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Proof-Functional Semantics for Relevant Implication

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This thesis reports original work carried out by me.

The contribution of others is duly acknowledged in the text.

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Peter S. Lavers

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Abstract

In this thesis I provide a theory of implication from within the Gentzen/Curry formalist constructivist tradition. Formal consecution and natural deduction systems, which satisfy the formalist and also the intuitionist desiderata for constructivity (including Lorenzen's principle of inversion), are provided for all implication logics. The similar— but simplified— binary relational ("Kripke-style") semantics are also given. The driving force behind this research has been the desire to provide an explanatory semantics for relevant implication in terms of "use as a subproof in a proof". To this end *relevant* consecution systems which exploit various precisely characterised notions of *use* are described.

The basis of this work has been the development of a way of describing the shapes of proofs in the "object language". In chapter 2 I motivate and introduce the basic machinery used to describe proofs, and show how thereby to capture *use*. This involves a more detailed consideration of the internal structure of formal systems than exploited by Curry in his *epitheory of formal systems.*

In chapter 3 the completely general "cloned" consecution systems are described, and it is shown that every logic with an axiomatic formulation is captured by such a system.

In chapter 4 the corresponding natural deduction systems are described and it is shown that Lorenzen's principle of inversion holds for them by proving the appropriate reduction theorem. Thus every implication logic has a formulation which satisfies the intuitionist formal criterion for constructivity.

In chapter 5 we return to the business of providing explanatory semantics for relevant implication, using the similar style of consecution system as in chapter 3, but with list (proof-description) manipulation rules which capture *use.*

In chapter 6 "cloned" binary relation semantics are described which also capture every

logic with an axiomatic formulation. These don't quite correspond to the consecution systems of chapter 3 in that they exploit a dramatic simplification of the list machinery (but do involve other complications). The similar relevant semantics using *use* rules is also given.

The corresponding "simplified" consecution and natural deduction systems are described in appendix B.2. These systems do not satisfy the Lorenzen principle of inversion and so are not constructive.

Chapter 7 rounds off and offers some thoughts about possible further developments.

Appendix A shows an early attempt to capture relevant implication, and is notable as the most complex formulation of intuitionist implication ever devised.[1]

[1] Thanks are due to Bob Meyer who showed that this system is somewhat stronger than I had earlier thought.

Contents

Chapter 1

Introduction

In this chapter the philosophical perspective towards logic and semantics which motivates this work is described, and then a more detailed introduction is provided to the technical results reported in this thesis.

First it behooves me to deny what is generally taken as a truism by logicians and philosophers alike, that the central task of Logic is to provide an account of reasoning, of Good Argument, that is of deduction in the non-technical sense of 'deduction'. The view that such is the purpose of logic provides a distorted perspective which subtly leads into error and false logic. For when considering logical entailment, that view propels one towards imagining a dialectical situation with a protagonist putting forward an argument towards a conclusion, and considering whether or not the argument establishes that conclusion. And surely, if the protagonist begins with true premisses and each inference step is truth-preserving, then the conclusion is true. How easy is the slide, from Good Argument to Truth-Preservation.

Logic is, however, concerned with that aspect of Good Argument whereby some arguments are "good" in virtue of their form alone. But this requires an account of the form of the sentences comprising such arguments, and this is best achieved by shunting aside the Good Argument picture and thinking about form in a more general setting.

The central task of logic is, then, to provide a formal theory of language (rather, of that aspect of language which is amenable to formal treatment).[1] To carry out this task one develops a *canonical language*, a formal counterpart of language which is an abstraction to be treated as part of mathematics, together with a *theory of meaning*, a *semantics*, for the formal language.

Having put aside the Good Argument view as to the role of logic the slippery slide into Truth-Preservation looks decidedly less appealing. The idea that our image of a protagonist holding forth is the right way to capture a theory of meaning seems very peculiar. Why should closure under Good Argument be sufficient? Even where Good Argument is pared down to Truth-Preservation there is no reason to suppose that it is fine-grained enough to capture meaning, that what marches together under Truth-Preservation must also be synonymous.

The approach here follows that of Curry and Gentzen, and can be described as a formalist strand of constructivism.

Constructivism is the view that the meaning of a connective is determined by its rules for introduction into discourse. Since we are working in a formal setting where "discourse" corresponds to "proof", and since I use representations of proofs in my semantics, I call the semantics presented here *proof-functional* semantics.

Formalism denotes a particular attitude towards mathematics where

> 'mathematics is characterized more by its method than its subject matter; its objects either are unspecified or, if they are specified, are such that their

[1] A recent historical concern has been to do this for a language sufficiently rich in which to do mathematics, using formal rigor so as to banish paradox.

exact nature is irrelevant, so that certain sorts of changes can be made in them without affecting the truth of the theorems.' (Curry [5] p.8)

Now some regard mathematics as just a game involving symbol manipulation, with nothing further to its content. This "empty symbol" view is not to be confused with formalism. There are two senses in which mathematics has meaning for the formalist. The first is that meaning is constituted in those transformations on arbitrary entities. And the second is that meaning can be conferred by providing an interpretation. But what is crucial to the formalist is that the actual business of doing mathematics must be able to be carried out with complete neutrality as to interpretation of the subject matter (not that it *must* be so carried out).

Curry (in [5]) contrasts formalism with "contensivism", the view that the objects of mathematical discourse are taken to exist as they are ordinarily understood, so that mathematical truth is determined by the facts. He further distinguishes between two forms of contensivism: *platonism* and *critical contensivism*. The principal form of the latter is intuitionism, which takes it that the mathematical universe is constructed by the human mind, and hence that mathematical reality is limited by the bounds of what is so "constructible". For both forms of contensivism the central notion of mathematical proof is seen as establishing Mathematical Truth in this realm of mathematical facts, and so resembles somewhat closely our old friend Good Argument. So that picture of a protagonist propounding good argument, which leads so easily to Truth-Preservation, is highly consonant with these views (even though for the different notions of Truth associated with them). But from the formalist perspective that image of carving out the truths of some independently existing realm is quite inappropriate, so that there is no temptation to turn towards the Good Argument slippery-slide into Truth-Preservation when seeking a theory of meaning.[2] Truth plays no part in the semantic theory.

[2]Note that Truth-Preservation is not a necessary encumberment to contensivism. For example even the platonist could quite consistently question whether Good (even Mathematical) Argument is too coarse-grained to capture meaning. He could adopt the constructivist notion of meaning for the logical

For Curry the basic semantic vehicle is the consecution systems. In these systems the entailment relation is determined purely by introduction rules for each of the logical connectives. The semantic importance Curry attaches to these systems is made explicit in his introduction to the chapter on negation in *The Foundations of Mathematical Logic* where he describes them as being rather abstruse from the point of view of ordinary mathematics but very natural from the semantic point of view, and continues:—

> 'Now that this semantic point of view has become familiar, it agrees with the theme of this book to take it as fundamental.' ([5] p.254)

Curry's and Gentzen's work shows that consecution systems have a central place in the logic arena, from a philosophical perspective as well as the technical. They are not merely means to solving decision problems and the like.

The constructive meaning of the connectives is captured in the consecution systems by showing how to generate an entailment relation for the language incorporating connectives from that for an arbitrary elementary system (which can be supposed to involve only the atomic language). This is done by determining introduction-on-the-right rules for each connective by considering the conditions under which they can be introduced into discourse, and then determining introduction-on-the-left rules from these by "inversion". Thus that an entailment holds can be broken down into facts about the entailment relation of the original elementary system. Since the theory of logic is generated from the properties of arbitrary formal systems, Curry describes logic as the *epitheory of formal systems.*

For example consider the implication operator (which Curry calls 'ply'). The introduction-on-the-right rule is motivated by the idea that, having $A \rightarrow B$ on assumptions Γ is warranted by being able to establish B on the assumptions Γ together with A. And this is

connectives but strengthen the resulting logical entailment, in order to capture mathematical proof and other forms of Good Argument, by adding disjunctive syllogism and excluded middle.

formally expressed by (leaving aside various issues concerning interpretation):—

$$\frac{\Gamma,\ A\ \Vdash B}{\Gamma\ \Vdash A{\rightarrow}B}$$

The introduction-on-the-left rule then takes the inverse of this idea, considering what we can suppose to follow from $A{\rightarrow}B$ given our knowledge of the above condition for its assertion.

> 'unless A is present, we can infer nothing from $A{\rightarrow}B$ that we cannot infer from it as a wholly unanalyzed proposition, but if A is present, we can infer from $A{\rightarrow}B$ any C which we can infer from B alone' ([5] p. 187)

And this is formally expressed by:—

$$\frac{\Gamma\ \Vdash A \qquad \Gamma,\ B\ \Vdash C}{\Gamma,\ A{\rightarrow}B\ \Vdash C}$$

The crucial test that the introduction-on-the-left rules are the correct inversions of the corresponding introduction-on-the-right rules is whether the resulting consecution system is closed under *cut*, that is, if we have deductions of

$\Gamma_1\ \Vdash\ A$ and $\Gamma_2,\ A\ \Vdash\ B$

then we also have a deduction of

$\Gamma_1,\ \Gamma_2\ \Vdash\ B$

For this ensures that we have correctly matched with the warrant for assertion of each compound wff (introduction-on-the-right) what we can suppose to follow from it (introduction-on-the-left).

This whole procedure is similar to the intuitionist approach towards determining introduction rules and their corresponding elimination rules for natural deduction systems. Lorenzen's principle of inversion (see section 4.4) is an important feature of intuitionist semantics and captures the idea that " you can't get out more than you put in." Clearly this principle is important from the present perspective where we wish to restrict logical entailment to that due to bare meaning (with meaning determined by introduction rules). (At the same time it is not appropriate to place such a stricture upon formal renderings of Good Argument, which may involve more coarse-grained closure notions.) The sense of inversion as captured by Lorenzen's principle is mirrored by the above-described inversion procedure for determining the introduction-on-the-left rules for consecution systems.

Thus closure under *cut* has the equivalent philosophical status for consecution systems as reduction-to-normal-form theorems (establishing Lorenzen's principle of inversion) have for natural deduction systems.

Note that Curry puts forward constructive semantics for material implication and classical logic, namely their respective multiple consecution systems.

Let me summarise and comment upon the main points so far.

I began by denying that a central task of logic is to provide an account of reasoning/Good Argument. Rather, the concern of logic is to provide a formal theory of language, so in particular of the meaning of the logical connectives. I suggested that the interdeducibility relation corresponding to Good Argument is too coarse-grained to capture synonymy, and that we ought adopt the formalist constructivist approach as exemplified by Curry and Gentzen.

Having developed the theory of the formal language one is then in a position to tackle Good Argument. In some cases Truth-Preservation is appropriate for its capture, such as in those dialectical contexts where a body of theory is asserted as true. But the prior task is to provide the theory of the formal language in which such Good Argument is to be presented.

A significant feature of this perspective is that Truth does not play a semantic role. Critical contensivists such as intuitionists and dialetheists adhere to the Good Argument/Truth-Preservation approach, yet modify the notion of truth allowing that it may be incomplete or inconsistent (or both) respectively, so obtaining "deviant" logics. Here it is rather denied that Truth-Preservation is a sufficient condition for logical entailment. Thus, for example, it does not follow from the failure of excluded middle to be logically valid that the bivalence of truth is discarded. So "deviant" logics may be obtained without deviant truth. My own view is that truth is consistent and complete (with respect to meaningful statements). Truth remains unsullied. However the point is that truth just doesn't figure in the theory of logic.

At this point I should say something about *entailment, theories* and *deduction.* Since we seek to develop a theory of the logical connectives which captures their meaning as determined by the rules for their introduction into discourse, our entailment relation will correspond to that due to bare meaning inclusion. This will not capture the entailment relation corresponding to some senses of Good Argument— there is a gap between the two to be filled in by considerations about the nature of the world, such as its consistency. Now 'theory' is appropriately applied to those classes of sentences determined by commitment in virtue of logical entailment (and so in virtue of meaning alone). The "protagonist holding forth in a dialectic" image is incongruous when applied to formal theories. So it is not appropriate to equate formal theories with the broader closure notions associated with Good Argument. Theories are neutered entities not of themselves asserting their own truth. Similarly I use 'deduction' in the corresponding denatured sense. So having eschewed Good Argument, preformal deducibility and correlated notions I nevertheless retain the usual terminology, with just this advertisement that perhaps it doesn't have some of the usual connotation. Sometimes 'asserted' is used when discussing interpretation in contexts where 'is true' or 'holds' tend to be used. The intended sense is that of holding in a context in virtue of meaning alone. The epistemic setting is a good way to think about logical entailment because there the certitudes associated with truth are removed. So we adopt a new image, of a protagonist carving out *beliefs* in a shifty epistemic context.

The final more general philosophical issue we consider is that concerning formal semantics. The contensive role of formal semantics is as a guide to how well our formal language captures parts of our natural language (or perhaps a somewhat less natural mathematical language). Where a canonical language connective is intended to approximate a natural language connective, a formal semantics may capture some aspects of the function of the natural language connective, so providing a bridge between the canonical and its intended interpretation. This can only be useful where those aspects captured by the semantics are more basic building blocks underpinning the use of the connective, so that the formal semantics isn't just a redescription of the canonical connective. The formal semantics ought to correspond to an *explanation* of the function of the natural language connective, where the explanands is itself accounted for.

In my opinion the semantics so far proposed for relevant logics fail to meet this criterion. In the case of the Urquhart semilattice semantics, the basic idea is that of joining pieces of information ([28]). This joining operation is to be distinguished from extensional conjoining, ~~and now tends to be referred to as 'fusion'.[3]~~ But a clear account of ~~fusion's~~ its natural language counterpart function remains to be given.[4] (How does the linguist involved in radical translation tell whether the native is ~~fusing~~ joining or merely conjoining pieces of information?) Also, the Routley-Meyer ternary relation semantics simply raises more questions, concerning the interpretation of the ternary relation, ~~which again seem to hinge on intensional fusing.~~ From the mathematical point of view the Routley-Meyer semantics are probably the neatest relational semantics one can hope to get to capture the relevant logics, however the concern here is with the philosophical role of the formal

~~[3]'fusion' was first coined by Fine in his [7].~~

[4]So I am appropriately stung by Michaelis Michael [14], and his timely criticisms provoked my thinking on these matters.

semantics. I should point out that I am similarly suspicious of the modal "accessibility" relation— more particularly as to how one chooses between the competing candidates for capturing it.

The usual paradigm for formal semantics is Kripke-style "worlds" semantics. However this is less attractive from the present perspective where Truth does not figure (let alone truth at "other worlds"). Nevertheless Kripke-style semantics, with the properties of the *points* loosened up to give them an epistemic flavour, do play a significant semantically explanatory role. As also do consecution systems and natural deduction systems (and, for that matter, algebraic semantics). For these can all provide a bridge between a logic and our intended interpretation. I don't think one can really pick out one from amongst the Kripke-style, consecution and natural deduction systems, as semantically paramount, especially since one can convert one type of system into the others.

The most important basic constituent of the semantics I propose for relevant implication is a notion of "use" which is made formally precise in terms of use as a subproof in a proof. And this does seem to have a "natural" language counterpart, at least in mathematical discourse.

Before going on to describe my semantic approach in more detail, let me indicate another feature of my formal systems which seems of philosophical interest. I show how to incorporate in the "object-language" a description of the shapes of proofs. This is done by introducing a subproof operator into the language. Since a sentence holding is interpreted as assertion in an epistemic context, there is no incongruity in regarding arguments/proofs in the same way. This enrichment of the formal language to include proof-descriptions adds some weight against the view that deep philosophical significance is to be attached to the object-language/metalanguage distinction.

Having set the general scene I now turn to the key ideas underlying this work.

In order to demonstrate the meaning of a connective it suffices to show how to introduce that connective into the language. That is, it suffices to provide a procedure whereby given a language not containing the connective, correct use of the language augmented with the connective is determined in terms of more basic linguistic functions. If one can do this then presumably one is in a position to teach the linguist attempting radical translation that aspect of one's language. My formal construing of this task is to suppose we begin with an "atomic language" and provide a mechanism for determining, for each atomic theory, whether a "new" sentence is assertable in the corresponding theory of the augmented language. Curry's presentation of the consecution systems can be seen in this light: The starting point is a deducibility relation on an *elementary system*, and using introduction rules for both sides of the turnstile determined by the previously discussed semantic method, Curry shows how to augment that deducibility relation to capture that for the expanded language.

From this perspective the Kripke-style semantics for intuitionist implication are exactly like the standard Kripke semantics but with the points interpreted as theories, the accessibility relation as theory-inclusion and the initial assignment interpreted as showing the atomic sentences which hold at each theory (so a model structure with just initial valuation corresponds to a set of theories in the atomic language). And the assignment clauses are the rules which we need to use to determine whether we have a warrant for asserting the corresponding compound sentence at that theory. The assignment clause for absolute implication

$$\begin{aligned}&I(T, A{\rightarrow}B) = 1 \text{ iff for all } T' \text{ such that } \quad TST' \text{ and } \quad I(T', A) = 1,\\ &\text{we have } \quad I(T', B) = 1\end{aligned}$$

is interpreted: $A{\rightarrow}B$ holds at the theory T iff every extension T' of T containing A also contains B. The picture is one of "generating" (in tandem) those more complex sentences which are assertable at each theory.[5]

[5]Note that with this epistemic perspective it would be a mistake to imagine that the usual clause for negation could capture *negation*, rather it captures *non-assertion*.

There is no reason to suppose that the atomic language is a featureless plain. Languages have structure and it is open to us to exploit that structure in our semantics. On Curry's approach the apparatus of formal systems is already incorporated in the elementary systems (and so at the level of the atomic language). And it is these properties of formal systems which provide the contensive underpinning of logic. However Curry only makes use of the surface-level properties of the elementary formal systems, which results in extensional logic. To fully develop the theory of logic as the epitheory of formal systems it is necessary to make use of the "internal" structure of formal systems. From the mathematical perspective the relevant aspects of that structure are axioms, rules and proofs. So I view an atomic theory as comprising atomic axioms and rules, the class of proofs generated using these, and the atomic theorems— those sentences for which there is a proof using the axioms and rules. The above-sketched semantics for absolute implication ignores this structure, it is concerned only with which atomic theorems hold at an atomic theory. Information as to the interconnections between the atomic theorems is not incorporated and so cannot be used. But this structure *is* available to the language user and so it is perfectly reasonable to suppose that such aspects of the structure play a role in the assertability conditions of complex sentences, and therefore should be included in the semantics.

Returning to the relational semantics setting, upon incorporating more information about the structure of the theories (points) into the model, such as what *proofs* hold at a theory, we are in a position to modify the $\rightarrow$ clause to the stronger: $A \rightarrow B$ holds at the theory T iff every extension T' of T containing A also contains B *and* contains B in virtue of an irredundant proof which has a subproof of A. That is, we can modify the $\rightarrow$ clause to include a *use* requirement, where *use* has a completely objective definition in terms of the structure of the proofs holding at a point.

For example, suppose an atomic theory T has

axioms $p,\ q$

rule $p \Rightarrow r$

Then every extension of T contains both p and q so that the clause for intuitionist implication makes $p \rightarrow q$ assertable at T. But there is no irredundant proof of q in T containing a subproof of p, so T provides a counter-example to the stronger clause and for that sense of implication $p \rightarrow q$ is not assertable at T. However T and all its extensions contain the proof

p	*axiom*
r	$p \Rightarrow r$

So for the stronger *relevant* sense of implication we do have $p \rightarrow r$ assertable at T.

Obviously to capture nested implication things get much more complicated, and most of my work involved developing the machinery to cater for those complications.

I use a list notation to capture the structure of proofs, with a list $\langle A\,B\rangle$ representing a proof of B with a subproof of A.[6] For relevant implication the intended interpretation of $\langle A\,B\rangle$ is a proof of B which *uses* a subproof of A. Thus the clause for $A \rightarrow B$ can be rewritten:—

> $A \rightarrow B$ holds at the theory T iff every extension T' of T containing A also contains $\langle A\,B\rangle$.

Assignment clauses involving just the list structure are used to generate the descriptions of proofs holding at a point. In the case of relevant implication these clauses are intended to capture various senses of *use*.

The modification to the $\rightarrow$ introduction rules for the consecution systems is similar. The warrant for $A \rightarrow B$ to hold given assumptions Γ is that upon assumptions Γ together with

[6] It is *not* the fusion of A with B.

A we have a proof of B which uses a subproof of A, that is we have $\langle A\,B \rangle$, and this is expressed by:—

$$\frac{\Gamma,\ A\ \Vdash \langle A\,B\rangle}{\Gamma\ \Vdash A \rightarrow B}$$

As before the introduction-on-the-left rule then takes the inverse of this idea, considering what we can suppose to follow from $A \rightarrow B$ given the above condition for its assertion. And again, without A we can infer nothing from $A \rightarrow B$ that we cannot already infer from it as an unanalyzed proposition, but if A is present, we can infer from $A \rightarrow B$ any C which we can infer from $\langle A\,B \rangle$ alone. And this is expressed by:—

$$\frac{\Gamma\ \Vdash A \qquad \Gamma,\ \langle A\,B\rangle\ \Vdash C}{\Gamma,\ A \rightarrow B\ \Vdash C}$$

Of course things are somewhat more complicated, as already noted to capture nested $\rightarrow$'s requires nested lists (subproofs), and that's only part of the story.

Since the list notation provides a way of describing the shapes of proofs in the "object language", it seems reasonable to suppose that one could capture any axiomatic formulation of any logic using the above type of consecution system, by including list manipulation rules which mimic the shapes of the proofs generated using the axioms. Indeed this is so, and although this "cloning" of axiom systems doesn't generally provide explanatory semantics, it is of philosophical interest that *every* implication logic has a constructive formulation. Furthermore from the formal perspective these cloned semantics *do* provide an objective characterization of the meaning of $\rightarrow$ connectives (the function of the canonical connective is formally well-defined) in terms of the possible shapes of proofs in the corresponding axiom systems.

Chapter 2

Preliminaries

In order to demonstrate the meaning of a connective it suffices to show how to introduce that connective into the language in a way which doesn't use the connective, the corresponding operator or correlated notions, but rather exploits more explanatorily basic constituents of its meaning. To do so requires a formal representation of an arbitrary language, to which logical connectives such as $\rightarrow$ are to be added. I refer to this language as the "atomic language". However this may be a misnomer, the atomic language may have a rich structure even perhaps including logical connectives, the point is that it represents an arbitrary language and can be considered atomic with respect to our concerns. This role played by the atomic language corresponds to that of Curry's *elementary systems* in his development of a theory of logical particles. But the only information Curry uses about the *elementary systems* is what (elementary) theorems hold. This corresponds to regarding the points of Kripke-style semantics as being equivalent to sets of sentences. Similarly in the Routley-Meyer semantics for relevant implication ([21]) the points can also be regarded as sets of sentences. The tendency has been to do the semantic work at a level external to theories, treating them as just sets of sentences. Here the key to capturing relevant $\rightarrow$ is to exploit more of the structure of theories than just what sentences hold; we also keep track of the connections (in terms of proofs) between sentences.

This chapter shows how to provide a formal representation of the atomic language which facilitates doing this, and so permits a formal characterization of *use*.

2.1 Atomic Theories

For Curry a theory of logic is a theory about the properties of arbitrary formal systems. So the properties of logical connectives are completely determined by those of formal systems, and the formal method itself. Thus he calls his theory of logic simply the *epitheory* of *elementary systems*. His work delivers intuitionist implication as the minimal implication logic. And from this logic-is-epitheory perspective concern about the paradoxes of implication appears misguided, and simply indicates a lack of attention to formal method— so much the worse for natural language and our "intuitions". However the rather narrow outcome of Curry's work is a result of his lack of attention to some of the properties of *elementary systems*. Specifically, Curry ignores their "internal" structure. Taking account of the key features of formal systems— rules and proofs— allows the epitheory of formal systems to blossom in great diversity. Consequently questions as to which logics best capture natural language are no longer closed off; formal method as espoused by Curry *is* compatible with a real debate over 'which logic?.' In this section we delve into this further structure of formal systems.

Some definitions to keep us going: I use p, q, r and s, perhaps with subscripts and superscripts, as variables ranging over atomic sentences. I use A, B, C, and D as variables ranging over wff. *Lists* are an important new category (see below) and I use lower-case Greek letters as variables ranging over them, perhaps with subscripts and superscripts. Further variables and notation are introduced along with their defined entities. Yet further variables are introduced (without any ado) via schemata which indicate the form of objects those variables range over. For example in the following definition $A{\rightarrow}B$ ranges over entities formed by concatenating a wff together with an arrow together with another wff.

Definition 2.1.1 *wff* is defined inductively as follows:—

- an atomic sentence is a wff
- if A and B are wff, then $A \to B$ is a wff.

Definition 2.1.2 *list* is defined inductively as follows:—

- a wff is a list
- if α and β are lists, then $\langle \alpha \beta \rangle$ is a list.

I call the occurrences of wff comprising a list its *constituents.*
Note that every list α is of the form $\langle \alpha_1 \langle \alpha_2 \ldots \langle \alpha_n A \rangle \cdots \rangle \rangle$.
The rightmost constituent A of α is the *guard* of α.
α itself, together with those sublists β such that $\alpha = \langle \alpha_1 \langle \alpha_2 \ldots \langle \alpha_r \beta \rangle \cdots \rangle \rangle$ where $1 \leq r \leq n$ (for example $\beta = \langle \alpha_n A \rangle$) are *guard lists* of α.
A sublist φ in a location $\langle \ldots \langle \varphi \delta \rangle \ldots \rangle$ is an *antecedent* sublist, while such δ is a *consequent* sublist.
The sublists $\alpha_1, \ldots \alpha_n$ of $\alpha = \langle \alpha_1 \langle \alpha_2 \ldots \langle \alpha_n A \rangle \cdots \rangle \rangle$ are the *major* antecedent sublists of α.
For example the list $\langle p \ \langle \langle p p \rangle \ p \to q \rangle \rangle$ has constituents the (occurrences of) p and $p \to q$, major antecedents p and $\langle p p \rangle$, guard lists the whole list itself, $\langle \langle p p \rangle \ p \to q \rangle$ and $p \to q$ which is also the guard.

I will abbreviate an ellipsis of right angle-brackets by '$\rfloor$', so that a list $\langle \alpha_1 \langle \alpha_2 \ldots \langle \alpha_n A \rangle \cdots \rangle \rangle$ may be denoted by $\langle \alpha_1 \langle \alpha_2 \ldots \langle \alpha_n A \rfloor$. The conventions of Church [3] are followed when denoting wff.

We suppose that formal systems in the atomic language have a structure comprising rules and proofs, which can be characterized as follows.

Definition 2.1.3 An *atomic rule* is a rule, denoted $p_1, \ldots p_n \Rightarrow q$, where the left-hand side denotes a sequence of atomic sentences which may be the null sequence.

So the order and multiplicity of the premisses of an atomic rule are significant, and the atomic axioms are captured by including the case that there may be no premisses.

Atomic proofs will be represented by trees with atomic sentences at nodes. The notion of a tree will be familiar to the reader, however in order to keep track of the order and multiplicity of use of premisses an order is here imposed upon the branches above a node.

Definition 2.1.4 A *tree* is a pair (N, S) comprising a set N of nodes and a sequence S of successor relations upon N satisfying

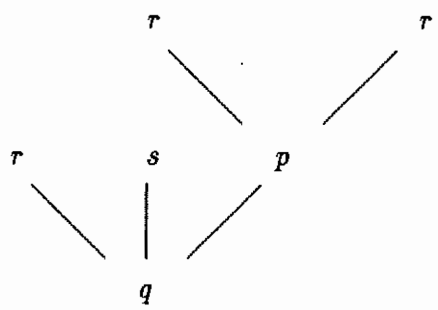
- if for $S \in \mathrm{S}$ and $a \in N$ there is some $b \in N$ such that aSb, then for every S' before S in S there is some $b' \in N$ such that $aS'b'$,
- if for $S \in \mathrm{S}$ and $a \in N$ there is some $b \in N$ such that aSb, then that b is unique, i.e. if aSc then $c = b$.

Definition 2.1.5 An *atomic proof* is a tree (N, S) where N is a finite set of indexed atomic sentences, and each sentence q is justified by an atomic rule with premiss sequence corresponding to the sequence of its successors and conclusion the sentence (at the node) q itself.

So an atomic proof is a tree of atomic sentences at nodes, where each sentence is connected by "branches" to sentences above it and is justified by an atomic rule with premisses identical to such sentences above it, respecting order and multiplicity. For example suppose we have the rules

$$\Rightarrow r; \ \Rightarrow s; \ r, r \Rightarrow p; \ r, s, p \Rightarrow q$$

then the following is an atomic proof which uses them



Definition 2.1.6 A *root* of a tree $(N, \mathbf{S})$ is an element a of N such that for no $S \in \mathbf{S}$ is there an element $b \in N$ such that bSa.

Note that the above definition of a tree permits there to be more than one root node, so that a tree may comprise several quite distinct subtrees which themselves are trees. This corresponds to allowing atomic proofs to contain irrelevant subproofs. Although extraordinarily unnatural, this is necessary to capture a standard notion of formal proof according to which a proof is a sequence of sentences where each is either an axiom or follows from earlier sentences by the application of a rule. We define a notion of proof which is a relevant restriction of this, in that it has the further requirement that every member of the tree (proof) except one, the conclusion, is used in the application of a rule giving a "later" sentence:—

$\Rightarrow r;\ \ \Rightarrow t;\ \ t,r \Rightarrow p;\ \ r,p \Rightarrow q;\ \ r \Rightarrow s$

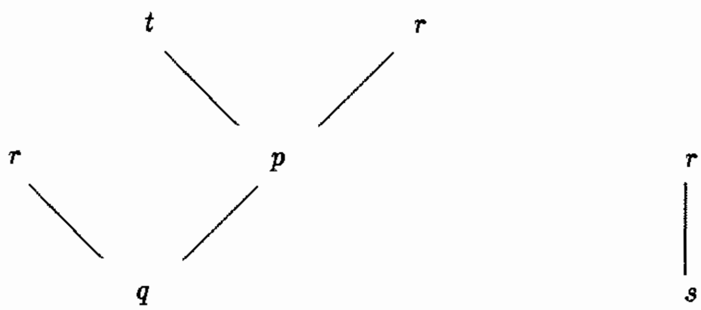


Figure 2.1: An irrelevant atomic proof using the indicated rules.

Definition 2.1.7 A *relevant proof* is an atomic proof with exactly one root.

Figure 2.1 shows an example of an irrelevant atomic proof which comprises two relevant subproofs.

Definition 2.1.8 Let $\Re$ be a set of atomic rules, then q is deducible from a sequence $[p_1, \ldots p_n]$, denoted $p_1, \ldots p_n \vdash_{\mathcal{R}} q$, iff there is an atomic proof with root node q using a subset of the rules of $\Re$ union the rules $\Rightarrow p_i$.

Definition 2.1.9 An *atomic theory* $\mathcal{A}$ is comprised of axioms and rules over the atomic language, and the deducibility relation generated by these axioms and rules. Letting $\Re_{\mathcal{A}}$ denote the set of axioms and rules, I denote $\mathcal{A}$ by the pair $\langle \Re_{\mathcal{A}}, \vdash_{\mathrm{A}} \rangle$.

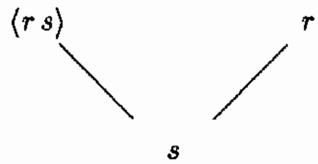
2.2 Introduction to hedges

Associated with each atomic theory in an atomic system is the set of proofs "in" the atomic theory. We now consider how to provide a formal representation of such proofs keeping in mind the aim of capturing *use*. The intention is for a list $\langle \alpha\, \beta \rangle$ to represent a proof β which *uses* a subproof α, so that α really contributes to the proof of the conclusion of β. In example 2.1 s fails to be used in the proof of q, whereas p does contribute to the proof of q. Thus we would want $\langle p\, q \rangle$ to "hold" but not $\langle s\, q \rangle$. Also the first subproof, which satisfies description $\langle p\, q \rangle$, uses a subproof r too and so satisfies the description $\langle r\ \langle p\, q \rangle \rangle$. So where we have the application of the atomic rule $r, p \Rightarrow q$ we have an atomic proof which satisfies the correspondingly shaped list description. Similarly from the second subproof in figure 2.1 we have $\langle r\, s \rangle$ "holding". More generally, for each list corresponding to a rule $p_1, \ldots p_n \Rightarrow q$ used to prove q we want $\langle p_1 \ldots \langle p_n\, q \rfloor$ to hold; for $\langle p_n\, q \rangle$ holds as it represents a proof of q which uses a subproof of p_n, and each $\langle p_r \ldots \langle p_n\, q \rfloor$ represents a proof satisfying description $\langle p_{r+1} \ldots \langle p_n\, q \rfloor$ which uses a proof of p_r, etc.. Thus in example 2.1 we want the following lists at least to "hold":—

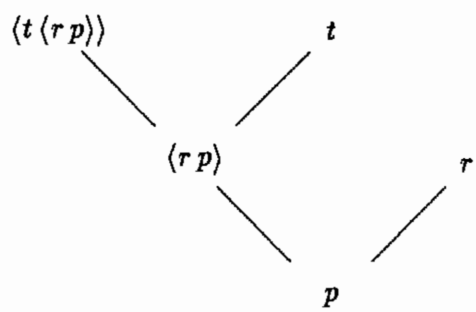
$$r,\ t,\ \langle t \langle r\, p \rangle \rangle,\ \langle r\, p \rangle,\ p,\ \langle r \langle p\, q \rangle \rangle,\ \langle p\, q \rangle,\ q,\ \langle r\, s \rangle,\ s$$

As things stand, from the above list representations we don't know that the p of $\langle r\ \langle p\, q \rangle \rangle$ is the same as that in $\langle t \langle r\, p \rangle \rangle$, from which observation it follows that the proof of q also uses a subproof of t (assuming that "use of a subproof" is transitive), so that we should also have $\langle t\, q \rangle$ holding; nor even do we know that $\langle t \langle r\, p \rangle \rangle$ and $\langle r\, p \rangle$ are generated from the same relevant atomic proof. Thus the lists, by themselves, only provide an incomplete picture as to what is going on in the atomic theory, leaving out information about the underlying connections which hold between them. It would be nice, in the case of a relevant atomic proof, to be able to capture all of this information about the lists it generates in a single structure. What is required is a structure which captures the total description of an atomic proof together with that of its subproofs, their subproofs and so on. This is achieved using the notion of a hedge, which is similar to a tree with two successor relations.

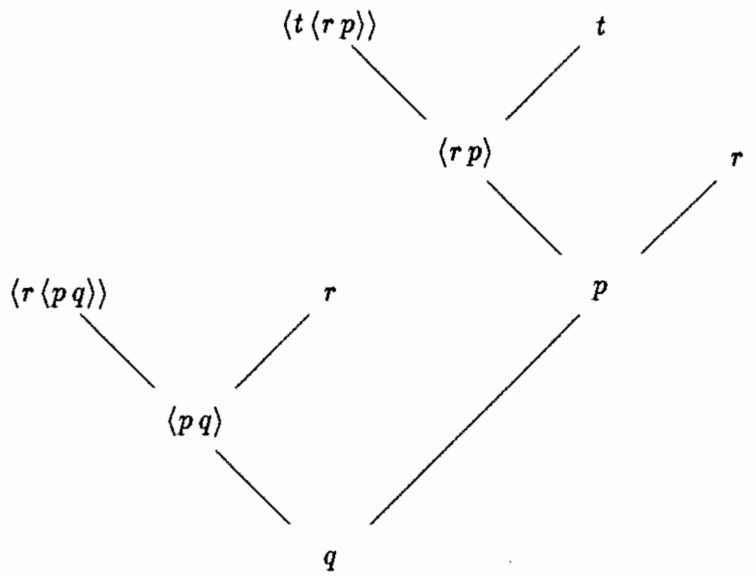
To motivate the definition of a hedge consider once more the atomic proof shown in figure 2.1. First consider the rather simple relevant subproof of s. The subproof of r is just in virtue of it being an axiom, so there is no further subproof information to be incorporated and just a single node with r assigned to it suffices as its hedge representation. For the proof of s the following hedge with three nodes is used:—



The bottom node tells us it's a proof of s, the left successor (looking up for successors) tells us it's a proof of s with a subproof of r, and the corresponding right successor has a subhedge which is the total description of that subproof of r (namely the single node r). Now consider the subproof of p in figure 2.1. This has a total description:—



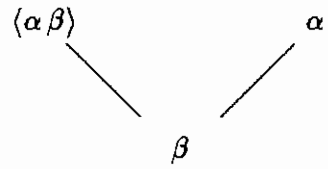
The "backbone" corresponds to the application of the atomic rule $t, r \Rightarrow p$ with the backbone tip having the same shape list and each right successor leading to the total description of the corresponding "peeled off" antecedent list (which in this case are just single nodes). And applying the same procedure to the final step in the atomic proof of q we have:—



The backbone (q and its left successors) corresponds to the application of $r, p \Rightarrow q$ and the total description of the subproof of p is given by the subhedge generated from q's right successor p, whilst that of r is given by the subhedge generated from $\langle p\, q\rangle$'s right successor. So the idea is that where we have the application of a rule $p_1, \ldots p_n \Rightarrow q$ we form a new backbone which has as tip $\langle p_1 \ldots \langle p_n\, q \rfloor$, nodes $\langle p_r \ldots \langle p_n\, q \rfloor$, and root q, and the right successors lead to subhedges which are total descriptions of the subproofs of the premisses $p_1, \ldots p_n$.

So a *hedge* is intended to capture a total description of a proof together with that of its subproofs, their subproofs and so on, where

- a list $\langle \alpha\, \beta \rangle$ represents a (sub-)proof satisfying description β which has a contributing subproof satisfying description α, and
- a node corresponding to $\langle \alpha\, \beta \rangle$ has a left predecessor corresponding to the β, which itself has a right successor corresponding to the α (as shown in the following diagram) whose subhedge describes the subproof of α.



A hedge is similar to a binary tree in that it comprises nodes with left and right successor relations. However there may be more than one root, and there is a lack of symmetry in the two successor relations whereby moving rightwards corresponds to taking a proper subhedge whereas moving leftwards does not. (A subhedge is generated by taking successors under both relations as well as left predecessors. The smallest subhedge containing the above β captures its total description (and includes the $\langle\alpha\,\beta\rangle$), and the smallest subhedge containing the α captures its total description (and excludes $\langle\alpha\,\beta\rangle$ and β).)

Definition 2.2.1 *hedge* is a class of structures $H = (K, S_L, S_R)$ where K is a set of nodes and S_L, S_R are relations on K, determined inductively as follows:—

- The structure with $K = \{X\}$ any singleton set and S_L, S_R both the empty relation, is a *hedge* with *backbone* $\{X\}$.
- For arbitrary n, given n *hedges* $H_1 = (K_1, {S_L}^1, {S_R}^1), \ \ldots \ H_n = (K_n, {S_L}^n, {S_R}^n)$ where $K_i \bigcap K_j = \emptyset$ for $i \neq j$, we can compose these to form a new *hedge* $H = (K, S_L, S_R)$ as follows:—
  - For each $H_i = (K_i, {S_L}^i, {S_R}^i)$ $1 \leq i \leq n$ choose $X_i \in K_i$ where X_i is in the *backbone* of H_i.
  - Choose $n{+}1$ arbitrary entities $Z_1, \ldots Z_{n+1}$ which are not members of $\bigcup_{i=1}^{n} K_i$ and put $K = \{Z_1, \ldots Z_{n+1}\} \bigcup (\bigcup_{i=1}^{n} K_i)$.
  - $\{Z_1, \ldots Z_{n+1}\}$ comprise the *backbone* of K.

- S_L is determined as follows.

  For elements Z_r and Z_m of the backbone, $Z_r\, S_L\, Z_m$ iff $m = r - 1$.

  S_L does not hold between elements of the backbone and the other members of K, and vice-versa.

  For the other members of K, $X\, S_L\, Y$ iff X and Y are members of K_i for some $1 \leq i \leq n$, and $X\, {S_L}^i\, Y$.

- S_R is determined as follows.

  For no element Z_i of the backbone do we have any Y such that $Y\, S_R\, Z_i$.

  For Z_i in the backbone we have $Z_i\, S_R\, Y$ iff $2 \leq i \leq n+1$ and $Y = X_{i-1}$ (the chosen member of the backbone of K_{i-1}).

  For the other members of K, $X\, S_R\, Y$ iff X and Y are members of K_i for some $1 \leq i \leq n$, and $X\, {S_R}^i\, Y$.

This completes the definition of the structure $H = (K, S_L, S_R)$ together with its *backbone*, and so the inductive clause of our definition of the category *hedge*.

Given a hedge $H = (K, S_L, S_R)$ then where $a\, S_L\, b$, b is the *successor* of a under S_L and a is the *predecessor* of b under S_L, and similarly for S_R.

A *tip* is an element of K which is not a predecessor under either relation.

A *root* is an element of K which is not a successor under either relation.

Having defined the general notion of a hedge we now adapt it to our present purposes by associating lists with the nodes.

Definition 2.2.2 A *list-hedge* is a hedge $H = (K, S_L, S_R)$ together with a function f with domain K and range the set of lists, where:—

- For arbitrary $X \in K$ if $f(X) = \langle \alpha\, \beta \rangle$ then there are $Y, Z \in K$ such that $Y\, S_L\, X$, $Y\, S_R\, Z$, $f(Y) = \beta$ and $f(Z) = \alpha$.

- If for arbitrary $X \in K$ there is $Y \in K$ such that $Y \; S_L \; X$ then $f(X) = \delta$ where δ is of the form $\langle \alpha \, \beta \rangle$.

Thus a list-hedge is simply a hedge with a list assigned to each node in a manner which gives the "triangle"

$$\begin{tikzcd}[arrows=dash] \langle \alpha \, \beta \rangle \arrow[dr] & & \alpha \arrow[dl] \\ & \beta & \end{tikzcd}$$

for each assigned list of the form $\langle \alpha \, \beta \rangle$; and for every such "triangle" in the hedge the corresponding nodes have lists of the above form assigned to them.

Note that it is not possible to define list-hedges by simply allowing the set of nodes of the corresponding hedge to be equal to the appropriate set of lists (the image of the assignment function f). For the same list may be assigned to distinct nodes as in the earlier (and the following) example of a hedge used to represent an atomic proof.

Before further consideration of the properties of hedges we formalise the idea of a hedge representing an atomic proof:—

Definition 2.2.3 A *hedge representation of a relevant atomic proof at an atomic theory* $\mathcal{A}$ is a list-hedge E determined as follows:—

- Associate with a proof comprising a single sentence p (so an axiom of $\mathcal{A}$) a singleton list-hedge with p assigned to the node.

- Where a sentence q is obtained by the application of a rule $p_1, \ldots p_n \Rightarrow q$ of $\mathcal{A}$, form a hedge by applying the inductive step of definition 2.2.1 to each of the hedges of the *hedge representations* of the proofs of each p_i, taking as the *chosen* element the node p_i is assigned to, and assign to the *backbone* tip the list $\langle p_1 \ldots \langle p_n\, q\rfloor$, and to the *backbone* nodes each $\langle p_r \ldots \langle p_n\, q\rfloor$ in descending order (i.e. taking the left predecessor), so that the *backbone* root is assigned q.

As a further example figure 2.2 shows the hedge representation of an atomic proof.

2.3 Properties of hedges

Now follows a detailed consideration of hedges as formally defined, together with some associated notions.

Lemma 2.3.1

1. Each hedge $H' = (K', S'_L, S'_R)$ used in the inductive procedure for generating a hedge $H = (K, S_L, S_R)$ or [1] which is equal to H, is retained in H in the sense that
   - $K' \subseteq K$
   - the closure of K' in H taking left predecessors, left successors and right successors is K' itself
   - the restrictions of S_L and S_R to K' are S'_L and S'_R respectively.
2. Suppose that H' is used in the inductive procedure for generating a hedge H, then a node of H which is in H' is a tip in H iff it is a tip in H'.
3. The backbone of a hedge is the unique maximal set of elements which is closed under S_L and contains no S_R successors.
4. If an element has no left successor then it has no right successor (and so is a tip).

[1] Allowing the uninteresting case where H' equals H facilitates the induction proof.

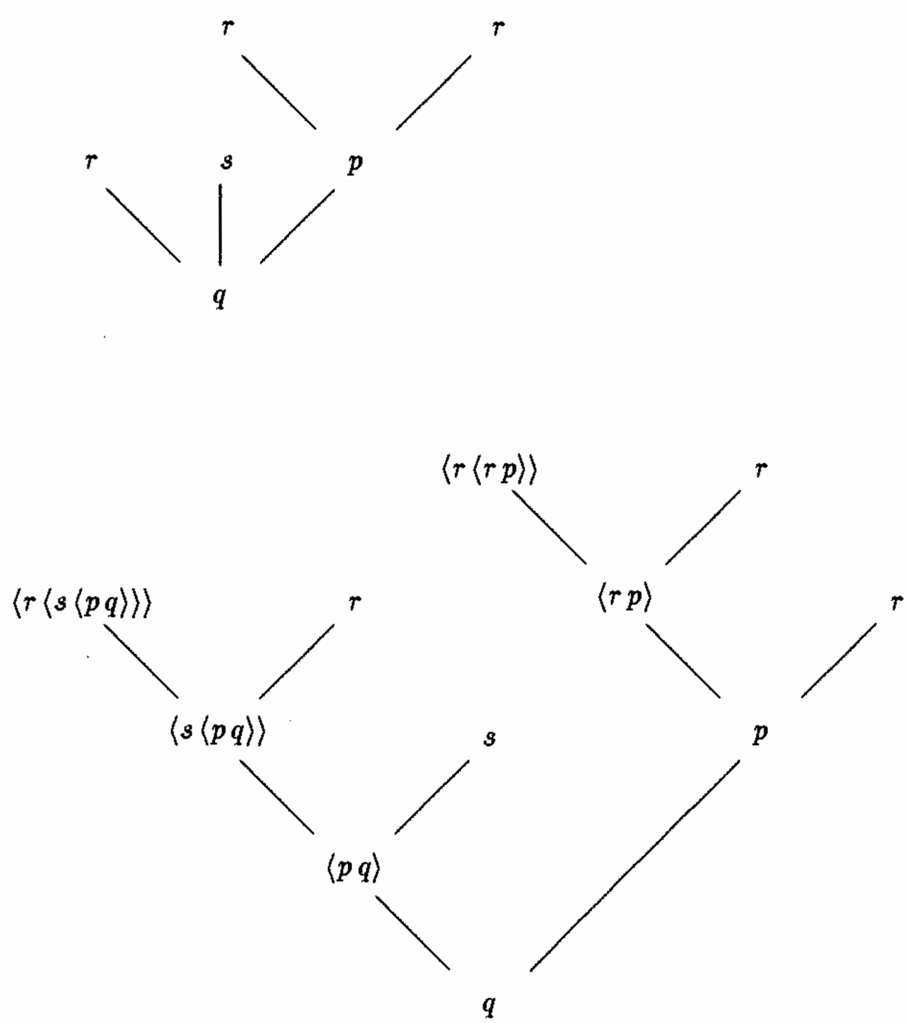


Figure 2.2: An atomic proof and its hedge representation.

5. Let a, b be any two elements of a hedge $H = (K, S_L, S_R)$ where a is in its backbone, then there is a sequence of elements $a_1, \ldots, a_m$ of K such that $a_1 = a$, $a_m = b$ and for each pair a_i, a_{i+1}, either one is the successor of the other under S_L, or $a_i \, S_R \, a_{i+1}$.

Proof

1. The proof is by induction on definition 2.2.1.
   Clearly a singleton hedge H satisfies (1) as the only hedge in the inductive procedure is H itself.
   Let H be obtained by applying the procedure of definition 2.2.1 to $H_1, \ldots H_n$. First let H' be used in the inductive procedure for generating H. So H' is used in the inductive procedure for generating H_k for a particular $1 \leq k \leq n$, or is equal to H_k. By the induction hypothesis H' is preserved in H_k in the desired fashion. Moreover H_k is preserved in H in the desired fashion, so that H' is too, as required. In the case that H' equals H (1) obviously holds.
   Thus the induction step is proved and hence we have (1).

2. (2) is a simple corollary of (1).

3. (3) is proved by a simple induction on definition 2.2.1; uniqueness is established by noting that in the procedure of definition 2.2.1 each of the previous backbones of the H_i lose the required property in virtue of the chosen X_i being a successor under S_R.

4. (4) and (5) are also proved by induction on definition 2.2.1.

Corollary From (1) and (5) an earlier hedge H' used to generate a hedge H is determined by taking a member of the backbone of H' and closing under (per H) left predecessors, left successors and right successors.

The fourth fact above shows that the tips of a hedge are just those elements with no left successor: the "left tips".

The lemma and its corollary suggest two distinct notions of subhedge:—

1. (The hedge itself or) an earlier hedge used in the inductive procedure of its generation.

2. A subset of nodes containing a "backbone element" and closed under left predecessors, left successors and right successors.

We observe here that these two senses of subhedge are equivalent. The above corollary tells us that the first is included in the second. That the second sense is included in the first follows from noting that given such a subset of nodes, it must first be included into the hedge where an earlier hedge containing the particular "backbone element" in its backbone is included, and by the corollary that earlier hedge is identical with the given subset of nodes.

Now under the intuitive interpretation of a list-hedge the idea is that where we have $\beta\, S_L\, \langle \alpha\, \beta \rangle$ and $\beta\, S_R\, \alpha$ (see figure 2.3),[2] the list $\langle \alpha\, \beta \rangle$ represents a proof of structure β which uses a subproof of structure α, and the list-hedge gives total descriptions of these corresponding to the sub-list-hedges generated from the elements β and α respectively. That is the α and β are thought of as being the same (in the sense of describing the same proofs) as the corresponding sublists of the $\langle \alpha\, \beta \rangle$. As example consider the hedge representation of the example atomic proof given in figure 2.2. The atomic proof contains just three distinct subproofs of r; the leftmost one is represented in the list-hedge by the r in $\langle r\, \langle s\, \langle p\, q \rangle \rangle \rangle$ as well as the r assigned to the corresponding right-branch. Similarly the atomic proof contains just one distinct subproof of p which is referred to by every occurrence of p in the list-hedge . It is convenient to use a located sublist relation on the

[2] Here abusing the notation in the obvious and natural way— strictly speaking we have three nodes of the underlying hedge to which are assigned the corresponding lists.

elements of a hedge which captures these identifications. This located sublist relation is obtained by extending the ordinary located sublist relation (in virtue of occurrence of a token) upon each list assigned to a node, to make the extra needed identifications: Where we have $\beta\ S_L\ \langle \alpha\, \beta \rangle$ and $\beta\ S_R\ \alpha$ every sublist of the first β is identified with the corresponding sublist in $\langle \alpha\, \beta \rangle$, and every sublist of the α is identified with the corresponding sublist in $\langle \alpha\, \beta \rangle$.

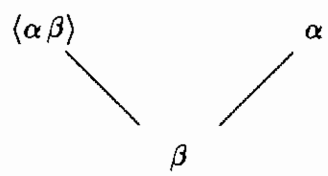


Figure 2.3: An element of a list-hedge with its left and right successors.

So in our example list-hedge (figure 2.2) we have exactly the following located sublists:

$$\langle r\, \langle s\, \langle p\, q \rangle \rangle \rangle,\ \ \textit{1st}\ r,\ \ \langle s\, \langle p\, q \rangle \rangle,\ \ s,\ \ \langle p\, q \rangle,\ \ q,\ \ p,\ \ \langle r\, p \rangle,\ \ \textit{2nd}\ r,\ \ \langle r\, \langle r\, p \rangle \rangle,\ \ \textit{3rd}\ r$$

Note that according to this located sublist relation the $\underline{p}$ is a located sublist of *both* $\underline{\langle r\, \langle s\, \langle p\, q \rangle \rangle \rangle}$ and $\underline{\langle r\, \langle r\, p \rangle \rangle}$. Thus the class of located sublists of a list-hedge does not in general correspond to the class of (occurrences of) sublists of some single list.

The class of located sublists of a list-hedge H is called the *domain* of H. As used above, an underline such as in '$\underline{\varphi}$' denotes a located sublist of a list-hedge , as well as occurrence in more usual settings.

A formal definition of the located sublist relation is provided at the end of this section.[3]

For every element $\underline{\varphi}$ in the domain of a list-hedge with underlying hedge $H = (K, S_L, S_R)$, there is a unique corresponding member X of K to which is assigned φ. For in virtue

[3] As formal definitions of such notions as "occurrence" tend to be, it is highly unilluminating.

of definition 2.2.2 every complex list $\langle \alpha\, \beta \rangle$ is "unpacked" giving nodes to which α and β are assigned; so that the domain of located sublists is isomorphic with K. The notions *left successor, right successor, backbone,* etc. carry over to members of the domain of the list-hedge, using this fact in the obvious way.

Since the domain of a list-hedge is isomorphic with the set of nodes of its underlying hedge, the distinction between them can be felicitously blurred. In what follows list-hedges will tend to be referred to simply as hedges.

As a further example let the domain of H be the set of located sublists of $\langle p\ \langle\langle p\, p\rangle\ p \rightarrow q\rangle\rangle$. Figure 2.4 shows the hedge for this domain. Figure 2.7 shows a more complex hedge— the antecedent sublist $\langle p\, r\rangle$ of $\big\langle\langle p\, r\rangle\ \langle\langle p\, p\rangle\ p\rightarrow q\rangle\big\rangle$ is identified with the guard list of $\langle p\ \langle p\, r\rangle\rangle$.

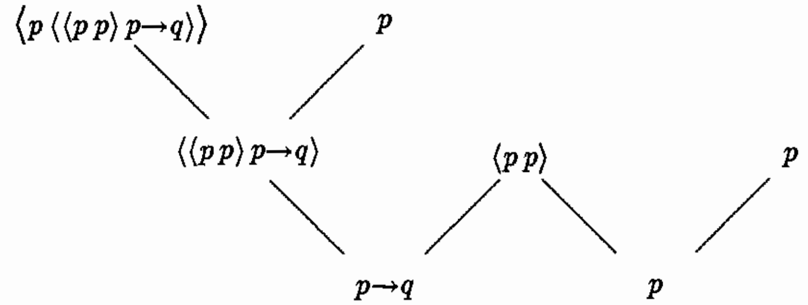


Figure 2.4: Example of a hedge.

Definition 2.3.2 Let α be a list, then the *identity hedge I_α for α* is the list-hedge whose images under the assignment function are exactly the located sublists of α.

Note that the identity hedge I_α is like the construction tree for α, but turned on its side and with a change in the branching structure where triangles

$$\begin{tikzcd}[column sep=small]
\alpha \arrow[dr, no head] & & \beta \arrow[dl, no head] \\
& \langle \alpha\, \beta \rangle &
\end{tikzcd}$$

are replaced by that of figure 2.3.

By lemma 2.3.1 (4) an element of the domain of a hedge H is a tip iff it has no left successor, that is iff, for any list-hedge with underlying hedge H, that element is not assigned a guard list of any element in the domain (other than itself). In particular the tips of the identity hedge on α are α itself and each antecedent sublist of α. Figure 2.4 is an example of an identity hedge and figure 2.6 shows a list-hedge which is not an identity hedge. As already noted, observe that the domain of a list-hedge does not in general correspond to the set of located sublists of a single list. So in general identity hedges do not suffice in order to represent atomic proofs— we do need the more general notion of list-hedge.

Definition 2.3.3 Suppose we have a hedge $H = (K, S_L, S_R)$ with $X \in K$. Then the *restriction of H to X* is the earliest hedge in the inductive procedure for generating H which contains X.

Definition 2.3.4 Suppose we have a list-hedge H with $\underline{\varphi}$ in its domain. Then the *restriction of H to $\underline{\varphi}$*, denoted $H\restriction_{\underline{\varphi}}$, is the list-hedge with underlying hedge the restriction of H to X where X is the node corresponding to $\underline{\varphi}$, and with assignment function the corresponding restriction of the original.[4]

[4]Using the formal definition of a located sublist (definition 2.3.9) $\underline{\varphi}$ is the equivalence class generated by $(\varphi, \emptyset, X)$.

Thus $H|_{\underline{\varphi}}$ is the smallest subhedge of H containing $\underline{\varphi}$. Also note that $\underline{\varphi}$ must be in the backbone of $H|_{\underline{\varphi}}$, for otherwise it occurs in an earlier hedge of the inductive procedure for generating $H|_{\underline{\varphi}}$, contradicting the above definition.

In terms of our intuitive interpretation, $H|_{\underline{\varphi}}$ is that restriction of the hedge H containing the total description of the subproof $\underline{\varphi}$ (and its subproofs); so that $H|_{\underline{\varphi}}$ has as domain those located sublists determined by $\underline{\varphi}$ in a manner intended to capture those elements upon which $\underline{\varphi}$ depends, what they depend upon, and so on, and similarly for each sublist of $\underline{\varphi}$.

Lemma 2.3.5 The domain of the restriction of a list-hedge H to $\underline{\varphi}$ is determined by:—

- $\underline{\varphi}$ is in the domain,
- for each located sublist of form $\underline{\langle\gamma\,\lambda\rangle}$ in the domain of H if $\underline{\lambda}$ is in the restricted domain then $\underline{\langle\gamma\,\lambda\rangle}$ is too,
- every located sublist of an element in the restricted domain is also in the restricted domain.

Proof

By the corollary to lemma 2.3.1 closure under left predecessors, left successors and right successors generates exactly the required subhedge, but in virtue of definition 2.2.2 this corresponds to the above inductive clauses.

I define a further notion of restriction of the domain of a hedge which is needed in the definitions of $(\rightarrow||)$ and *(mix)* to follow. The notion is similar to that of a filter generated by an element of a lattice except that we need to capture the whole hedge rightwards too.

Definition 2.3.6 The *bush* generated by a located sublist $\underline{\varphi}$ of a hedge $H = (K, S_L, S_R)$ is the structure (K', S'_L, S'_R), denoted $H\left\{\underline{\varphi}\right\}$, where K' is the subset of K which contains

- the $\underline{\varphi}$
- the S_L successor of $\underline{\varphi}$, its S_L successor, and so on
- the S_R successors $\underline{\delta}$ of all the above (except the last, which doesn't have one) together with all elements of the subhedges generated by each such $\underline{\delta}$;

and S'_L, S'_R are the restrictions to K' of S_L, S_R.

$$
\begin{tikzcd}
\langle p \, \langle \langle p \, p \rangle \, p \rightarrow q \rangle \rangle \arrow[dr, dash] & & p \arrow[dl, dash] \\
& \langle \langle p \, p \rangle \, p \rightarrow q \rangle &
\end{tikzcd}
$$

$$H\left\{\underline{\langle \langle p \, p \rangle \, p \rightarrow q \rangle}\right\}$$

Figure 2.5: A bush from our earlier example of a hedge (figure 2.4).

So $H\left\{\underline{\varphi}\right\}$ is that part of the hedge with domain that of $H|_{\underline{\varphi}}$ *minus* those elements which are proper sublists of $\underline{\varphi}$, and *minus* those elements which are in the domain of $H|_{\underline{\alpha}}$ for the major antecedents $\underline{\alpha}$ of $\underline{\varphi}$; $H\left\{\underline{\varphi}\right\}$ is the top of $H|_{\underline{\varphi}}$ and ignores all of the "internal" structure of $\underline{\varphi}$. Another way to think about a bush $H\left\{\underline{\varphi}\right\}$ is that it is the hedge obtained from H by supposing that $\underline{\varphi}$ has no structure, so that replacing $\underline{\varphi}$ by p in H gives a hedge isomorphic to $H\left\{\underline{\varphi}\right\}$. Figure 2.5 shows the bush $H\left\{\underline{\langle \langle p \, p \rangle \, p \rightarrow q \rangle}\right\}$ from figure 2.4.

Example 2.7 shows a more complicated bush, for the hedge of figure 2.6, which is not an identity hedge.

A hedge $H|_{\underline{\alpha}}$ can be thought of as the identity hedge $\mathcal{I}_\alpha$ with the bushes $H\left\{\underline{\varphi}\right\}$ stuck on to each of its tips $\underline{\varphi}$. For example the hedge of figure 2.7 can be formed by taking the identity hedge for $\langle\langle p\,p\rangle\, p{\rightarrow}q\rangle$ and sticking on $H\left\{\underline{\langle\langle p\,p\rangle\, p{\rightarrow}q\rangle}\right\}$, $H\left\{\underline{\langle p\,p\rangle}\right\}$ and $H\left\{\underline{p}\right\}$ (the latter two bushes contain just one element so don't augment the structure).

It is necessary to make precise the idea of "sticking" a bush onto a hedge. So suppose we have a hedge H with a tip $\underline{\varphi}$ and a bush $B\left\{\underline{\varphi}\right\}$ with backbone root $\underline{\varphi}$ to be stuck onto the hedge at that tip. Now $\underline{\varphi}$ is the backbone tip of a subhedge H' of the given hedge. Take the "premiss" hedges $H_1',\ldots,H_n'$ of H', and take the subhedges $B_1,\ldots,B_r$ generated by the right successors of the members of the backbone of $B\left\{\underline{\varphi}\right\}$, and at the step where H' is generated in the inductive production of H instead form $H^\dagger$ as follows:

- Apply the inductive step of definition 2.2.1 to the hedges $B_1,\ldots,B_r,\ H_1',\ldots,H_n'$,
- assign to the backbone tip the list assigned to that of the bush $B\left\{\underline{\varphi}\right\}$ (which is of the form $\langle\beta_1\ \ldots\langle\beta_r\ \varphi\rfloor\rangle$ (this determines the assignment to the remaining nodes of the backbone), and
- preserve the same assignment of lists for the subhedges.

Continue the inductive generation as per H but with $H^\dagger$ in place of H' to obtain the required hedge.

Lemma 2.3.7 Let H be a hedge containing $\underline{\alpha}$ in its backbone (so that $H = H|_{\underline{\alpha}}$). Then H is equal to the structure formed by taking the identity hedge on α, $\mathcal{I}_\alpha$ and sticking on to each of its tips $\underline{\varphi}$ the bush $H\left\{\underline{\varphi}\right\}$.

Proof

Since $\underline{\alpha}$ is in the backbone, by lemma 2.3.1 (5) every other element of H can be reached from $\underline{\alpha}$ by a series of moves taking left successors, left predecessors and right successors.

So we check that each such "move" is included within the sublists of $\underline{\alpha}$ or in the domain of a bush $H\left\{\underline{\varphi}\right\}$ for a tip $\underline{\varphi}$ of the identity hedge for $\underline{\alpha}$.

Consider a sublist $\underline{\lambda}$ of $\underline{\alpha}$.

A left predecessor of $\underline{\lambda}$ corresponds to a sublist of $\underline{\lambda}$, hence it is also a sublist of $\underline{\alpha}$ and so is included.

If it has a right successor $\underline{\gamma}$ then there is $\underline{\langle\gamma\,\lambda\rangle}$ in the domain and $\underline{\lambda}\, S_L\, \underline{\langle\gamma\,\lambda\rangle}$. If $\underline{\langle\gamma\,\lambda\rangle}$ is a sublist of $\underline{\alpha}$ then so is $\underline{\lambda}$ and hence it is included. Otherwise $\underline{\lambda}$ must be a tip of $\underline{\alpha}$, but then $\underline{\langle\gamma\,\lambda\rangle}$ and $\underline{\gamma}$ are in the domain of $H\{\underline{\lambda}\}$, so the latter is included as required.

If $\underline{\lambda}$ has a left successor then it is $\underline{\langle\gamma\,\lambda\rangle}$ as above, and so is included by the obvious modification of the above argument.

Consider a sublist $\underline{\lambda}$ in the domain of a bush $H\left\{\underline{\varphi}\right\}$ for some tip $\underline{\varphi}$ of $\underline{\alpha}$.

Bushes are closed under left and right successors, and left predecessors except where the left predecessor is a sublist of the generating $\underline{\varphi}$; but in that case the left predecessor is a sublist of $\underline{\alpha}$ and so is included.

Thus every member of the domain of H is included as required.

As a notational convention I use $[\underline{\alpha}\backslash\beta]$ to denote replacing a located list $\underline{\alpha}$ by β. In the case of hedges and associated structures, for each sublist of form $\langle A\,B\rangle$ replaced by $A{\to}B$, its left predecessor $\underline{B}$ and the right successor $\underline{A}$ of that are deleted, with all three replaced by the single node $\underline{A{\to}B}$; where this procedure may have to be done to successively larger sublists. So $H\left[\underline{\langle A\,B\rangle}\backslash A{\to}B\right]$ denotes the hedge obtained by replacing every occurrence of $\underline{\langle A\,B\rangle}$ and its "triangle"

$$\begin{tikzcd}[arrows=dash]
\langle A\,B\rangle \arrow[dr] & & A \arrow[dl] \\
& B &
\end{tikzcd}$$

in the hedge H by the single node $A \rightsquigarrow B$ (so that the subhedge generated by A is also "lost").

We now turn to the formal definition of *located sublist.* Let us represent the location of an occurrence of a sublist of a list by a sequence, the *location sequence*, of 1s and 2s where 1 denotes the first member and 2 the second member of a list. A location sequence applied to a list is "processed" by applying the rightmost element (1 or 2) first, and then repeating. For example 21 applied to $\big\langle\langle q\, p\rangle\, \langle p\, \langle r\, p\rangle\rangle\big\rangle$ denotes the left-most p, while the r is given by 122. Thus the location sequence can be thought of as providing the path followed in the construction tree of the whole list to find the location of that occurrence of a sublist.

Definition 2.3.8 Suppose we have a list-hedge with underlying hedge $H = (K, S_L, S_R)$ and assignment function f. The set K' of *sublists-at-nodes* is the set of triples formed as follows:

- for each $X \in K$ let
  $K_X = \{(\delta, e, X) : \delta \text{ is a sublist of } f(X) \text{ with location sequence } e\}$
- and put $K' = \bigcup_{X \in K} K_X$.

Definition 2.3.9 Suppose we have a list-hedge with underlying hedge $H = (K, S_L, S_R)$. Let K' be its set of sublists-at-nodes. Then the *located sublist relation* for the list-hedge is the smallest equivalence relation on K' which is generated from:

- the identity relation on K',
- where we have $X\, S_L\, Y$ and $X\, S_R\, Z$ identify each (δ, e, X) with $(\delta, e2, Y)$ and each (δ, e, Z) with $(\delta, e1, Y)$.

The set of equivalence classes under this relation is the *domain* of the list-hedge.

Note that every member of the domain, i.e. every equivalence class under the located sublist relation, contains a unique element $(\delta, \emptyset, X)$ where $\delta = f(X)$, in virtue of the definition of a list-hedge (definition 2.2.2). So each member of the domain can be denoted by $\underline{\delta}$, with δ as above.

2.4 Linked sequences

An alternative picture of a list-hedge H is that of a sequence of lists corresponding to the tips of the hedge, beginning with the backbone tip, and linking up major antecedents $\underline{\alpha}$ with the appropriate (sublist of the) later-occurring list corresponding to the tip of the backbone of $H|_{\underline{\alpha}}$. For example the hedge of figure 2.6 can be represented by

$$\Big\{\big\langle\langle p\, r\rangle\, \langle\langle p\, p\rangle\, p {\rightarrow} q\rangle\big\rangle,\ \langle p\, \langle p\, r\rangle\rangle\Big\}$$

The major antecedent $\underline{\langle p\, r\rangle}$ is linked to its location as a guard list in its subhedge which is the identity hedge for $\langle p\, \langle p\, r\rangle\rangle$.

Before defining this alternative notion we consider the associated ordering property of hedges.

Definition 2.4.1 The *order* on the domain of a hedge H is the total order determined as follows:

- order the element of a singleton hedge in the only possible way
- where H is formed from $H_1, \ldots, H_n$ put the backbone tip first, then its left predecessor, and so on until the backbone root is reached, then put the elements of H_1 and then those of H_2, etc., preserving their internal order.[5]

[5]The hedges considered here are always finite in virtue of definition 2.2.1, but it is evident how to

Lemma 2.4.2 The order on the domain of a hedge H satisfies:—

- For each $\underline{A}$ the sublists with guard $\underline{A}$, as well as $\underline{A}$ itself, are ordered superlists first,
- for each major antecedent $\underline{\alpha_i}$ of a list $\underline{\alpha} = \langle\alpha_1 \ldots, \langle\alpha_n\ A]$ $\underline{A}$ comes before every member of the domain of $H|_{\underline{\alpha_i}}$,
- and every located sublist in each $H|_{\underline{\alpha_i}}$ domain comes before every located sublist in the domain of $H|_{\underline{\alpha_{i+1}}}$ if there is such a later occurring major antecedent.

Proof

The proof is by induction on the procedure of definition 2.2.1.

Definition 2.4.3 A *linked sequence* is a pair (E, L) where E is a finite sequence of lists and L is a 1-1 relation on the occurrences of sublists of E (which "links" them) satisfying:—

- If $\underline{\alpha}\, L\, \underline{\beta}$ then in fact $\underline{\alpha}$ is list-identical with $\underline{\beta}$, and $\underline{\alpha}$ is a major antecedent of a list in E with $\underline{\beta}$ a guard list of a later occuring list in the sequence.
- If $\underline{\alpha_1}\, L\, \underline{\beta}$ and $\underline{\alpha_2}\, L\, \underline{\gamma}$ where $\underline{\alpha_1}$ and $\underline{\alpha_2}$ are major antecedents occurring in that order of a list in E, then the list containing $\underline{\beta}$ comes before that containing $\underline{\gamma}$ in E.
- Every list other than the first contains a unique guard list $\underline{\beta}$ such that for some $\underline{\alpha}$ we have $\underline{\alpha}\, L\, \underline{\beta}$.

A linked sequence is an abbreviated representation of a hedge, as the following two lemmas show. It is abbreviated in that guard lists are not denoted separately from the list they occur in, and where a major antecedent $\underline{\alpha}$ generates a subhedge which is just

modify the definition to allow them to be infinitary structures. The recipe of this definition has in mind a way of indexing the domain of a hedge.

its identity hedge (so with domain just the located sublists of $\underline{\alpha}$) it too is not denoted separately from the list it occurs in. Thus a linked sequence representation of a hedge is simply the sequence of backbone tips of its component subhedges, with "links" used to capture the located sublist relation.

Lemma 2.4.4 Every linked sequence (E, L) determines a hedge H where the order of lists in E is the appropriate suborder of the order on the domain of H.

Proof

Given a linked sequence (E, L) generate a hedge as follows.

1. Begin with those members $\underline{\gamma}$ of the sequence E not containing any $\underline{\alpha}$ which is a predecessor under L. Associate with the singleton sequence containing such $\underline{\gamma}$ the identity hedge on γ.

2. Choose a member $\underline{\gamma}$ of E which satisfies: For all (major antecedents) $\underline{\alpha}$ of $\underline{\gamma}$, if $\underline{\alpha}\, L\, \underline{\beta}$ then $\underline{\beta}$ is a sublist of an element which is the first member of a subsequence of E to which we have already assigned a hedge.

   Associate with each major antecedent of $\underline{\gamma}$ which is not a predecessor under L its identity hedge.

   Associate with the remainder the hedge assigned to the subsequence they are linked to.

   Obtain the hedge assigned to the larger subsequence of E with first member $\underline{\gamma}$ followed by the subsequence linked to its first major antecedent and then the next and so on, by applying the inductive step of definition 2.2.1 to these hedges with backbone tip γ, and using the above associated hedges. Note that this preserves the order (where we take the $\underline{\gamma}$ in E as coming before each of its guard lists and guard, which themselves are ordered superlist first and come before the major antecedents of $\underline{\gamma}$ as well as whatever follows in E).

3. Repeating the above eventually exhausts E, providing a unique hedge correspond-

ing to (E, L) with the corresponding order.

Lemma 2.4.5 Every hedge determines a linked sequence whose members occur in the same order as the order of the corresponding elements in the hedge.

Proof

1. Associate with a hedge containing one element (a wff) the singleton sequence containing that wff and empty link relation.

2. Suppose we have the application of the inductive step of definition 2.2.1 to form a new hedge where linked sequences are associated with each H_i. Where α is the list assigned to the backbone tip of the new hedge we put it as first member of the sequence of lists.

   Take each major antecedent in order. If its corresponding "premiss" hedge is an identity hedge we add nothing to the sequence and that sublist is not linked to any later sublist. Otherwise add the linked sequence associated with its corresponding "premiss" hedge, linking that major antecedent with the corresponding guard list or whole of the first member. This produces a linked sequence which preserves the correct suborder on its members.

So by deductive induction on the definition of hedge, to each there corresponds the required unique linked sequence.

Thus (list-)hedges can be thought of as linked sequences, and vice-versa. The "linked sequence" picture is a helpful way to view and represent the consecution systems defined in the next chapter.

2.5 The Support Function

In order to capture implication we need to incorporate a further feature into our hedge representation of proofs, viz: dependency. In a natural deduction setting $\rightarrow$ is captured using a hypothesis introduction rule and keeping track of which hypotheses each line of a proof depends upon. In Kripke-style semantics this corresponds to taking an arbitrary extension of a point to a further point where the hypothesis holds, with location at a point keeping track of dependency. The above are means for capturing the core "deduction equivalence" aspect of the sense of $\rightarrow$. But where whole proofs are represented as opposed to just sentences, we also need to keep track of the differing dependencies of subproofs.

For example suppose we wish to establish that a proof of $A\rightarrow B$ using a subproof α holds at a point a (in the Kripke-style semantics setting), i.e. $\langle \alpha\, A\rightarrow B\rangle$ holds at a. So, following the modified $\rightarrow$ assignment clause of the previous chapter, we consider an arbitrary extension a' of a containing A and check whether $\langle \alpha \langle A\, B\rangle\rangle$ holds at a' (i.e. whether we have the required proof of B which uses a subproof of A in that context—viz: also using a subproof α). But what if $\langle \alpha \langle A\, B\rangle\rangle$ does hold at a' where, however, the $\underline{\alpha}$ *also* depends upon A? (It could be that $\alpha = A$.) Then it would be wrong to suppose, in general, that the subproof α obtains at a, and hence that $\langle \alpha\, A\rightarrow B\rangle$ holds at a. We need a way of ensuring that nothing other than the $\underline{A}$ in $\langle \alpha \langle A\, B\rangle\rangle$ at a' depends upon this hypothesized proof of A, in order to safely infer that $\langle \alpha\, A\rightarrow B\rangle$ holds back at a. The following notion of *support hedge* does this.

Definition 2.5.1 A *support hedge* is a pair $(H, \mathcal{F})$ comprising a list-hedge H and a function $\mathcal{F}$ on the domain of H, called the *support function*, where:—

- The codomain of $\mathcal{F}$ is the power set of some given set I (the image-set).
- For $\underline{\langle \alpha\, \beta\rangle}$ in the domain of H $\quad F\left(\underline{\beta}\right) = F\left(\underline{\alpha}\right) \bigcup F\left(\underline{\langle \alpha\, \beta\rangle}\right)$

Note that a support function is completely determined by the values assigned to the tips (i.e. to the lists in the domain which are not guard lists of further elements). The image-set I depends upon the type of system: for Kripke-style semantics it is the set of points of the model, and for the consecution systems it is the set of occurrences of sublists in the antecedent of the consecution.

The union property of support functions is remniscent of the join operation in Urquhart's semilattice semantics ([28]) and also of the dependency manipulations upon subscripts in Anderson and Belnap's natural deduction systems ([1]).

Definition 2.5.2 Let $(H,\mathcal{F})$ be a support hedge with $\underline{\varphi}$ an element of the domain of H. Then the *restriction* of $(H,\mathcal{F})$ to $\underline{\varphi}$, denoted $(H,\mathcal{F})|_{\underline{\varphi}}$, is the support hedge $(H',\mathcal{F}')$ where $\quad H' = H|_{\underline{\varphi}}$ and $\quad \mathcal{F}'$ is the restriction of $\mathcal{F}$ to the domain of $H|_{\underline{\varphi}}$.

Definition 2.5.3 Let $(H,\mathcal{F})$ be a support hedge with $\underline{\varphi}$ an element of the domain of H. Then the *bush* of $(H,\mathcal{F})$ *determined by* $\underline{\varphi}$, denoted $(H,\mathcal{F})\left\{\underline{\varphi}\right\}$, is the structure $(H',\mathcal{F}')$ where $\quad H' = H\left\{\underline{\varphi}\right\}$ and $\quad \mathcal{F}'$ is the restriction of $\mathcal{F}$ to the domain of $H\left\{\underline{\varphi}\right\}$.

The following defined entity is the basis of the consecution and natural deduction systems to follow.

Definition 2.5.4 A *support triple* $[\alpha, E, \mathcal{F}]$ *corresponding to* a support hedge $(H,\mathcal{F})$ satisfies:—

- E is the linked sequence determined by H.
- α is either the first list of E (that is the tip of the backbone of H), or a guard list of the first list of E.
- $\mathcal{F}$ is the support function of the support hedge.

A support triple $[\alpha, E, \mathcal{F}]$ is intended to represent a proof of enthymematic description α whose total description is the hedge captured by E, with dependencies as per $\mathcal{F}$. The linked sequence representation of the hedge allows the domain to be explicitly shown, which is helpful in the consecution and natural deduction systems. For example the following support triple has the hedge of figure 2.4:—

$$\Big[\langle\langle p\,p\rangle\, p{\rightarrow}q\rangle,\ \Big\{\Big\langle p\,\langle\langle p\,p\rangle\, p{\rightarrow}q\rangle\Big\rangle\Big\}\ ,\ \mathcal{F}\Big]$$

And the following have the hedge of figure 2.6:—

$$\Big[\langle\langle p\,p\rangle\, p{\rightarrow}q\rangle,\ \Big\{\Big\langle\langle p\,r\rangle\,\langle\langle p\,p\rangle\, p{\rightarrow}q\rangle\Big\rangle,\ \langle p\,\langle p\,r\rangle\rangle\Big\}\ ,\ \mathcal{F}\Big]$$

$$\Big[p{\rightarrow}q,\ \Big\{\Big\langle\langle p\,r\rangle\,\langle\langle p\,p\rangle\, p{\rightarrow}q\rangle\Big\rangle,\ \langle p\,\langle p\,r\rangle\rangle\Big\}\ ,\ \mathcal{F}\Big]$$

Lemma 2.5.5 If $[\alpha, E, \mathcal{F}]$ is a support triple corresponding to a hedge H then $H = H|_{\underline{\alpha}}$.

Proof

This is immediate from the fact that $\underline{\alpha}$ is the first list in E or a guard list of it, and so is in the backbone of the hedge H.

2.6 Atomic Systems

We return to the atomic theory setting to illustrate support functions and support triples.

Given two atomic theories $\mathcal{A}$ and $\mathcal{B}$ there are various respects in which they can be compared. One basic respect is in terms of their axioms and rules. (An alternative would be in terms of the set of sentences provable in each.) The following definition of inclusion reflects this feature.

Definition 2.6.1 Suppose we have two atomic theories $\mathcal{A}$ and $\mathcal{B}$, then $\mathcal{B}$ *includes* $\mathcal{A}$, denoted $\mathcal{A} \subseteq \mathcal{B}$, iff $\Re_{\mathcal{A}} \subseteq \Re_{\mathcal{B}}$.

So $\mathcal{A}$ and $\mathcal{B}$ may have the same set of theorems, but with neither included in the other. Inclusion depends solely on containing the explicitly given axioms and rules.

Definition 2.6.2 An *atomic system* S is a triple $\langle \mathcal{O}, K, \subseteq \rangle$ where K is a set of atomic theories, $\mathcal{O} \in K$, $\subseteq$ is the inclusion relation on K and $\mathcal{O}$ is the least element in K under $\subseteq$.

The atomic systems are intended to be the formal rendering of a language to which a logical connective (here $\rightarrow$) is to be added. K corresponds to a set of extensions of an atomic theory $\mathcal{O}$, and the atomic theories are intended to capture reasoning contexts or linguistic contexts from the perspective of the atomic language.

Associated with each atomic theory in an atomic system is the set of proofs "in" that atomic theory. We have already seen how to determine the hedge description of a relevant atomic proof (definition 2.2.3), so it remains to incorporate the support function aspect. The idea is simply to associate with each tip $\langle p_1 \ldots \langle p_n\, q]$ corresponding to a rule $p_1, \ldots p_n \Rightarrow q$ a singleton set containing an included atomic theory which has that rule. So the support value of a tip indicates where the corresponding rule is grounded, and in virtue of the union property of support functions the support value of a list $\underline{\delta}$ is a set of atomic theories such that every theory which is an extension of all of them contains the subproof corresponding to $\underline{\delta}$. Thus the image-set I of the support function is the set of atomic theories K in the particular atomic system.

Definition 2.6.3 A *support hedge representation of a relevant proof at an atomic theory* $\mathcal{A}$ *in an atomic system* $\mathcal{S}$ is a support hedge $(E, \mathcal{F})$ determined as follows:—

- Associate with a proof comprising a single sentence p (so an axiom of the theory) the singleton hedge with $\mathcal{F}\left(\underline{p}\right) = \{\mathcal{B}\}$ where $\mathcal{B}$ is any atomic theory such that $\mathcal{B} \subseteq \mathcal{A}$ and $\Rightarrow p \in \Re_{\mathcal{B}}$.

- Where a sentence q is obtained by the application of a rule $p_1, \ldots p_n \Rightarrow q$, form a *backbone* which has as tip $\langle p_1 \ldots \langle p_n\, q\rfloor$, nodes $(p_r \ldots \langle p_n\, q\rfloor$, and root q. For each of these, except the tip, form a right-branch to p_{r-1} whose support subhedge is the support hedge representation of its subproof.
  Put $\mathcal{F}\left(\underline{\langle p_1 \ldots \langle p_n\, q\rfloor}\right) = \{\mathcal{B}\}$ where $\mathcal{B}$ is any atomic theory in $\mathcal{S}$ such that $\mathcal{B} \subseteq \mathcal{A}$ and $p_1, \ldots p_n \Rightarrow q \in \Re_{\mathcal{B}}$.
  The support values of the remaining new nodes (just in the backbone) are determined by the union property required of support functions on hedges.

Finally, the support triple representations of a relevant atomic proof are each $[\alpha, E, \mathcal{F}]$ where $(E, \mathcal{F})$ is a support hedge representation of the proof and $\underline{\alpha}$ is any member of the backbone of E.[6]

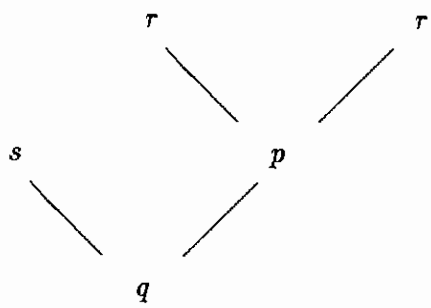
Such first list α is an enthymematic description of the given atomic proof while the hedge E provides a total description of it.

For example consider an atomic system containing $\mathcal{A}, \mathcal{B}$ such that

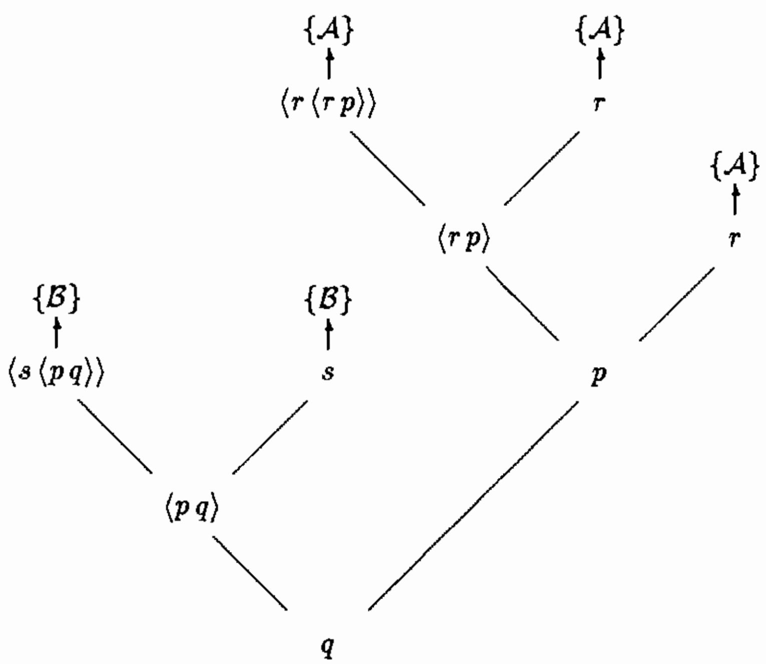
$$\Re_{\mathcal{B}} = \{s, p \Rightarrow q; \quad \Rightarrow s\}$$
$$\Re_{\mathcal{A}} = \{\Rightarrow r; \quad r, r \Rightarrow p; \quad s, p \Rightarrow q; \quad \Rightarrow s\}$$

We have the following proof in $\mathcal{A}$

[6] Here and in what follows I blur the distinction between list-hedges and linked sequences for the obvious reason.



some of whose list representations have the following support hedge $(E,\ \mathcal{F})$ (the remainder have one or both tip-images $\{\mathcal{B}\}$ replaced by $\{\mathcal{A}\}$)



So the following triples "hold" at $\mathcal{A}$:—

$$[\langle s\,\langle p\,q\rangle\rangle,\ E,\ \mathcal{F}] \qquad [\langle p\,q\rangle,\ E,\ \mathcal{F}] \qquad [q,\ E,\ \mathcal{F}]$$

That is to say, a proof of q with total description E and support $\mathcal{F}$, which is also a proof of q *using* a subproof of p with total description E and support $\mathcal{F}$, which is also a proof of q *using* a subproof of p and a subproof of s, with total description E and support $\mathcal{F}$.

This illustrates how the first-list part of the description of a proof describes subproofs which are used, but not necessarily all of them— it is an enthymematic description of the proof— whilst the hedge part of the description displays all.

2.7 Atomic proofs and *use*

Having defined a formal representation of the structure of proofs we are now able to show how manipulation of these structures captures various senses of *use*. This is done using rules for transforming support hedges which I call the *use* rules.

Definition 2.7.1 The *use* rules for support hedges are:—

(use)
$$\frac{(E, \mathcal{F}) \in \mathcal{A}}{(E', \mathcal{F}') \in \mathcal{A}}$$

Where E has backbone tip $\langle \alpha_1 \ldots \langle \alpha_n \langle \beta\, \gamma \rfloor$ and $\underline{\beta}$ is a guard list of a list $\underline{\langle \delta_1 \ldots \langle \delta_m\, \beta \rfloor}$ in E (i.e. $E|_{\underline{\beta}}$ has a backbone node $(\underline{\delta_1 \ldots \langle \delta_m\, \beta \rfloor})$ and

- E' has as backbone tip $\langle \alpha_1 \ldots \langle \alpha_n \langle \delta_1 \ldots \langle \delta_m\, \gamma \rfloor$, and each major antecedent has as support subhedge that of its ancestor in $(E, \mathcal{F})$,
- the remaining values of $\mathcal{F}'$ (just those for the backbone elements) are determined by assigning to the backbone tip $\underline{\langle \alpha_1 \ldots \langle \alpha_n \langle \delta_1 \ldots \langle \delta_m\, \gamma \rfloor}$ the union of $\mathcal{F}\left(\underline{\langle \alpha_1 \ldots \langle \alpha_n \langle \beta\, \gamma \rfloor}\right)$ with $\mathcal{F}\left(\underline{\langle \delta_1 \ldots \langle \delta_m\, \beta \rfloor}\right)$.

(merge) [7]

$$\frac{(E, \mathcal{F}) \in \mathcal{A}}{(E', \mathcal{F}') \in \mathcal{A}}$$

[7] Bob Meyer suggested this emendation of the *(use)* rule to capture suffixing without resort to full permutation, similar to the Belnap-Dunn merge rule for consecution systems.

Where E has backbone tip $\langle\alpha_1 \ldots \langle\alpha_n \langle\beta\,\gamma\rfloor$ and $\underline{\beta}$ is a guard list of a list $\underline{\langle\delta_1 \ldots \langle\delta_m\,\beta\rfloor}$ in E and

- E' has as backbone tip $\langle\varphi_1 \ldots \langle\varphi_{n+m}\,\gamma\rfloor$, where $\varphi_1, \ldots, \varphi_{n+m}$ is a permutation of $\alpha_1, \ldots, \alpha_n, \delta_1 \ldots, \delta_m$ and each major antecedent has as support subhedge that of its ancestor in $(E, \mathcal{F})$,
- the remaining values of $\mathcal{F}'$ (just those for the backbone elements) are determined by assigning to the backbone tip $\underline{\langle\varphi_1 \ldots \langle\varphi_{n+m}\,\gamma\rfloor}$ the union of $\mathcal{F}\left(\underline{\langle\alpha_1 \ldots \langle\alpha_n \langle\beta\,\gamma\rfloor}\right)$ with $\mathcal{F}\left(\underline{\langle\delta_1 \ldots \langle\delta_m\,\beta\rfloor}\right)$.

$$(id) \qquad \frac{(E, \mathcal{F}) \in \mathcal{A}}{(E', \mathcal{F}') \in \mathcal{A}}$$

Where

- the backbone tip of E is α,
- the backbone tip of E' is $\langle\alpha\ \alpha\rangle$ and each major antecedent has as support subhedge that of its ancestor in $(E, \mathcal{F})$,
- and the remaining values of $\mathcal{F}'$ are determined by putting $\mathcal{F}'\left(\underline{\langle\alpha\ \alpha\rangle}\right) = \{\}$.

$$(perm) \qquad \frac{(E, \mathcal{F}) \in \mathcal{A}}{(E', \mathcal{F}') \in \mathcal{A}}$$

Where

- E has backbone tip $\langle\alpha_1 \ldots \langle\alpha_n \langle\beta\langle\gamma\,\delta\rfloor$,
- E' has backbone tip $\langle\alpha_1 \ldots \langle\alpha_n \langle\gamma\langle\beta\,\delta\rfloor$,
- the support subhedge of each major antecedent list is that of its ancestor, and
- the support value of the backbone tip is the same as that of the premiss.

$$(con) \qquad \frac{(E, \mathcal{F}) \in \mathcal{A}}{(E', \mathcal{F}') \in \mathcal{A}}$$

Where

- E has backbone tip $\langle\alpha_1 \ldots \langle\alpha_n \langle\beta\langle\beta\ \gamma\rfloor$,

- E' has backbone tip $\langle \alpha_1 \ldots \langle \alpha_n \langle \beta \, \gamma \rfloor$,
- the support subhedges of each of the $\underline{\beta}$'s are identical,
- the support subhedge of each major antecedent list equals that of its ancestor, and
- the backbone tip has the same value as that of the premiss.

(mingle) $$\frac{(E, \mathcal{F}) \in \mathcal{A}}{(E', \mathcal{F}') \in \mathcal{A}}$$

Where

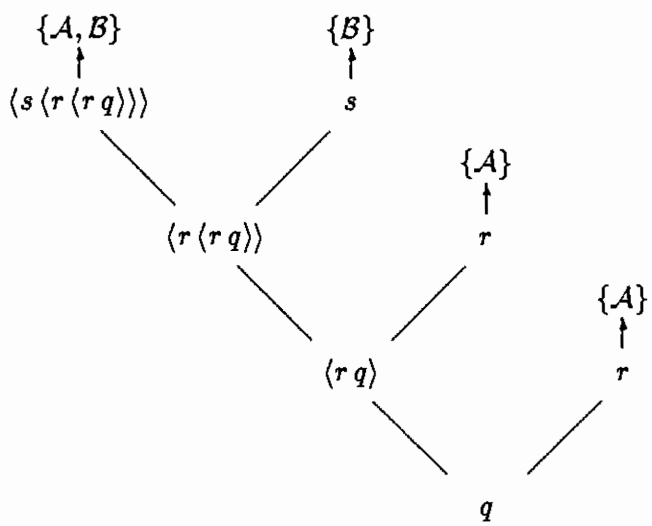
- E has backbone tip $\langle \alpha_1 \ldots \langle \alpha_n \langle \beta \, \gamma \rfloor$,
- E' has backbone tip $\langle \alpha_1 \ldots \langle \alpha_n \langle \beta \langle \beta \; \gamma \rfloor$,
- the support subhedge of each major antecedent list equals that of its ancestor, and
- the backbone tip has the same value as that of the premiss.

The above *use* rules are motivated as follows:—

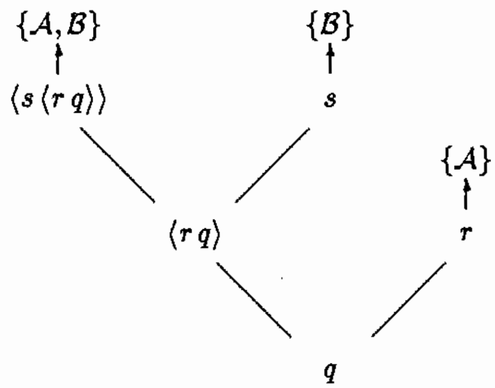
The *(use)* rule captures transitivity of use: where a subproof β is used in a proof of γ, and subproofs δ_i are used in the proof of β, the δ_i are taken to be *used* in the proof of γ.
The *(merge)* rule is just an adaptation of *(use)*, building in some permutation.
The *(id)* rule depends upon supposing that a proof uses itself, (which seems somewhat counter-intuitive to me); *(id)* will not be included where one seeks to capture the stricter sense of use as a subproof which only permits proper subproofs being regarded as really used.
The *(perm)* rule captures the idea that order of use of subproofs is not significant.
The *(con)* rule captures the idea that multiple use of subproofs can be regarded as single use.
The *(mingle)* rule captures the idea that single use of subproofs can be regarded as multiple use.

From this perspective the core notion of *use* is captured by the *(use)* rule. Various of the other rules may be added depending upon the sense of *use* we wish to capture. As example further consider our example of a support hedge representation of an atomic proof from the previous section:—

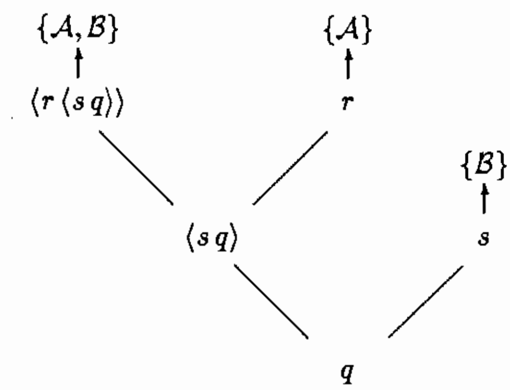
Now we can apply *(use)* to its support hedge to obtain



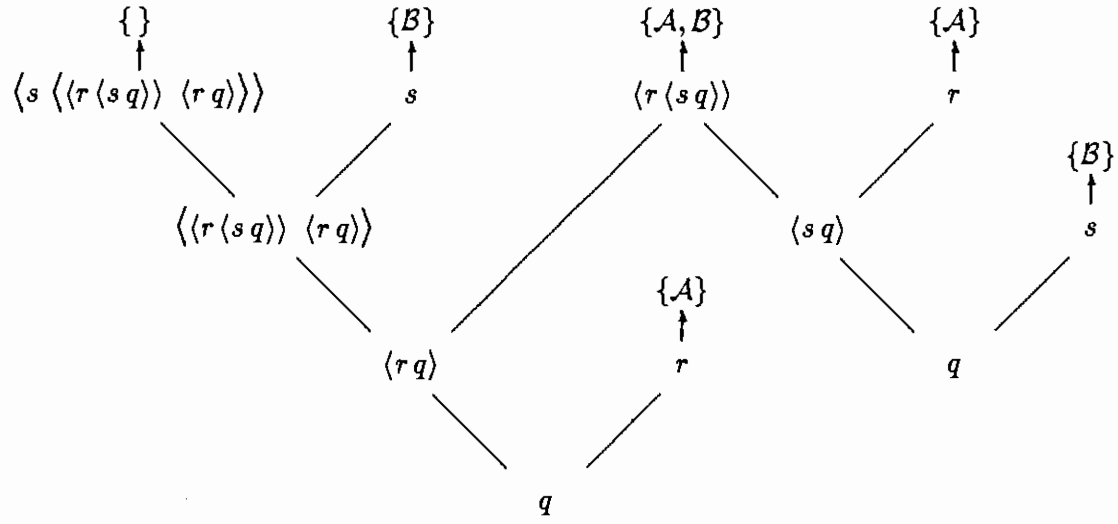
we may apply *(con)* to obtain



we may apply *(perm)* to obtain



And finally an *(id)* and then two further *(perm)*'s gives



The following formulation of Mingle, i.e. **RM0**$_{\rightarrow}$, is "close" to the above *use* rules and so is to be preferred from the point of view of this work.

Definition 2.7.2 The *use* formulation of Mingle has axiom:—

<u>Identity</u> $A \rightarrow A$

And rule schemes:—

<u>Use</u> $A_1 \rightarrow \ldots . A_n \rightarrow . B \rightarrow C \, , \quad D_1 \rightarrow \ldots . D_m \rightarrow B \;\Rightarrow\; A_1 \rightarrow \ldots . A_n \rightarrow . D_1 \rightarrow \ldots . D_m \rightarrow C$

Permutation $A_1 \rightarrow \ldots . A_n \rightarrow .B \rightarrow .C \rightarrow D \;\Rightarrow\; A_1 \rightarrow \ldots . A_n \rightarrow .C \rightarrow .B \rightarrow D$

Contraction $A_1 \rightarrow \ldots . A_n \rightarrow .B \rightarrow .B \rightarrow C \;\Rightarrow\; A_1 \rightarrow \ldots . A_n \rightarrow .B \rightarrow C$

Mingle $A_1 \rightarrow \ldots . A_n \rightarrow .B \rightarrow C \;\Rightarrow\; A_1 \rightarrow \ldots . A_n \rightarrow .B \rightarrow .B \rightarrow C$

Modus Ponens $A,\ A \rightarrow B \;\Rightarrow\; B$

Where possibly $n = 0$.

Use is simply a form of directed replacement of antecedent subwff. Various subsystems of Mingle can be obtained by dropping either the axiom or some rules, and if we don't have Permutation, by replacing Use by:—

Merge $A_1 \rightarrow \ldots . A_n \rightarrow .B \rightarrow .C\ ,\quad D_1 \rightarrow \ldots . D_m \rightarrow B \;\Rightarrow\; P_1 \rightarrow \ldots . P_{n+m} \rightarrow C$

Where $P_1, \ldots, P_{n+m}$ is any permutation of $A_1, \ldots, A_n,\ D_1, \ldots, D_m$.

It is easy to check that this formulation captures $\mathbf{RM0}_{\rightarrow}$. For example the proof of prefixing is:—

$A \rightarrow B \rightarrow .\ A \rightarrow B$	Identity
$C \rightarrow A \rightarrow .\ C \rightarrow A$	Identity
$A \rightarrow B \rightarrow .\ C \rightarrow A \rightarrow .\ C \rightarrow B$	Use

2.8 Summary

The main aim of this chapter has been to provide a foundation for an account of logic from within the Curry perspective— as the epitheory of formal systems— but with that perspective greatly broadened by taking a close look at the elementary systems (atomic theories), and taking due account of their internal structure in light of the fact that it is intended that they represent arbitrary formal systems. Curry did not do this and took the elementary systems as givens about which nothing further was known, resulting in $\rightarrow$ being captured by (as a minimum) intuitionist implication. I have supposed that formal systems are characterized (at least in part) by the property of having axioms, rules and proofs, so that these features may also be utilized in a formalist theory of logic.

To this end formal tools have been motivated and defined for describing atomic proofs. These tools also enable one to precisely characterize various senses of *use* as "use as a subproof in a proof", thus we also have the basis for a semantic account of relevant implication in terms of use.

While capturing relevance has been the driving force behind this research, the next two chapters are concerned with showing that formal consecution and natural deduction systems can be given for arbitrary implication logics. For having shown how to describe proofs we can build into these systems rules for manipulating the shapes of proofs, from which the $\rightarrow$ properties follow, seemingly putting the cart before the horse. These general "cloned" systems do not provide satisfactory explanatory semantics, but they do display singular consecution and natural deduction systems for arbitrary implication logics which satisfy Lorenzen's principle of inversion, and so are constructive in the formal sense that "constructivity" is characterized by the intuitionist, as well as for the formalist.

Consequently the notion of inversion, in particular Lorenzen's principle of inversion, is *alone* insufficient for providing explanatory semantics.

Finally, the example "arbitrary implication logic" used to illustrate the "cloned" systems is the following formulation of **S**:—

<u>Prefixing</u> $A \rightarrow B \rightarrow .C \rightarrow A \rightarrow .C \rightarrow B$

<u>Suffixing</u> $A \rightarrow B \rightarrow .B \rightarrow C \rightarrow .A \rightarrow C$

And rule <u>modus ponens</u> $A,\ A \rightarrow B \Rightarrow B$

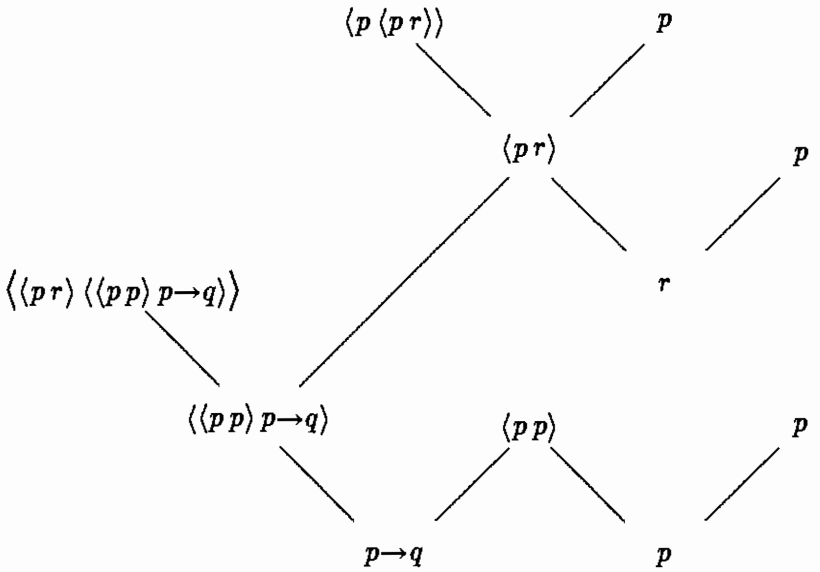
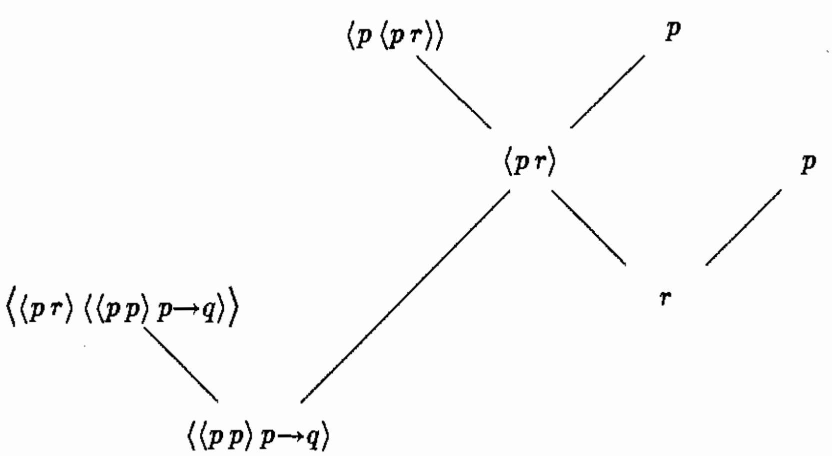


Figure 2.6: Example of a hedge H



$$H\left\{\underline{\langle\langle p\,p\rangle\,p\to q\rangle}\right\}$$

Figure 2.7: A more complex bush.

Chapter 3

The Consecution Systems

In this chapter the Curry semantic program is extended, using the machinery developed in the previous chapter, so as to take account of the internal structure of formal systems in our *epitheory of formal systems.* A general recipe is provided for obtaining consecution systems corresponding to arbitrary implication logics. First the latter Hilbert-system-cloning is shown in detail using **S**, to illustrate the general approach and introduce the novel features of these consecution systems in a fairly simple setting. The "cloned"systems are based on the idea of capturing the shapes of proofs in the corresponding Hilbert systems, with a list rule for each axiom capturing how that axiom modifies the possible shapes of proofs. The interpretation proof is quite simple and most of the hard work is done in section 3.4 in proving invertibility (lemma 3.4.2) and closure under cut (lemma 3.4.7). A payoff of the hard work is that corresponding natural deduction systems can be shown to be formally constructive, so that we have, for example, a constructive formulation of classical implication (chapter 4). And since every implication logic has a constructive formulation this shows that Lorenzen's formal criterion of constructivity is not itself alone suitable as a philosophical demarcation between logics.

The original motivation for this research— relevance— is returned to in chapter 5.

3.1 Definition of the consecution systems.

Here a consecution has as antecedent a sequence of lists and as succedent a single support triple $[\alpha,\ E,\ \mathcal{F}]$ where the codomain of $\mathcal{F}$ is the power set of the set of sublists of lists in the antecedent of the consecution.[1] The "bunching" operation upon the antecedent of the consecution is extensional.

The intuitive idea behind the rules, and the consecution system, is that a consecution represents a statement about what shapes of proofs (corresponding to the succedent) are provable in theories of the respective Hilbert system, on the hypothesis that certain other shapes of proofs are available (corresponding to the antecedent); where a list $\langle \alpha\, \beta \rangle$ is intended to represent a proof described by β which has a subproof described by α. The formal interpretation is given in section 3.3.

The system **GS** comprises axioms and the rules $(C\|)$, $(\|pref)$, $(\|suff)$, $(\rightarrow\|)$ and $(\|\rightarrow)$. Here I motivate the axioms and rules, which are defined immediately after.

The axioms are simply atomic identities, but also with the enthymematic sense of the first list in the succedent list triple built in, so that any guard list of the backbone (i.e. of the particular atomic list) is the highlighted first list.[2]

$(C\|)$ is well-motivated by the extensional bunching of the lists in the antecedent.[3]

As previously discussed, $(\|\rightarrow)$ captures the idea that a sufficient warrant for $\ldots A \rightarrow B \ldots$ is that in every extension with A the corresponding proof of B which has a subproof of A holds in the same context, that is $\ldots \langle A\, B \rangle \ldots$ holds, and we need to take the ap-

[1] In fact the support function only takes as values sets of sublists where those sublists are antecedent located sublists or whole lists in the antecedent of the consecution.

[2] This is captured in earlier versions by the separate rule $(\|iK)$.

[3] Extensional contraction can also be incorporated, but as it does no *logical* work here I have ommited it from this version—so simplifying *cut*. It is, however, needed in the **GRL** systems of chapter 5.

propriate care of dependencies.

$(\to\|)$ simply captures the corresponding form of modus ponens in this setting. This is obtained by the inversion procedure also previously described (in chapter 1).

$(\|pref)$ and $(\|suff)$ are intended to capture the shapes of proofs available where we have the corresponding axioms. In the general "cloned" systems there is a similar class of axiom-based list rules. Here too the enthymematic sense of the first list is built in. This is further discussed in the next section.

Definition 3.1.1 The axioms and rules for the example system **GS** are as follows.

Axioms $\alpha \Vdash [\delta, \{\alpha\}, \mathcal{I}]$

Where α is an atomic list, i.e. all its constituents are atomic sentences, the succedent support hedge $(I_\alpha, \mathcal{I})$ is the *identity support hedge on* α with for each tip $\underline{\varphi}$ of it having $\mathcal{I}\left(\underline{\varphi}\right)$ equal to the singleton set containing the corresponding located sublist $\underline{\varphi}$ in the antecedent α, and the first list δ is either α or a guard list of α (so that δ is any backbone list of the hedge, since α is the backbone tip).

$$(C\|)\ \frac{\Gamma \ \Vdash\ [\delta,\ E,\ \mathcal{F}]}{\Gamma' \ \Vdash\ [\delta,\ E,\ \mathcal{F}]}$$

Where Γ' is any permutation of Γ, and the tip-images follow their descendants (so the succedent is essentially unchanged).

$$(\to\|)\qquad \frac{\Gamma_1 \Vdash [A, G, \mathcal{H}] \qquad\qquad \Gamma_2, \langle \ldots \langle A\, B\rangle \ldots\rangle \Vdash [\delta, E, \mathcal{F}]}{\Gamma_1, \Gamma_2, \langle \ldots A\to B \ldots\rangle \ \Vdash\ [\delta, E', \mathcal{F}']}$$

Where E' is formed by, for each E tip $\underline{\varphi}$ with $\underline{A} \in \mathcal{F}\left(\underline{\varphi}\right)$, replacing the support bush $(E, \mathcal{F})\left\{\underline{\varphi}\right\}$ (which contains just $\underline{\varphi}$ since it is a tip) by the structure $(G, \mathcal{H})\, [\underline{A}\backslash\varphi]$.

Roughly speaking every tip of the support hedge $(E, \mathcal{F})$ with image containing the antecedent $\underline{A}$ is replaced by the hedge $\mathcal{H}$ to form the support hedge $(E', \mathcal{F}')$.[4]

[4]The image of a tip is always here a singleton set, so that in this situation $\mathcal{F}\left(\underline{\varphi}\right) = \{\underline{A}\}$.

$$(\Vert\!\!\rightarrow) \qquad (\Vert\!\!\rightarrow)\frac{\Gamma,\, A \;\Vert\!\!\vdash\; [\alpha,\, E,\, \mathcal{F}]}{\Gamma \;\Vert\!\!\vdash\; [\alpha',\, E',\, \mathcal{F}']}$$

Where

- $\underline{\alpha}$ is not permitted to be the (to-be-lost) $\underline{B}$ (in the case that $\underline{B}$ is the guard wff of the backbone tip),
- if the $\underline{\langle A\, B\rangle}$ occurs in the located sublist $\underline{\alpha}$ of the hedge E identified with the the first list of the list-triple then $\alpha' = \alpha\left[\underline{\langle A\, B\rangle}\backslash A{\rightarrow}B\right]$,
- otherwise the first list (and corresponding member of the backbone of the hedge) remains unchanged.

And further:—

- E' is like E but with that $\underline{\langle A\, B\rangle}$ and its "triangle" replaced by the single node $\underline{A{\rightarrow}B}$,
- the $\underline{A}$ in the succedent is a tip with tip-image *that* antecedent $\underline{A}$ which is dropped to obtain the conclusion (i.e. $\mathcal{F}(\underline{A}) = \{A\}$),
- the antecedent $\underline{A}$ is not a tip-image of any other tip of E,
- and $\quad (E', \mathcal{F}') = (E, \mathcal{F})\left[\underline{\langle A\, B\rangle}\backslash A{\rightarrow}B\right]$ where the tip-images are the descendants (in Γ) of those of the deductive ancestor tips.

So for $\underline{\varphi}$ of the conclusion succedent $\mathcal{F}'\left(\underline{\varphi}\right)$ equals $\mathcal{F}\left(\underline{\varphi'}\right)$ where $\varphi' = \varphi\left[\underline{A{\rightarrow}B}\backslash\underline{\langle A\, B\rangle}\right]$.

$$(\Vert pref) \qquad \frac{\begin{array}{l}\Gamma_1 \;\Vert\!\!\vdash\; [\langle\alpha\,\beta\rangle,\, E_1,\, \mathcal{F}_1] \\ \Gamma_2 \;\Vert\!\!\vdash\; [\langle\gamma\,\alpha\rangle,\, E_2,\, \mathcal{F}_2] \\ \Gamma_3 \;\Vert\!\!\vdash\; [\gamma, E_3, \mathcal{F}_3] \\ \Gamma_4 \;\Vert\!\!\vdash\; [\beta_1, E_4, \mathcal{F}_4] \\ \dots \\ \Gamma_{n+3} \;\Vert\!\!\vdash\; [\beta_n, E_{n+3}, \mathcal{F}_{n+3}]\end{array}}{\Gamma_1, \dots \Gamma_{n+3} \;\Vert\!\!\vdash\; [\delta,\, E,\, \mathcal{F}]}$$

Where $\beta = \langle\beta_1 \ldots \langle\beta_n B\rfloor$, δ is either $\big\langle\langle\alpha\,\beta\rangle\,\big\langle\langle\gamma\,\alpha\rangle\,\langle\gamma\,\beta\rangle\big\rangle\big\rangle$ or any guard list of it, and the conclusion support hedge E is formed by applying the inductive step of definition 2.2.1 to $E_1, \ldots E_{n+3}$ with backbone tip assigned the list $\big\langle\langle\alpha\,\beta\rangle\,\big\langle\langle\gamma\,\alpha\rangle\,\langle\gamma\,\beta\rangle\big\rangle\big\rangle$ (this is illustrated in figure 3.1); and $\mathcal{F}$ is determined by putting:—

- $\mathcal{F}\left(\underline{\big\langle\langle\alpha\,\beta\rangle\,\big\langle\langle\gamma\,\alpha\rangle\,\langle\gamma\,\beta\rangle\big\rangle\big\rangle}\right)$ is the empty set,
- $(E, \mathcal{F})|_{\underline{\langle\alpha\,\beta\rangle}} = (E_1, \mathcal{F}_1)$
- $(E, \mathcal{F})|_{\underline{\langle\gamma\,\alpha\rangle}} = (E_2, \mathcal{F}_2)$
- $(E, \mathcal{F})|_{\underline{\gamma}} = (E_3, \mathcal{F}_3)$
- for the major antecedents of the guard $\underline{\beta}$ $\quad (E, \mathcal{F})|_{\underline{\beta_i}} = (E_{i+3}, \mathcal{F}_{i+3})$
- and the remaining located sublists have values determined by the above and the union requirement upon support functions, since they are just the other superlists of the guard B (see figure 3.1).

$$(\|suff) \qquad \frac{\begin{array}{l} \Gamma_1 \;\Vdash\; [\langle\alpha\,\beta\rangle,\, E_1,\, \mathcal{F}_1] \\ \Gamma_2 \;\Vdash\; [\langle\beta\,\gamma\rangle,\, E_2,\, \mathcal{F}_2] \\ \Gamma_3 \;\Vdash\; [\alpha, E_3, \mathcal{F}_3] \\ \Gamma_4 \;\Vdash\; [\gamma_1, E_4, \mathcal{F}_4] \\ \ldots \\ \Gamma_{n+3} \;\Vdash\; [\gamma_n, E_{n+3}, \mathcal{F}_{n+3}] \end{array}}{\Gamma_1, \ldots \Gamma_{n+3} \;\Vdash\; [\delta,\, E,\, \mathcal{F}]}$$

Where $\gamma = \langle\gamma_1 \ldots \langle\gamma_n C\rfloor$, δ is any guard list or the whole of $\big\langle\langle\alpha\,\beta\rangle\big\langle\langle\beta\,\gamma\rangle\langle\alpha\,\gamma\rangle\big\rangle\big\rangle$, the conclusion support hedge E is formed by applying the inductive step of definition 2.2.1 to $E_1, \ldots E_{n+3}$, with backbone tip assigned the list $\big\langle\langle\alpha\,\beta\rangle\big\langle\langle\beta\,\gamma\rangle\langle\alpha\,\gamma\rangle\big\rangle\big\rangle$; and $\mathcal{F}$ is determined by:—

- $\mathcal{F}\left(\underline{\big\langle\langle\alpha\,\beta\rangle\,\big\langle\langle\beta\,\gamma\rangle\,\langle\alpha\,\gamma\rangle\big\rangle\big\rangle}\right)$ is the empty set,
- $(E, \mathcal{F})|_{\underline{\langle\alpha\,\beta\rangle}} = (E_1, \mathcal{F}_1)$
- $(E, \mathcal{F})|_{\underline{\langle\beta\,\gamma\rangle}} = (E_2, \mathcal{F}_2)$

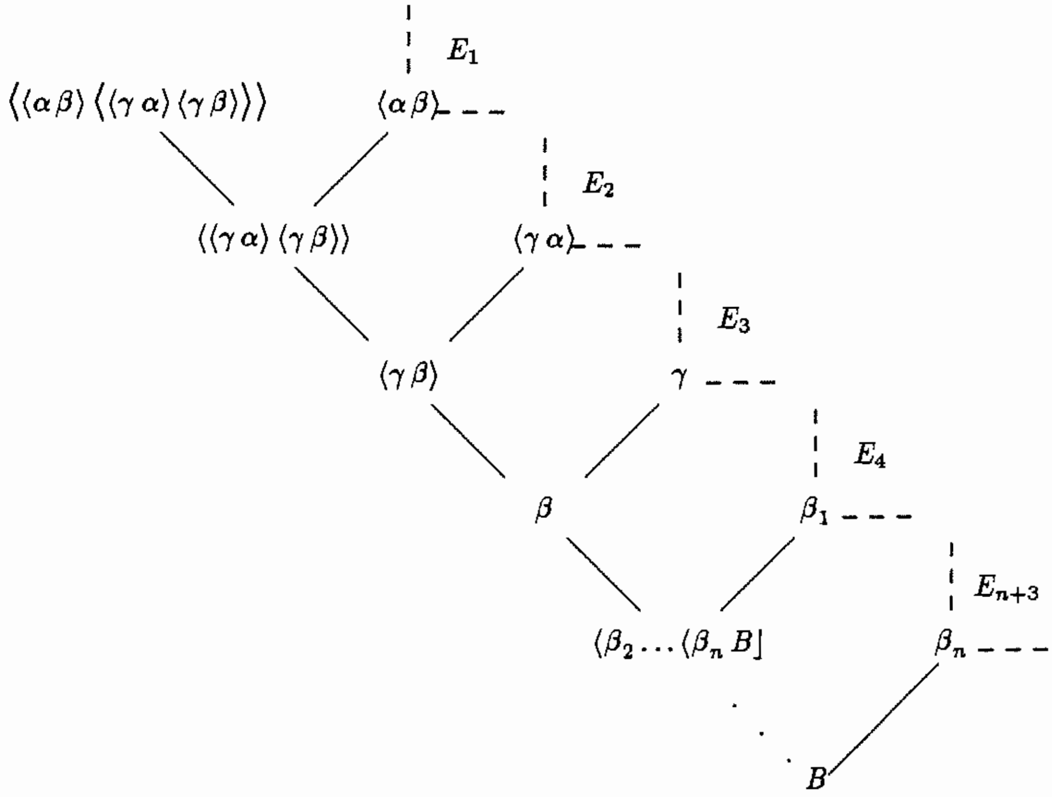


Figure 3.1: The hedge for ($\|$ *pref*)

- $(E,\mathcal{F})|_{\underline{\alpha}} = (E_3,\mathcal{F}_3)$
- for the major antecedents of the guard $\underline{\gamma}$ $(E,\mathcal{F})|_{\underline{\gamma_i}} = (E_{i+3},\mathcal{F}_{i+3})$

Two ancestor relations are defined below for these rules. The first is appropriate for a cloned system such as GS and relates to the form of the corresponding axioms, whereas the second is perhaps more natural.

Definition 3.1.2 The *schematic ancestor relation* is the transitive closure of the following:—

axioms Each located list of the succedent is its sole *schematic ancestor*.

($\|$*pref*)
- Sublists of either $\underline{\alpha}$ in the conclusion have the corresponding sublist in the first two premisses as *schematic ancestor*,
- a sublist of either $\underline{\beta}$ has the corresponding sublist of the first premiss as *schematic ancestor*, and where it is a sublist of a major antecedent β_i it also has the corresponding sublist of the $i+3$rd premiss as *schematic ancestor*,
- a sublist of either $\underline{\gamma}$ has the corresponding sublist of the second and third premisses as *schematic ancestor*,
- each remaining sublist of the conclusion has the corresponding sublist of the premiss hedge it came from as *schematic ancestor*.

($\|$*suff*)
- Sublists of either $\underline{\beta}$ in the conclusion have the corresponding sublist in the first two premisses as *schematic ancestor*,
- a sublist of either $\underline{\alpha}$ has the corresponding sublist of the first and third premisses as *schematic ancestor*,
- a sublist of either $\underline{\gamma}$ has the corresponding sublist of the second premiss as *schematic ancestor*, and where it is a sublist of a major antecedent γ_i it also has the corresponding sublist of the $i+3$rd premiss as *schematic ancestor*,

- each remaining sublist of the conclusion has the corresponding sublist of the premiss hedge it came from as *schematic ancestor.*

($\rightarrow\|$) Each sublist of the conclusion succedent has as *schematic ancestor* the corresponding sublist of the premiss.

($\|\rightarrow$) Each sublist of the conclusion has as *schematic ancestor* the corresponding sublist of the premiss (possibly with $\underline{A \rightarrow B}$ replaced by $\langle A\,B \rangle$).

Definition 3.1.3 The *direct ancestor relation* is defined on both the antecedent and succedent of the consecutions comprising a proof, with the obvious definition for the antecedent of the conclusion of each rule and for the succedent it is the transitive closure of the following:—

axioms Each located list of the succedent is its sole *direct ancestor.*

($\|$*pref*) The new $\big\langle \langle \alpha\,\beta \rangle \, \big\langle \langle \gamma\,\alpha \rangle \, \langle \gamma\,\beta \rangle \big\rangle \big\rangle$ and each of its guard lists has itself as sole *direct ancestor*, and each remaining sublist of the conclusion has the corresponding sublist of the premiss hedge it came from as *direct ancestor.*

($\|$*suff*) The new $\big\langle \langle \alpha\,\beta \rangle \, \big\langle \langle \beta\,\gamma \rangle \, \langle \alpha\,\gamma \rangle \big\rangle \big\rangle$ and each of its guard lists has itself as sole *direct ancestor*, and each remaining sublist of the conclusion has the corresponding sublist of the premiss hedge it came from as *direct ancestor.*

($\rightarrow\|$) Each sublist of the conclusion succedent has as *direct ancestor* the corresponding sublist of the premiss.

($\|\rightarrow$) Each sublist of the conclusion has as *direct ancestor* the corresponding sublist of the premiss (possibly with $\underline{A \rightarrow B}$ replaced by $\langle A\,B \rangle$).

The *direct ancestor* relation is the more natural of the two since for a given sublist it just takes the corresponding sublist of the premiss whose support hedge is that used to determine the subhedge of the conclusion in which it occurs. Note that the *direct ancestor* relation is 1-1 between the sublists of the premisses and those of the conclusion of each rule.

The *direct descendant* relation is the converse of the *direct ancestor* relation, and similarly for the *schematic descendant* relation.

The basis of the consecution systems is the axioms and the rules $(C\|)$, $(\|\rightarrow)$ and $(\rightarrow\|)$. The "list rules" (in this case $(\|\mathit{pref})$ and $(\|\mathit{suff})$) provide the variations upon the theme, determining which logic is captured.

3.2 Some properties and example deductions of GS.

The rules have been streamlined by building the following $(\|iK)$ and $(\|\mathit{splice})$ into the the axioms, and $(\|\mathit{pref})$ and $(\|\mathit{suff})$.

The idea behind $(\|iK)$ is that if we have a proof satisfying description $\langle\alpha\,\beta\rangle$ then we have a proof satisfying enthymematic description β, but with the same total description and hence the same support hedge.

$(\|\mathit{splice})$ corresponds to the idea that if one has a proof satisfying description δ which has in its total description a subproof satisfying description α, and one has an alternative proof satisfying description α, then one can replace the first subproof by the alternative. This generalizes the idea that one can replace a subproof of a sentence by an alternative proof of that sentence.

So both of these are intuitively motivated by our interpretation of support hedges and GS consecutions.

Definition 3.2.1 $(\|iK)$ and $(\|\mathit{splice})$ are defined as follows:—

$$(\|iK)\ \frac{\Gamma\ \Vdash\ [\langle\alpha\,\beta\rangle,\,E,\,\mathcal{F}]}{\Gamma\ \Vdash\ [\beta,\,E,\,\mathcal{F}]}$$

Where the support hedge of the conclusion succedent is equal to that of the premiss.

$$(\|splice)\ \frac{\Gamma_1\ \Vdash\ [\alpha,E,\mathcal{F}] \qquad \Gamma_2\ \Vdash\ [\delta,G,\mathcal{H}]}{\Gamma_1,\Gamma_2\ \Vdash\ [\delta,G',\mathcal{H}']}$$

Where δ or another element of G contains a designated antecedent sublist $\underline{\alpha}$ called the *splice* sublist, and the conclusion support hedge is formed by replacing the restricted support hedge $(G,\mathcal{H})|_{\underline{\alpha}}$ by $(E,\mathcal{F})$ (this means using E instead of $G|_{\underline{\alpha}}$ at the appropriate step of the procedure generating G, with appropriate modification of the support function). i.e.the subhedge $G|_{\underline{\alpha}}$ is replaced by the hedge E with a corresponding change in the support function, while the rest of the support hedge $(G',\mathcal{H}')$ remains the same.

We prove that **GS** is closed under $(\|iK)$ since this property is needed for the proof of closure under modus ponens.

Lemma 3.2.2 **GS** is closed under $(\|iK)$.

Proof

The property is proved by deductive induction.

Clearly the axioms are closed under $(\|iK)$ as we may simply choose the appropriate first list for the succedent triple.

Obviously $(C\|)$ preserves the property.

For $(\rightarrow\|)$ if we have

$$\frac{\Gamma_1\ \Vdash\ [A,G,\mathcal{H}] \qquad \Gamma_2,\langle\ldots\langle A\,B\rangle\ldots\rangle\ \Vdash\ [\langle\alpha\,\beta\rangle,E,\mathcal{F}]}{\Gamma_1,\Gamma_2,\langle\ldots A\rightarrow B\ldots\rangle\ \Vdash\ [\langle\alpha\,\beta\rangle,E',\mathcal{F}']}$$

then by the induction hypothesis the second premiss with β in place of $\langle\alpha\,\beta\rangle$ is provable so that $(\rightarrow\|)$ can be applied

$$\frac{\Gamma_1 \ \Vdash [A, G, \mathcal{H}] \qquad\qquad \Gamma_2, \langle \ldots \langle A\,B \rangle \ldots \rangle \ \Vdash [\beta, E, \mathcal{F}]}{\Gamma_1, \Gamma_2, \langle \ldots A \rightarrow B \ldots \rangle \ \ \Vdash \ \ [\beta, E', \mathcal{F}']}$$

obtaining the required conclusion.

For $(\| \rightarrow)$ if we have

$$(\| \rightarrow) \, \frac{\Gamma, \, A \ \ \Vdash \ \ [\langle \alpha \, \beta \rangle, \, E, \, \mathcal{F}]}{\Gamma \ \ \Vdash \ \ [\langle \alpha' \, \beta' \rangle, \, E', \, \mathcal{F}']}$$

Then by the induction hypothesis the premiss with β in place of $\langle \alpha \, \beta \rangle$ is provable so that we have

$$(\| \rightarrow) \, \frac{\Gamma, \, A \ \ \Vdash \ \ [\beta, \, E, \, \mathcal{F}]}{\Gamma \ \ \Vdash \ \ [\beta', \, E', \, \mathcal{F}']}$$

For $(\| \mathit{pref})$ and $(\| \mathit{suff})$ it is once again simply a matter of choosing the appropriate first list for the succedent triple of the conclusion.

This completes the possible cases whence the $(\| iK)$ property holds.

Lemma 3.2.3 Let $\Gamma \ \Vdash \ [\alpha, E, \mathcal{F}]$ be provable in **GS**, then $[\alpha, E, \mathcal{F}]$ is a support triple.

Proof

It is simply necessary to check that the first list does correspond to a guard list (or the whole) of the first list of E (i.e. of the backbone tip of the list-hedge). This is a simple deductive induction on the definition of **GS**.

For a consecution $\Gamma \ \Vdash \ [\alpha, E, \mathcal{F}]$ α is called the *succedent list*, E the *succedent hedge* and $\mathcal{F}$ the *succedent support function.*

Lemma 3.2.4 Let $\underline{\varphi}$ be a tip of E where $[\alpha, E, \mathcal{F}]$ is a succedent support triple, then $\mathcal{F}\left(\underline{\varphi}\right)$ is either the empty set or a singleton $\{\lambda\}$ with $\varphi^* = \lambda^*$.

Proof is by deductive induction.

Clearly the axioms satisfy this property.

$(C\|)$, $(\|pref)$ and $(\|suff)$ obviously preserve the property since those tips of the conclusion with non-empty images are identical with their direct ancestors, as are the corresponding images (in the antecedent).

For $(\rightarrow\|)$

$$\frac{\Gamma_1 \;\|\!\!- [A, G, \mathcal{H}] \qquad\qquad \Gamma_2, \langle \ldots \langle A\, B \rangle \ldots \rangle \;\|\!\!- [\delta, E, \mathcal{F}]}{\Gamma_1, \Gamma_2, \langle \ldots A \rightarrow B \ldots \rangle \;\|\!\!- \; [\delta, E', \mathcal{F}']}$$

Those tips of the right premiss with tip-image not equal to the $\underline{A}$ have the descendant tip-image in the conclusion which is either the same, or with the $\underline{\langle A\, B \rangle}$ replaced by $\underline{A \rightarrow B}$, so by hypothesis they satisfy the property since $\langle A\, B \rangle^* = A \rightarrow B^*$. The remaining tips of the conclusion correspond to tips of the left premiss support function $(G, \mathcal{H})$ but with $\underline{A}$ replaced by $\underline{\lambda}$ where $\underline{\lambda}$ is a tip with tip-image $\underline{A}$ in the right premiss. But by hypothesis $\lambda^* = A$ so the *wff values are unchanged and this with the hypothesis re $(G, \mathcal{H})$ ensures the required property.

For $(\|\rightarrow)$, the images of the conclusion are identical with those of the ancestral tips but with ancestral tips possibly having $\underline{\langle A\, B \rangle}$ in place of $\underline{A \rightarrow B}$, so the property is preserved. This completes the induction step and so the lemma is proved.

Where a tip $\underline{\varphi}$ has $\mathcal{F}\left(\underline{\varphi}\right) = \{\lambda\}$ $\underline{\lambda}$ is called the *tip-image* of $\underline{\varphi}$.

Lemma 3.2.5 Let $\underline{\varphi}$ be a sublist in either the antecedent or succedent hedge of a consecution in a proof. Then every (direct or schematic) descendant $\underline{\lambda}$ of $\underline{\varphi}$ satisfies $\varphi^* = \lambda^*$.

Proof

By deductive induction.

For all the rules other than $(\rightarrow\|)$ inspection shows that a descendant of a sublist always has the same *wff value.

In the case of $(\rightarrow\|)$ it is obvious for the descendants of the right premiss and of the antecedent of the left premiss. Descendants of the left premiss succedent have $\underline{A}$ replaced by φ where $\underline{\varphi}$ is a tip of the right premiss with tip-image the antecedent $\underline{A}$. But by the above lemma 3.2.4 $\varphi^* = A^* = A$ so this change preserves the *wff values of the descendants as required.

This completes the induction step and so the lemma is proved.

Lemma 3.2.6 Tip-images are always antecedent sublists or whole lists of the antecedent of the consecution.

Proof

By deductive induction: The axioms satisfy the property and the rules obviously preserve it.

I now, by way of example proofs, show that **GS** contains the axioms of the system **S**.

Lemma 3.2.7 The prefixing and suffixing axioms are provable:—

$$\Vdash p{\rightarrow}q{\rightarrow}.r{\rightarrow}p{\rightarrow}.r{\rightarrow}q \qquad \Vdash p{\rightarrow}q{\rightarrow}.q{\rightarrow}r{\rightarrow}.p{\rightarrow}r$$

Proof

As in the previous section I will represent the linked sequence of the succedent by links between lists of the succedent. And I will represent the support function by arrows, putting, where a tip $\underline{\alpha}$ has $\mathcal{F}(\underline{\alpha}) = \{\beta\}$, an arrow from $\underline{\alpha}$ to $\underline{\beta}$. In the following proof the succedent sequence remains just the singleton containing the first list, hence I represent the succedent by a single list plus arrows. For visual clarity I will not display all the arrows all the time.

$$\frac{\langle p\,q\rangle \Vdash \langle p\,q\rangle \qquad \langle r\,p\rangle \Vdash \langle r\,p\rangle \qquad r \Vdash r}{\langle p\,q\rangle,\ \langle r\,p\rangle,\ r \Vdash \big\langle \langle p\,q\rangle \big\langle \langle r\,p\rangle\, \langle r\,q\rangle \big\rangle \big\rangle}$$

$$(\Vert\!\to)\ \frac{\langle p\,q\rangle,\ \langle r\,p\rangle,\ r \Vdash \big\langle \langle p\,q\rangle \big\langle \langle r\,p\rangle\, \langle r\,q\rangle \big\rangle \big\rangle}{\langle p\,q\rangle,\ \langle r\,p\rangle \Vdash \big\langle \langle p\,q\rangle \big\langle \langle r\,p\rangle r\to q \big\rangle \big\rangle}$$

$$(\to\!\Vert)\ \frac{r \Vdash r \qquad \langle p\,q\rangle\,,\ \langle r\,p\rangle \Vdash \big\langle \langle p\,q\rangle \big\langle \langle r\,p\rangle r\to q \big\rangle \big\rangle}{r,\ \langle p\,q\rangle\,,\ r\to p \Vdash \big\langle \langle p\,q\rangle \big\langle \langle r\,p\rangle r\to q \big\rangle \big\rangle}$$

After a $(C\Vert)$:—

$$(\to\!\Vert)\ \frac{p \Vdash p \qquad \langle p\,q\rangle\,,\ r\to p,\ r \Vdash \big\langle \langle p\,q\rangle \big\langle \langle r\,p\rangle r\to q \big\rangle \big\rangle}{p,\ p\to q\,,\ r\to p,\ r \Vdash \big\langle \langle p\,q\rangle \big\langle \langle r\,p\rangle r\to q \big\rangle \big\rangle}$$

$$(\Vert\!\to)\ \frac{p,\ p\to q\,,\ r\to p,\ r \Vdash \big\langle \langle p\,q\rangle \big\langle \langle r\,p\rangle r\to q \big\rangle \big\rangle}{p,\ p\to q\,,\ r\to p \Vdash \big\langle \langle p\,q\rangle \big\langle r\to p\ r\to q \big\rangle \big\rangle}$$

$$(\Vert\!\to)\ \frac{p,\ p\to q\,,\ r\to p \Vdash \big\langle \langle p\,q\rangle \big\langle r\to p\ r\to q \big\rangle \big\rangle}{p,\ p\to q \Vdash \big\langle \langle p\,q\rangle\ r\to p\to .r\to q \big\rangle}$$

After a $(C\Vert)$:—

$$(\Vert\!\to)\ \frac{p\to q\,,\ p \Vdash \big\langle \langle p\,q\rangle\ r\to p\to .r\to q \big\rangle}{p\to q \Vdash \big\langle p\to q\ r\to p\to .r\to q \big\rangle}$$

And finally:—

$$(\Vert\!\rightarrow)\frac{p\rightarrow q \ \Vert\!\vdash \ \langle p\rightarrow q \ r\rightarrow p\rightarrow .r\rightarrow q\rangle}{\Vert\!\vdash \ p\rightarrow q \rightarrow . \ r\rightarrow p\rightarrow .r\rightarrow q}$$

The suffixing axiom is similarly provable using $(\Vert suff)$.

We now check that the class of such provable sentences is closed under universal substitution. Since one can always do such substitution first, it suffices to show that all identities $\alpha \ \Vert\!\vdash \ [\alpha, \{\alpha\}, \mathcal{I}]$ are provable with identity hedge and support function as per the axioms.

Lemma 3.2.8 All identities (with *identity support hedge*) are provable.

Proof

The proof is by induction on the sum of degrees of the constituents of the list. For the base case the list has only atomic constituents, hence the required consecution is an axiom. Assume that the result holds for lists with sum of degrees of constituents $\leq n$ and consider

$$\alpha \ \Vert\!\vdash \ [\alpha, \{\alpha\}, \mathcal{I}]$$

of degree $n+1$. Choose a constituent of form $A\rightarrow B$. Letting α' be α with that constituent replaced by the list $\langle A\,B\rangle$ we have by induction hypothesis

$A \ \Vert\!\vdash \ [A, \{A\}, \mathcal{I}']$ and $\alpha' \ \Vert\!\vdash \ [\alpha', \{\alpha'\}, \mathcal{I}'']$. So we can carry out the proof:—

$$(\rightarrow\!\Vert)\ \frac{A \ \Vert\!\vdash A \qquad \langle\ldots\langle A\,B\rangle\ldots\rangle \ \Vert\!\vdash \langle\ldots\langle A\,B\rangle\ldots\rangle}{A, \langle\ldots A\rightarrow B\ldots\rangle \ \Vert\!\vdash \langle\ldots\langle A\,B\rangle\ldots\rangle}$$

After a $(C\Vert)$:—

$$(\|\!\!\rightarrow)\ \frac{\langle \ldots A \rightarrow B \ldots \rangle,\ A\ \Vdash \langle \ldots \langle A\, B \rangle \ldots \rangle}{\langle \ldots A \rightarrow B \ldots \rangle\ \Vdash \langle \ldots A \rightarrow B \ldots \rangle}$$

As required.

I now do an example which illustrates the use of the inbuilt $(\|iK)$. I show that Wrightson's Folly $(q \rightarrow .p \rightarrow p) \rightarrow .q \rightarrow .q \rightarrow .p \rightarrow p$ is provable in $\mathbf{G}(\mathbf{R}-(id))$. The following extra rules, the clones of the permutation and contraction axioms, are added to **GS** to obtain $\mathbf{G}(\mathbf{R}-(id))$.

$$(\|perm)\quad \frac{\begin{array}{c} \Gamma_1 \Vdash [\langle \alpha \langle \beta\, \gamma \rangle \rangle,\ E_1,\ \mathcal{F}_1] \\ \Gamma_2 \Vdash [\beta,\ E_2,\ \mathcal{F}_2] \\ \Gamma_3 \Vdash [\alpha, E_3, \mathcal{F}_3] \\ \Gamma_4 \Vdash [\gamma_1, E_4, \mathcal{F}_4] \\ \ldots \\ \Gamma_{n+3} \Vdash [\gamma_n, E_{n+3}, \mathcal{F}_{n+3}] \end{array}}{\Gamma_1, \ldots \Gamma_{n+3} \Vdash [\delta,\ E,\ \mathcal{F}]}$$

Where $\gamma = \langle \gamma_1 \ldots \langle \gamma_n\, C \rfloor$, δ is any guard list or the whole of $\big\langle \langle \alpha \langle \beta\, \gamma \rangle \rangle\ \langle \beta \langle \alpha\, \gamma \rangle \rangle \big\rangle$ and the conclusion support hedge E is formed by applying the inductive step of definition 2.2.1 to $E_1, \ldots E_{n+3}$ with backbone tip assigned the above list; and $\mathcal{F}$ is determined by putting:—

- $\mathcal{F}\Big(\underline{\big\langle \langle \alpha \langle \beta\, \gamma \rangle \rangle\ \langle \beta \langle \alpha\, \gamma \rangle \rangle \big\rangle}\Big)$ is the empty set,
- $(E, \mathcal{F})|_{\underline{\langle \alpha \langle \beta\, \gamma \rangle \rangle}} = (E_1, \mathcal{F}_1)$
- $(E, \mathcal{F})|_{\underline{\beta}} = (E_2, \mathcal{F}_2)$
- $(E, \mathcal{F})|_{\underline{\alpha}} = (E_3, \mathcal{F}_3)$
- for the major antecedents of the guard $\underline{\gamma}$ $\quad (E, \mathcal{F})|_{\underline{\gamma_i}} = (E_{i+3}, \mathcal{F}_{i+3})$

- and the remaining located sublists have values determined by the above and the union requirement upon support functions.

$$(\|con)\quad \frac{\begin{array}{c}\Gamma_1 \Vdash [\langle\alpha\,\langle\alpha\,\beta\rangle\rangle,\, E_1,\, \mathcal{F}_1] \\ \Gamma_2 \Vdash [\alpha,\, E_2,\, \mathcal{F}_2] \\ \Gamma_2 \Vdash [\beta_1, E_2, \mathcal{F}_2] \\ \cdots \\ \Gamma_{n+2} \Vdash [\beta_n, E_{n+2}, \mathcal{F}_{n+2}]\end{array}}{\Gamma_1, \ldots \Gamma_{n+2} \Vdash [\delta,\, E,\, \mathcal{F}]}$$

Where $\beta = \langle\beta_1 \ldots \langle\beta_n\, B\rfloor$, δ is any guard list or the whole of $\big\langle\langle\alpha\,\langle\alpha\,\beta\rangle\rangle\;\langle\alpha\,\beta\rangle\big\rangle$ and the conclusion support hedge E is formed by applying the inductive step of definition 2.2.1 to $E_1, \ldots E_{n+2}$ with backbone tip assigned the above list; and $\mathcal{F}$ is determined by putting:—

- $\mathcal{F}\left(\underline{\big\langle\langle\alpha\,\langle\alpha\,\beta\rangle\rangle\;\langle\alpha\,\beta\rangle\big\rangle}\right)$ is the empty set,
- $(E, \mathcal{F})|_{\underline{\langle\alpha\,\langle\alpha\,\beta\rangle\rangle}} = (E_1, \mathcal{F}_1)$
- $(E, \mathcal{F})|_{\underline{\alpha}} = (E_2, \mathcal{F}_2)$
- for the major antecedents of the guard $\underline{\beta}$ $\quad (E, \mathcal{F})|_{\underline{\beta_i}} = (E_{i+2}, \mathcal{F}_{i+2})$
- and the remaining located sublists have values determined by the above and the union requirement upon support functions.

Now for the proof. Begin with the following instance of most of the proof of prefixing:—

$$p{\to}p \Vdash \langle p{\to}p\; p{\to}p{\to}.p{\to}p\rangle$$

Apply $(\|pref)$ to this and the identities $\langle q\; p{\to}p\rangle \Vdash \langle q\; p{\to}p\rangle$ and $q \Vdash q$:—

$$p{\to}p,\; \langle qp{\to}p\rangle,\; q \Vdash \big\langle\langle qp{\to}p\rangle\langle qp{\to}p{\to}.p{\to}p\rangle\big\rangle\;,\;\; \big\langle\langle p{\to}p\; p{\to}p{\to}.p{\to}p\rangle\;\big\langle\langle q\; p{\to}p\rangle\;\langle q\; p{\to}p{\to}.p{\to}p\rangle\big\rangle\big\rangle$$

Note that the succedent first list above is a guard list of the backbone tip. Apply $(\|{\rightarrow})$:—

$$\langle qp{\rightarrow}p\rangle,\ q \Vdash \big\langle \langle qp{\rightarrow}p\rangle,\ q\langle qp{\rightarrow}p{\rightarrow}.p{\rightarrow}p\rangle\big\rangle\ ,\quad \big\langle p{\rightarrow}p{\rightarrow}.p{\rightarrow}p{\rightarrow}.p{\rightarrow}p\ \big\langle\langle qp{\rightarrow}p\rangle\langle qp{\rightarrow}p{\rightarrow}.p{\rightarrow}p\rangle\big\rangle\big\rangle$$

I don't continue to show the linked sequence as the extra tip $\underline{p{\rightarrow}p{\rightarrow}.p{\rightarrow}p{\rightarrow}.p{\rightarrow}p}$ is a theorem with empty support, thus:—

$$\langle q\,p{\rightarrow}p\rangle,\ q \Vdash \big\langle\langle q\,p{\rightarrow}p\rangle\ \langle q\,p{\rightarrow}p{\rightarrow}.p{\rightarrow}p\rangle\big\rangle$$

Apply $({\rightarrow}\|)$ with left premiss $q\Vdash q$, then two $(\|{\rightarrow})$'s to "fill in" the q's:—

$$q{\rightarrow}.p{\rightarrow}p \Vdash \big\langle q{\rightarrow}.p{\rightarrow}p\,q{\rightarrow}.p{\rightarrow}p{\rightarrow}.p{\rightarrow}p\big\rangle \tag{3.1}$$

I use the abbreviations $A^2 = A{\rightarrow}A$ and $Q = q{\rightarrow}.p{\rightarrow}p$ in what follows. Do the following instance of most of the proof of prefixing:—

$$p^{2^2},\ q{\rightarrow}p^2 \Vdash \big\langle p^{2^2}\ \big\langle q{\rightarrow}p^2\ q{\rightarrow}p^2\big\rangle\big\rangle$$

Apply $(\|pref)$ to this and the identities $\langle q\,p^{2^2}\rangle \Vdash \langle q\,p^{2^2}\rangle$, $q \Vdash q$, $Q \Vdash Q$ (here again exploiting the inbuilt $(\|iK)$):—

$$p^{2^2},\ Q,\ \langle q\,p^{2^2}\rangle,\ q,\ Q\ \Vdash \big\langle\langle q\,p^{2^2}\rangle\ \langle q\,\langle Q\,Q\rangle\rangle\big\rangle\ ,\quad \big\langle\langle p^{2^2}\,\langle Q\,Q\rangle\rangle\ \big\langle\langle q\,p^{2^2}\rangle\ \langle q\,\langle Q\,Q\rangle\rangle\big\rangle\big\rangle$$

Apply two $(\|{\rightarrow})$'s:—

$$p^{2^2},\ Q,\ \langle q\,p^{2^2}\rangle \Vdash \big\langle\langle q\,p^{2^2}\rangle\ q{\rightarrow}.Q{\rightarrow}Q\big\rangle\ ,\quad \big\langle\langle p^{2^2}\,\langle Q\,Q\rangle\rangle\ \big\langle\langle q\,p^{2^2}\rangle\ q{\rightarrow}.Q{\rightarrow}Q\big\rangle\big\rangle$$

And a further two $(\|{\rightarrow})$'s, after the appropriate $(C\|)$'s:—

$$\langle q\, p^{2^2}\rangle \Vdash \big\langle \langle q\, p^{2^2}\rangle\; q{\rightarrow}.Q{\rightarrow}Q\big\rangle\,,\;\; \big\langle p^{2^2}{\rightarrow}.Q{\rightarrow}Q\; \big\langle \langle q\, p^{2^2}\rangle\; q{\rightarrow}.\langle Q{\rightarrow}Q\rangle\big\rangle\big\rangle$$

And $(\rightarrow\Vert)$ with left premiss $q \Vdash q$, and then $(\Vert\rightarrow)$ (using T to denote the theorem $p^{2^2} \rightarrow .Q\rightarrow Q$):—

$$q{\rightarrow}p^{2^2} \Vdash \big\langle q{\rightarrow}p^{2^2}\; q{\rightarrow}.Q{\rightarrow}Q\big\rangle\,,\;\; \big\langle T\, \big\langle q{\rightarrow}p^{2^2}\; q{\rightarrow}.\langle Q{\rightarrow}Q\rangle\big\rangle\big\rangle \tag{3.2}$$

(I won't continue to show this linked sequence as the extra T is a theorem with empty support.)

Do the following part of the instance of the proof of permutation (similar to prefixing, using $(\Vert perm)$):—

$$q{\rightarrow}.Q{\rightarrow}Q,\; Q \;\Vdash\; \big\langle q{\rightarrow}.Q{\rightarrow}Q\; \langle Q\; q{\rightarrow}Q\rangle\big\rangle \tag{3.3}$$

Now apply $(\Vert suff)$ to (3.1) and (3.2) and $Q \Vdash Q$:—

$$Q,\; q{\rightarrow}p^{2^2},\; Q \Vdash \big\langle Q\; q{\rightarrow}.Q{\rightarrow}Q\big\rangle\,,\;\; \big\langle \langle Q\; q{\rightarrow}p^{2^2}\rangle\; \big\langle \langle q{\rightarrow}p^{2^2}\; q{\rightarrow}.Q{\rightarrow}Q\rangle\; \big\langle Q\; q{\rightarrow}.Q{\rightarrow}Q\big\rangle\big\rangle\big\rangle$$

Apply two $(\Vert\rightarrow)$'s:—

$$Q \;\Vdash\; \big\langle Q\; q{\rightarrow}.Q{\rightarrow}Q\big\rangle\,,\;\; \big\langle Q{\rightarrow}.q{\rightarrow}p^{2^2}\; \big\langle q{\rightarrow}p^{2^2}{\rightarrow}.q{\rightarrow}.Q{\rightarrow}Q\; \big\langle Q\; q{\rightarrow}.Q{\rightarrow}Q\big\rangle\big\rangle\big\rangle$$

Again it is not necessary to show the rest of the sequence other than the first list as the remaining tips are theorems with empty support, so:—

$$Q \;\Vdash\; \big\langle Q\; q{\rightarrow}.Q{\rightarrow}Q\big\rangle$$

Apply $(\Vdash pref)$ to (3.3) and the above and two $Q \Vdash Q$:—

$$q \to .Q \to Q,\ Q,\ Q,\ Q, Q \Vdash \langle Q\ \langle Q\ q \to Q\rangle\rangle,\ \langle\langle q \to .Q \to Q\ \langle Q\ q \to Q\rangle\rangle\ \langle\langle Q\ q \to .Q \to Q\rangle\ \langle Q\ \langle Q\ q \to Q\rangle\rangle\rangle\rangle$$

Apply $(\Vdash\to)$:—

$$q \to .Q \to Q,\ Q,\ Q, Q \Vdash \langle Q\ \langle Q\ q \to Q\rangle\rangle,\ \langle\langle q \to .Q \to Q\ Q \to .q \to Q\rangle\ \langle\langle Q\ q \to .Q \to Q\rangle\ \langle Q \langle Q\ q \to Q\rangle\rangle\rangle\rangle$$

And a further two $(\Vdash\to)$'s:—

$$Q,\ Q \Vdash \langle Q\ \langle Q\ q \to Q\rangle\rangle,\ \langle (q \to .Q \to Q) \to (Q \to .q \to Q)\ \langle Q \to .q \to .Q \to Q\ \langle Q \langle Q\ q \to Q\rangle\rangle\rangle\rangle$$

Again we may disregard the remaining sequence:—

$$Q,\ Q \Vdash \langle Q\ \langle Q\ q \to Q\rangle\rangle$$

Apply *(con)* to the above and $Q \Vdash Q$:—

$$Q,\ Q,\ Q \Vdash \langle Q\ q \to Q\rangle,\ \langle\langle Q\ \langle Q\ q \to Q\rangle\rangle\ \langle Q\ q \to Q\rangle\rangle$$

Applying three $(\Vdash\to)$'s gives the required result:—

$$\Vdash Q \to .q \to Q,\ \langle Q \to .Q \to .q \to Q\ Q \to .q \to Q\rangle$$

(With only the first member of the linked sequence shown.)

Note that the proof is just an adaptation of a proof for the axiomatic system. These cloned consecution systems don't confer any advantage, it seems, in trying to check whether a wff is a theorem. They describe too closely what is going on in the corresponding Hilbert systems.

3.3 The formal interpretation of GS

The following lemma goes halfway towards showing that **GS** captures the weak system S. The result is easily adapted for the general cloned systems. The intuitive idea behind the interpretation is that lists and hedges represent proofs, and the support function tells us which antecedent proofs underpin subhedges (subproofs) of the succedent. We have already proved the core of this result in section 3.2. In what follows 'theory' is used to denote what is usually called a regular, detached theory— that is to denote a set of wff which contains the axioms of the logic under consideration and is closed under modus ponens.

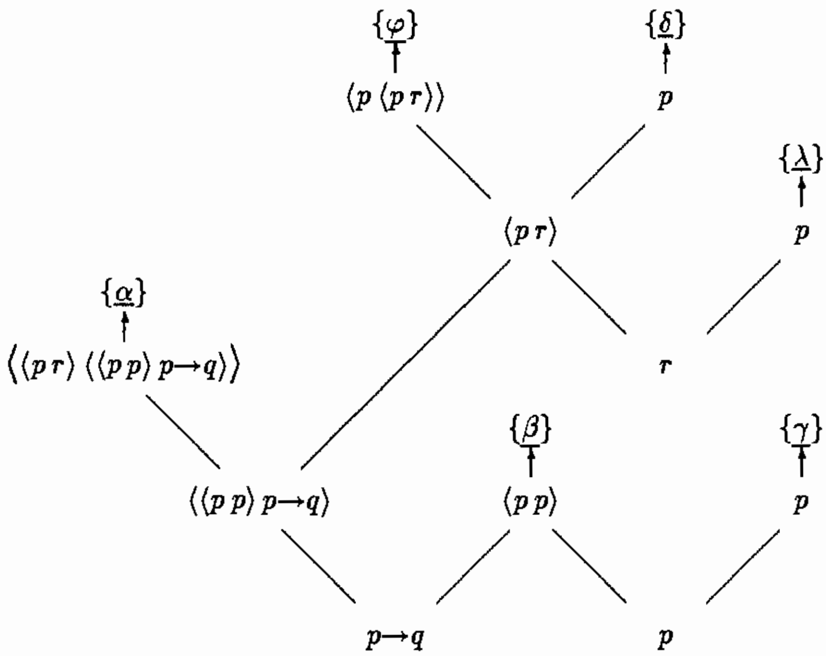
Lemma 3.3.1 If $\Gamma \mid\!\vdash_{\mathbf{GS}} [\alpha,\, E,\, \mathcal{F}]$ then for every located sublist $\underline{\varphi}$ of the succedent and every theory $\mathbf{T}$ of the logic S, if for every $\underline{\delta} \in F\left(\underline{\varphi}\right)$ we have $\vdash_{\mathrm{T}} \delta^*$ then $\vdash_{\mathrm{T}} \varphi^*$. Figure 3.2 shows an example.

Proof

The proof is by induction on the depth of $\underline{\varphi}$ in the hedge, where the depth of a tip is zero and the depth of the remaining located lists is the sum of those of their successors plus one.

The property holds of every tip $\underline{\varphi}$ which has a non-empty image, by lemma 3.2.4. Also, those tips $\underline{\varphi}$ with null image can only have been introduced by a $(\|pref)$ or a $(\|suff)$, and so φ^* is an instance of either prefixing or suffixing (as the rules preserve the *wff values of located lists—lemma 3.2.5) and hence is in every theory. Thus the required property holds of the tips.

Assume it obtains for all located lists with depth $\leq n$ and consider a list $\underline{\varphi}$ with depth $n+1$. Now $\underline{\varphi}$ has a left successor $\underline{\langle\gamma\,\varphi\rangle}$ and a right successor $\underline{\gamma}$, each having depth $\leq n$. Now $\mathcal{F}\left(\underline{\varphi}\right) = \mathcal{F}\left(\underline{\langle\gamma\,\varphi\rangle}\right) \bigcup \mathcal{F}\left(\underline{\gamma}\right)$ so by our induction hypothesis every theory $\mathbf{T}$ with $\vdash_{\mathrm{T}} \delta^*$ for all $\underline{\delta} \in F\left(\underline{\varphi}\right)$ has $\vdash_{\mathrm{T}} \langle\gamma\,\varphi\rangle^*$ and $\vdash_{\mathrm{T}} \gamma^*$. But $\langle\gamma\,\varphi\rangle^* = \gamma^* {\rightarrow} \varphi^*$ so by modus ponens the required wff φ^* holds in $\mathbf{T}$. Thus by induction the required property



Sublist γ of the hedge	$\mathcal{F}\left(\underline{\gamma}\right)$
$\langle\langle p\, r\rangle\, \langle\langle p\, p\rangle\, p \rightarrow q\rangle\rangle$	$\{\underline{\alpha}\}$
$\langle\langle p\, p\rangle\, p \rightarrow q\rangle$	$\{\underline{\alpha}, \underline{\varphi}, \underline{\delta}\}$
$p \rightarrow q$	$\{\underline{\alpha}, \underline{\varphi}, \underline{\delta}, \underline{\beta}\}$
r	$\{\underline{\varphi}, \underline{\delta}, \underline{\lambda}\}$

Every theory containing the *wffs of the members of $\mathcal{F}\left(\underline{\gamma}\right)$ also contains γ^*, for example every theory containing α^*, φ^*, δ^* and β^* must also contain $p \rightarrow q$.

Note that $\alpha^* = p \rightarrow r \rightarrow .p \rightarrow p \rightarrow .p \rightarrow q\,,\quad \varphi^* = p \rightarrow .p \rightarrow r\,,\quad \beta^* = p \rightarrow p\,,\quad \delta = \lambda = \gamma = p$

Figure 3.2: Example of a support hedge $(H, \mathcal{F})$ with some located sublists of the domain and their support images displayed.

holds of every located sublist of the succedent hedge.

And we have as a corollary the interpretation theorem:—

Theorem 3.3.2 If $\Vdash_{\mathbf{GS}} [A,\ E,\ \mathcal{F}]$ then $\vdash_{\mathrm{S}} A$.

Proof

Suppose that $\Vdash_{\mathbf{GS}} [A,\ E,\ \mathcal{F}]$. Since the antecedent is empty $\mathcal{F}(\underline{A})$ is the empty-set and so by the lemma every theory contains A^*, that is A, hence in particular S does.

3.4 The other way: GS contains S.

We have seen that the prefixing and suffixing axioms are provable in **GS** and that the class of such provable sentences is closed under uniform substitution. All that remains is to prove that this class is also closed under modus ponens. To do so I prove that the consecution system is closed under *cut*.

Definition 3.4.1 *(cut)* is defined:—

$$(cut)\ \frac{\Gamma \ \Vdash\ [\alpha, H, \mathcal{K}] \qquad \Lambda \ \Vdash\ [\delta, E, \mathcal{F}]}{\Gamma,\ \Lambda^{\circ} \ \Vdash\ [\delta, E^{\circ}, \mathcal{F}^{\circ}]}$$

Where

- Λ contains a whole list α,
- Λ° is like Λ but with that α deleted, α is called the *cut list*,
- $(E^{\circ}, \mathcal{F}^{\circ})$ is obtained from $(E, \mathcal{F})$ and $(H, \mathcal{K})$ in a manner similar to $(\rightarrow\Vert)$: For each E tip $\underline{\varphi}$ with tip-image $\underline{\lambda}$ occurring in α replace $(E,\mathcal{F})\{\underline{\varphi}\}$ by $(H,\mathcal{K})\{\underline{\lambda}\}\,[\underline{\lambda}\backslash\varphi]$.

The intuitive picture is that the support hedge of the right premiss "grows" at those of its tips $\underline{\varphi}$ with tip-images $\underline{\lambda}$ in the cut list, with the new growth corresponding to the support bush of the left premiss for the corresponding $\underline{\lambda}$.

For example with left premiss obtained from the appropriate substitution instance of the second to last line of the proof of prefixing, modified to incorporate $(\Vdash iK)$ at the $(\Vdash pref)$ step, (lemma 3.2.7):—

$$p{\rightarrow}q \Vdash r{\rightarrow}p{\rightarrow}.r{\rightarrow}q\ ,\quad \langle p{\rightarrow}q\ r{\rightarrow}p{\rightarrow}.r{\rightarrow}q\rangle$$

And right premiss the appropriate substitution instance of the third to last line of the proof of prefixing:—

$$r{\rightarrow}p,\ r{\rightarrow}p{\rightarrow}.r{\rightarrow}q \Vdash \langle\langle r{\rightarrow}p\, r{\rightarrow}q\rangle\ s{\rightarrow}.r{\rightarrow}p{\rightarrow}.s{\rightarrow}.r{\rightarrow}q\rangle$$

Application of *(cut)* delivers:—

$$p{\rightarrow}q,\ r{\rightarrow}p \Vdash \langle\langle r{\rightarrow}p\, r{\rightarrow}q\rangle\ s{\rightarrow}.r{\rightarrow}p{\rightarrow}.s{\rightarrow}.r{\rightarrow}q\rangle\ ,\quad \langle p{\rightarrow}q\ \langle r{\rightarrow}p\, r{\rightarrow}q\rangle\rangle$$

I now prove the invertibility of $(\Vdash\rightarrow)$. This fact is a key part of the strategy for the proof of closure under *(cut)*, for in the right rank $=1$ case where the right upper comes from a connective $(\rightarrow)$ introduction in the antecedent step, it enables us to suppose that the left upper came from the corresponding introduction in the consequent, so that the usual strategy for such case can apply. Invertibility is also needed in the proof that closure under cut delivers closure under modus ponens.

Lemma 3.4.2 The invertibility property holds: If $\Gamma \Vdash [\langle \ldots A{\rightarrow}B \ldots\rangle, E, \mathcal{F}]$ where $A{\rightarrow}B$ is a constituent of the succedent hedge— not necessarily in the first list— then there is a deduction of this consecution in which that $\to$ is introduced last, that is with last step

$$(\Vdash\!\to)\ \frac{\Gamma,\ A\ \ \Vdash\ \ [\langle\ldots\langle A\,B\rangle\ldots\rangle,\ E',\ \mathcal{F}']}{\Gamma\ \ \Vdash\ \ [\langle\ldots A\!\to\! B\ldots\rangle,\ E,\ \mathcal{F}]}$$

(Or with first list unchanged where it doesn't contain the $\underline{\langle A\,B\rangle}$).

Proof

Consider a proof of a consecution $\Gamma \Vdash [\alpha, E, \mathcal{F}]$ of such form. I show how to convert such a proof into one satisfying the required property (if it doesn't already). The idea is that one simply refrains from doing the corresponding ancestral $\to$ introductions but otherwise carry through the same proof. Call the earliest *schematic* ancestors (in this deduction) of $\underline{A\!\to\! B}$ still of form $A\!\to\! B$ the *initial ancestors.*

I prove that for each descendant consecution $\Lambda \Vdash [\delta, G, \mathcal{H}]$ of those containing initial ancestors, the corresponding $\Lambda, A, \ldots A \Vdash [\delta^{\triangleright}, G^{\triangleright}, \mathcal{H}^{\triangleright}]$ is also provable, where $\delta^{\triangleright}, G^{\triangleright}$ is δ, G with all schematic ancestral $\underline{A\!\to\! B}$'s replaced by $\langle A\,B\rangle$, and for each such opened up tip $\underline{A}$ $\mathcal{H}^{\triangleright}(\underline{A})$ is the singleton containing a new antecedent $\underline{A}$ (exactly one for each new tip), and each tip $\underline{\lambda}$ of the rest of the domain of G has $\mathcal{H}^{\triangleright}(\underline{\lambda}) = \mathcal{H}(\underline{\lambda'})$ where $\lambda' = \lambda\left[\underline{\langle A\,B\rangle}\backslash A\!\to\! B\right]$.

Thus the premiss for "multiple" $(\Vdash\!\to)$ introduction of these $\to$'s is provable.

The proof is by deductive induction.

Base is the consecutions containing the earliest initial ancestors. Since ancestors of a constituent $A\!\to\! B$ in the succedent still of the form $A\!\to\! B$ can only occur in the succedent too, the earliest initial ancestors can only have been obtained by $(\Vdash\!\to)$. So a consecution containing an earliest initial ancestor $\Lambda \Vdash [\delta, G, \mathcal{H}]$ has a premiss in the given proof of the required form: $\Lambda,\ A \Vdash [\delta^{\triangleright}, G^{\triangleright}, \mathcal{H}^{\triangleright}]$.

Now check that the rules preserve this property.

Suppose we have a descendant obtained by:—

$$(C\|)\,\frac{\Lambda \ \ \Vdash \ \ [\delta,\, G,\, \mathcal{H}]}{\Lambda' \ \ \Vdash \ \ [\delta,\, G,\, \mathcal{H}]}$$

By hypothesis we have

$$\Lambda,\ A, \ldots A \Vdash [\delta^{\triangleright}, G^{\triangleright}, \mathcal{H}^{\triangleright}]$$

and application of the corresponding $(C\|)$ delivers the required consecution. Suppose we have a descendant obtained by $(\|pref)$:—

$$\begin{array}{l}
\Lambda_1 \ \Vdash \ [\langle\alpha\,\beta\rangle,\, G_1,\, \mathcal{H}_1] \\
\Lambda_2 \ \Vdash \ [\langle\gamma\,\alpha\rangle,\, G_2,\, \mathcal{H}_2] \\
\Lambda_3 \ \Vdash \ [\gamma, G_3, \mathcal{H}_3] \\
\Lambda_4 \ \Vdash \ [\beta_1, G_4, \mathcal{H}_4] \\
\ldots \\
\Lambda_{n+3} \ \Vdash \ [\beta_n, G_{n+3}, \mathcal{H}_{n+3}] \\
\hline
\Lambda_1, \ldots \Lambda_{n+3} \ \Vdash \ [\delta,\, G,\, \mathcal{H}]
\end{array}$$

By hypothesis we have

$$\begin{array}{l}
\Lambda_1,\ A, \ldots A \ \Vdash \ [\langle\alpha\,\beta\rangle^{\triangleright},\, G_1^{\triangleright},\, \mathcal{H}_1^{\triangleright}] \\
\Lambda_2,\ A, \ldots A \ \Vdash \ [\langle\gamma\,\alpha\rangle^{\triangleright},\, G_2^{\triangleright},\, \mathcal{H}_2^{\triangleright}] \\
\Lambda_3,\ A, \ldots A \ \Vdash \ [\gamma^{\triangleright}, G_3^{\triangleright}, \mathcal{H}_3^{\triangleright}] \\
\Lambda_4,\ A, \ldots A \ \Vdash \ [\beta_1^{\triangleright}, G_4^{\triangleright}, \mathcal{H}_4^{\triangleright}] \\
\ldots \\
\Lambda_{n+3},\ A, \ldots A \ \Vdash \ [\beta_n^{\triangleright}, G_{n+3}^{\triangleright}, \mathcal{H}_{n+3}^{\triangleright}]
\end{array}$$

depending on the location of ancestral $A{\rightarrow}B$'s (possibly with no extra A's in the antecedent where in fact the corresponding premiss contained no schematic ancestors of the $\underline{A{\rightarrow}B}$). There are two cases to consider, depending upon whether the guard wff of

$\underline{\beta}$ is an ancestral $A{\rightarrow}B$ to be replaced by $\langle A\,B\rangle$, or not.

First suppose not, then applying $(\|pref)$ to the above consecutions (and possibly a $(C\|)$), we get:—

$$\Lambda_1, \ldots \Lambda_{n+3},\ A, \ldots A \ \ \|\!\!- \ [\delta'', G'', \mathcal{H}'']$$

Where there is exactly one extra 'A' in the antecedent for each "opened" $\underline{\langle A\,B\rangle}$ in the succedent hedge. Note that it is in virtue of the schematic sense of ancestor that the above-modified premisses are of the correct form for application of $(\|pref)$. Now the above succedent list might have twice too many opened $\langle A\,B\rangle$'s since some of the sibling $\underline{A{\rightarrow}B}$'s (which have all been "opened") may not be schematic ancestors. For example an ancestral $A{\rightarrow}B$ might have occurred in just *one* of the $\underline{\alpha}$'s. In this event apply $(\|{\rightarrow})$, repeating as necessary, to "fill in" the unwanted $\underline{\langle A\,B\rangle}$'s. So eventually we get the required consecution.

Now consider the case where the guard wff of $\underline{\beta}$ *is* an ancestral $A{\rightarrow}B$ to be replaced by $\langle A\,B\rangle$. To obtain the required conclusion simply apply $(\|pref)$ to the above-modified premisses together with, as an extra final premiss (corresponding to the extra final major antecedent $\underline{A}$ of $\beta^{\triangleright}$) the identity: $A \ \|\!\!- \ [A, \{A\}, \mathcal{I}]$. And adjust as before to obtain the required consecution.

Suppose we have a descendant obtained by $(\|suff)$ then the analogous argument to that for the above case shows we have the desired property.

Suppose we have a descendant obtained by:—

$$(\rightarrow\|)\,\frac{\Lambda_1 \ \|\!\!- \ [C, L, \mathcal{K}] \qquad\qquad \Lambda_2, \langle \ldots \langle C\,D\rangle \ldots\rangle \ \|\!\!- \ [\delta, G, \mathcal{H}]}{\Lambda_1, \Lambda_2, \langle \ldots C{\rightarrow}D \ldots\rangle \ \ \|\!\!- \ \ [\delta, G', \mathcal{H}']}$$

By hypothesis we have:—

$$\Lambda_2, \langle \ldots \langle C\,D\rangle \ldots\rangle,\ A, \ldots A \ \|\!\!- [\delta^{\triangleright}, G^{\triangleright}, \mathcal{H}^{\triangleright}]$$

and so can do

$$(\to\|)\ \frac{\Lambda_1 \ \|\!\vdash [C, L, \mathcal{K}] \qquad \Lambda_2, A, \ldots A,\ \langle \ldots \langle C\ D \rangle \ldots \rangle \ \|\!\vdash [\delta^{\triangleright}, G^{\triangleright}, \mathcal{H}^{\triangleright}]}{\Lambda_1, \Lambda_2, A, \ldots A,\ \langle \ldots C \to D \ldots \rangle \ \|\!\vdash [\delta^{\triangleright}, G^{\triangleright\prime}, \mathcal{H}^{\triangleright\prime}]}$$

Obtaining the required consecution (with adjustment $(C\|)$'s) since the order of the associated changes to the succedent support hedge is not significant.

Suppose we have a descendant obtained by $(\|\to)$:—

$$(\|\to)\ \frac{\Lambda, C \ \ \|\!\vdash \ \ [\delta', G', \mathcal{H}']}{\Lambda \ \ \|\!\vdash \ \ [\delta, G, \mathcal{H}]}$$

Where a $\underline{\langle C\ D \rangle}$ is replaced by $C \to D$ in the succedent hedge. By hypothesis we have:—

$$\Lambda,\ C,\ A, \ldots A \ \|\!\vdash \ [\delta'^{\triangleright}, G'^{\triangleright}, \mathcal{H}'^{\triangleright}]$$

In the case that the $\underline{C \to D}$ in fact equals an ancestral $\underline{A \to B}$ then the above is already of the required form (and $\underline{C \to D}$ was an initial ancestor). Otherwise, do the corresponding $(\|\to)$ (after a $(C\|)$):—

$$(\|\to)\ \frac{\Lambda,\ A, \ldots A, C \ \ \|\!\vdash \ \ [\delta'^{\triangleright}, G'^{\triangleright}, \mathcal{H}'^{\triangleright}]}{\Lambda,\ A, \ldots A \ \ \|\!\vdash \ \ [\delta^{\triangleright}, G^{\triangleright}, \mathcal{H}^{\triangleright}]}$$

obtaining the required consecution since the order of the two changes to the succedent support hedge once again does not matter.

This completes the induction cases and so the proof of the property.
Applying the property to the last line of our supposed proof $\Gamma,\ A \ \|\!\vdash [\alpha^{\triangleright}, E^{\triangleright}, \mathcal{F}^{\triangleright}]$ is provable, where the consecution is of correct form to apply $(\|\to)$ to obtain $\Gamma \ \|\!\vdash [\alpha, E, \mathcal{F}]$.

Thus that $\to$ can be introduced last, and the lemma is proved.

The following lemma shows that closure under cut ensures closure under modus ponens.

Lemma 3.4.3 If *(cut)* is admissible in **GS** then **GS** is closed under modus ponens: If $\Vdash [A,\ H,\ \mathcal{K}]$ and $\Vdash [A\rightarrow B,\ E,\ \mathcal{F}]$ then for some $M, \mathcal{N}$ we have $\Vdash [B, M, \mathcal{N}]$.

Proof

Suppose that we have both $\Vdash [A,\ H,\ \mathcal{K}]$ and $\Vdash [A\rightarrow B,\ E',\ \mathcal{F}']$. Now by invertibility there is a proof of the second consecution with that $\rightarrow$ introduction last, i.e. with last step:—

$$(\Vdash\rightarrow)\ \frac{A\ \Vdash\ [\langle A\,B\rangle,\ E,\ \mathcal{F}]}{\Vdash\ [A\rightarrow B,\ E',\ \mathcal{F}']}$$

So using the above premiss and closure under *(cut)*:—

$$(cut)\ \frac{\Vdash\ [A, H, \mathcal{K}] \qquad A\ \Vdash\ [\langle A\,B\rangle, E, \mathcal{F}]}{\Vdash\ [\langle A\,B\rangle, E^{\circ}, \mathcal{F}^{\circ}]}$$

And then by closure under $(\Vdash iK)$ (lemma 3.2.2):—

$$\Vdash\ [B, E^{\circ}, \mathcal{F}^{\circ}]$$

as required.

We are now in a position to prove closure under *(cut)*, but first some ancillary definitions.

Definition 3.4.4 The *degree* of a wff is defined:

- the degree of an atomic wff is zero,
- the degree of $A\rightarrow B$ is the sum of that of A plus that of B, plus one.

Definition 3.4.5 The *degree* of a list is the sum of the degrees of its constituent wffs.

Definition 3.4.6 Suppose we have proofs in **GS** of left and right premisses of appropriate form for an application of *(cut)*. We call these the *left upper* and *right upper* respectively.

The *right rank* is the maximum length of a path in the deduction, above and including the right upper, containing an ancestor of the cut α list-identical with α.

Lemma 3.4.7 **GS** is closed under *(cut)*.

Proof

The proof is by a double induction, on the degree of the cut list α and on the *right* rank.[5] The base case for degree is where the cut list is atomic. All possible such cases are particular subcases of:—

1. Right Rank= 1: 1.1
2. Right Rank> 1 All subcases 2.1–2.5.

So the base case for degree is established in passing by the proof of these cases. The induction step for degree is proved using 1.2 plus all subcases of 2, and so also is proved in passing through all of the above cases. Both are proved by an inner induction on right rank with the right rank induction hypothesis used in the right rank induction step (case 2).

1.Right rank = 1

I consider the two possible cases for the last step in the deduction of the right upper.

1.1 Right upper an axiom

[5]Along the lines of Gentzen's proof [26]. Thanks are due to Ross Brady who pointed out the redundancy of 2(b) in my original proof. Curry makes the observation that where a rule is invertible we can dispense with the corresponding (left or right) side of the rank induction (p.209 [5]).

$$(cut)\ \frac{\Gamma \ \Vdash\ [\alpha, H, \mathcal{K}] \qquad \alpha \ \Vdash\ [\delta, \{\alpha\}, \mathcal{I}]}{\Gamma \ \Vdash\ [\delta, \{\alpha\}^{\circ}, \mathcal{I}^{\circ}]}$$

We check that the support hedge of the left upper is already equal to that of the conclusion. So we need to show that $(H, \mathcal{K}) = (\{\alpha\}^{\circ}, \mathcal{I}^{\circ})$. Every tip $\underline{\lambda}$ in the antecedent α of the right premiss is required to be replaced by the bush $(H, \mathcal{K})\{\underline{\lambda}\}$ (noting that in this case the tips are identical with their tip-images). But by lemma 2.3.7 this ensures that $H = \{\alpha\}^{\circ}$. And since every tip of α is replaced, the tips and their images are as per the left upper so that $\mathcal{I}^{\circ} = \mathcal{K}$.

Thus the required *(cut)* conclusion is equal to the left upper, except that possibly the required first list δ is a guard sublist of α. In this event closure under $(\|iK)$ (lemma 3.2.2) ensures that the required consecution is derivable.

1.2 Right upper from $(\rightarrow\|)$

$$\frac{\Gamma \ \Vdash\ [\alpha, H', \mathcal{K}'] \qquad \dfrac{\Lambda_1 \ \Vdash\ [A, M, \mathcal{N}] \qquad \Lambda_2, \langle \ldots \langle A\, B \rangle \ldots \rangle \ \Vdash\ [\delta, E, \mathcal{F}]}{\Lambda_1, \Lambda_2, \langle \ldots A \rightarrow B \ldots \rangle \ \Vdash\ [\delta, E', \mathcal{F}']}}{\Gamma,\ \Lambda_1,\ \Lambda_2 \ \Vdash\ [\delta,\ E'^{\circ},\ \mathcal{F}'^{\circ}]}$$

Where $\alpha = \langle \ldots A \rightarrow B \ldots \rangle$ and we put $\alpha' = \langle \ldots \langle A\, B \rangle \ldots \rangle$. By invertibility (lemma 3.4.2) there is a deduction of the left upper doing *that* $\rightarrow$ introduction last, i.e. ending:—

$$(\|\rightarrow)\ \frac{\Gamma,\ A \ \Vdash\ [\langle \ldots \langle A\, B \rangle \ldots \rangle,\ H,\ \mathcal{K}]}{\Gamma \ \Vdash\ [\langle \ldots A \rightarrow B \ldots \rangle,\ H',\ \mathcal{K}']}$$

So by the induction hypothesis we can apply the following lower degree *(cut)*'s:—

$$
(cut)\ \dfrac{(cut)\ \dfrac{\Lambda_1 \ \Vdash\ [A, M, \mathcal{N}] \qquad \Gamma,\, A \ \Vdash\ [\alpha', H, \mathcal{K}]}{\Lambda_1,\, \Gamma \ \Vdash\ [\alpha', H^\diamond, \mathcal{K}^\diamond]} \qquad \Lambda_2,\, \alpha' \ \Vdash\ [\delta, E, \mathcal{F}]}{\Lambda_1,\, \Gamma,\, \Lambda_2 \ \Vdash\ [\delta,\, E^\diamond,\, \mathcal{F}^\diamond]}
$$

With a $(C\|)$ to get the antecedent right.

We need to check that we have obtained the correct modification of the support function. The required modifications to $(E, \mathcal{F})$ are: To form $(E', \mathcal{F}')$ copies of $(M, \mathcal{N})\,[\underline{A}\backslash\varphi]$ are added in place of $(E, \mathcal{F})\big\{\underline{\varphi}\big\}$ for those tips $\underline{\varphi}$ with tip-image the antecedent A; and to form $(E'^\diamond, \mathcal{F}'^\diamond)$ copies of $(H', \mathcal{K}')\{\underline{\lambda}\}\,[\underline{\lambda}\backslash\varphi]$ are added, for those tips $\underline{\varphi}$ of $(E', \mathcal{F}')$ with tip-image $\underline{\lambda}$ in the cut list α.

Now the first *(cut)* above replaces $(H, \mathcal{K})\{\underline{A}\}$ of the A in α' by $(M, \mathcal{N})$, and on the second *(cut)* every $(E, \mathcal{F})$ tip $\underline{\varphi}$ with tip-image the antecedent A has it replaced by $(M, \mathcal{N})\,[\underline{A}\backslash\varphi]$, which is exactly the first of the above required modifications.

The remaining $(E, \mathcal{F})$ tips $\underline{\varphi}$ with tip-image $\underline{\lambda}$ (other than $\underline{A}$) in α' are just the ancestors of those of $(E', \mathcal{F}')$ with corresponding tip-image $\lambda' = \lambda\left[\underline{\langle A\,B\rangle}\backslash A{\to}B\right]$ in α. In the first *(cut)* the corresponding $\underline{\lambda}$ were left unchanged, $(H^\diamond, \mathcal{K}^\diamond)\{\underline{\lambda}\} = (H, \mathcal{K})\{\underline{\lambda}\}$ (only the A has tip-image the antecedent A in the "new" premiss of the right upper), so on the second *(cut)* each such $\underline{\varphi}$ has $(E, \mathcal{F})\big\{\underline{\varphi}\big\}$ replaced by $(H, \mathcal{K})\{\underline{\lambda}\}\,[\underline{\lambda}\backslash\varphi]$. But such $(H, \mathcal{K})\{\underline{\lambda}\} = (H', \mathcal{K}')\{\underline{\lambda'}\}\,[\underline{A{\to}B}\backslash\langle A\,B\rangle]$ where $\lambda' = \lambda\left[\underline{\langle A\,B\rangle}\backslash A{\to}B\right]$. So replacing $\underline{\lambda}$ by φ in $(H, \mathcal{K})\{\underline{\lambda}\}$ gives the same result as replacing $\underline{\lambda'}$ by φ in $(H', \mathcal{K}')\{\underline{\lambda'}\}$, and we have exactly the second of the above required modifications.

So the succedent support hedge is indeed that required for the cut conclusion.

These are the only two possible cases for a right upper with right rank$=1$. Thus we have completed the proof of the base case of the rank induction.

2 Right Rank > 1

2.1 Right upper from $(C\|)$

$$\dfrac{\Gamma \Vdash [\alpha, H, \mathcal{K}] \qquad (C\|)\dfrac{\Lambda \Vdash [\delta,\, E,\, \mathcal{F}]}{\Lambda' \Vdash [\delta,\, E,\, \mathcal{F}]}}{\Gamma,\, \Lambda^{\circ} \Vdash [\delta,\, E^{\circ},\, \mathcal{F}^{\circ}]}\;(cut)$$

Do the lower rank *(cut)* with the premiss of the right upper and then the corresponding $(C\|)$.

2.2 Right upper from $(\|pref)$

$$\dfrac{\Gamma \Vdash [\varphi, H, \mathcal{K}] \qquad \dfrac{\begin{array}{l}\Lambda_1 \Vdash [\langle \alpha\, \beta\rangle,\, E_1,\, \mathcal{F}_1] \\ \Lambda_2 \Vdash [\langle \gamma\, \alpha\rangle,\, E_2,\, \mathcal{F}_2] \\ \Lambda_3 \Vdash [\gamma, E_3, \mathcal{F}_3] \\ \Lambda_4 \Vdash [\beta_1, E_4, \mathcal{F}_4] \\ \ldots \\ \Lambda_{n+3} \Vdash [\beta_n, E_{n+3}, \mathcal{F}_{n+3}]\end{array}}{\Lambda_1, \ldots \Lambda_{n+3} \Vdash [\delta,\, E,\, \mathcal{F}]}}{\Gamma,\, \Lambda_1^{\circ},\, \Lambda_{n+3} \Vdash [\delta,\, E^{\circ},\, \mathcal{F}^{\circ}]}\;(cut)$$

Where one of the Λ_i's contains the cut φ, say for example Λ_1. Do the lower rank *(cut)*, with the corresponding (in this case first) premiss of the $(\|pref)$:—

$$\dfrac{\Gamma \Vdash [\varphi, H, \mathcal{K}] \qquad \Lambda_1 \Vdash [\langle \alpha\, \beta\rangle,\, E_1,\, \mathcal{F}_1]}{\Gamma,\, \Lambda_1^{\circ} \Vdash [\langle \alpha\, \beta\rangle,\, E_1^{\circ},\, \mathcal{F}_1{}^{\circ}]}$$

And then apply $(\|pref)$:—

$$\dfrac{\begin{array}{l}\Gamma,\ \Lambda_1^\circ\ \Vdash\ [\langle\alpha\,\beta\rangle,\ E_1^\circ,\ \mathcal{F}_1^\circ]\\ \Lambda_2\ \Vdash\ [\langle\gamma\,\alpha\rangle,\ E_2,\ \mathcal{F}_2]\\ \Lambda_3\ \Vdash\ [\gamma, E_3, \mathcal{F}_3]\\ \Lambda_4\ \Vdash\ [\beta_1, E_4, \mathcal{F}_4]\\ \ldots\\ \Lambda_{n+3}\ \Vdash\ [\beta_n, E_{n+3}, \mathcal{F}_{n+3}]\end{array}}{\Gamma,\ \Lambda_1^\circ,\ \Lambda_2,\ldots\Lambda_{n+3}\ \Vdash\ [\delta,\ E^\circ,\ \mathcal{F}^\circ]}$$

And adjust the antecedent using $(C\|)$ (in the case that Γ is not in the first premiss and so needs to be moved leftwards).

A tip-image is in the cut list iff the corresponding tip in the premiss has the same sublist in the ancestral cut list as tip-image, so we do have the required succedent hedge since the order of modification of the support hedge is not significant.

2.3 Right upper from $(\|\mathit{suff})$

Similar to the above case.

2.4 Right upper from $(\rightarrow\|)$

$$\dfrac{\Gamma\ \Vdash [\alpha, H, \mathcal{K}] \qquad (\rightarrow\|)\,\dfrac{\Lambda_1\ \Vdash [A, M, \mathcal{N}] \qquad \Lambda_2, \langle\ldots\langle A\,B\rangle\ldots\rangle\ \Vdash [\delta, E, \mathcal{F}]}{\Lambda_1, \Lambda_2, \langle\ldots A\rightarrow B\ldots\rangle\ \ \Vdash\ \ [\delta, E', \mathcal{F}']}}{\Gamma,\ \Lambda_1^\circ,\ \Lambda_2^\circ,\ \langle\ldots A\rightarrow B\ldots\rangle \Vdash [\delta,\ E'^\circ,\ \mathcal{F}'^\circ]}$$

Since the right rank is > 1 the cut list occurs in either Λ_1 or Λ_2. Do the lower rank *(cut)*, and then apply the corresponding $(\rightarrow\|)$ (here is shown the case where Λ_2 contains the cut list):—

$$\dfrac{\Lambda_1\ \Vdash\ [A, M, \mathcal{N}] \qquad \dfrac{\Gamma\ \Vdash [\alpha, H, \mathcal{K}] \qquad \Lambda_2, \langle\ldots\langle A\,B\rangle\ldots\rangle\ \Vdash [\delta, E, \mathcal{F}]}{\Gamma, \Lambda_2^\circ, \langle\ldots\langle A\,B\rangle\ldots\rangle\ \Vdash [\delta, E^\circ, \mathcal{F}^\circ]}}{\Lambda_1,\ \Gamma,\ \Lambda_2^\circ,\ \langle\ldots A\rightarrow B\ldots\rangle \Vdash [\delta,\ E^{\circ\prime},\ \mathcal{F}^{\circ\prime}]}$$

And adjust the antecedent using $(C||)$.

It is necessary to check that the above provides the correct modification of the support hedge. This amounts to showing that the two modifications to the support hedge $(E, \mathcal{F})$ can be done in any order.

Consider an E' tip $\underline{\varphi}$ with tip-image $\underline{\lambda}$ in the antecedent Λ_2. Such a tip has ancestor $\underline{\varphi}$ in E with tip-image the corresponding ancestral $\underline{\lambda'}$.

First suppose the tip-image is not in the cut list. We require these to remain the same in the cut. In our modified deduction, the ancestors of such tips are left undisturbed by the (if any) righthand cut and the various ensuing adjustments. And $\underline{\varphi}$ has tip-image in $\langle \ldots A \rightarrow B \ldots \rangle$ iff its ancestor has corresponding tip-image in $\langle \ldots \langle A\, B \rangle \ldots \rangle$ and these are left undisturbed since the $\langle \ldots A \rightarrow B \ldots \rangle$ isn't a cut list, to be replaced by the required sublist of same in the $(\rightarrow||)$.

Suppose the tip-image $\underline{\lambda'}$ is in a cut list.

If in Λ_2 then $\lambda = \lambda'$ and the righthand *(cut)* replaces $(E, \mathcal{F})\left\{\underline{\varphi}\right\}$ by $(H, \mathcal{K})\{\underline{\lambda}\}\,[\underline{\lambda}\backslash\varphi]$ to obtain $(E^\circ, \mathcal{F}^\circ)$, and the remaining steps preserve this change; which is as required since the tip's descendant in E' is list-identical so the same modification of $(H, \mathcal{K})\{\underline{\lambda}\}$ was required to be added.

Consider an E' tip $\underline{\varphi}$ with tip-image $\underline{\lambda}$ in the antecedent Λ_1. Such a tip is part of a $(M, \mathcal{N})\{\underline{A}\}\,[\underline{A}\backslash\gamma]$ "appended" to a tip $\underline{\gamma}$ originally with tip-image the $\underline{A}$.

First suppose the tip-image is not in a cut list. The (if any) lefthand cut leaves the ancestral tip-image undisturbed, and the $(\rightarrow||)$ preserves this when adding the copy of $(M^\circ, \mathcal{N}^\circ)\{\underline{A}\}\,[\underline{A}\backslash\gamma]$ to $(E^\circ, \mathcal{F}^\circ)$ at the tip $\underline{\gamma}$.

Suppose the tip-image $\underline{\lambda}$ is in a cut list. In this case the cut is made with the left premiss. The lefthand cut replaces such tip-image by $(H, \mathcal{K})\{\underline{\lambda}\}\,[\underline{\lambda}\backslash\varphi]$, to obtain $(M^\circ, \mathcal{N}^\circ)$, and this modification is preserved in the copies of this appended in place of the tip-image $\underline{A}$ by the $(\rightarrow||)$ as is required.

Thus we have the required succedent support hedge.

2.5 Right upper from $(\Vdash\!\rightarrow)$

$$
(cut)\ \dfrac{\Gamma \Vdash [\alpha, H, \mathcal{K}] \qquad (\Vdash\!\rightarrow)\ \dfrac{\Lambda, A \ \Vdash\ [\langle \ldots \langle A\,B\rangle \ldots\rangle,\ E,\ \mathcal{F}]}{\Lambda \ \Vdash\ [\langle \ldots A\!\rightarrow\! B \ldots\rangle,\ E',\ \mathcal{F}']}}{\Gamma,\ \Lambda^{\circ} \ \Vdash\ [\langle \ldots A\!\rightarrow\! B \ldots\rangle, E'^{\circ}, \mathcal{F}'^{\circ}]}
$$

(The $\underline{\langle A\,B\rangle}$ need not occur in the first list as shown above.)

Do the *(cut)* on the premiss of the right upper first, and then apply the $(\Vdash\!\rightarrow)$:—

$$
(\Vdash\!\rightarrow)\ \dfrac{(cut)\ \dfrac{\Gamma \Vdash [\alpha, H, \mathcal{K}] \qquad \Lambda, A \Vdash [\delta', E, \mathcal{F}]}{\Gamma, \Lambda^{\circ}, A \Vdash [\delta', E^{\circ}, \mathcal{F}^{\circ}]}}{\Gamma,\ \Lambda^{\circ} \Vdash \left[\delta, E^{\circ'}, \mathcal{F}^{\circ'}\right]}
$$

Note that each tip $\underline{\varphi}$ of the right upper with tip-image $\underline{\lambda}$ in the antecedent has as ancestor a tip $\underline{\varphi'}$ (possibly with $\langle A\,B\rangle$ in place of $A\!\rightarrow\!B$) with tip-image the ancestral $\underline{\lambda}$ in Λ.
For such tip-image $\underline{\lambda}$ in the cut list the above cut "appends" to its tip $\underline{\varphi'}$ the bush $(H, \mathcal{K})\{\underline{\lambda}\}\,[\underline{\lambda}\backslash\varphi']$, and then the $(\Vdash\!\rightarrow)$ replaces the $\underline{\langle A\,B\rangle}$ by $\underline{A\!\rightarrow\!B}$ (possibly in $\underline{\varphi'}$), so we have the correct transformation— $(H, \mathcal{K})\{\underline{\lambda}\}\,[\underline{\lambda}\backslash\varphi]$— appended at the tip $\underline{\varphi}$.
For tip-images not in the cut list there is no change (other than the corresponding tip $\underline{\varphi'}$ replaced by $\underline{\varphi}$), as is required.
So this modifies the support hedge correctly.

This completes the (right) rank induction step, and so the lemma is proved.

This was the last element needed for:—

Theorem 3.4.8 **GS** contains **S**, i.e. if $\vdash_{\mathrm{S}} A$ then $\Vdash_{\mathrm{GS}} [A,\ E,\ \mathcal{F}]$ for some support hedge $(E, \mathcal{F})$ (where the only value taken by $\mathcal{F}$ is the empty set).

Proof

By the lemmas 3.2.7, 3.2.8, 3.4.3 and 3.4.7.

3.5 Generalization to all implication logics

In this section I show that there is a similar such **GL** system for every implication logic **L** which has a Hilbert formulation with axioms and the rule modus ponens. I do this by providing a recipe for list rules corresponding to (i.e. "clones" of) each axiom of the logic.

Definition 3.5.1 Let A be a wff. Each instance of the *list rule associated with* $A = A_1 \rightarrow .A_2 \rightarrow .\ldots A_n \rightarrow p$ is determined from A as follows:—

Let $\alpha^{\dagger}$ be the atomic list with $\alpha^{\dagger *} = A$, and let $\alpha = \langle\alpha_1 \ldots \langle\alpha_n\, \beta\rfloor$ be a list formed by uniformly replacing all constituents of $\alpha^{\dagger}$ by lists (so that the p is replaced by β), then the following is the instance of the *list rule associated with* A determined by α:—

$$\frac{\begin{array}{l}\Gamma_1 \ \Vdash\ [\alpha_1,\, E_1,\, \mathcal{F}_1] \\ \Gamma_2 \ \Vdash\ [\alpha_2,\, E_2,\, \mathcal{F}_2] \\ \ldots \\ \Gamma_n \ \Vdash\ [\alpha_n, E_n, \mathcal{F}_n] \\ \Gamma_{n+1} \ \Vdash\ [\beta_1, E_{n+1}, \mathcal{F}_{n+1}] \\ \ldots \\ \Gamma_{n+m} \ \Vdash\ [\beta_m, E_{n+m}, \mathcal{F}_{n+m}]\end{array}}{\Gamma_1, \ldots \Gamma_n,\ \Gamma_{n+1}, \ldots \Gamma_{n+m} \ \Vdash\ [\delta,\, E,\, \mathcal{F}]}$$

Where $\beta = \langle\beta_1 \ldots \langle\beta_m\, B\rfloor$, δ is any guard list or the whole of α, and the conclusion support hedge E is formed by applying the inductive step of definition 2.2.1 to $E_1, \ldots E_{n+m}$ with backbone tip assigned the list α; and $\mathcal{F}$ is determined by putting:—

- $\mathcal{F}\,(\underline{\alpha})$ is the empty set,
- for $1 \leq i \leq n$, $(E, \mathcal{F})|_{\underline{\alpha_i}} = (E_i, \mathcal{F}_i)$,

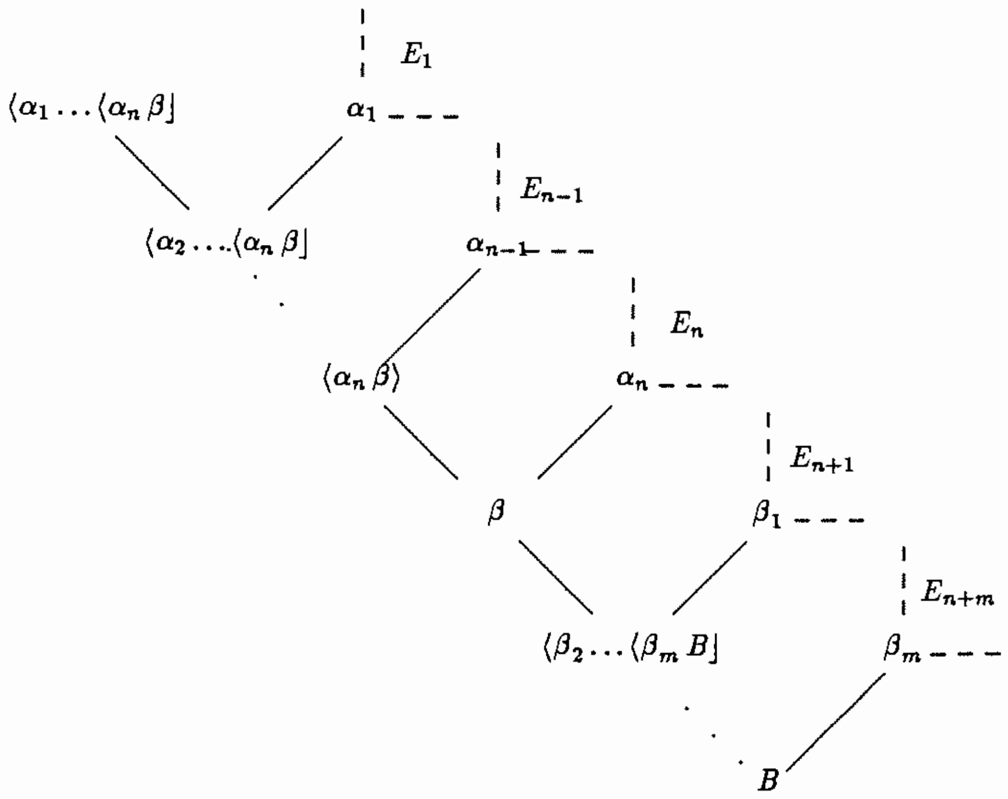


Figure 3.3: The hedge for the list rule associated with A.

- for $1 \leq i \leq m, \quad (E, \mathcal{F})|_{\underline{\beta_i}} = (E_{n+i}, \mathcal{F}_{n+i})$,
- and the remaining located sublists have values determined by the above and the union requirement upon support functions, since they are just the guard lists of α (see figure 3.3).

Thus the general rule cloned from a wff A is the straightforward adaptation of $(\|pref)$ and $(\|suff)$.

The corresponding clauses for the two ancestor relations are:—

- For the *schematic* ancestor relation: Each located sublist $\underline{\varphi}$ of α, part of a list substituted for some q in $\alpha^\dagger$, has as its *schematic* ancestors *all the φ's in the premisses which are also in substituends of q's in the major antecedents of $\alpha^\dagger$*; and the remaining elements of E have as sole ancestor the corresponding list of that premiss which generates the subhedge containing it.

- For the *direct* ancestor relation: Each sublist of the conclusion succedent has the corresponding sublist of the premiss hedge it came from as *direct ancestor.*

For example the rule associated with positive paradox $p \rightarrow . q \rightarrow p$ is:—

$$\frac{\begin{array}{l} \Gamma_1 \Vdash [\alpha,\, E_1,\, \mathcal{F}_1] \\ \Gamma_2 \Vdash [\beta,\, E_2,\, \mathcal{F}_2] \\ \Gamma_3 \Vdash [\alpha_1, E_3, \mathcal{F}_3] \\ \ldots \\ \Gamma_{n+2} \Vdash [\alpha_n, E_{n+2}, \mathcal{F}_{n+2}] \end{array}}{\Gamma_1, \ldots \Gamma_{n+2} \Vdash [\delta,\, E,\, \mathcal{F}]}$$

Where $\alpha = \langle \alpha_1 \ldots \langle \alpha_n A \rfloor$, δ is any guard list or the whole of $\langle \alpha \langle \beta \alpha \rangle \rangle$ and $\mathcal{F}$ is determined by putting:—

- $\mathcal{F}\left(\underline{\langle \alpha \langle \beta \alpha \rangle \rangle}\right)$ is the empty set,
- $(E, \mathcal{F})|_{\underline{\alpha}} = (E_1, \mathcal{F}_1)$,
- $(E, \mathcal{F})|_{\underline{\beta}} = (E_2, \mathcal{F}_2)$,
- $(E, \mathcal{F})|_{\underline{\alpha_i}} = (E_{i+2}, \mathcal{F}_{i+2})$,
- and the remaining located sublists have values determined by the above and the union requirement upon support functions, since they are just the guard lists of $\langle \alpha \langle \beta \alpha \rangle \rangle$.

I now define the general Gentzen systems, using $(C\Vert)$, $(\rightarrow\Vert)$ and $(\Vert\rightarrow)$ as before, and with the cloned list rules associated with the axioms.

Definition 3.5.2 Let **L** be a logic (in the implication vocabulary) formulated with axioms and the rule modus ponens. Then the *general Gentzen list formulation of* **L**, denoted **GGL**, (corresponding to this axiomatic formulation, presumed henceforth fixed) is the consecution system with:—

- Axioms as for **GS** (definition 3.1.1).
- $(C\|)$ as for **GS**.
- $(\rightarrow\|)$ as for **GS**.
- $(\|\rightarrow)$ as for **GS**.
- For each axiom-scheme A of **L**, the list rule-scheme associated with A.

The proof that **GGL** captures **L** has the same structure as that for **GS** and **S**, with much of the detail exactly the same.

The tip-images and *wff properties— lemmas 3.2.4 and 3.2.5— are proved exactly as for **GS**, with the new cases for the associate rules as trivial as for $(\|\mathit{pref})$ in the original proof. The interpretation lemma follows from these exactly as before (lemma 3.3.1), with corollary the interpretation theorem:—

Theorem 3.5.3 If $\Vdash_{\mathbf{GGL}} [A, E, \mathcal{F}]$ then $\vdash_{\mathbf{L}} A$.

That **GGL** contains **L** is also proved exactly as for **S**. All identities are provable in each **GGL** system, and the following lemma shows that each axiom is too. The proof of the lemma simply rests on the general routine which can be distilled from our earlier proof for the case of prefixing.

Lemma 3.5.4 Each axiom A of **L** is provable in **GGL**.

Proof

Let $A^\dagger$ be an axiom-scheme of **L** with an instance A and let $\alpha^\dagger$ be the atomic list with $\alpha^{\dagger *} = A^\dagger$, and let $\alpha = \langle\alpha_1 \ldots \langle\alpha_n\,\beta\rfloor$ be the uniform substitution-instance of $\alpha^\dagger$ with $\alpha^* = A$.

Apply the associate rule for A to the following identities:—

$$\frac{\begin{array}{l}\alpha_1 \;\Vdash\; [\alpha_1,\, \{\alpha_1\},\, \mathcal{I}_1] \\ \ldots \\ \alpha_n \;\Vdash\; [\alpha_n, \{\alpha_n\}, \mathcal{I}_n] \\ \beta_1 \;\Vdash\; [\beta_1, \{\beta_1\}, \mathcal{I}_{n+1}] \\ \ldots \\ \beta_m \;\Vdash\; [\beta_m, \{\beta_m\}, \mathcal{I}_{n+m}]\end{array}}{\alpha_1, \ldots, \alpha_n,\ \beta_1, \ldots, \beta_m \;\Vdash\; [\langle\alpha_1 \ldots \langle\alpha_n\,\beta\rfloor,\ E,\ \mathcal{I}]}$$

Where $\beta = \langle\beta_1 \ldots \langle\beta_m\,B\rfloor$ and the final list-sequence E is in fact the singleton $\{\alpha\}$ since it corresponds to the identity support hedge for α (in virtue of definition 3.5.1).

For the right-most antecedent sublist or whole list of the antecedent which is a wff D: If not a whole list it must be in a $\ldots\langle D\,C\rangle\ldots$ location and do the $(\rightarrow\Vert)$:—

$$(\rightarrow\Vert)\ \frac{D \Vdash [D, \{D\}, \mathcal{I}] \qquad\qquad \alpha_1, \ldots, \langle\ldots\langle D\,C\rangle\ldots\rangle \Vdash [\alpha, \{\alpha\}, \mathcal{I}]}{D, \alpha_1, \ldots, \langle\ldots D{\rightarrow}C\ldots\rangle \;\Vdash\; [\alpha, \{\alpha\}', \mathcal{I}']}$$

And then fill in the corresponding $\rightarrow$ in the succedent (after a $(C\Vert)$):—

$$(\Vert\rightarrow)\ \frac{\alpha_1, \ldots, \langle\ldots D{\rightarrow}C\ldots\rangle, D \;\Vdash\; [\alpha, \{\alpha\}, \mathcal{I}']}{\alpha_1, \ldots, \langle\ldots D{\rightarrow}C\ldots\rangle \;\Vdash\; [\alpha', \{\alpha\}', \mathcal{I}'']}$$

If D is a whole list just do the last step (in this case $D = \alpha_i{}^*$ for one of the major antecedents of α).

Repeat this procedure.

It eventually stops when no lists remain in the antecedent of the consecution, and we have A the succedent list.

To be sure we can apply the $(\|\!\rightarrow)$'s, we need to check that the D to be "carried over" is the tip-image of that to be "filled in". Since the premisses are axioms every antecedent sublist and whole list of the succedent α_i's is a tip with tip-image the corresponding sublist of the antecedent, and this fact is preserved by the *associated* rule (but with the whole succedent α having no tip-image). The above procedure preserves this relationship between the descendant tip and tip-images; for the (on hypothesis) tip-image $\underline{D}$ of the corresponding succedent $\underline{D}$ is replaced by the D introduced by the $(\rightarrow\|)$ (if needed) so the $(\|\!\rightarrow)$ can be done, and by hypothesis each independent list and superlist of the succedent $\underline{\langle D\,C\rangle}$ of the ancestor satisfied this property and the tip-images just follow the descendants for these lists in the $(\rightarrow\|)$ (if needed), as well as in the $(\|\!\rightarrow)$.

It remains to show that the class of sentences so provable in **GGL** is closed under modus ponens. I do so (as for **GS**) by showing that these systems are closed under *(cut)*. The proof that this ensures closure under modus ponens is exactly as before (lemma 3.4.3), using *invertibility*. The following proof of invertibility treats the general cloned rules exactly along the lines as the treatment of $(\|pref)$ in our earlier proof (lemma 3.4.2).

Lemma 3.5.5 If $\Gamma \Vdash [\langle \ldots A\rightarrow B \ldots\rangle, E, \mathcal{F}]$ where $\underline{A\rightarrow B}$ is a constituent, then there is a deduction of this consecution in which that $\rightarrow$ is introduced last, that is with last step:—

$$(\|\!\rightarrow)\,\frac{\Gamma,\,A \;\;\Vdash\;\; [\langle \ldots \langle A\,B\rangle \ldots\rangle,\,E',\,\mathcal{F}']}{\Gamma \;\;\Vdash\;\; [\langle \ldots A\rightarrow B \ldots\rangle,\,E,\,\mathcal{F}]}$$

(Where we include the case that $\underline{\langle A\,B\rangle}$ and the resulting $\underline{A\rightarrow B}$ do not occur in the succedent first list.)

Proof

Exactly as for lemma 3.4.2 with the extra cases for the new rules associated with the axioms of **L** as follows.

Suppose we have a descendant obtained by:—

$$\begin{array}{l}
\Gamma_1 \;\Vdash\; [\alpha_1,\, E_1,\, \mathcal{F}_1] \\
\ldots \\
\Gamma_n \;\Vdash\; [\alpha_n, E_n, \mathcal{F}_n] \\
\Gamma_{n+1} \;\Vdash\; [\beta_1, E_{n+1}, \mathcal{F}_{n+1}] \\
\ldots \\
\Gamma_{n+m} \;\Vdash\; [\beta_m, E_{n+m}, \mathcal{F}_{n+m}] \\
\hline
\Gamma_1, \ldots \Gamma_n,\; \Gamma_{n+1}, \ldots \Gamma_{n+m} \;\Vdash\; [\delta,\, E,\, \mathcal{F}]
\end{array}$$

Where $\beta = \langle \beta_1 \ldots \langle \beta_m\, D \rfloor$. For each premiss containing (schematic) ancestral $A \rightarrow B$'s we have on hypothesis that the corresponding consecutions are provable:—

$$\Gamma_i,\; A, \ldots A \;\Vdash [\alpha_i{}^{\triangleright}, E_i{}^{\triangleright}, \mathcal{F}_i{}^{\triangleright}]$$

or

$$\Gamma_{n+i},\; A, \ldots A \;\Vdash [\beta_i{}^{\triangleright}, E_{n+i}{}^{\triangleright}, \mathcal{F}_{n+i}{}^{\triangleright}]$$

In the case that the guard wff D is not an ancestral $A \rightarrow B$ the definition of the schematic ancestor relation ensures that the succedent lists still correspond to a uniform substitution into the axiom, so we have the correct form of the lists for an application of the *associated* rule. Applying the rule to this modified set of premisses, with adjustment $(C\Vert)$'s, we obtain:—

$$\Gamma_1, \ldots, \Gamma_{n+m},\; A, \ldots A \;\Vdash\; [\delta'', E'', \mathcal{F}'']$$

In the case that the guard wff D is an ancestral $A \rightarrow B$ to be replaced by $\langle A\, B \rangle$ simply add to the above modified premisses a final premiss the identity $A \;\Vdash\; [A,\, \{A\},\, \mathcal{I}]$

corresponding to the extra required major antecedent A of $\beta^{\triangleright}$.

As for the list rule $(\Vert pref)$ of **GS**, the resulting succedent list might have too many "opened up" $\langle A\, B\rangle$'s which need to be "filled in" to obtain the required consecution with only the ancestral $A{\rightarrow}B$'s opened up. So we carry out the needed $(\Vert{\rightarrow})$'s to obtain the required consecution.

This is the only case needed to extend our earlier inductive proof, so the lemma is proved.

Lemma 3.5.6 **GGL** is closed under *(cut)*.

Proof

The proof is as for lemma 3.4.7, with all but one of the cases already proved. The only outstanding case is for Right Rank> 1, with the right upper from a new rule associated with an **L** axiom:—[6]

$$\Lambda_1 \Vdash [\gamma_1,\, E_1,\, \mathcal{F}_1]$$

$$\ldots$$

$$\Lambda_n \Vdash [\gamma_n, E_n, \mathcal{F}_n]$$

$$\Lambda_{n+1} \Vdash [\beta_1, E_{n+1}, \mathcal{F}_{n+1}]$$

$$\ldots$$

$$(cut)\;\dfrac{\Gamma \Vdash [\alpha, H, \mathcal{K}] \qquad \dfrac{\Lambda_{n+m} \Vdash [\beta_m, E_{n+m}, \mathcal{F}_{n+m}]}{\Lambda_1, \ldots \Lambda_n,\ \Lambda_{n+1}, \ldots \Lambda_{n+m} \Vdash [\delta,\ E,\ \mathcal{F}]}}{\Gamma,\ \Lambda_1^{\circ}, \ldots, \Lambda_{n+m}^{\circ} \Vdash [\langle \delta_1 \ldots \langle \delta_r\, D \rfloor, E^{\circ}, \mathcal{F}^{\circ}]}$$

Where one of the Λ_i has an α removed to obtain Λ_i°, the rest remaining unchanged. Exactly as in the earlier case for $(\Vert pref)$ use the appropriate lower rank *(cut)*, and then apply the *associated* rule plus adjustment $(C\Vert)$ to obtain the required consecution.

All other cases are proved exactly as for **GS**, and the lemma is proved.

[6]It might seem surprising that there is no need to deal with a case where the left upper arises from the application of such a new rule. The reason we don't is the structure of my *(cut)* proof and its use of invertibility—where the corresponding work is done.

Whence we have:—

Theorem 3.5.7 GGL contains L: If $\vdash_{\mathrm{L}} A$ then $\Vdash_{\mathbf{GGL}} [A,\ E,\ \mathcal{F}]$.

So we have shown that **GGL** does indeed exactly capture such **L**. Thus every implication logic **L** is captured by a consecution system which describes the **L**-proofs of the Hilbert system.

As further examples note the following associate rules which suffice to capture classical implication. The corresponding natural deduction system is a constructive formulation of classical implication in that it satisfies Lorenzen's principle of inversion.[7]

1. The rule associated with positive paradox as shown earlier.
2. The rule associated with self-distribution:—

$$\frac{\begin{array}{l} \Gamma_1 \Vdash [\langle\alpha\,\langle\beta\,\gamma\rangle\rangle,\ E_1,\ \mathcal{F}_1] \\ \Gamma_2 \Vdash [\langle\alpha\,\beta\rangle,\ E_2,\ \mathcal{F}_2] \\ \Gamma_3 \Vdash [\alpha, E_3, \mathcal{F}_3] \\ \Gamma_4 \Vdash [\gamma_1, E_4, \mathcal{F}_4] \\ \dots \\ \Gamma_{n+3} \Vdash [\gamma_n, E_{n+3}, \mathcal{F}_{n+3}] \end{array}}{\Gamma_1, \dots \Gamma_{n+3} \Vdash [\delta,\ E,\ \mathcal{F}]}$$

Where $\gamma = \langle\gamma_1 \dots \langle\gamma_n\, C\rfloor$, δ is any guard list or the whole of $\big\langle\langle\alpha\,\langle\beta\,\gamma\rangle\rangle\big\langle\langle\alpha\,\beta\rangle\langle\alpha\,\gamma\rangle\big\rangle\big\rangle$ and $\mathcal{F}$ is determined by putting:—

- $\mathcal{F}\left(\underline{\big\langle\langle\alpha\,\langle\beta\,\gamma\rangle\rangle\,\big\langle\langle\alpha\,\beta\rangle\,\langle\alpha\,\gamma\rangle\big\rangle\big\rangle}\right)$ is the empty set,
- $(E, \mathcal{F})\big|_{\underline{\langle\alpha\,\langle\beta\,\gamma\rangle\rangle}} = (E_1, \mathcal{F}_1)$

[7]The multiple-conclusion natural deduction system corresponding to the multiple-conclusion consecution system for classical logic also satisfies Lorenzen's principle of inversion.

- $(E,\mathcal{F})|_{\underline{\langle\alpha\,\beta\rangle}} = (E_2,\mathcal{F}_2)$
- $(E,\mathcal{F})|_{\underline{\alpha}} = (E_3,\mathcal{F}_3)$
- and for the major antecedents of the guard γ

$$(E,\mathcal{F})|_{\underline{\gamma_i}} = (E_{i+3},\mathcal{F}_{i+3})$$

3. The rule associated with Peirce's law is:—

$$\frac{\begin{array}{l}\Gamma_1 \ \Vdash\ [\langle\langle\alpha\,\beta\rangle\,\alpha\rangle,\ E_1,\ \mathcal{F}_1] \\ \Gamma_2 \ \Vdash\ [\alpha_1,\ E_2,\ \mathcal{F}_2] \\ \ldots \\ \Gamma_{n+1} \ \Vdash\ [\alpha_n, E_{n+1}, \mathcal{F}_{n+1}]\end{array}}{\Gamma_1,\ldots\ \Gamma_{n+1} \ \Vdash\ [\delta,\ E,\ \mathcal{F}]}$$

Where $\alpha = \langle\alpha_1 \ldots \langle\alpha_n\, A]$, δ is any guard list or the whole of $\big\langle\langle\langle\alpha\,\beta\rangle\,\alpha\rangle\,\alpha\big\rangle$ and $\mathcal{F}$ is determined by putting:—

- $\mathcal{F}\left(\underline{\big\langle\langle\langle\alpha\,\beta\rangle\,\alpha\rangle\,\alpha\big\rangle}\right)$ is the empty set,
- $(E,\mathcal{F})|_{\underline{\langle\langle\alpha\,\beta\rangle\,\alpha\rangle}} = (E_1,\mathcal{F}_1)$
- and for the major antecedents of the guard α

$$(E,\mathcal{F})|_{\underline{\alpha_i}} = (E_{i+1},\mathcal{F}_{i+1})$$

I illustrate the proofs of the corresponding axioms, using the appropriate associate rule, with the case of self-distribution. The other cases are similar rote applications of the general proof (lemma 3.5.4).

$$\frac{\langle A\,\langle B\,C\rangle\rangle \Vdash \langle A\,\langle B\,C\rangle\rangle \qquad \langle A\,B\rangle \Vdash \langle A\,B\rangle \qquad A \Vdash A}{\langle A\,\langle B\,C\rangle\rangle,\ \langle A\,B\rangle,\ A \Vdash \big\langle\langle A\,\langle B\,C\rangle\rangle\ \big\langle\langle A\,B\rangle\,\langle A\,C\rangle\big\rangle\big\rangle}$$

Apply $(\Vdash\!\rightarrow)$:—

$$\langle A\,\langle B\,C\rangle\rangle,\ \langle A\,B\rangle \Vdash \big\langle\langle A\,\langle B\,C\rangle\rangle\ \big\langle\langle A\,B\rangle\ A{\to}C\big\rangle\big\rangle$$

And continue in this manner:—

$$(\Vdash\to)\ \dfrac{(\Vdash\to)\ \dfrac{(\to\Vdash)\ \dfrac{A \Vdash A \qquad \langle A\,\langle B\,C\rangle\rangle,\ \langle A\,B\rangle \Vdash \big\langle\langle A\,\langle B\,C\rangle\rangle\ \big\langle\langle A\,B\rangle\ A{\to}C\big\rangle\big\rangle}{A,\ \langle A\,\langle B\,C\rangle\rangle,\ A{\to}B \Vdash \big\langle\langle A\,\langle B\,C\rangle\rangle\ \big\langle\langle A\,B\rangle\ A{\to}C\big\rangle\big\rangle}}{\langle A\,\langle B\,C\rangle\rangle,\ A{\to}B \Vdash \big\langle\langle A\,\langle B\,C\rangle\rangle\ \big\langle A{\to}B\ A{\to}C\big\rangle\big\rangle}}{\langle A\,\langle B\,C\rangle\rangle \Vdash \big\langle\langle A\,\langle B\,C\rangle\rangle\ A{\to}B{\to}.A{\to}C\big\rangle}$$

$$(\Vdash\to)\ \dfrac{(\to\Vdash)\ \dfrac{B \Vdash B \qquad \langle A\,\langle B\,C\rangle\rangle \Vdash \big\langle\langle A\,\langle B\,C\rangle\rangle\ A{\to}B{\to}.A{\to}C\big\rangle}{B,\ \langle A\,B{\to}C\rangle \Vdash \big\langle\langle A\,\langle B\,C\rangle\rangle\ A{\to}B{\to}.A{\to}C\big\rangle}}{\langle A\,B{\to}C\rangle \Vdash \big\langle\langle A\,B{\to}C\rangle\ A{\to}B{\to}.A{\to}C\big\rangle}$$

$$(\Vdash\to)\ \dfrac{(\Vdash\to)\ \dfrac{(\to\Vdash)\ \dfrac{A \Vdash A \qquad \langle A\,B{\to}C\rangle \Vdash \big\langle\langle A\,B{\to}C\rangle\ A{\to}B{\to}.A{\to}C\big\rangle}{A,\ A{\to}.B{\to}C \Vdash \big\langle\langle A\,B{\to}C\rangle\ A{\to}B{\to}.A{\to}C\big\rangle}}{A{\to}.B{\to}C \Vdash \big\langle A{\to}.B{\to}C\ A{\to}B{\to}.A{\to}C\big\rangle}}{\Vdash (A{\to}.B{\to}C) \to .\ A{\to}B{\to}.A{\to}C}$$

By the general results of this section it follows that the consecution system formulated with the above clones of positive paradox, self-distribution and Peirce, does indeed capture classical implication.

In this chapter we have seen that by using structures—support hedges— representing proofs rather than just sentences, one can provide consecution systems for all implication logics. This shows that *proof*, normally regarded as a meta-theoretic entity, can be formally treated in the "object language" alongside the usual connectives. These consecution systems are also constructive in that the meaning of the $\rightarrow$ wffs and lists are given by introduction rules corresponding to their interpretation as describing the shapes of proofs of the associated Hilbert Systems. So while these systems don't provide philosophically discerning semantics due to being derivative clones of the axiom systems, they do show that the formalist notion of *constructivity* is by itself insufficient as a philosophical criterion for distinguishing between logics.

Chapter 4

The Natural Deduction Systems and Constructivity

In this chapter I describe natural deduction systems corresponding to the consecution systems of chapter 3. I define these natural deduction systems in the first section, and show that they capture their corresponding logics by proving that they contain the Hilbert system (section 4.2) and that they are contained within the corresponding consecution system (section 4.3). Finally I show that the natural deduction systems are constructive by showing that they satisfy Lorenzen's "principle of inversion" (section 4.4); so that all implication connectives are formally intuitionistically constructive.

4.1 Definition of the Natural Deduction Systems.

TS, the system which captures S, is used as example for the general cloned systems **TL**.

A **TL** proof consists of lines comprising a support triple $[\alpha, E, \mathcal{F}]$ with side-bars to its left, which are introduced with a hypothesis introduction step. The support function's range is the power set of the set of side-bars, corresponding to that of the set of HYP wffs introduced along with the side-bars.

Definition 4.1.1 The rules for **TS** are as follows. (As an example lemma 4.2.1 shows a proof of prefixing.)

HYP A wff can be introduced by HYP, beginning a new vertical side-bar:—

$$\begin{array}{|l|l|l} \ldots & & \\ \ldots & & [A, \{A\}, \mathcal{I}] \end{array}$$

Where $\mathcal{I}\,(\underline{A})$ is the singleton containing the new side-bar.

REP A line can be repeated as long as the new line includes the side-bars of the original to its left (i.e. it is in the same subproof, or in a further subproof of it...).

$\rightarrow E$ There are two forms of $\rightarrow E$, the first is:—

$$\begin{array}{|l|l} \ldots & [A, E_1, \mathcal{F}_1] \\ \ldots & [\alpha, E_2, \mathcal{F}_2] \\ \cline{2-2} \ldots & [\alpha', E, \mathcal{F}] \end{array}$$

Where the linked sequence E_2 contains $\underline{A \rightarrow B}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ have no images in common, and $(E, \mathcal{F})$ is formed by taking $(E_2, \mathcal{F}_2)\,[\underline{A \rightarrow B} \backslash \langle A\, B \rangle]$ with $(E, \mathcal{F})|_{\underline{A}} = (E_1, \mathcal{F}_1)$.

If the highlighted first list α contains the $\underline{A \rightarrow B}$ then $\alpha' = \alpha\,[\underline{A \rightarrow B} \backslash \langle A\, B \rangle]$, otherwise $\alpha' = \alpha$.

And the second form of $\rightarrow E$ is:—

$$\begin{array}{|l|l} \ldots & [A, E_1, \mathcal{F}_1] \\ \ldots & [A \rightarrow B, E_2, \mathcal{F}_2] \\ \cline{2-2} \ldots & [B, E, \mathcal{F}] \end{array}$$

Where $(E, \mathcal{F})$ is formed exactly as above. Note that B must be the root of the backbone of E. This form of $\rightarrow E$ (which actually looks like ordinary arrow elimination!) is needed to obtain closure under iK.

$\rightarrow I$

$$
\begin{array}{l|l|l}
\ldots & & [A, \{A\}, \mathcal{I}] \qquad HYP \\
\ldots & & \vdots \\
\ldots & & [\alpha', E', \mathcal{F}'] \\
\hline
\ldots & [\alpha, E, \mathcal{F}] &
\end{array}
$$

Where E' contains $\underline{\langle A\, B\rangle}$ which may or may not be in the first list α' and the $\underline{A}$ has as support image the singleton containing the side-bar introduced by the first HYP A, that side-bar is not in an image of any other list and is the innermost side-bar, and the support function is determined exactly as was that for $(||\rightarrow)$ in **GS**, viz: $(E, \mathcal{F}) = (E', \mathcal{F}')\left[\underline{\langle A\, B\rangle}\backslash A \rightarrow B\right]$.

PREF

$$
\begin{array}{l|l}
\ldots & [\langle \alpha\, \beta\rangle,\ E_1,\ \mathcal{F}_1] \\
\ldots & [\langle \gamma\, \alpha\rangle,\ E_2,\ \mathcal{F}_2] \\
\ldots & [\gamma, E_3, \mathcal{F}_3] \\
\ldots & [\beta_1, E_4, \mathcal{F}_4] \\
\ldots & \ldots \\
\ldots & [\beta_n, E_{n+3}, \mathcal{F}_{n+3}] \\
\hline
\ldots & [\delta,\ E,\ \mathcal{F}]
\end{array}
$$

Where $\beta = \langle \beta_1 \ldots \langle \beta_n\, B \rfloor$, δ is any guard list or the whole of $\left\langle \langle \alpha\, \beta\rangle \left\langle \langle \gamma\, \alpha\rangle\, \langle \gamma\, \beta\rangle \right\rangle \right\rangle$ and the conclusion support hedge is obtained exactly like that for the consecution rule, with backbone tip $\underline{\left\langle \langle \alpha\, \beta\rangle \left\langle \langle \gamma\, \alpha\rangle\, \langle \gamma\, \beta\rangle \right\rangle \right\rangle}$ and:—

- $\mathcal{F}\left(\underline{\left\langle \langle \alpha\, \beta\rangle \left\langle \langle \gamma\, \alpha\rangle\, \langle \gamma\, \beta\rangle \right\rangle \right\rangle}\right)$ is the empty set,
- $(E, \mathcal{F})|_{\underline{\langle \alpha\, \beta\rangle}} = (E_1, \mathcal{F}_1)$

- $(E,\mathcal{F})|_{\underline{\langle\gamma\,\alpha\rangle}} = (E_2, \mathcal{F}_2)$
- $(E,\mathcal{F})|_{\underline{\gamma}} = (E_3, \mathcal{F}_3)$
- For the major antecedents of the guard $\underline{\beta}$ $\quad (E,\mathcal{F})|_{\underline{\beta_i}} = (E_{i+3}, \mathcal{F}_{i+3})$

$$
SUFF \qquad
\begin{array}{l|l}
\ldots & [\langle\alpha\,\beta\rangle,\ E_1,\ \mathcal{F}_1] \\
\ldots & [\langle\beta\,\gamma\rangle,\ E_2,\ \mathcal{F}_2] \\
\ldots & [\alpha,\ E_3, \mathcal{F}_3] \\
\ldots & [\gamma_1,\ E_4, \mathcal{F}_4] \\
\ldots & \ldots \\
\ldots & [\gamma_n,\ E_{n+3}, \mathcal{F}_{n+3}] \\ \cline{2-2}
\ldots & [\delta,\ E,\ \mathcal{F}]
\end{array}
$$

Where $\gamma = \langle\gamma_1 \ldots \langle\gamma_n\, C\rfloor$, δ is any guard list or the whole of $\Big\langle\langle\alpha\,\beta\rangle\Big\langle\langle\beta\,\gamma\rangle\langle\alpha\,\gamma\rangle\Big\rangle\Big\rangle$ and the conclusion support hedge is obtained as for the consecution rule:—

- $\mathcal{F}\left(\underline{\Big\langle\langle\alpha\,\beta\rangle\ \Big\langle\langle\beta\,\gamma\rangle\ \langle\alpha\,\gamma\rangle\Big\rangle\Big\rangle}\right)$ is the empty set,
- $(E,\mathcal{F})|_{\underline{\langle\alpha\,\beta\rangle}} = (E_1, \mathcal{F}_1)$
- $(E,\mathcal{F})|_{\underline{\langle\beta\,\gamma\rangle}} = (E_2, \mathcal{F}_2)$
- $(E,\mathcal{F})|_{\underline{\alpha}} = (E_3, \mathcal{F}_3)$
- For the major antecedents of the guard $\underline{\gamma}$ $\quad (E,\mathcal{F})|_{\underline{\gamma_i}} = (E_{i+3}, \mathcal{F}_{i+3})$

So **TS** incorporates multiple iK in *PREF* and *SUFF*, and single iK in the particular case of $\rightarrow E$ where the new $\underline{\langle A\,B\rangle}$ would be split by applying iK. The unusual feature of a varying number of premisses facilitates the incorporation of $(\|splice)$ in *PREF* and *SUFF*.

As for **GS** there are two senses of ancestor defined below.

Definition 4.1.2 The *schematic ancestor* relation is the transitive closure of the following:—

HYP The introduced wff is its sole *schematic ancestor.*

REP For every sublist of the conclusion the corresponding sublist of the premiss is its *schematic ancestor.*

PREF **and** *SUFF* Exactly as for **GS**, taking account of the form of the corresponding axiom.

$\rightarrow E$ Each sublist of the new support subhedge generated by the $\underline{A}$ has as *schematic ancestor* the corresponding sublist of the first premiss, and the remaining sublists have the corresponding sublist of the second premiss (possibly with $\underline{\langle A\, B\rangle}$ replaced by $A \rightarrow B$) as *schematic ancestor.*

$\rightarrow I$ Each sublist of the conclusion has as *schematic ancestor* the corresponding sublist of the premiss (possibly with $\underline{A \rightarrow B}$ replaced by $\langle A\, B\rangle$).

Definition 4.1.3 The *direct ancestor* relation is the similar transitive closure with each clause exactly as above except for *PREF* and *SUFF*, which are the obvious direct relation linking a list with just that corresponding list in the appropriate premiss (for those lists not on the new backbone).

To deal with arbitrary implication logics add rules also determined from the axioms in a manner analogous to *PREF* and *SUFF*. The general definition, which is virtually a transliteration of that for the consecution case, is given below. For example the **TL** rule associated with self-distribution is:—

$$
\begin{array}{l|l}
\ldots & [\langle\alpha\,\langle\beta\,\gamma\rangle\rangle,\ E_1,\ \mathcal{F}_1] \\
\ldots & [\langle\alpha\,\beta\rangle,\ E_2,\ \mathcal{F}_2] \\
\ldots & [\alpha,\ E_3,\ \mathcal{F}_3] \\
\ldots & [\gamma_1,\ E_4,\ \mathcal{F}_4] \\
\ldots & \vdots \\
\ldots & [\gamma_n,\ E_{n+3},\ \mathcal{F}_{n+3}] \\
\hline
\ldots & [\delta,\ E,\ \mathcal{F}]
\end{array}
$$

Where $\gamma = \langle\gamma_1 \ldots \langle\gamma_n\, C\rfloor$, and δ is a guard list or the whole of $\big\langle\langle\alpha\,\langle\beta\,\gamma\rangle\rangle\,\big\langle\langle\alpha\,\beta\rangle\langle\alpha\,\gamma\rangle\big\rangle\big\rangle$ which is the backbone tip, and the support hedge is determined by:—

- $\mathcal{F}\left(\underline{\big\langle\langle\alpha\,\langle\beta\,\gamma\rangle\rangle\,\big\langle\langle\alpha\,\beta\rangle\,\langle\alpha\,\gamma\rangle\big\rangle\big\rangle}\right)$ is the empty set,
- $(E,\mathcal{F})|_{\underline{\langle\alpha\,\langle\beta\,\gamma\rangle\rangle}} = (E_1,\mathcal{F}_1)$
- $(E,\mathcal{F})|_{\underline{\langle\alpha\,\beta\rangle}} = (E_2,\mathcal{F}_2)$
- $(E,\mathcal{F})|_{\underline{\alpha}} = (E_3,\mathcal{F}_3)$
- For the major antecedents of the guard $\underline{\gamma}$ $\quad (E,\mathcal{F})|_{\underline{\gamma_i}} = (E_{i+3},\mathcal{F}_{i+3})$

List rules which are clones of each axiom of a logic can be generated using the following recipe, which is the obvious adaptation from the consecution systems case.

Definition 4.1.4 Let A be a wff. Each instance of the *natural deduction rule associated with*

$A = A_1 {\rightarrow}.A_2{\rightarrow}.\ldots.A_n{\rightarrow}p$ is determined from A as follows:—

Let $\alpha^\dagger$ be the atomic list with $\alpha^{\dagger *} = A$, and let $\alpha = \langle\alpha_1 \ldots \langle\alpha_n\,\beta\rfloor$ be a list formed by uniformly replacing all constituents of $\alpha^\dagger$ by lists (so that the p is replaced by β), then the following is the instance of the *natural deduction rule associated with A* determined by α:—

$$
\begin{array}{l|l}
\ldots & [\alpha_1,\ E_1,\ \mathcal{F}_1] \\
\ldots & [\alpha_2,\ E_2,\ \mathcal{F}_2] \\
\ldots & \vdots \\
\ldots & [\alpha_n, E_n, \mathcal{F}_n] \\
\ldots & [\beta_1, E_{n+1}, \mathcal{F}_{n+1}] \\
\ldots & \vdots \\
\ldots & \underline{[\beta_m, E_{n+m}, \mathcal{F}_{n+m}]} \\
\ldots & [\delta,\ E,\ \mathcal{F}]
\end{array}
$$

Where $\beta = \langle\beta_1 \ldots \langle\beta_m\, B\rfloor$, δ is any guard list or the whole of α and the conclusion support hedge E is formed by applying the inductive step of definition 2.2.1 to $E_1, \ldots E_{n+m}$ with backbone tip assigned the list α; and $\mathcal{F}$ is determined by putting:—

- $\mathcal{F}\,(\underline{\alpha})$ is the empty set,
- for $1 \leq i \leq n$, $(E, \mathcal{F})|_{\underline{\alpha_i}} = (E_i, \mathcal{F}_i)$,
- for $1 \leq i \leq m$, $(E, \mathcal{F})|_{\underline{\beta_i}} = (E_{n+i}, \mathcal{F}_{n+i})$.

Definition 4.1.5 A support triple $[\alpha,\ E,\ \mathcal{F}]$ is *provable in* **TL**, denoted $\vdash_{\mathrm{TL}}\ [\alpha,\ E,\ \mathcal{F}]$, if there is a **TL** proof of that line with no side-bars to its left.

The following lemma shows that we have closure under iK.

Lemma 4.1.6 If we have a line $[\langle\alpha\,\beta\rangle,\ E,\ \mathcal{F}]$ in a **TS** proof, then so do we have the line $[\beta,\ E,\ \mathcal{F}]$, using the "same" **TS** proof but with the appropriate different choices of first lists as per the following argument.

Proof

We show that for every line of a **TS** proof the result of applying iK to it is also provable. A line introduced by HYP is vacuously closed under iK as it comprises a wff.

In the case of REP the hypothesis for the premiss ensures closure of the conclusion under iK.

Consider the case of $\rightarrow E$.

If the second form of the rule is applied then the first list of the conclusion triple is a wff, so it is not of correct form for the application of iK and we thus vacuously have closure under iK.

So suppose the first form of $\rightarrow E$ is applied.

In the case that applying iK to the conclusion results in the whole of $\underline{\langle A\,B\rangle}$ being either in or out of the first list, by the induction hypothesis we can apply iK to the major premiss (obtaining $\underline{A \rightarrow B}$ in or out accordingly). So we can apply the first form of $\rightarrow E$ to the modified premiss and obtain the required conclusion.

There remains the case where iK would "split" $\underline{\langle A\,B\rangle}$, so that it is impossible for the major premiss to take the needed form. This can only occur where $\underline{\langle A\,B\rangle}$ is the first list of the conclusion triple so that iK would result in first list B. But in that case the first list of the major premiss must be $A \rightarrow B$ and we can apply the second form of $\rightarrow E$ to obtain the desired conclusion.

For $\rightarrow I$, on the hypothesis that we have the result of applying iK to the major premiss we can apply $\rightarrow I$ to the so-modified major premiss obtaining the required conclusion.

And for $PREF$ and $SUFF$ one simply chooses the appropriate first list δ to achieve the effect of a further iK on the conclusion (this is where the effect of multiple iK's is needed).

It might seem that there is still elimination involved in applications of the $PREF$ and $SUFF$ rules of **TS**. For example in the following form of application of $PREF$ it may seem that we have an elimination from the first premiss.

$$
\begin{array}{l|l}
\ldots & [\langle \alpha\, \beta\rangle,\ E_1,\ \mathcal{F}_1] \\
\ldots & [\langle \gamma\, \alpha\rangle,\ E_2,\ \mathcal{F}_2] \\
\ldots & [\gamma, E_3, \mathcal{F}_3] \\
\ldots & [\beta_1, E_4, \mathcal{F}_4] \\
\ldots & \ldots \\
\ldots & [\beta_n, E_{n+3}, \mathcal{F}_{n+3}] \\ \cline{2-2}
\ldots & [\beta,\ E,\ \mathcal{F}]
\end{array}
$$

But this is an erroneous impression. The conclusion β generates a new, different support hedge and in general has a different value under the support function. It just happens that the new proof-descriptions *PREF* tells us can be generated from the premisses includes one with a first list (enthymematic description) which is list-identical to part of that for the first premiss. The α *looks* like a middle term but it is not— it remains as part of the hedge E which has backbone tip $\big\langle \langle \alpha\, \beta\rangle \big\langle \langle \gamma\, \alpha\rangle \langle \gamma\, \beta\rangle \big\rangle \big\rangle$.

Consider the simpler case

$$
\begin{array}{l|l}
\ldots & [\langle A\, B\rangle,\ E_1,\ \mathcal{F}_1] \\
\ldots & [\langle \gamma\, A\rangle,\ E_2,\ \mathcal{F}_2] \\
\ldots & [\gamma, E_3, \mathcal{F}_3] \\ \cline{2-2}
\ldots & [B,\ E,\ \mathcal{F}]
\end{array}
$$

The first premiss describes a proof of B with a subproof of A, of a particular form corresponding to $(E_1,\ \mathcal{F}_1)$, and the conclusion describes a *new* proof of B with form corresponding to $(E,\ \mathcal{F})$. It should be no surprise that proof-shape *introduction* rules might deliver new proofs with enthymematic descriptions (corresponding to the first list) in common with parts of those of their premisses.

4.2 TS contains S.

Lemma 4.2.1 Prefixing and suffixing are provable in **TS**.

$A{\rightarrow}B$	(1)	HYP
$C{\rightarrow}A$	(2)	HYP
A	(3)	HYP
C	(4)	HYP
$\langle A\,B\rangle$	(5)	$\rightarrow E$ 1,3
$\langle C\,A\rangle$	(6)	$\rightarrow E$ 2,4
C	(7)	HYP
$\langle\langle A\,B\rangle\,\langle\langle C\,A\rangle\,\langle C\,B\rangle\rangle\rangle$	(8)	$PREF$ 5,6,7
$\langle\langle A\,B\rangle\,\langle\langle C\,A\rangle\,C{\rightarrow}B\rangle\rangle$	(9)	$\rightarrow I$ 7,8
$\langle\langle A\,B\rangle\,\langle C{\rightarrow}A\,C{\rightarrow}B\rangle\rangle$	(10)	$\rightarrow I$ 4,9

$$\langle A{\rightarrow}B\ \langle C{\rightarrow}A\ C{\rightarrow}B\rangle\rangle \qquad (11)\quad {\rightarrow}I\,3,10$$

$$\langle A{\rightarrow}B\ C{\rightarrow}A{\rightarrow}.C{\rightarrow}B\rangle \qquad (12)\quad {\rightarrow}I\,2,11$$

$$A{\rightarrow}B{\rightarrow}.C{\rightarrow}A{\rightarrow}.C{\rightarrow}B \qquad (13)\quad {\rightarrow}I\,1,12$$

Suffixing is proved similarly.

As illustrated in the above proof I use an abbreviated form not explicitly showing all the *REP* steps, but instead indicating where the premisses for the application of a rule come from. Also the linked sequence is not shown as it just contains the single list of each line, with hedge its identity hedge.

Lemma 4.2.2 The class of sentences provable in **TS** is closed under modus ponens: Suppose we have a proof of $[A,\ E,\ \mathcal{F}]$ with no side-bars to the left of the line (i.e. no $\mathcal{F}$ tips have a tip-image) and a proof of $[A{\rightarrow}B,\ H,\ \mathcal{K}]$ also with no side-bars, then there is a proof of $[B,\ H',\ \mathcal{K}']$ with no side-bars.

Proof

On the above supposition we can put the two proofs together and apply REP to obtain premisses ripe for ${\rightarrow}E$, and apply the second form of ${\rightarrow}E$.

Theorem 4.2.3 If $\vdash_{\mathrm{S}} A$ then for some support hedge $(E, \mathcal{F})$ we have $\vdash_{\mathrm{TS}} [A,\ E,\ \mathcal{F}]$ (where the only image under $\mathcal{F}$ is the empty set).

Proof

By lemmas 4.2.1 and 4.2.2.

Modifying the proof to cater for arbitrary L simply requires showing that the appropriate axioms are provable using the corresponding list rules. It's essentially the same proof as that showing an axiom is provable using its associate rule in the consecution system. As in lemma 4.2.1 you just unpack the major antecedents of the axiom using HYP's and $\rightarrow E$'s, apply the associated rule to the resulting lists and then "fill in" the axiom using $\rightarrow I$'s. I state this observation as a theorem:—

Theorem 4.2.4 If $\vdash_{\mathrm{L}} A$ then for some enthymeme sequence E and support function $\mathcal{F}$ we have $\vdash_{\mathrm{TL}} [A, E, \mathcal{F}]$.

I cannot resist illustrating the above by the following *constructive proof of Peirce's law*[1]:—

$A \to B \to A$	(1)	HYP
$A \to B$	(2)	HYP
$\langle A \to B\ A \rangle$	(3)	$\to E$ 2, 1
A	(4)	HYP
$\langle\langle A\ B \rangle\ A \rangle$	(5)	$\to E$ 4, 3
$\langle\langle\langle A\ B \rangle\ A \rangle\ A \rangle$	(6)	$PEIRCE$ 5
$\langle\langle A \to B\ A \rangle\ A \rangle$	(7)	$\to I$ 4, 6
$\langle A \to B \to A\ A \rangle$	(8)	$\to I$ 2, 7
$A \to B \to A \to A$	(9)	$\to I$ 1, 8

4.3 TS is contained in S.

In this section I prove the remaining inclusion needed to show that TS captures S, and indicate how to prove the more general results concerning extensions of S and arbitrary implication logics. The proof is an adaptation of Curry's for absolute implication

[1] As John Slaney urged, we can do a similar *constructive* proof of that justification for double-negation, the "Axiom of Relativity" $A \to B \to B \to A$, which does nasty things in some logics since contraction is an instance of the antecedent.

(c.f. Curry [5] pp.217–219), and procedes by showing that **TS** is contained in **GS**. (Alternatively, one could directly interpret **TS** "in" S similarly to the consecution case.)

Definition 4.3.1 The *transform* of a line (with side-bars) $[\alpha,\ E,\ \mathcal{F}]$ in a **TL** proof is a consecution determined as follows:—

- The antecedent of the consecution comprises, for each side-bar which is a tip-image of $\mathcal{F}$, the corresponding *HYP* wff.
- The succedent is the triple $\left[\alpha,\ E,\ \mathcal{F}^{\dagger}\right]$ where for every element $\underline{\varphi}$ of the domain, $F^{\dagger}\left(\underline{\varphi}\right)$ is like $F\left(\underline{\varphi}\right)$ but with each tip-image a side-bar introduced along with a wff A (by *HYP*) replaced by the corresponding A in the antecedent of the consecution.

Lemma 4.3.2 Let $[\alpha,\ E,\ \mathcal{F}]$ (with side-bars) be a line of a **TS** proof, then the transform of it is a provable consecution in **GS**.

Proof

Proof is by deductive induction on the **TS** proof.

HYP The transform of an introduced line $[A,\ \{A\},\ \mathcal{I}]$ is $A \Vdash [A, \{A\}, \mathcal{I}]$ where $I\left(\underline{A}\right)$ is the singleton containing the antecedent A, and this is provable in **GS**.

REP The transform is that of the ancestor line, hence is provable by hypothesis.

$\rightarrow E$ Suppose we have an application of the first form:—

$$\begin{array}{l|l}
\ldots & [A,\ E_1,\ \mathcal{F}_1] \\
\ldots & \underline{[\alpha,\ E_2,\ \mathcal{F}_2]} \\
\ldots & [\alpha',\ E,\ \mathcal{F}]
\end{array}$$

Then by hypothesis the transforms of the premisses are provable:—

$$\Gamma_1 \Vdash \left[A,\ E_1,\ \mathcal{F}_1{}^{\dagger}\right] \qquad \Gamma_2 \Vdash \left[\langle \ldots A \rightarrow B \ldots \rangle,\ E_2,\ \mathcal{F}_2{}^{\dagger}\right]$$

(The $\underline{A \to B}$ is shown in the first list for simplicity only, it need not occur there.)

By invertibility[2] we also have $\Gamma_2,\ A \ \Vdash\ \left[\langle \ldots \langle A\,B\rangle \ldots\rangle,\ E_2',\ \mathcal{F}_2{}^{\dagger'}\right]$

So using closure under *(cut)* (lemma 3.4.7) we have:—

$$\frac{\Gamma_1 \Vdash \left[A,\ E_1,\ \mathcal{F}_1{}^{\dagger}\right] \qquad \Gamma_2,\ A \ \Vdash\ \left[\langle \ldots \langle A\,B\rangle \ldots\rangle,\ E_2',\ \mathcal{F}_2{}^{\dagger'}\right]}{\Gamma_1,\ \Gamma_2 \Vdash \left[\langle \ldots \langle A\,B\rangle \ldots\rangle,\ E,\ \mathcal{F}^{\dagger}\right]}$$

That the above is in fact the required support hedge for the transform of $(E, \mathcal{F})$ is easy to see, since the *(cut)* adds on to the tip $\underline{A}$ the support hedge $(E_1,\ \mathcal{F}_1{}^{\dagger})$. The case for the second form of $\to E$ is proved in exactly the same way.

$\to I$ Suppose we have:—

$$\begin{array}{l|l|l}
\ldots & & [A,\ \{A\},\ \mathcal{I}] \qquad HYP \\
\ldots & & \vdots \\
\ldots & & [\langle \ldots \langle A\,B\rangle \ldots\rangle,\ E,\ \mathcal{F}] \\
\hline
\ldots & \multicolumn{2}{l}{[\langle \ldots A \to B \ldots\rangle,\ E',\ \mathcal{F}']}
\end{array}$$

Then by hypothesis we have the transform of the major premiss:—

$$\Gamma,\ A \Vdash \left[\langle \ldots \langle A\,B\rangle \ldots\rangle, E, \mathcal{F}^{\dagger}\right]$$

Where the succedent $\underline{A}$ has as tip-image the antecedent A, so applying $(\Vert\!\to)$ to this delivers the required transform of the conclusion.

PREF Applying $(\Vert pref)$ to the transforms of the premisses delivers the required transform.

SUFF As for the above case.

This completes the induction steps, so by induction the lemma is proved.

[2]This use of invertibility was suggested by Bob Meyer and simplifies my original proof.

Theorem 4.3.3 If $\vdash_{\mathrm{TS}} [A,\ E,\ \mathcal{F}]$ then $\vdash_{\mathrm{S}} A$.

Proof

Suppose the antecedent obtains. Applying the above lemma 4.3.2 we have $\Vdash_{\mathrm{GS}} [A,\ E,\ \mathcal{F}]$ and so by lemma 3.3.1 $\vdash_{\mathrm{S}} A$.

The extension of this result to arbitrary **TL** is trivial, with the different list rules dealt with as in the third case of lemma 4.3.2. I state the general result:—

Theorem 4.3.4 If $\vdash_{\mathrm{TL}} [A,\ E,\ \mathcal{F}]$ then $\vdash_{\mathrm{L}} A$.

4.4 The Natural Deduction Systems are Constructive.

In this section I show that the **TL** systems satisfy the Lorenzen principle of inversion. This principle requires that one can't get more from the elimination rules of a formal system than is put in by the introduction rules.

> 'The inversion principle says in effect that nothing is "gained" by inferring a formula through introduction for use as a major premiss in an elimination. The principle thus suggests the following *inversion theorem*:
>
> *If* $\Gamma \vdash A$*, then there is a deduction of* A *from* Γ *in which no formula occurrence is both the consequence of an application of an (introduction)-rule and major premiss of an (elimination)-rule'.* (Prawitz [15], p.33.)[3]

This principle is the central property required of a constructive account of the logical connectives, such as is put forward by the Intuitionists.

Now with the standard $\rightarrow$ elimination and introduction rules the above-described undesirable situation occurs in the manner:—

[3] Also c.f. Curry [5] p.173.

$$
\begin{array}{l|l|ll}
\ldots & & A & HYP \\
\ldots & & \vdots & \\
\ldots & & B & \\
\hline
\ldots & & A \to B & \to I \\
\ldots & & \vdots & \\
\ldots & & A & \\
\ldots & & A \to B & REP \\
\hline
\ldots & & B & \to E
\end{array}
$$

Where $A \to B$ is the offending wff. And the required "reduction" of such deductions, to satisfy the inversion theorem, is obtained by systematically modifying such subproofs by doing the proof of A first and then continuing as per the initial proof of B, but with this A in place of the HYP A.

The approach here is similar, however it is necessary to cater for the greater complexity arising from the fact that further manipulation of the support hedge containing an introduced $\underline{A \to B}$ can be carried out before that $\underline{A \to B}$, as *part of* a new support hedge, is again opened up by $\to E$. So what is required is that *no wff* $\underline{A \to B}$ *arises as the consequence of an application of* $\to I$ *and has a direct descendant in a major premiss for application of* $\to E$ *which itself is "opened" by that* $\to E$.

This extra complexity is dealt with by using *invertibility* of the **TL** systems, from which it follows that we can, after all, leave till just before the offending $\to E$ step the corresponding $\to$I step. The proof of invertibility is similar to that for **GS** (lemma 3.4.2) and uses the two senses of ancestor, defined earlier.

Definition 4.4.1 A *sibling* of a located sublist in a line of a **TL** proof is any other located sublist in that line which has a common *schematic ancestor*.

An *initial ancestor* of a located sublist γ in a line of a **TL** proof is a latest schematic ancestor which is list-identical with γ, but whose ancestor in a premiss, if it has one, is not.

Lemma 4.4.2

1. A descendant (hence ancestor), under either relation, of a list γ is a list δ such that $\delta^* = \gamma^*$.

2. Every direct descendant of a wff introduced by a HYP is a tip of the hedge with support the corresponding side-bar.

Proof of these facts is a simple deductive deduction on the rules of **TS**; observe that the relations only hold between lists (and not proper subwff).

And now for invertibility.

Lemma 4.4.3 If a **TL** deduction contains a line $[\alpha,\ E,\ \mathcal{F}]$ with $\underline{A \to B}$ in the linked sequence E, then there is a **TL** proof of this with last step the introduction of *that* $\to$.

Proof

The proof is similar to that for the invertibility of **GS**.

But first we take that modification of the given proof which satisfies the property that every $\to$ of the conclusion is introduced by $\to I$: For each $\underline{C \to D}$ not satisfying this property (so whose ancestor is "in" a HYP wff) introduce a $HYP\ C$, apply $\to E$ and then $\to I$ to "return" the $\underline{C \to D}$.

Suppose that we have such a proof of $[\alpha,\ E,\ \mathcal{F}]$. Now the initial ancestors of $\underline{A \to B}$ in this (modified) deduction are introduced by $\to I$.
Do the similar deduction but without making these $\to I$ steps (unless already the last). Then for each further line of the deduction descendant from these, the corresponding line

with $\underline{A{\rightarrow}B}$ replaced by $\langle A\, B\rangle$, where the A is a tip with tip-image an extra side-bar (corresponding to the HYP introduced A) and where the support hedge is otherwise the same (with $\langle A\, B\rangle$ substituted for $\underline{A{\rightarrow}B}$), is provable. Ohviously HYP, REP and $\rightarrow E$ preserve this fact (the latter must involve constituents independent of the $\underline{\langle A\, B\rangle}$ as we only backtracked to the most recent introduction of $\underline{A{\rightarrow}B}$).

$\rightarrow I$ also obviously preserves this fact, but in the case that the introduced wff is also a schematic ancestor of $\underline{A{\rightarrow}B}$ to be left open simply refrain from applying the $\rightarrow I$.

For the list rules corresponding to the Hilbert system axioms, like $PREF$ and $SUFF$, since *schematic ancestors* of the $\underline{A{\rightarrow}B}$ have been "opened" we have the correct form of the premisses in order to apply the rule (using an extra HYP A in the case that the guard of the conclusion was also a schematic ancestor of $\underline{A{\rightarrow}B}$ to be "opened"). And finally fill in the extra sibling $\underline{\langle A\, B\rangle}$'s which do not correspond to schematic ancestors of the final $\underline{A{\rightarrow}B}$, using $\rightarrow I$. After these adjustments we are back to only the ancestral $\underline{A{\rightarrow}B}$ "opened up", as required.

Thus we eventually obtain the modification of the original proof, with $[\alpha',\ E',\ \mathcal{F}']$ where $(E', \mathcal{F}') = (E, \mathcal{F})[\underline{A{\rightarrow}B}\backslash\langle A\, B\rangle]$ and there is one side-bar corresponding to the initial HYP introduction of A, which is tip-image of the "opened" $\underline{A}$.

So that the $\underline{A{\rightarrow}B}$ can now be introduced as the last step to obtain the line $[\alpha,\ E,\ \mathcal{F}]$ as required.

Theorem 4.4.4 The inversion principle holds: Suppose that we have a **TL** deduction, then there is a corresponding normal deduction with the same conclusion but satisfying the property that no wff $\underline{A{\rightarrow}B}$ arises as the consequence of an application of $\rightarrow I$ and has a direct descendant in a major premiss for application of $\rightarrow E$ which itself is "opened" by the $\rightarrow E$.

Proof

First modify the deduction (as in the above proof of invertibility) so that every $\rightarrow$ wff of the conclusion is introduced by $\rightarrow I$.

It is simply necessary to carry out the following reduction procedure where such wff occur in the original proof, beginning with the innermost such occurrences.

Consider an innermost subproof where $\underline{A \to B}$ is introduced by $\to I$ and later its direct descendant is the "to-be-opened" part of the major premiss for an application of $\to E$:—

$$
\begin{array}{l|l|ll}
\ldots & & [A, \{A\}, \mathcal{I}] & HYP \\
\ldots & & \vdots & \\
\ldots & & [\delta', M', \mathcal{N}'] & \\
\cline{3-4}
\ldots & \multicolumn{2}{l}{[\delta, M, \mathcal{N}]} & \to I \\
\ldots & \multicolumn{2}{l}{\vdots} & \\
\ldots & \multicolumn{2}{l}{[A, H, \mathcal{K}]} & \\
\ldots & \multicolumn{2}{l}{\vdots} & \\
\ldots & \multicolumn{2}{l}{[\alpha, E, \mathcal{F}]} & \\
\cline{2-4}
\ldots & \multicolumn{2}{l}{\left[\alpha', E^{\dagger}, \mathcal{F}^{\dagger}\right]} & \to E
\end{array}
$$

Where M' contains $\underline{\langle A\,B\rangle}$ which is replaced by $\underline{A \to B}$ in M, which has as direct descendant the $\underline{A \to B}$ in E which is replaced by $\underline{\langle A\,B\rangle}$ again in $E^{\dagger}$. Now by invertibility we can suppose that the introduction of $\underline{A \to B}$ in E occurs as the last step, so that we have the simpler situation:—

$$
\begin{array}{l|l|ll}
\ldots & & [A, \{A\}, \mathcal{I}] & HYP \\
\ldots & & \vdots & \\
\ldots & & [\alpha', E', \mathcal{F}'] & \\
\cline{3-4}
\ldots & \multicolumn{2}{l}{[\alpha, E, \mathcal{F}]} & \to I \\
\ldots & \multicolumn{2}{l}{\vdots} & \\
\ldots & \multicolumn{2}{l}{[A, H, \mathcal{K}]} & \\
\ldots & \multicolumn{2}{l}{[\alpha, E, \mathcal{F}]} & REP \\
\cline{2-4}
\ldots & \multicolumn{2}{l}{\left[\alpha', E^{\dagger}, \mathcal{F}^{\dagger}\right]} & \to E
\end{array}
$$

Where E' contains $\underline{\langle A\,B\rangle}$ which is replaced by $\underline{A \to B}$ to form E and then by $\underline{\langle A\,B\rangle}$ again in $E^{\dagger}$.

Instead do the proof of $[A, H, \mathcal{K}]$ first and then continue with this in place of the HYP step:—

$$\begin{array}{l|l|l} \ldots & & \vdots \\ \ldots & & [A, H, \mathcal{K}] \\ \ldots & & \vdots \\ \ldots & & \left[\alpha', E^{\dagger}, \mathcal{F}^{\dagger}\right] \end{array}$$

To check the conclusion is as required check that each line of the modified subproof is like the corresponding one of the original, but where the original contains a tip $\underline{\varphi}$ with tip-image that side-bar (so φ is a descendant of the HYP A in the original proof) we instead have appended the bush $(H, \mathcal{K})\,[\underline{A}\backslash\varphi]$. This fact is obviously preserved by the rules and applying it to the last line of that subproof we do indeed have $\left[\alpha', E^{\dagger}, \mathcal{F}^{\dagger}\right]$ as required.

In the case that the second form of $\rightarrow E$ was applied (so that $\alpha = A \rightarrow B$ and the conclusion is $\left[B, E^{\dagger}, \mathcal{F}^{\dagger}\right]$) simply use lemma 4.1.6 and the corresponding modification of the subproof.

Thus every implication connective can be given an intuitionistically constructive formal characterisation.

Note the essential use of closure under iK (lemma 4.1.6) to cater for the second form of $\rightarrow E$ in the above theorem.

These results show that the inversion principle just does not distinguish between $\rightarrow$ logics, in terms of those that have it and those that don't. I think that my work can be generalised to capture arbitrary connectives so I suspect that this fact holds completely generally: Every logic satisfies Lorenzen's principle of inversion. Hence, of itself, this principle does no philosophical work for us. Intuitionist logic is far from *the* constructive logic. It seems that matters of interpretation— semantics— cannot be resolved by such a formal property. Incorporating the shapes of proofs into how one displays the meaning of a connective is a constructive procedure which cannot be ~~mitigated~~ argued against on *formal* grounds.

It might be argued that the "proof-shape" introduction rules such as *PEIRCE* don't fall under the rubric intended in the inversion principle. What would be required is an independent statement of the inversion principle together with what it is appropriate to call an introduction rule, which precludes such "proof-shape" introduction rules. I don't think this can faithfully be achieved.

Even though the inversion principle does not discriminate between logics, the constructivist can still require of competing semantics that they satisfy it, and then proceed to examine such semantics on their merits. The "proof-shape" semantics described in this and the previous chapter are completely derivative of the corresponding Hilbert systems and so have little to offer as philosophically explanatory *semantics* (unlike the relevant consecution systems described in chapter 5).

Chapter 5

Relevant Implication

In this chapter intuitively motivated consecution systems for relevant implication logics are described. These systems are of the same basic form as the **GL** systems described in chapter 3, but with list-triple ("proof-shape") introduction rules determined by various senses of *use of a subproof in a proof.* It is necessary to permit greater extensional manipulation of the antecedent of a sequent, with the result that a mix version of cut is needed, and *splice* is taken as a separate rule, but otherwise the technical argument proceeds exactly like that of chapter 3.

The various systems are defined by making an appropriate selection from the following axioms and rules, where all include the "core" system comprising the axioms, the extensional bunching rules $(C\|)$ and $(W\|)$, $(\|splice)$ and $(\rightarrow\|)$ and $(\|\rightarrow)$. (Where $(W\|)$ may be omitted if $(\|con)$ is.)

Definition 5.0.5 Axioms $\alpha \Vdash [\delta, \{\alpha\}, \mathcal{I}]$ As per definition 3.1.1.

$(C\|)$ As per definition 3.1.1.

$$(W\|)\ \frac{\Gamma,\ \alpha, \alpha\ \Vdash\ [\delta,\ E,\ \mathcal{F}]}{\Gamma,\ \alpha\ \Vdash\ [\delta,\ E,\ \mathcal{F}']}$$ Where tip-images follow descendants.

$$(\Vert splice)\quad \frac{\Gamma_1 \ \Vdash\ [\alpha, E, \mathcal{F}] \qquad \Gamma_2 \ \Vdash\ [\delta, G, \mathcal{H}]}{\Gamma_1, \Gamma_2 \ \Vdash\ [\delta, G', \mathcal{H}']}$$

Where δ or another element of G contains a designated antecedent sublist $\underline{\alpha}$ called the *splice* sublist, and the conclusion support hedge is formed by replacing the restricted support hedge $(G, \mathcal{H})|_{\underline{\alpha}}$ by $(E, \mathcal{F})$ (this means using E instead of $G|_{\underline{\alpha}}$ at the appropriate step of the procedure generating G, with corresponding modification of the support function). i.e.the support subhedge $(G, \mathcal{H})|_{\underline{\alpha}}$ is replaced by $(E, \mathcal{F})$, while the rest of the support hedge $(G', \mathcal{H}')$ remains the same as $(G, \mathcal{H})$.

$(\rightarrow\Vert)$ As per definition 3.1.1, but with the extra restriction on the right premiss that if the A to be "filled in" is in the support of a tip $\underline{\varphi}$ then $\varphi^* = A$.

$(\Vert\rightarrow)$ As per definition 3.1.1.

$$(\Vert use)\quad \frac{\Gamma \ \Vdash\ [\varphi,\ E,\ \mathcal{F}]}{\Gamma \ \Vdash\ [\lambda,\ E', \mathcal{F}']}$$

Where

- E has backbone tip $\langle\alpha_1 \ldots \langle\alpha_n\langle\beta\,\gamma\rfloor$ and φ is a guard list of it,
- $\underline{\beta}$ is a guard list of a list $\underline{\langle\delta_1 \ldots \langle\delta_m\,\beta\rfloor}$ in E (i.e. $E|_{\underline{\beta}}$ has a backbone node $\underline{\langle\delta_1 \ldots \langle\delta_m\,\beta\rfloor}$),
- $\mathcal{F}\left(\underline{\langle\delta_1 \ldots \langle\delta_m\,\beta\rfloor}\right) = \mathcal{F}\left(\underline{\langle\alpha_1 \ldots \langle\alpha_n\langle\beta\,\gamma\rfloor}\right)$,
- E' has as backbone tip $\langle\alpha_1 \ldots \langle\alpha_n\,\langle\delta_1 \ldots \langle\delta_m\,\gamma\rfloor$ with λ any guard list of it.

And the support hedge is determined as in *(use)* (definition 2.7.1). Note that in virtue of the above restriction on $\mathcal{F}$ values the support for the new backbone tip is equal to that of the premiss backbone tip.[1]

$$(\Vert merge)\quad \frac{\Gamma \ \Vdash\ [\varphi,\ E,\ \mathcal{F}]}{\Gamma \ \Vdash\ [\lambda,\ E', \mathcal{F}']}$$

[1]This restriction on $\mathcal{F}$ is needed to maintain the same style of treatment of the support function as already carried through for GS; for without it it would be possible for tips to have support images which are not just singleton sets.

Where

- E has backbone tip $\langle\alpha_1 \ldots \langle\alpha_n\langle\beta\,\gamma\rfloor$ and φ is a guard list of it,
- $\underline{\beta}$ is a guard list of a list $\underline{\langle\delta_1 \ldots \langle\delta_m\,\beta\rfloor}$ in E,
- $\mathcal{F}\left(\underline{\langle\delta_1 \ldots \langle\delta_m\,\beta\rfloor}\right) = \mathcal{F}\left(\underline{\langle\alpha_1 \ldots \langle\alpha_n\langle\beta\,\gamma\rfloor}\right)$,
- E' has as backbone tip $\langle\varphi_1 \ldots \langle\varphi_{n+m}\,\gamma\rfloor$ with λ any guard list of it, where $\varphi_1, \ldots, \varphi_{n+m}$ is a permutation of $\alpha_1, \ldots, \alpha_n, \delta_1 \ldots, \delta_m$.

And the support hedge is determined as in *(merge)* (definition 2.7.1).

$$(\|id) \qquad \frac{\Gamma \Vdash [\varphi,\, E,\, \mathcal{F}]}{\Gamma \Vdash [\lambda,\, E', \mathcal{F}']}$$

Where

- the backbone tip of E is α and φ is a guard list of it,
- the backbone tip of E' is $\langle\alpha\;\alpha\rangle$ with λ any guard list of it.

And the support hedge is determined as in (*id*).

$$(\|perm) \qquad \frac{\Gamma \Vdash [\varphi,\, E,\, \mathcal{F}]}{\Gamma \Vdash [\lambda,\, E', \mathcal{F}']}$$

Where

- E has backbone tip $\langle\alpha_1 \ldots \langle\alpha_n\,\langle\beta\langle\gamma\,\delta\rfloor$ and φ is a guard list of it,
- E' has backbone tip $\langle\alpha_1 \ldots \langle\alpha_n\,\langle\gamma\langle\beta\,\delta\rfloor$ with λ any guard list of it.

And the support hedge is determined as in *(perm)*.

$$(\|con) \qquad \frac{\Gamma \Vdash [\varphi,\, E,\, \mathcal{F}]}{\Gamma \Vdash [\lambda,\, E', \mathcal{F}']}$$

Where

- E has backbone tip $\langle\alpha_1 \ldots \langle\alpha_n\,\langle\beta\langle\beta\;\gamma\rfloor$ and φ is a guard list of it,
- $(E, \mathcal{F})|_{\underline{1st\,\beta}} = (E, \mathcal{F})|_{\underline{2nd\,\beta}}$

- E' has backbone tip $\langle\alpha_1 \ldots \langle\alpha_n \langle\beta\,\gamma\rfloor$ with λ any guard list of it.

And the support hedge is determined as in *(con)*.

$$(\|mingle) \qquad \frac{\Gamma \Vdash [\varphi,\, E,\, \mathcal{F}]}{\Gamma \Vdash [\lambda,\, E', \mathcal{F}']}$$

Where

- E has backbone tip $\langle\alpha_1 \ldots \langle\alpha_n \langle\beta\,\gamma\rfloor$ and φ is a guard list of it,
- E' has backbone tip $\langle\alpha_1 \ldots \langle\alpha_n \langle\beta \langle\beta\ \gamma\rfloor$ with λ any guard list of it.

And the support hedge is determined as in *(mingle)*.

So the list manipulation rules are just adaptations of the *use* rules (definition 2.7.1) with the same intuitive motivation. The only extra feature is the incorporation of iK (and its converse!) into the enthymematic— "highlighted" first list— part of the proof-descriptions embodied in the succedent triples. It may be of help to refer to the example applications of the *use* rules in section 2.7. I will refer to the "optional" list-manipulation rules collectively as the '*use* rules', and to the various consecution systems obtained from the above as the *relevant consecution systems*, denoted **RGL**; the corresponding relevant logics will be denoted **RL**.

As before two ancestor relations are defined for these rules, a *schematic* and a *direct* ancestor relation. These are the obvious adaptations of those for **GL** (c.f. the formulation of mingle and subsystems 2.7.2 for the schematic relation) so they are not restated in full here; for $(\|splice)$ the direct relation just follows the recipe as to where the support values come from, with the sublists of the splice list having the corresponding lists of the left premiss only as direct ancestors, while the schematic relation also allows the sublists of the splice list to have the corresponding list of the right premiss as ancestor too.

Lemma 5.0.6 The relevant consecution systems are closed under $(\|iK)$ (definition 3.2.1).

Proof

The property is proved, as before (lemma 3.2.2), by deductive induction.

Consider the extra cases.

$(W\|)$, like $(C\|)$, obviously preserves the property.

Clearly $(\|splice)$ preserves the property in virtue of the induction hypothesis for the right premiss.

And the *use* rules also obviously preserve the property since we can simply make the appropriate choice of the succedent first list of the conclusion.

Whence the iK property holds.

Note that, as before, all identity consecutions are provable. Furthermore the support-image of each tip of the succedent hedge of a provable consecution is a singleton set, the member of which is called the tip's tip-image. In general it is not the case here that the *wff of a tip equals that of its tip-image, as was the case for **GS**. This is why the extra restriction on $(\rightarrow\|)$ is needed.

We show that the consecution systems are contained within the logics. The interpretation property is the same as for **GS** but the proof is a little more complicated.

Lemma 5.0.7 If $\Gamma \Vdash_{\mathbf{RGL}} [\alpha, E, \mathcal{F}]$ then for every located sublist $\underline{\varphi}$ of the succedent and every theory $\mathbf{T}$ of the logic **RL**, if for every $\underline{\delta} \in \mathcal{F}\left(\underline{\varphi}\right)$ we have $\vdash_{\mathrm{T}} \delta^{*}$ then $\vdash_{\mathrm{T}} \varphi^{*}$.

Proof

We use the following fact, which was the basis of the proof of lemma 3.3.1:—

If the above interpretation property holds for every tip of the support hedge $(E, \mathcal{F})$ then it holds for every list $\underline{\varphi}$ of the hedge.

The proof of this fact is by induction on the depth of $\underline{\varphi}$ in the hedge, exactly as for lemma 3.3.1, using the closure of theories under modus ponens.

Thus it is only necessary to show that the interpretation property holds for the tips of the succedent hedge. The proof of this fact is by deductive induction:—

ad *axioms*

Clearly the property holds of the tips as they have identical tip-images.

ad $(C\|)$ and $(W\|)$

These obviously preserve the property since there is no change to the list-identities of the tip-images.

ad $(\|splice)$

The tips of the conclusion succedent hedge have the same support-images as their direct ancestors, hence by the induction hypothesis the property holds.

ad $(\rightarrow\|)$

$$\frac{\Gamma_1 \;\Vdash\; [A, G, \mathcal{H}] \qquad\qquad \Gamma_2, \langle \ldots \langle A\, B\rangle \ldots\rangle \;\Vdash\; [\delta, E, \mathcal{F}]}{\Gamma_1, \Gamma_2, \langle \ldots A \rightarrow B \ldots\rangle \;\Vdash\; [\delta, E', \mathcal{F}']}$$

Consider those E' tips $\underline{\lambda'}$ added as part of $(G, \mathcal{H})\,[\underline{A}\backslash\varphi]$ to an E tip $\underline{\varphi}$ with $\mathcal{F}\left(\underline{\varphi}\right) = \{\underline{A}\}$, $\mathcal{F}'\left(\underline{\lambda'}\right) = \mathcal{H}\left(\underline{\lambda}\right)$ where $\lambda' = \lambda\,[\underline{A}\backslash\varphi]$.

Now it is required that $\varphi * = A$, hence $\lambda'^* = \lambda^*$ and so the induction hypothesis for the left premiss ensures that the property holds of $\underline{\lambda'}$.

For the remaining tips with ancestors in the right premiss the only possible change in the support-image is a replacement of $\underline{\langle A\, B\rangle}$ by $\underline{A \rightarrow B}$, which leaves the *wff unchanged. Thus the induction hypothesis for the right premiss ensures the property holds of them.

ad $(\|\rightarrow)$

The property is obviously preserved since the conclusion succedent tips have the same support-images as their ancestors, but themselves may have $\underline{\langle A\, B\rangle}$ replaced by $\underline{A \rightarrow B}$ which leaves the *wff unaffected.

ad $(\| use)$

Consider the new backbone tip of the conclusion. Its support is equal to that for $\underline{\langle\alpha_1 \ldots \langle\alpha_n \langle\beta\,\gamma\rfloor}$ and that for $\underline{\langle\delta_1 \ldots \langle\delta_m\,\beta\rfloor}$ so that by the induction hypothesis every theory containing the *wff of its support also contains both $\alpha_1^* \rightarrow . \ldots \alpha_n^* \rightarrow . \beta^* \rightarrow \gamma^*$ and $\delta_1^* \rightarrow . \ldots . \delta_m^* \rightarrow \beta^*$. Hence applying Use delivers the required wff $\alpha_1^* \rightarrow . \ldots \alpha_n^* \rightarrow . \delta_1^* \rightarrow . \ldots . \delta_m^* \rightarrow \gamma^*$.

The remaining tips are all identical with their ancestors and have the same support-images, hence the induction hypothesis ensures the property holds of them.

ad $(\| merge)$

Similar to the above case.

ad $(\| id)$

The only "new" tip is the backbone tip $\underline{\langle\alpha\,\alpha\rangle}$ with empty support. The corresponding logics contain the identity axiom so the property holds of it, and the induction hypothesis ensures it holds of the remaining tips.

ad $(\| perm)$

Once again the only new tip is the backbone tip which has the same support as the premiss backbone tip, so the Perm rule ensures that the property holds.

ad $(\| con)$ and $(\| mingle)$

Similarly to the above case.

This completes the induction proof that the interpretation property holds of the tips of the succedent hedge of a provable consecution. Hence by the lemma the property holds of all located sublists in the hedge as required.

Hence we have the interpretation theorem:—

Theorem 5.0.8 If $\Vdash_{\mathbf{RGL}} [A,\, E,\, \mathcal{F}]$ then $\vdash_{\mathrm{RL}} A$.

Proof

Suppose that $\Vdash_{\mathbf{RGL}} [A,\ E,\ \mathcal{F}]$. Since the antecedent is empty $\mathcal{F}(\underline{A})$ is the empty-set and so by the lemma every theory contains A^*, that is A, hence in particular **RL** does.

Before proving the converse, an example proof of prefixing (using $(\Vert id)$ and $(\Vert use)$) is displayed:—

$$(\Vert id)\ \frac{\langle A\,B\rangle \Vdash \langle A\,B\rangle}{\langle A\,B\rangle \Vdash \big\langle \langle A\,B\rangle\,\langle A\,B\rangle\big\rangle}$$

And apply $(\Vert splice)$ to the first $\underline{A}$ of the succedent:—

$$(\Vert splice)\ \frac{A \Vdash A \qquad \langle A\,B\rangle \Vdash \big\langle \langle A\,B\rangle\,\langle A\,B\rangle\big\rangle}{A,\ \langle A\,B\rangle \Vdash \big\langle \langle A\,B\rangle\,\langle A\,B\rangle\big\rangle}$$

And $(\Vert\!\rightarrow)$ gives:—

$$\langle A\,B\rangle \Vdash \big\langle A \rightarrow B\ \langle A\,B\rangle\big\rangle$$

Similarly (but with first list the backbone root in the $(\Vert id)$ step) we obtain:—

$$\langle C\,A\rangle \Vdash A,\ \big\langle C \rightarrow A\ \langle C\,A\rangle\big\rangle$$

Applying $(\rightarrow\!\Vert)$ to these (with the above the left premiss) gives:—

$$\langle C\,A\rangle,\ A \rightarrow B \Vdash \big\langle A \rightarrow B\ \langle A\,B\rangle\big\rangle,\ \big\langle C \rightarrow A\ \langle C\,A\rangle\big\rangle$$

Now apply $(\Vert use)$:—

$$\langle C\,A\rangle,\ A \rightarrow B \Vdash \big\langle A \rightarrow B\ \big\langle C \rightarrow A\ \langle C\,B\rangle\big\rangle\big\rangle$$

$(\to\|)$ with left premiss the identity $C \Vdash C$ gives:—

$$C,\ C\to A,\ A\to B \ \Vdash\ \langle A\to B\ \langle C\to A\ \langle C\, B\rangle\rangle\rangle$$

And then three $(\|\to)$'s delivers the required result:—

$$\Vdash\ A\to B\to .C\to A\to .C\to B$$

After that pleasant interlude it is now time to tackle the task of showing that the relevant consecution systems capture the corresponding relevant logics. A fact which obviously holds in light of the above example! The strategy is as before, via proofs of invertibility and closure under mix (strengthened *(cut)*).

Lemma 5.0.9 The invertibility property holds: If $\Gamma \Vdash [\langle \ldots A\to B \ldots\rangle, E, \mathcal{F}]$ where $A\to B$ is a constituent of the succedent hedge— not necessarily the first list— then there is a deduction of this consecution in which that $\to$ is introduced last, that is with last step

$$(\|\to)\ \frac{\Gamma,\ A\ \ \Vdash\ \ [\langle \ldots \langle A\, B\rangle \ldots\rangle,\ E',\ \mathcal{F}']}{\Gamma\ \ \Vdash\ \ [\langle \ldots A\to B \ldots\rangle,\ E,\ \mathcal{F}]}$$

(Or with first list unchanged where it doesn't contain the $\underline{\langle A\, B\rangle}$).

Proof

The proof is exactly like that of lemma 3.4.2, with extra cases for the new rules:—

$(W\|)$ Like $(C\|)$ this obviously preserves the property.

$$(\|splice)\qquad \frac{\Lambda_1\ \Vdash\ [\beta, L, \mathcal{K}] \qquad \Lambda_2\ \Vdash\ [\delta, G, \mathcal{H}]}{\Lambda_1,\ \Lambda_2\ \ \Vdash\ \ [\delta, G', \mathcal{H}']}$$

By hypothesis we have at least one and possibly, where an ancestral $\underline{A\to B}$ occurs in the splice list $\underline{\beta}$, both of:—

$$\Lambda_1,\ A\ \Vdash [\beta^{\triangleright}, L^{\triangleright}, \mathcal{K}^{\triangleright}] \qquad \Lambda_2,\ A \Vdash [\delta^{\triangleright}, G^{\triangleright}, \mathcal{H}^{\triangleright}]$$

Thus we can apply the corresponding $(\|splice)$ to obtain the required consecution:—

$$(\|splice)\ \frac{\Lambda_1,\ [A]\ \Vdash [\beta^{\triangleright}, L^{\triangleright}, \mathcal{K}^{\triangleright}] \qquad \Lambda_2,\ A\ \Vdash [\delta^{\triangleright}, G^{\triangleright}, \mathcal{H}^{\triangleright}]}{\Lambda_1, \Lambda_2,\ A\ \Vdash\ [\delta^{\triangleright}, G^{\triangleright\prime}, \mathcal{H}^{\triangleright\prime}]}$$

(Where $(C\|)$ and $(W\|)$ are also applied to get one A in the case that an ancestral $A \rightarrow B$ occurred in the splice list so that both modified premisses provided a copy of the corresponding antecedent A.)

We need to check that this does indeed give the required support function.

Ancestral $A \rightarrow B$'s with ancestors in the right premiss but not in the splice list, have (by hypothesis) the ancestors in the right premiss replaced by $\langle A\, B\rangle$ with tip-image of $\underline{A}$ the antecedent A, and this change is preserved in the new $(\|splice)$. If the splice list contains one or more ancestral $A \rightarrow B$'s then in the new right premiss they are replaced by $\langle A\, B\rangle$ with tip-image of $\underline{A}$ the antecedent A. And the left premiss has the corresponding change in $\underline{\beta}$, so we can apply $(\|splice)$ with the change preserved in the support hedge $(L^{\triangleright}, \mathcal{K}^{\triangleright})$ added in place of $(G^{\triangleright}, \mathcal{H}^{\triangleright})|_{\underline{\beta^{\triangleright}}}$.

$(\rightarrow\|)$ The extra restriction on $(\rightarrow\|)$ does not affect the previous proof, since the modified deduction retains the same support for the original tips of the succedent (i.e. those other than the "new" $\underline{A}$'s), which retain the same *wff values.

$(\|use)$, $(\|merge)$, $(\|perm)$ Clearly application of the rule to the modified premiss gives the required modified conclusion.

$(\|id)$, $(\|mingle)$ In these cases application of the rule to the modified premiss almost gives the required conclusion. "almost" because a single opened $\langle A\, B\rangle$ of the premiss may have two descendants in the conclusion but with both A's having the same A in the antecedent as tip-image. Splice in an extra copy of A with $(\|splice)$ left premiss $A \Vdash A$ and splice list one of the $\underline{A}$'s. For each such $\langle A\, B\rangle$ of the premiss, if both $\langle A\, B\rangle$'s in the conclusion are schematic ancestors to be left open leave them be, otherwise simply fill in by $(\|\!\!\rightarrow)$ the non-schematic-ancestor. Thence the consecution is of the required form.

$(\|con)$ Here again application of the rule to the modified premiss almost gives the required conclusion. It may happen that two schematic ancestors $\langle A\,B\rangle$ are to be replaced by a single descendant $\langle A\,B\rangle$. In this case first apply $(W\|)$ to the two antecedent A's to obtain the correct form for application of $(\|con)$, and $(\|con)$ then delivers the required modified conclusion.

Whence as for lemma 3.4.2 the invertibility property holds.

Now for closure under *(mix)*, which is defined:—

Definition 5.0.10 $(mix)\,\dfrac{\Gamma \;\Vdash\; [\alpha, H, \mathcal{K}] \qquad \Lambda \;\Vdash\; [\delta, E, \mathcal{F}]}{\Gamma,\, \Lambda^\circ \;\Vdash\; [\delta, E^\circ, \mathcal{F}^\circ]}$

Where

- Λ contains at least one copy of α as a whole list,
- Λ° is like Λ but with some (and at least one) of its (whole list) copies of α deleted *but not necessarily all*— these deleted α's are called the *mix lists,*
- $(E^\circ, \mathcal{F}^\circ)$ is obtained from $(E, \mathcal{F})$ and $(H, \mathcal{K})$ in a manner similar to $(\rightarrow\!\|)$: For each E tip $\underline{\varphi}$ with tip-image $\underline{\lambda}$ occurring in a mix list replace $(E, \mathcal{F})\big\{\underline{\varphi}\big\}$ by $(H, \mathcal{K})\{\lambda\}\,[\lambda\backslash\varphi]$.

Recall that the intuitive picture is of the support hedge of the right premiss "growing" at those of its tips $\underline{\varphi}$ with tip-images $\underline{\lambda}$ in the mix lists, with the new growth corresponding to the support bush of the left premiss for $\underline{\lambda}$.

That closure under mix ensures closure under modus ponens is proved exactly like lemma 3.4.3, using invertibility and lemma 5.0.6.

Definition 5.0.11 Suppose we have proofs in GS of left and right premisses of appropriate form for an application of *(mix)*. We call these the *left upper* and *right upper* respectively.

For a particular mix α in the antecedent of the right upper, its *right rank* is the maximum length of a path in the deduction, above and including the right upper, containing an ancestor of that α list-identical with α.

The *right rank* of the right upper is the maximum of the right ranks of the mix lists in the antecedent of the right upper.

Lemma 5.0.12 GS is closed under *(mix)*.

Proof

The proof is by a double induction, on the degree of the mix list α and on the *right* rank.

<u>1.Right rank = 1</u>

<u>1.1 Right upper an axiom</u>

As for lemma 3.4.7.

<u>1.2 Right upper from $(\rightarrow\|)$</u>

Since the right rank is one, once again this is the same as applying *(cut)* so the proof is as for lemma 3.4.7.

<u>2 Right Rank > 1</u>

<u>2.1 Right upper from $(W\|)$</u>

$$(mix)\ \frac{\Gamma \ \|\!\!- [\alpha, H, \mathcal{K}] \qquad (W\|)\ \dfrac{\Lambda,\ \ \beta, \beta \ \ \|\!\!-\ \ [\delta,\ E,\ \mathcal{F}]}{\Lambda,\ \ \ \beta \ \ \|\!\!-\ \ [\delta,\ E,\ \mathcal{F}']}}{\Gamma,\ \Lambda^{\circ},\ \beta^{\circ} \ \ \|\!\!-\ \ [\delta,\ E^{\circ},\ \mathcal{F}'^{\circ}]}$$

Do the lower rank *(mix)* with the premiss of the right upper, with mix lists the ancestors of those in the right upper. Clearly tip-images in the Λ (i.e. parametric) get treated correctly, and almost as obviously those in the major constituent of the $(W\|)$ do as well— if β is a mix list (so equals α) then so are both ancestors in the premiss, and a tip-image in the conclusion β can arise only in virtue of a corresponding tip-image in one or both of the ancestral β's.

2.2 Right upper from $(C\|)$ Similarly to lemma 3.4.7.

2.3 Right upper from $(\|splice)$

$$
(mix)\ \dfrac{\Gamma \Vdash [\alpha, H, \mathcal{K}] \qquad (\|splice)\ \dfrac{\Lambda_1 \Vdash [\beta, M, \mathcal{N}] \qquad \Lambda_2 \Vdash [\delta, E, \mathcal{F}]}{\Lambda_1, \Lambda_2 \Vdash [\delta, E', \mathcal{F}']}}{\Gamma, \Lambda_1^\circ, \Lambda_2^\circ \Vdash [\delta, E'^\circ, \mathcal{F}'^\circ]}
$$

Do the needed lower rank (mix)'s, and then apply the corresponding $(\|splice)$:—

$$
\dfrac{\dfrac{\Gamma \Vdash [\alpha, H, \mathcal{K}] \qquad \Lambda_1 \Vdash [\beta, M, \mathcal{N}]}{\Gamma, \Lambda_1^\circ \Vdash [\beta, M^\circ, \mathcal{N}^\circ]} \qquad \dfrac{\Gamma \Vdash [\alpha, H, \mathcal{K}] \qquad \Lambda_2 \Vdash [\delta, E, \mathcal{F}]}{\Gamma, \Lambda_2^\circ \Vdash [\delta, E^\circ, \mathcal{F}^\circ]}}{\Gamma, \Lambda_1^\circ, \Lambda_2^\circ \Vdash [\delta, E^{\circ\prime}, \mathcal{F}^{\circ\prime}]}
$$

Where the splice list is the descendant of the original splice list (the domain of E° includes that of E — the tree just grows at some of its tips).

We need to check that this provides the correct transformation of the support hedge.

An $(E', \mathcal{F}')$ tip $\underline{\varphi}$ with tip-image $\underline{\lambda}$ in Λ_2 corresponds to an identical $(E, \mathcal{F})$ tip not in the splice bush.

For such $\underline{\varphi}$ without tip-image in a mix list, the righthand mix leaves it unchanged, as does the final $(\|splice)$, which is what is required; while for such $\underline{\varphi}$ with a tip-image $\underline{\lambda}$ in a mix list, the righthand mix replaces $(E, \mathcal{F})\{\underline{\varphi}\}$ by $(H, \mathcal{K})\{\underline{\lambda}\}[\underline{\lambda}\backslash\varphi]$, and the final splice leaves this added bush unchanged, as is required.

Consider a "new" tip $\underline{\varphi}$ of E' in the introduced splice bush (so with image empty or in Λ_1).

If its tip-image isn't in a mix list, we require no further change. The lefthand mix leaves such tip of $(M,\mathcal{N})$ unchanged, and the final splice adds it as part of the introduced splice bush as required.

Where the tip-image $\underline{\lambda}$ is in a mix list in Λ_1 we require $(M,\mathcal{N})\{\underline{\varphi}\}$ to be replaced by $(H,\mathcal{K})\{\underline{\lambda}\}\,[\underline{\lambda}\backslash\varphi]$. The lefthand mix does this replacement, which is preserved in the introduced splice bush of the final splice.

So we have the required transformation of the succedent support hedge.

2.4 Right upper from $(\rightarrow\|)$

$$\dfrac{\Gamma \Vdash [\alpha, H, \mathcal{K}] \qquad (\rightarrow\|)\dfrac{\Lambda_1 \Vdash [A, M, \mathcal{N}] \qquad \Lambda_2, \langle\ldots\langle A\,B\rangle\ldots\rangle \Vdash [\delta, E, \mathcal{F}]}{\Lambda_1, \Lambda_2, \langle\ldots A\rightarrow B\ldots\rangle \Vdash [\delta, E', \mathcal{F}']}}{\Gamma,\ \Lambda_1^\circ,\ \Lambda_2^\circ,\ \langle\ldots A\rightarrow B\ldots\rangle^\circ \Vdash [\delta,\ E'^\circ,\ \mathcal{F}'^\circ]}$$

Do the appropriate lower rank *(mix)* or *(mix)*'s, and then apply the corresponding $(\rightarrow\|)$:—

$$\dfrac{\dfrac{\Gamma \Vdash [\alpha, H, \mathcal{K}] \qquad \Lambda_1 \Vdash [A, M, \mathcal{N}]}{\Gamma,\ \Lambda_1^\circ \Vdash [A, M^\circ, \mathcal{N}^\circ]} \qquad \dfrac{\Gamma \Vdash [\alpha, H, \mathcal{K}] \qquad \Lambda_2, \langle\ldots\langle A\,B\rangle\ldots\rangle \Vdash [\delta, E, \mathcal{F}]}{\Gamma, \Lambda_2^\circ, \langle\ldots\langle A\,B\rangle\ldots\rangle \Vdash [\delta, E^\circ, \mathcal{F}^\circ]}}{\Gamma,\ \Lambda_1^\circ,\ \Gamma,\ \Lambda_2^\circ,\ \langle\ldots A\rightarrow B\ldots\rangle \Vdash [\delta,\ E^{\circ\prime},\ \mathcal{F}^{\circ\prime}]}$$

And adjust the antecedent using $(C\|)$ and $(W\|)$.

Now if $\langle\ldots A\rightarrow B\ldots\rangle$ is also a mix list (hence identical with α) we remove it by doing the further mix with only that list a mix list:—

$$(mix)\,\dfrac{\Gamma \Vdash [\alpha, H, \mathcal{K}] \qquad \Gamma,\ \Lambda_1^\circ,\ \Lambda_2^\circ,\ \langle\ldots A\rightarrow B\ldots\rangle \Vdash [\delta, E^{\circ\prime}, \mathcal{F}^{\circ\prime}]}{\Gamma,\ \Gamma,\ \Lambda_1^\circ,\ \Lambda_2^\circ \Vdash [\delta, E^{\circ\prime\circ}, \mathcal{F}^{\circ\prime\circ}]}$$

With $(C\|)$'s and $(W\|)$'s as needed to adjust the antecedent.

For this mix the right rank is one, so strictly less than the original right rank, hence it falls under our induction hypothesis.

That the above provides the correct modification of the support hedge is shown similarly to the corresponding case of lemma 3.4.7.

2.5 Right upper from $(\Vert\!\rightarrow)$

$$
(mix)\ \dfrac{\Gamma \Vdash [\alpha, \mathcal{H}, \mathcal{K}] \qquad (\Vert\!\rightarrow)\ \dfrac{\Lambda, A \ \Vdash\ [\langle \ldots \langle A\, B\rangle \ldots\rangle,\ E,\ \mathcal{F}]}{\Lambda \ \Vdash\ [\langle \ldots A \rightarrow B \ldots\rangle,\ E',\ \mathcal{F}']}}{\Gamma, \Lambda^{\circ} \ \Vdash\ [\langle \ldots A \rightarrow B \ldots\rangle, E'^{\circ}, \mathcal{F}'^{\circ}]}
$$

Do the *(mix)* on the premiss of the right upper first, and then apply the $(\Vert\!\rightarrow)$:—

$$
(\Vert\!\rightarrow)\ \dfrac{(mix)\ \dfrac{\Gamma \Vdash [\alpha, \mathcal{H}, \mathcal{K}] \qquad \Lambda, A \Vdash [\delta', E, \mathcal{F}]}{\Gamma, \Lambda^{\circ}, A \ \Vdash\ [\delta', E^{\circ}, \mathcal{F}^{\circ}]}}{\Gamma, \Lambda^{\circ} \ \Vdash\ \left[\delta, E^{\circ'}, \mathcal{F}^{\circ'}\right]}
$$

Where $\delta = \langle \ldots A \rightarrow B \ldots \rangle$ and δ' is its ancestor.

That this provides the correct modification of the support hedge is shown similarly to the corresponding case of lemma 3.4.7.

2.6 Right upper from a *use* rule

Each of these rules can be dealt with exactly like $(\Vert pref)$ for **GS**. One simply applies closure under *(mix)* before the particular *use* rule.

This completes case 2, and so the lemma is proved.

We now check that each system with a particular *use* rule captures the corresponding axiom or rule (c.f. definition 2.7.1).

Lemma 5.0.13 1. In those relevant consecution systems containing $(\Vert id)$, the consecution

$\Vdash [A \rightarrow A, \{A \rightarrow A\}, \mathcal{I}]$ is provable for all $A \rightarrow A$.

Proof

Applying $(\|id)$ to the identity $A \Vdash [A, \{A\}, \mathcal{I}]$ gives $A \Vdash [\langle A\, A\rangle, \{\langle A\, A\rangle\}, \mathcal{I}']$ and $(\|\rightarrow)$ delivers the required consecution.

2. Those systems containing $(\|use)$ are closed under

$$\frac{\Vdash [A_1 \rightarrow . \ldots A_n \rightarrow .B \rightarrow .C,\, E_1,\, \mathcal{F}_1] \qquad \Vdash [D_1 \rightarrow . \ldots D_m \rightarrow B,\, E_2,\, \mathcal{F}_2]}{\Vdash [A_1 \rightarrow . \ldots A_n \rightarrow .D_1 \rightarrow . \ldots D_m \rightarrow C,\, E,\, \mathcal{F}]}$$

This weak closure fact with empty antecedents is to ensure that the support value restriction for the application of $(\|use)$ (which follows) is met.

Proof

By invertibility (repeated) we have

$$A_1, \ldots A_n,\, B \Vdash [\langle A_1 \ldots \langle A_n \langle B\, C\rfloor,\, E_1',\, \mathcal{F}_1']$$

and

$$D_1, \ldots D_m \Vdash [\langle D_1 \ldots \langle D_m\, B\rfloor,\, E_2',\, \mathcal{F}_2']$$

Further by closure under iK (lemma 5.0.6) we have

$$D_1, \ldots D_m \Vdash [B,\, E_2',\, \mathcal{F}_2']$$

so by closure under *(mix)* we have

$$D_1, \ldots D_m,\, A_1, \ldots A_n \Vdash \left[\langle A_1 \ldots \langle A_n \langle B\, C\rfloor,\, E_1'^{\diamond},\, \mathcal{F}_1'^{\circ}\right]$$

Where the succedent tip B has had $(E_2', \mathcal{F}_2')$ stuck onto it, so that now the succedent is of correct form for applying $(\|use)$ (note that both $\underline{\langle A_1 \ldots \langle A_n \langle B\, C\rfloor}$ and $\underline{\langle D_1 \ldots \langle D_m\, B\rfloor}$ have empty support) to obtain

$$D_1, \ldots D_m,\, A_1, \ldots A_n \Vdash [\langle A_1 \ldots \langle A_n \langle D_1 \ldots \langle D_m\, C\rfloor,\, E',\, \mathcal{F}']$$

Finally repeated applications of $(\|\!\rightarrow)$ delivers the required conclusion.

3. The corresponding closure fact holds for those systems containing $(\|merge)$, with the proof similar to the above.

4. Those systems containing $(\|perm)$ are closed under

$$\frac{\Gamma \Vdash [A_1 \rightarrow . \ldots A_n \rightarrow . B \rightarrow . C \rightarrow D,\ E,\ \mathcal{F}]}{\Gamma \Vdash [A_1 \rightarrow . \ldots A_n \rightarrow . C \rightarrow . B \rightarrow D,\ E',\ \mathcal{F}']}$$

Proof

Given the premiss the conclusion can obviously be obtained by using invertibility, applying $(\|perm)$ and then repeated $(\|\!\rightarrow)$'s.

5. Those systems containing $(\|con)$ are closed under

$$\frac{\Gamma \Vdash [A_1 \rightarrow . \ldots A_n \rightarrow . B \rightarrow . B \rightarrow C,\ E,\ \mathcal{F}]}{\Gamma \Vdash [A_1 \rightarrow . \ldots A_n \rightarrow . B \rightarrow C,\ E',\ \mathcal{F}']}$$

Proof

Given the premiss by invertibility we have

$$A_1,\ \ldots A_n,\ B,\ B \Vdash \left[\langle A_1 \ldots \langle A_n \langle B \langle B\, C \rfloor,\ E^{\dagger},\ \mathcal{F}^{\dagger}\right]$$

and applying $(W\|)$ and then $(\|con)$ gives

$$A_1,\ \ldots A_n,\ B \Vdash \left[\langle A_1 \ldots \langle A_n \langle B\, C \rfloor,\ E^{\dagger\prime},\ \mathcal{F}^{\dagger\prime}\right]$$

and then repeated $(\|\!\rightarrow)$'s gives the required conclusion.

6. Those systems containing $(\|mingle)$ are closed under

$$\frac{\Gamma \Vdash [A_1 \rightarrow . \ldots A_n \rightarrow . B \rightarrow C,\ E,\ \mathcal{F}]}{\Gamma \Vdash [A_1 \rightarrow . \ldots A_n \rightarrow . B \rightarrow . B \rightarrow C,\ E',\ \mathcal{F}']}$$

Proof

Given the premiss by invertibility we have

$$A_1, \ldots A_n, B \Vdash \left[\langle A_1 \ldots \langle A_n \langle B\, C \rfloor, E^{\dagger}, \mathcal{F}^{\dagger}\right]$$

and applying $(\Vert mingle)$ gives

$$A_1, \ldots A_n, B \Vdash \left[\langle A_1 \ldots \langle A_n \langle B \langle B\, C \rfloor, E^{\dagger'}, \mathcal{F}^{\dagger'}\right]$$

And applying $(\Vert splice)$ using the identity $B \Vdash B$ and splice list one of the succedent $\underline{B}$'s provides the necessary copy of B in the antecedent so that application of $(\Vert\rightarrow)$'s delivers the required conclusion.

The above lemma together with closure under modus ponens establishes that each of the relevant consecution systems contains the corresponding relevant logic, which fact is stated in the following theorem:—

Theorem 5.0.14 **RGL** contains **RL**, i.e. if $\vdash_{\mathbf{RL}} A$ then $\Vdash_{\mathbf{RGL}} [A, E, \mathcal{F}]$ for some support hedge $(E, \mathcal{F})$ (where the only value taken by $\mathcal{F}$ is the empty set).

In this chapter I have provided *explanatory* semantics, in the Curry tradition, for various relevant implication connectives, corresponding to Mingle and subsystems. The basis of the semantics is a formal description of proofs which allows one to capture in a precise way the notion of "use as a subproof in a proof". Proof-descriptions are generated using *use* rules which correspond intuitively to particular aspects of this notion. Different senses of "use", corresponding to the different relevant logics, are obtained by taking the appropriate selection from amongst the *use* rules; for example that sense of "use" which only captures transitivity of use of a subproof, but rejects the notion that a proof uses itself, and keeps track of the multiple use of subproofs, is obtained by taking $(\Vert use)$ only. So our further development of Curry's approach towards logic— that logic is the epitheory of formal systems— shows that there is a solid intuitive foundation for relevant implication from within that tradition.

Chapter 6

Binary Relation Semantics

In this chapter I describe a two-valued binary relation semantics for S, and I provide a recipe for generalising the semantics to cater for all implication logics. The semantics are an adaptation of the Kripke semantics for Intuitionist Logic but involving the representation of *proofs* holding at a point as well as sentences. Such semantics can be provided which correspond exactly to the consecution systems of chapter 3, however presented here is a version of the semantics involving a dramatic simplification of the support structure, viz: dispensing with hedges, so that a support function is defined only on the located sublists of a *single* list. Thus only the enthymematic description corresponding to the first list of a list-triple is retained, dispensing with the *total* proof-description which includes that of all subproofs.

The corresponding simplified consecution systems and natural deduction systems are described in appendix B.2. In section 6.6 I show how to provide the similar constructive semantics for relevant implication connectives by incorporating formal conditions capturing "use as a subproof in an irredundant proof". The advantage of this approach is that the completeness proof for the semantics is a standard induction on degree. However it is not possible to use the strategy of chapter 4 to obtain closure of the natural deduction system under iK, so the full Lorenzen principle of inversion does not hold for these simplified systems. Furthermore the "simplified" version of *(use)* becomes a two-premiss rule and is less intuitively appealing.

The new definition of a support function follows.

Definition 6.0.15 A *support function $\mathcal{F}$ on a list α* is a function from the set of located sublists of α into a set of finite sets formed from a given class of entities I satisfying:—

- For each located sublist $\underline{\langle\gamma\,\delta\rangle}$ we have $\mathcal{F}\left(\underline{\delta}\right)$ formed by the union of $\mathcal{F}\left(\underline{\langle\gamma\,\delta\rangle}\right)$ and $\mathcal{F}\left(\underline{\gamma}\right)$.

Here the members of I will be the set of points of the model structure.

Definition 6.0.16 A *list-pair* $[\alpha, \mathcal{F}]$ is a pair comprising a list α and a support function $\mathcal{F}$ on α.

6.1 Models

In this section I define the models. These are in some respects similar to those of the Kripke semantics for Intuitionist Logic. But in other respects they are very non-standard—list-pairs are assigned to points rather than wff, which corresponds to assigning *proofs* to the points.

Definition 6.1.1 A *model structure* M is a quadruple $M = \langle o, \perp, K, S\rangle$ where K is a set of *points* with $o, \perp$ members of K, and S is a transitive and reflexive relation on K satisfying $\forall\, a \in K$, oSa and $aS\perp$.

An assignment is generated by an initial valuation of atomic list-pairs, where the support function images are sets of points (elements of K):—

Definition 6.1.2 An *initial valuation* on a model structure M is a function

$$v : K \times \mathcal{A} \to \{0,1\}$$

where $\mathcal{A}$ is the set of list-pairs $[\alpha, \mathcal{F}]$ with α an atomic list, satisfying:—

Images Condition If $v(a,[\alpha,\mathcal{F}]) = 1$ then for every point b in an image of $\mathcal{F}$, bSa.

Hereditary Condition If $v(a,[\alpha,\mathcal{F}]) = 1$ then for every $b \in K$, if aSb then $v(b,[\alpha,\mathcal{F}]) = 1$.

Sufficiency Condition If $v(a,[\alpha,\mathcal{F}]) = 1$ and for some located sublist $\underline{\delta}$ of α and some point $b \in K$, if cSb for every element c of all images of $\mathcal{F}|_{\underline{\delta}}$ (the restriction of $\mathcal{F}$ to the located sublists of $\underline{\delta}$) then $v(b, \left[\delta, \mathcal{F}|_{\underline{\delta}}\right]) = 1$.

Witness Condition For every atomic list α, $v(\perp,[\alpha,\mathcal{I}]) = 1$ where $\mathcal{I}$ is some support function on α.

(splice) **Condition** If $v(a,[\alpha,\mathcal{F}]) = 1$ and $v(a,[\delta,\mathcal{H}]) = 1$ where δ contains a designated antecedent located sublist $\underline{\alpha}$ called the *splice* sublist, then $v(a,[\delta,\mathcal{H}']) = 1$, with $\mathcal{H}|_{\underline{\alpha}}$ replaced by $\mathcal{F}$ so that:—

- $\underline{\alpha}$ has $\mathcal{H}'|_{\underline{\alpha}} = \mathcal{F}$,
- the remaining antecedent sublists φ of δ, both proper-superlist and independent of $\underline{\alpha}$, as well as δ itself, have $\mathcal{H}'\left(\underline{\varphi}\right) = \mathcal{H}\left(\underline{\varphi}\right)$,
- and the values of the remaining (consequent) sublists are determined by the union requirement upon support functions.

This condition captures the idea that one subproof of a proof may be replaced by another with the same subconclusion.

(pref) **Condition** [1]

$$\frac{[\langle \alpha\,\beta\rangle, \mathcal{F}_1] \in a \quad [\langle \gamma\,\alpha\rangle, \mathcal{F}_2] \in a \quad [\gamma, \mathcal{F}_3] \in a \quad [\beta_1, \mathcal{F}_4] \in a \quad \ldots [\beta_n, \mathcal{F}_{n+3}] \in a}{\Big[\big\langle \langle \alpha\,\beta\rangle \,\big\langle \langle \gamma\,\alpha\rangle\, \langle \gamma\,\beta\rangle \big\rangle \big\rangle, \mathcal{F}\Big] \in a}$$

Where $\beta = \langle \beta_1 \ldots \langle \beta_n\, q \rfloor$ and $\mathcal{F}$ is determined as follows:—

- $\mathcal{F}\left(\underline{\big\langle \langle \alpha\,\beta\rangle \,\big\langle \langle \gamma\,\alpha\rangle\, \langle \gamma\,\beta\rangle \big\rangle \big\rangle}\right)$ is the empty set,
- $\mathcal{F}|_{\underline{\langle \alpha\,\beta\rangle}}$ equals $\mathcal{F}_1$,
- $\mathcal{F}|_{\underline{\langle \gamma\,\alpha\rangle}}$ equals $\mathcal{F}_2$,
- $\mathcal{F}|_{\underline{\gamma}}$ equals $\mathcal{F}_3$,
- $\mathcal{F}|_{\underline{\beta_i}}$ equals $\mathcal{F}_{i+3}$,
- and the remaining located sublists have values determined by the above and the located sublist relation, using the union property required of support functions, since they are just the (consequent) superlists of the guard q.

(suff) **Condition**

$$\frac{[\langle \alpha\,\beta\rangle, \mathcal{F}_1] \in a \quad [\langle \beta\,\gamma\rangle, \mathcal{F}_2] \in a \quad [\alpha, \mathcal{F}_3] \in a \quad [\gamma_1, \mathcal{F}_4] \in a \quad \ldots [\gamma_n, \mathcal{F}_{n+3}] \in a}{\Big[\big\langle \langle \alpha\,\beta\rangle \,\big\langle \langle \beta\,\gamma\rangle\, \langle \alpha\,\gamma\rangle \big\rangle \big\rangle, \mathcal{F}\Big] \in a}$$

Where $\gamma = \langle \gamma_1 \ldots \langle \gamma_n\, q \rfloor$ and $\mathcal{F}$ is determined as follows:—

- $\mathcal{F}\left(\underline{\big\langle \langle \alpha\,\beta\rangle \,\big\langle \langle \beta\,\gamma\rangle\, \langle \alpha\,\gamma\rangle \big\rangle \big\rangle}\right)$ is the empty set,
- $\mathcal{F}|_{\underline{\langle \alpha\,\beta\rangle}}$ equals $\mathcal{F}_1$,
- $\mathcal{F}|_{\underline{\langle \beta\,\gamma\rangle}}$ equals $\mathcal{F}_2$,
- $\mathcal{F}|_{\underline{\alpha}}$ equals $\mathcal{F}_3$,
- $\mathcal{F}|_{\underline{\gamma_i}}$ equals $\mathcal{F}_{i+3}$,
- and the remaining located sublists have values determined by the above.

[1] Here I use the usual simpler '$\in$' notation.

Definition 6.1.3 An *assignment* I on a model structure M with initial valuation v is a function

$$I : K \times \{\text{list} - \text{pairs } [\alpha, \mathcal{F}]\} \rightarrow \{0, 1\}$$

defined as follows:—

- If $v(a, [\alpha, \mathcal{F}]) = 1$ then $I(a, [\alpha, \mathcal{F}]) = 1$.
- $(\rightarrow)$ introduction:—

  $$[\langle \ldots A \rightarrow B \ldots \rangle, \mathcal{F}] \in a$$

  where $A \rightarrow B$ is a constituent, if, for each image-point b of the support function $\mathcal{F}$ bSa, and for every a' such that aSa' and $[A, \mathcal{H}] \in a'$ we have:—

  $$[\langle \ldots \langle A\, B \rangle \ldots \rangle, \mathcal{F}'] \in a'$$

  Where for sublists δ independent or superlists of $\underline{\langle A\, B \rangle}$ $\mathcal{F}'(\underline{\delta'}) = \mathcal{F}(\underline{\delta})$ with $\delta' = \delta\,[\underline{A \rightarrow B} \backslash \langle A\, B \rangle]$, $\mathcal{F}'(\underline{A}) = \mathcal{H}(\underline{A})$ and the remaining consequent list $\underline{B}$ has $\mathcal{F}'(\underline{B}) = \mathcal{H}(\underline{A}) \cup \mathcal{F}'\left(\underline{\langle A\, B \rangle}\right) = \mathcal{H}(\underline{A}) \cup \mathcal{F}(\underline{A \rightarrow B})$.

Definition 6.1.4 A *model* $\langle M, I \rangle$ is a model structure $M = \langle o, \perp, K, S \rangle$ together with an assignment I on M.

6.2 Properties of the Models

In this section I show that the assignment is closed under the extension of the atomic list conditions to arbitrary lists. This property is crucial to the soundness proof as can be seen in the proof of validity of prefixing (lemma 6.3.3).

Lemma 6.2.1 The Hereditary Property holds:—

If in a model $\langle M, I\rangle$ $[\alpha, \mathcal{F}] \in a$ and aSb, then $[\alpha, \mathcal{F}] \in b$.

Proof

The proof is by induction on the degree of lists.

Base: the initial assignment satisfies the property by stipulation.

Assume the property holds for lists of degree $\leq n$ and suppose we have $[\langle \ldots A{\rightarrow}B \ldots\rangle, \mathcal{F}] \in a$ of degree $n+1$ in virtue of the $(\rightarrow)$ clause with respect to $\underline{A{\rightarrow}B}$, so for each image-point c of the support function $\mathcal{F}$, cSa, and for every a' such that aSa' and $[A, \mathcal{H}] \in a'$ we have $[\langle \ldots \langle A\, B\rangle \ldots\rangle, \mathcal{F}'] \in a'$ as per the $(\rightarrow)$ clause. Consider $b \in K$ such that aSb. First, all the tip-images c of $\mathcal{F}$ satisfy cSb by transitivity of the accessibility relation. And for every b' such that bSb' and $[A, \mathcal{H}] \in b'$, then since aSb' we have $[\langle \ldots \langle A\, B\rangle \ldots\rangle, \mathcal{F}'] \in b'$ by the induction hypothesis. And so the requirements are fulfilled for $[\langle \ldots A{\rightarrow}B \ldots\rangle, \mathcal{F}] \in b$. This completes the inductive step and so the proof.

Lemma 6.2.2 The Images Property holds:—

If $[\alpha, \mathcal{F}] \in a$ then for every image-point b of $\mathcal{F}$, bSa.

Proof

The proof is by induction on the assignment of lists to points.

The initial valuation satisfies the property by stipulation.

The property is preserved since it is just part of the $(\rightarrow)$ assignment clause.

So by induction the property holds of all lists assigned to points.

I now prove the corresponding invertibility result, which allows us to suppose, given $[\alpha, \mathcal{F}] \in a$ with a constituent $\underline{A{\rightarrow}B}$, that the list was assigned in virtue of *that* $\underline{A{\rightarrow}B}$; so that independent $\rightarrow$'s can be introduced in any order.

Lemma 6.2.3 If $[\langle \ldots A{\rightarrow}B \ldots\rangle, \mathcal{F}] \in a$ (where $\underline{A{\rightarrow}B}$ is a constituent), then for each image-point b of the support function $\mathcal{F}$, bSa, and for every a' such that aSa' and

$[A,\mathcal{H}] \in a'$, we have

$$[\langle \ldots \langle A\,B \rangle \ldots \rangle, \mathcal{F}'] \ \in a'$$

of correct form as per the $(\rightarrow)$ clause.

That is the converse of the $(\rightarrow)$ clause holds.

Proof

Suppose such a list-pair holds at some point in some model. Then, backtracking on the assignment of lists to points we reach the stages where the "ancestral" $\underline{A \rightarrow B}$ is introduced. I prove by deductive induction from these assignments that for ancestral assignments containing an ancestral $\underline{A \rightarrow B}$, $[\beta, \mathcal{N}] \in b$, the $(\rightarrow)$ clause holds w.r.t. the ancestral $\underline{A \rightarrow B}$.

Now in each case the image-points part of the clause holds since the list *does* hold at the point, and applying the Images Property (lemma 6.2.2).

So we just need to show for such assignments that for every b' such that bSb' and $[A,\mathcal{H}] \in b'$, the corresponding $[\beta', \mathcal{N}'] \in b'$ as per the $(\rightarrow)$ clause.

Note that these ancestral lists have the same "shape" as the conclusion $[\alpha, \mathcal{F}] \in a$, that is each $\beta^* = \alpha^*$.

Base is the initial ancestors where the lists hold in virtue of $(\rightarrow)$ w.r.t. the ancestral $\underline{A \rightarrow B}$. So the property holds of these.

So consider such $[\beta, \mathcal{N}] \in b$ assigned to a point by $(\rightarrow)$ in respect of some other $\underline{C \rightarrow D}$. Note that $\underline{A \rightarrow B}$ and $\underline{C \rightarrow D}$ must be independent. For arbitrary $b^\dagger$ such that $bSb^\dagger$ and $[C,\mathcal{J}] \in b^\dagger$ we have $[\beta^\dagger, \mathcal{N}^\dagger] \in b^\dagger$. And by the inductive hypothesis for arbitrary further $b^{\dagger\prime}$ such that $b^\dagger S b^{\dagger\prime}$ and $[A,\mathcal{H}] \in b^{\dagger\prime}$ we have the appropriate $[\beta^{\dagger\prime}, \mathcal{N}^{\dagger\prime}] \in b^{\dagger\prime}$.

Now for arbitrary b' such that bSb' and $[A,\mathcal{H}] \in b'$, for arbitrary further $b'^\dagger$ such that $b'Sb'^\dagger$ and $[C,\mathcal{J}] \in b'^\dagger$ we have $[\beta^{\dagger\prime}, \mathcal{N}^{\dagger\prime}] \in b'^\dagger$, since such $b'^\dagger = b^{\dagger\prime}$ (so to speak). But also $\mathcal{N}^{\dagger\prime} = \mathcal{N}'^\dagger$ (the order of modification of the support function makes no difference), so we have $[\beta', \mathcal{N}'] \in b'$ by $(\rightarrow)$ on $\underline{C \rightarrow D}$ (and the fact that all the tip-images of $\mathcal{N}'$ are those of $\mathcal{N}$ and $\mathcal{H}$, and lemma 6.2.2).

So we have the inductive step, whence the proof.

Hence in particular for every a' such that aSa' and $[A, \mathcal{H}] \in a'$, we have $[\langle \ldots \langle A\, B \rangle \ldots \rangle, \mathcal{F}'] \in a'$ as required.

Lemma 6.2.4 The *(pref)* Property holds: The interpretation is closed under unrestricted *(pref)*.

As does the corresponding *(suff)* Property.

Proof

The proof is by induction on the sum of the degrees of the "premiss" lists.

As above, the base case is ensured by the *(pref)* Condition on the initial valuation.

So suppose we have the "premisses" in virtue of an application of $(\rightarrow)$ to some $\underline{A \rightarrow B}$:—

$$[\langle \alpha\, \beta \rangle, \mathcal{F}_1] \in a \quad [\langle \gamma\, \alpha \rangle, \mathcal{F}_2] \in a \quad [\gamma, \mathcal{F}_3] \in a \quad [\beta_1, \mathcal{F}_4] \in a \quad \ldots [\beta_n, \mathcal{F}_{n+3}] \in a$$

Note that by the Images Property all image-points b of the desired *(pref)* conclusion (just those of the premisses) satisfy bSa. In every a' such that aSa' and $[A, \mathcal{H}] \in a'$ we have the corresponding "opened" list holding with $\underline{A \rightarrow B}$ replaced by $\langle A\, B \rangle$, etc.. The other premisses also hold at such a' by the Hereditary Property (lemma 6.2.1). There are two cases:—

1. The $\underline{A \rightarrow B}$ is not the guard of $\underline{\beta}$. Exactly one other copy has to be "opened" to get the correct form for applying the rule. So consider arbitrary a'' such that $a'Sa''$ and $[A, \mathcal{N}] \in a''$. By the Hereditary Property and Invertibility Property (lemma 6.2.3) we have:—

   $[\langle \alpha'\, \beta' \rangle, \mathcal{F}_1'] \in a'' \quad [\langle \gamma'\, \alpha' \rangle, \mathcal{F}_2'] \in a'' \quad [\gamma', \mathcal{F}_3'] \in a'' \quad [\beta_1', \mathcal{F}_4'] \in a'' \ldots [\beta_n', \mathcal{F}_{n+3}'] \in a''$

   With both $\underline{A \rightarrow B}$'s "opened", one with $\mathcal{H}$ and the other with $\mathcal{N}$. So we can apply (by inductive hypothesis) *(pref)* to obtain

   $$\left[\left\langle \langle \alpha\, \beta \rangle \left\langle \langle \gamma\, \alpha \rangle \langle \gamma\, \beta \rangle \right\rangle \right\rangle'', \mathcal{F}'' \right] \in a''$$

Which is as needed for

$$\left[\left\langle\langle\alpha\,\beta\rangle\,\left\langle\langle\gamma\,\alpha\rangle\,\langle\gamma\,\beta\rangle\right\rangle\right\rangle',\mathcal{F}'\right] \in a'$$

"filling in" the "opened" $\underline{\langle A\,B\rangle}$ which has $\mathcal{F}''(\underline{A}) = \mathcal{N}(\underline{A})$ using the $(\rightarrow)$ clause. (The tip-images satisfy the extra condition since they all correspond to tip-images of either $\mathcal{F}$ or $\mathcal{H}$ (and using lemma 6.2.2).) And this is what we needed for

$$\left[\left\langle\langle\alpha\,\beta\rangle\,\left\langle\langle\gamma\,\alpha\rangle\,\langle\gamma\,\beta\rangle\right\rangle\right\rangle,\mathcal{F}\right] \in a$$

"filling in" the other to obtain the required list-pair.

2. $\underline{A{\rightarrow}B}$ is the guard of $\underline{\beta}$. In this case the extra copy $A{\rightarrow}B$ of the corresponding constituent in the desired "conclusion" doesn't figure in the "premisses". Consider arbitrary a'' such that $a'Sa''$ and $[A,\mathcal{N}] \in a''$. In this case to apply *(pref)* we add an extra premiss corresponding to the extra (last) major antecedent list $\underline{A}$ of $\beta' = \beta\,[\underline{A{\rightarrow}B}\backslash\langle A\,B\rangle]$:—

   $[\langle\alpha\,\beta'\rangle,\mathcal{F}_1'] \in a''$ $\;[\langle\gamma\,\alpha\rangle,\mathcal{F}_2] \in a''$ $\;[\gamma,\mathcal{F}_3] \in a''$ $\;[\beta_1,\mathcal{F}_4] \in a''$
   $\ldots[\beta_n,\mathcal{F}_{n+3}] \in a''$ $\;[A,\mathcal{N}] \in a''$

   Where $\beta' = \langle\beta_1\,\langle\beta_2\ldots\langle\beta_n\,\langle A\,B\rfloor$. So again we can apply (by the induction hypothesis) *(pref)* to obtain

$$\left[\left\langle\langle\alpha\,\beta'\rangle\,\left\langle\langle\gamma\,\alpha\rangle\,\langle\gamma\,\beta'\rangle\right\rangle\right\rangle,\mathcal{F}''\right] \in a''$$

   And we can "fill in" the $\underline{A{\rightarrow}B}$'s as in the first case to obtain

$$\left[\left\langle\langle\alpha\,\beta\rangle\,\left\langle\langle\gamma\,\alpha\rangle\,\langle\gamma\,\beta\rangle\right\rangle\right\rangle,\mathcal{F}\right] \in a$$

Whence by induction the property holds. The *(suff)* Property is proved similarly.

Lemma 6.2.5 The *(splice)* Property holds:—
The interpretation is closed under unrestricted *(splice)*.

Proof

The proof is by induction on the sum of the degrees of the "premiss" lists.

Base: The initial valuation satisfies the property by stipulation.

Assume the property holds for sum of premiss degrees $\leq n$ and suppose we have such "premisses" with sum of degrees equal to $n+1$, so in virtue of an application of the $(\rightarrow)$ clause with respect to $\underline{A \rightarrow B}$ say:—

$$[\alpha, \mathcal{F}] \in a \quad [\delta, \mathcal{H}] \in a$$

Where δ contains a designated antecedent list $\underline{\alpha}$. In every a' such that aSa' and $[A, \mathcal{N}] \in a'$ we have the corresponding "opened" list holding with $\underline{A \rightarrow B}$ replaced by $\langle A\, B \rangle$. And if the $\underline{A \rightarrow B}$ was in the first premiss α or in the designated list $\underline{\alpha}$ of the second premiss, we also have the other copy "opened" too by Invertibility (lemma 6.2.3). So we have

$$[\alpha', \mathcal{F}'] \in a' \quad [\delta', \mathcal{H}'] \in a'$$

(Where $[\alpha', \mathcal{F}'] = [\alpha, \mathcal{F}]$ if the $\underline{A \rightarrow B}$ was in the second premiss but not the designated splice list.) Applying our induction hypothesis we obtain the corresponding spliced conclusion

$$[\delta', \mathcal{H}'^{\dagger}] \in a'$$

And the order of modification of $\mathcal{H}$ can be permuted $(\mathcal{H}'^{\dagger} = \mathcal{H}^{\dagger\prime})$ so we can apply $(\rightarrow)$ to obtain the required *(splice)* conclusion $[\delta, \mathcal{H}^{\dagger}] \in a$, completing the induction step and so the proof of the property.

Lemma 6.2.6 The Witness Property holds:—

For arbitrary α, $I(\perp, [\alpha, \mathcal{F}]) = 1$ where $\mathcal{F}$ is some support function on α.

Proof

By induction on the degree of α.

The base case is ensured by the condition on initial valuations.

So assume the property holds for lists of degree $\leq n$, and consider $\alpha = \langle \ldots A \rightarrow B \ldots \rangle$ of degree $n+1$. Then by hypothesis for $\alpha' = \langle \ldots \langle A\, B \rangle \ldots \rangle$ we have $I(\perp, [\alpha', \mathcal{F}']) = 1$, for some support function $\mathcal{F}'$. I show that $I(\perp, [\alpha, \mathcal{F}]) = 1$ where $\mathcal{F}\left(\underline{\varphi}\right) = \mathcal{F}'\left(\underline{\varphi'}\right)$ for $\varphi' = \varphi[\underline{A \rightarrow B} \backslash \langle A\, B \rangle]$. Clearly the first part of the needed $(\rightarrow)$ clause holds, for whatever the image-points of $\mathcal{F}$ are, they can "see" $\perp$. And for $\perp'$ such that $\perp S \perp'$ with $[A, \mathcal{K}] \in \perp'$ we also have $[\alpha', \mathcal{F}'] \in \perp'$ using the Hereditary Property. The required (for application of the $(\rightarrow)$ clause) $[\alpha', \mathcal{F}''] \in \perp'$ is obtained by applying the *(splice)* Property using splice list $\underline{A}$, to replace $\mathcal{F}'(\underline{A})$ by $\mathcal{K}(\underline{A})$. So $I(\perp, [\alpha, \mathcal{F}]) = 1$ by the $(\rightarrow)$ clause, completing the induction step.

And thus by induction the lemma is proved.

Lemma 6.2.7 The Sufficiency Property holds:—

If $I(a, [\alpha, \mathcal{F}]) = 1$ and for some located sublist $\underline{\varphi}$ of α and some point $b \in K$, if cSb for every element c of all images of $\mathcal{F}|_{\underline{\varphi}}$ then $I(b, \left[\varphi, \mathcal{F}|_{\underline{\varphi}}\right]) = 1$.

Proof

By induction on the degree of α.

The Initial Valuation satisfies the property by stipulation.

Let $[\langle \ldots A \rightarrow B \ldots \rangle, \mathcal{F}] \in a$ in virtue of that $\underline{A \rightarrow B}$. Suppose that for some sublist $\underline{\varphi}$ and some $b \in K$, every point c contained in the images of $\mathcal{F}|_{\underline{\varphi}}$ satisfies cSb. To show the sublist holds at b consider the two possible cases:—

1. $\underline{\varphi}$ doesn't contain the $\underline{A \rightarrow B}$.

   By the Witness Property for some $\mathcal{I}$ we have $[A, \mathcal{I}] \in \perp$ and so by $(\rightarrow)$ $[\langle \ldots \langle A\, B \rangle \ldots \rangle, \mathcal{F}'] \in \perp$. Now the corresponding $\underline{\varphi}$ is contained in this list and moreover $\mathcal{F}'|_{\underline{\varphi}} = \mathcal{F}|_{\underline{\varphi}}$. And by the induction hypothesis we have $\left[\varphi, \mathcal{F}'|_{\underline{\varphi}}\right] \in b$, which is to say $\left[\varphi, \mathcal{F}|_{\underline{\varphi}}\right] \in b$.[2]

2. $\underline{\varphi}$ contains the $\underline{A \rightarrow B}$.

   I show that the desired instance of the $(\rightarrow)$ clause for $\mathcal{F}|_{\underline{\varphi}}$ is satisfied. The image-points requirement is satisfied in virtue of our hypothesis concerning b. Consider arbitrary b' such that bSb' and $[A, \mathcal{H}] \in b'$. Now $aS\bot$ and by the Hereditary Property $[A, \mathcal{H}] \in \bot$ and so the appropriate $[\langle \ldots \langle A\, B\rangle \ldots\rangle, \mathcal{F}'] \in \bot$, applying the $(\rightarrow)$ clause. Putting $\varphi' = \varphi\,[\underline{A \rightarrow B} \backslash \langle A\, B\rangle]$, all of the image-points of $\mathcal{F}'|_{\underline{\varphi'}}$ are those of $\mathcal{F}|_{\underline{\varphi}}$ plus those of $\mathcal{H}$. By the Images Property all image-points of $\mathcal{H}$ "see" b' and those of $\mathcal{F}|_{\underline{\varphi}}$ "see" b' since they "see" b. Thus all image-points of $\mathcal{F}'|_{\underline{\varphi'}}$ "see" b', and applying the induction hypothesis we have $\left[\varphi', \mathcal{F}'|_{\underline{\varphi'}}\right] \in b'$. So the $(\rightarrow)$ clause is fulfilled for $\left[\varphi, \mathcal{F}|_{\underline{\varphi}}\right] \in b$ as is required, completing the induction step whence the proof.

6.3 Validity and Soundness

Definition 6.3.1 Let $\langle M, I\rangle$ be a model and let a be a point of the model structure, then $[\alpha, \mathcal{F}]$ holds at a, denoted $\models_a [\alpha, \mathcal{F}]$, iff $I(a, [\alpha, \mathcal{F}]) = 1$;
$[\alpha, \mathcal{F}]$ holds in the model $\langle M, I\rangle$, denoted $\models_{\langle M, I\rangle} [\alpha, \mathcal{F}]$, iff for all $a \in K$ we have $\models_a [\alpha, \mathcal{F}]$;
$[\alpha, \mathcal{F}]$ holds in the model structure M, denoted $\models_M [\alpha, \mathcal{F}]$, iff for all interpretations I on M we have $\models_{\langle M, I\rangle} [\alpha, \mathcal{F}]$;
and $[\alpha, \mathcal{F}]$ is valid, denoted $\models [\alpha, \mathcal{F}]$, iff for all model structures M $\models_M [\alpha, \mathcal{F}]$.

I now show that the class of valid wff includes **S**. (Throughout this section and the next, $\models$ indicates validity with respect to our models for **S**.) We need to check that the axioms

[2] This is the only place where the Witness Property, and so *(splice)* Property, is used. This could be diluted by restricting *(splice)* to just those points in the equivalence class of $\bot$. The corresponding consecution systems (B.2) get by without any *(splice)* analogue.

hold, and that validity is closed under modus ponens.

Lemma 6.3.2 If $\models [A,\mathcal{F}]$ and $\models [A\to B,\mathcal{K}]$, then $\models [B,\mathcal{N}]$ for some $\mathcal{N}$.

Proof

It suffices to show that if $\models_a [A,\mathcal{F}]$ and $\models_a [A\to B,\mathcal{K}]$ in an arbitrary model, then $\models_a [B,\mathcal{N}]$.

So suppose the antecedent holds. Then by the assignment clause $(\to)$ and reflexivity of S, we must have $\models_a [\langle A\,B\rangle,\mathcal{K}']$ with $\mathcal{K}'$ as per the $(\to)$ clause. But by the Images Property every image-point of $\mathcal{K}'|_{\underline{B}}$ "sees" a. So by the Sufficiency Property (lemma 6.2.7) we have $\models_a \left[B,\mathcal{K}'|_{\underline{B}}\right]$, giving the required result.

Lemma 6.3.3 The prefixing and suffixing axioms are valid.

Proof

Consider an arbitrary point a of an arbitrary model, to show:—

$$[A\to B\to .C\to A\to .C\to B,\mathcal{I}]\in a$$

Where $\mathcal{I}(\underline{A\to B\to .C\to A\to .C\to B})=\{\}$.

Consider an arbitrary point a_1 with aSa_1 and $[A\to B,\mathcal{K}_1]\in a_1$, to show:—

$$[\langle A\to B\; C\to A\to .C\to B\rangle,\mathcal{I}_1]\in a_1$$

As per the $(\to)$ clause.

Repeating, for arbitrary a_2 with a_1Sa_2 and $[C\to A,\mathcal{K}_2]\in a_2$, to show:—

$$[\langle A\to B\,\langle C\to A\; C\to B\rangle\rangle,\mathcal{I}_2]\in a_2$$

Again for arbitrary a_3 with a_2Sa_3 and $[C,\mathcal{K}_3]\in a_3$, to show:—

$$[\langle A \to B\, \langle C \to A\, \langle C\, B\rangle\rangle\rangle, \mathcal{I}_3] \in a_3$$

And for arbitrary a_4 with $a_3 S a_4$ and $[C, \mathcal{K}_4] \in a_4$, to show:—

$$[\langle A \to B\, \langle\langle C\, A\rangle\, \langle C\, B\rangle\rangle\rangle, \mathcal{I}_4] \in a_4$$

And finally for arbitrary a_5 with $a_4 S a_5$ and $[A, \mathcal{K}_5] \in a_5$, to show:—

$$[\big\langle\langle A\, B\rangle\, \big\langle\langle C\, A\rangle\, \langle C\, B\rangle\big\rangle\big\rangle, \mathcal{I}_5] \in a_5$$

But by $(\to)$ we must have, using $[C, \mathcal{K}_4] \in a_4$

$$[\langle C\, A\rangle, \mathcal{K}_2{}'] \in a_4$$

And using $[A, \mathcal{K}_5] \in a_5$ we have

$$[\langle A\, B\rangle, \mathcal{K}_1'] \in a_5$$

Thus we have the "premisses" for an application of the *(pref)* Property (lemma 6.2.4) (and using the Hereditary Property again):—

$$\frac{[\langle A\, B\rangle, \mathcal{K}_1'] \in a_5 \qquad [\langle C\, A\rangle, \mathcal{K}_2{}'] \in a_5 \qquad [C, \mathcal{K}_3] \in a_5}{[\big\langle\langle A\, B\rangle\, \big\langle\langle C\, A\rangle\, \langle C\, B\rangle\big\rangle\big\rangle, \mathcal{F}] \in a_5}$$

That the resultant support function is indeed $\mathcal{I}_5$ is obvious, keeping in mind that the value of $\mathcal{F}\left(\underline{\big\langle\langle A\, B\rangle\, \big\langle\langle C\, A\rangle\, \langle C\, B\rangle\big\rangle\big\rangle}\right)$ is the empty set and the remaining values are determined by the corresponding premiss-values.

Hence

$\models [A \to B \to .C \to A \to .C \to B, \mathcal{I}]$.

Similarly using the *(suff)* property we can prove

$\models [A \to B \to .B \to C \to .A \to C, \mathcal{I}]$

Theorem 6.3.4 Soundness

If $\vdash_S A$ then $\models [A, \mathcal{I}]$ where $\mathcal{I}(\underline{A}) = \{\}$.

Proof

By lemmas 6.3.2 and 6.3.3.

6.4 Semantic Completeness

As usual the proof that the notion of validity doesn't outgrow our logic is via a canonical model. But in order to do this we need a syntactic correlate of a list-pair holding at a point.

Definition 6.4.1 Let $\mathbf{T}$ be a theory of a logic $\mathbf{L}$, using the traditional notion of a theory—contains the axioms of $\mathbf{L}$ and is closed under the rule (modus ponens). Then a list-pair $[\alpha, \mathcal{F}]$ *holds at* $\mathbf{T}$, denoted $\vdash_{\mathrm{T}} [\alpha, \mathcal{F}]$, iff all image-points of $\mathcal{F}$ are theories contained within $\mathbf{T}$, and for every sublist $\underline{\varphi}$ with $\mathcal{F}\left(\underline{\varphi}\right) = \{\mathbf{T_1}, \mathbf{T_2}, \ldots \mathbf{T_n}\}$, for every $\mathbf{T}'$ containing all $\mathbf{T_i}$ $1 \leq i \leq n$ we have $\vdash_{\mathbf{T}'} \varphi^*$.

Note that since a support function is determined by the values assigned to the whole list and just the antecedent sublists, it would do just as well to make this restriction in the above definition.

For each sublist $\langle\beta_1 \langle\beta_2 \ldots \langle\beta_m \gamma\rfloor$ of a list in a list-pair which holds at a theory, the corresponding "similarly shaped" proof can be done in the theory beginning with major premiss

$$\langle\beta_1 \langle\beta_2 \ldots \langle\beta_m \gamma\rfloor^* = \beta_1{}^* \rightarrow . \beta_2{}^* \rightarrow . \ \ldots \beta_m{}^* \rightarrow \gamma^*$$

and peeling off each $\beta_i{}^*$ using modus ponens.

We are now in a position to define the canonical model.

Definition 6.4.2 The *canonical model for* S has model structure $M = \langle o, \perp, K, S \rangle$ where K is the set of theories of $\mathbf{S}$, o is $\mathbf{S}$, $\perp$ is the trivial theory and the accessibility relation S is just inclusion $\subseteq$. I denote this model structure by $\langle \mathbf{S}, \perp, K, \subseteq \rangle$.
And the interpretation is determined by the initial valuation which has

$$v(\mathbf{T}, [\alpha, \mathcal{F}]) = 1 \quad \text{iff} \quad \vdash_{\mathbf{T}} [\alpha, \mathcal{F}]$$

for all atomic lists α.

We check that the canonical model is well-defined:—

Lemma 6.4.3 The above-defined structure is a model.

Proof

Clearly $\langle \mathbf{S}, \perp, K, \subseteq \rangle$ is a model structure (definition 6.1.1).
So it remains to check that v is an initial valuation (definition 6.1.2).
The Images Condition is ensured by the restriction on image-theories in definition 6.4.1.
The Hereditary Condition and Sufficiency Condition are obviously satisfied.
The Witness Condition is satisfied since the trivial theory contains every wff.
The *(splice)* Condition is satisfied since the new set assigned to each antecedent sublist and the whole list is equal to that assigned by one of the "premisses", ensuring the required property of them, and this suffices to ensure it holds of the remaining (consequent) sublists using a simple induction on depth.
In the case of the *(pref)* and *(suff)* Conditions the new set assigned to each antecedent sublist is equal to that assigned by one of the "premisses". The new conclusion list with image { } has *wff an instance of either prefixing or suffixing which holds at every S-theory. So the antecedent sublists and whole list of the "conclusion" satisfy the required property whence as above they all do. Thus all of the conditions required for an initial valuation are satisfied, completing the proof of the lemma.

We need to show that the syntactic and semantic assignment of lists to theories correspond.

Lemma 6.4.4 A list holds at a theory iff it is assigned to the theory by the interpretation function of the canonical model:—

$$\vdash_{\mathbf{T}} [\alpha, \mathcal{F}] \text{ iff } \models_{\mathbf{T}} [\alpha, \mathcal{F}]$$

Proof

The proof is by induction on the degree of the list α.

By definition, the property holds of the atomic lists assigned by the canonical initial valuation.

Assume the property holds for lists of degree $\leq n$ and consider α of degree $n+1$.

If Suppose $I(\mathbf{T}, [\alpha, \mathcal{F}]) = 1$.

So the $(\rightarrow)$ clause holds for some constituent $\underline{A \rightarrow B}$: all image-points of $\mathcal{F}$ are subtheories of $\mathbf{T}$, and for every theory $\mathbf{T}'$ such that $\mathbf{T} \subseteq \mathbf{T}'$ and $I(\mathbf{T}', [A, \mathcal{K}]) = 1$, we have the appropriate $I(\mathbf{T}', [\alpha', \mathcal{F}']) = 1$. That is, by inductive hypothesis, for every theory $\mathbf{T}'$ such that $\mathbf{T} \subseteq \mathbf{T}'$ and $\vdash_{\mathbf{T}'} [A, \mathcal{K}]$, we have $\vdash_{\mathbf{T}'} [\alpha', \mathcal{F}']$.

To show that $\vdash_{\mathbf{T}} [\alpha, \mathcal{F}]$.

Now by the $(\rightarrow)$ clause every sublist $\underline{\varphi}$ of α corresponds to $\underline{\varphi'}$ of α' $(\varphi' = \varphi[\underline{A \rightarrow B} \backslash \langle A\ B \rangle]$ if appropriate) with the same image set. So the second requirement of definition 6.4.1 is satisfied, and since all the image-points are subtheories of $\mathbf{T}$ the first requirement is too, so that $\vdash_{\mathbf{T}} [\alpha, \mathcal{F}]$.

Only If Suppose that $\vdash_{\mathbf{T}} [\alpha, \mathcal{F}]$.

To show that $I(\mathbf{T}, [\alpha, \mathcal{F}]) = 1$.

Do so by showing that the $(\rightarrow)$ clause holds for an arbitrary constituent $\underline{A \rightarrow B}$. The image-point requirement is assured by the first part of definition 6.4.1. Suppose that $\mathbf{T} \subseteq \mathbf{T}'$ and $I(\mathbf{T}', [A, \mathcal{K}]) = 1$. By our induction hypothesis $\vdash_{\mathbf{T}'} [A, \mathcal{K}]$.

To show that $I(\mathrm{T}', [\alpha', \mathcal{F}']) = 1$ it suffices to show that $\vdash_{\mathrm{T}'} [\alpha', \mathcal{F}']$, again by the induction hypothesis. So we check that the requirements of definition 6.4.1 are satisfied. Now all image-theories of $\mathcal{F}'$ "see" T' since they are all image-theories of $\mathcal{F}$ or of $\mathcal{K}$. For sublists independent or superlists of $\underline{\langle A\, B\rangle}$ $\mathcal{F}'(\underline{\delta'}) = \mathcal{F}(\underline{\delta})$ with $\delta' = \delta\,[\underline{A \rightarrow B} \backslash \langle A\, B\rangle]$, so since $\vdash_{\mathrm{T}} [\alpha, \mathcal{F}]$ the second requirement obtains for these sublists of α' (noting that $\delta'^{*} = \delta^{*}$); $\mathcal{F}'(\underline{A}) = \mathcal{K}(\underline{A})$ so since $\vdash_{\mathrm{T}'} [A, \mathcal{K}]$ the second requirement also obtains for $\underline{A}$; and it obtains for the remaining consequent list $\underline{B}$ as $\mathcal{F}'(\underline{B}) = \mathcal{K}(\underline{A}) \cup \mathcal{F}(\underline{A \rightarrow B})$ so any supertheory of all $\mathcal{F}'(\underline{B})$ contains $A \rightarrow B$ and A. Thus we have $\vdash_{\mathrm{T}'} [\alpha', \mathcal{F}']$ and so the $(\rightarrow)$ clause is satisfied giving $I(\mathrm{T}, [\alpha, \mathcal{F}]) = 1$.

This completes the induction step, and so by induction the lemma is proved.

Completeness is a corollary of the above lemma:—

Theorem 6.4.5 If $\models [A, \mathcal{F}]$ then $\vdash_{\mathrm{S}} A$.

Proof

Suppose that $\models [A, \mathcal{F}]$, then in particular in the canonical model $I(\mathrm{S}, [A, \mathcal{F}]) = 1$ and by lemma 6.4.4 we have $\vdash_{\mathrm{S}} [A, \mathcal{F}]$. So in virtue of definition 6.4.1, $\vdash_{\mathrm{S}} A$.

Thus we have a Kripke-style binary relation semantics which captures S. However, unlike the Kripke semantics for intuitionist logic and modal logics, most of the work is done by features other than the accessibility relation (in particular the conditions upon an initial valuation).

6.5 Generalisation of the Semantics

In this section I show how to extend the semantics to cater for all logics in the $\rightarrow$ vocabulary with a Hilbert system comprising axioms and the rule modus ponens.

The procedure is to extend the semantics by adding atomic list conditions (corresponding to the *associated* rules of NGL (appendix B.2)) to the definition of a valuation on a model structure, with one corresponding to each axiom in a manner similar to the *(pref)* and *(suff)* conditions.

Definition 6.5.1 Let $A = A_1 \rightarrow . A_2 \rightarrow . \ldots A_n \rightarrow p$ be a wff. The *atomic list condition associated with the wff* A is:—

$$\frac{[\alpha_1, \mathcal{F}_1] \in a \quad [\alpha_2, \mathcal{F}_2] \in a \ \ldots [\alpha_n, \mathcal{F}_n] \in a \quad [\beta_1, \mathcal{F}_{n+1}] \in a \ \ldots [\beta_m, \mathcal{F}_{n+m}] \in a}{[\langle \alpha_1 \langle \alpha_2 \ldots \langle \alpha_n \beta \rfloor, \mathcal{F}] \in a}$$

Where $\alpha = \langle \alpha_1 \ldots \langle \alpha_n \beta \rfloor$ is a list formed by uniformly replacing all constituents of $\alpha^\dagger$ by atomic lists, where $\alpha^\dagger$ is the atomic list with $\alpha^{\dagger *} = A$, and $\beta = \langle \beta_1 \ldots \langle \beta_m \, q \rfloor$ is the list substituted for the guard of $\alpha^\dagger$.

And $\mathcal{F}$ is determined by:—

$\mathcal{F}(\underline{\alpha}) = \{\}$

$\mathcal{F}|_{\underline{\alpha_i}} = \mathcal{F}_i$

$\mathcal{F}|_{\underline{\beta_i}} = \mathcal{F}_{n+i}$

Let **L** be a logic (in the implication vocabulary) formulated with a hereafter fixed set of axioms and the rule modus ponens. An **L**-*initial valuation* on a model structure M is defined as earlier (definition 6.1.2) but without the *(pref)* and *(suff)* conditions, and with each *condition associated with* each axiom of **L**. An assignment is defined exactly as before (definition 6.1.3), as is a model, and the notions of validity.

That the Hereditary Property, Images Property, Invertibility, *(splice)* Property, Witness Property and Sufficiency Property hold for **L**-models is proved exactly as before.

To show that the Properties corresponding to each *associated condition* are satisfied by **L**-interpretations involves only a slight modification of the earlier *(pref)* case: The induction step is almost the same as that of lemma 6.2.4, except that in each case a differing number of copies of $A \rightarrow B$ may need to be "opened" depending on how many are in substituands of the same propositional variable of that **L** axiom. This involves taking the appropriate number of further arbitrary extensions of the point a.

That the class of **L**-valid wff is closed under modus ponens is proved exactly as in lemma 6.3.2.

The proof that each axiom A is valid is exactly according to the recipe of lemma 6.3.3.

The above two observations establish Soundness.

The canonical **L**-model is defined exactly as before (in section 6.4). The proof of well-definedness is exactly along the lines of the earlier proof for **S**, as is the proof that a list holds at a theory iff it is assigned to the theory by the interpretation function of the canonical model (lemma 6.4.4). And so Completeness holds.

Thus the **L**-models do indeed capture **L**, and we have a very general binary relation semantics for all implication logics.

While these semantics have a high degree of generality, they are also obviously purely formal. Presumably any justification of the *(pref)* Condition (say) on shapes of proofs amounts to a justification of the corresponding axiom in the Hilbert system. However the formal representation of proofs in the semantics does allow the definition of conditions upon the initial valuation which capture *use* as a subproof in an irredundant proof. In this way intuitive "worlds" semantics can be provided for relevant implication connectives. I show how to do this in the next section.

6.6 Binary Relation Semantics For Relevant Implication

In this chapter I describe a two-valued binary relation semantics for relevant implication which are a modification of those already presented, with list conditions which capture the various senses of *use* in place of the *(pref)* and *(suff)* conditions. We use the formulation of Mingle given in definition 2.7.2.

Model structures are defined as before, and the definition of an assignment is similar:—

Definition 6.6.1 An *initial valuation* on a model structure M is a function

$$v : K \times \mathcal{A} \to \{0,1\}$$

Where $\mathcal{A}$ is the set of list-pairs $[\alpha, \mathcal{F}]$ with α an atomic list. And satisfying:—

Images Condition As before.

Hereditary Condition As before.

Sufficiency Condition As before.

Witness Condition As before.

(splice) **Condition** As before.

(Id) **Condition** $\dfrac{[\alpha, \mathcal{F}] \in a}{[\langle\alpha\,\alpha\rangle, \mathcal{F}'] \in a}$

Where $\mathcal{F}'$ is determined as follows:—

- $\mathcal{F}'\left(\underline{\langle\alpha\,\alpha\rangle}\right)$ is the empty set,
- $\mathcal{F}'|_{\underline{1st\,\alpha}}$ equals $\mathcal{F}$,
- $\mathcal{F}'|_{\underline{\alpha_i}}$ equals $\mathcal{F}|_{\underline{\alpha_i}}$ for the major antecedents of the second α,

- and the remaining located sublists have values determined by the above and the union property required of support functions.

(Use) **Condition**
$$\frac{[\langle\alpha_1 \ldots \langle\alpha_n\langle\beta\,\gamma\rfloor, \mathcal{F}_1] \in a \quad [\langle\delta_1 \ldots \langle\delta_m\,\beta\rfloor, \mathcal{F}_2] \in a}{[\langle\alpha_1 \ldots \langle\alpha_n\,\langle\delta_1 \ldots \langle\delta_m\,\gamma\rfloor, \mathcal{F}] \in a}$$

Where $\mathcal{F}_1|_{\underline{\beta}} = \mathcal{F}_2|_{\underline{\beta}}$ and with $\mathcal{F}$ determined by assigning to each antecedent list the value assigned to its ancestor and assigning to the whole list the union of the values of the (whole) "premiss" lists.

(Perm) **Condition**
$$\frac{[\langle\alpha_1 \ldots \langle\alpha_n\,\langle\beta\langle\gamma\,\delta\rfloor, \mathcal{F}] \in a}{[\langle\alpha_1 \ldots \langle\alpha_n\,\langle\gamma\langle\beta\,\delta\rfloor, \mathcal{F}'] \in a}$$

Where $\mathcal{F}'$ is determined by assigning to each antecedent list and the whole list the value assigned to its ancestor by $\mathcal{F}$.

(Con) **Condition**
$$\frac{[\langle\alpha_1 \ldots \langle\alpha_n\,\langle\beta\langle\beta\,\gamma\rfloor, \mathcal{F}] \in a}{[\langle\alpha_1 \ldots \langle\alpha_n\,\langle\beta\,\gamma\rfloor, \mathcal{F}'] \in a}$$

Where $\mathcal{F}|_{\underline{1st\,\beta}} = \mathcal{F}|_{\underline{2nd\,\beta}}$ and $\mathcal{F}'$ is determined by assigning to each antecedent list and the whole list the value assigned to its ancestor.

(Mingle) **Condition**
$$\frac{[\langle\alpha_1 \ldots \langle\alpha_n\,\langle\beta\,\gamma\rfloor, \mathcal{F}] \in a}{[\langle\alpha_1 \ldots \langle\alpha_n\,\langle\beta\,\langle\beta\,\gamma\rfloor, \mathcal{F}'] \in a}$$

Where $\mathcal{F}'$ is determined by assigning to each antecedent list and the whole list the value assigned to its ancestor.

The above new *use* conditions are simply the list analogues of the corresponding *use* rules (definition 2.7.1). Each is obviously motivated by the corresponding intuitive sense of *use*, as discussed there (with the *(Use)* Condition also depending on the idea behind splice, so that we can suppose that the second premiss proof of β can be taken as being used in the subproof of β in the first premiss).

An *assignment* I on a model structure M with initial valuation v is defined as before (definition 6.1.3), as is a *model* $\langle M, I\rangle$.

We check that the assignment is closed under the extension of the atomic list conditions to arbitrary lists.

That the Hereditary Property, Images Property, semantic invertibility, *(splice)* Property, Witness Property and Sufficiency Property hold are proved as before (lemmas 6.2.1, 6.2.2, 6.2.3, 6.2.5, 6.2.6 and 6.2.7).

Lemma 6.6.2 The *(Id)* Property holds: The interpretation is closed under unrestricted *(Id)*.

Proof

The proof is by induction on the degree of the "premiss" list.

The base case is ensured by the *(Id)* Condition on the initial valuation.

So suppose we have the "premiss" $[\alpha, \mathcal{F}] \in a$ in virtue of an application of $(\rightarrow)$ to some $\underline{A \rightarrow B}$. Note that by the Images Property all image-points b of the desired *(Id)* conclusion (just those of the premisses) satisfy bSa. In every a' such that aSa' and $[A, \mathcal{H}] \in a'$ we have the corresponding "opened" list holding $[\alpha', \mathcal{F}'] \in a'$ with $\underline{A \rightarrow B}$ replaced by $\langle A\, B \rangle$, etc.. So we can apply (by inductive hypothesis) *(Id)* to obtain

$$[\langle \alpha' \, \alpha' \rangle, \mathcal{F}''] \in \, a'$$

Now for an arbitrary further extension a'' such that $a'Sa''$ and $[A, \mathcal{N}] \in a''$ we have

$$[\langle \alpha' \, \alpha' \rangle, \mathcal{F}'''] \in \, a''$$

where the first (say) copy of A has $\mathcal{F}''(\underline{A})$ replaced by $\mathcal{N}(\underline{A})$, by the *(splice)* Property (lemma 6.2.5). So we have both $\underline{A \rightarrow B}$'s "opened", one with $\mathcal{H}$ and the other with $\mathcal{N}$, which is as needed for $[\langle \alpha \, \alpha \rangle, \mathcal{F}'] \in \, a$ applying the $(\rightarrow)$ clause twice.

Whence by induction the property holds.

The corresponding *(Perm)*, *(Con)* and *(Mingle)* Properties also hold and their proofs are exactly similar to the above. In the case of *(Mingle)* it may be necessary to use the *(splice)* Property as above where the $\underline{A \rightarrow B}$ occurs in the duplicated sublist.

Lemma 6.6.3 The *(Use)* Property holds:—

$$\frac{[\langle \alpha_1 \ldots \langle \alpha_n \langle \beta\, \gamma \rfloor, \mathcal{F}_1] \in a \quad [\langle \delta_1 \ldots \langle \delta_m\, \beta \rfloor, \mathcal{F}_2] \in a}{[\langle \alpha_1 \ldots \langle \alpha_n \langle \delta_1 \ldots \langle \delta_m\, \gamma \rfloor, \mathcal{F}] \in a}$$

for arbitrary degree lists, where $\mathcal{F}_1|_{\underline{\beta}} = \mathcal{F}_2|_{\underline{\beta}}$.

Proof

The proof is by induction on the sum of degrees of the premiss lists, similar to the *(Id)* case. Base case is assured by the *(Use)* Condition. Assume the property holds for "premisses" with sum of degrees $\leq n$ and consider a pair with sum of degrees $= n+1$. At least one of them holds in virtue of the $(\rightarrow)$ clause, with respect to $\underline{A \rightarrow B}$ say, possibly occurring in the $\underline{\beta}$. Thus for every a' such that aSa' and $[A, \mathcal{H}] \in a'$ we have the corresponding "opened" lists holding:

$$[\langle \alpha'_1 \ldots \langle \alpha'_n \langle \beta'\, \gamma' \rfloor, \mathcal{F}'_1] \in a' \quad [\langle \delta'_1 \ldots \langle \delta'_m\, \beta' \rfloor, \mathcal{F}'_2] \in a'$$

Where that $\underline{A \rightarrow B}$ is replaced by $\underline{\langle A\, B \rangle}$ with $\mathcal{F}'_i(\underline{A}) = \mathcal{H}(\underline{A})$, and if it is in the $\underline{\beta}$ the corresponding list of the other premiss is opened too in virtue of invertibility. These also satisfy the requirements for the application of the Property, so by induction hypothesis we have the corresponding conclusion:

$$[\langle \alpha'_1 \ldots \langle \alpha'_n \langle \delta'_1 \ldots \langle \delta'_m\, \gamma' \rfloor, \mathcal{F}'] \in a'$$

Now there are two cases:—

1. The $\underline{A \rightarrow B}$ was not in $\underline{\beta}$.

   So the above list contains the corresponding $\underline{\langle A\, B \rangle}$, also all of the image-points of the required $\mathcal{F}$ see a since they are those of $\mathcal{F}_1$ and $\mathcal{F}_2$, so by the $(\rightarrow)$ clause we have the required

$$[\langle\alpha_1 \ldots \langle\alpha_n \langle\delta_1 \ldots \langle\delta_m \gamma\rfloor, \mathcal{F}] \in a$$

2. The $\underline{A \rightarrow B}$ was in $\underline{\beta}$.

   In this case we cannot use the $(\rightarrow)$ clause, since the copies of $\underline{\langle A\, B\rangle}$ are lost and we in fact have

$$[\langle\alpha_1 \ldots \langle\alpha_n \langle\delta_1 \ldots \langle\delta_m \gamma\rfloor, \mathcal{F}] \in a'$$

   But by the Witness Property such an a' exists, $a' = \perp$, and as above all of the image-points of $\mathcal{F}$ see a, so by the Sufficiency Property we have the required

$$[\langle\alpha_1 \ldots \langle\alpha_n \langle\delta_1 \ldots \langle\delta_m \gamma\rfloor, \mathcal{F}] \in a$$

Whence by induction the Property holds.

Validity is defined as before (definition 6.3.1). To show that the class of valid wff includes the system Mingle we need to check that the axiom holds, and that validity is closed under the "use" rules, as well as modus ponens. Closure under modus ponens is proved as before (lemma 6.3.2).

Lemma 6.6.4 The identity axiom is valid.

Proof

For an arbitrary point a of an arbitrary model, we show that $[A \rightarrow A, \mathcal{I}] \in a$ where $\mathcal{I}(\underline{A \rightarrow A}) = \{\}$. So consider a_1 with aSa_1 and $[A, \mathcal{H}] \in a_1$. We have

$$[\langle A\, A\rangle, \mathcal{I}_1] \in a_1$$

Where $\mathcal{I}_1(\underline{A}) = \mathcal{H}(\underline{A})$ for the first $\underline{A}$, and $\mathcal{I}_1\left(\underline{\langle A\, A\rangle}\right) = \{\}$, by the *(Id)* Property (lemma 6.6.2). But this establishes the needed $(\rightarrow)$ clause so that $[A \rightarrow A, \mathcal{I}] \in a$.

Lemma 6.6.5 The semantic rules corresponding to *(Perm)*, *(Con)* and *(Mingle)* hold:—

- If $\models_{Mingle} [A_1 \rightarrow . \ldots A_n \rightarrow .B \rightarrow .C \rightarrow D, \mathcal{I}]$ then $\models_{Mingle} [A_1 \rightarrow . \ldots A_n \rightarrow .C \rightarrow .B \rightarrow D, \mathcal{I}']$.
- If $\models_{Mingle} [A_1 \rightarrow . \ldots A_n \rightarrow .B \rightarrow .B \rightarrow C, \mathcal{I}]$ then $\models_{Mingle} [A_1 \rightarrow . \ldots A_n \rightarrow .B \rightarrow C, \mathcal{I}']$.
- If $\models_{Mingle} [A_1 \rightarrow . \ldots A_n \rightarrow .B \rightarrow C, \mathcal{I}]$ then $\models_{Mingle} [A_1 \rightarrow . \ldots A_n \rightarrow .B \rightarrow .B \rightarrow C, \mathcal{I}']$.

where in each case the value of the new list equals that of its "premiss" which has value $\{\}$.

Proof

I prove the *(Perm)* case, the other proofs being similar.

So suppose that some $[A_1 \rightarrow . \ldots A_n \rightarrow .B \rightarrow .C \rightarrow D, \mathcal{F}] \in a$. Consider arbitrary a_1 such that aSa_1 and $[A_1, \mathcal{H}_1] \in a_1$. We have $[\langle A_1\, A_2 \rightarrow . \ldots A_n \rightarrow .B \rightarrow .C \rightarrow D \rangle, \mathcal{F}_1] \in a_1$ by the $(\rightarrow)$ clause and invertibility.

Consider arbitrary a_2 such that a_1Sa_2 and $[A_2, \mathcal{H}_2] \in a_2$.

We have $[\langle A_1\, \langle A_2\, A_3 \rightarrow . \ldots A_n \rightarrow .B \rightarrow .C \rightarrow D \rangle \rangle, \mathcal{F}_2] \in a_2$.

Repeat this procedure for each A_i but reversing the order of B and C.

So eventually we are considering a point a_{n+2} with

$$[\langle A_1\, \langle A_2 \ldots \langle A_n\, \langle B\, \langle C\, D \rfloor, \mathcal{F}_{n+2}] \in a_{n+2}$$

and by the *(Perm)* Property we have

$$[\langle A_1\, \langle A_2 \ldots \langle A_n\, \langle C\, \langle B\, D \rfloor, \mathcal{F}'_{n+2}] \in a_{n+2}$$

So we can "fill in" each antecedent. That is, the $(\rightarrow)$ clause is satisfied for

$$[\langle A_1\, \langle A_2 \ldots \langle A_n\, \langle C\, B \rightarrow D \rfloor, \mathcal{F}'_{n+1}] \in a_{n+1}$$

Repeating this procedure we eventually get back to

$$[A_1 \rightarrow \ldots . A_n \rightarrow .C \rightarrow .B \rightarrow D, \mathcal{F}'] \in a$$

where $\mathcal{F}' \left(\underline{A_1 \rightarrow \ldots . A_n \rightarrow .C \rightarrow .B \rightarrow D}\right) = \mathcal{F} \left(\underline{A_1 \rightarrow \ldots . A_n \rightarrow .B \rightarrow .C \rightarrow D}\right)$ since the value assigned to the whole list remains unchanged through all of the above transformations. Thus if $\models_a [A_1 \rightarrow \ldots . A_n \rightarrow .B \rightarrow .C \rightarrow D, \mathcal{F}]$ then $\models_a [A_1 \rightarrow \ldots . A_n \rightarrow .C \rightarrow .B \rightarrow D, \mathcal{F}']$ in an arbitrary model, whence the required result follows.

The other cases are proved exactly similarly.

Lemma 6.6.6 The *(Use)* rule holds semantically:—

If $\models_{Mingle} [A_1 \rightarrow \ldots . A_n \rightarrow .B \rightarrow C, \mathcal{I}_1]$ and $\models_{Mingle} [D_1 \rightarrow \ldots . D_m \rightarrow B, \mathcal{I}_2]$ then $\models_{Mingle} [A_1 \rightarrow \ldots . A_n \rightarrow . D_1 \rightarrow \ldots . D_m \rightarrow C, \mathcal{I}]$.

Proof

Suppose that in some arbitrary model $[A_1 \rightarrow \ldots . A_n \rightarrow .B \rightarrow C, \mathcal{F}_1] \in a$ and $[D_1 \rightarrow \ldots . D_m \rightarrow B, \mathcal{F}_2] \in a$.

Apply the same procedure as in the previous lemma in the order $A_1 \ldots A_n, D_1 \ldots D_m$, eventually obtaining a point a_{n+m} with

$$[\langle A_1 \ldots \langle A_n \, B \rightarrow C \rfloor, \mathcal{F}_1{}'] \in a_{n+m}$$

and

$$[\langle D_1 \ldots \langle D_m \, B \rfloor, \mathcal{F}_2{}'] \in a_{n+m}$$

Now by the Sufficiency Property we have $[B, \mathcal{F}_2{}'|_B] \in a_{n+m}$. Hence by the invertibility of $(\rightarrow)$ we also have

$$[\langle A_1 \ldots \langle A_n \, \langle B \, C \rangle \rfloor, \mathcal{F}_1{}''] \in a_{n+m}$$

where the value of B is $\mathcal{F}'_2 (\underline{B})$. Thus we can apply the *(Use)* Property to obtain

$$[(A_1 \ldots (A_n\,(D_1 \ldots (D_m\,C], \mathcal{F}''] \in a$$

And then "fill in" the antecedent lists in reverse order applying the $(\rightarrow)$ clause to obtain the required

$$[A_1 \rightarrow \ldots . A_n \rightarrow . D_1 \rightarrow \ldots . D_m \rightarrow C, \mathcal{F}'] \in a_{n+m}$$

Thus if $\models_a [A_1 \rightarrow \ldots . A_n \rightarrow . B \rightarrow C, \mathcal{I}_1]$ and $\models_a [D_1 \rightarrow \ldots . D_m \rightarrow B, \mathcal{I}_2]$ then $\models_a [A_1 \rightarrow \ldots . A_n \rightarrow . D_1 \rightarrow \ldots . D_m \rightarrow C, \mathcal{I}]$, and so the result follows.

Theorem 6.6.7 Soundness

If $\vdash_{\text{Mingle}} A$ then $\models_{\text{Mingle}} [A, \mathcal{I}]$ where $\mathcal{I}(\underline{A}) = \{\}$.

Proof

By lemmas 6.3.2, 6.6.4, 6.6.5 and 6.6.6.

Completeness is proved as before, using the same definition of a list-pair *holding at* a theory (definition 6.4.1).

Definition 6.6.8 The *canonical model for Mingle* has model structure $M = \langle o, \bot, K, S \rangle$ where K is the set of theories of Mingle, o is Mingle, $\bot$ is the trivial theory and the accessibility relation S is just inclusion $\subseteq$. I denote this model structure by $\langle \mathbf{L}, \bot, K, \subseteq \rangle$. And the interpretation is determined by the initial valuation which has

$$v(\mathrm{T}, [\alpha, \mathcal{F}]) = 1 \quad \text{iff} \quad \vdash_{\mathrm{T}} [\alpha, \mathcal{F}]$$

for all atomic lists α.

The canonical model is well-defined:—

Lemma 6.6.9 The above-defined structure is a model.

Proof

Clearly $\langle \mathrm{L}, \perp, K, \subseteq \rangle$ is a model structure (definition 6.1.1).

So it remains to check that v is an initial valuation (definition 6.6.1).

The Images Condition, Hereditary Condition, Sufficiency Condition, Witness Condition and *(splice)* Condition are satisfied as before (lemma 6.4.3).

In the case of the *(Id)* Condition the set of points assigned to each antecedent sublist is equal to that assigned by the "premiss", so since the required property of definition 6.4.1 holds of it, these lists also satisfy it. The new conclusion list with image { } has *wff an instance of identity which holds in Mingle. So the antecedent sublists and whole list of the "conclusion" satisfy the required property whence they all do.

In the case of the *(Perm)* Condition the new set of points assigned to each antecedent sublist is equal to that assigned by the "premiss" function, so as above these satisfy the required property. The new whole list has set of points equal to that of the "premiss" and so satisfies the property, since by the Perm Rule of the logic every theory containing the one contains the other.

The *(Con)* and *(Mingle)* Conditions are shown to hold exactly as above.

The *(Use)* Condition is similar— the antecedent lists are treated as above, and the "conclusion" whole list has set of images the union of those of the "premisses", so the property holds since by the Use Rule of the logic every theory containing the "premisses" contains the "conclusion".

That the syntactic and semantic assignment of lists to theories correspond is proved exactly as before (lemma 6.4.4). From which completeness is immediate:—

Theorem 6.6.10 If $\models_{\mathbf{Mingle}} [A, \mathcal{F}]$ then $\vdash_{\mathbf{Mingle}} A$.

Note that one can use any subset of the *(Id)*, *(Perm)*, etc. conditions to capture the corresponding sublogics of Mingle. So, as already indicated, we can capture different senses of "use" corresponding to discounting a proof as using itself, or requiring the order of use of subproofs to be preserved, or not counting multiple use of a subproof as single use, or single use as multiple use. That the so-modified semantics captures the corresponding sublogic of Mingle is obvious, since there is no interplay between the different *use* conditions in the above.

The semantics presented here go quite a way towards providing an explanatory semantics for relevant implication. The initial valuation conditions are well-motivated as capturing "use", together with other properties of atomic proofs, and the ($\rightarrow$) assignment clause is just the appropriate adaptation of the usual one. However too much is packed into the initial valuation for comfort. The remaining step needed to provide intuitive, "ground up" semantics of this type for relevant implication connectives is to show how to generate an initial valuation from an atomic language where one has clauses in the definition of an assignment instead of the *(Id)*, *(Perm)*, etc. conditions upon the initial valuation, and where the sufficiency condition holds as a closure property, rather than being imposed upon the assignment. For these reasons I feel that the relevant consecution systems of chapter 5 provide a better explanatory semantics than the above "simplified", "worlds" semantics.[3]

[3]Another reason is that it seems the former can be modified so as to make splice a closure property. But this involves treating the list-forming operation as another connective and so permitting $\rightarrow$'s between lists, which is best left to another time.

Chapter 7

Afterword

The work reported in this thesis shows that intuitive semantics, in the formalist, constructivist tradition, can be provided for relevant implication. Furthermore it shows that formal semantics in the constructivist tradition, as well as relational ("Kripke-style") semantics, can be provided for *every* implication logic.

This has been achieved by further developing the Curry formalist perspective. According to Curry the meaning of the logical connectives, our theory of logic, is grounded in the properties of formal systems. Thus a compound statement is deemed to be, upon recursive decomposition, a statement of a property of arbitrary formal systems. Curry calls this theory of logic the *epitheory* of formal systems.

Taking account of the "internal" properties of formal systems has enabled the inclusion of facts about the shapes of proofs in this epitheory, providing the basis for the above-mentioned results. In particular it has permitted an objective characterisation of the notion of *use* as "use as a subproof of a proof", which has provided the intuitive basis for the semantics for relevant implication.

In this thesis two ways for capturing "proof-descriptions" have been given, list-triples and the simpler list-pairs. In the further development of this work it seems that the list-triples are to be prefered. For it is possible, by fully incorporating the list-forming operation into the language as another binary connective, to obtain $(\|splice)$ for the relevant consecution systems as a closure property. But in any case I feel that the full hedge picture as provided by the list-triples tells you what's really going on; this is highlighted by the interpretation lemma for the "simplified" consecution systems where the structure of a list as an identity hedge is used.

Finally let me mention some avenues for the further development of this research.

The obvious one is the application of these methods to other logical connectives. This seems straightforward for conjunction and disjunction, but negation seems to involve other issues, such as how best to capture denial— the dual of assertion— in formal systems.

As already noted $(\|splice)$ can be obtained as a closure property of the relevant consecution systems by modifying $(\|\rightarrow)$ to allow the simultaneous filling in of siblings. However this requires fully introducing the list-forming operator into the language as another binary operator.

Another project stems from the feeling that a more natural theory of the internal structure of formal systems could be obtained by using proof-descriptions which don't have as their basic constituent a *binary* operator. The binary feature of lists is imposed by the aim of capturing a binary implication operator. However if one puts that aside and just considers how best to characterise atomic proofs, it seems that a generalized implication operator which permits multiple antecedents— something more like a rule— is more appropriate. This idea seems to be supported by the importance of fusion in understanding the structure of relevant logics, for such a generalised implication could be considered akin to a fusion of antecedent wffs implying the consequent. But this really is something to be left until another time.

Appendix A

The Crude Systems

I define the "crude" consecution systems and state some facts concerning them. The corresponding Kripke style semantics are defined in the obvious way, so that their tableaux representation turned upside-down is the consecution system. These systems have the benefit of a less complex succedent— its just a list with no extra support tree structure. However the $(\Vdash\rightarrow)$ is more complicated, involving multiple premisses to make sure that we don't get something from nothing. Dr. Meyer showed that the strongest of these systems in fact captures absolute implication. But this is not true of the weaker systems, which may be of interest as they give yet more constructive implication connectives. I won't show the attendant proof of positive paradox here, but I nevertheless claim it is the longest irredundant extant proof of it, and that the consecution system is the most complex formulation of absolute implication anyone has ever been unhappy enough to devise.

Definition A.0.11 The *Crude Consecution Systems* are defined:—

Axioms $\quad \alpha \Vdash \alpha \quad$ where α is atomic

Extensional contraction $(W\|)\ \dfrac{\Gamma,\ \ \alpha,\alpha\ \ \Vdash\ \beta}{\Gamma,\ \ \ \alpha\ \ \Vdash\ \beta}$

Extensional permutation $(C\|)\ \dfrac{\Gamma\ \Vdash\ \beta}{\Gamma'\ \Vdash\ \beta}$

Extensional thinning $(K\|)\ \dfrac{\Gamma\ \Vdash\ \beta}{\Gamma,\alpha\ \Vdash\ \beta}$

Intensional thinning $(\|iK)\ \dfrac{\Gamma\ \Vdash\ \langle\alpha\,\beta\rangle}{\Gamma\ \Vdash\ \alpha}\qquad(\|iK)\ \dfrac{\Gamma\ \Vdash\ \langle\alpha\,\beta\rangle}{\Gamma\ \Vdash\ \beta}$

$$(\rightarrow\|)\ \frac{\Gamma_1\ \Vdash\ A\qquad\qquad \Gamma_2,\ \langle\ldots\langle A\,B\rangle\ldots\rangle\ \Vdash\ \delta}{\Gamma_1,\ \Gamma_2,\ \langle\ldots A\rightarrow B\ldots\rangle\ \Vdash\ \delta}$$

$$(\|\rightarrow)\frac{\Gamma\Vdash\delta_{i\,[1\le i\le n]}\qquad \Gamma,\ A\Vdash\langle\ldots\langle A\,B\rangle\ldots\rangle}{\Gamma\Vdash\langle\ldots A\rightarrow B\ldots\rangle}$$

Where $\{\delta_i : 1\le i\le n\}$ is the set of all proper sublists of the succedent of the conclusion. Thus the number of left-premisses varies.

With further rules any selection of:—

Prefixing $(\|pref)\dfrac{\Gamma_1\Vdash\langle\alpha\,\beta\rangle\qquad\Gamma_2\Vdash\langle\gamma\,\alpha\rangle}{\Gamma_1,\ \Gamma_2\Vdash\big\langle\langle\alpha\,\beta\rangle\,\big\langle\langle\gamma\,\alpha\rangle\,\langle\gamma\,\beta\rangle\big\rangle\big\rangle}$

Suffixing Similar to above.

Identity $\dfrac{\Gamma\ \Vdash\alpha}{\Gamma\ \Vdash\langle\alpha\,\alpha\rangle}$

Permutation $\dfrac{\Gamma\ \Vdash\langle\alpha\,\langle\beta\,\gamma\rangle\rangle}{\Gamma\ \Vdash\big\langle\langle\alpha\,\langle\beta\,\gamma\rangle\rangle\,\big\langle\beta\,\langle\alpha\,\gamma\rangle\big\rangle\big\rangle}$

Contraction $\dfrac{\Gamma\ \Vdash\langle\alpha\,\langle\alpha\,\beta\rangle\rangle}{\Gamma\ \Vdash\big\langle\langle\alpha\,\langle\alpha\,\beta\rangle\rangle\,\langle\alpha\,\beta\rangle\big\rangle}$

$$\textbf{Mingle} \qquad \frac{\Gamma \Vdash \langle \alpha\, \beta \rangle}{\Gamma \Vdash \big\langle \langle \alpha\, \beta \rangle \, \big\langle \alpha \, \langle \alpha\, \beta \rangle \big\rangle \big\rangle}$$

In the crude systems, because we are not keeping track of dependencies, $(\Vert iK)$ can get us into a lot of trouble. It can do much of the work of contraction, which is derivable if we have identity and permutation (just drop the α off!). Without the left-premisses for $(\Vert\!\rightarrow)$ we would get triviality, since the carried over A might still be needed to support a part of the sublist (subproof) of the succedent other than B.

The idea behind these systems was to try and capture a *necessary* condition for relevance, using the left-premiss device to prevent an explosion. For the formal interpretation associate with a list α the wff $\alpha_1{}^* \wedge \alpha_2{}^* \wedge \ldots \wedge \alpha_n{}^*$ where $\{\alpha_i : 1 \leq i \leq n\}$ is the set of all sublists of α.

These systems are closed under a rule I call 'Mangle', and Dr. Meyer showed that mangle plus R permits a proof of positive paradox. So the system with prefixing, suffixing, identity and permutation in fact captures absolute implication. After Hard Labour I managed to produce a direct proof of this fact. Mangle turned out to be very aptly named.

That the above systems are closed under *cut* is proved in the same manner as for **GS**, exploiting an invertibility lemma to save work on the left-upper cases.

Appendix B

The Simplified Consecution and Natural Deduction Systems

Here simplified forms of the consecution and natural deduction systems of chapters 3 and 4 are presented. The consecution systems are almost the upside-down tableaux systems of the corresponding binary relation two-valued semantics described in chapter 6. The simplification involves dispensing with hedges, so that a support function is defined only on the located sublists of a single list. This means that we no longer have the *total* proof-description, with all subproofs retained, and as a consequence it is not possible to use the strategy of chapter 4 to obtain closure of the natural deduction system under iK. Consequently the Lorenzen principle of inversion does not hold for these simplified systems due to the presence of iK, which in this context *is* an elimination rule because without the hedge structure the "dropped" antecedent list really is eliminated.

B.1 The Simplified Consecution Systems

Support function and *list-pair* are defined as in chapter 6, with the image-set I equal to the set of located sublists of the antecedent of a consecution.

Here a *consecution* has as antecedent a sequence of lists and as succedent a list-pair $[\alpha, \mathcal{F}]$ where $\mathcal{F}$ has as its range the power set of the set of located sublists of the antecedent of the consecution.

Definition B.1.1 The axioms and rules for the basic system **NGS** are as follows:—

Axioms $\alpha \Vdash [\alpha, \mathcal{I}]$

Where α is atomic, i.e. all its constituents are atomic sentences, and $\mathcal{I}$ is the "identity" support function on α where each antecedent sublist and α itself has the singleton set containing the corresponding located sublist of the antecedent α as its image.

$$(W\Vert)\ \frac{\Gamma,\ \alpha,\alpha\ \Vdash\ [\delta,\ \mathcal{F}]}{\Gamma,\ \alpha\ \Vdash\ [\delta,\ \mathcal{F}']}$$

Where $\mathcal{F}'$ is determined from $\mathcal{F}$ by replacing in each image a located sublist of one of the major α's by its descendant.

$$(C\Vert)\ \frac{\Gamma\ \Vdash\ [\delta,\ \mathcal{F}]}{\Gamma'\ \Vdash\ [\delta,\ \mathcal{F}]}$$

Where Γ' is any permutation of Γ, and the images follow their descendants.

$$(\Vert iK)\ \frac{\Gamma\ \Vdash\ [\langle\alpha\,\beta\rangle, \mathcal{F}]}{\Gamma\ \Vdash\ \left[\beta,\ \mathcal{F}|_{\underline{\beta}}\right]}$$

Here $\mathcal{F}|_{\underline{\beta}}$ is the restriction of $\mathcal{F}$ to the set of located sublists of β.

$$(\rightarrow\Vert)\quad \frac{\Gamma_1\ \Vdash\ [A, \mathcal{H}] \qquad\qquad \Gamma_2, \langle\ldots\langle A\,B\rangle\ldots\rangle\ \Vdash\ [\delta, \mathcal{F}]}{\Gamma_1, \Gamma_2, \langle\ldots A\rightarrow B\ldots\rangle\ \Vdash\ [\delta, \mathcal{F}']}$$

Where $\mathcal{F}'$ is formed by replacing *the* $\underline{A}$ by $\mathcal{H}(\underline{A})$, specifically:—

- In the case that $\underline{\varphi}$ has $\underline{A} \in \mathcal{F}\left(\underline{\varphi}\right)$,

$$\mathcal{F}'\left(\underline{\varphi}\right) = \left(\mathcal{F}\left(\underline{\varphi}\right)[\underline{\langle A\,B\rangle}\backslash A{\rightarrow}B] - \{\underline{A}\}\right) \cup \mathcal{H}(\underline{A})$$

that is the remaining whole-list descendants of $\mathcal{F}\left(\underline{\varphi}\right)$ union (the descendants of) $\mathcal{H}(\underline{A})$.

- Otherwise $\mathcal{F}'\left(\underline{\varphi}\right) = \mathcal{F}\left(\underline{\varphi}\right)[\underline{\langle A\,B\rangle}\backslash A{\rightarrow}B]$, that is the descendants of $\mathcal{F}\left(\underline{\varphi}\right)$.

$$(\|{\rightarrow})\ \frac{\Gamma,\ A \quad \Vdash \quad [\langle\ldots\langle A\,B\rangle\ldots\rangle,\ \mathcal{F}]}{\Gamma \quad \Vdash \quad [\langle\ldots A{\rightarrow}B\ldots\rangle,\ \mathcal{F}']}$$

- Where the succedent $\underline{A}$ of the premiss has image the singleton containing *that* antecedent $\underline{A}$ which is dropped to obtain the conclusion,
- the antecedent $\underline{A}$ is not in the image of any other antecedent sublist or the whole succedent list under $\mathcal{F}$,
- for $\underline{\varphi}$ of the conclusion succedent $\mathcal{F}'\left(\underline{\varphi}\right)$ equals (the descendants of) $\mathcal{F}\left(\underline{\varphi'}\right)$ where $\varphi' = \varphi\,[\underline{A{\rightarrow}B}\backslash\langle A\,B\rangle]$.

$(\|\mathit{pref})$

$$\frac{\Gamma_1 \Vdash [\langle\alpha,\beta\rangle,\mathcal{F}_1] \quad \Gamma_2 \Vdash [\langle\gamma,\alpha\rangle,\mathcal{F}_2] \quad \Gamma_3 \Vdash [\gamma,\mathcal{F}_3] \quad \Gamma_4 \Vdash [\beta_1,\mathcal{F}_4] \quad \ldots\Gamma_{n+3} \Vdash [\beta_n,\mathcal{F}_{n+3}]}{\Gamma_1,\ldots\Gamma_{n+3} \Vdash \left[\big\langle\langle\alpha\,\beta\rangle\,\big\langle\langle\gamma\,\alpha\rangle\,\langle\gamma\,\beta\rangle\big\rangle\big\rangle,\mathcal{F}\right]}$$

Where $\beta = \langle\beta_1\ldots\langle\beta_n\,B\rfloor$ and $\mathcal{F}$ is determined as follows:—

- $\underline{\big\langle\langle\alpha\,\beta\rangle\,\big\langle\langle\gamma\,\alpha\rangle\,\langle\gamma\,\beta\rangle\big\rangle\big\rangle}$ has the empty-set as image,
- $\mathcal{F}|_{\underline{\langle\alpha\,\beta\rangle}} = \mathcal{F}_1$,
- $\mathcal{F}|_{\underline{\langle\gamma\,\alpha\rangle}} = \mathcal{F}_2$,
- $\mathcal{F}|_{\underline{\gamma}} = \mathcal{F}_3$,
- $\mathcal{F}|_{\underline{\beta_i}} = \mathcal{F}_{i+3}$,
- and the remaining located sublists have values determined by the above (they are just the superlists of the guard B).

$(\|suff)$

$$\frac{\Gamma_1 \Vdash [\langle\alpha,\beta\rangle,\mathcal{F}_1] \quad \Gamma_2 \Vdash [\langle\beta,\gamma\rangle,\mathcal{F}_2] \quad \Gamma_3 \Vdash [\alpha,\mathcal{F}_3] \quad \Gamma_4 \Vdash [\gamma_1,\mathcal{F}_4] \quad \ldots\Gamma_{n+3} \Vdash [\gamma_n,\mathcal{F}_{n+3}]}{\Gamma_1,\ldots\Gamma_{n+3} \Vdash \left[\left\langle\langle\alpha\,\beta\rangle\left\langle\langle\beta\,\gamma\rangle\,\langle\alpha\,\gamma\rangle\right\rangle\right\rangle,\mathcal{F}\right]}$$

Where $\gamma = \langle\gamma_1 \ldots \langle\gamma_n\, C\rfloor$ and

- $\underline{\left\langle\langle\alpha\,\beta\rangle\left\langle\langle\beta\,\gamma\rangle\,\langle\alpha\,\gamma\rangle\right\rangle\right\rangle}$ has the empty-set as image,
- $\mathcal{F}\rceil_{\underline{\langle\alpha\,\beta\rangle}} = \mathcal{F}_1$,
- $\mathcal{F}\rceil_{\underline{\langle\beta\,\gamma\rangle}} = \mathcal{F}_2$,
- $\mathcal{F}\rceil_{\underline{\alpha}} = \mathcal{F}_3$,
- $\mathcal{F}\rceil_{\underline{\gamma_i}} = \mathcal{F}_{i+3}$,
- and the remaining located sublists have values determined by the above.

Closure under $(\|iK)$ cannot be obtained as in chapter 3; for in the $(\|\!\rightarrow)$ case of the closure argument the antecedent $\underline{A}$ cannot be removed where the $\underline{A\rightarrow B}$ is in the antecedent list required to be "dropped" and the induction hypothesis removes its ancestral $\underline{\langle A\,B\rangle}$.

That all identities (with identity support function) are provable is proved exactly as for lemma 3.2.8. That prefixing and suffixing are valid is also proved exactly as for **GS**.

As for **GS**, the intuitive idea behind the interpretation is that lists represent proofs (where $\langle\alpha\,\beta\rangle$ is a proof of description β which has a subproof of description α), and the support function images tell us which antecedent proofs underpin sublists (subproofs) of the succedent. The difference from **GS** is that a record of all of the proofs used is not retained; the enthymemes, corresponding to that part of the hedge of a list triple not in the first list, are dropped entirely.

The following sense of depth of a sublist is needed in the interpretation lemma.

Definition B.1.2 The *depth* of a sublist of a list α is defined:—

- The depth of the whole list α, and of every antecedent sublist, is zero,
- for a consequent sublist $\underline{\delta}$ in $\underline{\langle\gamma\,\delta\rangle}$, the *depth* of $\underline{\delta}$ is that of $\underline{\langle\gamma\,\delta\rangle}$ plus one.

This notion of depth just corresponds to depth on the identity *hedge* for the list. Although hedges have been dispensed with they underpin what is going on, so I feel that for the full picture the systems incorporating hedges are to be preferred.

The proof that **S** contains **NGS** is similar to that for **GS**, using the following interpretation lemma. As previously a *theory* is a set of wff containing the axioms of the logic and closed under the rule(s), i.e. a theory is taken to be "regular" and "detached" in the usual terminology.

Lemma B.1.3 If $\Gamma \Vdash_{\mathbf{NGS}} [\alpha, \mathcal{F}]$ then, for every located sublist $\underline{\varphi}$ of α and every theory **T** of the logic **S**, if for every element $\underline{\delta}$ of $\mathcal{F}\left(\underline{\varphi}\right)$ we have $\vdash_{\mathrm{T}} \delta^*$, then $\vdash_{\mathrm{T}} \varphi^*$.

Proof

We use the following fact as in lemma 5.0.7:—

If the above interpretation property holds for the succedent whole list α and every antecedent sublist of it then it holds for every sublist of the succedent.

The proof of this fact is by induction on the depth of $\underline{\varphi}$ in α, exactly as for lemma 3.3.1, using the closure of theories under modus ponens.

Thus it is only necessary to show that the interpretation property holds for the tips of the succedent hedge. The proof of this fact is by deductive induction:—

Axioms For the whole succedent list and antecedent sublists of it $\mathcal{F}\left(\underline{\varphi}\right) = \{\underline{\varphi}\}$, so the property holds.

$(W\|)$, $(C\|)$, $(\|iK)$ These obviously preserve the property, as the images for the conclusion contain the same lists as those for the premiss.

$$(\rightarrow\|)\quad \frac{\Gamma_1 \;\|\!\!-\; [A,\mathcal{H}] \qquad\qquad \Gamma_2, \langle \ldots \langle A\, B\rangle \ldots\rangle \;\|\!\!-\; [\delta, \mathcal{F}]}{\Gamma_1, \Gamma_2, \langle \ldots A \rightarrow B \ldots\rangle \;\|\!\!-\; [\delta, \mathcal{F}']}$$

For those succedent sublists whose ancestor doesn't contain the $\underline{A}$ in its image, the image is unchanged except possibly for replacing $\underline{\langle A\, B\rangle}$ by $\underline{A \rightarrow B}$, and since $A \rightarrow B^* = \langle A\, B\rangle^*$ the property is preserved for these.

For those succedent sublists φ whose ancestor contains the $\underline{A}$ in its image $\mathcal{F}'\left(\underline{\varphi}\right) = \left(\text{Descendentsof}\, \mathcal{F}\left(\underline{\varphi}\right)\right) \cup \mathcal{H}\left(\underline{A}\right)$. But then if a theory contains all the *wffs of the elements in $\mathcal{F}'\left(\underline{\varphi}\right)$, by the induction hypothesis it contains $A^* = A$ in virtue of the subset $\mathcal{H}\left(\underline{A}\right)$, and it also contains the *wffs of all the other members of $\mathcal{F}\left(\underline{\varphi}\right)$ (again using the fact that $A \rightarrow B^* = \langle A\, B\rangle^*$), and so by the induction hypothesis for the right premiss it contains φ^* as required.

$(\|\rightarrow)$ Here the property is obviously preserved, again using the fact that $A \rightarrow B^* = \langle A\, B\rangle^*$.

$(\|pref)$, $(\|suff)$ As before for lemma 3.3.1.

And we have as a corollary the interpretation theorem:—

Theorem B.1.4 If $\|\!\!-_{\mathbf{NGS}} [A, \mathcal{F}]$ then $\vdash_{\mathbf{S}} A$.

The proof that **NGS** contains **S** is similar to that for **GS**, using a version of *cut*, together with *invertibility*, to prove closure under modus ponens.

Definition B.1.5 *(mix)* is defined:—

$$(mix)\,\frac{\Gamma \;\|\!\!-\; [\alpha, \mathcal{K}] \qquad \Lambda \;\|\!\!-\; [\delta, \mathcal{F}]}{\Gamma,\, \Lambda^\circ \;\|\!\!-\; [\delta, \mathcal{F}^\circ]}$$

Where

- Λ contains at least one copy of α,
- Λ° is like Λ but with some (and at least one) of its copies of α deleted *but not necessarily all*; these deleted α's are called the *mix lists*,
- $\mathcal{F}^\circ$ is obtained from $\mathcal{F}$ and $\mathcal{K}$ in a manner similar to $(\rightarrow\|)$:—
  - For those $\underline{\varphi}$ with $\mathcal{F}\left(\underline{\varphi}\right)$ not containing any sublist of a mix list, $\mathcal{F}^\circ\left(\underline{\varphi}\right)$ contains just the descendants of $\mathcal{F}\left(\underline{\varphi}\right)$.
  - For each $\underline{\varphi}$ with image containing $\underline{\lambda}$ which occurs in a mix list, replace $\underline{\lambda}$ by (the descendants of) $\mathcal{K}(\underline{\lambda})$ for the corresponding $\underline{\lambda}$ of the left premiss; that is $\mathcal{F}^\circ\left(\underline{\varphi}\right)$ contains those descendants of $\mathcal{F}\left(\underline{\varphi}\right)$ which remain, union the descendants of $\mathcal{K}(\underline{\lambda})$ for each such lost $\underline{\lambda}$.

That closure under *(mix)* together with invertibility ensures closure under modus ponens is proved exactly as for **GS** (lemma 3.4.3), using $(\|iK)$.

Proof of the invertibility result is exactly like that of lemma 3.4.2; so I state it without proof.

Lemma B.1.6 If $\Gamma \Vdash [\langle \ldots A{\rightarrow}B \ldots\rangle, \mathcal{F}]$ where $A{\rightarrow}B$ is a constituent, then there is a deduction of this consecution in which that $\rightarrow$ is introduced last, that is with last step

$$(\|\rightarrow)\ \frac{\Gamma,\ A\ \Vdash [\langle\ldots\langle A\,B\rangle\ldots\rangle,\ \mathcal{F}']}{\Gamma\ \Vdash [\langle\ldots A{\rightarrow}B\ldots\rangle,\ \mathcal{F}]}$$

The proof of closure under *(mix)* is also virtually identical to the earlier proof (lemma 5.0.12). However it is not quite a transliteration so it is shown in full.

Lemma B.1.7 **GS** is closed under *(mix)*.

Proof

The proof is by a double induction, on the degree of the mix list α and on the right rank.

1. Right Rank = 1

1.1 Right upper an axiom

$$(mix)\ \frac{\Gamma \Vdash [\alpha, \mathcal{K}] \qquad \alpha \Vdash [\alpha, \mathcal{I}]}{\Gamma \Vdash [\alpha, \mathcal{I}^\circ]}$$

We check that the left upper is already equal to the conclusion. So we need to show that $\mathcal{K} = \mathcal{I}^\circ$. Each antecedent $\underline{\lambda}$ in the succedent α has $\mathcal{I}(\underline{\lambda})$ replaced by $\mathcal{K}(\underline{\lambda})$, as does the whole list α, but this ensures that $\mathcal{K} = \mathcal{I}^\circ$. Thus the required *(mix)* conclusion is indeed equal to the left upper.

1.2 Right upper from $(\rightarrow\Vert)$

$$\frac{\Gamma \Vdash [\alpha, \mathcal{K}'] \qquad \dfrac{\Lambda_1 \Vdash [A, \mathcal{N}] \qquad \Lambda_2, \langle \ldots \langle A\,B \rangle \ldots \rangle \Vdash [\delta, \mathcal{F}]}{\Lambda_1, \Lambda_2, \langle \ldots A \rightarrow B \ldots \rangle \Vdash [\delta, \mathcal{F}']}}{\Gamma,\ \Lambda_1,\ \Lambda_2 \Vdash [\delta,\ \mathcal{F}'^\circ]}$$

Where $\alpha = \langle \ldots A \rightarrow B \ldots \rangle$ and we put $\alpha' = \langle \ldots \langle A\,B \rangle \ldots \rangle$. By invertibility there is a deduction of the left upper doing *that* $\rightarrow$ introduction last, i.e. ending:—

$$(\Vert\rightarrow)\ \frac{\Gamma,\ A \Vdash [\langle \ldots \langle A\,B \rangle \ldots \rangle,\ \mathcal{K}]}{\Gamma \Vdash [\langle \ldots A \rightarrow B \ldots \rangle,\ \mathcal{K}']}$$

So by the degree induction hypothesis we can apply the following lower degree *(mix)*'s:—

$$(mix)\ \frac{(mix)\ \dfrac{\Lambda_1 \Vdash [A, \mathcal{N}] \qquad \Gamma,\ A \Vdash [\alpha', \mathcal{K}]}{\Lambda_1,\ \Gamma \Vdash [\alpha', \mathcal{K}^\circ]} \qquad \Lambda_2,\ \alpha' \Vdash [\delta, \mathcal{F}]}{\Lambda_1,\ \Gamma,\ \Lambda_2 \Vdash [\delta,\ \mathcal{F}^\circ]}$$

With a $(C\Vert)$ to get the antecedent right.

We need to check that we have obtained the correct modification of the support function. The required modifications to $\mathcal{F}$ are: For those sublists with image containing the antecedent A, A is replaced by $\mathcal{N}(\underline{A})$, and for those $\underline{\varphi}$ of the domain of $\mathcal{F}'$ with image containing a $\underline{\lambda}$ in the mix list α, this $\underline{\lambda}$ is replaced by $\mathcal{K}'(\underline{\lambda})$.

Now the first *(mix)* above replaces the image $\{A\}$ of the A in α' by $\mathcal{N}(\underline{A})$ and on the second *(mix)* every $\underline{\varphi}$ with image containing the antecedent A has it replaced by $\mathcal{N}(\underline{A})$, which is exactly the first of the above required modifications.

$\underline{\varphi}$ with image $\mathcal{F}\left(\underline{\varphi}\right)$ containing $\underline{\lambda}$ in α' other than $\underline{A}$ are just the ancestors of the corresponding φ with $\mathcal{F}'\left(\underline{\varphi}\right)$ containing $\underline{\lambda'} = \underline{\lambda}\left[\underline{\langle A\,B\rangle}\backslash A{\rightarrow}B\right]$. On the second *(mix)* each such $\underline{\lambda}$ is replaced by $\mathcal{K}^{\circ}(\underline{\lambda})$ and for such sublists $\mathcal{K}^{\circ}(\underline{\lambda}) = \mathcal{K}(\underline{\lambda}) = \mathcal{K}'(\underline{\lambda'})$, so we have exactly the second of the above required modifications.

2 Right Rank > 1

2.1 Right upper from $(W\|)$

$$\textit{(mix)}\ \frac{\Gamma \Vdash [\alpha, \mathcal{K}] \qquad (W\|)\,\dfrac{\Lambda,\ \ \beta, \beta \ \Vdash\ [\delta, \mathcal{F}]}{\Lambda,\ \ \beta \ \Vdash\ [\delta, \mathcal{F}']}}{\Gamma,\ \Lambda^{\circ},\ [\beta]\ \Vdash\ [\delta,\ \mathcal{F}'^{\circ}]}$$

Do the lower rank *(mix)* with the premiss of the right upper with mix lists the ancestors of those in the right upper, and then the corresponding $(W\|)$ if need be.

2.2 Right upper from $(C\|)$

$$\textit{(mix)}\ \frac{\Gamma \Vdash [\alpha, \mathcal{K}] \qquad (C\|)\,\dfrac{\Lambda \ \Vdash\ [\delta, \mathcal{F}]}{\Lambda' \ \Vdash\ [\delta, \mathcal{F}]}}{\Gamma,\ \Lambda'^{\circ}\ \Vdash\ [\delta,\ \mathcal{F}^{\circ}]}$$

Similarly to the above case, do the lower rank *(mix)* with the premiss of the right upper and then the corresponding $(C\|)$.

2.3 Right upper from $(\|iK)$

$$
(mix)\ \dfrac{\Gamma \Vdash [\alpha, \mathcal{K}] \qquad (\|iK)\ \dfrac{\Lambda \Vdash [\langle \delta\, \lambda\rangle, \mathcal{F}]}{\Lambda \Vdash \left[\lambda,\ \mathcal{F}\lfloor_{\underline{\lambda}}\right]}}{\Gamma,\ \Lambda^{\circ} \Vdash [\lambda,\ \mathcal{F}\lfloor_{\underline{\lambda}}{}^{\circ}]}
$$

Do the lower rank *(mix)* with the premiss of the right upper and then the corresponding $(\|iK)$. The support function of the right upper is equal to the restriction of its premiss, so the support functions are correctly modified.

2.4 Right upper from $(\|pref)$

$$
\dfrac{\Gamma \Vdash [\lambda, \mathcal{K}] \qquad \dfrac{\Lambda_1 \Vdash [\langle \alpha\beta\rangle, \mathcal{H}_1] \quad \Lambda_2 \Vdash [\langle \gamma\alpha\rangle, \mathcal{H}_2] \quad \Lambda_3 \Vdash [\gamma, \mathcal{H}_3] \quad \ldots \quad \Lambda_{n+3} \Vdash [\beta_n, \mathcal{H}_{n+3}]}{\Lambda_1, \ldots \Lambda_{n+3} \Vdash \left[\left\langle \langle \alpha\,\beta\rangle \left\langle \langle \gamma\,\alpha\rangle \langle \gamma\,\beta\rangle \right\rangle \right\rangle, \mathcal{H}\right]}}{\Gamma,\ \Lambda_1^{\circ},\ \Lambda_2^{\circ}, \ldots \Lambda_{n+3}^{\circ} \Vdash \left[\left\langle \langle \alpha\,\beta\rangle \left\langle \langle \gamma\,\alpha\rangle \langle \gamma\,\beta\rangle \right\rangle \right\rangle,\ \mathcal{H}^{\circ}\right]}
$$

Do the lower rank *(mix)*'s with the premisses and then apply the corresponding $(\|pref)$:—

$$
(mix)\ \dfrac{\Gamma \Vdash [\lambda, \mathcal{K}] \qquad \Lambda_1 \Vdash [\langle \alpha\,\beta\rangle, \mathcal{H}_1]}{\Gamma,\ \Lambda_1^{\circ} \Vdash [\langle \alpha\,\beta\rangle, \mathcal{H}_1^{\circ}]}
$$

...

$$
(mix)\ \dfrac{\Gamma \Vdash [\lambda, \mathcal{K}] \qquad \Lambda_{n+3} \Vdash [\beta_n, \mathcal{H}_{n+3}]}{\Gamma,\ \Lambda_{n+3}^{\circ} \Vdash [\beta_n, \mathcal{H}_{n+3}^{\circ}]}
$$

$$
\dfrac{\Gamma,\ \Lambda_1^{\circ} \Vdash [\langle \alpha\,\beta\rangle, \mathcal{H}_1^{\circ}] \quad \Gamma,\ \Lambda_2^{\circ} \Vdash [\langle \gamma\,\alpha\rangle, \mathcal{H}_2^{\circ}] \quad \Gamma,\ \Lambda_3^{\circ} \Vdash [\gamma, \mathcal{H}_3^{\circ}] \quad \ldots \quad \Gamma,\ \Lambda_{n+3}^{\circ} \Vdash [\beta_n, \mathcal{H}_{n+3}^{\circ}]}{\Gamma,\ \Lambda_1^{\circ},\ \Gamma,\ \Lambda_2^{\circ} \ldots \Gamma,\ \Lambda_{n+3}^{\circ} \Vdash \left[\left\langle \langle \alpha\,\beta\rangle \left\langle \langle \gamma\,\alpha\rangle \langle \gamma\,\beta\rangle \right\rangle \right\rangle,\ \mathcal{H}'\right]}
$$

And adjust the antecedent using $(C\|)$ and $(W\|)$.

An image-list of the conclusion of the $(\|pref)$ is in a mix list iff the corresponding ancestor is in the ancestral mix list, so we do have the required modification of the succedent support function.

2.5 Right upper from $(\Vert suff)$

Similar to the above case.

2.6 Right upper from $(\rightarrow\Vert)$

$$\frac{\Gamma \Vdash [\alpha, \mathcal{K}] \qquad (\rightarrow\Vert)\dfrac{\Lambda_1 \Vdash [A, \mathcal{N}] \qquad \Lambda_2, \langle \ldots \langle A\, B \rangle \ldots \rangle \Vdash [\delta, \mathcal{F}]}{\Lambda_1, \Lambda_2, \langle \ldots A \rightarrow B \ldots \rangle \Vdash [\delta, \mathcal{F}']}}{\Gamma,\ \Lambda_1^\circ,\ \Lambda_2^\circ,\ \langle \ldots A \rightarrow B \ldots \rangle^\circ \Vdash [\delta,\ \mathcal{F}'^\circ]}$$

Do the two lower rank *(mix)*'s, and then apply the corresponding $(\rightarrow\Vert)$:—

$$\frac{\dfrac{\Gamma \Vdash [\alpha, \mathcal{K}] \qquad \Lambda_1 \Vdash [A, \mathcal{N}]}{\Gamma,\ \Lambda_1^\circ \Vdash [A, \mathcal{N}^\circ]} \qquad \dfrac{\Gamma \Vdash [\alpha, \mathcal{K}] \qquad \Lambda_2, \langle \ldots \langle A\, B \rangle \ldots \rangle \Vdash [\delta, \mathcal{F}]}{\Gamma, \Lambda_2^\circ, \langle \ldots \langle A\, B \rangle \ldots \rangle \Vdash [\delta, \mathcal{F}^\circ]}}{\Gamma,\ \Lambda_1^\circ,\ \Gamma,\ \Lambda_2^\circ,\ \langle \ldots A \rightarrow B \ldots \rangle \Vdash [\delta,\ \mathcal{F}^{\circ\prime}]}$$

And adjust the antecedent using $(C\Vert)$ and $(W\Vert)$.

Now if $\langle \ldots A \rightarrow B \ldots \rangle$ is also a mix list (hence identical with α) we remove it by doing the further mix with only that list a mix list:—

$$(mix)\,\frac{\Gamma \Vdash [\alpha, \mathcal{K}] \qquad \Gamma,\ \Lambda_1^\circ,\ \Lambda_2^\circ,\ \langle \ldots A \rightarrow B \ldots \rangle \Vdash [\delta, \mathcal{F}^{\circ\prime}]}{\Gamma,\ \Gamma,\ \Lambda_1^\circ,\ \Lambda_2^\circ \Vdash [\delta, \mathcal{F}^{\circ\prime\circ}]}$$

With $(C\Vert)$'s and $(W\Vert)$'s as needed to adjust the antecedent.
For this mix the right rank is one, so strictly less than the original right rank, hence the rank is strictly lower and so it falls under our induction hypothesis.

It is necessary to check that the above provides the correct modification of the support function.

Consider $\mathcal{F}'\left(\underline{\varphi}\right)$ containing $\underline{\lambda}'$ in the antecedent Λ_2, $\langle\ldots A{\rightarrow}B\ldots\rangle$. Such a list has ancestor $\underline{\varphi}$ with $\mathcal{F}\left(\underline{\varphi}\right)$ containing $\underline{\lambda}$ the ancestor of $\underline{\lambda}'$ (in its place).

First suppose the $\underline{\lambda}'$ is not in a mix list. We require these to remain in the mix. In our modified deduction, the ancestors $\underline{\lambda}$ of such image-lists in Λ_2 are left undisturbed by the righthand mix and the various ensuing adjustments. And $\underline{\varphi}$ has image-list in $\langle\ldots A{\rightarrow}B\ldots\rangle$ iff its ancestor has corresponding image-list in $\langle\ldots\langle A\,B\rangle\ldots\rangle$, and these are left undisturbed when the $\langle\ldots A{\rightarrow}B\ldots\rangle$ isn't a mix list, to be replaced by the required sublist of same in the (in that case) final $(\rightarrow\|)$.

Suppose the $\underline{\lambda}'$ is in a mix list.

If in Λ_2 then $\lambda = \lambda'$ and the righthand *(mix)* replaces $\underline{\lambda}$ by $\mathcal{K}(\underline{\lambda})$ to obtain $\mathcal{F}^{\diamond}$, and the remaining steps preserve this change, which is as required.

If in $\langle\ldots A{\rightarrow}B\ldots\rangle$ then the ancestral $\underline{\lambda}$ in $\langle\ldots\langle A\,B\rangle\ldots\rangle$ is left undisturbed by the righthand *(mix)* and $\underline{\lambda}'$ itself is instated by the $(\rightarrow\|)$, and the required replacement of this by $\mathcal{K}(\underline{\lambda}')$ is made in the final *(mix)*, which change is preserved by the ensuing adjustments.

Consider a $\underline{\varphi}$ in the domain of $\mathcal{F}'$ with image-list $\underline{\lambda}$ in the antecedent Λ_1. Such a $\underline{\varphi}$ is part of an "added" $\mathcal{N}(\underline{A})$ originally with image-list the $\underline{A}$.

First suppose $\underline{\lambda}$ is not in a mix list. The lefthand mix leaves the ancestral $\underline{\lambda}$ undisturbed, and the $(\rightarrow\|)$ preserves this when adding the $\mathcal{N}^{\diamond}(\underline{A})$ in place of $\underline{A}$ to $\mathcal{F}^{\diamond}\left(\underline{\varphi}\right)$ to form $\mathcal{F}^{\diamond\prime}$, and this remains undisturbed by the final *(mix)* (if needed).

Suppose the $\underline{\lambda}$ is in a mix list. Then the lefthand mix replaces such image-list by $\mathcal{K}(\underline{\lambda})$ to obtain $\mathcal{N}^{\diamond}$, and this modification is preserved in the copies of this added in place of the image-list $\underline{A}$ by the $(\rightarrow\|)$, and such image-lists in the Γ remain undisturbed in the (if needed) final *(mix)*, the $(C\|)$'s, and get appropriately replaced in the $(W\|)$'s.

Thus we have the required succedent support function.

2.7 Right upper from $(\|\rightarrow)$

$$
(mix)\ \frac{\Gamma \Vdash [\alpha, \mathcal{K}] \qquad (\Vert\!\!\rightarrow)\,\dfrac{\Lambda,\, A \ \Vdash\ [\langle \ldots \langle A\, B\rangle \ldots\rangle,\, \mathcal{F}]}{\Lambda \ \Vdash\ [\langle \ldots A{\rightarrow}B \ldots\rangle,\, \mathcal{F}']}}{\Gamma,\, \Lambda^{\circ} \ \Vdash\ [\langle \ldots A{\rightarrow}B \ldots\rangle, \mathcal{F}'^{\circ}]}
$$

Do the *(mix)* on the premiss of the right upper first, and then apply the $(\Vert\!\!\rightarrow)$:—

$$
(\Vert\!\!\rightarrow)\ \frac{(mix)\,\dfrac{\Gamma \Vdash [\alpha, \mathcal{K}] \qquad \Lambda,\, A \Vdash [\delta', \mathcal{F}]}{\Gamma, \Lambda^{\circ}, A \Vdash [\delta', \mathcal{F}^{\circ}]}}{\Gamma,\, \Lambda^{\circ} \Vdash \left[\delta, \mathcal{F}^{\circ\prime}\right]}
$$

Where $\delta = \langle \ldots A{\rightarrow}B \ldots\rangle$ and δ' is its ancestor.

Note that each $\underline{\varphi}$ of the right upper with image-list $\underline{\lambda}$ in the antecedent has as ancestor $\underline{\varphi'}$ (possibly with $\langle A\, B\rangle$ in place of $A{\rightarrow}B$) with image-list the ancestral $\underline{\lambda}$ in Λ.

For such image-list $\underline{\lambda}$ in a mix list the above mix replaces it by $\mathcal{K}(\underline{\lambda})$, and then the $(\Vert\!\!\rightarrow)$ replaces the $\underline{\langle A\, B\rangle}$ by $\underline{A{\rightarrow}B}$ (possibly in $\underline{\varphi'}$), so we have the correct transformation.

For images-lists not in a mix list there is no change, as is required.

So the above proof modifies the support function correctly.

This completes the induction step of the proof, and so the lemma is proved.

And so we have the inclusion theorem:—

Theorem B.1.8 **NGS** contains **S**, i.e. if $\vdash_{\mathbf{S}} A$ then $\Vdash_{\mathbf{NGS}} [A, \mathcal{F}]$ (where $\mathcal{F}(\underline{A}) = \{\}$).

As before, there is a similar such **NGL** system for every implication logic **L** which has a Hilbert formulation with axioms and the rule modus ponens. The following definition provides a recipe for list rules corresponding to each axiom of a logic.

Definition B.1.9 Let A be a wff. Instances of the *list rule associated with*

$A = A_1 \rightarrow . A_2 \rightarrow . \ldots A_n \rightarrow p$ are:—

$$\frac{\Lambda_1 \Vdash [\alpha_1, \mathcal{F}_1] \ \ldots \ \Lambda_n \Vdash [\alpha_n, \mathcal{F}_n] \quad \Lambda_{n+1} \Vdash [\beta_1, \mathcal{F}_{n+1}] \ \ldots \ \Lambda_{n+m} \Vdash [\beta_m, \mathcal{F}_{n+m}]}{\Lambda_1, \ldots, \Lambda_{n+m} \Vdash [\langle \alpha_1 \ldots \langle \alpha_n \, \beta \rfloor, \mathcal{F}]}$$

Where $\alpha = \langle \alpha_1 \ldots \langle \alpha_n \, \beta \rfloor$ is a list formed by uniformly replacing all constituents of $\alpha^\dagger$ by lists, where $\alpha^\dagger$ is the atomic list with $\alpha^{\dagger *} = A$, and $\beta = \langle \beta_1 \ldots \langle \beta_m \, B \rfloor$ is substituted for p. The ancestor relation is defined: each located sublist $\underline{\varphi}$ of α, part of a list substituted for q in $\alpha^\dagger$, has as its ancestors *all the corresponding* φ*'s in the premisses which also correspond to substituands of* q*'s in* $\alpha^\dagger$. The whole list α has the empty-set as image and the values of $\mathcal{F}$ for the remaining elements are determined by the following recipe:—

- $\mathcal{F}\restriction_{\underline{\alpha_i}} = \mathcal{F}_i$
- $\mathcal{F}\restriction_{\underline{\beta_i}} = \mathcal{F}_{n+i}$

Define the general Gentzen systems, using $(W\Vert)$, $(C\Vert)$, $(\Vert iK)$, $(\rightarrow\Vert)$ and $(\Vert\rightarrow)$ as before:—

Definition B.1.10 Let **L** be a logic (in the implication vocabulary) formulated with axioms and the rule modus ponens. Then **NGL** (corresponding to a fixed axiomatic formulation) is the consecution system with:—

- Axioms as for **NGS** (definition 3.1.1).
- $(W\Vert)$ as for **NGS**.
- $(C\Vert)$ as for **NGS**.
- $(\Vert iK)$ as for **NGS**.

- $(\rightarrow\|)$ as for NGS.
- $(\|\rightarrow)$ as for NGS.
- For each axiom A of L, the list rule associated with A.

The proof that NGL captures L has the same structure as that for GS and S, with much of the detail exactly the same.

The proof of the interpretation lemma is exactly like that of lemma 3.3.1, but replacing the $(\|pref)$ and $(\|suff)$ cases by the (similarly proved) list rules associated with axioms cases. I state the interpretation theorem:—

Theorem B.1.11 If $\|\!\vdash_{\mathbf{NGL}} [A,\ \mathcal{F}]$ then $\vdash_{\mathrm{L}} A$.

The other way— NGL contains L— is also straightforward repetition of the earlier results.

As before, all identities are provable in each NGL system. Proof of the axioms is just like the GL case (lemma 3.5.4).

It remains to show that the class of sentences so provable in NGL is closed under modus ponens. This is done, as before, by showing that these systems are closed under *(mix)*. The proof that this ensures closure under modus ponens is exactly as for (lemma 3.4.3), using *invertibility*:—

Lemma B.1.12 If $\Gamma\ \|\!\vdash_{\mathbf{NGL}} [\langle \ldots A{\rightarrow}B \ldots\rangle,\ \mathcal{F}]$ where $\underline{A{\rightarrow}B}$ is a constituent, then there is a deduction of this consecution in which that $\rightarrow$ is introduced last, that is with last step:—

$$(\|\rightarrow)\ \frac{\Gamma,\ A\ \|\!\vdash [\langle \ldots \langle A\,B\rangle \ldots\rangle,\ \mathcal{F}']}{\Gamma\ \|\!\vdash [\langle \ldots A{\rightarrow}B \ldots\rangle,\ \mathcal{F}]}$$

Proof

Exactly as for lemma 3.4.2 with the cases for the new rules associated with the axioms of **L** as follows.

Suppose we have a descendant obtained by:—

$$\frac{\Lambda_1 \;||\!\!- [\alpha_1, \mathcal{F}_1] \;\ldots\; \Lambda_n \;||\!\!- [\alpha_n, \mathcal{F}_n] \;\; \Lambda_{n+1} \;||\!\!- [\beta_1, \mathcal{F}_{n+1}] \;\ldots\; \Lambda_{n+m} \;||\!\!- [\beta_m, \mathcal{F}_{n+m}]}{\Lambda_1, \ldots, \Lambda_{n+m} \;||\!\!- [\langle\alpha_1 \ldots \langle\alpha_n\, \beta], \mathcal{F}]}$$

For each α_i/β_j containing ancestral $A \rightarrow B$'s we have on hypothesis that the corresponding lists are provable

$$\Lambda_i,\, A, \ldots A \;||\!\!- [\alpha_i{}^{\triangleright}, \mathcal{F}_i{}^{\triangleright}] \quad OR \quad \Lambda_{n+j},\, A, \ldots A \;||\!\!- [\beta_j{}^{\triangleright}, \mathcal{F}_{n+j}{}^{\triangleright}]$$

The definition of the ancestor relation in definition B.1.9 ensures that the premisses still correspond to a uniform substitution in the atomic list corresponding to the axiom, so we have correct form of the lists for application of the *associated* rule; possibly also needing an extra last premiss $A \;||\!\!- [A, \mathcal{I}]$ in the case that an ancestral $\underline{A \rightarrow B}$ is the guard of a $\underline{\beta}$ with an ancestral $\underline{\beta}$ determining the guard of the conclusion succedent. Applying the rule to this set of premisses, with adjustment $(C||)$'s and $(W||)$'s, we obtain:—

$$\Lambda_1, \ldots, \Lambda_n, \Lambda_{n+1}, \ldots \Lambda_{n+m},\, A, \ldots A \;||\!\!- [\alpha'', \mathcal{H}'']$$

As for the list rules such as $(||pref)$ of the **GS** systems, the resulting succedent list might have too many "opened up" $\langle A\, B\rangle$'s which need to be "filled in" to obtain the required consecution. So as before apply $(||\!\!\rightarrow)$ to fill in the required $A \rightarrow B$'s.

Lemma B.1.13 **NGL** is closed under *(mix)*.

Proof

The proof is as for lemma B.1.7, with all but one of the cases already proved. The only outstanding case is for Right Rank> 1 (2), with the right upper from a rule associated with an L axiom:—

$$\dfrac{\Gamma \Vdash [\delta, \mathcal{K}] \qquad \dfrac{\Lambda_1 \Vdash [\alpha_1, \mathcal{F}_1] \ \ldots\ \Lambda_n \Vdash [\alpha_n, \mathcal{F}_n] \ \ \Lambda_{n+1} \Vdash [\beta_1, \mathcal{F}_{n+1}] \ \ldots\ \Lambda_{n+m} \Vdash [\beta_m, \mathcal{F}_{n+m}]}{\Lambda_1, \ldots, \Lambda_{n+m} \Vdash [(\alpha_1 \ldots (\alpha_n \beta), \mathcal{F}]}}{\Gamma,\ \Lambda_1^\circ, \ldots, \Lambda_{n+m}^\circ \Vdash [(\alpha_1 \ldots (\alpha_n \beta), \mathcal{F}^\circ]}$$

Do the lower rank *(mix)*'s, and then apply the corresponding *associated* rule plus adjustment $(C\|)$'s and $(W\|)$'s, exactly as for $(\|pref)$. This is easily seen to deliver the required conclusion as in that case for **NGS**.

All other cases are proved exactly as for **NGS**, and the lemma is proved.

This was the last element needed for:—

Theorem B.1.14 **NGL** contains **L**, i.e. if $\vdash_{\mathbf{L}} A$ then $\Vdash_{\mathbf{NGL}} [A, \mathcal{F}]$.

So the generalized **NGL** versions of the simplified form of consecution system are adequate in that they exactly capture the corresponding logic **L**. Nevertheless there is no intuitively satisfactory way of doing without $(\|iK)$, unlike the **GL** systems. It *is* possible to eliminate $(\|iK)$, it seems, by using the appropriate forms of the list rules associated with the axioms, and adding a further rule:—

$$\frac{\Lambda,\ \alpha \Vdash [\delta, \mathcal{F}]}{\Lambda \Vdash [\delta, \mathcal{F}]}$$

Where α contains no members of images under $\mathcal{F}$.

This rule allows the proof of the inductive case of closure under $(\|iK)$ for $(\|\rightarrow)$ to go through, and satisfies our interpretation, but it involves a gross form of elimination.

B.2 Simplified Natural Deduction Systems

Natural deduction systems corresponding to the simpler consecution systems can also be defined. That they capture their corresponding logics is proved exactly as before for the case of **TS** (subsection 4.2 and subsection 4.3). These simplified natural deduction systems are "partially" constructive in that they satisfy Lorenzen's principle of inversion relative to $\rightarrow$. In this section a brief description of these systems is given.

A **NTL** proof consists of lines comprising a support pair $[\alpha, \mathcal{F}]$ with side-bars to its left, which are introduced with a hypothesis introduction step. The support function's range is the power set of the set of side-bars, corresponding to the set of HYP wffs introduced along with a side-bar.

Definition B.2.1 The rules for **NTS** are as follows.

HYP A wff can be introduced by *HYP*, beginning a new vertical side-bar:—

$$\begin{array}{|l|l|l}
\ldots & & \\
\ldots & & [A, \mathcal{I}]
\end{array}$$

Where $\mathcal{I}(\underline{A})$ is the singleton containing the new side-bar.

REP A line can be repeated as long as the new line includes the side-bars of the original to its left.

iK

$$\begin{array}{|l|l}
\ldots & \dfrac{[\langle \alpha\, \beta \rangle, \mathcal{F}]}{[\beta, \mathcal{F}|_{\underline{\beta}}]} \\
\ldots &
\end{array}$$

This is the same transformation of the support-pair as for the corresponding rule in the consecution systems.

$\rightarrow E$

$$
\begin{array}{l|l}
\ldots & [A,\ \mathcal{F}_1] \\
\ldots & \underline{[\langle \ldots A \rightarrow B \ldots \rangle,\ \mathcal{F}_2]} \\
\ldots & [\langle \ldots \langle A\, B \rangle \ldots \rangle,\ \mathcal{F}]
\end{array}
$$

For the whole list and antecedent sublists $\underline{\varphi'}$ $\mathcal{F}\left(\underline{\varphi'}\right) = \mathcal{F}_2\left(\underline{\varphi}\right)$ where $\varphi = \varphi'\left[\underline{\langle A\, B \rangle}\backslash A \rightarrow B\right]$, and $\mathcal{F}(\underline{A}) = \mathcal{F}_1(\underline{A})$.

$\rightarrow I$

$$
\begin{array}{l|l}
\ldots & \quad [A,\ \mathcal{I}] \qquad HYP \\
\ldots & \quad \vdots \\
\ldots & \quad \underline{[\langle \ldots \langle A\, B \rangle \ldots \rangle,\ \mathcal{F}]} \\
\ldots & [\langle \ldots A \rightarrow B \ldots \rangle,\ \mathcal{F}']
\end{array}
$$

Where the $\underline{A}$ has as image the singleton containing the side-bar introduced by the first *HYP A*, that side-bar is not in an image of any other list except the $\underline{B}$, and the support function is determined exactly as was that for $(\|\!\rightarrow)$ in **NGS**.

PREF

$$
\begin{array}{l|l}
\ldots & [\langle \alpha\, \beta \rangle,\ \mathcal{F}_1] \\
\ldots & [\langle \gamma\, \alpha \rangle,\ \mathcal{F}_2] \\
\ldots & [\gamma, \mathcal{F}_3] \\
\ldots & [\beta_1, \mathcal{F}_4] \\
\ldots & \ldots \\
\ldots & \underline{[\beta_n, \mathcal{F}_{n+3}]} \\
\ldots & \left[\left\langle \langle \alpha\, \beta \rangle \left\langle \langle \gamma\, \alpha \rangle \langle \gamma\, \beta \rangle \right\rangle \right\rangle,\ \mathcal{F}\right]
\end{array}
$$

With the same transformation of the support-pair as for the corresponding rule in the consecution systems

SUFF

$$\begin{array}{l|l} \ldots & [\langle\alpha\,\beta\rangle,\ \mathcal{F}_1] \\ \ldots & [\langle\beta\,\gamma\rangle,\ \mathcal{F}_2] \\ \ldots & [\alpha, \mathcal{F}_3] \\ \ldots & [\gamma_1, \mathcal{F}_4] \\ \ldots & \ldots \\ \ldots & [\gamma_n, \mathcal{F}_{n+3}] \\ \hline \ldots & \left[\left\langle\langle\alpha\,\beta\rangle\left\langle\langle\beta\,\gamma\rangle\,\langle\alpha\,\gamma\rangle\right\rangle\right\rangle,\ \mathcal{F}\right] \end{array}$$

Ditto.

To deal with arbitrary implication logics add rules also determined in the same way from the consecution rules *associated with* the axioms. For example that corresponding to Peirce's Law is:—

$$\begin{array}{l|l} \ldots & [\langle\langle\alpha\,\beta\rangle\,\alpha\rangle,\ \mathcal{F}_1] \\ \ldots & [\alpha_1,\ \mathcal{F}_2] \\ \ldots & \ldots \\ \ldots & [\alpha_n,\ \mathcal{F}_{n+1}] \\ \hline \ldots & \left[\left\langle\langle\langle\alpha\,\beta\rangle\,\alpha\rangle\,\alpha\right\rangle,\ \mathcal{F}\right] \end{array}$$

With $\mathcal{F}$ determined exactly as for the consecution rule (all that has changed is the range of the support function).

Definition B.2.2 A pair $[\alpha,\ \mathcal{F}]$ is *provable in* **NTL**, denoted $\vdash_{\mathrm{NTL}} [\alpha, \mathcal{F}]$, iff there is a **NTL** proof of that line with no side-bars to its left.

That prefixing and suffixing are provable in **NTS** is proved exactly as for **TS**.

In this case closure under modus ponens hinges upon $(\|iK)$:—

Lemma B.2.3 The class of sentences provable in **NTS** is closed under modus ponens: Suppose we have a proof of $[A,\ \mathcal{F}]$ with no side-bars to the left of the line and a proof of $[A{\rightarrow}B,\ \mathcal{K}]$ also with no side-bars, then there is a proof of $[B,\ \mathcal{K}']$ with no side-bars.

Proof

On the above supposition we can put the two proofs together and apply *REP* to obtain premisses ripe for $\rightarrow E$ and continue:—

$$\begin{array}{ll}
[A,\ \mathcal{F}] & \\
[A{\rightarrow}B,\ \mathcal{K}] & \\
\left[\langle A\,B\rangle,\ \mathcal{K}^{\dagger}\right] & \rightarrow E \\
[B,\ \mathcal{K}'] & iK
\end{array}$$

And so we have:—

Theorem B.2.4 If $\vdash_{\mathrm{S}} A$ then we have $\vdash_{\mathrm{NTS}} [A,\ \mathcal{F}]$.

Extending the proof to cater for arbitrary **L** simply requires showing that the appropriate axioms are provable using the corresponding list rule, exactly as previously. I state this observation as a theorem:—

Theorem B.2.5 If $\vdash_{\mathrm{L}} A$ then $\vdash_{\mathrm{NTL}} [A,\ \mathcal{F}]$ where $\mathcal{F}(\underline{A}) = \{\}$.

The proof that **NTS** is included in **S** is similar to that for **TS**. The case for $\rightarrow E$ in the transform lemma is simpler than that case in lemma 4.3.2, but since it is different it is shown below.

Definition B.2.6 The *transform* of a line (with side-bars) $[\alpha,\ \mathcal{F}]$ in a **NTL** proof is a consecution determined as follows:—

- The antecedent of the consecution comprises all those wff introduced by *HYP* whose side-bars are in images of $\mathcal{F}$.

- The succedent is $\left[\alpha,\ \mathcal{F}^{\dagger}\right]$ where for every element $\underline{\varphi}$ of the domain, $\mathcal{F}^{\dagger}\left(\underline{\varphi}\right)$ is like $\mathcal{F}\left(\underline{\varphi}\right)$ but with each element a side-bar introduced along with A (by HYP) replaced by the corresponding A in the antecedent of the consecution.

Lemma B.2.7 Let $[\alpha,\ \mathcal{F}]$ (with side-bars) be a line of a **NTS** proof, then the transform of it is a provable consecution in **NGS**.

Proof

The proof is by deductive induction on the **NTS** proof.

HYP The transform of an introduced line $[A,\ \mathcal{I}]$ is $A \Vdash [A, \mathcal{I}]$ where $\mathcal{I}(\underline{A})$ is the singleton containing the antecedent A, and this is provable in **NGS**.

REP The transform is that of the ancestor line, hence is provable by hypothesis.

iK, PREF, SUFF The modification of the premiss support functions to obtain that of the conclusion is in each case exactly the same as for the corresponding rule in the consecution system **NGS**. So applying the corresponding rule to the transforms of the premisses delivers the required transform.

$\rightarrow E$ Suppose we have:—

$$
\begin{array}{|c|l}
\ldots & [A,\ \mathcal{F}_1] \\
\ldots & [\langle \ldots A \rightarrow B \ldots \rangle,\ \mathcal{F}_2] \\
\hline
\ldots & [\langle \ldots \langle A\, B \rangle \ldots \rangle,\ E,\ \mathcal{F}]
\end{array}
$$

Then by hypothesis the transforms of the premisses are provable:—

$$\Gamma_1 \Vdash \left[A,\ {\mathcal{F}_1}^{\dagger}\right] \qquad \Gamma_2 \Vdash \left[\langle \ldots A \rightarrow B \ldots \rangle,\ {\mathcal{F}_2}^{\dagger}\right]$$

So using the identity $\langle \ldots \langle A\, B \rangle \ldots \rangle \Vdash [\langle \ldots \langle A\, B \rangle \ldots \rangle,\ \mathcal{I}]$ and using closure under *(mix)* (lemma B.1.7) we have:—

$$\dfrac{\Gamma_2 \Vdash \left[\langle \ldots A{\rightarrow}B \ldots \rangle,\ \mathcal{F}_2{}^{\dagger}\right] \qquad \dfrac{\Gamma_1 \Vdash \left[A,\ \mathcal{F}_1{}^{\dagger}\right] \qquad \langle \ldots \langle A\, B\rangle \ldots \rangle \Vdash \langle \ldots \langle A\, B\rangle \ldots \rangle}{\Gamma_1,\ \langle \ldots A{\rightarrow}B \ldots \rangle \Vdash [\langle \ldots \langle A\, B\rangle \ldots \rangle,\ \mathcal{I}']}}{\Gamma_1,\ \Gamma_2 \Vdash [\langle \ldots \langle A\, B\rangle \ldots \rangle,\ \mathcal{I}'']}$$

Now the $(\rightarrow\Vert)$ replaces the A of the antecedent in $\mathcal{I}(\underline{A})$ by $\mathcal{F}_1{}^{\dagger}(A)$, with the rest of $\mathcal{I}'$ just like the identity support function $\mathcal{I}$ (but with images having the $\underline{\langle A\, B\rangle}$ replaced by $A{\rightarrow}B$). And the *(mix)* replaces the image-lists of the remaining antecedent sublists of $\langle \ldots \langle A\, B\rangle \ldots \rangle$ by the corresponding $\mathcal{F}_2{}^{\dagger}\left(\underline{\varphi}\right)$. So $\mathcal{I}''$ is the required support function for the transform of $\mathcal{F}$.

$\rightarrow I$ Suppose we have:—

$$\begin{array}{l|l|l}
\ldots & & [A,\ \mathcal{I}] \qquad HYP \\
\ldots & & \vdots \\
\ldots & & \underline{[\langle \ldots \langle A\, B\rangle \ldots \rangle,\ \mathcal{F}]} \\
\ldots & \multicolumn{2}{l}{[\langle \ldots A{\rightarrow}B \ldots \rangle,\ \mathcal{F}']}
\end{array}$$

Then by hypothesis we have the transform of the major premiss:—

$$\Gamma,\ A \Vdash \left[\langle \ldots \langle A\, B\rangle \ldots \rangle, \mathcal{F}^{\dagger}\right]$$

Where the succedent $\underline{A}$ (only) has as image singleton containing the antecedent A, so applying $(\Vert\rightarrow)$ to this delivers the required transform of the conclusion.

This completes the induction steps, so by induction the lemma is proved.

Theorem B.2.8 If $\vdash_{\mathrm{NTS}} [A,\ \mathcal{F}]$ then $\vdash_{\mathrm{S}} A$.

Proof

Suppose the antecedent obtains. Applying the above lemma we have $\Vdash_{\mathbf{NGS}} [A,\ \mathcal{F}]$ and so $\vdash_{\mathrm{S}} A$.

The extension of this result to arbitrary **NTL** is trivial, with the different list rules dealt with as in the third case of the above lemma.

The **NTL** systems satisfy the Lorenzen principle of inversion with respect to $\rightarrow$, but since the **NTL** systems include *iK*, this does not suffice to establish the Lorenzen principle of inversion, for in these systems the dropped antecedent list really is "lost" so that *iK* functions as an elimination rule.[1] The strategy suggested for removing *iK* from **NGL** can also be applied to **NTL**. The corresponding rule is one which allows a side-bar not in any image of the support function to be removed. This looks awful as a rule in a Fitch-style natural deduction system, and still involves a form of elimination.

Finally, note that corresponding relevant systems can be defined using the appropriate list-pair *use* rules, as in section 6.6.

[1] Here Allen Hazen's criticism clearly does apply.

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