Storage and Manipulation of Optical Information Using Gradient Echo Memory in Warm Vapours and Cold Ensembles

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Part I

Introduction to Quantum Memories
Chapter 1

Introduction: Quantum Memories for Fun and Profit

Chuck: “Hey, ... here are a couple of things you might need to know, or maybe you just forgot.” - “Chuck Versus the Ex” Chuck, NBC, 10 November 2008, TV series.

1.1 Introduction to the Introduction

Quantum information processing (QIP) encompasses the fields of quantum key distribution, quantum computing and quantum metrology. What draws all these fields together is their combination of quantum principles, such as superposition and entanglement, with information theory. I feel this is best articulated by a quote from N. Gisin et al. in their review of quantum key distribution in 2002 [14]:

“One could characterise quantum information processing as the science of turning quantum conundrums into potentially useful applications.”

For those potentially useful applications quantum information processing offers great promise, from providing provably-secure long-distance communication, while at the same time offering the potential for breaking our current, classical, security algorithms. However, to become experimentally operational, many aspects of QIP require a quantum memory - a device that can store quantum states without measuring them. Not only would a quantum memory have practical uses, but the study of light-matter interactions entailed in their development are of fundamental scientific interest.

It is impossible to achieve the functionality of a quantum memory by classical, measure-and-reproduce, means due to the “conundrum” of wave-function collapse. In the last decade or so much progress has been made towards the development of a quantum memory and in this chapter we will review the main methods used and results achieved in Sec. 1.5. But first we will discuss the main quantum “conundrums” in Sec. 1.2 and motivate the need for quantum memories by examining its uses in quantum information processing in Sec. 1.3. We will also discuss what makes a “good” quantum memory in Sec. 1.4 and finish the chapter with an overview of the best results achieved from quantum memories in Sec. 1.6, as well as outlining the structure of the rest of the thesis.
1.2 Quantum “Conundrums” and Resources

In this section we will briefly review some of the “odd” properties of quantum mechanics alluded to by Gisin et al. in the quote above, most of which are resources relevant for quantum information processing. For a more mathematical treatment of some of these resources see Chs 2 and 3.

1.2.1 Quanta

The name “quantum mechanics” derives from the fact that, unlike classical mechanics, everything in quantum mechanics is discretised into quanta. The most famous example of this is the photo-electric effect, where Einstein postulated that light was made up of individual particles, now termed photons, that carried a set amount of energy proportional to their frequency [15]. Bohr also showed that the internal energy levels of atoms are quantised, explaining their discretised absorption and emission spectra [16]. These are the main quantisation phenomena that we will make use of in this thesis.

1.2.2 Uncertainty, Superposition and Measurement

Another major difference between classical and quantum mechanics is that we can never know everything about an arbitrary quantum state. This stems from the Heisenberg uncertainty principle which states that certain pairs of system properties (most famously position and momentum) cannot be measured simultaneously past a certain uncertainty limit [17].

A quantum system is also probabilistic rather than deterministic. This means that a quantum system can be in multiple states at the same time. However, if we were to measure the system we would always find it in one state as the act of measuring a quantum system will cause its probability amplitude, or wavefunction, to collapse into one of the possible states. Therefore measurement will alter a quantum system. An example that will be of use later in this chapter is polarisation of light. Let us suppose that the polarisation of the light is at 45° to the horizontal (i.e., diagonal - D). Mathematically we can treat this as the light being in a superposition of horizontal (H) and vertical (V) polarisation states with equal probability, i.e., \( |D\rangle = |H\rangle + |V\rangle \) (ignoring normalisation). If we then measure the polarisation of the light in this horizontal-vertical basis, we will find that 50% of the time it will be horizontal and 50% of the time it will be vertical. This is illustrated in Fig. 1.1(a)(i) and can be used to create a quantum random number generator [18].

The fact that measurement will perturb the quantum state has deep implications for quantum information processing. For instance, it led W. K. Wootters and W. H. Zurek to determine that single quanta cannot be cloned [19]. This, in turn, leads to the differentiation between classical and quantum memories. Classical memories use measurement and reproduction (i.e., cloning) to faithfully store quantum states. Quantum memories, on the other hand, must use anything but measurement to localise a quantum state for a controllable period of time. This difference is illustrated in Fig. 1.1.
### 1.2 Quantum “Conundrums” and Resources

**Figure 1.1: Classical Vs Quantum Memory.** (a) A classical memory with (i) measurement (M) and (ii) reproduction (R). This scheme will only work for a set of orthogonal states. If this is not the case, as here with the diagonal (D) photons being measured with a horizontal-polarisation filter (H Filter) - a superposition of horizontal (H) and vertical (V) photons as shown in the inset - then the superposition will collapse upon measurement, thus removing quantum information. (b) A quantum memory (QM) (i) stores information, without measurement, and (ii) retrieves the information when triggered (T).

#### 1.2.3 Phase and Coherence

Not only can quantum systems be in a multitude of states at the same time, as discussed above, but there is also a phase between these states. This is why, when G. I. Taylor performed his famous single-photon double-slit experiment in 1909, he was able to see interference at the output \[20\]. If the phase between the two states (i.e., left- and right-slit states) was not well defined, then the interference pattern would not be visible. This is an illustration of the importance of the preservation of phase, otherwise known as coherence, in quantum systems. For instance, if the phase between the superposition of horizontal and vertical polarisation states in Fig. 1.1(b) was lost inside the memory, it could only perform as well as its classical equivalent. Another, rather dramatic, illustration of the link between quantum systems and coherence was the experiment by M. R. Andrews et al., who showed interference between two macroscopic ensembles of atoms in a Bose-Einstein condensate, a state of matter where all atoms occupy one quantum state \[22\].

#### 1.2.4 Quantum Bits

Combining the quantisation, superpositions and phase properties led to the development of the quantum bit, or qubit. In classical computing a bit is a piece of information that can be either a 0 or a 1. In quantum mechanics, however, we can have a system which is partly in a 0 state and partly in a 1 state, with some relative phase between them. This gives an idea of how useful quantum mechanics can be for improving computation processes.

To illustrate all the information contained in a qubit we can use a unit sphere, called the Bloch sphere, where the height of the vector represents the relative probability of being...
**Figure 1.2: Qubit.** A Bloch sphere representation, with the probability of the qubit being in state 0 or 1 given by how high the vector points and the relative phase between the two states given by the azimuthal angle (superposition states not normalised).

in state $|1\rangle$, as opposed to state $|0\rangle$. The azimuthal angle, meanwhile, tells us about the relative phase between the two. This is shown in Fig. 1.2.

Experimentally, qubits can be superpositions of polarisation, spatial-mode or time-bin states [23], or other states such as photon-number [24] or path [25]. Normally photons are used as qubits as they preserve their state for long times due to their lack of interaction with the environment. They also allow travel at the speed of light, as fast as any information is allowed to travel in our universe, and therefore allow for fast data transfer.

### 1.2.5 Entanglement

Entanglement is the quantum mechanical phenomenon whereby two quantum states are correlated in a non-local manner. This means that if, for instance, one of the two entangled particles is measured, it will also cause the *instantaneous* collapse of the other particle’s wavefunction. Like most counter-intuitive quantum mechanical effects, early critics of quantum theory used entanglement to conclude that the theory was not complete [26]. Nearly 80 years later, however, entanglement is now a key component of quantum teleportation [27], quantum key distribution [28] and quantum computing protocols following experiments by A. Aspect et al. in 1981 [29].

Pairs of entangled photons are, these days, generally created via type-II spontaneous parametric down-conversion, first demonstrated by P. G. Kwiat et al. in 1995 [30]. This is where a strong pump laser is sent into a non-linear crystal. There is then a possibility that pump photons will split into two photons of roughly half the frequency, being entangled in momentum, energy and polarisation. This is a probabilistic process.

### 1.2.6 Squeezing

The Heisenberg uncertainty principle, discussed above, gives us a bound on how much information we can simultaneously know about certain pairs of properties of a quantum system. This is known as the standard quantum limit for pairs of properties such as energy and momentum, or amplitude and phase. This does not mean, however, that we can’t
Figure 1.3: Squeezed State. Ball-on-stick representation of a squeezed state of light, with dashed blue circle showing the initial, symmetric, Heisenberg uncertainty limit of a coherent state (see Sec. 2.3.3), with an amplitude determined by the size of the stick. The solid green ellipse shows the phase-squeezed final state, with the arrows indicating the direction of squeezing.

make a measurement below the quantum limit, as long as the overall uncertainty remains the same (i.e., an increase in uncertainty in the other observable). This is known as squeezing (see Ref. [31] and other references in the same issue), as the circular Heisenberg limit is squashed into an ellipse, as illustrated in Fig. 1.3.

Squeezing, first demonstrated by R. E. Slusher et al. in 1985 [32] using four-wave mixing (see Sec. 1.5.6), is useful for precision metrology applications where only one of the two properties is of interest, for instance the phase in a gravitational-wave interferometer (see Ref. [33] and references therein). Some of the best squeezing results have been demonstrated using optical parametric processes, with up to 12.7 dB of noise suppression below the Heisenberg limit at high frequencies [34] and 10 dB down to 10 Hz [35].

1.3 Quantum Information Processing

We will now discuss real-world applications of these quantum phenomena in quantum information processing, focusing on the role played by quantum memories.

1.3.1 Quantum Key Distribution, Quantum Repeaters, and Quantum Networks

Quantum key distribution (QKD) is perhaps the first practical use of single-photon quantum phenomena. The original QKD proposal came from C. H. Bennett and G. Brassard in 1984, who suggested using two non-orthogonal bases for polarised photons (i.e., horizontal and vertical, and diagonal and anti-diagonal) to create an encryption key for a one-time pad [36]. It had already been shown, by C. Shannon in 1949, that if a key was completely random and the same length as the message to be encoded, then it could not be broken
However, without QKD there are practical issues with attempting to safely distribute such keys over large distances (i.e., trusting couriers etc.).

Bennett and Brassard’s idea was to use quantum information encoded onto photons to create a key between two parties, Alice and Bob. For instance, if Alice wanted to send an encoded message to Bob, they would first create their secret key. To do this, Alice encodes a 0 or a 1 onto the polarisation of a photon in one of the bases, for instance the horizontal-vertical basis. She then sends this photon to Bob who proceeds to measure it in one of the two bases. If Alice and Bob use the same basis (determined through a classical communication channel after Bob’s measurements) then they can say with certainty that the bit was either a 1 or 0 and include it in their secret key. However, if they measure in different bases then, due to the measurement Bob performed in the “wrong” bases, the photon he received will have collapsed with equal probability into a 0 or 1 (see Fig. 1.1(a)). They therefore discard these photons from their sifted key. Once they have enough bits measured in the same basis they then have their shared key that Alice can use to encode her information to send to Bob over a classical channel, and Bob can use the same key to decode the message.

The fact that measurement affects the quantum state is also important if any eavesdropper, Eve, decides to try and spy on Alice and Bob (as is her wont). To be able to glean information, Eve will have to measure the quantum state being sent from Alice to Bob. However, if she does so, there is no way she can make an identical copy of the state due to the no-cloning theorem [19]. Therefore, if Alice and Bob were to compare some bits of their sifted key and noticed large discrepancies between the 1s and 0s encoded and measured, they could deduce the presence of Eve. The proof of security for this scheme was proved by P. W. Shor and J. Preskill in 2000 [38].

An alternate QKD protocol was developed by A. K. Ekert in 1991 where, instead of Alice sending a pre-prepared photon to Bob, a polarisation-entangled photon source would be placed between them and one photon sent to Alice and one to Bob [28]. This is shown in Fig. 1.4(a). The beauty of this scheme is that, by testing the non-classicality of the correlations between the photon pairs, Alice and Bob can guarantee that Eve is not present.

The first experimental demonstration of QKD was over 32 cm of free-space in 1992 by C. H. Bennett et al. [39]. Since then distances up to 144 km in free-space [40, 41], and between 250-260 km in fibres [42, 43] have been achieved. Apart from single-photons, and entangled photon pairs, QKD has also been demonstrated with continuous variables (see review in Ref. [44]).

Extending the distance of QKD is problematic as quantum states are sensitive to loss and noise in the system. These will include dark counts and inefficiencies of the detectors but, in terms of scaling, the biggest issue is loss in the quantum channel. For instance, at its optimal wavelength of around 1500 µm, optical fibre has a loss of approximately 0.2 dBkm⁻¹ [45]. This means that, for a total distance of 1000 km and an initial key creation rate of 1 GHz, it would take on average over 3000 years for a photon to reach the other end. Obviously this is far too slow for any practical purpose.

Meanwhile, free-space communication is limited by line-of-sight and the weather. To a certain extent these limitations can be overcome with combination of ground bases and satellites [46, 47]. However, apart from the prohibitive cost, if measurement were to occur on the satellites it would involve a third party, Charlie, who would be needed to create one key with Alice and one with Bob. This would provide Eve with a chance to again use nefarious classical means to obtain Alice’s and Bob’s secret message.
Figure 1.4: Quantum Repeater Protocol. (a) Ekert QKD protocol, where one each of a pair of entangled photons (1 and 2) are sent separately to Alice and Bob, who perform measurements on them in non-orthogonal bases. Key rates reduce exponentially with distance. (b) Simplified quantum repeater protocol - (i) the distance between Alice and Bob is divided by placing quantum repeater (QR) nodes containing quantum memories (QMs) between them. Multiple entangled-photon pairs 1,2 and 3, 4 and 5,6 are probabilistically generated and deterministically stored in QMs. (ii) Bell-state measurement (B) is performed at one node, and the result is fed forward classically (CFF) to the next node to teleport the quantum state of Photon 5 onto Photon 3. (iii) The process is repeated until the quantum state of Photon 5 has been teleported to Photon 1. This is then equivalent to (a), but with a polynomial decrease in key rate with distance.
To attempt to overcome the exponential loss of information and avoid any third party involvement, the idea of a quantum repeater was proposed by H.-J. Briegel et al. in 1998 [48]. Their idea was that, by dividing up the distance from Alice to Bob into many sub-lengths separated by nodes, the signal decay could be made polynomial with distance as opposed to exponential. The basics of the quantum repeater protocol are illustrated in Fig. 1.4(b). A quantum repeater is made up of three parts: an entangled (Bell-state) photon source, a detection set-up; and a quantum memory. In the first step of the quantum repeater protocol, a pair of entangled photons is separated and sent to different quantum repeater nodes and stored in the quantum memories located there. Once two photons from two different entangled pairs (i.e., Photons 4 and 5 from Fig. 1.4(b)) are stored in a node, they are removed from the memory and a Bell-state measurement is performed [49, 50]. This measurement gives information about the relationship between the two photons measured. This information is then sent through a classical channel from one quantum repeater to the next so that operations can be performed on the second photon from one of the pairs in the first repeater to make it indistinguishable to the photon from the other pair (i.e., to make Photon 3 indistinguishable from Photon 5). This is called quantum teleportation, first demonstrated by C. H. Bennett et al. in 1993 [27]. By this method entanglement can be swapped down the quantum repeater chain from Bob to Alice [51], with Photon 3 being teleported to Photon 1 so that we end up with Photons 1 and 6 being entangled. These photons are then retrieved from their quantum memories and sent to Alice and Bob, equivalent to the Ekert protocol.

Without quantum memories, this protocol would not scale polynomially due to the probabilistic nature of entangled-photon-pair generation. If this were the case, the protocol would only work if all photon sources emitted a pair of photons at exactly the same time, and even then we are assuming 100% efficient detectors and communication lines. Quantum memories mean that once a photon pair is produced it can be stored until required. And, if a detection fails, then not all entangled pairs need be reproduced. Many different quantum repeater protocols have been developed, with the most famous being the Duan-Lukin-Cirac-Zoller (DLCZ) protocol [1], which will be discussed in more detail in Sec. 1.5.4. Most protocols also involve some form of entanglement purification [52].

Apart from the goal of globally provably-secure communication via QKD, quantum memories will also allow for the development of quantum networks. Such a network will consist of quantum-repeater-style interconnected nodes, where quantum memories are used to store information, which is then processed and teleported along as necessary [53, 11]. These networks will have the potential to allow distributed quantum computing, as well as multi-party quantum communication that could be put to such uses as quantum secret sharing [54].

For more information on QKD see review articles Ref.s [14] and [55].

1.3.2 Quantum Computing

Quantum computing, i.e., computing using qubits instead of bits, could offer significant advantages over its classical counterpart. In particular, there are certain problems that could be solved much more efficiently (i.e., polynomial resources as opposed to exponential). These include L. K. Grover’s algorithm for database searching [57, 58], P. W. Shor’s algorithm for factoring large numbers into two primes [2], as well as the ability to simulate quantum dynamics [59]. The second of these three abilities has, perhaps, the most
Figure 1.5: Quantum Computing. (a) Simplified Knill et al. protocol for linear-optic quantum computation [25]. One photon out of each of two Bell-state entangled pairs (B2 and B3) are used to attempt a two-qubit gate (2QG). At the same time the two qubits Q1 and Q2 are stored in quantum memories (QMs). Once the gate has been achieved, a trigger (T) is sent to the quantum memories and the states of Q1 and Q2 are teleported onto B2 and B3 by performing Bell-state measurements (B) and sending classical feedforward signals (CFF - black dashed lines) to one-qubit gates (1QG). (b) A four-qubit entangled state is used for one-way quantum computing via feedforward. Q3 and Q4 are stored in QMs to allow for synchronisation. A detection protocol (D) is performed on qubits Q1 and Q2 and logic (L) performed on the outcomes of these measurements is used to determine operations to apply to Q3 before it, too, is detected. The measurement is also used to trigger QM₁. The combined logic from detections Q1-Q3 is used to alter Q4 at another 1QG and trigger QM₂. This is a simplified and modified version of the protocol presented in Ref. [56].
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relevance to our daily lives, as most of the encrypted information we send and receive over public channels, such as the internet, is protected by the difficulty classical computers have in factoring large numbers [60].

A quantum computer needs to be able to perform single-quibit gates such as polarisation rotation, phase shifters and beam splitters, as well as two-quibit gates, namely a controlled-not gate [61]. There are many proposed protocols for a quantum computer, such as trapped ions [62, 63, 64] and solid state ensembles [65]. Here we will focus on linear-optic-based quantum computing (LOQC). The advantage of LOQC is that photons, as mentioned above, are great carriers of information, and performing single-quibit gates (such as polarisation rotations, beam-splitters and phase shifters) is straight-forward. Using photons would also allow for linking between LOQC and quantum communication channels. However, the fact that photons interact only very weakly with their environment means that performing two-quibit gates is quite difficult. These operations use the fact that, after approaching a beam-splitter, two indistinguishable photons will choose to both travel down the same path, experimentally demonstrated by C. K. Hong et al. in 1987 [66]. Unfortunately, with only these elements, deterministic gates cannot be formed and the system will be probabilistic. This will mean that it will need exponential resources, due to the nature of compounded probabilistic gates, and therefore offer no advantage over a classical computer.

There are two main methods for overcoming this. The first of these was proposed in 2001 by E. Knill et al. using detectors and teleportation [25]. Their idea was that each two-quibit gate is still probabilistic, but now heralded so that a successful gate is identified. Even so, for the times when the gates are unsuccessful, the two qubits would still be lost. To overcome this issue, they combined their idea with work by D. Gottesman et al. who, in 1999, proposed taking the gates “offline” and then teleporting them in [67]. This means that, though the gates will still fail with the same probability, the qubits of interest will not be destroyed and the non-unity probability of success will only reduce the gate creation rate.

An illustration of this scheme is shown in Fig. 1.5(a). Here two photons from two different Bell states (B2 and B3) are used to attempt the two-quibit gate, while the qubits Q1 and Q2 wait inside quantum memories until the gate has succeeded. Once success has been heralded, the states of Q1 and Q2 are teleported onto B2 and B3 by performing Bell-state measurements with their partner photons B1 and B4 and various single-quibit operations. Though this scheme greatly improves the possibility of success for a gate (for instance, for the controlled-phase gate described in Ref. [25] the total probability of success is 1/16), it will still be non-unity, as with only linear-optics components we cannot completely distinguish between the four Bell states [68].

The other method of performing LOQC is with one-way, or cluster-state, quantum computing [69, 70, 71]. These protocols rely on entanglement, measurement and feed-forward techniques to process the initial state. In this situation quantum memories would be used as synchronisation devices so that feed-forward operations could be timed properly. For instance, in 2007 R. Prevedel et al. demonstrated fast LOQC using feed-forward, where fibre delay-lines were used to allow for feed-forward measurements [56]. This is illustrated in Fig. 1.5(b), modified to include quantum memories in place of the delay-lines. These one-way feed-forward schemes would be more robust if controllable delays were available via quantum memories, especially if more qubits were to be processed in parallel.

For more information on quantum computing, see review articles in Ref.s [72] and [73].
1.3 Single-Photon Sources

While not a quantum information process, deterministic single-photons are an important resource for both QKD (where extra photons can be used by Eve to attempt to remain undetected) and in quantum computing (where extra photons can lead to measurement and gate errors), as well as other processes such as quantum random number generation [18]. Many attempts have been made to produce an on-demand single-photon “gun” using single atoms [74] or ions [75] in cavities, quantum dots [76, 77] or nitrogen-vacancy (NV) centres in diamond [78]. However, all of these methods suffer from issues such as complex and expensive set-ups (single atoms/ions), the need for spectral filters that reduce efficiency [79], or issues involving dark states (NV centres) [80].

Another option is to use heralded entangled photon pairs, such as those produced via spontaneous parametric down-conversion [81, 21], and a quantum memory. This is illustrated in Fig. 1.6. In this example, the detection of one photon indicates that the other is stored in the quantum memory and therefore we can turn the probabilistic process (with efficiencies of $10^{-7} \rightarrow 10^{-11}$ [80]) into a deterministic one, allowing us to recall the photon from the memory at any desired time.

For more information on single-photons, their production and uses, see Ref. [80].

1.3.4 Quantum Metrology

Finally, we will briefly discuss quantum metrology. This is a field where quantum phenomena such as squeezing and entanglement are used to allow for more precise measurements than are allowed for by classical theory. There is no better example of this than atomic frequency standards being used to define the second [82, 83]. Other areas where quantum processes can improve measurements include quantum lithography to beat the classical diffraction limit [84, 85], quantum positioning and clock synchronisation based on shared prior entanglement [86, 87] and ghost imaging where an image of an object is built (though it is never directly viewed) by using entangled photon pairs [88, 89].

Quantum memories can be useful for quantum metrology applications as their mapping of the quantum state from light to atoms means that measurements can be performed on stationary qubits, rather than ones travelling at the speed of light. For instance,
spin-squeezed states of atoms improve signal-to-noise for measurements in sensing and metrology [90]. Also, precision magnetometry can be implemented with entangled atoms [91].

For more information on quantum metrology see Ref. [3]. For more information on quantum images see Ref.s [92] and [93]. For examples of quantum memories improving metrology see Sec. 1.5.2.

1.4 Quantum Memory Metrics

Now that we have established the motivation for the development of a quantum memory we will discuss the necessary properties for a quantum memory to be useful for the tasks discussed above.

1.4.1 Efficiency

Efficiency is perhaps one of the easiest properties of a quantum memory to measure and understand. It is simply the ratio of output to input energy. For single-photon storage, the efficiency translates to the probability of photon re-emission from the memory. It is important to have high efficiencies (> 90%) for single-photon states to allow for high key creation rates with quantum repeaters [55] as well as for deterministic single-photon sources. For multi-photon and continuous-variable state storage efficiency is also important, as a non-unit efficiency will lead to coupling of the vacuum into the quantum state, leading to its degradation. Care must be taken when measuring the efficiency as other, detrimental, effects such as amplification of the state may occur, leading to misleadingly good results.

Efficiency is normally linked to how strongly the photons interacts with the storage medium. This can be characterised by the optical depth (OD) of a memory. The OD is not the whole story, as different techniques require more or less OD to achieve the same efficiencies. Efficiencies down to 50% can still be useful for applications such as quantum computing if error correction protocols are used [94].

1.4.2 Fidelity

The fidelity of a quantum memory is a measure of the overlap of the recalled state with the input state. This is illustrated in Fig. 1.7 for a Wigner function state representation [95]. Measuring the fidelity is one way of determining whether noise has been added to the output state. Below 50% fidelity recall, a classical measure-and-reproduce memory could work just as well [96]. If we can achieve fidelities over 68% for coherent states we pass the no-cloning limit [97]. This is where the output is guaranteed to be the best possible copy of the input state. For single-photons we can decouple efficiency and fidelity by looking at the conditional fidelity, i.e., only using measurement outcomes where a photon is detected.

One issue with fidelity is that it is state dependent. As can be seen from Fig. 1.7, for a constant non-unit efficiency, overlap of two uncertainty “balls” (determining the fidelity) will be different depending on the size of the original amplitude “stick”. Perhaps a more accurate, state-independent, measure of how well the output state matches the input state is using a T-V diagram [98, 99].
1.4 Quantum Memory Metrics

Figure 1.7: Fidelity. Wigner function ball-on-stick state representation for two coherent states (see next chapter) with different amplitudes: IP - input state (dashed); OP - output state (solid); F - fidelity (shaded regions). Both output states have same percentage reduction of stick length.

Both fidelity and T-V diagrams will be discussed in more detail in Sec. 3.2.1.

1.4.3 Storage Time

The coherence time of a memory is a measure of how quickly the stored information is lost, normally measured as the 1/e decay point. Even a quantum memory that has an efficiency approaching unity for short storage times will be useless if the information decays too quickly for its intended application. For quantum repeaters, for instance, we require a coherence time much longer than the time it takes information to travel between several nodes [100]. This is normally on the order of milliseconds. As an upper bound, we would ideally want information to last within a quantum repeater until final entanglement distillation is complete. This can be up to hundreds of seconds [55]. We also want to be able to control the storage time and be able to recall information from the memory as quickly as possible after triggering the release.

1.4.4 Bandwidth

The bandwidth of a memory will determine how narrow a pulse can be stored and therefore limit the repetition and bit rates for a system. The required bandwidth will depend on the application as well as the source. For instance, single photons generated by down-conversion normally have bandwidths on the order of nanometers [101], though using cavities these can be reduced to under ten megahertz [102, 103].
1.4.5 Delay-Bandwidth Product

Apart from the absolute value of coherence times and bandwidths of a memory, another important metric is the delay-bandwidth product (DBP). Simply put, this is the ratio of storage time to pulse length. To be useful we want a large DBP to ensure the coherence time is much longer than the input pulse duration. The DBP also determines the number of pulses, or bits of information, that could be stored in the memory at one time. However, a memory may have a large DBP while not being able to store multiple modes (see next section).

1.4.6 Multi-Mode Capacity

The multi-mode capacity of a memory determines how many different modes can be stored in the memory at one time. These could be different spatial, temporal, or frequency modes. Apart from improving bit rates, the multi-mode capacity of a memory can also be used to help improve quantum repeater protocols. For instance, with a multi-mode memory, the required storage time for full entanglement distillation reduces from hundreds of seconds down to seconds.

1.4.7 Wavelength

Another property to consider for a quantum memory is the wavelength at which it operates. For long-distance communication we would ideally want a memory that functioned at around 1500 nm, the lowest absorption band for optical fibres.

1.4.8 Storage Media

Though not technically a metric of performance, different storage media lend themselves towards certain applications over others. Normally ensembles of atoms are used as the storage medium, though single atoms can also be used (see Sec. 1.5.1). These are usually contained either in solid state ensembles cooled down to cryogenic temperatures, or alkali atoms, either in warm vapour cells or cooled and trapped using lasers. Memories using nitrogen-vacancy centres in diamond are also being touted for potential future memories at room temperature.

When choosing a storage medium we want one that has a high OD to allow for high efficiencies. The storage medium will have a large impact on the coherence times that can be achieved. For instance, in some media atoms are free to move, limiting coherence times due to diffusion and inter-atomic collisions, while others hold atoms rigidly in place. The storage medium will also determine which wavelengths can be used. In the end, the choice of storage medium is nearly as important as the memory scheme used for satisfying the criteria listed above, as will be seen in the next section.
Having defined the properties we desire in a quantum memory, we will now view some of the main quantum memory candidates. As will be noticed by the reader, these candidates are composed of a vast array of different beasts. Like a trip to any zoo, there is not enough time to view all the animals, but we will endeavour to highlight the main genera and a few of the more exotic varieties. We will only present species that have been experimentally sighted, avoiding the mythical beasts that have been theoretically proposed but have of yet not been demonstrated.

### 1.5.1 Delay Lines and Cavities

Perhaps the simplest organism in the quantum memory world is a loop of fibre. The advantages of using such a delay line are that it is readily available, easy to use and has a high bandwidth. However, it does not allow for controllable storage times. Even long delay times are not possible as fibres have a minimum absorption of 0.2 dB km$^{-1}$ at 1500 nm \[^{45}\] and therefore 50% of the light will be lost after 15 km, a storage time of roughly 70 µs. For storage times on the order of a millisecond, efficiencies will be effectively zero. Despite these issues, fibres have been used to delay entanglement for 125 µs \[^{107}\], and theoretical proposals made to improve their performance using feedback and error correction \[^{108}\].

A slightly more evolved version of the fibre loop was demonstrated by T. B. Pittman et al. in 2002 where they used a free-space loop containing dynamic polarisation control to be able to recall 700 nm single-photons at multiples of the initial cycle time (13 ns), with approximately 20% loss per cycle \[^{109}\]. This led to a quasi-coherence time of 50 ns.

Taking this idea a step further leads to cavity-based quantum memories. The idea is very similar to a fibre loop, with light being coupled into the cavity and then left to bounce around inside for a set amount of time. The number of cycles the light will undergo inside a cavity will depend on its quality-factor (Q). Cavities, as with fibre-loops, suffer from fixed storage times and there are limitations on the modes that satisfy resonance conditions. However, if the Q can be controlled, then storage times can be altered, as was the case in the experiment of T. Tanabe et al., where they could delay an approximately 1 ns pulse by 1.45 ns and then tune the Q of their photonic crystal cavity ($\approx 10^6$) with pump pulses \[^{110}\]. Because of the small round-trip distance in such a cavity (on the order of micrometers), very large finesse will be needed for useful storage times. And, while Qs up to $4 \cdot 10^{10}$ have been achieved with microtoroids \[^{111}\], there will be a trade-off between finesse and coupling efficiency \[^{112}\].

However, if we were to put an atom inside a cavity then we have a different story. In fact, some of the first proposals for quantum memories relied on the strong coupling that can be achieved between light and atoms when they are surrounded by a cavity \[^{113}\]. J. I. Cirac et al. proposed using atoms in cavities as nodes in a quantum network with entangled single-photons in 1997 \[^{53}\] and it was soon after, in 1999, that the first experiment demonstrating the localisation of an atom in a cavity was performed by J. Ye et al. \[^{114}\]. Since that time, atoms have been localised inside a cavity for over ten seconds \[^{115, 116}\]. Using this method, H. P. Specht et al. were able to demonstrate storage with 9.3% efficiency (limited by atom-cavity coupling), with a coherence time of 180 µs (measured from the fidelity) for single-photon storage at 780 nm using a $^{87}$Rb atom \[^{117}\]. However, this technique is very difficult to implement, and will only work at the few-photon level.
Introduction: Quantum Memories for Fun and Profit

1.5.2 Electromagnetically-Induced Transparency

Electromagnetically-induced transparency (EIT) is one of the most studied, and therefore most developed, quantum memory candidates. The basic EIT scheme is illustrated in Fig. 1.8(a). As can be seen, it is based around an ensemble of quantised three-level atoms, with ground states $|1\rangle$ and $|2\rangle$ and an excited state $|3\rangle$. If we were to send in our quantum signal encoded on a weak (probe) pulse of light resonant with the $|1\rangle \rightarrow |3\rangle$ transition then, if we had a large number of atoms present, the whole quantum signal would be absorbed. This would not be useful as we would have no way of then retrieving the excitation. However, if we were to also shine a strong (coupling) light field resonant with the $|2\rangle \rightarrow |3\rangle$ transition onto the atoms, we would create a transparency window around the probe transition. This is shown in Fig. 1.8(b), and the corresponding change in refractive index is shown in Fig. 1.8(c). The theory behind this effect will be discussed in more detail in Ch. 3.

As can be seen from Fig. 1.8(c), with the coupling field present there is a sharp change in refractive index around the transmission window for the probe and this will lead to a corresponding reduction in the group velocity of the information. Using EIT, velocities down to the order of tens of meters per second have been achieved [118, 119]. By slowly reducing the amount of coupling light present we can reduce the width of the transparency window and therefore decrease the group velocity further. If we adiabatically reduce the coupling field power to zero, it follows that the probe pulse will become trapped inside the atomic cloud. This was the idea for EIT as a quantum memory proposed by M. Fleischhauer et al. in 2000 [120]. We can characterise this as a stopped-light effect, with the information being stored in a coherence between the two ground states.

The first experimental demonstrations of classical light storage with EIT were by C. Liu et al. and D. F. Phillips et al. in 2001, using cold atomic ensembles [121] and warm vapour cells [122] respectively. The first demonstration using a solid state system was also performed in the same year by A. V. Turukhin [123]. Since then efficiencies in EIT have
reached up to 43% with warm vapours \cite{124} and 78% with cold atomic ensembles \cite{13}; both of these made use of optimisation techniques \cite{125} and using light between 780-795 nm. The latter experiment had a coherence time of 98 \(\mu\)s and a DBP of 74 for greater than 50% efficiency. A coherence time of 2.3 s for EIT was demonstrated by J. J. Longdell et al. in 2005 using rephasing pulses in a solid state ensemble with light at approximately 600 nm \cite{5}.

Apart from storage of classical light, EIT has also been used to store single-photons \cite{126,127}, as well as the delay \cite{128} and storage \cite{129,130,131} of squeezed vacuum pulses. Entanglement has been stored using a pair of EIT memories by K. S. Choi et al. in 2008 \cite{132}. EIT has also been used in quantum metrology as a frequency standard \cite{133,134} and in high precision magnetometry \cite{135}.

As well as achieving these remarkable results, there are a few issues associated with light storage using EIT. Firstly, it has been found both theoretically \cite{136,137} and experimentally \cite{138} that EIT adds noise to the signal, most probably due to coupling-field-induced scattering \cite{139}. There are also issues with achieving large bandwidths in EIT, as a pulse must both spectrally fit within the EIT transmission window and spatially fit within the memory length. The first condition sets a minimum pulse length (maximum bandwidth) and the second condition sets a maximum pulse length (minimum bandwidth). This trade-off also limits the multi-mode ability for EIT, with it scaling with the square root of the OD \cite{140}. To achieve high bandwidths and allow for multi-mode storage therefore requires large ODs (\(\gg 1\)). Also, with high ODs and a resonant \(\Lambda\)-system, four-wave mixing becomes an issue. This will be discussed more in Sec. 5.3.5.

### 1.5.3 Raman Memory

Raman memory, first proposed by J. Nunn et al. in 2007 \cite{141}, uses a \(\Lambda\)-system and additional coupling field like EIT but, unlike EIT, this time both fields are detuned from resonance. This is illustrated in Fig. 1.9. By being off-resonance the behaviour of the system without the coupling field is completely different, in that the probe will pass
straight through the memory. The presence of the coupling field allows a coherence to be built between the two ground states (i.e., a spin-wave). Storing information in the ground states of a real atom is normally desirable as excited states have relatively short coherence times due to spontaneous emission. The information can also only be recalled with the application of a second coupling pulse, making it a controllable process.

The input pulse will have a non-zero bandwidth, inversely proportional to its temporal width. Therefore, to allow the full pulse to be absorbed when there is a small inhomogeneous linewidth, the “write” coupling field must also be pulsed. If the two are mode-matched then a large storage bandwidth can be achieved. The information is then retrieved with a second “read” coupling pulse.

This technique was first demonstrated using a warm vapour cell in 2010, when K. F. Reim et al. were able to store a 300 ps pulse, with a bandwidth of approximately 1.5 GHz [7]. This work was extended further, with single-photon-level storage with efficiencies of 30% and a coherence time of 1.2 µs [142]. This gave them a DBP of 2,500. Both experiments used light at 850 nm. It was also found that, with the off-resonant nature of the memory, a beam-splitter operation could be performed on the stored information [143].

The efficiencies achieved were limited by lack of coupling field power to increase the effective OD, as well as reabsorption issues for retrieval in the forwards direction (this will be discussed more in Sec. 1.5.5). Though a large DBP can be achieved with the Raman memory, it is effectively single-mode in its simplest form [144]. This is because, to write another pulse into the memory requires turning on the coupling field, and that will lead to recall of the already-stored information. Noise measurements performed on the Raman scheme showed that coupling-field-induced scattering added thermal photons to the detection [142].

1.5.4 The Duan-Lukin-Cirac-Zoller Protocol

The Duan-Lukin-Cirac-Zoller (DLCZ) protocol, named after its architects, is designed to create long-lived, long-distance entanglement between memories, with quantum repeater applications in mind [1]. As with the previous two schemes presented, the DLCZ protocol uses an ensemble of Λ-type atoms and, like the Raman memory, it is detuned from resonance. The basic scheme is illustrated in Fig. 1.10(a). The system is prepared by sending a weak off-resonance “write” pulse, near the |1⟩ → |3⟩ transition, into the memory. This will probabilistically create a spin excitation, whereby one atom is moved from the |1⟩ state to state |2⟩, heralded by the emission of a Stokes photon. To retrieve the excitation, a “read” pulse, off-resonance from the |2⟩ → |3⟩ transition, is sent into the memory to create an anti-Stokes photon.

To create entanglement between different memories, beam-splitters are used to combine the anti-Stokes output ports from two memories, and read pulses applied to both. In this situation, if there is a photon detected we will know that one memory has fired but not know which, leaving them entangled. This is illustrated in Fig. 1.10(b). One advantage of this protocol over other quantum-memory-based repeater protocols (i.e., see Fig. 1.4) is that entangled photon sources are not required; instead detectors and beam-splitters should be placed between the memories. Also entanglement purification is built into the scheme [1].

First demonstrated in 2003 for both many-photons in a warm vapour [145] and for
1.5 A Trip to the Quantum Memory Zoo

Figure 1.10: DLCZ Scheme. (a) The DLCZ level scheme with (i) writing operation where a write pulse (red, solid) detuned from resonance is sent into an ensemble of atoms (A) causing the excitation of a Stokes photon (orange, dashed), the detection of which indicates the memory is “loaded.” (ii) The reading operation where a read pulse (orange, solid) detuned from resonance is sent into the ensemble to cause the emission of an anti-Stokes photon (red, dashed). (b) Entangling two quantum memories (QMs) by firstly loading them and then applying read pulses. The outputs from the QMs are combined on a beam-splitter (BS) so that a detection event (D) indicates that one memory has fired its anti-Stokes photon, but without knowing which one.

single-photons in a cold atomic ensemble [146], retrieval efficiencies of up to 50% for 30-ns-long pulses stored for 300 ns have been demonstrated without a cavity, using cold atomic ensembles at 850 nm [147]. With a cavity to define the mode of the anti-Stokes photons, this has been increased to 84% for another cold atomic ensemble with a coherence time on the order of hundreds of nanoseconds also at 850 nm [148]. Another cavity arrangement with cold atoms has led to a slightly lower recall efficiency of 73%, but a much longer coherence time of 3.2 ms at 795 nm [149].

As can be seen, the DLCZ protocol lends itself to producing deterministic single-photon sources. It can also be used to create pairs of entangled photons [150]. Moving towards quantum repeater applications, entanglement between two remote atomic ensembles has been demonstrated [151], and even four at once [152]. One issue with the DLCZ protocol is that it is single-mode and care must be taken to ensure only one atom is excited during the write process.

1.5.5 Photon Echo Techniques

The idea for a photon echo memory stemmed from the concept of nuclear-spin echoes proposed by E. L. Hahn in 1950 [153] and was first implemented by N. A. Kurnit et al. in 1964 using a ruby crystal [154]. Photon echo memories are based around the idea of reversible absorption from an ensemble of atoms. If we imagine, as illustrated in Fig. 1.11, an ensemble of two-level atoms initially in the ground state, then all their dipole vectors will be aligned downwards. If we were to excite the ensemble with enough light to put it in an equal superposition of its ground and excited states, called a $\pi/2$-pulse, then the dipoles should still be aligned if the absorption was coherent. However, due to the inhomogeneous broadening of the ensemble, each atom will then start to rotate, or precess, around the Bloch sphere with a frequency determined by its detuning from resonance. If this were all
we were to do then the excitation will remain trapped inside the ensemble until it decayed via spontaneous emission. However, if we were to apply another pulse that rotated the dipole vectors by 180° (i.e., inverted the population), called a π-pulse, then the dipoles will realign and, by a reversal of the absorption process, build a macroscopic coherence to emit a coherent pulse of light - a photon echo.

There are, however, major concerns about using this two-pulse photon echo for quantum applications. Namely, the π-pulse that is used to invert the population can lead to spontaneous emission from the ensemble and therefore amplification and added noise [155]. This is especially an issue for storing single-photons, which we have seen are so useful for quantum information processing. There is also a reabsorption issue as, if the ensemble has a high optical depth, then light attempting to leave the memory will have a high likelihood of being absorbed again. This leads to a maximum retrieval efficiency of 54% in the forwards direction [156], and is an issue for most transmissive memories. However, 100% recall can be achieved in the backwards direction.

As with many organisms facing extinction, photon echoes have had to evolve. We will discuss some of these variants below. However, before this we should note that the extra noise generated by the two-pulse photon echo can be useful for quantum repeater applications. In 2010, P. M. Ledingham et al. proposed a scheme that would use the extra noise to create a DLCZ-type quantum repeater [157]. This technique was named rephased amplified spontaneous emission (RASE), where the extra noise is rephased and a stream of photons are stored and recalled, rather than just one. This was first implemented in 2012 by S. E. Beavan et al. in a solid state ensemble [158]. Around the same time, P. M. Ledingham et al. were able to demonstrate correlations between the amplified spontaneous emission and RASE fields [159]. RASE has an advantage over the DLCZ protocol in terms of its multi-mode ability and large bandwidth.
A Trip to the Quantum Memory Zoo

Three-Pulse Photon Echo

In 1982 T. M. Mossberg et al. described a way to store arbitrary pulses using a three-pulse photon echo technique [160]. Their idea being that a write pulse, with area $\pi/2$ and a bandwidth as large as the information to be stored, is sent into the ensemble first to prepare the memory. The information is then sent in and, after a time $\tau$, another $\pi/2$-pulse is sent in. As with the $\pi$-pulse in the two-level photon echo scheme, this will cause rephasing of the dipoles and the emission of a photon echo. This is illustrated in Fig. 1.12.

Using this technique, storage has been demonstrated for up to 3.8 hours in a solid solid state system at 580 nm [161], up to 4000 pulses have been stored at one time in another solid state ensemble at 790 nm [162] and signal processing at gigahertz rates has been achieved using yet another solid state system, also at approximately 790 nm [163]. However, this scheme suffers from the same amplification and noise issues as the two-level scheme, meaning that high efficiencies cannot be achieved without population inversion and the corresponding spontaneous emission. Modified photon echo schemes have produced published efficiencies of 20% in 2012 [164].

In 2011, two groups proposed and demonstrated a photon-echo technique that works without these drawbacks by suppressing the first echo and then rephasing it. D. L. McAuslan et al. used electric fields to suppress the first echo intensity by over 99% in a solid state ensemble at 606 nm [165]. The efficiency for the final echo, however, was only a few percent in this case. V. Damon et al. used spatial phase mismatch to suppress the first echo [166]. They named this technique ROSE - rephasing of silenced echo - and demonstrated a recall efficiency of 10% after storage for 25 $\mu$s in a solid state ensemble at 790 nm.

Atomic Frequency Combs

Atomic Frequency Combs (AFCs), proposed by M. Afzelius et al. [167], are another way of rephasing the atomic dipoles without the noise and amplification issues discussed above. In its simplest form, AFC uses an ensemble of two-level atoms that is prepared with an inhomogeneous broadening resembling a comb pattern, with a set spacing $\delta_c$ between each comb. This is shown in Fig. 1.13(a). When we then send a pulse to excite the storage medium, each dipole will precess with a speed that is a multiple of the other dipoles around it. This means that, after a time $t_s = 2\pi/\delta_c$, all the dipoles will realign and therefore emit a photon echo in the forwards direction.
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**Figure 1.13: AFC Scheme.** (a) Comb structure of AFC absorption window in frequency space, with $\delta_c$ being the frequency spacing between combs. (b) Delay-line AFC storage. After the absorption of the probe (P) the dipoles precess at a rate proportional to their detuning (i.e., multiples of $\delta_c$). This means that at $t = \frac{2\pi}{\delta_c}$ they will realign to emit an echo (E) from the ensemble. (c) Controllable storage level structure for AFC, where coupling pulses are used to transfer the coherence to and from an auxiliary state $|\text{Aux}\rangle$.

The preparation of a storage medium with a comb structure has meant that, to date, only solid state ensembles have been used for AFC memories. The first demonstration of AFC was with weak coherent pulses by H. de Riedmatten et al. in 2008 [168] and since then efficiencies up to 35% have been achieved at 605 nm [169]. One of the main advantages of the AFC scheme is that it is highly multi-mode, scaling with the number of teeth in the comb (which, in turn, is limited by the inhomogeneous broadening and the excited state decay rate) [167]. With this multi-mode capacity, and bandwidths up to 5 GHz at 795 nm [170], 1060 pulses have been stored at once [8]. Entanglement storage with true single-photons has been demonstrated with AFCs [171], as has entanglement between two memories [172] and storage of time-bin qubits [6].

However, there are some issues with this two-level AFC scheme. Firstly, recall in the forward direction is still limited to 54%, due to reabsorption, and backwards retrieval is not possible. Secondly, the storage time is fixed so that it is more a delay-line than a quantum memory. Finally, storing light in an excited state will limit storage times due to the higher excited-state decay rate. All these issues can be overcome by adding an auxiliary ground-state $|\text{aux}\rangle$ and using coupling pulses to move the excitation from the upper state to the auxiliary state before the rephasing time $t_s$ and, by using counter-propagating pulses, the echo could be recalled backwards. Three-level storage was first experimentally demonstrated by M. Afzelius et al. in 2010, where they showed controllable storage with a maximum efficiency of 1% with two modes and a coherence time of approximately 20 $\mu$s for recall in the forwards direction at approximately 600 nm [173]. Very recent results have improved on this to have 6% efficiency and demonstrate storage of five modes, again at 600 nm [174].

**Controlled Reversible Inhomogeneous Broadening**

Another way of producing an echo is via controlled reversible inhomogeneous broadening (CRIB). The idea here is that, if the inhomogeneous broadening of each atom could be somehow reversed, then we could time-reverse the initial absorption process exactly to produce a photon echo with high efficiency. The original proposal, by S. A. Moiseev et al. in 2001 [175], envisaged a warm vapour medium and used the fact that the Doppler
Figure 1.14: CRIB Scheme. (a) The level structure of the atomic ensemble (A) and pulses applied (and times) for the CRIB scheme. The coherence between the states $|1\rangle$ and $|3\rangle$ created by a probe (P) pulse is transferred to state $|2\rangle$ with a $\pi$-pulse and then transferred back with a counter-propagating $\pi$-pulse (assuming a Doppler-broadened ensemble) to cause re-emission of an echo (E) in the backwards direction. (b) Bloch sphere representation of absorption and emission. (i) Input, with dipoles precessing at a frequency determined by their Doppler-shifted detuning. (ii) When coherence is transferred back to the $|1\rangle \rightarrow |3\rangle$ transition the detunings will be reversed due to nature of the Doppler shift, causing realignment of the dipoles.
shift of light seen by a moving atom (in the rest frame of the lab) is opposite for counter-propagating lasers. If we were to use a Λ-style atomic ensemble again, and send in the information to be stored on the $|1\rangle \rightarrow |3\rangle$ transition, a pair of counter-propagating $\pi$-pulses would allow for storage and retrieval, respectively, from state $|2\rangle$. As long as the time between the input pulse and initial $\pi$-pulse is much smaller than the spontaneous emission rate from state $|3\rangle$, then the opposite Doppler shifts due to the two $\pi$-pulses will reverse the direction of the dipole precession on the $|1\rangle \rightarrow |3\rangle$ transition and cause an echo to be emitted. This is illustrated in Fig. 1.14.

This idea was then extended to solid state systems (where atoms have no velocity) by controlling the inhomogeneous broadening of the atoms externally, for instance with electric or magnetic fields [176, 177]. The counter-propagating $\pi$-pulses were still required to allow for up to 100% recall efficiency in the backwards direction. It was found that a two-level implementation without the $\pi$-pulses was also possible, but that retrieval in the forwards direction was limited to 54% [156]. In terms of multi-mode ability, it was found that the number of modes scales linearly with the OD, better than EIT but lower than AFC [140].

The first demonstration of this two-level, frequency-gradient-reversal technique was by A. L. Alexander et al. in 2006 in a solid state ensemble using reversible electric fields for rephasing. They improved on their results a year later with up to four pulses stored, as well as demonstrating conservation of phase inside the memory, but with efficiencies on the order of $10^{-5}$ at 580 nm [178]. In 2010, B. Lauritzen et al. were able to use the same method with photons at telecom wavelengths (around 1550 nm), again in a solid state ensemble [179]. They were able to demonstrate storage with 0.2% efficiency after approximately 300 ns.

**Gradient Echo Memory**

From a taxonomy standpoint the gradient echo memory (GEM) scheme should be classed as longitudinal CRIB. In any case, as it is the main focus of this thesis we will discuss it separately. GEM follows directly from the realisation that only a two-level atom and a reversible frequency gradient are required for the emission of a photon echo. However, it was the discovery that a monotonic gradient placed along the memory could lead to 100% recall efficiency in the forwards direction that sets it apart. This was first put forward by G. Hétet et al. in 2008 when they performed another gradient-reversing echo experiment in a solid state ensemble with an efficiency of 15% after approximately 3 $\mu$s at 600 nm [180]. This reversal mechanism is illustrated in Fig. 1.15.

The frequency gradient is really the key to GEM, not just causing the rephasing of the atomic dipoles, but the bandwidth of the memory is determined by the product of the gradient $\eta$ and the length of the memory. The monotonicity of the gradient ensure that the recalled light will not be absorbed as it leaves the ensemble and therefore allows for the 100% forward recall efficiency. It also makes GEM a frequency-encoding memory, as different frequencies of the pulse will be stored at different spatial locations along the memory (i.e., $z$-direction).

Much experimental work has been undertaken on GEM since these first experiments. In its initial incarnation, i.e., two-level solid state scheme, efficiencies up to 69% have been achieved, with no added noise to place its performance above the no-cloning limit at 600 nm [181]. This was achieved by M. P. Hedges et al. in 2010. The bandwidth of the
memory was approximately 1 MHz.

A three-level version of GEM, known as Λ-GEM, has also been developed using a warm rubidium-87 vapour cell [182, 183]. The initial reason for the change was the convenience of using vapour cells, in terms of ease of use, laser wavelengths (795 nm), and the high ODs that can be achieved. The change in level structure was to avoid using the excited state of $^{87}$Rb that would otherwise limit the memory lifetime to tens of nanoseconds [184]. The detuned three-level structure with the addition of a coupling field, similar to that shown in Fig. 1.9(a), also allows for decoupling of the OD and coherence time on the same parameter. This quasi-two-level arrangement should, in principle, allow for long storage times with high efficiencies. Efficiencies up to 87% have been demonstrated so far, making Λ-GEM the most efficient quantum memory to date [4]. Again the bandwidth was on the order of 1 MHz. Up to twenty pulses have also been stored using Λ-GEM and a coherence time of approximately 20 µs measured in the same experiment.

Performing operations on stored information using both GEM and Λ-GEM have also been theoretically proposed, such as manipulation using the gradient [185]. Simple operations such as stretching or compressing a pulse by recalling with a steeper or shallower gradient have already been demonstrated [9]. A frequency offset has also been added [4]. On top of this, the extra “knob” for Λ-GEM in terms of the coupling field allows us the ability to efficiently resequence pulses inside the memory in Λ-GEM [9], as well as perform controllable beam-splitter operations [186, 10]. Other operations such as using GEM as an optical router [187] or all-optical multiplexer [188] have also been put forward, demonstrating the flexibility of the scheme.

Fully quantum states have not yet been stored using GEM, though tomography
was performed on two-level GEM using weak coherent states to characterise the noise properties. One issue with storing states such as single-photons in GEM is the relatively small bandwidth of roughly one megahertz. The current limitation on the bandwidth is OD, as the effective broadened OD scales as 1/bandwidth. Also, in Λ-GEM the effective two-level OD is much lower than the OD of the $|1\rangle \rightarrow |3\rangle$ transition, meaning that high ODs are needed for efficient storage. The multi-mode capability of GEM scales linearly with the OD, the same as CRIB.

1.5.6 Other Species

Apart from the “big four” above (EIT, Raman, AFC and CRIB/GEM), there are many other quantum memories being developed. Here we will take a look at a select few to give a flavour of what is out there.

Four-Wave Mixing

Four-wave mixing (FWM) can be both a very useful process, as well as a detrimental one in other quantum memory protocols (see Sec. 5.3.5). FWM has been used to generate entanglement (see, for example, Ref.s 189 and 190) and squeezing [191, 192] from warm atomic vapours at wavelengths and bandwidths compatible with quantum memories. Not many experiments, however, have used FWM alone to store information. FWM takes place in a Λ-system similar to EIT, where the coupling field can also interact with the $|1\rangle \rightarrow |3\rangle$ transition and produce an extra light field. As with EIT, delay of information has been demonstrated [193], most notably with entangled images [194]. Storage has also been demonstrated in a warm vapour cell, with efficiencies of a few percent and a coherence time of 120 $\mu$s at 795 nm [195].

Off-Resonance Faraday Interactions

The first demonstration of mapping quantum properties of light into an ensemble of atoms was performed by B. Julsgaard et al. in 2004 [196]. The authors were able to map the polarisation state of photons onto atoms using two vapour cells, a polarisation-sensitive detection scheme, and feedback onto the atoms by applying a magnetic field. It was shown that the fidelity of the writing operation was 70%, therefore above the classical limit, for up to 4 ms at 850 nm. Using a similar method of measurement and feedback, K. Jensen et al. demonstrated storage of continuous-variable squeezed-states in warm vapour cells with a lifetime of 1 ms [197]. However, retrieving the information from the memory in these cases could prove difficult, potentially requiring some form of atom-light teleportation (i.e., see Ref. [198]).

Stimulated Brillouin Scattering

If we were to pass a high intensity laser through an optical fibre it would cause the fibre to change shape due to the non-uniform intensity distribution. This will create phonons than can then interact with other photons. By sending in strong write and read pulses, counter-propagating with the information pulses, information can be controllably stored and retrieved using these phonons. With this method, efficiencies up to 29% have been
Table 1.1: Quantum Memory Performance Summary. N.B. the value of 40 for DBP for A-GEM stems from the fact that in the 20-pulse storage experiment the first pulse sent in was the last pulse recalled. Bracketed terms in DBP refer to the DBP with recall above the classical 50% efficiency limit. av. - averaged, cond. - conditional.

<table>
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<th>Efficiency</th>
<th>Fidelity</th>
<th>Coherence Time</th>
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<td>97.5% (av., cond.)</td>
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<th>Bandwidth</th>
<th>DBP</th>
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<td>5 GHz</td>
<td>2,500 GHz [142]</td>
<td>1,060 [8]</td>
</tr>
<tr>
<td>Raman (EIT)</td>
<td>Warm Vapour (Cold Atoms)</td>
<td>850 nm (780 nm)</td>
<td>793 nm</td>
<td></td>
<td>AFC</td>
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<tr>
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<td>40 (11 &gt; 50%)</td>
<td>20</td>
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</tbody>
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Achieved for approximately 2 ns pulses with a coherence time on the order of nanoseconds [199]. One benefit of this method is that it allows for storage at telecom wavelengths, as fibres are used as the storage medium. They were also able to store up to three pulses in the memory at one time.

Nuclear Spin Memory

Nuclear spins are ideal for storage of information as they are much more isolated than electron spins and therefore decay at a much slower rate. Using a single nuclear spin of a $^{13}$C atom in a diamond nitrogen-vacancy centre at room temperature, P. C. Maurer et al. were able to demonstrate a coherence time of over one second [200]. This was achieved using a green laser field and a sequence of radio-frequency pulses. Published at the same time, M. Steger used a $^{28}$Si atom’s nuclear spin to show a coherence time of three minutes at 2 K [201]. One issue with using the nuclear spin is the low electron-to-nuclear spin coupling.

1.6 Conclusions and Thesis Outline

I hope that by the end of this chapter the reader is convinced of the importance of developing a quantum memory as it is the key piece of quantum hardware at the heart of long-distance provably-secure communication, while at the same time being able to help break current encryption methods and having the potential to allow us to measure quantities better than ever before. I also hope that the reader can appreciate that there are many different schemes out there, each with their own strengths and weaknesses. Tab. 1.1 gives a summary of the best results achieved for the metrics discussed in Sec. 1.4. As can be seen, if we were able to combine the best of these together we would already have developed a useful quantum memory.
Introduction: Quantum Memories for Fun and Profit

Sadly, not all these properties are available in one memory and so we must do our best to find a candidate that satisfies as many metrics as possible and attempt to improve it. This is the goal of my thesis, using Λ-GEM. The best performance metrics for Λ-GEM are also shown Tab. 1.1. As can be seen, Λ-GEM is a promising candidate due to the high efficiencies achieved.

As such, the experiments presented in my thesis are split into two main parts: in Ch.s 5-8 we investigate ways of improving the functionality of Λ-GEM in warm vapour cells, from digitising the set-up and performing noise measurements, to spectral manipulation experiments and demonstrating Λ-GEM's ability to store multiple spatial modes. Then in Ch.s 9-12 we propose and implement a move from warm to cold atoms to allow for alternate gradient creation techniques, and attempt to improve Λ-GEM’s performance in terms of longer storage times while maintaining high efficiencies, we also take a slight detour to investigate cold atom storage with EIT. Before all of this, we present the development of digital control software for quantum optics experiments in Ch. 4, and conclude the introductory part of the thesis by presenting all relevant theory in Ch.s 2 and 3. This outline is illustrated in Fig. 1.16.

The majority of this thesis has been published, submitted to, or accepted for publication in international journals. Some selected articles resulting from the work done during my Ph.D. and included in this thesis are:

Figure 1.16: Thesis Outline


Other articles published during my Ph.D. include:


Quantum Theory of Atom-Light Interactions


### 2.1 Introduction

In this chapter we will develop the quantum mechanical formalism required to understand the workings of electromagnetically-induced transparency and the gradient echo memory, which will be presented in the next chapter. We will also add mathematical scaffolding to better understand some of the issues related to quantum information processing presented in the previous chapter. We will start by introducing quantum mechanical formalism in Sec. 2.2, before discussing different quantum states of light (Sec. 2.3), atoms (Sec. 2.4) and their interactions (Sec. 2.5).

This is not designed to be a comprehensive theoretical introduction to quantum atom-optics. Interested readers looking for more detail may wish to read Ref.s [203, 204, 205, 206, 207, 208, 209] as well as the thesis of G. Hétet [210].

### 2.2 Quantum Formalism

This section details the relevant formalism needed to understand the issues associated with measurement in quantum mechanics, as well as the time-evolution of states.

#### 2.2.1 Quantum States and Operators

In quantum mechanics we can represent the system using state vectors $|\psi\rangle$ (in Dirac’s bracket notation). These are complex vectors that live within a complex vector space called the Hilbert space, the dimensions of the Hilbert space being the number of degrees of freedom of the property described by the state (or equivalently the number of orthogonal basis vectors). An inner product $\langle \psi'|\psi \rangle$ is defined on this Hilbert space. We can represent an arbitrary state in the Hilbert space as a sum or orthonormal basis states $\{|\psi_j\rangle\}$, i.e., $|\psi\rangle = \sum_j c_j |\psi_j\rangle$ where $c_j$ are complex numbers, $|c_j|^2$ is the probability of finding the system in state $|\psi_j\rangle$ and $\sum_j |c_j|^2 = 1$ for normalisation. We call this a coherent superposition of the...
basis states. We can also define a state as being a statistical mixture of the basis states with probability $|c_j|^2$, i.e., $|\psi\rangle = \{ |c_j|^2, |\psi_j\rangle \}$.

The difference between a coherent superposition and a statistical mixture may not be readily apparent. However, if we define a density matrix $\hat{\rho}$ to be

$$\hat{\rho} = |\psi\rangle \langle \psi| = \sum_{j,k} c_{jk} |\psi_j\rangle \langle \psi_k|,$$

(2.1)

where $c_{jk} = c_j \cdot c_k^*$ - the star denoting the complex conjugate. We can now differentiate between the two because, though both have populations $c_{jj} = |c_j|^2$, a statistical mixture will have $c_{jk} = 0$ for all $j \neq k$, while this is not the case for a superposition state. We call the $c_{jk}$ terms the coherences, and they determine the phase between the states. This means that a coherent superposition is simultaneously in multiple states at once, and we can see interference between these (think, for example, of a single photon going through two slits and interfering with itself). A statistical mixture, meanwhile, is either in one state or another and, without any coherence terms, no interference can occur. We can therefore see how quantum-mechanical phenomena are linked to these coherences, as mentioned in the introduction.

Two important properties of the density matrix are

$$\text{Tr} \{ \hat{\rho} \} = \sum_j \langle \phi_j | \hat{\rho} | \phi_j \rangle = 1 \quad (2.2)$$

$$\text{Tr} \{ \hat{\rho}^2 \} = \sum_j \langle \phi_j | \hat{\rho}^2 | \phi_j \rangle \leq 1, \quad (2.3)$$

where the trace (Tr) is over an arbitrary basis $\{|\phi_j\rangle \}$. If the trace in Eq. 2.3 equals one, then we have a pure state where the system is in one of the eigenstates $|\phi_j\rangle$ with unit probability. We can also define operators $\hat{O}$ that act on states in the Hilbert space as follows

$$\hat{O} = \sum_{j,k} c_{jk} |\phi_j\rangle \langle \phi_k|.$$  

(2.4)

If $o_{jk} = o_{kj}^*$ then $\hat{O} = \hat{O}^\dagger$. We call these operators Hermitian.

### 2.2.2 Quantum Measurements and Uncertainty

Before measuring a quantum system we first have to know what properties we can observe. We can associate these observables with a set of Hermitian measurement operators $\{\hat{M}_j\}$, where $j$ refers to the measurement outcome, i.e.,

$$\hat{M} = \sum_j m_j \hat{M}_j = \sum_j m_j |\psi_j\rangle \langle \psi_j|,$$

(2.5)

where $\{m_j\}$ are a set of real eigenvalues corresponding to the measurement eigenstates $\{|\psi_j\rangle\}$. The average, or expectation value, for the measurement will be given by

$$\langle \hat{M} \rangle = \text{Tr} \{ \hat{M} \hat{\rho} \} = \sum_j m_j c_{jj},$$

(2.6)
for the state described by the density matrix in Eq. 2.1. After a measurement with result $j$ the system will no longer be in a superposition of the measurement states, but solely in the state $|\psi_j\rangle$. This is known as the collapse of the wavefunction, and illustrates how performing a measurement on a quantum state will alter it. This will be the case except for a pure state (in the measurement basis).

Apart from wavefunction collapse, there is also the issue of inherent uncertainty in a quantum state. For instance, if we wanted to obtain information about two observables, represented by measurement operators $M$ and $M'$, we cannot measure both to an arbitrary degree of certainty if the following is true

$$[\hat{M}, \hat{M}'] = \hat{M}\hat{M}' - \hat{M}'\hat{M} \neq 0.$$  

(2.7)

If this is the case then we say the two observables are non-commuting. This is encapsulated in the Heisenberg uncertainty principle (HUP), which states that for two non-commuting observables the following relation holds

$$\Delta \hat{M}\Delta \hat{M}' \geq \frac{\hbar}{2},$$  

(2.8)

where $\Delta \hat{M}$ is the uncertainty in measurement of $\hat{M}$, mathematically given by

$$\Delta \hat{M} = \sqrt{\langle \hat{M}^2 \rangle - \langle \hat{M} \rangle^2}.$$  

(2.9)

It is now obvious why a classical measure-and-reproduce memory is not adequate for the uses presented in the previous chapter: firstly performing a measurement leads to the collapse of the state; and secondly there is a finite limit on how much we can know about a state through measurement.

### 2.2.3 Pictures and Time-Evolution

Perhaps the most important operator of all is the Hamiltonian $\hat{H}$. This is an Hermitian operator that is used to determine the total energy $E$ of a system via

$$\hat{H} |\psi\rangle = E |\psi\rangle.$$  

(2.10)

It also determines the time evolution of the system. In this thesis we are mostly concerned with the evolution of operators and will therefore use the Heisenberg picture where

$$\frac{d}{dt} \hat{O} = \frac{1}{i\hbar} [\hat{O}, \hat{H}] + \frac{\partial}{\partial t} \hat{O}.$$  

(2.11)

The partial derivative in the above equation will equal zero if the operator does not contain explicit time-dependence. We can also calculate the evolution of the density matrix via the master equation

$$\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}] + \mathcal{L} [\hat{\rho}],$$  

(2.12)

where $\mathcal{L} [\hat{\rho}]$ is the Liouvillian of the system, describing the effects of coupling the system to an external reservoir.

To make calculations as simple as possible, we can split the Hamiltonian into two parts: $\hat{H} = \hat{H}_o + \hat{V}$, where $\hat{H}_o = \hat{H}_d + \hat{H}_a$ is the Hamiltonian of the individual systems.
(in our case the light and the atoms respectively) and $\hat{V}$ is the interaction Hamiltonian, describing their interactions. In the remainder of this chapter we will discuss the forms of $\hat{H}_l$, $\hat{H}_a$ and $\hat{V}$.

### 2.3 Quantum Theory of Light

In this section we will present the quantum mechanical description of light, focusing mainly on those states that we will use later in this thesis.

#### 2.3.1 Number States, Creation and Annihilation Operators

The building blocks of quantum states of light are photons, and the easiest way of representing a state with a definite number of photons is, perhaps unsurprisingly, the number state basis. In this basis, a state with $n$ photons in mode $k$ (with angular frequency $\omega_k = c/k$, $c$ being the speed of light) is described by $|n_k\rangle$. Two important operators that act on number states in mode $k$ are the creation ($\hat{a}_k^\dagger$) and annihilation ($\hat{a}_k$) operators, following the rules

\[
\hat{a}_k^\dagger |n_k\rangle = \sqrt{n_k + 1} |n_k + 1\rangle \\
\hat{a}_k |n_k\rangle = \sqrt{n_k} |n_k - 1\rangle \\
[\hat{a}_k, \hat{a}_j^\dagger] = \delta_{kj},
\]

where $\delta_{kj}$ is the Kronecker delta ($\delta_{kk} = 1$, $\delta_{kj} = 0$ for all $k \neq j$). These operators are non-Hermitian and therefore do not represent observables of the system. We can, however, create a very useful observable from them. This is the number operator $\hat{N}_k = \hat{a}_k^\dagger \hat{a}_k$ that acts on a number state, without altering it, to determine the number of photons present. I.e.,

\[
\hat{N}_k |n_k\rangle = n_k |n_k\rangle.
\]

The Hamiltonian for a single-mode number state, taking the energy of the vacuum state ($|0\rangle$) to have zero energy (as we are only interested in the change of energy between states), gives

\[
\hat{H}_l = \hbar \omega_k \hat{N}_k.
\]

#### 2.3.2 Indistinguishability, Entangled States and Teleportation

Single-photons in the same mode are indistinguishable and this can lead to some interesting results. For instance, if we image two single photons, indistinguishable apart from their position denoted by the modes $a$ and $b$, approaching a 50:50 beam-splitter at the same time (as shown in Fig. 2.1) then we can determine their paths after the beam splitter to be

\[
|1_a\rangle \rightarrow \frac{1}{\sqrt{2}} (|1_c\rangle + |1_d\rangle) \\
|1_b\rangle \rightarrow \frac{1}{\sqrt{2}} (|1_c\rangle - |1_d\rangle),
\]
where the negative sign for the second mode comes from the need to conserve energy. Now if we look at the total state (ignoring normalisation) we have

\[
|1_a\rangle |1_b\rangle \rightarrow (|1_c\rangle + |1_d\rangle) (|1_c\rangle - |1_d\rangle)
\]

\[
= (|2_c\rangle - |2_d\rangle + |1_c\rangle |1_d\rangle - |1_d\rangle |1_c\rangle),
\]

(2.20)

(2.21)

and as modes \(c\) and \(d\) are orthogonal and the photons indistinguishable we can commute then to remove the last two terms and produce the state

\[
\frac{1}{\sqrt{2}} (|2_c\rangle - |2_d\rangle).
\]

(2.22)

This property of photons was discussed in the previous chapter as a way of creating two-photon gates for a linear-optic-based quantum computer.

Another useful two-photon state is an entangled state, this is where we have correlated photon pairs. Mathematically we can determine if a two-photon state is entangled or not by seeing if we can factorise the combined state. For instance, if we were to use the tried-and-true example of two photons, in modes \(a\) and \(b\), in either horizontal (\(|H\rangle\)) or vertical (\(|V\rangle\)) polarisation-states, then the following state would be entangled

\[
|\Phi^+_{ab}\rangle = \frac{1}{\sqrt{2}} (|H_a\rangle |H_b\rangle + |V_a\rangle |V_b\rangle).
\]

(2.23)

Another way of viewing this is that, if we were to perform a polarisation measurement on the photon in mode \(a\) and find it to be horizontal, then we know that the photon in mode \(b\) must also be horizontal. This is how Alice and Bob can generate their key from an entangled photon pair in the Ekert quantum key distribution protocol discussed in Sec. 1.3.1.

The state shown above is one of four maximally-entangled Bell states, the others are given below

\[
|\Phi^-_{ab}\rangle = \frac{1}{\sqrt{2}} (|H_a\rangle |H_b\rangle - |V_a\rangle |V_b\rangle)
\]

(2.24)
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Figure 2.2: Coherent, Squeezed and Thermal States. Wigner function ball-on-stick representations of (i) vacuum state (blue), (ii) coherent state (red), (iii)thermal state (yellow, dashed line), and (iv) squeezed state (green).

\[ |\Psi^{\pm}_{ab}\rangle = \frac{1}{\sqrt{2}} (|H_a\rangle |V_b\rangle \pm |V_a\rangle |H_b\rangle). \]  

These states can also be used to teleport information. For instance, imagine we had a qubit in state \( |\psi_q\rangle = c_H |H_q\rangle + c_V |V_q\rangle \), assuming normalisation. If we were to combine this qubit with one photon from a Bell-state pair, lets take \( |\Psi^{-}\rangle \), then the state of the whole system will be given by

\[ |\psi_{abq}\rangle = \frac{1}{\sqrt{2}} \left( (c_H |H_q\rangle + c_V |V_q\rangle) (|H_a\rangle |V_b\rangle \pm |V_a\rangle |H_b\rangle) \right) \]

\[ = \frac{1}{2} \left( (|\Phi^+_{qa}\rangle (c_H |V_b\rangle - c_V |H_b\rangle) + |\Phi^-_{qa}\rangle (c_H |V_b\rangle + c_V |H_b\rangle) \right. \]
\[ + \left. |\Psi^+_{qa}\rangle (-c_V |H_b\rangle + c_H |V_b\rangle) + |\Psi^-_{qa}\rangle (c_V |H_b\rangle + c_H |V_b\rangle) \right). \]  

From the above equation we can see that if we are able to distinguish between the four Bell-states then we will know what operation (i.e., phase flip on one polarisation state, or rotation of the total polarisation by \( \pi/2 \)) is required on the photon in mode \( b \) to make it indistinguishable from the initial qubit. A measurement that can determine the exact Bell-state of the entangled pair of photons is, unsurprisingly, called a Bell-state measurement. It can be seen from Eq. 2.23-2.25 above that using only polarising beam-splitters we can determine between two of the four in any given measurement (i.e., both photons through one port, or one through each).
2.3.3 Coherent and Squeezed States

Fixed-number photons states are difficult to produce experimentally and we will therefore look slightly further afield to describe other, more easily producible, quantum states. The most common form of a single-mode light states is the coherent state. These states are eigenstates of the annihilation operator and can be described in terms of number states to be

\[ |\alpha_k\rangle = \text{Exp}\left[ -\frac{|\alpha|^2}{2} \right] \sum_{n=0}^{\infty} \frac{\alpha^n}{(n!)^{1/2}} |n_k\rangle. \] (2.27)

From this definition we can quickly determine that the mean number of photons in a coherent state is given by \[ \langle \hat{N} \rangle = |\alpha|^2. \] For \( \alpha = 0 \) the coherent state and number state representations are the same. Indeed, we can build coherent states from this vacuum state \( |0\rangle \) using the displacement operator \( \hat{D}_k(\alpha) = \text{Exp}\left[ \alpha \hat{a}^\dagger_k - \alpha^* \hat{a}_k \right] \) as follows

\[ \hat{D}_k(\alpha) |0\rangle = |\alpha_k\rangle. \] (2.28)

A state similar to a coherent state is a thermal state. However, a thermal state is a statistical mixture of number states of the form

\[ \hat{\rho}_\text{th} = \frac{1}{G} \left( |0\rangle \langle 0| + \frac{G-1}{G} |1\rangle \langle 1| + \left( \frac{G-1}{G} \right)^2 |2\rangle \langle 2| + \ldots \right), \] (2.29)

where \( G \) is a parameter dependent on the temperature of the thermal field, becoming larger as the temperature increases. The other difference between a coherent and a thermal state is that a coherent state follows Poissonian statistics for photon number, while a thermal state has a super-Poissonian distribution.

For both coherent and thermal states we can define amplitude and phase operators in terms of the creation and annihilation operators. They are respectively defined as

\[ \hat{X}_{1,k} = \frac{1}{2} \left( \hat{a}_k + \hat{a}_k^\dagger \right) \] (2.30)
\[ \hat{X}_{2,k} = -\frac{i}{2} \left( \hat{a}_k - \hat{a}_k^\dagger \right). \] (2.31)

Using the commutation relation for \( \hat{a}_k \) and \( \hat{a}_k^\dagger \) from Eq. 2.15 we can determine the commutator for these two observables to be

\[ \left[ \hat{X}_{1,k}, \hat{X}_{2,j} \right] = \frac{i}{2} \delta_{kj}. \] (2.32)

We therefore know, by the HUP, that there will be an uncertainty relation between the amplitude and phase “quadratures” for a coherent state. For a coherent state we can determine that the uncertainty in both quadratures is the same, that is \( \Delta X_{1,k} = \Delta X_{2,k} \), and both are at the Heisenberg limit. This is true regardless of the value of \( \alpha \), even for the vacuum state. For a thermal state the uncertainties are also equal, but larger than the Heisenberg limit. These are shown in Fig. 2.2. Coherent states, as pure states with minimum uncertainty, are useful for applications such as quantum tomography (discussed in Sec. 6.3) as well as for the generation of Schrödinger cat states (discussed more in Sec. 4.5) where a superposition of two of these “quasi-classical” states \( |\alpha\rangle \) and \( |-\alpha\rangle \) can be thought of as the dead-alive superposition of Schrödinger’s ill-fated pet.
As stated in the introduction, we can reduce the uncertainty in one of these quadrates if we are prepared for a corresponding increase in the other. This is achieved via a squeezing operator \( \hat{S}_k(\zeta) = \exp \left[ \frac{1}{2} \zeta^* \hat{a}_k^\dagger - \frac{1}{2} \zeta \hat{a}_k \right] \), where \( \zeta \) is the squeezing parameter. As we will not be working with squeezing in this thesis we will not discuss it in detail except to say that the squeezed state is then given by

\[
|\alpha_k, \zeta\rangle = \hat{D}_k(\alpha) \hat{S}_k(\zeta) |0_k\rangle,
\]

and this is again illustrated in Fig. 2.2.

### 2.3.4 The Quantised Multi-Mode Electric Field

In the above sections we have only considered single-mode fields. We will now expand this to a continuum of fields. Firstly, solving the Maxwell equations in the Coulomb gauge gives an equation for the quantised, multi-mode electric field in terms of the creation and annihilation operators

\[
\hat{E}(\mathbf{r}, t) = i \int_{-\infty}^{\infty} dk \sqrt{\frac{\hbar \omega_k}{2 \epsilon_0 V}} \left( \hat{a}_k u_k(\mathbf{r}) \exp[-i \omega_k t] - \hat{a}_k^\dagger u_k^*(\mathbf{r}) \exp[i \omega_k t] \right),
\]

where \( k = (\omega_k/c) \mathbf{r} \) is the wave-vector in the direction of the unit vector \( \mathbf{r} \), \( c \) being the speed of light in a vacuum, \( \epsilon_0 \) is the permittivity in a vacuum, \( V = L^3 \) is the quantisation volume for a box of length \( L \), and \( u_k(\mathbf{r}) = \exp[i \mathbf{k} \cdot \mathbf{r}] \) are the unnormalised orthogonal transverse spatial modes. We will discuss spatial modes in more detail below.

We can simplify the above equation if we assume that there are only a small range of frequencies \( \delta \omega \) present in our light field, centred around \( \omega_o = c |k_o| \), and that \( L \ll 1/|k| \). This allows us to define a multi-mode slowly-varying envelope operator

\[
\hat{\tilde{E}}(\mathbf{r}, t) = i \int_{-\infty}^{\infty} dk \hat{a}_k u_k(\mathbf{r}).
\]

The electric field operator then becomes

\[
\hat{E}(\mathbf{r}, t) = \sqrt{\frac{\hbar \omega_o}{2 \epsilon_0 V}} \left( \hat{\tilde{E}}(\mathbf{r}, t) \exp[-i \omega_o t] - \hat{\tilde{E}}^\dagger(\mathbf{r}, t) \exp[i \omega_o t] \right).
\]

The commutation relation for \( \hat{\tilde{E}}(\mathbf{r}, t) \) is given by

\[
\left[ \hat{\tilde{E}}(\mathbf{r}, t), \hat{\tilde{E}}^\dagger(\mathbf{r}', t') \right] = L \delta(\mathbf{r} - \mathbf{r}') \delta(t - t'),
\]

where \( \delta(t - t_o) \) is the Dirac delta function (which equals zero unless \( t = t_o \) in which case it equals one) and the \( L \) comes from the normalisation of the \( u_k(\mathbf{r}) \) modes. We can also define new multi-mode amplitude and phase operators to be, respectively,

\[
\hat{X}^+(\mathbf{r}, t) = \frac{1}{2} \left( \hat{\tilde{E}}(\mathbf{r}, t) + \hat{\tilde{E}}^\dagger(\mathbf{r}, t) \right),
\]

\[
\hat{X}^-(\mathbf{r}, t) = -\frac{i}{2} \left( \hat{\tilde{E}}(\mathbf{r}, t) - \hat{\tilde{E}}^\dagger(\mathbf{r}, t) \right).
\]

The Hamiltonian for this multi-mode field will be given by an extension of the single-mode
We will now return to discuss the different spatial modes \( u_k(r) = \text{Exp} \left[ i \mathbf{k} \cdot \mathbf{r} \right] \). These must satisfy the following equation for a wave travelling in the \( z \)-direction and bounded in a volume \( V \), following from classical electromagnetism

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - 2ik \frac{\partial}{\partial z} \right) u_k(r) = 0. \tag{2.41}
\]

One solution to this equation is the Hermite-Gauss traverse electromagnetic modes (TEM). These are of the form \[211\]

\[
\text{TEM}_{mn}(x, y, z) = \sqrt{\frac{2}{\pi w(z)^2}} \frac{1}{\sqrt{m! n! 2^{m+n}}} H_m \left( \frac{\sqrt{2}x}{w(z)} \right) H_n \left( \frac{\sqrt{2}y}{w(z)} \right) \text{Exp} \left[ -\frac{x^2 + y^2}{w(z)^2} \right] \times \text{Exp} \left[ -ik \frac{x^2 + y^2}{2R(z)} + i (m + n + 1) \phi_G(z) - i k z \right], \tag{2.42}
\]

where the transverse beam radius, radius of curvature of the beam wave-front, Rayleigh length and Gouy phase shift are given by, respectively,

\[
w(z) = w_o \sqrt{1 + (z/z_r)^2} \tag{2.43}
\]

\[
R(z) = z \left( 1 + \frac{z^2 R_o}{z_r^2} \right) \tag{2.44}
\]
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Figure 2.4: Heterodyne Detection Set-Up. The weak probe field $E$ is combined with the strong local oscillator (LO) on a 50:50 beam-splitter (BS). The intensity signals measured at detectors D1 and D2 are subtracted to produce the heterodyne signal, supplying both amplitude and phase information about the probe.

\[
z_R = \frac{\pi w_o^2}{\lambda} \quad (2.45)
\]

\[
\phi_G(z) = \tan^{-1}\left[\frac{z}{z_R}\right] \quad (2.46)
\]

where $w_o$ is the waist of the laser beam, with wavelength $\lambda = 2\pi/k$. The $H_{m,n}$ terms are the Hermite polynomials of the form

\[
H_m(x) = (-1)^n \exp\left[\frac{x^2}{2}\right] \frac{d^n}{dx^n} \exp\left[-\frac{x^2}{2}\right]. \quad (2.47)
\]

Some of these modes are plotted in Fig. 2.3.

2.3.5 Detection

As most current detectors cannot detect signals oscillating at $\omega_o$, we will use the multimode slowly-varying envelope operator to define detection. As we defined the number operator to determine how many photons there were in a mode, we can similarly define a photo-current operator

\[
\hat{i}(t) = \hat{E}(t)\dagger \hat{E}(t). \quad (2.48)
\]

Now, for a pulse of information we can simplify $\hat{E}(t)$ by splitting it into its amplitude and phase quadrature components, $\hat{E}_+^p(t)$ and $\hat{E}_-^p(t)$ respectively. The photocurrent will therefore be

\[
I(t) = \langle \hat{i}(t) \rangle = \langle \hat{E}_+^p(t)\dagger \hat{E}_+^p(t) \rangle = \langle \hat{E}_+^p(t)^2 \rangle + \langle \hat{E}_-^p(t)^2 \rangle.
\]

As can be seen, a single photo-detector can only measure the intensity of the pulse. If we are interested in measuring the amplitude or phase quadratures separately we need another technique. For example, if we were to combine our weak probing pulse with a strong classical local oscillator field $|E_{LO}| \gg |E_p|$, with a frequency difference $\Delta \omega_{LO}$ and a relative phase $\phi_{LO}$ on a 50:50 beam-splitter, as shown in Fig. 2.4, we obtain the following
for the two output ports
\[
\begin{align*}
I_1(t) &= \frac{1}{2} \mathcal{E}_{LO}^2 \\
&\quad + \mathcal{E}_{LO} \left( \mathcal{E}_p^+(t) \cos(\Delta \omega_{LO} t + \phi_{LO}) + \mathcal{E}_p^-(t) \sin(\Delta \omega_{LO} t + \phi_{LO}) \right) \quad (2.50) \\
I_2(t) &= \frac{1}{2} \mathcal{E}_{LO}^2 \\
&\quad - \mathcal{E}_{LO} \left( \mathcal{E}_p^+(t) \cos(\Delta \omega_{LO} t + \phi_{LO}) + \mathcal{E}_p^-(t) \sin(\Delta \omega_{LO} t + \phi_{LO}) \right). \quad (2.51)
\end{align*}
\]

If we were to then subtract the two signals we will produce
\[
I_{het} = \mathcal{E}_{LO} \left( \mathcal{E}_p^+(t) \cos(\Delta \omega_{LO} t + \phi_{LO}) + \mathcal{E}_p^-(t) \sin(\Delta \omega_{LO} t + \phi_{LO}) \right). \quad (2.52)
\]

If \( \Delta \omega_{LO} = 0 \) then we have homodyne detection, where we obtain the shape of our probe multiplied by the amplitude of the local oscillator and we can view the amplitude or phase quadrature by altering the angle \( \phi_{LO} \). If \( \Delta \omega_{LO} \neq 0 \) then this will be equivalent to a constant modulation of our signal which will scan from the amplitude quadrature to the phase quadrature and back, known as heterodyne detection. In this thesis we will mostly work with classical pulses of light and can therefore ignore the noise terms.

### 2.4 Quantising Atoms

We will not present a derivation of the quantisation of atomic electron orbitals here. For interested readers see, for example, Ref. [204] and a more detailed description of the energy levels of the atoms of interest in this thesis are presented in App. A. Here we will start from the second quantisation of the atomic Hamiltonian, namely where we have quantised atomic levels \( \ket{j} \), with corresponding energies \( \hbar \omega_j \), to give a Hamiltonian
\[
\hat{H}_a = \sum_j \hbar \omega_j \ket{j} \bra{j}, \quad (2.54)
\]
where \( \bra{j}k\rangle = \delta_{jk} \). For simplicity we can define atomic operators
\[
\hat{\sigma}_{jk} = \ket{j} \bra{k}, \quad (2.55)
\]
with terms of the form \( \hat{\sigma}_{jj} \) being the atomic populations and \( \hat{\sigma}_{jk}, j \neq k \) are the coherences. The time-dependence of these operators can be calculated by Eq. 2.11 to be
\[
\hat{\sigma}_{jk}(t) = \hat{\sigma}_{jk} \exp \left[ -i \omega_{jk} t \right], \quad (2.56)
\]
where \( \hbar \omega_{jk} \) is the energy difference between the atomic states \( \ket{k} \) and \( \ket{j} \).

In this thesis we will be performing experiments using ensembles of atoms. In this case we need to define locally-averaged atomic operators by taking a cross-section of the ensemble with length \( \delta z \), area \( A \), and with an atomic density \( n \) to give [212]
\[
\hat{\sigma}_{jk}(z,t) = \frac{1}{nA \delta z} \sum_{z' \in \delta z} \hat{\sigma}_{jk}^t(z'), \quad (2.57)
\]
where $nA\delta z = N_z$, the atomic population in the slice $z$. The total number of atoms in an ensemble of length $L$ is therefore given by $N = LN_z$.

2.5 The Interaction of Atoms and Light

We now come to the interaction of atoms with the light field, starting with determining the form of the interaction Hamiltonian.

2.5.1 The Interaction Hamiltonian

We can approximate the interaction Hamiltonian as the potential energy of the atom’s electric dipole in the light field to give

$$\hat{\mathcal{V}} = \sum_j e \mathbf{r}_j \cdot \hat{E}(\mathbf{r}_j, t) \tag{2.58}$$

$$= -\sum_j \left( \sum_k d_{jk} \hat{\sigma}_{jk}(\mathbf{r}_j, t) \right) \cdot \hat{E}(\mathbf{r}_j, t), \tag{2.59}$$

where $d_{jk} = -e \langle j | \hat{r} | k \rangle$ is the dipole moment for the $|j\rangle \rightarrow |k\rangle$ transition and equals zero if $j = k$. If we assume that the light field contains only a small range of frequencies detuned from an atomic transition, say $\omega_{12} + \Delta$, and we insert the time-dependence of $\hat{\sigma}$ from Eq. 2.56 with the definition of the electric field in terms of $\hat{E}$ from Eq. 2.36 we will find for a single atom that

$$\hat{\mathcal{V}} = -i\hbar \sqrt{\frac{\omega_{12}}{2\hbar\epsilon_o V}} \left( -d_{12} \hat{\sigma}_{12} \hat{E} \exp[i\Delta t] + d_{21} \hat{\sigma}_{21} \hat{E} \exp[-i\Delta t] 
- d_{12} \hat{\sigma}_{12} \hat{E} \exp[-2i\omega_{12} t - i\Delta t] + d_{21} \hat{\sigma}_{21} \hat{E} \exp[2i\omega_{12} t + i\Delta t] \right)$$
2.5 The Interaction of Atoms and Light

Figure 2.6: Dressed States. (a) Bare and (b) dressed state picture in the presence of a strong field with Rabi frequency $\Omega_n = g\sqrt{n+1}$, also shows Mollow triplet formation with different colours indicating different frequencies.

\[
\frac{\Delta t}{\Delta t} + \frac{\Delta t}{\Delta t} = g_{12} \hat{E}_1 \hat{E} \exp\left[-i\Delta t\right] + g_{12} \hat{E} \hat{E} \exp\left[i\Delta t\right],
\]

where we have performed the rotating-wave approximation to remove the fast oscillating terms $(2i\omega_{12}t)$ and rotated into a frame where $d_{12} = i|d_{12}|$ to produce the Jaynes-Cummings interaction Hamiltonian, with a atom-light coupling strength

\[
g_{12} = \frac{\omega_{12}}{2\hbar\epsilon_o V} |d_{12}|. \quad (2.61)
\]

If we were to have $N$ two-level atoms in a length $L$ in the $z$-direction (i.e., the direction of propagation of the light field) then, in the continuum limit, we will have

\[
\hat{V} = -\frac{\hbar}{L} \int_0^L dz (\hat{\sigma}_{21}(z,t)\hat{\sigma}(z,t) \exp[-i\Delta t] + \hat{\sigma}_{12}(z,t)\hat{\sigma}_1(z,t) \exp[i\Delta t]) . \quad (2.62)
\]

Similarly, if we have two light fields, $\hat{E}_{23}$ interacting with the $|2\rangle \rightarrow |3\rangle$ transition, detuned from resonance by $\Delta_{23}$, and another field $\hat{E}_{13}$ interacting with the $|1\rangle \rightarrow |3\rangle$ transition, detuned from resonance by $\Delta_{13}$, then we can describe the situation with the following Hamiltonian

\[
\hat{V} = -\frac{\hbar}{L} \int_0^L dz (\hat{\sigma}_{32}(z,t)\hat{\sigma}_2(z,t) \exp[-i\Delta_{23}t] + g\hat{\sigma}_{31}(z,t)\hat{\sigma}_{13}(z,t) \exp[-i\Delta_{13}t] + \text{H.C.}),
\]

where H.C. is the Hermitian conjugate. This is illustrated in Fig. 2.5 where we define $\Omega_c = g_{23}\langle \hat{E}_{23} \rangle$ as a classical Rabi frequency that be treated as a complex number (as its phase is not important), label $\Delta = \Delta_{23}$ as the one-photon detuning and $\delta = \Delta_{23} - \Delta_{13}$ as the two-photon detuning. We also assume that state $|3\rangle$ is the excited state, with the two ground-states $|1\rangle$ and $|2\rangle$ being much closer in energy, for reasons that will become clear later.
2.5.2 The Dressed State Picture

If we revert for a minute to a single two-level atom with a single-mode light field then we can write the total system Hamiltonian as

\[
\hat{H} = \hat{H}_l + \hat{H}_a + \hat{V}
\]

\[
= \hbar \omega_{12} \hat{\sigma}_{22} + \hbar \omega \hat{\sigma}_{\alpha} + g \hat{\sigma}_{21} + g \hat{\sigma}_{12},
\]

(2.64)

where \( \omega_l \) is the frequency of the light field. If we assume \( \omega_l = \omega_{12} \) then we see that the above Hamiltonian will couple states of the form \( |n\rangle_l |1\rangle_a \) and \( |n+1\rangle_l |2\rangle_a \) via

\[
\langle n|_l \langle 1|_a \hat{V} |n+1\rangle_l |2\rangle_a = g \sqrt{n+1},
\]

(2.65)

where the subscripts \( l \) and \( a \) refer to the light and atomic states respectively, both states having the same energy. We can therefore write the energy eigenstates of the free Hamiltonian (i.e., \( \hat{H}_l + \hat{H}_a \)) as

\[
|+n\rangle = \frac{1}{\sqrt{2}} (|n\rangle_l |1\rangle_a + |n+1\rangle_l |2\rangle_a)
\]

(2.66)

\[
|-n\rangle = \frac{1}{\sqrt{2}} (|n\rangle_l |1\rangle_a - |n+1\rangle_l |2\rangle_a),
\]

(2.67)

with relative energies \( E_{\pm} = \pm \hbar g \sqrt{n+1} \) from the initial state energy (i.e., \( \hbar \omega_{12} \) for the excited state). This is known as the dressed-state picture and this level-splitting effect responsible for such phenomena as Mollow triplets [213]. This is illustrated in Fig. 2.6. It can also be explained in terms of the ac Stark effect (see Ch. 10).

2.5.3 Time-Evolution

Now that we have the form of the interaction Hamiltonian, we can determine the time-evolution of the atomic and electric field operators. Using the fact that the individual and continuum atomic operators have the commutation relation

\[
[\hat{\sigma}_{jk}(z), \hat{\sigma}_{kl}(z')] = \frac{1}{N_z} \delta_{jl}(z) \delta (z - z')
\]

(2.68)

\[
[\hat{\sigma}_{jk}(z), \hat{\sigma}_{kj}(z')] = \frac{1}{N_z} (\hat{\sigma}_{kk}(z) - \hat{\sigma}_{kk}(z)) \delta (z - z')
\]

(2.69)

we can find the expectation values of the atomic operators using the Heisenberg-Langevin equations from Eq. 2.12 using \( \hat{V} \) to give

\[
\frac{\partial}{\partial t} \hat{\sigma}_{11} = \gamma \hat{\sigma}_{33} + \gamma_c (\hat{\sigma}_{22} - \hat{\sigma}_{11}) + ig \left( \hat{E}^\dagger \hat{\sigma}_{13} - \hat{E} \hat{\sigma}_{31} \right) + \hat{F}_{11}
\]

(2.70)

\[
\frac{\partial}{\partial t} \hat{\sigma}_{22} = \gamma \hat{\sigma}_{33} + \gamma_c (\hat{\sigma}_{11} - \hat{\sigma}_{22}) + i (\Omega_c^* \hat{\sigma}_{23} - \Omega_c \hat{\sigma}_{32}) + \hat{F}_{22}
\]

(2.71)

\[
\frac{\partial}{\partial t} \hat{\sigma}_{33} = -2\gamma \hat{\sigma}_{33} + i \left( g \hat{E} \hat{\sigma}_{31} + \Omega_c \hat{\sigma}_{32} - \text{H.C.} \right) + \hat{F}_{33}
\]

(2.72)

\[
\frac{\partial}{\partial t} \hat{\sigma}_{12} = - (\gamma_o + \gamma_c + i\delta) \hat{\sigma}_{12} + i \left( \Omega_c^* \hat{\sigma}_{13} - g \hat{E} \hat{\sigma}_{32} \right) + \hat{F}_{12}
\]

(2.73)

\[
\frac{\partial}{\partial t} \hat{\sigma}_{13} = - (\gamma + \gamma_o/2 + \gamma_c/2 + i\Delta) \hat{\sigma}_{13} + ig \hat{E} (\hat{\sigma}_{11} - \hat{\sigma}_{33}) + i\Omega_c \hat{\sigma}_{12} + \hat{F}_{13}
\]

(2.74)
\[
\frac{\partial}{\partial t} \hat{\sigma}_{32} = -\left(\gamma + \gamma_o/2 + \gamma_c/2 + i(\Delta - \delta)\right) \hat{\sigma}_{32} + i\Omega^r_2 (\hat{\sigma}_{33} - \hat{\sigma}_{22}) - ig\hat{E}^\dagger \hat{\sigma}_{12} + \hat{F}_{2k2.75}
\]

where we have dropped the explicit time- and spatial-dependence for ease of viewing. The \(\gamma\), \(\gamma_o\), and \(\gamma_c\) terms denote, respectively, the decay rate from \(|3\rangle\) to the other two states (taken to be the same for the \(|3\rangle \rightarrow |1\rangle\) and \(|3\rangle \rightarrow |2\rangle\) transitions), the ground-state decoherence rate, and ground-state population exchange (through collisions etc.). These are illustrated in Fig. 2.5. The decay rate depends on the atom-light coupling strength \(g\), and is much larger than the two ground-state decoherence terms \(\gamma_o \gg \gamma_c\). Also, \(\gamma_o\) is normally much larger than \(\gamma_c\) \[139\] and we will therefore ignore \(\gamma_c\) in future analysis.

The Langevin operators \(\hat{F}_{jk}\) are required to preserve the commutation relations for the system in the presence of spontaneous emission-induced damping. Here we will ignore these terms, as they solely add vacuum noise in the regimes of interest to us \[214, 137\].

We can determine the dynamics of \(\hat{\mathcal{E}}_{13}\) using the Heisenberg equation of motion (Eq. 2.11) with the relevant Hamiltonian being the sum of the interaction Hamiltonian (Eq. 2.63) and the multi-mode light field Hamiltonian (Eq. 2.40) to give

\[
\frac{d}{dt} \hat{\mathcal{E}}_{13} = \frac{-i}{\hbar} \left[ \hat{\mathcal{E}}_{13}, \hat{\mathcal{H}}_{li} + \hat{V} \right]
\]

\[
= \frac{-i}{\hbar} \left( i \int dk' \left( \sum_k \hbar \omega_k \exp[ik' \cdot r] \left[ \hat{a}_{k'}, \hat{a}_{k'}^\dagger \hat{a}_k \right] \right) - \frac{N}{L} \int dz \left( g \hat{\sigma}_{13} \left[ \hat{\mathcal{E}}_{13}, \hat{\mathcal{E}}_{13}^\dagger \right] \right) \right),
\]

where we have kept only the terms whose commutator is non-zero and used the definition of \(\hat{\mathcal{E}}\) from Eq. 2.35. Now if we use the commutator relations for \(\hat{\mathcal{E}}\) (Eq. 2.37) and \(\hat{a}_k\) (Eq. 2.15) we obtain

\[
\frac{d}{dt} \hat{\mathcal{E}}_{13} = -\omega_{13} \hat{\mathcal{E}}_{13} + igN\hat{\sigma}_{13}.
\]

(2.77)

We can also determine the \(z\) derivative of \(\hat{\mathcal{E}}_{13}\) as follows

\[
\frac{\partial}{\partial z} \hat{\mathcal{E}}_{13} = i \int dz \left( \hat{a}_k \frac{\partial}{\partial z} \exp[ik \cdot r] \right)
\]

\[
= (ik_z) \hat{\mathcal{E}}_{13}.
\]

(2.78)

We can now combine these two equations to give us what is known as the Maxwell equation for \(\hat{\mathcal{E}}_{13}\)

\[
\frac{d}{dt} \hat{\mathcal{E}}_{13} + c \frac{\partial}{\partial z} \hat{\mathcal{E}}_{13} = igN\hat{\sigma}_{13} + i (\omega_{13} - ck_z) \hat{\mathcal{E}}_{13}
\]

\[
= igN\hat{\sigma}_{13} + \frac{ic}{2\kappa_{13}} \nabla_{x,y}^2 \hat{\mathcal{E}}_{13},
\]

(2.79)

where we have made the paraxial approximation \(k_z \gg k_x, k_y\) and therefore that \(k_{13} \approx k_z\). We can then approximate

\[
\omega_{13} - ck_z = c \left( k_{13} - \frac{k_z}{\sqrt{k_x^2 + k_y^2 + k_z^2}} \right)
\]
Figure 2.7: Bloch Sphere. (a) Bloch sphere showing the interaction between a two-level atom and a light field (i) on resonance (blue) and (ii) off resonance (red). (b) Bloch vectors showing the effects of (i) population decay with time constant $T_1$ (blue) and (ii) decoherence with time constant $T_2^*$ (red). (c) Absorption spectrum of an atomic ensemble with individual, homogeneous, linewidths $\gamma$ and overall, inhomogeneous, linewidth $\delta\omega$.

\[ \approx \frac{1}{2k_{13}} \left( k_x^2 + k_y^2 \right). \]

For most of this thesis we will make a plane-wave approximation, i.e., $\exp[i\mathbf{k} \cdot \mathbf{r}] = \exp[ik_z z]$, and therefore the second term from Eq. 2.79 will vanish, leaving us with

\[ \left( \frac{d}{dt} + c \frac{\partial}{\partial z} \right) \tilde{E}(z, t) = igN\tilde{\sigma}_{13}(z, t). \]  

(2.80)

We will use this equation to help determine the time-evolution of single-mode storage in both EIT and GEM in the next chapter. However, the $\nabla^2_{x,y}\tilde{E}_{13}$ term from Eq. 2.79 will be important for TEM modes greater than TEM$_{00}$, and therefore image storage (see Ch. 8).

### 2.5.4 Optical Bloch Equations

The optical Bloch equations and corresponding Bloch sphere visualisations, already seen in the previous chapter, are an elegant way of describing the interaction of a light field with two-level atoms. We start by defining a Bloch vector $\rho = (u, v, w)$, where the components are described in terms of the atomic populations and coherences by [204]

\[ u = \sigma_{12} + \sigma_{21} \]  

(2.81)

\[ v = i(\sigma_{21} - \sigma_{12}) \]  

(2.82)

\[ w = \sigma_{11} - \sigma_{22}, \]  

(2.83)

where we have removed the hats for ease of notation. If we have a light field $\mathcal{E}(t)$ incident on our two-level atom, detuned an amount $\Delta$ from resonance, the evolution will be described by its interaction with the torque vector $\mathbf{\Omega} = (-g\mathcal{E}(t), 0, -\Delta)$ in a frame rotating at the atomic transition frequency $\omega_{12}$ by

\[ \frac{d}{dt}\rho = \mathbf{\Omega} \times \rho. \]  

(2.84)
This gives the following equations for the time-evolution of the individual components

\[
\begin{align*}
\frac{d}{dt}u &= -\Delta v \\
\frac{d}{dt}v &= \Delta u - g\mathcal{E}(t)w \\
\frac{d}{dt}w &= g\mathcal{E}(t)v.
\end{align*}
\] (2.85)

(2.86)

(2.87)

From this set of equations it can be seen that if the atom is subjected to a constant light field, on resonance, then it will simply oscillate between the ground and excited states at the Rabi frequency \(\Omega_c = g\mathcal{E}\). This type of behaviour is called Rabi flopping and is illustrated in Fig. 2.7(a)(i). This assumes no spontaneous emission or decoherence.

If, on the other hand, we were to excite the atom with an on-resonance pulse of area \(A(t) = \int_{-\infty}^{t} dt' g\mathcal{E}(t')\) then, after time \(t\), we will find the system in the state

\[
\begin{align*}
u(t) &= 0 \quad (2.88) \\
v(t) &= \text{Sin} \left[ A(t) \right] \quad (2.89) \\
w(t) &= \text{Cos} \left[ A(t) \right]. \
\end{align*}
\] (2.90)

(2.91)

It follows from these equations that, if \(A(t) = \pi/2\) then \(\rho \left( t \right) = (0, 1, 0)\) and, if \(A(t) = \pi\) then \(\rho \left( t \right) = (0, 0, 1)\). These are the \(\pi/2\)- and \(\pi\)-pulses mentioned in the introduction.

If the light is not on resonance (i.e., \(\Delta \neq 0\)) we can perform a coordinate change to help with determining the system dynamics by rotating by an angle \(\phi = \text{Tan}^{-1} \left[ \Delta / g\mathcal{E}(t) \right] \) about the \(v\) axis. The time-evolution in these new coordinates is given by

\[
\begin{align*}
\frac{d}{dt}u' &= 0 \\
\frac{d}{dt}v' &= -\Omega_c(t, \Delta) w \\
\frac{d}{dt}w' &= \Omega_c(t, \Delta) v, 
\end{align*}
\] (2.92)

(2.93)

(2.94)

where \(\Omega_c(t, \Delta) = \sqrt{g^2\mathcal{E}^2(t) + \Delta^2}\) is the off-resonance Rabi frequency. The evolution of \(\rho\) for a constant detuned field is shown in Fig. 2.7(a)(ii).

We will now ask the question: what will happen to the Bloch vector dynamics once the light field is removed? Initially assuming that the excitation pulse was on resonance, the Bloch vector will remain stationary according to the above equations. This is not the case in practice, however, due to spontaneous emission with rate \(\gamma\) and decoherence with rate \(\gamma_o\). As stated in the previous section, \(\gamma\) is responsible for population decay from \(|2\rangle \rightarrow |1\rangle\) (also called longitudinal decay) with a time-constant \(T_1\). \(\gamma_o\), meanwhile, is responsible for dephasing (transverse decay, due to scattering from the surface of the sphere) with time-constant \(T_2^*\). We can phenomenologically add these terms to the Bloch evolution equation to produce

\[
\begin{align*}
\frac{d}{dt}u &= -\Delta v - \frac{u}{T} \\
\frac{d}{dt}v &= \Delta u - g\mathcal{E}(t)w - \frac{v}{T}. 
\end{align*}
\] (2.95)

(2.96)
where $T = T_1 + T_2^*$. The $T_1$ and $T_2^*$ processes are illustrated in Fig. 2.7(b). As can be seen, decoherence does not lead to a loss of population, but leads to a reduction in the length of the Bloch vector, while spontaneous emission leads to population decay as well as a change in length of the Bloch vector. The reduction of the length of the Bloch vector is significant, as coherent superpositions states sit on the surface of the sphere while statistical mixtures sit inside the sphere.

If, for a moment, we neglect the damping terms and assume the atom was illuminated by off-resonance $\pi/2$-pulse of length $\tau$, short compared to the $1/\Delta$, we find that

$$\rho(t) = (\sin[\Delta t], \cos[\Delta t], 0). \quad (2.98)$$

That is, the Bloch vector rotates around the $u-v$ plane at a rate determined by the detuning of the light.

Now let us consider an ensemble of two-level atoms. The absorption profile of an individual atom is Lorentzian with a full-width-half-maximum (FWHM) at its lowest limit of $\gamma$, i.e.,

$$L(\omega) = \frac{\gamma}{2\pi(\omega - \omega_{12})^2 + (\gamma/2)^2}. \quad (2.99)$$

This is known as the homogeneous broadening of the atom, as the width will be the same for all atoms. However, the centre absorption frequency will vary from atom to atom due to, for example, Doppler shifts or inhomogeneous electric or magnetic fields. This leads to an absorption bandwidth for an ensemble of atoms that is greater than the bandwidth of an individual atom. This is called inhomogeneous broadening. The difference between homogeneous and inhomogeneous broadening is illustrated in Fig. 2.7(c). Inhomogeneous broadening is desirable for the absorption of light, as we would like to absorb as large a range of frequencies as possible. However, if we consider inhomogeneous broadening in terms of Eq. 2.98, where each individual dipole will rotate at a slightly different frequency due to the inhomogeneous broadening, then we see that the sum of all the Bloch vectors, known as the macroscopic coherence, will destructively interfere and head to zero. We can investigate the macroscopic coherence mathematically by looking at the ensemble polarisation

$$\mathcal{P}(t) \propto \int d\Delta' \text{Re} \left\{ (u(t, \Delta') + iv(t, \Delta')) \exp[i\omega_{12}] \right\} I(\Delta'), \quad (2.100)$$

where $I(\Delta)$ is the inhomogeneous broadening function. If we were to assume a Lorentzian inhomogeneous broadening with FWHM $\delta\omega$ and centre frequency $\omega_{12} + \Delta$ then the above equation simplifies to

$$\mathcal{P}(t) = \mathcal{P}(0) \cos[(\omega_{12} + \Delta) t] \exp[-(\delta\omega)t]. \quad (2.101)$$

As can be seen, there are two components to the polarisation: a cosine oscillating close to the atomic transition frequency and a slow decay, determined by the inhomogeneous linewidth. If we could find a way to cause the individual Bloch vectors to realign then we could regain the macroscopic coherence and, by a reversal of the absorption process, cause the coherent emission of a pulse from the ensemble. This is the basis of the photon

\footnote{A $\pi/2$-pulse is not necessary to observe this behaviour, but is used for simplicity.}
echo techniques discussed in the next chapter.

2.6 Conclusions

In this chapter we have presented the basic theory needed to understand the ideas presented in the previous chapter with regards to quantum phenomena including coherence, entanglement and teleportation, as well as providing an introduction to atom-light interaction theory as a spring-board for the development of quantum memory theory presented in the next chapter.
Chapter 3

Quantum Memory Theory


3.1 A Memorable Introduction

In this chapter we will use the basic theory presented in the previous chapter to construct the workings of the two quantum memory protocols used in this thesis: Electromagnetically-induced transparency (EIT) and photon-echo-based memories. These are presented in Sec. 3.3 and Sec. 3.4 respectively. But first we will take a more in-depth look at a few of the quantum memory metrics discussed in the introduction.

3.2 Quantum Memory Metrics Take II

Here we will give a more solid mathematical basis for some of the quantum memory performance metrics presented in Sec. 1.4.

3.2.1 Efficiency, Fidelity and T-V Diagrams

Let us start by defining efficiency as the ratio of the number of photons in the output pulse compared to the input, i.e.,

$$\varepsilon_m = \frac{\langle \hat{E}_{\text{out}} \hat{E}_{\text{out}} \rangle}{\langle \hat{E}_{\text{in}} \hat{E}_{\text{in}} \rangle}.$$  (3.1)

Classically, a non-unit efficiency will simply lead to a loss of signal, which could be recovered by amplification. However, from a quantum-mechanical picture, loss in a system is equivalent to placing a beam-splitter in the optical path with transmissivity $\sqrt{\varepsilon_m}$. This is

![Figure 3.1: Loss from Quantum Systems](image)

Classically, a non-unit efficiency will simply lead to a loss of signal, which could be recovered by amplification. However, from a quantum-mechanical picture, loss in a system is equivalent to placing a beam-splitter in the optical path with transmissivity $\sqrt{\varepsilon_m}$. This is

Figure 3.1: Loss from Quantum Systems, equivalent to a beam-splitter (BS) coupling in vacuum noise $X_{\text{vac}}$. For quantum memories the transmissivity of the BS will be given by $\sqrt{\varepsilon_m}$, where $\varepsilon_m$ is the memory efficiency.
an issue for the recalled quantum state as a quanta of vacuum noise $X_{\text{vac}}$ will be introduced through the other entry port of the beam-splitter, degrading the output state. This is illustrated in Fig. 3.1. We also cannot deterministically amplify an arbitrary state without increasing the noise further [215]. In a quantum optics experiment losses may come from absorption or scatter off optics, or from inefficient detection (both the intrinsic detector efficiency and visibility for heterodyne/homodyne measurements). Except when performing noise measurements, in this thesis we will concern ourselves only with the memory efficiency. However, if we were to investigate the storage of, for example, squeezed states then these other sources of loss would be an issue.

The unconditional fidelity of a state is linked to the efficiency. This was illustrated in Fig. 1.7 for coherent states. Mathematically we can describe the fidelity of a memory by the overlap of the input and output states, i.e.,

$$F = |\langle \psi_{\text{out}} | \psi_{\text{in}} \rangle|^2.$$  \hfill (3.2)

Fidelity is a state-dependent measure. For instance, for a coherent state with mean photon number $|\alpha|^2 = n_p$, the fidelity of the output state as a function of the memory efficiency, assuming a linear memory process, will be [216]

$$F_{n_p}(\epsilon_m) = \frac{1}{1 + n_p (1 - \sqrt{\epsilon_m})^2}.$$  \hfill (3.3)

From this equation it can be seen that, even for different sized coherent states, the fidelity of a fixed-efficiency memory will be different, decreasing as $n_p$ increases. Even so, much theoretical work has been done investigating bounds with fidelity. The classical fidelity bound - the maximum fidelity for a classical measure-and-reproduce memory - for coherent states is given by [217, 218]

$$F_c(n_p) \leq \frac{1 + n_p}{2n_p + 1}.$$  \hfill (3.4)

For large mean photon numbers this tends to 50%, the classical limit discussed in the introduction. To break the no-cloning limit for coherent states we need a fidelity of greater than 68% [97]. Using Eq. 3.3 we can see that for a coherent state with $n_p = 10$ we require an efficiency of 60% and for $n_p = 100$ we require 87%.

T-V diagrams [219] were proposed as a state-independent method of measuring how well an output state resembles an input state, originally for quantum non-demolition measurements [98] followed by quantum teleportation [99]. We start here by assuming that a light field has a coherent information term and a much smaller fluctuating noise term, i.e.,

$$X_{\text{in/out}}^\pm = \mathcal{E}_{\text{in/out}}^\pm + \delta \hat{X}_{\text{in/out}}^\pm,$$  \hfill (3.5)

where the $\mathcal{E}_{\text{in}}^\pm$ etc. terms refer to the amplitude or phase quadrature of the light state. We can then define the variance of these fields $V^\pm$ using Eq. 2.9. Once we have these terms, we can define the signal to noise ratio of the state

$$\mathcal{R}_{\text{in/out}} = \frac{4\left(\mathcal{E}_{\text{in/out}}^\pm\right)^2}{V_{\text{in/out}}^\pm}.$$  \hfill (3.6)
Next we define the conditional variances and signal transfer coefficients to be, respectively,

\[ V_\pm = \frac{V_{\pm}^{\text{out}} - \langle \hat{X}_{\pm}^{\text{in}} \hat{X}_{\pm}^{\text{out}} \rangle^2}{V_{\pm}^{\text{in}}} \]  

(3.7)

\[ T_\pm = \frac{R_{\pm}^{\text{out}}}{R_{\pm}^{\text{in}}} \]  

(3.8)

Finally, we define \( V = \sqrt{V_+ V_-} \) and \( T = T^+ + T^- \). To produce the best possible copy of the input state (i.e., to be in the no-cloning regime) we need to have high efficiency without adding any noise. On a T-V diagram this region is represented by \( V < 1 \) (low noise) and \( T > 1 \) (high efficiency) [96], while a classical measure-and-reproduce memory cannot beat either of these bounds [99].

### 3.2.2 Bandwidths and Coherence Times

We will now have a very quick look into mathematically describing the bandwidth and coherence times of a memory. In this thesis we will mostly work with Gaussian pulses, i.e., pulses with a temporal profile

\[ \mathcal{E}(t) = \mathcal{E} \exp \left[ -\frac{t^2}{2\sigma_t^2} \right], \]  

(3.9)

where \( \sigma_t \) is the Gaussian temporal standard deviation. The full-width-half-maximum (FWHM) of such a pulse is given by \( 2\sigma_t \sqrt{2\ln 2} \). By performing a Fourier transform of such a pulse we can determine its bandwidth. As the Fourier transform of a Gaussian is another Gaussian, with a new standard deviation \( \sigma_\omega = 1/2\pi\sigma_t \), we normally define the pulse’s bandwidth \( B_p \) as its FWHM in frequency, i.e.,

\[ B_p = \frac{\sqrt{2\ln 2}}{\pi\sigma_t}. \]  

(3.10)

Of course, in practise we want to store all frequency components of a pulse. To store 99% of the pulse would require a memory bandwidth of \( 6\sigma_\omega \).

We found in Sec. 2.5.4 that a state will decay exponentially due to spontaneous emission and decoherence, i.e., \( \exp\left[ -t/T \right] \), where \( T = T_1 + T_y^* \). Other effects, such as diffusion of atoms out of the interaction volume defined by the light field, are expected to follow a Gaussian decay, i.e., \( \exp\left[ -t^2/T^2 \right] \). The total decay will be a combination of the two, i.e., \( \exp\left[ -t/T_e \right] \cdot \exp\left[ -t^2/T_y^2 \right] \). Normally one decay term is much greater than the other, for instance \( T_e \gg T_y^* \), and so the decay of a state can be described with just one of these terms. We then define the coherence time of the memory by this limiting time. We can also define the delay-bandwidth product (DBP) using these terms as \( B_p T \). Alternative versions of the delay-bandwidth product simply use the ratio of maximum storage time \( t_s \) and pulse length \( t_p \), or place limits on the efficiency for \( t_s \) (i.e., \( \epsilon_m(t < t_s) > 50\% \)).
3.3 Electromagnetically-Induced Transparency Theory

We now return to studying the mathematical behaviour of quantum memories, firstly looking at electromagnetically-induced transparency (EIT).

3.3.1 Slowing Light with EIT

The level scheme for EIT was presented in Fig. 1.8(a). It requires two ground-states $|1\rangle$ and $|2\rangle$, an excited-state $|3\rangle$ and two optical fields, the weak probe $\hat{E}_p$ on the $|1\rangle \rightarrow |3\rangle$ transition and a coupling field with Rabi frequency $\Omega_c$ on the $|2\rangle \rightarrow |3\rangle$ transition. Firstly, it should be noted that EIT is a coherent population trapping effect (for instance, see Ref. [134]), caused by destructive Fano interference between the light and atomic fields \[220\], and not due to dressed-state splitting caused by the coupling field (see Sec. 2.5.2). The latter effect results in what is known as Autler-Townes splitting \[221, 222\] and, while both lead to enhanced transmission for the probe, they should be considered separately\(^1\)

To analyse the situation mathematically, let us start from the time-evolution equations for the atomic operators and probe field (Eq. 2.70-2.75 and Eq. 2.80 respectively). If we start our analysis by assuming the system is in a steady state (i.e., $\partial \hat{\sigma} / \partial t = 0$) in a reference frame moving at the speed of light ($t' = t - z/c$), and Fourier transforming our operators using

$$\hat{O}(\omega) = \frac{1}{2\pi} \int_0^\infty dt \hat{O}(t) \text{Exp} \left[-i\omega t\right],$$

we obtain the following equation for the spatial evolution of $\hat{E}_p$

$$\frac{\partial}{\partial z} \hat{E}_p(z, \omega) = -\mathcal{T}(\omega) \hat{E}_p(z, \omega)$$

giving a solution

$$\hat{E}_p(L, \omega) = \hat{E}_p(0, \omega) \text{Exp} \left[-\mathcal{T}(\omega)L\right].$$

$\mathcal{T}(\omega)$ is the transfer function for the medium. The susceptibility of the EIT medium (i.e., where $\Delta = 0$) is given by $c \mathcal{T} / igN$ to be \[223\]

$$\chi(\omega) = g \left[ \frac{4\delta \left( |\Omega_c|^2 - 4\delta^2 \right) - 4\delta\gamma_0^2 + i \left( 8\delta^2 (\gamma + \gamma_0) + 2\gamma_0 \left( |\Omega_c|^2 + \gamma_0\gamma \right) \right)}{\left| |\Omega_c|^2 + (\gamma + \gamma_0 + 2i\delta) (\gamma_0 + 2i\delta) \right|^2} \right],$$

assuming that we are in the weak probe regime where $\langle \hat{\sigma}_{11} \rangle \approx 1$ and $g^2\langle \hat{\xi} \rangle^2 \approx 0$. The susceptibility is very important for EIT, as it tells us a lot about the dynamics of the system. For instance, the imaginary component determines the absorption of the system, via Eq. 3.13, and the real component determines the refractive index, via

$$\mathcal{R}(\omega) = 1 + \frac{1}{2} \text{Re} \left[ \chi(\omega) \right],$$

and therefore the dispersion. These are shown in Fig. 1.8(b) and (c) respectively. As can be seen, when there is a coupling field present we have enhanced transmission of the probe, accompanied by a sharp change in refractive index, indicating a large drop in probe group

\(^1\)For a more detailed discussion of the differences between the two, see Sec. 11.3.3.
velocity \( v_p \). To determine the group velocity mathematically we again make the adiabatic and weak-probe approximations, as well as ignore the decay terms for simplicity and assuming \( \Delta = \delta = 0 \), to obtain

\begin{align*}
\hat{\sigma}_{13} &= -i \frac{\partial}{\partial t} \hat{\sigma}_{12} \quad (3.16) \\
\hat{\sigma}_{12} &= g\hat{E}(z,t) \Omega_c(t) \quad (3.17) \\
\left( \frac{d}{dt} + c \frac{\partial}{\partial z} \right) \hat{E}(z,t) &= -g^2 N \frac{\partial}{\partial t} \left( \frac{\hat{E}(z,t)}{\Omega_c(t)} \right) \quad (3.18)
\end{align*}

If we maintain the assumption that the coupling field is constant, then Eq. 3.18 becomes

\begin{equation}
\left( \frac{d}{dt} + c \frac{\Omega_c^2}{1 + g^2 N} \frac{\partial}{\partial z} \right) \hat{E}(z,t) = 0, \quad (3.19)
\end{equation}

and therefore the group velocity will be

\begin{equation}
v_p = \frac{c \Omega_c^2}{1 + g^2 N} \approx \frac{c \Omega_c^2}{g^2 N} = \frac{\Omega_c^2 L}{\gamma \text{OD}}, \quad (3.20)
\end{equation}

where we have assumed that \( g^2 N/\Omega_c^2 \ll 1 \), with \( L \) being the length of the memory, and defining the optical depth (OD) of the ensemble to be

\begin{equation}
\text{OD} = \frac{g^2 NL}{c\gamma}. \quad (3.21)
\end{equation}

We can also determine the transmission bandwidth of the EIT memory from the linewidth of the transparency feature shown in Fig. 1.8(b) to be

\begin{equation}
B_{EIT} = \gamma_0 + \frac{\Omega_c^2}{\gamma \sqrt{\text{OD}}}. \quad (3.22)
\end{equation}

From these equations we can see that as we decrease the Rabi frequency (proportional to the intensity of the coupling field) we both reduce the group velocity, allowing longer pulses to fully fit within the memory length, and reduce the frequency range that will fit inside the memory. This was the issue mentioned in the introduction with regards to storing multiple pulses using EIT: to store many pulses we need the total temporal length \( \tau \propto 1/v_p \) to physically fit inside the memory while the bandwidth would have to scale as \( n/\tau \), where \( n \) is the number of temporal modes in the pulse train. Another way of viewing this is that the product \( B_{EIT}/v_p \propto \sqrt{\text{OD}} \) and therefore large ODs are needed to be able to store many modes.

### 3.3.2 EIT Polariton and Storing Light

We now discuss the mathematics behind the storage of light using EIT, as this requires a dynamic \( \Omega_c(t) \). From the above information one can imagine that if, after a pulse has entered the storage medium with enhanced transparency, and is travelling at a group velocity \( v_p \propto \Omega_c^2(t) \) then, if we were to reduce \( \Omega_c(t) \rightarrow 0 \), the pulse would cease to move and therefore be stored in the memory. If we return to Eq. 3.18, we can solve this without
removing the time dependence of $\Omega_c(t)$ by introducing the states [120]

$$\hat{\Psi}_\pm(z,t) = \cos \left[ \theta(t) \right] \hat{E}(z,t) \pm \sin \left[ \theta(t) \right] \sqrt{N} \hat{\sigma}_{12}(z,t)$$

(3.23)

$$\cos \left[ \theta(t) \right] = \frac{\Omega_c(t)}{\sqrt{\Omega_c^2(t) + g^2 N}}$$

(3.24)

$$\sin \left[ \theta(t) \right] = \frac{g N}{\sqrt{\Omega_c^2(t) + g^2 N}}.$$ 

(3.25)

We call these states polaritons (bosonic quasi-particles) and they obey

$$\left[ \frac{d}{dt} + c \cos^2 \left[ \theta(t) \right] \frac{\partial}{\partial z} \right] \hat{\Psi}_-(z,t) = 0$$

(3.26)

$$\hat{\Psi}_+(z,t) = 0.$$ 

(3.27)

The first of these is called the dark-state polariton, the second is called the bright-state polariton and is a trivial solution. Again, it can be seen that the group velocity $v_g = c \cos^2 \left[ \theta(t) \right]$. As $\Omega_c(t) \rightarrow 0$ we see firstly that $v_g \rightarrow 0$, along with the cosine term, while the sine term tends to one. The cosine and sine represent the light and atomic components of the polariton respectively and, therefore, reducing the coupling power causes information to be removed from the light field and stored in the atomic component of the polariton. Similarly, increasing the coupling power from zero leads to mapping of the information from the atoms to the light field. Polaritons are an elegant and useful way of representing a light-atom system, as we will see again when discussing GEM.

### 3.4 Gradient Echo Memory Theory

Here we present the theory of light storage using the gradient echo memory (GEM) scheme, starting by focusing on two-level systems.

#### 3.4.1 Echo Memories

The goal in all echo-memory-style technique is to recreate the initial macroscopic coherence of dipoles, discussed in Sec. 2.5.4, to time-reverse the absorption process and therefore create an echo of the stored information. The workings of this type of memory stem from the atomic polarisation (Eq. 2.100), which can be rewritten and related to the Maxwell equation for $\hat{E}$ as follows

$$\left( \frac{d}{dt} + c \frac{\partial}{\partial z} \right) \hat{E}_{13}(z,t) = igN \int d\Delta I(\Delta) \hat{\sigma}_{13}(z,t,\Delta),$$

(3.28)

where $I(\Delta)$ is the inhomogeneous broadening function. As was discussed in Sec. 2.5.4, the inhomogeneous broadening will lead to a loss of macroscopic coherence and therefore the integral will head to zero.

There are a few ways of realigning the dipoles and we will quickly run through these here. Firstly, the Hahn photon echo scheme. Here the ensemble is excited by a $\pi/2$-pulse. From Eq.s 2.88-2.90 we see that after the absorption the Bloch vector will be $\rho(0) = (0,1,0)$. Then, the different detunings will lead to rotation around the Bloch
§3.4 Gradient Echo Memory Theory

sphere meaning that after time $\tau$ the Bloch vector will be

$$\rho(\tau, \Delta) = (\sin[\Delta \tau], \cos[\Delta \tau], 0),$$

(ignoring decoherence effects). Now, if we were to apply a $\pi$-pulse $(u, v, 0) \rightarrow (u, -v, 0)$. Evolution after this pulse will be described by

$$\rho(t > \tau, \Delta) = (\sin[\pi - \Delta \tau + \Delta(t - \tau)], \cos[\pi - \Delta \tau + \Delta(t - \tau)], 0),$$

and therefore, after a total time $2\tau$, the system will be back in its initial state and an echo will be emitted. Another method of realigning the dipoles is using the atomic frequency comb (AFC) scheme. Here each dipole is separated in frequency by $\Delta$ and, therefore, the system after initial excitation will be given by

$$\rho_n(t, \Delta) = (\sin[n \Delta t], \cos[n \Delta t], 0),$$

where $\rho_n$ is the Bloch vector for the $n$th dipole. Now it can be seen that, after a time $t_s = 2\pi/\Delta$ all dipoles will again realign. Finally, there is the controlled reversible inhomogeneous broadening (CRIB)/gradient echo memory (GEM) method whereby, at time $\tau$, we reverse the detunings of all dipoles so that the Bloch vector will be

$$\rho(t > \tau, \Delta) = (\sin[-\Delta(t - \tau) + \Delta t], \cos[-\Delta(t - \tau) + \Delta t], 0).$$

Again, it can be seen that after time $2\tau$ the dipoles will realign.

3.4.2 Two-Level Gradient Echo Memory

We will discuss the two-level GEM scheme before extending to three-levels. As mentioned in the introduction, GEM relies on a linear frequency gradient $\eta$ placed along the length of the memory $L$, giving a memory bandwidth

$$\mathcal{B}_{GEM} = \eta L.$$  

The frequency-detuning for an atom at position $z$ is determined by the gradient, i.e., $\Delta(t) = \eta(t)z$, and therefore a reversal of the gradient $\eta \rightarrow -\eta$ will lead to a reversal of $\Delta$ about the centre of the ensemble and the rephasing discussed in the previous section. To mathematically analyse the two-level GEM behaviour we use Eqs 2.74 and 2.80 and relabel $\hat{\sigma}_{13} \rightarrow \hat{\sigma}_{12}$ to give

$$\frac{\partial}{\partial t} \hat{\sigma}_{12} = -\left(\gamma/2 + i\eta(t)z\right) \hat{\sigma}_{12} + ig \hat{E}(\hat{\sigma}_{11} - \hat{\sigma}_{22})$$

and

$$\left(\frac{d}{dt} + c \frac{\partial}{\partial z}\right) \hat{\mathcal{E}} = igN\hat{\sigma}_{12}. $$

In the weak probe regime we can assume $\langle \hat{\sigma}_{11} \rangle \approx 1$ and, moving into a reference frame moving at the speed of light while Fourier transforming the operators as per Eq. 3.11, we

\footnote{We continue to assume excitation by a $\pi/2$-pulse here for ease. However, AFC and GEM both work in the weak probe regime where $A \ll \pi/2$.}
obtain the following equation for the propagation of $\hat{E}$

$$\frac{\partial}{\partial z} \hat{E}(z, \omega) = -\frac{g^2 N}{c} \hat{E}(z, \omega) \frac{1}{\gamma - i (\eta z + \omega)}.$$ (3.36)

This can be solved to give

$$\hat{E}(z, \omega) = \hat{E}(-z_o, \omega) \mathcal{J}(z, \omega)$$ (3.37)

$$\mathcal{J}(z, \omega) = \left[ \frac{\gamma - i (\eta z + \omega)}{\gamma - i (\omega - \eta z_o)} \right]^{\text{OD}}.$$ (3.38)

where $z_o = -L/2$ and the two-level broaden Optical depth is given by

$$\text{OD}_{2l} = \frac{g^2 N}{c \eta}.$$ (3.39)

Eq.s 3.37-3.38 contain a discontinuity for $\gamma = 0$. A correction for this, found in Ref. [186], gives $\hat{E}$ at the end of the memory (now moving to a memory from $z = 0 \rightarrow L$ for ease of notation) as

$$\hat{E}(L, \omega) = \hat{E}(0, \omega) \text{Exp} \left[ -\pi \text{OD}_{2l} \text{Hs} \left[ \frac{\omega - \omega_o}{\eta L} \right] \right],$$ (3.40)

where Hs[\omega] is the heavy-side step function (1 for $-0.5 \leq \omega \leq 0.5$ and 0 otherwise) and $\omega_o$ is the centre frequency. This is plotted in Fig. 3.2 for a range of bandwidths. As can be seen, as the gradient is increased the bandwidth increases (as per Eq. 3.33) while the absorption decreases (as per Eq. 3.40). If the pulse bandwidth fits within the memory bandwidth we can ignore Hs[\omega] and see that the shape of the pulse will be preserved, with an amount Exp $[-\pi \text{OD}_{2l}]$ leaking through the memory. If the pulse bandwidth does not fit well within the memory bandwidth there will be large dispersion at the edges of the memory where the absorption changes sharply, as we saw with EIT earlier. We can define the efficiency of the storage process (remembering efficiency $\propto |\hat{E}|^2$) to be

$$\varepsilon_s = 1 - \text{Exp} [-2\pi \text{OD}_{2l}].$$ (3.41)
As GEM is a time-reversal process, the recall efficiency will be the same as the storage efficiency, giving a total efficiency of

$$\varepsilon_t = (1 - \text{Exp}[-2\pi\text{ODl}])^2.$$  \hspace{1cm} (3.42)

For this reason, GEM can be conceptually represented as a beam-splitter process \cite{186}, with transmissivity $\text{Exp}[-\pi\text{ODl}]$ and can therefore be used for interference phenomena, as demonstrated experimentally by G. T. Campbell et al. in 2012 \cite{10}. It can be seen that as $\text{ODl} > 1$ the total efficiency will approach unity, not limited by reabsorption issues in the forwards direction due to the monotonicity of the gradient.

There is also a phase shift associated with GEM storage, stemming from Eq. 3.38 with \cite{224}

$$\Delta \phi(t) = \text{ODl} \ln \left( \text{ODl} + \frac{B}{2}t \right),$$  \hspace{1cm} (3.43)

with $t$ being the storage time, assuming the pulse entered the memory at $t = 0$. This phase shift can be split into three terms, a constant phase shift (first term in the logarithm), a constant frequency shift and a frequency chirp. The latter two being due to the linear and nonlinear components of the second term in the logarithm. The frequency shift is proportional to $1/B\tau$ and, as long as the storage time is much longer than the inverse of the memory bandwidth, and the OD is not too high, the frequency chirp will be negligible. This is the case in the experimental work presented in this thesis.

As GEM is a frequency-encoding memory we will not encounter the same pulse-bandwidth/length trade-off that was shown to exist for EIT. However, with the optical depth being inversely proportional to the bandwidth, increasing the bandwidth will lower the efficiency of the memory. It has been found that the multi-mode capacity of GEM scales with $\text{ODl}^{1/2}$ \cite{140}, a factor of $\sqrt{\text{ODl}}$ better than EIT.

### 3.4.3 GEM Polariton

When discussing EIT we saw that defining a polariton, a mixture of optical and atomic modes, helped with determining the properties of the memory. We can create a similar polariton with GEM. To do so, however, involves taking a plane-wave decomposition of the optical and atomic fields by using a spatial Fourier transform of Eq. 3.34 and 3.35 to obtain \cite{225}

$$\frac{\partial}{\partial t} \hat{\sigma}_{12} = -i\eta z \hat{\sigma}_{12} + ig \hat{\mathcal{E}},$$  \hspace{1cm} (3.44)

$$\hat{\mathcal{E}}(k,t) = \frac{gN}{ck} \hat{\sigma}_{12},$$  \hspace{1cm} (3.45)

where we have ignored decoherence terms. From these relations we can obtain the following equation for the atomic polarisation

$$\left( \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial k} - \frac{ig^2N}{ck} \right) \hat{\sigma}_{12} = 0.$$  \hspace{1cm} (3.46)

This, plus a corresponding equation for $\hat{\mathcal{E}}$, can be solved by creating polariton states of the form

$$\hat{\Psi}_\pm(k,t) = k \hat{\mathcal{E}}(k,t) \pm \frac{gN}{c} \hat{\sigma}_{12}(k,t),$$  \hspace{1cm} (3.47)
Figure 3.3: GEM Polaritons. (a) Plot of atomic polarisation (see legend) as a function of position inside memory ($-3 \mu m < z < 3 \mu m$) during storage. Insets show (i) temporal profile of pulse, (ii) cross-section of atomic excitation (which is the Fourier transform of the pulse). Adapted from Ref. [225]. (b) Absolute value of polariton for “normal” first-in-last-out storage. (i) Temporal profile on input pulses (blue) and (ii) output pulses (red). Adapted from Ref. [9]. (c) Pulse resequencing using Λ-GEM. (i) Shows normalised coupling field power (left hand scale shaded) and size of gradient (right hand scale, green dashed line), (ii) shows polariton evolution (see scale on right), and (iii) shows temporal profile of input pulses and echoes. Adapted from Ref. [9].
solving the equations

\[ \hat{\Psi}_- = 0 \] \hspace{1cm} (3.48)

\[ \left( \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial k} - \frac{ig^2 N}{c k} \right) \hat{\Psi}_+ = 0, \] \hspace{1cm} (3.49)

where the solution for the negative state can be seen from Eq. 3.45.

The first piece of information we can take from the polaritonic description is the frequency-encoding nature of GEM storage, gleaned from the fact that the polariton is in \( k \)-space rather than \( z \)-space. This, in turn, means that the pulse is stored as its spatial Fourier transform along the memory. This is shown in Fig. 3.3(a). The polariton propagates in \( k \)-space\(^3\), starting at \( k = 0 \), at a rate determined by \( \eta \). As the polariton travels to higher values of \( k \), the light field component decreases due to dephasing of the dipoles discussed in Sec. 2.5.4. Again it can be seen that, if we reverse \( \eta \), we will reverse the direction of propagation of the polariton in \( k \)-space, leading to re-emission at \( k = 0 \) (corresponding to realignment of the dipoles).

If we recall with a steeper (sharper) gradient the polariton will pass through \( k = 0 \) for a shorter (longer) time, leading to pulse compression (stretching). There will be a corresponding change in optical depth and efficiency via Eq.s 3.39 and 3.42. If we were to turn off the gradient after the pulse had entered the memory, then the polariton would stop evolving in \( k \)-space. This is illustrated by the “Hold” labels in Fig. 3.3(c). It can also be seen from the evolution of the polariton in Fig. 3.3(b) that GEM acts as a first-in-last-out memory, reflecting its time-reversal nature. With non-unit retrieval efficiencies, left-over energy remains in the memory and this (much smaller) polariton can be reversed again to recall in a first-in-first-out manner. However, if we were to have a “knob” on the memory that would allow us to suppress emission at \( k = 0 \) on the first pass, then we would be able to achieve high efficiency first-in-first-out recall, as well as the ability to resequence pulses.

### 3.4.4 Stepping Up a Level - Three-Level GEM

There are a few reasons for moving from a two-level version of GEM to a three-level, or \( \Lambda \), version. The first of these is that the excited-state decay rate \( \gamma \) depends on the coupling strength between the two levels, as does the optical depths and therefore efficiency. This means that we can either have long storage times, or high efficiencies, but not both. Secondly, with a three-level system we have an extra “knob” to play with (see below).

Let us start with a \( \Lambda \)-system similar to that for EIT, except with a large excited-state one-photon detuning \( \Delta \), shown in Fig. 3.4(a). We will again use Eq.s 2.70-2.75 and Eq. 2.80 while making the adiabatic approximation (\( \partial \delta_{13} / \partial t \ll \gamma \)), which is equivalent to \( 1/T \ll \gamma \), where \( T \) is the fastest time-scale of the system, and \( \Delta \gg \gamma \). For an ensemble memory, these conditions become \( \text{OD}_{2l} \gg 1/T \gamma \) and \( \text{OD}_{2l} \ll \Delta / \gamma \), leading to \( \Omega_c / \Delta \ll 1 \) [214]. With these assumptions, while still assuming a weak probe and moving to a frame travelling at the speed of light, we find

\[ \frac{\partial}{\partial t} \hat{\sigma}_{12} = - \left( \gamma_o + i \eta(t) \frac{\Omega_c^2}{\Delta} \right) \hat{\sigma}_{12} - i \frac{g \Omega_c \hat{E}}{\Delta} \] \hspace{1cm} (3.50)

\(^3\)Technically a new polariton, with its own commutation relations, is created at each new time interval \( \delta t \) [225].
Figure 3.4: Two-Level, Three-Level Equivalence. (a) Three-level atom used for Λ-GEM, with: probe field $\hat{E}$; atom-light coupling strength $g$; one-photon detuning $\Delta$; two-photon detuning $\delta$ depending on the position of the atom in the ensemble ($z$) and the size of the gradient $\eta$; coupling field with Rabi frequency $\Omega_c$; and decoherence rate $\gamma_o$. (b) Equivalent two-level atom, with: excited-state decay rate $\gamma_o$; detuning $\delta$; and modified coupling strength $g' = (\Omega_c/\Delta)g$.

\[
\frac{\partial}{\partial z} \hat{E} = \frac{ig}{c} N \frac{\Omega_c}{\Delta} \hat{\sigma}_{12} + \frac{ig^2 N}{c} \hat{\sigma}_{12}, \tag{3.51}
\]

where the two-photon detuning $\delta(z, t) = \eta(t)z$. The $\Omega_{c}^{2}/\Delta$ term in Eq. 3.50 is due the an ac Stark shift from the coupling field. If we were to choose a coupling field frequency to compensate for this term, and move into a frame where the speed of light in the medium is normalised by $ig^2 N \hat{E}/c$ then these equations become

\[
\begin{align*}
\frac{\partial}{\partial t} \hat{\sigma}_{12} &= -\left(\gamma_o + i\delta(z, t)\right) \hat{\sigma}_{12} - i\frac{g \Omega_c \hat{E}}{\Delta} \\
\frac{\partial}{\partial z} \hat{E} &= \frac{ig}{c} N \frac{\Omega_c}{\Delta} \hat{\sigma}_{12}, \tag{3.53}
\end{align*}
\]

formally equivalent to Eqs 3.34 and 3.35 if $g \to g \Omega_c/\Delta$ and $\gamma \to \gamma_o$. This is shown in Fig. 3.4(b). The new OD for the system will then be

\[
\text{OD}_{3l} = \frac{g^2 N}{c \eta} \left(\frac{\Omega_c}{\Delta}\right)^2. \tag{3.54}
\]

With Λ-GEM we can therefore alter the OD during the storage process by changing the coupling field power. If we were to turn off the coupling field completely we would have an OD of zero and therefore no emission would occur, even if the polariton passed through $k = 0$. This is the extra “knob” we use to allow for pulse sequencing and beam-splitting operations in Λ-GEM (see Fig. 3.3(c)) [9]. By altering the coupling field frequency we can also change the frequency of the output pulse, allowing us to compensate for the constant frequency shift discussed in Sec. 3.4.2.

Another benefit of moving from two to three levels is that the decay is now limited by $\gamma_o \ll \gamma$. Experimentally this opens up a range of readily available storage medium options, with large numbers of addressable atoms at convenient laser wavelengths, that
would otherwise be unusable due to their fast excited-state decay rates (for instance, see App. A). One disadvantage of using three levels is that, as $\Omega_c/\Delta \ll 1$, $OD_3l \ll OD_2l$ assuming the same coupling strength.

3.5 Conclusions

In this chapter we have presented the theory behind the two memory techniques we will use in this thesis: electromagnetically-induced transparency; and the three-level gradient echo memory. We showed how an EIT-based memory stores light by reducing its group velocity to zero and back again. GEM, in comparison, is based around reversing the initial absorption process. We discussed some of the properties of $\Lambda$-GEM that make it a promising quantum memory candidate: high efficiencies for large ODs; the coherence time being determined by the ground-state decoherence rate; the frequency-encoding nature of the storage process that allows for multi-mode storage; as well as the ability to manipulate the stored information using the gradient and coupling field.
Aside
Digital Locking
A Digital Locking System for use on Quantum Optics Experiments

4.1 Introduction

Examples of the use of digital control algorithms, rather than analog electronics, to lock the frequency of single [226] and multiple [227] lasers date back to 1998. This progression has been motivated by the fact that a digital system can offer distinct advantages over its analog counterpart. These include cost savings (as features can be programmed rather than purchased), as well as space savings (as most of the features can be included in the code they only take up as much space as the hardware needed to run it). Mostly, however, the advantage of digital locking over analog systems lies in their flexibility: the ability to change the functionality of the system by altering the control code rather than having to purchase new equipment or physically modify old equipment. This makes a digital system easily reproducible, as well as having the ability to add complex logic that would be difficult to include with analog circuits alone (for examples see Ref. [228]). Using the added power of digital control has lead many groups to develop novel methods for frequency stabilising lasers (see, for example, Ref.s [229, 230, 231, 232, 233, 234]).

Complex logic and functionality are not entirely the domain of digital technology, with both G. J. Koch [235] and S. K. Lee et al. [236] having developed methods using analog electronics to automatically lock lasers to gas absorption lines and a Fabry-Perot cavity respectively, with S K. Lee’s group being able to unlock and relock the cavity at will. However, with digital control this technology can be improved with, for instance, L. Dong et al. [237] using LabVIEW and an alternate method of scanning and noting of frequencies to allow their system to lock to any selected gas absorption line.

Quantum optics is one branch of science that can benefit from digitisation and computer control. One component common to many quantum optics experiments is the cavity. For instance, mode-cleaner-style ring cavities to select only one spatial mode of an input field, or Fabry-Perot cavities to select a specific frequency. Cavities are not just used for filtering, however, and can also be an integral part of an experiment. For instance, cavities containing non-linear optical crystals (known as optical parametric oscillators - OPOs) are used to produce entanglement [26] and squeezing [238, 239], two powerful quantum resources. These effects are extremely sensitive to noise or loss in the system. Therefore
the properties of the system used to lock an OPO must be stringently monitored. As well as the use of cavities, another issue of concern is the relative phase between various fields. For instance, homodyne detection using a local oscillator [240, 241].

Quantum optics experiments can require many locking loops and sophisticated data acquisition, and therefore these experimental systems can become highly complex. As an example, in 2003 W. Bowen et al. demonstrated the teleportation of the quadrature amplitudes of two light fields [242]. To achieve this the experiment required a frequency doubling cavity, a high finesse ring cavity used to seed a pair of OPOs, as well as two homodyne detector set-ups requiring phase control.

More recently, another example of the complexity of quantum optics experiments is the work carried out by M. Yukawa et al. in 2008 to generate four-mode cluster states for use in quantum computing [243]. This experiment required a frequency doubler, which was used to pump four OPOs, and four homodyne set-ups were required for the measurements giving a total of nine independent locking loops, each requiring long-term stability. From these examples it can be seen that a digital control system for quantum optics experiments would need to be extendible to many locks, flexible in the style of locks it can control, as well as be as effective, or better, than current analog controllers in reducing noise.

As more complex experiments are developed to investigate further into the quantum realm, the practicality of digital control will become more apparent. This is because digital control allows for integration of all relevant information about the system to be accessed, and controlled, from one location. This, in turn, allows the system to take into account the sequential nature of the locks comprising the experiment. Also, the complete system information can be used for conditional data acquisition, to ensure that data is only recorded when the system is in the desired state. As well as this, digital interfaces for triggering experimental components are a fast, user-friendly method to control timings for experiments (these will be discussed more in the next chapter).

In this chapter we present a code designed for quantum optics experiments using field programmable gate arrays (FPGAs) programmed with National Instruments (NI) LabVIEW software. This code is freely available to be downloaded and to be modified, as both a pedagogical tool, and to allow individual users to tailor it to their specific needs (see App. B).

The remainder of this chapter is structured as follows: an introduction to control theory is presented in Sec. 4.2, followed by an overview of the hardware (Sec. 4.3.1) and software features (Sec. 4.3.2) of the code. In Sec. 4.4 we show how the inbuilt locking analysis tools, including a white-noise network analyser, can be used to help optimise individual locks, and verify the long term stability of the digital system against an analog proportional-integral controller. Finally, in Sec. 4.5, we present an example of the benefits of digital locking for quantum optics by applying the code to a specific experiment used to characterise optical Schrödinger cat states.

Most of the work presented in this chapter has been published in the journal articles:


and its applications to reconstructing non-Gaussian states, Physical Review A 84, 050302(R) (2011).

4.2 Control Theory

This section provides an overview of the relevant control theory, for a more in depth description see, for example, Ref. [244]. Figure 4.1(a) shows the basics of a closed-loop feedback system. A plant - the system to be controlled - having a transfer function (i.e., frequency response) $G(\omega)$ produces some form of error signal $Y(\omega)$. This is passed to the controller, having a transfer function $H(\omega)$, used to suppress the noise added to the system $X(\omega)$.

The system, or closed-loop, transfer function $T(\omega)$ can be determined from $G(\omega)$ and $H(\omega)$ by using the definition of a transfer function, namely

$$ T(\omega) = \frac{Y(\omega)}{X(\omega)} $$

(4.1)

From Fig. 4.1(a) and the fact that, for transfer functions, components in series multiply, we can see that the actuating signal $E(\omega)$ depends on the controlling signal and the input noise via

$$ E(\omega) = X(\omega) - R(\omega) = X(\omega) - H(\omega) \cdot Y(\omega), $$

(4.2)

and also that the output signal depends on the actuating signal and the plant transfer function via

$$ Y(\omega) = E(\omega) \cdot G(\omega). $$

(4.3)

Substituting Eq. 4.2 into Eq. 4.3 gives

$$ Y(\omega) = \frac{X(\omega) - H(\omega) \cdot Y(\omega)}{1 + G(\omega) \cdot H(\omega)} \cdot G(\omega) $$

$$ [1 + G(\omega) \cdot H(\omega)] \cdot Y(\omega) = X(\omega) \cdot G(\omega) $$

$$ \frac{Y(\omega)}{X(\omega)} = \frac{G(\omega)}{1 + G(\omega) \cdot H(\omega)}, $$

(4.4)

which is the definition of $T(\omega)$ given by Eq. 4.1.

For quantum optics experiments, as with all locking servo applications, we are interested in three properties of the controller: how well it can suppress noise at low frequencies; what frequency range it can achieve this over; and how stable the lock is. These properties can be gleaned from the equation above. Firstly, the larger $H(\omega)$ the smaller $T(\omega)$ will be. This means that the noise entering the system $E(\omega)$ is being suppressed, with more suppression occurring as $H(\omega)$ increases, and the amount of suppression determined by the difference between $T(\omega)$ and $G(\omega)$. Conversely, as $H(\omega) \to 0$, $T(\omega) \to G(\omega)$ and therefore there will be no noise suppression. The frequencies for which noise suppression occurs define the bandwidth of the system - the range of frequency for which the system can compensate for the input noise. This is normally defined mathematically as the point where $|GH(\omega_B)|$, known as the open-loop transfer function, equals one. This is also known as the unity gain frequency $\omega_B$. Finally, as $G(\omega) \cdot H(\omega)$ approaches $-1$, $T(\omega) \to \infty$ and therefore the system becomes unstable. The stability of the system is usually measured
Figure 4.1: Feedback Control System. (a) Block diagram of a basic feedback control system. Here $G(\omega)$ and $H(\omega)$ are the transfer functions of the plant and controller respectively, $X(\omega)$ is the input noise, $Y(\omega)$ is the output of the system, $R(\omega)$ is the feedback signal, and $E(\omega)$ is the actuating error signal. Also shown is the placement of a network analyser, with $X'(\omega)$ the extra noise added to the system in the form of a swept sine wave (Sine) or white noise (WN). (b) Experimental set-up for measuring $G(\omega)$ for a ring cavity with: Cav - ring cavity; E - electro-optic modulator; BS - 50:50 beam-splitter; PZT - piezo-electric transducer; D - detector; and HV AMP - high voltage amplifier. (c) Bode plot of the transfer function of a ring cavity for (i) a network analyser (model MS4630B from Anritsu) and (ii) a white noise generator included in the code.
Figure 4.2: Controller Properties. (a) The Bode plot of (i) $G(\omega)$ and (ii) $T(\omega)$ for the system shown in Fig. 4.1 and (b) the corresponding open-loop transfer function with: $\omega_B$ - system bandwidth; $\omega_G$ - frequency satisfying $\angle GH(\omega_G) = \pi$; $\omega_R$ - system resonance frequency; NS - low-frequency noise suppression; PM - phase margin; GM - gain margin.
using the Nyquist stability criterion \[ [245] \] via the phase margin (PM) defined by

\[
PM = \pi + \angle GH(\omega_B).
\]  

That is, the clearance of the phase of \( GH(\omega_B) \) from \(-\pi\). Conversely, if we define \( \omega_G \) to be the frequency at which \( \angle GH(\omega_G) = \pi \), then we can define another stability criterion, the gain margin (GM), as

\[
GM = \frac{1}{|GH(\omega_G)|}.
\]  

These properties are illustrated in Fig. 4.2.

Ideally we would like to make the bandwidth, as well as the size of \( H(\omega) \), as large as possible for maximum noise suppression over all frequencies. However, most systems have some resonance, at frequency \( \omega_R \), above which it is not possible to suppress noise. Delay within in a feedback system will also limit the frequency response of a controller. The frequency response of each system to be locked should, therefore, be investigated to allow for optimal locking. One way of investigating the system transfer function is to use a network analyser to add known noise \( X'(\omega) \gg X(\omega) \) to the system and measuring the frequency response. For commercial network analysers (such as the Anritsu model MS4630B used here) this extra noise is usually in the form of a swept sine wave, though white noise - that is noise with the same amplitude at all frequencies - can also be used. The placement of a network analyser to measure the system transfer function is shown in Fig. 4.1(a). We investigated the unlocked transfer function for a ring cavity, used for mode-cleaning applications, where we drove a high-voltage (HV) amplifier that, in turn, drove a piezo-electric transducer (PZT) on the back mirror of the cavity (see figure) to change the cavity’s length. To take data for an unlocked system it was necessary to build a Michelson interferometer at the rear of our cavity, using the back cavity mirror as one of the Michelson arms. This is shown in Fig. 4.1(b).

A Bode plot shows both the magnitude \( (10 \cdot \log_{10} |T(\omega)|) \) and phase delay \( (\angle |T(\omega)|) \). Examples of Bode plots for the ring cavity system are shown in Fig. 4.1(c). Note the flatness of \( G(\omega) \) at frequencies below 10 kHz and compare this to the suppression of noise shown in Fig. 4.2(a). It can also be seen that the phase delay \( \Delta \phi \) (which is determined by the time delay \( \Delta t \) via \( \Delta \phi = \omega \cdot \Delta t \)) reaches \(-\pi\) at approximately \( \omega_R = 50 \) kHz. Once \( \Delta \phi > -\pi \) feedback control will no longer work as the controlling and output signals will be \( \pi \) rad (or 180°) out of phase and therefore when the feedback signal should be positive it will be negative and vica-versa, causing the system to resonate. This limits the controller bandwidth to some frequency below the plant resonance.

In practice we must take into account the fact that \( G(\omega) \) is composed of the responses of the cavity, the photo-diode detector, the analog-to-digital converter (ADC), the digital demodulation and filtering, as well as the digital-to-analog converter (DAC), the HV amplifier and the PZT, not to mention the time delay due to the cables and through the digital controller. However, apart from the PZT-cavity system \( G'(\omega) \), the responses of these components in the frequency range we are interested in (< 100 kHz) should be approximately flat and therefore only add constant gain to the system (i.e. \( G(\omega) \approx K \cdot G'(\omega) \) where \( K \) is a constant).

Another measure of the efficacy of a lock is the deviation of the error signal from its desired value over time. This is usually measured using the root mean squared (RMS)
method, which is expressed mathematically as
\[
\text{RMS} = \sqrt{\frac{\sum_{i=1}^{n} (S_i - S_{\text{des}})^2}{n}}.
\]
(4.7)
Here \(S_i\) is the value of the error signal at a particular time \(i\), \(S_{\text{des}}\) is the desired value of the error signal (usually zero), and \(n\) is the number of points the RMS was measured over.

### 4.3 The Digital Control System

The digital control system presented here is based on the Pound-Drever-Hall (PDH) locking technique \([246, 247]\), presented in App. C. This technique uses phase modulation of the laser frequency to produce an anti-symmetric error signal (see Fig. 4.3(a) inset) that is then fed back to the actuating mechanism to keep the system locked to a desired position. The code allows for automatic, sequential locking, as well as analysis of the locks comprising the experiment to facilitate optimizing procedures and these will all be discussed here. More details on the code can be found in App. B.

#### 4.3.1 The Hardware

Figure 4.3(a) shows the basic setup for a single lock (in this case a mode-cleaner cavity). A frequency generation (FG) signal at 80 MHz (NI PXI-5404) is split in two (MiniCircuits splitter ZSC-2-1), with one output being sent to a clock generator board (CGB - AD9959 from Analog Devices) and one to clock a high-speed analog-to-digital convertor (HS ADC - AD9460BSVZ-80 from Analog Devices at 80 M Samples s\(^{-1}\)). The output from the CGB is controlled via an FPGA card (NI PXI-7852R), and sends a sine wave of a desired frequency to some form of modulation device - here an electro-optic modulator (EOM - Newport free-space model 4002-M) - to modulate the phase of the input field.

Once the modulated signal passes through the cavity, part of it is reflected to a photo-detector that has two output ports - one ac-coupled and one dc-coupled. The ac component, containing the modulation signal, is sent to the HS ADC. The dc component, if there is one, is sent to a low-speed analog input (AIP \(\approx 1\) M Samples s\(^{-1}\)) of the FPGA. The controller algorithm is discussed in Sec. 4.3.2, with the output signal sent through a low-speed analog output (AOP), via a high-voltage (HV) amplifier to the piezo-electric transducer (PZT) controlling the round-trip cavity length.

This system has been designed with scalability in mind to cater to experiments with large numbers of locks. Using the same principle as above, up to eight locks can be implemented using two FPGAs, two CGs, one FG, and eight HS ADCs. This is achieved by splitting the 80 MHz FG signal as many times as necessary to send it to clock all HS ADCs and one signal to each CGB (both controlled by one of the FPGAs). Also, the 16-bit signals from pairs of HS ADCs are combined into one 32-bit signal before being sent to an FPGA. This is because the FPGA cards used here have only two 40-bit digital input/output (DIO) connections, and in this way four locks can be implemented on each FPGA. Each FPGA also has eight low-speed AIPs and eight low-speed AOPs, and another 16 DIO lines all located on a mixed IO connector. These are more than enough for the four dc input and four controller output signals required per FPGA, as well as extra digital lines to control the CGBs.
Figure 4.3: Digital Locking System Set-Up. (a) Digital locking hardware set-up for a mode-cleaner ring cavity (Cav) with: MSPS - M Samples s$^{-1}$; PXI - PXI chassis; FPGA - field programmable gate array card; Freq. Gen. - frequency generator; RT - real time controller; AIP - analog input to FPGA (sampling frequency < 1 MSPS); AOP - analog output from FPGA (sampling frequency < 1 MSPS); HS ADC - high-speed analog-to-digital converter (80 MSPS); D - detector; Clock Gen. - digital clock generation board; fmod - modulation frequency for lock; E - electro-optic modulator; PZT - piezo-electric transducer; HV AMP - high-voltage amplifier. Inset shows the different signals that are used for locking (DC - dc reflection signal; Error - Error signal; Scan - scan function) as a function of PZT position. (b) Depiction of LabVIEW code used to program FPGA cards. CIC - cascaded-integrator-comb filter; PII controller - proportional, integral, double integral controller; Scan T - scan threshold; Lock T - lock threshold; T - true; F- false.
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Figure 4.4: Digital Locking Racks. (a) ADC rack with: Clock - 80 MHz input; inputs - IPx.1 and IPx.2; and combined outputs OPx that is then sent to a DIO terminal on the PXI chassis. (b) Clock rack with: CLK-IN - 80 MHz input; CTRL - FPGA control input; and DAC-x - frequency outputs.

Using this method up to 16 locks can be implemented on one PXI chassis (here a NI PXI-1042Q). We took the set-up one step further by having rack-mountable boxes designed for both the ADCs and CGBs. These are shown in Fig. 4.4

4.3.2 The Software

The code was developed using LabVIEW 2010 (32 bit). An overview of the FPGA code is shown in Fig. 4.3(b) and consists of two loops: a high-speed loop running at 80 M Samples s\(^{-1}\); and a low-speed loop running at approximately 750 k Samples s\(^{-1}\) (limited by the low-speed AIP/AOP). In the high speed loop the modulated signal from the HS ADC is demodulated with a cosine function generated using a look-up table at the same frequency as the modulation (i.e. \(f_{mod}\)). This passes through a cascaded-integrator-comb (CIC) filter to produce an error signal that is then sent to the low-speed loop.

There are two main components to the low-speed loop: a lock mode (consisting of a P, I and I\(^2\) - PII - controller, discussed in more detail in Sec. 4.3.3); and a scan mode (consisting of a saw tooth scan function). Once directed by the user to lock, the code will move from scan mode to lock mode once the dc signal drops below the user defined scan threshold (determined as a fraction of the peak size measured by the code, see Fig. 4.3(b)). The code will then remain in lock mode unless either the user disengages locking or the dc input rises above the lock threshold (set higher than the scan threshold). In both cases the system will revert to scan mode. In the latter case, however, the system will move back to lock mode once the dc input again drops below the lock threshold, unless the Do Not Relock option is engaged\(^1\). If the lock has no dc component (i.e. a phase-style lock) the absolute value of the error signal can be used to set the thresholds instead. Alternatively, if desired, both thresholds can be set to zero and the user can manually lock the system as they would with an analog controller.

The low-speed loop includes a white noise generator, and this is used to measure the system transfer function as described in Sec. 4.2.

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\(^1\)Useful for situations where two peaks are of approximately equal height, as we will discover in the next chapter.
A master code runs on the odd-numbered FPGAs and a slave code runs on the even-numbered FPGAs. The codes differ in that the master code contains an internal PXI trigger for the FG (only on the first FPGA) and also runs the CGB programming, ensuring that the $f_{\text{mod}}$ used for demodulation in the high-speed loop is the same frequency as that sent to drive the modulator, and with a fixed phase relation.

A LabVIEW Real Time (RT) code acts as the interface between the FPGAs and the user, as well as to allow communication between FPGAs, with all FPGA cards being controlled from the one RT code. The RT code contains logic that does not need to be included in the FPGA code, for instance a Beam Blocked function and a lock Time-Out function. If the beam is blocked before the lock while the code is running, then the minimum value for the dc signal will be set to zero. When it becomes unblocked the lock will more than likely no longer be on resonance and therefore the system will start scanning as discussed above. However, as the minimum registered value is now zero for both the scan and lock thresholds, the dc value at which the code will move into lock mode will be much lower than before meaning that, in most cases, the system cannot relock. To overcome this problem, if the user knows that they are going to block the light before the cavity, the Beam Blocked button can be activated. This freezes the cavity in its current position and doesn’t register any new dc values until the button is released. Following from this, the idea for the Time-Out function is that, if the laser is blocked inadvertently or the power drifts over time (or any other event that would alter the maximum and minimum values), the program will try to re-lock for a certain amount of time (specified by the user). After this time, if it is unsuccessful, the minimum and maximum values are reset, allowing the system to lock again with new thresholds.

The ability for the RT code to communicate between FPGAs allows for sequential locking logic to be implemented across all FPGAs. Here the user can specify the order of the locks in their experiment, and which locks are dependent on which others. The locks are grouped into sets, where each lock in a set depends on the lock before it in a linear
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The Digital Control System

The Digital Control System

The simplest controller design imaginable is to multiply the error signal by some constant gain $k_P$ (either positive or negative), known as a proportional (P) controller. This does not allow much room for optimisation, however, with the same controller gain at all frequencies. For instance, this method limits the amount of noise suppression at low frequencies as, after a critical value of $k_P$, any additional proportional gain will lead the system to resonate. This is why usually a proportional-integral-derivative (PID) controller is used, with the integral (I) component (with gain $k_I$) for extra gain at low frequencies and the derivative (D) component doing the same at high frequencies.

In this experiment we are more concerned with noise at low frequencies rather than those above the cavity resonance (with $\omega_R$ on the order of 10 kHz) and therefore we decided to use a proportional-integral-double integral (PII) controller\(^2\) with the double-integrator

\(^2\)Though of course, as stated previously in this chapter, one of the advantages of digital locking is the flexibility to choose any desired controller algorithm.
(II) having gain $k_{II}$. For a continuous system the PII controller has the following transfer function \[ H(s) = k_P + \frac{k_I}{s} + \frac{k_{II}}{s^2}. \] (4.8)

where $s$ is the Laplace variable that can be expanded in terms of the imaginary number $i$ and frequency $f = \omega/2\pi$ to give $s = 1/if$. However, a digital system is discretised rather than continuous and, in this case, the transfer function becomes

\[ H(z) = k_P + k_I \cdot \frac{z}{1 + z^2} + k_{II} \cdot \left(\frac{z}{1 + z^2}\right)^2, \] (4.9)

where $z = \text{Exp}[if/fs]$, with $fs$ being the sampling frequency of the controller system. Bode plots for the three different components of the PII controller are shown in Fig. 4.6. As already stated, the P component has the same gain at all frequencies, while the I and II have more gain at lower frequencies. It is also worth noting that both the I and II components have a phase delay associated with them ($\pi/2$ and $\pi$ radians respectively) that can affect the stability of the system if their bandwidth extents close to the resonant frequency of the system. At higher frequencies, where the I and II components no longer suppress noise, this phase delay decreases.

For ease of use in situations where the ratio of the gains $k_P$, $k_I$ and $k_{II}$ is optimised but the system gain needs to be increased/decreased (for instance, due to an increase in error signal size), an extra controller gain parameter $k_H$ has been added to the code to scale all other gains (i.e. $H_{act}(z) = k_H \cdot H(z)$).

### 4.4 Locking Analysis and Optimisation

In this section we present an experimental analysis of our locking system acting on both a ring cavity, and bow-tie OPO cavity containing a periodically poled potassium titanyl phosphate (PPKTP) crystal and co-resonant with light at 1064 and 532 nm.

#### 4.4.1 Measuring System Responses

It is important, for both the optimisation and stability of a lock, to know the frequency response of the system in question. This normally requires a network analyser. However, these devices are expensive pieces of equipment (usually on the order of $10,000$ AUD) and have a minimum sine amplitude which can, in some cases, be enough to unlock the system being measured (even with extra external attenuation), adversely affecting the results. Another option, and one that is simple to implement in LabVIEW thanks to an inbuilt waveform generator, is to use white noise - a signal with the same amplitude at all frequencies. Not only is it easy and costs nothing to implement on top of LabVIEW and PXI set-up costs, but the output signal amplitude can be as low as $3 \cdot 10^{-4}$ V and, as an added extra, the code can be set to only take data when the system of interest is locked, and has been locked for a set amount of time to allow it to settle, removing the possibility of this affecting the results. One area where a network analyser has an advantage over white noise generation is resolution at low frequencies as, due to memory limitations on the FPGA cards, only approximately half a seconds worth of data can be taken at any one time. This can, however, be improved by averaging multiple traces, with $2^{10}$ traces of $2^{15}$
Figure 4.7: Approximating $G$ and Simulating $H$. (a) Approximation of $G(\omega)$ by using very small controller values and measured using a white noise generator. (b)(i) A comparison of $|H|$ calculated from Eq. 4.10, with $G$ and $T$ measured using LabVIEW (red), and (ii) $|H|$ simulated from Eq. 4.9 (blue).
points and $f_s \approx 747$ kHz used for all analysis below. A comparison of the two methods is shown in Fig. 4.1(b).

Both methods require timescales on the order of hundreds of seconds to obtain sufficient resolution at low frequencies. This proves to be a problem for measuring $G$ for all but the most stable of cavities as, if the system is not very close to resonance, there will be no output signal (see inset Fig. 4.3(a)). One way around this, as mentioned previously, is to use a Michelson interferometer (Fig. 4.2(b)) as a signal incident on the detector will be a sine wave (and therefore the system will never be in a range where the output signal will remain constant for a small perturbation to the system). However, this will only work if there is a ring PZT at the back of the cavity to allow the laser to reflect off the back of the mirror. This will not always be possible and, even if it were, ideally we would use the same set-up for all measurements.

Another option is to measure the closed-loop transfer function, as shown in Fig. 4.2(a), where the input for the network analyser is taken from just before the controller, and the output is combined with the output of the controller. If only very small gain is used, especially on the I controller as it also affects the phase of the measurement, we can approximate $H \approx 0$ and therefore we obtain $T \approx G$ from Eq. 4.4. This can be seen by comparing Fig. 4.2(b) with Fig. 4.7(a), with traces of $G$ measured using a Michelson method, and $T$ using minimal controller gain, respectively (though for different cavities). Both traces show the same characteristic features of the plant: flat frequency response at low frequencies, for both magnitude and phase, before the magnitude increases towards the system resonance while the phase heads towards $-\pi$.

The reason for measuring $G$ becomes apparent when the stability and bandwidth of the system need to be determined. Both of these depend on $GH$ and, thankfully, $H$ is easy to simulate. Figure 4.7(b) shows a comparison of $H$ calculated from $T$ and $G$ and a version simulated from Eq. 4.9. Once $G$ and $T$ are measured they can be used to examine the lock. This will be discussed in the following section. One other advantage of measuring $G$ is to help with calibration. In Sec. 4.2 it was discussed how, hopefully, the only “interesting” element of $G$ (at least as far as frequency responses go) is the cavity itself $G'$ and so we could write $G = K \cdot G'$. This $K$ will lead to a constant offset on the Bode plot for $G'$, and there will be a similar effect for $H$ and $T$. One way of negating this is to determine the value of these constants, which turns out to be quite easy if $G$ has been measured. This is because, at low frequencies, $G' = 1$ and therefore $K$ can be determined by inspection from $G$ bode plot at these frequencies. This value should be the same for $T$ and should not change over time. The same is true for $H$, and the constant offset here can be determined by rearranging Eq. 4.4 to give

$$H(\omega) = \frac{G - T}{G \cdot T}.$$  \hspace{1cm} (4.10)

This can then be compared to the simulated version and the extra gain inferred (see Fig. 4.7(b)). These normalising options are built into the program and were used for the following analysis.

The other method for characterising the lock response is RMS noise. This information is easily attained from the RT scope function for any lock.
Figure 4.8: Lock Optimisation Protocol. (a) Magnitude plot of system transfer function for an OPO cavity with controller gain increasing from (i) to (iv). (b) Normalised RMS values for these traces for both the error (ES) and dc (DC) signals. Error bars derived from standard deviation across ten runs used for each point. Also shown is the sequence in which the different gains are added, the bandwidth of the system ($\omega_B$) and phase margin (PM) measured by the code.
4.4.2 Lock Optimisation

In our analysis we used the two tools mentioned above to investigate the individual locks: the system transfer function and the RMS of the error and dc signals, both measured internally using the code. For the transfer functions, the three key indicators were the amount of noise suppression at low frequencies, the range of frequencies for which noise was suppressed (i.e., bandwidth) and the stability of the lock as measured by the phase margin, as discussed in Sec. 4.2.

Figure 4.8 illustrates an optimisation procedure developed for a generic lock applied to the OPO:

1. **Measuring G.** This need only be undertaken once for each lock and then reloaded into the code later if necessary. As mentioned previously, to measure $G$ we need to use as little gain as possible, especially with I gain. Normally, the minimal I gain needed to lock the system delivers the best approximations of $G$, with the option of adding P gain and decreasing I gain available if the output Bode plots do not display the same behaviour as Fig. 4.2(b) or Fig. 4.7(a). This step should also be used to determine the offset of $T$.

2. **I Gain.** As the system has no memory of its current position apart from where the scan stopped (which will be closer to the sides of the peak than the middle) I gain is important. This is because I gain has memory, meaning that if the value of the error signal is zero the I value will not change from its previous value, hopefully keeping the system at the centre of the peak. P gain, on the other hand, would be zero at the same point, moving the system away from the centre. I gain is also handy in that, if the correct amount of gain is used, it reduces noise at low frequencies while not exciting any system resonances. The amount of gain also determines the bandwidth of the system ($GH = 1$), therefore the I gain should be increased as much as possible until it either excites the system resonance or the phase margin drops below its desired value (normally taken to be $30^\circ$).

3. **P Gain.** Once I gain has been increased as much as possible it is time to add P gain. Normally this is simply added until it excites the resonance above a certain value or the system is seen to visibly oscillate. This P gain may make it possible to increase the I gain further, making Steps 3 and 4 an iterative process.

4. **I^2 Gain.** Once the values for I and P gains have been optimised, $I^2$ gain can be added to further reduce the system noise at low frequencies. It is important to be aware of any system instabilities that can be caused by increasing the gain due to the $-\pi$ phase delay.

5. **RMS Measurements.** To determine whether these increases in noise suppression are actually having an effect on the power fluctuation of the laser output noise measurements should be performed. As with the LabVIEW network analyser it is best to take a number of traces to make sure that the values are consistent. The sampling time for these RMS values (on the order of one second) depends on the sampling rate of the digital scope and therefore can only really be used for comparison purposes.

Figure 4.8(a) shows the system transfer functions for this process and the reduction in noise that occurs at low frequencies. Figure 4.8(b) shows the average RMS for these
traces taken over ten runs, illustrating the reduction in noise as more gain is added to the system. Also shown on this figure is the bandwidth and phase margin measured by the system. As can be seen, though the noise suppression measured on the transfer function traces does not increase significantly after the initial gain is added, the bandwidth does increase while the RMS decreases with increasing gain. In all cases the phase margin was kept above \( \pi/6 \).

Though the amount and ratios of the gains required to optimise each individual lock will differ due to different plant resonances etc., the combination of the P, I, and I^2 controllers allow for flexibility when optimising.

### 4.4.3 Long-Term Stability and Comparison with Analog PI Controller

The long-term stability of an OPO cavity was investigated over a period of one hour. To accomplish this, both the error and dc signals for the OPO were measured at one second intervals using the code. The OPO was locked firstly with the internally programmed PII controlled and then with an external analog PI controller developed in-house. These analog PIs have been used previously on many quantum optics experiments, including the one described in Ref. [242]. The results of this comparison are shown in Fig.s 4.9(a) and (b) for the error and dc signals respectively, with the former converted to frequency and the latter normalised to the maximum peak height.

As can be seen, both the analog and digital controllers are stable over the period of one hour, though at one point, approximately 2200 s into the run, it can be seen that the analog system jumped slightly, but not enough to lose lock. This stability was despite the fact that during an hour the laser resonant frequency was found to move by approximately 175 MHz (measured by monitoring the PII controller output during the digital stability run). This is shown in Fig. 4.9(c).

The amount of noise suppression over this time was again measured using transfer functions and RMS values, with the former shown in Fig. 4.9(d). From the RMS values included in Fig.s 4.9(a)(i) and (ii) it can be seen that the RMS for the digital error signal trace was approximately 0.3 MHz, as opposed to the 0.4 MHz for the analog controller. For the dc RMS, even removing the perturbed region around 2200 s from the analog trace, the RMS was approximately the same \((S_i\text{ being normalised to the total size of the peak and } S_{dev} \text{ is taken to be the peak minima})\). Also, from Fig. 4.9(d) it can be seen that the digital controller has 50 dB greater noise suppression at low frequencies.

These results will vary depending on the different analog and digital controllers developed and the gains used. For instance, the extra noise suppression at lower frequencies for the digital controller can, in part, be attributed to the I^2 component, which was not included in its analog counterpart. However, it is much easier to alter a digital, rather than an analog, controller and this comparison verifies the efficacy and long-term stability of this general digital controller.

### 4.4.4 Other Optimisations

Apart from the locking parameters themselves there are many other ways to improve the overall quality of the lock. Some of these are briefly discussed below.
Figure 4.9: Analog and Digital Controller Comparison. (a) Error signal, measured in megahertz, and (b) dc signal, normalised to total peak height, for (i) a digital PII and (ii) analog PI controller, measured over a one hour period. (c) The digital PII controller signal measured during the digital run showing the frequency drift of the laser that was compensated for. (d) Magnitude plots of the system transfer function for (i) no locking, (ii) for locking with the analog PI controller, and (iii) for locking with the digital PII controller.
§4.4 Locking Analysis and Optimisation

Low-Pass Filters and Resistors

One way to reduce the ringing at higher frequencies associated with increasing $P$ gain is to include a low-pass filter (LPF) between the controller and the PZT. As a low-pass filter consists of a resistor and capacitor in series, and as a typical PZT has an inherent capacitance on the order of 1 nF, adding a resistor can also be approximated to a LPF, with the cut-off frequency $f_c$ being given by [249]

$$f_c = \frac{1}{2\pi CR}, \quad (4.11)$$

where $C$ is the capacitance of the system and $R$ the resistance. Figure 4.10(a) shows the effect of adding different resistors between the HV amplifier and the PZT. As can be seen, the higher the resistance, the lower the magnitude of $T$ at frequencies higher than $f_c$. However the effective capacitance seems to decrease with increasing $R$ (from approximately $10^{-7}$ to $10^{-9}$ F for the range of resistances shown) and therefore adding a resistor after the HV amplifier must also be having some other effect. For instance, increasing $R$ decreases the voltage supplied to the PZT and, therefore, the system gain. It will also increase the controllers sensitivity by decreasing the range of the HV amplifier.

Figure 4.10: Low-Pass Filtering. (a) Bode plot of $T$ for a ring cavity with different resistors attached after the HV amplifier (see key). (b) Magnitude-only plot of infinite-impulse-response (IIR) filter created in LabVIEW for two frequencies, (i) 124.7 kHz and (ii) 124 kHz, with a cut-off frequency of 124.5 kHz.
A Digital Locking System for use on Quantum Optics Experiments

Using a LPF (with capacitance on the order of a microfarad) is a better alternative as it allows for more exact control of $f_c$ and will not alter controller gain below the cut-off frequency. The choice of an optimal value for $f_c$ (by choosing the correct combination of $R$ and $C$) is an interesting one and depends on what are the main characteristics desired for the lock. For instance, if reducing noise at low frequencies (less than a kilohertz) is important, then a filter with $f_c \approx 10$ Hz is a good option. If this is not a priority, choosing $f_c$ anywhere up to the resonant frequency will help the lock by allowing for maximum gain at the lower frequencies. One side-effect of using a LPF is that it can interfere with the autolock mechanism, as the abrupt change from scanning to locking functionality can be interfered with if $f_c$ is too low (or, for the case of using a resistor, $R$ is too high). This issue can be overcome by reducing the scan speed.

An alternative to a physical LPF is to use a digital one programmed in LabVIEW. Again this is made easy as LabVIEW contains an inbuilt tool to determine parameters for infinite-impulse-response (IIR)-type filters. However, these parameters only work for $f_c > 0.166 f_s$, as shown in Fig. 4.10(b), which for the code described above would be approximately 124.5 kHz. This is not a useful cut-off frequency for the systems of interest to us and therefore this digital filtering method was not used. If a digital filter was desired, again due to its flexibility and ease of use, the code would have to be run at one quarter of its current speed to obtain a “useful” $f_c \approx 30$ kHz, though care must be taken with this approach as reducing $f_s$ would increase the phase delay at higher frequencies.

Digital Noise

Unfortunately there are some disadvantages to go along with the many advantages of digital locking, and one of these is digitisation noise. This is due to the finite sensitivity of the system for both the input and output signals and may introduce additional noise into the system. In the case of the FPGA, $2^{16}$ bits are used for both signals, each with a voltage range of $\pm 10$ V. Each bit therefore represents $20/2^{16} = 3.05 \cdot 10^{-4}$ V. For a “typical” error signal entering the FPGA, the peak-to-peak range is on the order of $V_{pp} = 0.5 - 1$ V, meaning that the percentage error caused by the digitisation is $3.05 \cdot 10^{-4}/1 \cdot 100 = 0.03\%$. This does not seem very large, though it can be further reduced by increasing the size of the error signal to the maximum possible ($V_{pp} = 20$ V). The difference in dc RMS values for error signals ranging in size from 0.3 to 18 V are shown in Fig. 4.11(a). To keep the actual controller gain ($k_H \cdot V_{pp}$) constant, whenever the error signal height was changed (by approximately a factor of two each time) $k_H$ was doubled/halved as appropriate. As can be seen from Fig. 4.11(a), there is a negligible effect within the error bars (though a trend towards higher RMS can be seen as $V_{pp} \rightarrow 0$) and therefore this effect can be ignored except for the most sensitive of applications.

Similar to the input scenario, the output controller signal should also span the full $\pm 10$ V to minimise digitisation noise. However, we also want the scanning range of the signal to be at least one free spectral range (FSR). These two features cannot both be achieved simultaneously through one output of the FPGA. If we satisfy the second but not the first, then the sensitivity of the locking will be limited as, if the finesse of the cavity is approximately 1000, the peak full-width-half-maximum (FWHM) will only be $2^{16}/1000 \approx 65$ points (see Fig. 4.11(b)). One solution to this problem is to use two outputs for each lock: one for the scan and one for the lock. The idea here is to set the
Figure 4.11: Digital Noise. (a) RMS measurements of the dc signal for variously sized error signals. (b) Schematic of the dc signal as a function of cavity position showing the width, in points, of both an FSR and the FWHM of one peak (not to scale). (c) RMS measurements of the dc signal for varying attenuation applied after the LabVIEW output and before the HV amplifier.
scan range to be one, or better two, FSRs (ensuring that at all times at least one peak is not on the outskirts of the scan range), and using analog attenuation for the lock output to increase the sensitivity of the locking. This method works well in principle with the scan and lock system: the scan output, with its large range, can always find the peak; once this happens it will set the dc offset around which the system will lock with the attenuated lock signal; and, if long term drift of the system proves too much for the lock signal to negate over a long period of time (i.e., hours), then the system will simply rellock and start again. In this case (assuming a scan range of three FSR) the number of points $p$ for the FWHM will be given by

$$p = 2^{16} / 2000 \cdot 10^{att/10} \approx 33 \cdot 10^{att/10}, \quad (4.12)$$

where $att$ is the analog attenuation applied to the locking signal in dB. The effect on the RMS of the signal with varying amounts of attenuation is shown in Fig. 4.11(c) (again with increased $k_H$ to compensate for loss in controller gain due to the attenuation) and, as can be seen, this again has a negligible effect on the locking quality within the error bars. A slight trend towards lower RMS values can be seen as $att$ increased, however, more data points would be needed to determine if this is a real effect and the system refused to lock with $att > 21$ dB. This means that the two output method is unnecessary for most applications, leaving open the possibility of using the remaining output for other functions such as a trigger signal for an external scope, etc.

### 4.5 Quantum Measurements

So far we have presented results which are not specific to the field of quantum optics. In this section, a bow-tie OPO cavity was inserted into a larger quantum optics experiment designed to characterise optical Schrödinger cat states (see, for instance, Ref. [250]). This experiment, described in detail in Ref. [251] and shown in Fig. 4.12(a), consisted of two mode-cleaner cavities (one at 532 nm and one at 1064 nm), the OPO - which was used to produced squeezing on the 1064 nm beam, two phase locks, a double homodyne, and a tomographic homodyne lock to allow locking to any quadrature value. This was all controlled using the hardware and software described in Sec. 4.3 and integrated into one user-controlled RT code.

As in many other quantum optics experiments, it was desirable to only take data when all components of the system were locked. Also, in this case, vast quantities of data (hundreds of gigabits) were required to be able to produce the necessary tomographic reconstructions of the optical Schrödinger cats, requiring many hours of acquisition. It was also necessary to compare the squeezing data traces with dark noise (DN - signal and local oscillator blocked) and shot noise (SN - only signal blocked) traces at various times during data collection for normalisation purposes.

To accommodate these requirements, a second RT code was developed to take data using two data acquisition cards (NI PXI-5124). The data acquisition code communicated with the locking code to determine whether all components were locked, as well as to change locking parameters as desired. For instance, the phase of the single homodyne lock had to be changed to allow for the tomographic reconstruction, and blocking of the various fields present using motors controlled via the locking FPGA was necessary for collecting DN and SN traces. The data was saved to the hard drive of the PXI controller and later extracted for examination.
Figure 4.12: Schrödinger Cat Experiment. (a) Schematic of cavity set-up with: MC - mode-cleaner cavity (R - red, G - green); BT - bow-tie cavity; PZT - piezo-transducer phase lock; HOM - homodyne lock (T - tomographic, x 2 - double homodyne); AmM - amplitude modulator; PhM - phase modulator; Sh - mechanical shutter; SQZ Data - squeezing data; D - lock dependencies (0 = no dependencies); F - FPGA lock is controlled from; Ext. Demod. - external demodulation carried out for these locks to produce the error signal; AIP - analog input; DAQ - data acquisition card; Clk. Sig. - clock signal; and LO - local oscillator. (b) Flow diagram of data acquisition procedure used to take squeezing data (SQZ) for various angles of the tomographic lock $\phi$, as well as take dark noise (DN) and shot noise (SN) traces. Values for the number of experimental runs $N$, number of squeezing traces for each tomographic angle $M$, and change in tomographic lock angle between runs $\Delta \phi$ are defined by the user. (c) Plot of variance in noise normalised to the quantum noise limit (QNL), as a function of tomographic lock angle taken using the above procedure with a total of 1.2 million points per angle, and $\Delta \phi = \pi/36$. Circles correspond to data points, the dashed line corresponds to a theoretical fit.
A flow diagram of the data collection protocol is shown in Fig. 4.12(b). Here the user determined the number of data traces to take for each tomographic lock angle $M$, as well as the change in tomographic lock angle between measurements $\Delta \phi$, the maximum tomographic lock angle $\phi_{max}$, and the number of experimental runs $N$ for $\phi = 0 \rightarrow \phi_{max}$. It should be noted that the waiting time programmed into the code was to allow time for the shutters to close or open, and the system check included such functions as resetting error signal offsets so that all locks remained stable during the hours of data collection. If a lock did fail during the measurement of one angle, the protocol would reset and was attempted again for that angle until it was successful. Figure 4.12(c) shows an example of squeezing data collected from one such run with $\Delta \phi = \pi/36$. Approximately 1.5 dB of noise suppression below the quantum noise limit (QNL) could be seen for a tomographic angle of $\phi = 90^\circ$, accompanied by an increase of 1.8 dB in the opposite quadrature.

This implementation of a digital control algorithm on a uniquely quantum system highlights the benefits digital control can bring to quantum optics experiments: the integration of information allowed for data to only be taken when all system components were locked to ensure only useful data was collected; the automation of the locks meant that the experiment could be left unattended for many hours to collect the required amount of data as quickly as possible; and the added flexibility allowed for automated acquisition of SN and DN, as well as the ability to alter the homodyne angle to perform the tomographic measurements.

4.6 Conclusions

In conclusion, we have presented a digital locking system that allows for automatic and sequential locking and is easily scalable. This code was programmed using LabVIEW software and is free to download. We have shown how the inbuilt locking analysis tools can be used to help optimise individual locks, analysed the code and its behaviour on multiple types of cavities in great detail, and demonstrated the efficacy and long term stability of the digital controller. Finally, we used an example of an experiment used to characterise optical Schrödinger cat states to illustrate the advantages of digital control for quantum optics experiments.