}, \]

\( \exp(\mathcal{N}_q) \) is dense in \( T^q(G) \) (with \( \exp \) given by (2.5.8)). Using (2.5.69)

\[ \mathfrak{x}_p \circ \exp(\mathcal{N}_q) = \exp_0 \circ d\mathfrak{x}_p(\mathcal{N}_q) \]

\[ = \exp_0(\tilde{N}_0) \]

whence \( \exp_0(\tilde{N}_0) \) is dense in \( \mathcal{G}_0 \) and hence \( \text{Ad}^*(\exp_0(\tilde{N}_0))\beta_0 \) is dense in \( \Omega_0 \). Then \( \mathcal{U} \) is dense in \( \Omega_{[0,1]} \) and this completes the proof.

We now show that the maps \( \tilde{\theta}_M^{-1}(t) \) define a generalised Saletan contraction of \( \mathfrak{g} \):

**Lemma 5.1.9(a)**

Let \( \tilde{\theta}_M : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be given by the proof of (4.1.9) with respect to a fixed basis \( \{X_\alpha\} \) of \( \mathfrak{g} \). Then, letting

\[ \tilde{d}^q = \{ x^\alpha X_\alpha \mid x \in C^q_{\mathbb{R}^n}[0,1], \lim_{t \to 0} \tilde{\theta}_M^{-1}(t)x(t) \exists \}, \]

and letting \( \tilde{\mathcal{G}}_0 \) denote the Saletan contraction (1.5.1) generated by \( \tilde{\theta}_M \), we have

\[ \tilde{d}^q = d^q \quad \text{and} \quad \tilde{\mathcal{G}}_0 \text{ isomorphic to } \mathcal{G}_0. \]

**Proof:**

We first show that \( \tilde{d}^q = d^q \):

Now for \( X(t) = x^\alpha(t)X_\alpha, \lim_{t \to 0} \tilde{\theta}_M^{-1}(t)x(t) \exists \) if and only if \( \lim_{t \to 0} \tilde{\theta}_M^{-1}(t)X(t) \exists \).

Let \( X \in \tilde{d}^q \).
Now in the notation of the proof of (5.1.6),
\[ dX(Y^\alpha) = \lim_{t \to 0} d\Phi(t)Y^\alpha(t) \]
\[ = \lim_{t \to 0} d\Phi(t) \circ \tilde{\theta}(t)(X^\alpha) \]
by the definition of \( \tilde{\theta} \) in the proof of (5.1.6).

Then
\[ \lim_{t \to 0} d\Phi(t) \circ \tilde{\theta}(t)(Z), \text{ exists, } \forall Z \in g. \]

Now,
\[ d\Phi(t)(X(t)) = (d\Phi(t) \circ \tilde{\theta}(t)) \circ (\tilde{\theta}^{-1}(t)X(t)) \]
and therefore
\[ \lim_{t \to 0} d\Phi(t)(X(t)) \text{ exists and so } \overset{\sim}{g} \subset d^q. \]

Now let \( X \in d^q \). We have,
\[ \tilde{\theta}^{-1}(t)X(t) = \tilde{\theta}^{-1}(t) \circ d\Phi^{-1}(t) \circ d\Phi(t)X(t) \]
\[ = (d\Phi(t) \circ \tilde{\theta}(t))^{-1} \circ d\Phi(t)X(t) \]

Now without loss of generality, we may view \( g \) as a subspace of \( E \) (see (2.4.6)). Further, the inverse map is continuous on \( GL(E, R) \), so if \( d\Phi(t) \circ \tilde{\theta}(t) \) can be extended to an invertible map \( E \to E \), for \( t \in [0,1] \), then as
\[ \lim_{t \to 0} d\Phi(t) \circ \tilde{\theta}(t) \text{ exists,} \]

\[ \lim_{t \to 0}(\Phi(t) \circ \tilde{\theta}(t))^{-1} \text{ will exist.} \]

We now show that \( d\Phi(t) \circ \tilde{\theta}(t) \) can always be so extended:
Let \( \dim g = n, \dim E = N \), and let \( \xi \) be a map
\[ \xi : [0,1] \times \mathbb{R}^n \to \mathbb{R}^N \]
continuous in its first argument and linear in its second argument with \( \xi(t) \) injective for \( t \in [0,1] \). Representing \( \xi(t) \) as an \( N \times N \) matrix, \( \xi(t) \) has first \( n \) columns of rank
n. Given the matrix $\xi(0)$, choose $N - n$ column vectors $\eta_i^0$ such that the columns of $\xi(0)$ together with the vectors $\{\eta_i^0\}$, form a linearly independent set. Let $\{\eta_i(t)\}$ be $N - n$ continuous curves of column vectors on $[0, 1]$ such that $\eta_i(t) = \eta_i^0$. Form the matrix $\xi | \eta(t)$ consisting of the first $n$ columns being the columns of $\xi(t)$ and last $N - n$ columns being the column vectors $\{\eta_i(t)\}$. By definition, $\det(\xi | \eta(0)) \neq 0$ and by the continuity of $\det$ and $\xi | \eta$, $\det(\xi | \eta(t)) \neq 0$ for $t \in [0, \epsilon)$ for some $\epsilon > 0$.

We may take $\epsilon = 1$, again without loss of generality. Then $\xi | \eta(t)$ is in $GL(E, \mathbb{R})$ for $t \in [0, 1]$. Expressing $d\Phi(t) \circ \tilde{\theta}(t)$ in matrix form, this procedure establishes that $d\Phi(t) \circ \tilde{\theta}(t)$ can be extended as asserted.

Hence
\[
\lim_{t \to 0} \tilde{\theta}^{-1}(t)X(t) \text{ exists as } \lim_{t \to 0} (d\Phi(t) \circ \tilde{\theta}(t))^{-1} \text{ exists and }
\]
\[
\lim_{t \to 0} d\Phi(t)X(t) \text{ exists. Then } \tilde{d} = d^r.
\]

We now give the isomorphism of $\tilde{g}_0$ and $g_0$:

Let $(d\Phi \circ \tilde{\theta})^\circ = \lim_{t \to 0} d\Phi(t) \circ \tilde{\theta}(t)$. Now, for the usual contraction,

\[
[X_0, Y_0] = \lim_{t \to 0} d\Phi(t)[\tilde{\theta}(t)X, \tilde{\theta}(t)Y],
\]

where

\[
X_0 = \lim_{t \to 0} d\Phi(t) \circ \tilde{\theta}(t)X,
\]

\[
Y_0 = \lim_{t \to 0} d\Phi(t) \circ \tilde{\theta}(t)Y, \quad \forall X, Y \in g,
\]

while, for Saletan contraction (1.5.1),

\[
[X, Y]_0 = \lim_{t \to 0} \tilde{\theta}^{-1}(t)[\tilde{\theta}(t)X, \tilde{\theta}(t)Y].
\]

Therefore,

\[
[X_0, Y_0]_0 = [(d\Phi \circ \tilde{\theta})^\circ X, (d\Phi \circ \tilde{\theta})^\circ Y]_0
\]

\[
= \lim_{t \to 0} (d\Phi(t) \circ \tilde{\theta}(t)) \circ \tilde{\theta}^{-1}(t)[\tilde{\theta}(t)X, \tilde{\theta}(t)Y]
\]

\[
= (d\Phi \circ \tilde{\theta})^\circ [X, Y]_0, \text{ whence }
\]

259
$(d\Phi \circ \tilde{\theta})^\circ : \tilde{g}_0 \to g_0$ is the required homomorphism. $(d\Phi \circ \tilde{\theta})^\circ$ is clearly 1–1 and onto for contractions where $\dim g = \dim g_0$. This completes the proof.

The following lemma completes the proof of (5.1.9):

**Lemma 5.1.9(b)**

Let the map $d\mathcal{X}$ be given by $d\mathcal{X} : d\mathcal{g} \to \tilde{g}_0$, $d\mathcal{X} : X \mapsto \lim_{t \to 0} \tilde{\theta}(t)X(t)$, and the map $(d\Phi \circ \tilde{\theta})^\circ : \tilde{g}_0 \to g_0$ by the proof of (5.1.9(a)). Then the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{g}^q & \xrightarrow{\mathcal{d}\mathcal{X}} & \tilde{g}_0 \\
\downarrow I & & \downarrow (d\Phi \circ \tilde{\theta})^\circ \\
\mathcal{g}^q & \xrightarrow{d\mathcal{X}} & g_0
\end{array}
\]

**Proof:**

Now for $X \in \mathcal{g}^q$,

\[
d\mathcal{X}(X) = \lim_{t \to 0} d\Phi(t)X(t)
\]

\[
= \lim_{t \to 0} (d\Phi(t) \circ \tilde{\theta}(t)) \circ \tilde{\theta}^{-1}(t)X(t)
\]

\[
= (d\Phi \circ \tilde{\theta})^\circ \circ d\mathcal{X}(X).
\]

**Notation 5.1.9(c)**

From the proof of (5.1.6) we may conveniently regard $\Omega_{[0,1]}$ as a subset of $\mathcal{g}_0^*$. Let $\| \cdot \|$ be a fixed norm on $\mathcal{g}_0^*$.

Equipped with a chart which is almost global on $\Omega_{[0,1]}$, we give a proof of (5.1.1). In the following theorem, we assume that the universal character formula ((1.3.46), (1.3.47), (1.3.50)) applies:

---

260
Theorem 5.1.10

Suppose that conditions (i), (ii), (iii) and (iv) of Lemma (5.1.9) are satisfied by $\mathcal{G}$ and the contraction process. Let $(\mathcal{C}, \mathcal{U}, \lambda)$ be the chart of (5.1.9) and suppose further that $\Omega_{[0,1]}$ satisfies the condition,

$$\|C^{-1}(\lambda, t)\| \geq M_j|\lambda_j|, \ j = 1, \ldots, m$$  \hspace{1cm} (5.1.10(a))

for some constants $M_j > 0$, and $t \in (0, 1]$.

Then, for any $f \in C_0^\infty(\mathcal{G}_{[0,1]}(0))$,

$$\lim_{t \to 0} \chi_{\Omega_t}(f) = \chi_{\Omega_0}(f), \quad \text{as in (5.1.1)}.$$

Remarks (5.1.10(b))

In all examples of contraction, even if the $\Omega_t$ are compact for $t \in (0, 1]$, the $\Omega_t$ deform to an unbounded $\Omega_0$ as $t \to 0$. The unboundedness of $\Omega_0$ corresponds directly to the unboundedness of $\tilde{\Theta}_M^{-1}(0)$, as can be seen from the definition of the chart $(\mathcal{C}, \mathcal{U}, \lambda)$ in (5.1.9), and originates from the existence of a nilpotent ideal of $\mathcal{G}_0$ on which $\exp_0$ is unbounded. It is always possible, as we shall see in the example below, to choose the special subset $\{Y^\alpha\} \subset \mathcal{G}^q$ as in (5.1.6) so that a minimal number of the coordinates $\{\lambda^j\}$ are unbounded. What (5.1.10(a)) says is that with respect to these unbounded coordinates, the "divergence" of $\Omega_{[0,1]}$ happens at a certain minimal rate. Notwithstanding the restriction (5.1.10(a)), we here recover the corresponding result already obtained for the cases covered by Dooley and Gaudry (1986) and Dooley and Ricci (1985).

Proof of (5.1.10):

Now,

$$\chi_{\Omega_t}(f) = \int_{\Omega_t} \int_{\mathcal{G}} e^{i(\alpha_t, X_t)} f(X_t)(P_t(X_t))^{-1} dX_t d\mu_t(\alpha_t),$$  \hspace{1cm} (5.1.11)
from (1.3.46) and (1.3.47). We re-write the above integrand in coordinates given by the chart (5.1.9), denoted $(\mathcal{C}, \mathcal{U}, \lambda^i)$.

$
\mu_t
$ is the Kirillov measure on $\Omega_t$ given by (1.3.35),

$$
\mu_t = \frac{1}{(m^2)!} \omega \wedge \cdots \wedge \omega
$$

where $m$ is the dimension of the $\Omega_t$ and $\omega$ is the 2-form given also by (1.3.35).

Using the notation of Proposition (5.1.6), the expression for the 2-form $\omega$ on $\Omega_t$ at $\text{Ad}^*(g_t)\beta_t$, $g_t \in \mathcal{G}_t$, is, with respect to the chart $(\mathcal{C}, \mathcal{U}, \lambda^i)$,

$$
\omega_{\text{Ad}^*(g_t)\beta_t} \left( \frac{\partial}{\partial \lambda^i}, \frac{\partial}{\partial \lambda^j} \right) = \langle \text{Ad}^*(g_t)\beta_t, [d\Phi(t)Y_i(t), d\Phi(t)Y_j(t)] \rangle
$$

where $g_t = \Phi(g, t)$.

$$
= \langle \text{Ad}^*(g)\beta(t), [Y_i(t), Y_j(t)] \rangle,
$$

where $g_t = \Phi(g, t)$.

$$
= \langle \text{Ad}^*(g)\beta(t), \theta(t)(Y_i^0), \theta(t)(Y_j^0) \rangle
$$

where $\theta(t)$ is given by (5.1.7), $Y_i(t)$ by the proof of (4.1.6) with $Y_i^0 = d\Phi(t)$.

Observe that the expression $\theta^{-1}(t)[\theta(t)(Y_i^0), \theta(t)(Y_j^0)]$ defines the generalised Saletan contraction (see Lemmas (5.1.9(a)) and (5.1.9(b))) associated to $d\Phi(t)$, and

$$
\lim_{t \to 0} \theta^{-1}(t)[\theta(t)(Y_i^0), \theta(t)(Y_j^0)] = [Y_i^0, Y_j^0]_0
$$

while

$$
\lim_{t \to 0} \theta^*(t)\text{Ad}^*(g(t))\beta(t) = \text{Ad}^*(g_0)\beta_0
$$

for $g \in \mathcal{D}^g$. Write $\eta_{ij}(t) = \theta^{-1}(t)[\theta(t)(Y_i^0), \theta(t)(Y_j^0)]$, and $\eta_{ij}(0) = \lim_{t \to 0} \eta_{ij}(t)$.

The expression for the Kirillov measure on $\Omega_t$ in coordinate form is,

$$
\mu_t(\lambda) = \frac{1}{(m^2)!} \epsilon^{i_1 \cdots i_m} \langle \theta^*(t)\text{Ad}^*(\exp(\lambda^i Y_i^i(t)))\beta(t), \eta_{i_1 i_2}(t) \rangle \times \cdots
$$
\[ \times (\theta^*(t) \text{Ad}^*(\exp(\lambda^i Y^i(t)))\beta(t), \eta_{i_{m-1}, i_m}(t))d\lambda^1 \cdots d\lambda^m \quad \text{for} \quad t \in (0, 1] \]

and

\[ \mu_0(\lambda^i) = \lim_{t \to 0} \mu_t(\lambda^i). \]

We give here an inequality for \( \mu_t \), to be used in a later application of the Lebesgue dominated convergence theorem.

With respect to the norm (5.1.9(e)) on \( g_0^* \) and a norm \( \| \cdot \|_0 \) on \( g_0 \), we have

\[ |\mu_t(\lambda^i)| \leq K \| \theta^*(t) \text{Ad}^*(\exp(\lambda^i Y^i(t)))\beta(t)\|_2 \times \]

\[ \times \varepsilon^{i_1 \cdots i_m} \| \eta_{i_1 i_2}(t)\|_0 \cdots \| \eta_{i_{m-1}, i_m}(t)\|_0 \ d\lambda^1 \cdots d\lambda^m \]

for \( t \in (0, 1] \) and some positive constant \( K \).

In order to take the limit \( t \to 0 \) in (5.1.11), we re-write it in coordinate form:

We make the following convenient abbreviations:

\[ \beta_\theta(\lambda^i, t) = \theta^*(t) \text{Ad}^*(\exp(\lambda^i Y^i(t)))\beta(t) \quad \text{for} \quad t \in (0, 1], \]

and

\[ \beta_\theta(\lambda^i, 0) = \lim_{t \to 0} \beta_\theta(\lambda^i, t) \]

\[ \eta(t) = \varepsilon^{i_1 \cdots i_m} \| \eta_{i_1 i_2}(t)\|_0 \cdots \| \eta_{i_{m-1}, i_m}(t)\|_0. \]

The inner integral in (5.1.11) is a \( C^\infty \) function on \( \Omega_{[0,1]} \), which we abbreviate using (5.1.4(a)):

\[ F(\lambda^i, t) = \int_{\mathbb{R}^n} e^{i\beta_\theta(\lambda^i, t), \zeta} f(d\Phi(t)\theta(t)\zeta)(P_d(d\Phi(t)\theta(t)\zeta))^{-1} \left| \frac{\partial d\Phi(t)\theta(t)\zeta}{\partial \zeta} \right| d\zeta \]

\[ F(\lambda^i, 0) = \lim_{t \to 0} F(\lambda^i, t). \]

Then,

\[ \chi_{\pi_1}(f) = \]

\[ \frac{1}{(\frac{m}{2})!} \int_{\delta_{M_S}^{-1}(t)N^2} F(\lambda^i, t)e^{i_1 \cdots i_m} \langle \beta_\theta(\lambda^i, t), \eta_{i_1, i_2}(t) \rangle \cdots \langle \beta_\theta(\lambda^i, t), \eta_{i_{m-1}, i_m}(t) \rangle d\lambda^1 \cdots d\lambda^m \]

263
\[
\frac{1}{(m^2)!} \int_{\mathbb{R}^m} K_{\tilde{\theta}_M^{-1}(t)} N_2(\lambda) F(\lambda, t) e^{i\lambda_1 \cdots i\lambda_m} \langle \beta_\theta(\lambda, t), \eta_{i_{m-1} i_m}(t) \rangle \times \cdots
\]

\[
\cdots \times \langle \beta_\theta(\lambda, t), \eta_{i_{m-1} i_m}(t) \rangle d\lambda_1 \cdots d\lambda_m
\] (5.1.13)

where \( K_{\tilde{\theta}_M^{-1}(t)} N_2(\lambda) \) is the characteristic set function of \( \tilde{\theta}_M^{-1}(t) N_2 \).

Observe that if we take the limit inside the integral of (5.1.13), the resulting integral over \( \mathbb{R}^m \) will give \( \chi_{\pi_0}(f) \).

We need an application of the Lebesgue dominated convergence theorem, in order to be able to take the limit, \( \lim_{t \to 0} \chi_{\pi_t}(f) \) inside the integral (5.1.13) thus establishing that \( \lim_{t \to 0} \chi_{\pi_t}(f) = \chi_{\pi_0}(f) \).

To find a function which dominates the integrand, we have

\[
|F(\lambda, t) e^{i\lambda_1 \cdots i\lambda_m} \langle \beta_\theta(\lambda, t), \eta_{i_{m-1} i_m}(t) \rangle| \\
\leq K |F(\lambda, t)| \cdot \|\beta_\theta(\lambda, t)\|^{\frac{m}{2}} \cdot |\eta(t)| \\
\leq K |F(\lambda, t)| \cdot \|\beta_\theta(\lambda, t)\|^{\frac{m}{2}} \cdot (\max_{t \in [0,1]} |\eta(t)|)
\]

By the Paley-Wiener Theorem (1.1.89), applied to (5.1.12), for any \( N = 0, 1, 2, 3, \ldots \), there exists \( K_N \geq 0 \) such that

\[
|F(\lambda, t)| \leq K_N (1 + \|\beta_\theta(\lambda, t)\|)^{-N},
\]

as the integrand in (5.1.12) is a continuous function of \( t \) on the compact interval \([0,1]\), and \( f \) is of compact support. Then,

\[
|F(\lambda, t)| \leq K_N^{1+\frac{m}{2}} (1 + \|\beta_\theta(\lambda, t)\|)^{-(N+\frac{m}{2})} \\
\leq K_N^{1+\frac{m}{2}} \|\beta_\theta(\lambda, t)\|^{-\frac{m}{2}} \cdot (1 + \|\beta_\theta(\lambda, t)\|)^{-N}
\]

Therefore

\[
|F(\lambda, t) e^{i\lambda_1 \cdots i\lambda_m} \langle \beta_\theta(\lambda, t), \eta_{i_{m-1} i_m}(t) \rangle| \\
\leq L (1 + \|\beta_\theta(\lambda, t)\|)^{-N},
\]

264
for some positive constant \( L \).

By the definition of \( C \), \( C^{-1}(\lambda^i, t) = \beta_\theta(\lambda^i, t) \), and applying the assumption of the theorem,

\[
(1 + \|\beta_\theta(\lambda^i, t)\|)^{-N} \leq (1 + \sum_j M_j|\lambda^j|)^{-N}.
\]

Then the integrand of (5.1.13) is dominated by the function \( L(1 + \sum_j M_j|\lambda^j|)^{-N} \) (as the integral is a function globally defined on \( \mathbb{R}^m \), but zero on the complement of \( \tilde{\vartheta}_M^{-1}(t)\mathcal{N}_2 \) for \( t \in [0, 1] \)) which is integrable over \( \mathbb{R}^n \). This completes the proof.\( \blacksquare \)

**Example 5.1.14**

To demonstrate how the special condition (5.1.10(a)) is fulfilled in practice, we consider the contraction of the generalised character formula for the case \( SO(3) \to M(2) \). In this example, we identify the elements of \( \mathfrak{so}(3)^* \) with vectors \( v \times w, v, w \in \mathbb{R}^3 \), acting on \( \mathfrak{so}(3) \)

\[
(v \times w, X) = (v, Xw)
\]

where \( X \in SO(3) \) is represented as a 3 \( \times \) 3 matrix and \((\cdot, \cdot)\) is the usual inner product on \( \mathbb{R}^3 \).

The co-adjoint action of \( SO(3) \) is given by

\[
\text{Ad}^*(S)(v \times w) = (S^{-1}v) \times (S^{-1}w)
\]

and the contraction of \( SO(3) \) to \( M(2) \) is given by Example (4.2.1). In order to give \( \Omega_{[0,1]} \) explicitly in coordinate form (5.1.9), we need to choose the \( \{Y^\alpha\} \) and hence generate \( \theta(t) \):

Take

\[
Y^1(t) = w_1, \ Y^2(t) = w_2t, \ Y^3(t) = w_3t
\]

where \( \{w_i\} \) is the basis of \( \mathfrak{so}(3) \) in (4.2.7).
Let \( \{ e_i \} \) be the basis of \( M(2) \) given by (4.2.1). With respect to these bases,

\[
\theta(t) = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}, \quad \theta^*(t) = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}.
\]

We choose the sequence of functionals

\[
\beta(t) = \left(0, \frac{r}{t}, 0\right) = \frac{r}{t}(0, 0, 1) \times (1, 0, 0).
\]

The curve of vectors \( Y(t) = rtw_2 \) satisfies (4.1.14) and so \( \beta \in d_q^* \).

Then

\[
\Omega_t = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} SO(3) \left(0, \frac{r}{t}, 0\right), \quad t \in (0, 1].
\]

As \( t \) varies from 1 to 0, \( \Omega_t \) deforms from a sphere of radius \( r \), to "cigars" which become longer as \( t \to 0 \), limiting to a cylinder of radius \( r \) with axis the \( e_1 \)-axis.

The stabiliser of \( (0, r, 0) \) is the group of rotations in the \( e_1 e_3 \)-plane.

Then the chart of (5.1.9) is:

\[
\left(1 \quad t \quad 0\right) e^{(\lambda_1 w_3 + t\lambda_3 w_1)}(0, \frac{r}{t}, 0) \mapsto (\lambda_1, \lambda_3, t)
\]

\[
\lim_{t \to 0} \left(1 \quad t \quad 0\right) e^{(\lambda_1 w_3 + t\lambda_3 w_1)}(0, \frac{r}{t}, 0) \mapsto (\lambda_1, \lambda_3, 0)
\]

for \( (\lambda_1, \lambda_3, t) \) in the domain given by

\[
\sqrt{\lambda_1^2 + t^2 \lambda_3^2} < \pi, \quad \lambda_1 \neq 0, \quad t \in [0, 1].
\]

Here \( e^{(\lambda_1 w_3 + t\lambda_3 w_1)} \) is the exponential power series of the matrix \( (\lambda_1 w_3 + t\lambda_3 w_1) \).

This power series can be computed explicitly, so that the chart of (5.1.9) becomes,

\[
C : \mathcal{U} \to \mathbb{R}^{2+1}
\]
\( C : r \left( \begin{array}{c}
\frac{\lambda_3 \sin \sqrt{\lambda_1^2 + t^2 \lambda_3^2}}{\sqrt{\lambda_1^2 + t^2 \lambda_3^2}} \\
\frac{\cos \sqrt{\lambda_1^2 + t^2 \lambda_3^2}}{\sqrt{\lambda_1^2 + t^2 \lambda_3^2}} \\
\frac{-\lambda_1}{\sqrt{\lambda_1^2 + t^2 \lambda_3^2}} \cdot \sin \frac{\sqrt{\lambda_1^2 + t^2 \lambda_3^2}}{t}
\end{array} \right) \mapsto (\lambda_1, \lambda_3, t) \)

for \( \sqrt{\lambda_1^2 + t^2 \lambda_3^2} < \pi, \lambda_1 \neq 0, t \in [0,1] \).

Observe that as \( t \) varies from 1 to 0, the set \( C(U \cap \Omega_t) \) deforms from an open disk of radius \( \pi \), through elliptical disks of increasing major axis lengths on the \( \lambda_3 \)-axis to a “strip”, \(-\pi < \lambda_1 < \pi\), with the \( \lambda_3 \)-axis removed from all these figures.

In order to prove an inequality of the form (5.1.10(a)), it is more convenient to take two overlapping charts on \( \Omega_{[0,1]} \) and to compute the limit in (5.1.13) by using a \( C^\infty \) partition of unity on \( \Omega_{[0,1]} \).

The first chart is given by,

\[
C_1 : U_1 \rightarrow \mathbb{R}^{2+1}
\]

\[
C_1 : r \left( \begin{array}{c}
\frac{\lambda_3 \sin \sqrt{\lambda_1^2 + t^2 \lambda_3^2}}{\sqrt{\lambda_1^2 + t^2 \lambda_3^2}} \\
\frac{\cos \sqrt{\lambda_1^2 + t^2 \lambda_3^2}}{\sqrt{\lambda_1^2 + t^2 \lambda_3^2}} \\
\frac{-\lambda_1}{\sqrt{\lambda_1^2 + t^2 \lambda_3^2}} \cdot \sin \frac{\sqrt{\lambda_1^2 + t^2 \lambda_3^2}}{t}
\end{array} \right) \mapsto (\lambda_1, \lambda_3, t) \]

for \( t \in [0,1], \epsilon^2 < \lambda_1^2 + t^2 \lambda_3^2 < (\pi - \epsilon)^2 \),

for some \( \epsilon > 0 \). We now “rotate this chart by \( \frac{\pi}{2} \) about the \( e_1 \)-axis”, giving:

\[
C_2 : U_2 \rightarrow \mathbb{R}^{2+1}
\]

\[
C_1 : r \left( \begin{array}{c}
\frac{\lambda_3 \sin \sqrt{\lambda_1^2 + t^2 \lambda_3^2}}{\sqrt{\lambda_1^2 + t^2 \lambda_3^2}} \\
\frac{-\lambda_1 \sin \sqrt{\lambda_1^2 + t^2 \lambda_3^2}}{\sqrt{\lambda_1^2 + t^2 \lambda_3^2}} \\
\frac{-\cos \sqrt{\lambda_1^2 + t^2 \lambda_3^2}}{\sqrt{\lambda_1^2 + t^2 \lambda_3^2}}
\end{array} \right) \mapsto (\lambda_1, \lambda_3, t) \]

for \( t \in [0,1], \epsilon^2 < \lambda_1^2 + t^2 \lambda_3^2 < (\pi - \epsilon)^2 \).
By choosing $\epsilon$ small enough, these two charts over $\Omega_{[0,1]}$ completely.

Now

$$\| C_i^{-1}(\lambda, t) \| = \left( \frac{\lambda_1^2 + \lambda_3^2}{\lambda_1^2 + t^2 \lambda_3^2} \sin^2 \sqrt{\lambda_1^2 + t^2 \lambda_3^2} + \cos^2 \sqrt{\lambda_1^2 + t^2 \lambda_3^2} \right)^{\frac{1}{2}}$$

$$\geq \left( \frac{\lambda_1^2 + \lambda_3^2}{\lambda_1^2 + t^2 \lambda_3^2} \sin^2 \sqrt{\lambda_1^2 + t^2 \lambda_3^2} \right)^{\frac{1}{2}}$$

$$\geq \frac{\sin \epsilon}{\epsilon} (\lambda_1^2 + \lambda_3^2)^{\frac{1}{2}},$$

whence

$$\| C_i^{-1}(\lambda, t) \| \geq \frac{\sin \epsilon}{\epsilon} | \lambda_j |, \ j = 1, 3 \quad (5.1.14(a))$$

To construct a $C^\infty$ partition of unity, let $\eta$ be a positive real number so small that the $C^i$ still cover $\Omega_{[0,1]}$ with $\epsilon$ replaced by $\epsilon + \eta$. Let $f$ be a $C^\infty$ function on the region

$$\{(\lambda_1, \lambda_3, t) \mid \epsilon^2 < \lambda_1^2 + t^2 \lambda_3^2 < (\pi - \epsilon)^2, \ t \in [0, 1] \}$$

which is non-zero on the region

$$\{(\lambda_1, \lambda_3, t) \mid (\epsilon + \eta)^2 < \lambda_1^2 + t^2 \lambda_3^2 < (\pi - \epsilon - \eta)^2, \ t \in [0, 1] \},$$

and zero on the region

$$\left\{ (\lambda_1, \lambda_3, t) \mid \epsilon^2 < \lambda_1^2 + t^2 \lambda_3^2 < (\epsilon + \eta_1)^2, \right.$$

$$(\pi - \epsilon - \eta_1)^2 < \lambda_1^2 + t^2 \lambda_3^2 < (\pi - \epsilon)^2, \ t \in (0, 1) \},$$

with $0 < \eta_1 < \eta$ and such that $f(\lambda_1, \lambda_3, 0)$ approaches zero as $\lambda_3 \to \pm \infty$.

Define the $C^\infty$ functions $f_i$ on $\Omega_{[0,1]}$ by

$$f_i = f \circ C_i \text{ on } U_i$$

$$f_i = 0 \text{ elsewhere.}$$
A $C^\infty$ partition of unity is given by the functions,

$$F_i = \frac{f_i}{f_1 + f_2}.$$ 

The following lemma is useful at this point:

**Lemma 5.1.15**

Let $\{W_\alpha\}$ be a cover of $[0,1]$, with each $W_\alpha$ homeomorphic to a neighbourhood of zero in the half space

$$\mathbb{R}^{(m+1)^+} = \{(x_1, \ldots, x_{m+1}) \mid x_{m+1} > 0\}$$

and let $\{f_\alpha\}$ be a corresponding $C^\infty$ partition of unity. Then $\{f_\alpha|_{\Omega_t}\}$ is a $C^\infty$ partition of unity for the cover $\{W_\alpha \cap \Omega_t\}$ of $\Omega_t$.

**Proof:**

Let $A'_t = \{\alpha \mid W_\alpha \cap \Omega_t \neq \emptyset\}$.

We claim that $\{f_\alpha|_{\Omega_t} \mid \alpha \in A'_t\}$ is a $C^\infty$ partition of unity for $\Omega_t$.

(i) Each $f_\alpha$ is $C^\infty$ on $\Omega_t$ and is zero outside of $W_\alpha \cap \Omega_t$.

(ii) The requirement $0 \leq f_\alpha|_{\Omega_t} \leq 1$ on $\Omega_t$ is immediately satisfied.

(iii) Now, for $\gamma_t \in \Omega_t$,

$$1 = \sum_\alpha f_\alpha(\gamma_t) = \sum_\alpha f_\alpha|_{\Omega_t}(\gamma_t)$$

$$= \sum_{\alpha \in A'_t} f_\alpha|_{\Omega_t}(\gamma_t),$$

thus establishing the lemma.

Re-writing (5.1.13) as

$$\chi(\pi_t(f) = \int_{\mathbb{R}^2} \mathcal{K}_{\tilde{\theta}_M^{-1}(t),\mathcal{N}_2}(\lambda) \mathcal{F}(\lambda^1, \lambda^2, t) d\lambda^1 d\lambda^2,$$
we have

\[ \chi_{\pi_t}(f) = \sum_{j=1}^{2} \int_{\mathbb{R}^2} F_i(\lambda^1, \lambda^2, t) K_j(\mu_j \cap \Omega_t)(\lambda^1, \lambda^2, t) \mathcal{F}(\lambda^1, \lambda^2, t) d\lambda^1 d\lambda^2. \]

Computing the limit inside the above integrals, we obtain \( \chi_{\pi_0}(f) \). We may then apply the Lebesgue dominated convergence theorem using (5.1.14(a)). The functions \( F_i \) do not impede our finding a dominating function, as they satisfy \( F_i(\lambda, t) \in [0,1] \).

Then for this particular contraction of \( SO(3) \) to \( M(2) \) with \( \beta(t) = (0, \frac{t}{t}, 0) \) we have shown that \( \lim_{t \to 0} \chi_{\pi_t}(f) = \chi_{\pi_0}(f) \).
§5.2 Contractions of Infinitesimal Characters

In proposition (1.3.63), we saw that for any Lie group $G$, if it had a unitary irreducible representation $\mathcal{R}$ (constructed by the method of orbits) with generalised character $\chi_Z$ (1.3.46); then the infinitesimal character (1.3.58) could be expressed as a homomorphism from the polynomials on $g^*$ invariant under the co-adjoint action of $G$ (denoted $\mathcal{P}^n_T(g^*)$), to the complex numbers. Further, this expression for the infinitesimal character had an explicit form: Let $\Omega$ be the co-adjoint orbit associated to $\mathcal{R}$ by (1.3.38). Then the infinitesimal character $\lambda_\Omega$ is given by:

$$\lambda_\Omega : \mathcal{P}^n_T(g^*) \rightarrow \mathbb{C}$$

$$\lambda_\Omega : p \mapsto p(\beta), \text{ for any } \beta \in \Omega.$$  \hspace{1cm} (5.2.1)

This is a particularly simple and convenient form of the infinitesimal character, and our task in this section is to develop the theory to contract a homomorphism of the type (5.2.1) by a well-defined process, and to show that this process gives all possible maps of the style of (5.2.1) for the contracted Lie group $G_0$ with Lie algebra $g_0$.

The point is that it is not currently known whether the infinitesimal character (1.3.58) can be given in the form (1.3.63) for unitary representations of arbitrary Lie groups. In particular it is certainly not known whether (1.3.63) holds for general contractions of semisimple Lie groups because it is not yet known whether a generalised character $\chi_Z$ exists in the form (1.3.46) for unitary representations $\mathcal{R}$ of such Lie groups, and this condition is a pre-requisite of Proposition (1.3.63). We will show in (5.2.9) and (5.2.10), that the formula (5.2.1) can be meaningfully contracted to a formula of the same type (and that all possible maps $\lambda_{\Omega_0} : \mathcal{P}^n_T(g_0^*) \rightarrow \mathbb{C}$ are obtained), and this will lend weight to the conjecture that (5.2.1) is universal for arbitrary unitary representations of arbitrary Lie groups.

This result leads to a theorem (Theorem (5.2.12)) limiting the infinitesimal characters $\lambda_{\mathcal{R}_t}$, $t \neq 0$ corresponding to unitary representations $\mathcal{R}_t : G_t \rightarrow \mathcal{U}(H_t)$ (see
The conclusions to be drawn about contracting polynomials on $\mathfrak{g}^*$ are analogous to the results of Example (4.1.5) for the contraction of linear functionals on $\mathfrak{g}$, as expected:

Let $p$ be a polynomial on $\mathfrak{g}^*$, $p : \mathfrak{g}^* \to \mathbb{C}$, and let $\beta \in \mathcal{D}_{\mathfrak{g}}^\ast$. By (4.1.14) and (4.1.16),

$$\lim_{t \to 0} (d\Phi^{-1})^* \beta(t) = d\mathfrak{x}^*(\beta).$$

Not surprisingly (in view of (4.1.5)), if one tries to compute the limit,

$$\lim_{t \to 0} p \circ d\Phi^*((d\Phi^{-1})^* \beta(t))$$

to obtain a value $p_0(d\mathfrak{x}^*(\beta))$ where $p_0$ is a polynomial on $\mathfrak{g}^*_0$, this limit won’t in general exist, since

$$\lim_{t \to 0} p \circ d\Phi^*((d\Phi^{-1})^* \beta(t)) = \lim_{t \to 0} p(\beta(t)),$$

and $\beta$ has the form $\beta(t) = \frac{\gamma(t)}{t^q}, \gamma \in C^\infty_{\mathfrak{g}}[0,1]$.

In keeping then, with the philosophy of Chapter 2 and the results of §4.1, the by now standard way of circumventing this difficulty is to construct a scenario...
in which the polynomials on \( g^* \) will be contracted by defining a mapping whose
domain will be a space of suitably differentiable curves of polynomials on \( g^* \) and
whose image will be the polynomials on \( g_0^* \).

In order to define the domain of this map we need some notation:

**Convention 5.2.3**

By (2.3.2), we have \( d\Phi(t) = \left( \frac{d\phi(t)}{t^q}, t \right) \) where \( d\phi \) is a map \( d\phi : [0, 1] \times g \rightarrow E \).

Following Convention (4.1.11) (which implies that \( g \) is a subspace of \( E \)), this may
be extended to a map \( d\phi : [0, 1] \times E \rightarrow E \), linear on \( E \). Similarly, if \( p \) is a polynomial
on \( g^* \), \( p \) has an extension to a polynomial \( p : E \rightarrow C \).

We give the domain of the contraction mapping for the polynomials on \( g^* \):

**Definition 5.2.4**

Let \( P^n(g^*) \) denote the linear space of all polynomials \( p : g^* \rightarrow C \) for any Lie
algebra \( g \), and let Convention (5.2.3) apply to \( P^n(g^*) \) and \( d\Phi \).

Let \( g^*_{p^*} \) denote the subspace of \( C^q_{P^n(g^*)}[0, 1], \)

\[
g^*_{p^*} = \{ p \mid p \in C^q_{P^n(g^*)}[0, 1], \lim_{t \to 0} p(t) \circ d\Phi^*(t)X_E \text{ exists for all } X_E \in E \}.\]

We now define the contraction mapping:

**Definition 5.2.5**

Let the **contraction mapping** for polynomials be denoted by the linear map
\( dX_{p^*} : g^*_{p^*} \rightarrow P^n(g_0^*) \), given by,

\[
dX_{p^*}(p)(dX^*(\beta)) = \lim_{t \to 0} p(t) \circ d\Phi^*(t)((d\Phi^{-1})^*\beta(t)),
\]

\[
= \lim_{t \to 0} p(t)(\beta(t)), \quad \forall \beta \in g^*_{p^*}.\]

(Remark: To see that the range of \( dX_{p^*} \) is contained in \( P^n(g_0^*) \), note that as
\( p \in g^*_{p^*} \) and \( \beta \in g^*_q \), the above equation defines a function on \( g_0^* \). By the form of
the limit; \( \lim_{t \to 0} p(t)(\beta(t)) = \lim_{t \to 0} p(t)(\frac{2(t)}{t^2}) \); \( d\mathcal{X}_{p^*}(p) \) will be a polynomial on \( \mathcal{G}_0^* \), and the image of \( d\mathcal{X}_{p^*} \) is contained in \( \mathcal{P}^n(\mathcal{G}_0^*) \).

It is important that all polynomials on \( \mathcal{G}_0^* \) can be obtained by the contraction process in order that the infinitesimal character for \( \mathcal{G}_0 \) may be defined on this domain. With a very mild condition on the map \( d\Phi \), this is always the case:

**Proposition 5.2.6**

Suppose that the map \( d\Phi^{-1} \) is extendable to a \( C^q \) map, \( d\Phi^{-1} : [0, 1] \times E \to E \), linear on \( E \). Then the contraction mapping \( d\mathcal{X}_{p^*} \) is onto its image \( \mathcal{P}^n(\mathcal{G}_0^*) \).

**Remark 5.2.7**

If \( d\Phi \) is given by \( d\Phi(t) = (U^{-1}(t), t) \) where \( U \) is a \( C^\infty \) map \( U : [0, 1] \times E \to E \) linear on \( E \) with \( U(0) \) singular, then \( d\Phi^{-1} \) will be extendable as above, by (2.6.7). A contraction with respect to \( d\Phi \) in this form is clearly still very general, encompassing Saletan contractions (2.4.6), for example.

**Proof of (5.2.6):**

Let \( p_0 \) be any polynomial on \( \mathcal{G}_0^* \), and let \( p \) be its extension to a polynomial on all of \( E \). As \( d\Phi^{-1} \) is extendable to a \( C^q \) map \( d\Phi^{-1} : [0, 1] \times E \to E \), let \( \tilde{p}(t) \) denote the curve of polynomials \( \tilde{p}(t) = p \circ (d\Phi^{-1})^*(t) \). It is an element of \( C^q_{\mathcal{P}^n(\mathcal{G}_0^*)}[0, 1] \) and \( \lim_{t \to 0} \tilde{p}(t) \circ d\Phi^*(t)X_E \) clearly exists and hence \( \tilde{p} \in \mathcal{G}_0^* \). Computing \( d\mathcal{X}_{p^*}(\tilde{p}) \); let \( \beta \in d\mathcal{X}_{p^*} \), then

\[
d\mathcal{X}_{p^*}(\tilde{p}) = \lim_{t \to 0} \tilde{p}(t)(\beta(t))
\]

\[
= \lim_{t \to 0} p \circ (d\Phi^{-1})^*(t)(\beta(t))
\]

\[
= p(\lim_{t \to 0}(d\Phi^{-1})^*(t)(\beta(t))
\]

\[
= p_0(d\mathcal{X}^*(\beta))
\]

by (4.1.14) and (4.1.16). Hence the map \( d\mathcal{X}_{p^*}(\tilde{p}) \) is onto.
We now show how all invariant polynomials on $g_0^*$ may be obtained by the contraction process:

**Contracting Invariant Polynomials**

We will show that the invariant polynomials on $g_0^*$ are the image by the contraction mapping of a subspace of $d_p^q$. We first define this domain:

**Definition 5.2.8**

Let $d_p^q_{\mathcal{I}}$ denote the linear subspace of $d_p^q$ given by

$$
d_p^q_{\mathcal{I}} = \{ p | p \in d_p^q, \lim_{t \to 0} p(t)(\text{Ad}^*g(t)\beta(t)) = \lim_{t \to 0} p(t)(\beta(t)), \forall g \in D^q, \forall \beta \in g_0^* \}.
$$

**Lemma 5.2.9**

Suppose that the map $d\Phi^{-1}$ is extendable to a $C^q$ map $d\Phi^{-1} : [0,1] \times E \to E$, linear on $E$. Then in the notation of (1.3.62),

$$
d\mathcal{X}_p\cdot(d_p^q_{\mathcal{I}}) = P_{\mathcal{I}}^p(g_0^*).
$$
Proof:

Let \( p_0 \) be an invariant polynomial on \( g_\Phi^* \), and let \( \tilde{p} \in d_{P^*}^g \) be as given in the proof (5.2.6). Then, for any \( g \in D^q, \beta \in d_{P^*}^g \),

\[
\lim_{t \to 0} \tilde{p}(t)(\text{Ad}^* g(t) \beta(t)) = \lim_{t \to 0} p \circ (d\Phi^{-1})^*(t)(\text{Ad}^* g(t) \beta(t))
\]

\[
= \lim_{t \to 0} p(\text{Ad}^* \Phi(g(t))(d\Phi^{-1})^* \beta(t))
\]

\[
= p(\lim_{t \to 0} \text{Ad}^* \Phi(g(t))(d\Phi^{-1})^* \beta(t))
\]

\[
= p(\text{Ad}^* \mathfrak{X}(g)d\mathfrak{X}^*(\beta)) \text{ by } (4.1.20),
\]

\[
= p_0(\text{Ad}^* \mathfrak{X}(g)d\mathfrak{X}^*(\beta))
\]

\[
= p_0(d\mathfrak{X}^*(\beta))
\]

\[
= \lim_{t \to 0} p((d\Phi^{-1})^*(t)(\beta(t)))
\]

\[
= \lim_{t \to 0} \tilde{p}(t)(\beta(t)).
\]

Then \( \tilde{p} \in d_{P^*}^g \) by (5.2.8), proving the lemma.

Contrasting Infinitesimal Characters

We first give a proposition relating homomorphisms,

\[
\lambda_{\Omega_t} : \mathcal{P}^n_T(d\Phi(g,t)^*) \to C
\]
to homomorphisms, \( \lambda_{\Omega_0} : \mathcal{P}^n_T(g_0^*) \to C \), where the form of the homomorphism is given by (1.3.63). In the following proposition, the domain of \( \lambda_{\Omega_t} \) for \( t \neq 0 \) has been expanded to the algebra \( \mathcal{P}^n(d\Phi(g_,t)^*) \):

Proposition 5.2.10

Let \( \beta \in d_{P^*}^g \), and let \( \Omega_t \) be the co-adjoint orbit of \( \Phi(g,t) \) through \((d\Phi^{-1})^* \beta(t))\), with \( \Omega_0 \) the co-adjoint orbit of \( G_0 \) through \( d\mathfrak{X}^*(\beta) \). Suppose that the map \( d\Phi^{-1} \) is extendable to a \( C^1 \) map \( d\Phi^{-1} : [0,1] \times E \to E \), linear on \( E \). Then, in the notation of (1.3.63),

\[
\lambda_{\Omega_0}(d\mathfrak{X}^*(p)) = \lim_{t \to 0} \lambda_{\Omega_t}(p(t) \circ d\Phi^*(t)), \forall p \in d_{P^*}^g.
\]

276
Proof:

Let \( p \in \mathcal{D}_p^2 \), then

\[
\lambda_{\Omega_t}(p(t) \circ d\Phi^*(t)) = p(t) \circ d\Phi^*(t)((d\Phi^{-1})^*\beta(t)).
\]

By (5.2.4) and (5.2.8),

\[
\lim_{t \to 0} \lambda_{\Omega_t}(p(t) \circ d\Phi^*(t)) = \lim_{t \to 0} p(t) \circ d\Phi^*(t)((d\Phi^{-1})^*\beta(t))
\]

\[
= d\mathcal{X}_{p^*}(p)(d\mathcal{X}^*(\beta))
\]

(on using (4.1.14) and (4.1.16) also)

\[
= \lambda_{\Omega_0}(d\mathcal{X}_{p^*}(p)),
\]

establishing the proposition. ■

In Proposition (5.2.10) the functions \( \lambda_{\Omega_t} \) for \( t \neq 0 \) have been extended to the algebra \( \mathcal{P}^n(d\Phi(g, t)^*) \), however they only coincide with the infinitesimal characters \( \lambda_{\mathcal{R}_t} \) of (1.3.58) (for representations \( \mathcal{R}_t : \mathcal{G}_t \to \mathcal{U}(\mathcal{H}_t) \) (4.3.6) corresponding to orbits \( \Omega_t \) as in (1.3.38)) when their domain is restricted to \( \mathcal{P}_T^n(d\Phi(g, t)^*) \).

Consequently, in order to give a theorem showing \( \lambda_{\mathcal{R}_0} \) as a limit of the \( \lambda_{\mathcal{R}_t} \) in a similar sense to (5.2.10), we need to show that \( C^q \) curves on \( \mathcal{P}_T^n(g^*) \) contract by (5.2.5) to elements of \( \mathcal{P}_T^n(g^*_0) \):

**Proposition 5.2.11**

\[
d\mathcal{X}_{p^*}(d\mathcal{D}_p^q \cap C^q_{\mathcal{P}_T^0(g^*)}[0, 1]) \subset \mathcal{P}_T^n(g^*_0).
\]
Proof:

Let \( \beta \in \mathcal{P}^* \), and \( p \in \mathcal{P} \cap C^*_{\mathcal{P}}(\mathcal{P}^*)[0,1] \). Then,

\[
\lim_{t \to 0} p \circ d\Phi^*(Ad^*\Phi(g(t))(d\Phi^{-1})^*\beta(t)) = \lim_{t \to 0} p \circ d\Phi^*((d\Phi^{-1})^*(Ad^*g(t)\beta(t)))
= \lim_{t \to 0} p(Ad^*g(t)\beta(t))
= \lim_{t \to 0} p(\beta(t))
= dX_p(p)(dX^*(\beta)).
\]

If we can now show that

\[
\lim_{t \to 0} p \circ d\Phi^*(Ad^*\Phi(g(t))(d\Phi^{-1})^*\beta(t)) = dX_p(p)(Ad^*\Phi(g)\beta(p))
\]

then \( dX_p(p) \) will be invariant and the proof will be complete. The proof is simplified by using a concrete realisation of functionals on \( g_t \). By Convention (4.1.11) we may regard each \( g_t \) as a subspace of \( E \), \( \forall t \). Elements of \( g_t^* \) may be represented by the inner product of an element of \( g_t \) with all vectors in \( g_t \).

Now \( d\Phi \) is a \( C^\infty \) map \( d\Phi : [0,1] \times g \to E \), linear on \( g \). We first extend \( d\Phi \) to a \( C^\infty \) map \( d\Phi : [0,1] \times E \to E \), linear on \( E \). Our aim is to next extend the domain of the map \( Ad\Phi(g,t) \), \( g \in \mathcal{G} \) to a continuous map \( Ad : \mathcal{G}[0,1] \times E \to E \), linear on \( E \).

Now \( Ad : \mathcal{G} \times g \to g \) is a linear map on \( g \) and can be extended to a \( C^\infty \) map \( Ad : \mathcal{G} \times E \to E \) linear on \( E \). Then \( d\Phi \circ Ad(g) \) is a linear map \( d\Phi \circ Ad(g) : E \to E \), \( \forall g \in \mathcal{G} \), and for \( X_t \in d\Phi(g,t) \) it is an extension of the map \( Ad\Phi(g,t) : d\Phi(g,t) \to d\Phi(g,t) \), since \( d\Phi(t) \circ Ad(g) = Ad\Phi(g,t) \) on \( d\Phi(g,t) \). In this way, the map

\[
Ad : \Phi(g,t) \times E \to E, \ Ad : (\Phi(g,t),X_E) \mapsto d\Phi(t) \circ AdgX_E,
\]

\( g \in \mathcal{G}, \ X_E \in E \) extends to a \( C^\infty \) map,

\[
Ad : \mathcal{G}[0,1] \times E \to E, \ Ad : (\Phi(g,t),X_E) \mapsto d\Phi(t) \circ AdgX_E
\]

278
Now \( \mathcal{G}[0,1] \) is certainly dense in \( \mathcal{G}_{[0,1]} \), so \( \text{Ad} \) has a continuous unique extension to a map,

\[
\text{Ad} : \mathcal{G}[0,1] \times E \to E
\]

which agrees with the usual definition of \( \text{Ad} \) on \( g_t : \mathcal{G}_t \times g_t \to g_t, \forall t \). (An alternative way of obtaining this result is to apply the manifold structure of \( \mathcal{G}[0,1] \) as given in the proof of (4.3.1).) With the existence of this extension of \( \text{Ad} \), we have the result,

\[
\lim_{t \to 0} \text{Ad}(\Phi(g(t)))X_E = \text{Ad}(\lim_{t \to 0} \Phi(g(t)))X_E = \text{Ad}(\mathcal{X}(g))X_E.
\]

Now let \((\cdot, \cdot)\) be the inner product on \( E \), and let \( Y \in \mathfrak{g}^q \) correspond to \( \beta \) as in (4.1.14).

Let \( A^T \) denote the transpose of any linear map \( A : E \to E \) with respect to this inner product.

Then

\[
\lim_{t \to 0} p \circ d\Phi^*(\text{Ad}^*\Phi(g(t))(d\Phi^{-1})^*\beta(t))
\]

\[
= \lim_{t \to 0} p \circ d\Phi^T((\text{Ad}\Phi(g(t)))^Td\Phi(Y(t)))
\]

\[
= d\mathcal{X}p^*(p)((\text{Ad}\mathcal{X}(g))^Td\mathcal{X}(Y))
\]

\[
= d\mathcal{X}p^*(p)(\text{Ad}\mathcal{X}(g)d\mathcal{X}^*(\beta)), \forall g \in \mathcal{D}^q,
\]

as we wished to show. This establishes the proposition.

We now give the theorem corresponding to Proposition (5.2.10), for infinitesimal characters proper:

**Theorem 5.2.12**

Let \( \beta \in \mathfrak{g}_q^* \) with \( \Omega \) specified by (5.2.10) \( \forall t \). Let \( \xi_t : \mathcal{P}_T(g_t^*) \to Z(g_t) \) be the isomorphism of (1.3.62).
Given unitary representations $\mathcal{R}_t : G_t \to \mathcal{U}(\mathcal{H}_t)$, $\forall t$, corresponding to $\Omega_t$ in the sense of (1.3.38); suppose that each $\mathcal{R}_t$ has a generalised character given by $\chi_{\mathcal{R}_t}$.

Then,

$$\lambda_{\mathcal{R}_0}(\xi_0(p \circ \Phi^* t)) = \lim_{t \to 0} \lambda_{\mathcal{R}_t}(\xi_t(p(t) \circ d\Phi^* t)), $$

$$\forall p \in D_{p^*} \cap C_p^q(\Omega^* )[0, 1],$$

where $\lambda_{\mathcal{R}_t}$ is given by (1.3.58), $\forall t$.

**Proof:** Follows immediately from (5.2.10) and (5.2.11). Observe that, in view of (5.2.11), the condition that $d\Phi^{-1}$ be extendable to $E$, is not necessary. ■
§6 Appendix

§A.1

In the proof of (4.3.1) it is shown how two overlapping charts defined at points of $G_0$ in $G_{[0,1]}$ are $C^\infty$-related. This can put more clearly by the following argument:

Let $(\chi_2, N_2)$ and $(\chi_3, N_3)$ be overlapping charts corresponding to elements $g_2, g_3$ of $D^q$ as in (4.3.3). Then

$$\chi_3 \circ \chi_2^{-1}(\lambda, t) = (\theta(t))^{-1} \circ \chi(g_3^{-1}(t)g_2(t)g(\lambda, t))$$

$$= (\theta(t))^{-1} \circ \exp^{-1}(g_3^{-1}(t)g_2(t)g(\lambda, t)).$$

Now $g_2, g_3, g \in D^q$ and $(g_3^{-1}g_2g)[0,1]$ is in the domain of $\exp^{-1}$. Therefore, by (2.5.69), there is an $X_\lambda \in D^q$ such that $\exp X_\lambda(t) = g_3^{-1}(t)g_2(t)g(\lambda, t)$ and $X_\lambda(t)$ is a $C^\infty$ function of $\lambda$, as well as a $C^\infty$ function of $t \in [0, \epsilon)$. Then

$$\chi_2 \circ \chi_1(\lambda, t) = (\theta(t))^{-1} X_\lambda(t)$$

$$= (d\Phi(t) \circ \theta(t))^{-1}(d\Phi(t)X_\lambda(t)).$$

Now $X_\lambda(t)$ is a $C^\infty$ function of $t$, so $d\Phi(t)X_\lambda(t)$ extends to a $C^\infty$ function of $(\lambda, t)$ by (3.1.8). That $(d\Phi(t) \circ \theta(t))^{-1}$ extends to a $C^\infty$ function of $t$, follows from the proof of Lemma (5.1.9(a)).

Then $\chi_2 \circ \chi_1$ is a $C^\infty$ function of $(\lambda, t)$. \[\square\]
§A.2

Owing to Lemma (5.1.9(a)) and Lemma (5.1.9(b)) we have the following, much-improved version of Theorem (2.4.17):

**Theorem A.2.1**

Suppose that \( \dim \mathfrak{g} = \dim \mathfrak{g}_0 \).

The following statements are equivalent

1. \( \mathfrak{d}^q(\mathfrak{d}^q) \) is a subalgebra of \( \mathcal{P}^q(\mathfrak{g}) \).
2. \( \mathfrak{d}^q \) is a Lie algebra.
3. \( \mathfrak{s}^q \) is a Lie algebra with bracket as in Definition (2.4.13).
4. \( \mathfrak{g}_0 \) is a Lie algebra with bracket induced unambiguously as in Definition (2.4.13) from any two curves in \( \mathfrak{s}^q \) with endpoints on two elements of \( \mathfrak{g}_0 \).
5. There is a set of \( n \) sections \( S_t \in \mathfrak{s}^q \) which for \( t \in [0, \epsilon) \) (for some \( 0 < \epsilon \leq 1 \)) form a basis of the fibres of \( \mathfrak{g}_{(0,1)} \) over each \( t \), and of \( \mathfrak{g}_0 \); and if \( d \Psi(t) \) has matrix \( u(t) \) with respect to these bases and a basis \( \{X_i\} \) of \( \mathfrak{g} \), then

\[
\lim_{t \to 0} u(t)_i C^l_m u^{-1}(t)_j m u^{-1}(t)_k \text{ exists.}
\]

**Proof:**

The proof of (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) is given the proof of (2.4.17). By (5.1.9(a)), (5.1.9(b)), (1.5.1) and Corollary (2.4.8), \( \mathfrak{g}_0 \) is a Lie algebra if and only if \( \mathfrak{d}^q \) is a Lie algebra, establishing (2) \( \iff \) (4). The proof of (4) \( \Rightarrow \) (5) and (5) \( \Rightarrow \) (1) is also given in the proof of Theorem (2.4.17). The proof will be completed by showing (3) \( \Rightarrow \) (2):

Let \( X, Y \in \mathfrak{d}^q \), with \( U = d \Phi_s X \), \( V = d \Phi_s Y \). Now \( \alpha U + \beta V \in \mathfrak{s}^q \), and since \( d \Phi_s \) is \( 1-1 \) and linear, \( \alpha X + \beta Y \in \mathfrak{d}^q \).

Now \( [U, V] \in \mathfrak{s}^q \), and so \( d \Phi_s^{-1}[U, V] \in \mathfrak{d}^q \). But by (2.4.13)

282
\[ d\Phi^{-1}_s[U, V](t) = d\Phi^{-1} \circ d\Phi[d\Phi^{-1}U(t), d\Phi^{-1}V(t)] \]
\[ = [X, Y](t), \quad \text{for} \quad t \in (0, 1). \]

Taking the limit \( t \to 0 \), we have

\[ d\Phi^{-1}_s[U, V](t) = [X, Y](t), \quad \text{for} \quad t \in [0, 1], \]

and \([X, Y] \in \mathfrak{g}^q\) as required.

The corresponding theorem for the global case is awkward and unenlightening, and to spare the reader any unpleasantness we give a slightly neater, albeit less extensive theorem than (2.5.30):

**Theorem A.2.1**

Suppose that \( dP(d^q) \) is a Lie algebra, \( P(D^q) \) a group, \( P(D^q) \) is connected, and \( \text{Ker}\mathcal{X}_P \) is generated by a nucleus of \( \text{Exp}(\text{Ker}d\mathcal{X}_P) \).

Then:

- \( D^q \) is a Banach-Lie group and a Banach-Lie subgroup (1.2.8) of the Banach-Lie group \( C^q_0[0, 1] \).
- \( P(D^q) \) is a closed, analytic subgroup of \( T^q(G) \);
- \( G_0 \) is a Lie group with manifold structure inherited from \( P(D^q) \) by \( \mathcal{X}_P \);
- \( S^q \) is a Banach-Lie group;
- \( \mathcal{X}, \Phi_s, P \) and \( \epsilon \) are \( C^\infty \)-Fréchet homomorphisms;
- \( \mathcal{X}_P \) is an analytic homomorphism;

and the following diagram commutes:
Proof:

By the first section of the proof of Theorem (2.5.69), \( \text{Exp}(dP(d^q)) \subseteq P(D^q) \).

Since \( P(D^q) \) is connected, and is a subgroup of \( T^q(G) \), it must be the connected Lie subgroup of \( T^q(G) \) with Lie algebra \( dP(d^q) \).

By Theorem (A.2.1) and Theorem (2.4.17), \( \text{Ker} \mathfrak{X}_P \) is an ideal of \( dP(d^q) \).

By the assumption of the theorem, \( \text{Ker} \mathfrak{X}_P \) must be the normal Lie subgroup of \( P(D^q) \) with Lie algebra \( \text{Ker} \mathfrak{X}_P \). The theorem now follows by an application of Theorem (2.5.24) and Theorem (2.5.30).
§7 Bibliography


G.C. Hegerfeldt and J. Henning (1968), Coupling of Space-Time and Internal Symmetry, Fortschr. Physik, 16, pp 491-544.


R. Hermann (1966), Lie Groups for Physicists, Benjamin/Cummings, Reading, Massachusetts.


288


§8 Index to Notation and Subjects

In this index, subjects are listed in alphabetical order. Symbols from foreign alphabets are listed in alphabetical order according to their English spelling. Mathematical symbols are listed in the miscellaneous section.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Ad(h)</td>
<td>1.1.46</td>
</tr>
<tr>
<td>A</td>
<td>adjoint representation</td>
<td>1.1.46</td>
</tr>
<tr>
<td>A</td>
<td>adj</td>
<td>3.1.7</td>
</tr>
<tr>
<td>A</td>
<td>ad(X)</td>
<td>1.1.46</td>
</tr>
<tr>
<td>A</td>
<td>Ad*</td>
<td>1.1.48</td>
</tr>
<tr>
<td>A</td>
<td>Ad*</td>
<td>4.1.19</td>
</tr>
<tr>
<td>A</td>
<td>affine connection</td>
<td>1.4.1</td>
</tr>
<tr>
<td>A</td>
<td>arc length</td>
<td>1.4.5(a)</td>
</tr>
<tr>
<td>B</td>
<td>Baker-Campbell-Hausdorff formula</td>
<td>1.1.33</td>
</tr>
<tr>
<td>B</td>
<td>Banach algebra</td>
<td>1.3.39</td>
</tr>
<tr>
<td>B</td>
<td>Banach-Lie group</td>
<td>1.2.6</td>
</tr>
<tr>
<td>B</td>
<td>Banach-Lie subgroup</td>
<td>1.2.8</td>
</tr>
<tr>
<td>B</td>
<td>Banach space</td>
<td>1.2.1</td>
</tr>
<tr>
<td>B</td>
<td>β-derivative</td>
<td>1.3.11</td>
</tr>
<tr>
<td>B</td>
<td>Bootstrap Theorem</td>
<td>4.3.13</td>
</tr>
<tr>
<td>B</td>
<td>Bracket</td>
<td>1.1.21</td>
</tr>
<tr>
<td>B</td>
<td>βt</td>
<td>5.1.5(b)</td>
</tr>
<tr>
<td>B</td>
<td>B(X,Y)</td>
<td>1.1.73</td>
</tr>
<tr>
<td>C</td>
<td>canonical coordinates</td>
<td>1.1.32</td>
</tr>
<tr>
<td>C</td>
<td>Cartan decomposition</td>
<td>1.1.74</td>
</tr>
<tr>
<td>C</td>
<td>Cartan involution</td>
<td>1.1.67(a)</td>
</tr>
<tr>
<td>C</td>
<td>Cartan involution of G</td>
<td>1.1.67(a)</td>
</tr>
<tr>
<td>C</td>
<td>Cartan involution of a Lie algebra</td>
<td>1.1.75(a)</td>
</tr>
<tr>
<td>C</td>
<td>Cartan subalgebra</td>
<td>1.1.75</td>
</tr>
<tr>
<td>C</td>
<td>centre</td>
<td>1.1.37</td>
</tr>
<tr>
<td>C</td>
<td>character</td>
<td>1.3.7</td>
</tr>
<tr>
<td>C</td>
<td>c</td>
<td>§2.3</td>
</tr>
<tr>
<td>C</td>
<td>C∞ cross-section</td>
<td>1.1.19</td>
</tr>
<tr>
<td>C</td>
<td>C∞ functions</td>
<td>1.1.11</td>
</tr>
<tr>
<td>C</td>
<td>C∞ homomorphism</td>
<td>1.1.11</td>
</tr>
<tr>
<td>C</td>
<td>Cg∞</td>
<td>1.1.15</td>
</tr>
<tr>
<td>Term</td>
<td>Page</td>
<td></td>
</tr>
<tr>
<td>------</td>
<td>------</td>
<td></td>
</tr>
<tr>
<td>$C^\infty(\mathcal{G})$</td>
<td>1.1.22</td>
<td></td>
</tr>
<tr>
<td>$C^\infty$-manifold</td>
<td>1.1.1</td>
<td></td>
</tr>
<tr>
<td>$C^\infty$-manifold with boundary</td>
<td>1.1.1(a)</td>
<td></td>
</tr>
<tr>
<td>$C^\infty$-related</td>
<td>1.1.1</td>
<td></td>
</tr>
<tr>
<td>$C^\infty(\mathcal{G}, \mathcal{G}_1, \mathcal{R}_1)$</td>
<td>1.3.4</td>
<td></td>
</tr>
<tr>
<td>$C^\infty(\mathcal{G}, \mathcal{G}_\beta, \rho; \mathcal{H}(\beta))$</td>
<td>1.3.12</td>
<td></td>
</tr>
<tr>
<td>$C^\infty(\mathcal{G}, \mathcal{G}_2, \mathcal{R}_2 \uparrow \mathcal{G}_1)$</td>
<td>1.3.19</td>
<td></td>
</tr>
<tr>
<td>co-adjoint representation</td>
<td>1.1.48</td>
<td></td>
</tr>
<tr>
<td>cocompact nilradical</td>
<td>1.1.70</td>
<td></td>
</tr>
<tr>
<td>compact Lie algebra</td>
<td>1.1.73</td>
<td></td>
</tr>
<tr>
<td>compact support</td>
<td>1.3.45</td>
<td></td>
</tr>
<tr>
<td>compact symmetric pair</td>
<td>1.1.67(a)</td>
<td></td>
</tr>
<tr>
<td>connected component</td>
<td>1.1.36(d)</td>
<td></td>
</tr>
<tr>
<td>connected component of $\mathcal{G}_1$ in $\mathcal{G}$</td>
<td>1.1.36(b)</td>
<td></td>
</tr>
<tr>
<td>contracted Lie algebra</td>
<td>1.5.1</td>
<td></td>
</tr>
<tr>
<td>contraction of $\mathcal{G}$</td>
<td>1.5.1</td>
<td></td>
</tr>
<tr>
<td>contraction of $\mathcal{G}$</td>
<td>2.3.5</td>
<td></td>
</tr>
<tr>
<td>contraction mapping</td>
<td>2.3.7</td>
<td></td>
</tr>
<tr>
<td>convolution</td>
<td>1.3.52</td>
<td></td>
</tr>
<tr>
<td>covariant derivative</td>
<td>1.4.4</td>
<td></td>
</tr>
<tr>
<td>$C^{r,\infty}_{\mathcal{T},\Sigma}$</td>
<td>2.5.35</td>
<td></td>
</tr>
<tr>
<td>$C^q_\mathcal{G}[0,1]$</td>
<td>2.3.2ff</td>
<td></td>
</tr>
<tr>
<td>Current Algebra</td>
<td>2.4.22</td>
<td></td>
</tr>
<tr>
<td>$\mathcal{X}$</td>
<td>2.3.7</td>
<td></td>
</tr>
<tr>
<td>$\mathcal{X}_P$</td>
<td>2.5.22</td>
<td></td>
</tr>
<tr>
<td>$(\chi, \mathcal{N})$</td>
<td>1.1.1</td>
<td></td>
</tr>
<tr>
<td>$\chi_\pi$</td>
<td>1.3.43</td>
<td></td>
</tr>
<tr>
<td><strong>D</strong></td>
<td>1.1.17</td>
<td></td>
</tr>
<tr>
<td>derivative</td>
<td>1.1.17</td>
<td></td>
</tr>
<tr>
<td>de Sitter groups</td>
<td>§2.1</td>
<td></td>
</tr>
<tr>
<td>diffeomorphism</td>
<td>1.1.11</td>
<td></td>
</tr>
<tr>
<td>differential form</td>
<td>1.1.58</td>
<td></td>
</tr>
<tr>
<td>differential $j$-form</td>
<td>1.1.59</td>
<td></td>
</tr>
<tr>
<td>dimension</td>
<td>1.1.9</td>
<td></td>
</tr>
<tr>
<td>discrete series</td>
<td>1.3.49</td>
<td></td>
</tr>
<tr>
<td>domain of the contraction mapping</td>
<td>2.3.7</td>
<td></td>
</tr>
<tr>
<td>dual</td>
<td>1.1.47</td>
<td></td>
</tr>
<tr>
<td>dual tangent vector space</td>
<td>1.1.54</td>
<td></td>
</tr>
<tr>
<td>$\mathcal{D}^d$-continuous</td>
<td>4.3.9</td>
<td></td>
</tr>
<tr>
<td>$\frac{\partial}{\partial x^i}</td>
<td>_g$</td>
<td>1.1.16</td>
</tr>
</tbody>
</table>
\( d\varphi \big|_g, d\varphi \)
\( d\ell_h \)
\( \Delta \)
\( \Delta^+ \)
\( \tilde{\nabla}_Z \)
\( \Delta g \)
\( \delta \)
\( \nabla \)
\( \mathcal{D}^q \)
\( d\Phi \)
\( \mathfrak{g}^q \)
\( d\mathfrak{X} \)
\( dP \)
\( d\Phi_s \)
\( de \)
\( \mathfrak{g}^q(P) \)
\( \mathfrak{g}^q_\infty \)
\( dP \) (generalised)
\( \mathcal{D}^q_\infty \)
\( (d\Phi^{-1})^* \)
\( \beta_0 \)
\( \mathfrak{g}^*_q \)
\( d\mathfrak{X}^* \)
\( (D^q)_\beta \)
\( dP^* \)
\( d\mathfrak{X}^*_P \)
\( (\mathfrak{g}^q)_P^\beta \)
\( d\mathfrak{X}^*_P \)
\( \mathfrak{g}^q_{P,I} \)

\( E \)
\( E(\mathcal{M}) \)
\( E(V) \)
\( \text{Exp} \)
\( \text{Exp}_q \)
\( \exp_q \)

exponential Lie group
exponential map (manifolds)
exponential mapping
exterior algebra bundle
<table>
<thead>
<tr>
<th>Term</th>
<th>Section</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>1.1.57</td>
</tr>
<tr>
<td>fibre</td>
<td>1.4.16</td>
</tr>
<tr>
<td>fibre bundle</td>
<td></td>
</tr>
<tr>
<td>finitely connected</td>
<td>1.1.36(d)</td>
</tr>
<tr>
<td>$F_i(X_j, X_k)$</td>
<td>2.5.4</td>
</tr>
<tr>
<td>Fréchet $C^\infty$ homomorphism</td>
<td>1.2.9</td>
</tr>
<tr>
<td>Fréchet $C^\infty$ manifold</td>
<td>1.2.5</td>
</tr>
<tr>
<td>Fréchet $C^k$, Fréchet-$C^\infty$</td>
<td>1.2.4</td>
</tr>
<tr>
<td>Fréchet diffeomorphism</td>
<td>1.2.9</td>
</tr>
<tr>
<td>Fréchet-differentiable</td>
<td>1.2.2</td>
</tr>
<tr>
<td>$F(X, Y)$</td>
<td>1.1.33</td>
</tr>
<tr>
<td>functionals</td>
<td>1.1.47</td>
</tr>
<tr>
<td>$X$</td>
<td>2.3.7</td>
</tr>
<tr>
<td>$X_P$</td>
<td>2.5.22</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Term</th>
<th>Section</th>
</tr>
</thead>
<tbody>
<tr>
<td>G</td>
<td>1.3.53</td>
</tr>
<tr>
<td>Gårding space</td>
<td></td>
</tr>
<tr>
<td>Gauge theories</td>
<td>2.4.22</td>
</tr>
<tr>
<td>generalised character</td>
<td>1.3.43</td>
</tr>
<tr>
<td>generalised function</td>
<td>1.3.51</td>
</tr>
<tr>
<td>generalised (global) Saletan contraction</td>
<td>2.4.6</td>
</tr>
<tr>
<td>generalised I-W contraction</td>
<td>1.5.7</td>
</tr>
<tr>
<td>generalised Saletan contraction</td>
<td>1.5.1</td>
</tr>
<tr>
<td>geodesic</td>
<td>1.4.8</td>
</tr>
<tr>
<td>global I-W contraction of $G$</td>
<td>3.1.1</td>
</tr>
<tr>
<td>$g^*$</td>
<td>1.1.47</td>
</tr>
<tr>
<td>$G_\beta$</td>
<td>1.1.53</td>
</tr>
<tr>
<td>$g_\lambda$</td>
<td>1.1.75(d)</td>
</tr>
<tr>
<td>$g^\alpha$</td>
<td>1.1.76</td>
</tr>
<tr>
<td>$g' = [g, g]$</td>
<td>1.1.82</td>
</tr>
<tr>
<td>$\Gamma_{ij}^k$</td>
<td>1.4.2</td>
</tr>
<tr>
<td>$g_0$</td>
<td>1.5.1</td>
</tr>
<tr>
<td>$G_{(0,1]}$</td>
<td>2.3.3</td>
</tr>
<tr>
<td>$G_0$</td>
<td>2.3.5</td>
</tr>
<tr>
<td>$G_L$</td>
<td>2.3.8</td>
</tr>
<tr>
<td>$G_{(0,1]}$</td>
<td>2.4.1</td>
</tr>
<tr>
<td>$G_0$</td>
<td>2.4.3</td>
</tr>
<tr>
<td>$G'$</td>
<td>2.5.12</td>
</tr>
<tr>
<td>$G^\infty$</td>
<td>2.5.12</td>
</tr>
<tr>
<td>$(G_0)_{\theta_0}$</td>
<td>4.2.3</td>
</tr>
</tbody>
</table>

294
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>(G_{[0,1]})</td>
<td>Haar measure</td>
<td>1.3.21</td>
</tr>
<tr>
<td>(G_t)</td>
<td>Hilbert space</td>
<td>1.3.1</td>
</tr>
<tr>
<td>(L_{[0,1]})</td>
<td>Homomorphism of Lie algebras</td>
<td>1.1.38</td>
</tr>
<tr>
<td>(h(\beta)) (finite dimensions)</td>
<td>Inönü-Wigner contraction</td>
<td>See polarization</td>
</tr>
<tr>
<td>(h(\beta)) for (\beta \in \mathcal{A}_q^*)</td>
<td>Infinitesimal character</td>
<td>4.2.14</td>
</tr>
<tr>
<td>(h(\beta_0))</td>
<td>Induced (\mathcal{A})-module</td>
<td>4.2.16</td>
</tr>
<tr>
<td>(\mathcal{H}_t)</td>
<td>Induced representation</td>
<td>4.3.6</td>
</tr>
<tr>
<td></td>
<td>Ideal</td>
<td>1.1.37</td>
</tr>
<tr>
<td></td>
<td>Induced (\mathcal{A})-module</td>
<td>1.3.4</td>
</tr>
<tr>
<td></td>
<td>Induced representation</td>
<td>1.3.26</td>
</tr>
<tr>
<td></td>
<td>(\text{Ind}(\mathcal{A}, \mathcal{A}_2, \mathcal{R}_1 \uparrow \mathcal{A}_q))</td>
<td>1.3.19</td>
</tr>
<tr>
<td></td>
<td>(\text{Ind}\mathcal{L}_2(\mathcal{A}, \mathcal{A}_c, \mathcal{R}_c))</td>
<td>1.3.26</td>
</tr>
<tr>
<td></td>
<td>Infinitesimal character</td>
<td>1.3.58</td>
</tr>
<tr>
<td></td>
<td>Inönü-Wigner contraction</td>
<td>1.5.5ff</td>
</tr>
<tr>
<td></td>
<td>Integral</td>
<td>1.1.61</td>
</tr>
<tr>
<td></td>
<td>Integral curve</td>
<td>1.4.3</td>
</tr>
<tr>
<td></td>
<td>Integral functional</td>
<td>1.3.7</td>
</tr>
<tr>
<td></td>
<td>Invariant (C^\infty) vector field</td>
<td>1.1.52</td>
</tr>
<tr>
<td></td>
<td>Invariant differential (q)-form</td>
<td>1.1.62</td>
</tr>
<tr>
<td></td>
<td>(T^q)</td>
<td>2.5.32</td>
</tr>
<tr>
<td></td>
<td>Irreducible</td>
<td>1.3.5</td>
</tr>
<tr>
<td></td>
<td>I-W</td>
<td>1.5.7</td>
</tr>
<tr>
<td></td>
<td>Iwasawa decomposition</td>
<td>1.1.80</td>
</tr>
<tr>
<td>(J)</td>
<td>Jet bundle</td>
<td>1.4.18</td>
</tr>
<tr>
<td></td>
<td>Jet of (f)</td>
<td>1.4.20</td>
</tr>
<tr>
<td>(J^q(\mathcal{M}_1, \mathcal{M}_2))</td>
<td>1.4.18</td>
<td></td>
</tr>
<tr>
<td>(J^q([0, 1]^n, \mathcal{A}))</td>
<td>3.2.13</td>
<td></td>
</tr>
<tr>
<td>(K)</td>
<td>Killing form</td>
<td>1.1.73</td>
</tr>
<tr>
<td>(L)</td>
<td>Laurent map</td>
<td>2.3.2</td>
</tr>
<tr>
<td></td>
<td>Left invariant</td>
<td>1.1.25</td>
</tr>
<tr>
<td>Term</td>
<td>Page</td>
<td></td>
</tr>
<tr>
<td>-------------------------------------------</td>
<td>------</td>
<td></td>
</tr>
<tr>
<td>Lie algebra</td>
<td>1.1.21</td>
<td></td>
</tr>
<tr>
<td>Lie group</td>
<td>1.1.2</td>
<td></td>
</tr>
<tr>
<td>Lie subalgebra</td>
<td>1.1.21</td>
<td></td>
</tr>
<tr>
<td>Lie subgroup</td>
<td>1.1.5</td>
<td></td>
</tr>
<tr>
<td>local $C^\infty$ homomorphism</td>
<td>1.1.13</td>
<td></td>
</tr>
<tr>
<td>local $C^\infty$ isomorphism</td>
<td>1.1.13</td>
<td></td>
</tr>
<tr>
<td>Lorentz group</td>
<td>§2.3</td>
<td></td>
</tr>
<tr>
<td>${L_i}$</td>
<td>1.1.16</td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td>1.1.15</td>
<td></td>
</tr>
<tr>
<td>$\mathcal{L}$</td>
<td>1.1.20</td>
<td></td>
</tr>
<tr>
<td>$\ell_h$</td>
<td>1.1.25</td>
<td></td>
</tr>
<tr>
<td>$\mathcal{L}(C^\infty(\mathcal{G},\mathcal{G}_1,\mathcal{R}_1))$</td>
<td>1.3.4</td>
<td></td>
</tr>
<tr>
<td>$\mathcal{L}^2(\mathcal{G},\mathcal{G}_c,\mathcal{R}_c)$</td>
<td>1.3.25</td>
<td></td>
</tr>
<tr>
<td>$\lambda_{\mathcal{R}}$</td>
<td>1.3.58</td>
<td></td>
</tr>
<tr>
<td>$\lambda_{\Omega}$</td>
<td>1.3.63</td>
<td></td>
</tr>
<tr>
<td>$\mathcal{L},\mathcal{L}^1$</td>
<td>§2.3</td>
<td></td>
</tr>
<tr>
<td>$\mathbf{M}$</td>
<td>1.1.1.</td>
<td></td>
</tr>
<tr>
<td>manifold</td>
<td>1.3.22</td>
<td></td>
</tr>
<tr>
<td>measurable</td>
<td>1.3.48</td>
<td></td>
</tr>
<tr>
<td>minimal parabolic subgroup</td>
<td>1.3.25</td>
<td></td>
</tr>
<tr>
<td>$M(4 \times 4)$</td>
<td>§2.3</td>
<td></td>
</tr>
<tr>
<td>$M(2)$</td>
<td>4.2.1</td>
<td></td>
</tr>
<tr>
<td>$\mathfrak{m}(2)$</td>
<td>4.2.1</td>
<td></td>
</tr>
<tr>
<td>$\mathbf{N}$</td>
<td>1.1.12</td>
<td></td>
</tr>
<tr>
<td>neighbourhood</td>
<td>1.1.68</td>
<td></td>
</tr>
<tr>
<td>nilpotent</td>
<td>1.1.69</td>
<td></td>
</tr>
<tr>
<td>nilradical</td>
<td>1.4.6</td>
<td></td>
</tr>
<tr>
<td>normal coordinates</td>
<td>1.4.6</td>
<td></td>
</tr>
<tr>
<td>$\mathbf{O}$</td>
<td>1.3.41</td>
<td></td>
</tr>
<tr>
<td>operator norm</td>
<td>1.1.49</td>
<td></td>
</tr>
<tr>
<td>orbit</td>
<td>5.1.5(b)</td>
<td></td>
</tr>
<tr>
<td>$\Omega_t$</td>
<td>5.1.5(b)</td>
<td></td>
</tr>
<tr>
<td>$\Omega_{[0,1]}$</td>
<td>5.1.5(b)</td>
<td></td>
</tr>
<tr>
<td>$\mathbf{P}$</td>
<td>1.1.89</td>
<td></td>
</tr>
<tr>
<td>Paley-Wiener Theorem</td>
<td>1.1.36(a)</td>
<td></td>
</tr>
<tr>
<td>path-connected</td>
<td>1.3.9</td>
<td></td>
</tr>
<tr>
<td>polarization</td>
<td>1.3.48</td>
<td></td>
</tr>
<tr>
<td>positive roots with respect to $C^+$</td>
<td>1.1.75(g)</td>
<td></td>
</tr>
<tr>
<td>Principal Series representation</td>
<td>1.3.48</td>
<td></td>
</tr>
<tr>
<td>Term</td>
<td>Page</td>
<td></td>
</tr>
<tr>
<td>-------------------------------------</td>
<td>------</td>
<td></td>
</tr>
<tr>
<td>projection map</td>
<td>1.1.18</td>
<td></td>
</tr>
<tr>
<td>projection map</td>
<td>1.1.57</td>
<td></td>
</tr>
<tr>
<td>Pukansky condition</td>
<td>1.3.15</td>
<td></td>
</tr>
<tr>
<td>$\pi$</td>
<td>1.1.18</td>
<td></td>
</tr>
<tr>
<td>$\pi_E$</td>
<td>1.1.57</td>
<td></td>
</tr>
<tr>
<td>$P^+$</td>
<td>1.1.80</td>
<td></td>
</tr>
<tr>
<td>$P\mu$</td>
<td>1.3.47</td>
<td></td>
</tr>
<tr>
<td>$P_T^n(g^*)$</td>
<td>1.3.62</td>
<td></td>
</tr>
<tr>
<td>$\mathcal{P}$</td>
<td>§2.3</td>
<td></td>
</tr>
<tr>
<td>$\phi$</td>
<td>2.3.2</td>
<td></td>
</tr>
<tr>
<td>$\Psi$</td>
<td>2.3.2</td>
<td></td>
</tr>
<tr>
<td>$\Phi$</td>
<td>2.3.2</td>
<td></td>
</tr>
<tr>
<td>$P^i(g)$</td>
<td>2.4.9</td>
<td></td>
</tr>
<tr>
<td>$P_L(g)$</td>
<td>2.4.22</td>
<td></td>
</tr>
<tr>
<td>$\psi_g$</td>
<td>2.5.4</td>
<td></td>
</tr>
<tr>
<td>$\Psi_{ij}$</td>
<td>2.5.8ff</td>
<td></td>
</tr>
<tr>
<td>$\Phi_s$</td>
<td>2.5.17</td>
<td></td>
</tr>
<tr>
<td>$P$</td>
<td>2.5.19</td>
<td></td>
</tr>
<tr>
<td>$P_L^n(g)$</td>
<td>3.2.21</td>
<td></td>
</tr>
<tr>
<td>$(P(D^i))_{dP^*(\beta)}$</td>
<td>4.2.5</td>
<td></td>
</tr>
<tr>
<td>$P^n(g^*)$</td>
<td>5.2.4</td>
<td></td>
</tr>
</tbody>
</table>

**Q**

- $q$-jet bundle                           1.4.18
- $q$-jet of $f$                           1.4.20
- Quantum chromodynamics (QCD)             2.4.22
- quarks                                  §2.1

**R**

- radical                                 1.1.65
- rank                                    1.1.75
- real rank                               1.1.75(c)
- reductive                                1.1.67
- reductive                                1.5.8
- regular                                  1.1.78
- Riemannian manifold                      1.4.5
- Riemannian metric                        1.4.5
- root of $\mathfrak{g}$ with respect to $h_p$  1.1.75(d)
- roots, non-zero roots                     1.1.76
- $r(g)$                                   1.1.52
- $\mathcal{R}^i\mathfrak{g}$              1.1.11(a)
<table>
<thead>
<tr>
<th>Term</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{\rho}$</td>
<td>1.3.20</td>
</tr>
<tr>
<td>$r(g_1, g)$</td>
<td>1.3.26</td>
</tr>
<tr>
<td><strong>S</strong></td>
<td></td>
</tr>
<tr>
<td>Saletan contraction</td>
<td>1.5.5</td>
</tr>
<tr>
<td>Seed function</td>
<td>4.3.14</td>
</tr>
<tr>
<td>semi-direct product</td>
<td>1.1.71</td>
</tr>
<tr>
<td>semisimple</td>
<td>1.1.37</td>
</tr>
<tr>
<td>simple representation</td>
<td>1.3.61</td>
</tr>
<tr>
<td>solvable</td>
<td>1.1.64</td>
</tr>
<tr>
<td>Special chart in $T^q(G)$</td>
<td>2.5.3ff</td>
</tr>
<tr>
<td>Spontaneous symmetry breaking</td>
<td>2.4.22</td>
</tr>
<tr>
<td>structure constants</td>
<td>1.1.27</td>
</tr>
<tr>
<td><strong>Submanifold</strong></td>
<td></td>
</tr>
<tr>
<td>$\Sigma$</td>
<td>1.1.75(e)</td>
</tr>
<tr>
<td>$\Sigma^+$</td>
<td>1.1.75(g)</td>
</tr>
<tr>
<td>$S^\infty$</td>
<td>1.3.36</td>
</tr>
<tr>
<td>$\sigma_R$</td>
<td>1.3.42</td>
</tr>
<tr>
<td>$S^1$</td>
<td>2.3.2ff</td>
</tr>
<tr>
<td>$S^q$</td>
<td>2.3.4</td>
</tr>
<tr>
<td>$SO(3,2,\mathbb{R})$</td>
<td>2.3.10</td>
</tr>
<tr>
<td>$\mathfrak{s}^q$</td>
<td>2.4.2</td>
</tr>
<tr>
<td>$\mathfrak{sker}$</td>
<td>2.4.17</td>
</tr>
<tr>
<td>$S_{\text{ker}}$</td>
<td>2.4.24</td>
</tr>
<tr>
<td>$SO(3)$</td>
<td>4.2.1</td>
</tr>
<tr>
<td>$\mathfrak{so}(3)$</td>
<td>4.2.1</td>
</tr>
<tr>
<td><strong>T</strong></td>
<td></td>
</tr>
<tr>
<td>tangent cundle</td>
<td>1.1.18</td>
</tr>
<tr>
<td>tangent vector</td>
<td>1.1.15</td>
</tr>
<tr>
<td>tangent vector (for Banach manifold)</td>
<td>1.2.11</td>
</tr>
<tr>
<td>trace class</td>
<td>1.3.43</td>
</tr>
<tr>
<td>$T(G)$</td>
<td>1.1.18</td>
</tr>
<tr>
<td>$T,(G), T_g$</td>
<td>1.1.16</td>
</tr>
<tr>
<td>$T^*_m$</td>
<td>1.1.54</td>
</tr>
<tr>
<td>$T^q(G)$</td>
<td>1.4.15</td>
</tr>
<tr>
<td>$T^q(G)$</td>
<td>2.5.1</td>
</tr>
<tr>
<td>$\mathfrak{j}$</td>
<td>2.5.37</td>
</tr>
<tr>
<td>$\mathfrak{t}_q(g)$</td>
<td>2.5.38</td>
</tr>
<tr>
<td>$\tilde{\theta}$</td>
<td>5.1.7(c)ff</td>
</tr>
<tr>
<td>$\tilde{\theta}_M$</td>
<td>5.1.9ff</td>
</tr>
<tr>
<td><strong>U</strong></td>
<td></td>
</tr>
</tbody>
</table>
unitary representation 1.3.2
universal enveloping algebra 1.1.88
$uC_g^\infty[0,1]$ 3.1.11
$uC_g^\infty(0)[0,1]$ 3.1.11
$u_q$ 3.1.14
$\mathcal{U}_q$ 3.1.14
$\mathbf{V}$
vector bundle 1.4.12
vector field 1.1.19

**W**
Weyl chambers 1.1.75(f)

**X**
$X_{x_q^{-1}(x^j)}$ 1.1.20
$\mathbf{x}$
$\mathbf{x}_p$
2.5.22

**Y**
Yang-Mills 2.4.22
$Z(g)$ 1.3.57

**MISCELLANEOUS**
$[X, Y]$ 1.1.21
$\langle \beta, X \rangle$ 1.1.47
$\{ \partial \}$ 1.1.55
$\{ dx^i \}$ 1.1.55
$u \wedge v$ 1.1.56
$dx^{i_1} \wedge \ldots \wedge dx^{i_j}$ 1.1.57
$\mathcal{J}_\mathcal{M}$ 1.1.61
$G_1 \ltimes N$ 1.1.70
$[X, Y]$ (infinite dimensions) 1.2.15
$\| X \|$ 1.3.1
$\sim_{q,m_1}$ 1.4.17
$[X, Y]_0$ 1.5.1
$\{ x \}$ 2.4.9
$(\nabla, \ldots, \nabla^q)$ 2.5.3
$(\nabla^0, \nabla, \ldots, \nabla^q)$ 2.5.5ff
$\| V \|$ 2.5.38
$\| (V, W) \|_1$ 2.5.41
$\{ X \}_q$ 2.6.13
$(0, 1]^\sigma$, $[0, 1]^\sigma$
$(v \times w, X)$

3.2.1
5.1.14