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# Proof-Functional Semantics for Relevant Implication

Peter Lavers

A thesis submitted for the degree of Doctor of Philosophy of the  
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This thesis reports original work carried out by me.  
The contribution of others is duly acknowledged in the text.

A handwritten signature in black ink, appearing to read 'P. Lavers', written in a cursive style.

Peter S. Lavers

## Acknowledgements

I wish to thank my supervisors Dr Richard Sylvan, Prof Michael McRobbie and Dr Bob Meyer. Dr Sylvan's support and encouragement for my project has been invaluable, as has been the provision of an excellent research environment by Dr McRobbie and Dr Meyer. The reader will appreciate the tremendous contribution Dr Meyer has made in showing me how to make this thesis more digestible (I hope!). Naturally I am responsible for any remaining obfuscation.

I also thank Dr Errol Martin for the help he provided when I considered the application of my results to the  $P - W$  problem, and John Barlow and Gustav Meglicki for keeping the machines going.

One of the essential ingredients to doing logic is colleagues who will listen to half-baked ideas and provide constructive criticism. In that regard I have been extraordinarily lucky in the group of research students and fellows in the philosophy and logic community at ANU. I must thank in particular Andre Fuhrmann for his help, friendship and enthusiasm about logic, Michaelis Michael for teasing me about the existing semantics for relevant logics, and Dr Kim Sterelny who I almost didn't fail to convince as to how to approach these issues.

I wish to emphasise the substantial beneficial impact upon my research work due to the contribution of the overseas student members of our Department. They are making a tremendous contribution to the Australian research effort, which I am ashamed to say appears not to be recognised by the present Australian government.

Finally I thank Heather and Sally for being extremely patient.

## Abstract

In this thesis I provide a theory of implication from within the Gentzen/Curry formalist constructivist tradition. Formal consecution and natural deduction systems, which satisfy the formalist and also the intuitionist desiderata for constructivity (including Lorenzen's principle of inversion), are provided for all implication logics. The similar—but simplified—binary relational (“Kripke-style”) semantics are also given. The driving force behind this research has been the desire to provide an explanatory semantics for relevant implication in terms of “use as a subproof in a proof”. To this end *relevant* consecution systems which exploit various precisely characterised notions of *use* are described.

The basis of this work has been the development of a way of describing the shapes of proofs in the “object language”. In chapter 2 I motivate and introduce the basic machinery used to describe proofs, and show how thereby to capture *use*. This involves a more detailed consideration of the internal structure of formal systems than exploited by Curry in his *epitheory of formal systems*.

In chapter 3 the completely general “cloned” consecution systems are described, and it is shown that every logic with an axiomatic formulation is captured by such a system.

In chapter 4 the corresponding natural deduction systems are described and it is shown that Lorenzen's principle of inversion holds for them by proving the appropriate reduction theorem. Thus every implication logic has a formulation which satisfies the intuitionist formal criterion for constructivity.

In chapter 5 we return to the business of providing explanatory semantics for relevant implication, using the similar style of consecution system as in chapter 3, but with list (proof-description) manipulation rules which capture *use*.

In chapter 6 “cloned” binary relation semantics are described which also capture every

logic with an axiomatic formulation. These don't quite correspond to the consecution systems of chapter 3 in that they exploit a dramatic simplification of the list machinery (but do involve other complications). The similar relevant semantics using *use* rules is also given.

The corresponding "simplified" consecution and natural deduction systems are described in appendix B.2. These systems do not satisfy the Lorenzen principle of inversion and so are not constructive.

Chapter 7 rounds off and offers some thoughts about possible further developments.

Appendix A shows an early attempt to capture relevant implication, and is notable as the most complex formulation of intuitionist implication ever devised.<sup>1</sup>

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<sup>1</sup>Thanks are due to Bob Meyer who showed that this system is somewhat stronger than I had earlier thought.

# Contents

<b>1</b>	<b>Introduction</b>	<b>6</b>
<b>2</b>	<b>Preliminaries</b>	<b>19</b>
2.1	Atomic Theories . . . . .	20
2.2	Introduction to hedges . . . . .	24
2.3	Properties of hedges . . . . .	31
2.4	Linked sequences . . . . .	43
2.5	The Support Function . . . . .	47
2.6	Atomic Systems . . . . .	49
2.7	Atomic proofs and <i>use</i> . . . . .	53
2.8	Summary . . . . .	58
<b>3</b>	<b>The Consecution Systems</b>	<b>61</b>
3.1	Definition of the consecution systems. . . . .	62

3.2	Some properties and example deductions of GS. . . . .	69
3.3	The formal interpretation of GS . . . . .	81
3.4	The other way: GS contains S. . . . .	83
3.5	Generalization to all implication logics . . . . .	97
<b>4</b>	<b>The Natural Deduction Systems and Constructivity</b>	<b>109</b>
4.1	Definition of the Natural Deduction Systems. . . . .	109
4.2	TS contains S. . . . .	118
4.3	TS is contained in S. . . . .	121
4.4	The Natural Deduction Systems are Constructive. . . . .	124
<b>5</b>	<b>Relevant Implication</b>	<b>131</b>
<b>6</b>	<b>Binary Relation Semantics</b>	<b>149</b>
6.1	Models . . . . .	150
6.2	Properties of the Models . . . . .	153
6.3	Validity and Soundness . . . . .	160
6.4	Semantic Completeness . . . . .	163
6.5	Generalisation of the Semantics . . . . .	166
6.6	Binary Relation Semantics For Relevant Implication . . . . .	169



<b>7</b>	<b>Afterword</b>	<b>179</b>
<b>A</b>	<b>The Crude Systems</b>	<b>181</b>
<b>B</b>	<b>The Simplified Consecution and Natural Deduction Systems</b>	<b>184</b>
B.1	The Simplified Consecution Systems . . . . .	185
B.2	Simplified Natural Deduction Systems . . . . .	201