USE OF THESES

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TOWARDS FOUNDATIONS FOR
THE LOGIC OF DISTINCTIONS

by

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AT

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(Peter Jablon)
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This work is very much an independent effort. However, we consider that we are unusually indebted to our supervisor, without whose remarkable combination of tolerant perspicacity and concise frankness we may well never have managed to acquire the art of critical rigour.
ABSTRACT

This thesis is a contribution to the philosophy of logic and the foundations of metaphysics, and not to logic proper.

We suggest how the fundamental concepts of predicate logic and set theory may be reduced entirely, or almost entirely, to relational concepts. We also urge and begin to introduce the use of a new type of interpretation of logic (and we call this type of interpretation a "Chinese-mystical" one). Against this background, we look briefly at a variety of aspects of possible beginnings for the logic of (intensions of) relations (equivalently, for the logic of distinctions).

In Chapter 1 we introduce nearly all of the primitive notions and the symbolic vocabulary in terms of which, in Chapter 1 and also later, we attempt to explain all the fundamental concepts of predicate logic and set theory and to explore the logic of contexts and tokenicity. We also remark on some of the differences between our conception of the foundations of logic and the conception presupposed in, or commonly associated with, PM, with special emphasis on the deficiencies of the latter.

In Chapter 2 we display various senses in which the properties of the true/false distinction can be said to be generalized to those of certain other distinctions. That is, we suggest that there are certain distinctions any one of which, for the purposes of logic, will effectively achieve all the work the true/false dichotomy can do, and other work besides. And we emphatically suggest that the scope of modern logic should be broadened.

In Chapter 3 we attempt to introduce some inchoate aspects of the logic of relations proper (that is, of the intensions of relations). There we also introduce "Chinese-mystical" metaphysics; a "Chinese-mystical" metaphysics can be described as a relational metaphysics
which denies any reality to "thing" concepts.

In Chapter 4 we explore some of the rudiments of how to attempt to replace the propositional calculus by an analogue which would formalize the logical characteristics of in general unasserted relations instead of those of propositions.

In Chapter 5 we briefly explain some of the philosophical inadequacies of contemporary set theory.

In Chapter 6 we suggest how set theory can be altered to make it less unacceptable as a formalization of the concept of "manyness" and foundation for mathematics.

In Chapter 7 we digress to briefly survey something of the diversity of theories of what are the bearers of truth and falsity.

In Chapter 8 we attempt to indulge in a little pioneering work on foundations for the logic of tokenicity.

In Chapter 9 we suggest a clearcut way in which the paradoxes of mysticism may be explained logically.
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Philosophy and celebration

We believe that, at its best, philosophy ought to be, among other things, a celebration of life. A decent celebration is an end in itself, an experience of aesthetic pleasure, a playful occasion, a stimulating event that delights greatly. We believe that at its best philosophy ought to be, among other things, a creative activity, one where before one's eyes new conceptual creatures are born or reborn, and where one becomes aware of new dimensions of knowledge which had already all the time been accessible to those who knew how to open their eyes to it.

The greatest philosophers have always been extremely original, and we are not clear about what else one could say they have all been without exception. We too are attempting to be very original in this work. Hence the emphasis in this work is on stimulating the reader rather than indoctrinating the reader. We are therefore exploiting the privilege of much greater freedom of scope which distinguishes Masters degree requirements from Ph.D. degree requirements. In this work we are continually dipping into different aspects of a very broad field indeed, and therefore nothing is covered here in a genuinely complete way.

Moreover, the emphasis in this work is on the display of originality rather than on any compilation of information on the work or the views of other writers.

Also, there is a deliberate avoidance of formality, as far as that is possible in the philosophy of logic.
Logic and concrete objects

For Aristotle, whatever was real was individual and particular. The province of logic, by direct contrast, is the abstract and general. It follows (one would suppose!) that the Aristotelian concept of material reality, and with it such concomitant Aristotelian concepts as those of substance and material existence, and indeed any closely similar concepts, are hardly ideal candidates for being identified among the foundational concepts on which logic is built.

In this work we presuppose that logic is wholly made up of activities, not objects. Such a view may be defended on the ground that logic is one part of philosophy and on the adoption of the Wittgensteinian view that all philosophy is an activity and not a doctrine (a doctrine being one type of abstract "object"), which, Gellner claims (pp.63-64), is a central feature of "Linguistic Philosophy". Instead of looking for abstract "realities", therefore, we simply seek to examine the activities which are involved in doing logic, and try to forget that we have been conditioned into looking for objects where, according to Aristotle (Metaphysics, Z 13), none really exist. As will become apparent, we are principally and fundamentally concerned with those activities which involve, or use, or instantiate, or introduce, or remove, or significantly manipulate, or presuppose distinctions of any sort.

Logic, in the sense in which we use that term in this chapter and in much of this work, does not itself have any business in the matter of relating itself to material reality. One might identify logic in our sense as "pure logic", as distinct from "applied logic", the latter dealing with any questions which are commonly described as issues of "logic" and which involve reference or concrete
interpretation. Thus "pure logic" is related to reality by means of "applied logic", but excludes any mention of concrete objects.

**Logic without real objects**

The attitude we are adopting in this work to any talk of abstract "objects" therefore is such that, for all we are concerned, it makes no difference to regard all such entities as fictional. It may be useful to stress here that we consider this attitude does not in any way commit us to regarding such entities as being nonsensical. As Khatchadourian emphatically points out (p. 78):

"...we do not ordinarily say that we are talking about nothing, that we are not talking about anything, when we make statements that purport to be about centaurs or golden mountains.... Further, we do not ask or try to find out whether something, X, being (as we say) talked about, actually exists in order to decide whether it would be proper to say that X is indeed being talked about instead of only being apparently talked about. And... when someone asks 'What are you talking about?' and he receives the reply: 'About centaurs', he does not automatically assume - falsely - that centaurs exist."

We state our position more fully in (3.3), to which the reader may find it useful to refer if anything we have said so far seems puzzling.

**Symbols in logic**

It goes without saying that logic as commonly known is impossible without symbols, and yet symbols are one kind of concrete object. On the other hand, of course, the use of symbols is invariably an activity. Moreover, it is largely the use to which
a symbol is put (and the fact that it is actually being used) that identifies it, and not its shape, which is after all purely a matter of convention. In the last analysis, the concrete objecthood of symbols in logic is significant only in so far as activities are symbolically represented, for convenience, as concrete objects. In logic, on our view, therefore, symbols do not represent the object of an activity; rather, they represent the fact that the activity in question is being carried out. Wherever in this work we refer to the meaning of instances of symbols or of particular categories of symbols, this should always be understood as sheer shorthand for reference to the appropriate activities; this will hopefully become clearer below.

So let us now begin to concentrate solely on the activities that make up logic, in our sense of "logic". Logical form and material content certainly are not activities, and hence they cannot be among the activities that constitute "logic". This is particularly significant because the primitive concepts of propositional logic are those of a propositional form and of certain relations between or operators on such forms (cf PM, pp. 91-93), so that we are unavoidably committed to seeking a fundamentally new account of the fundamental concepts of logic.

"The primitive concepts of logic"

Definition: by "the primitive/fundamental/foundational concepts/ideas/notions of logic" we mean the most general, or universal, abstract ideas which are in common use or which are in use among scientists, mathematicians, or logicians.

An inquiry into the foundational concepts of logic is always of special importance, because the entire edifice of logic and mathematics
depends vitally on the soundness and the explicability of such concepts. It is in fact quite possible that the tiniest advance in research into the foundational ideas of logic may have gargantuan repercussions throughout formal logic and mathematics. In our opinion such a development is very likely to occur some time in the future; later in this work we try to secure this opinion at least a little credibility through our observations on set theory and on the analysis of relations (Chapters 2, 3, 4, 5, 6, 8). In our expectation, in fact, some time in the future it may well turn out that the most sophisticated and fascinating results of modern logic (such as undefinability and undecidability results, for example) will be seen merely to reflect the peculiar logical consequences of the presupposition of one particular collection of primitive ideas and arbitrary conventions for logic.

***Forms, moulds, and patterns***

Conventionally, logic is comprised of activities and of objects as well. The most important of these objects are known as "logical forms", or as "abstract forms". Thus Weyl (p. 27), in giving a general description of logic and pure mathematics, says that logic (and with it pure mathematics) "develops the theory of logical 'molds' without binding itself to one or the other among the possible concrete interpretations." Again, Langer (p. 17) says the following of the different branches of logic:

"Underlying them all is the principle of generality which culminates in the attainment of abstractions. The several branches of logic are so many studies in generalization. The aims of logical research may vary with the interests of different investigators - one may be interested in the
validity of Aristotelian logic, another in that of mathematics, a third in the canons of science, a fourth in the relation between mathematics and science or mathematics and classical logic, etc., etc. - but the procedure is everywhere the same: it is progressive systematization and generalization. Likewise, the criterion of success is the same: it is the discovery of abstract forms. The latter interest distinguishes logic from natural science, which is in search of general but not abstract formulae, i.e. formulae for concrete facts." (her italics)

We prefer to speak of abstract "patterns" - in a very broad sense of the word - rather than of "forms" or "moulds"; and of course despite common usage we emphatically prefer to talk about the activities of abstract "patterning" or "forming" or "moulding" rather than of the corresponding objects (which we have banished to "applied logic" anyway). In practice, however, we shall often use the customary "object talk" to avoid needless pedantry.

Inevitably, therefore, the account of some of the activities on which logic is grounded which we shall eventually give ought to provide an explication of at least some of the most important aspects of the activity of abstract patterning, and in some detail.

Western logic has nearly always confined its attention entirely to propositional patternings, even though this fact is not mentioned in the above statements of Weyl's and Langer's. We fail to see the necessity for such a confinement of the scope of logic. Why should every abstract patterning be always judgable as "right" or "wrong" (or true or false)? We shall have much more to say on this in Chapters 2, 3 and 4.
WHAT MAY WE SUPPOSE MAKES A SATISFACTORY NOTATION?

"Syntactical" requirements

The first question that needs to be settled in the construction of any adequate foundations for logical theory is that of what, for the purposes of logical theory as it is to be expounded, is to constitute a symbolic notation. The (symbolic) inscriptions used in communicating written English are what they are partly through historical accident and the needs of convenience. Therefore, it is something of an open question what sort of symbolic inscriptions are best suited for the needs of logic and analysis.

We do not claim to possess any definitive solutions to the problem of what is the optimal symbolic notation(s) for logical theory, but we do adopt a number of prejudices or presuppositions on this problem.

We assume throughout this work that in any symbolic notation we shall use, any given inscription can be copied an indefinite number of times - that is, that there is no difficulty involved in writing down any finite number of replicas of any given inscription.

Goodman gives two "syntactical" requirements for a satisfactory notation. He describes the first one on pp. 132-133.

"A necessary condition for a notation ... is character-indifference among the instances of each character. Two marks are character-indifferent if each is an inscription (i.e., belongs to some character) and neither one belongs to any character the other does not.... A character in a notation is ... a class of marks such that every two are character-indifferent and such that no mark outside the class is character-indifferent with every member of it....
That the characters must thus be disjoint may not seem very important or striking; but it is an absolutely essential and, I think, rather remarkable feature of notations...
The disjointness of the characters is... somewhat surprising since we have in the world not a realm of inscriptions neatly sorted into clearly separate classes but, rather, a bewildering miscellany of marks differing from each other in all ways and degrees. To impose a partitioning into disjoint sets seems a willful even though needful violence.

He acknowledges, therefore, that this requirement of character-indifference is merely one of convention and as far as he can see is motivated by practical considerations rather than by any logical or a priori ones. This is made clear on p. 134:

"There is no way of... ensuring that due caution will protect against all mistakes in identifying a mark as belonging or not belonging to a given character. But this trouble is not peculiar to notations; it is a pervasive and inescapable fact of experience."

Findlay (p.176) might be interpreted as suggesting that even this requirement should be relinquished in any satisfactory notation underlying what he conceives as the logic of mysticism (which is supposed to be a universal logic, embracing contemporary logic as a special case - cf 'The logic of mysticism' ibid.):

"The forms of our common utterance are by no means vacuous and innocuous: though they may not say that the world consists of certain types and ranges of elements and no others, or that it permits of certain sorts of treatment and no others, they may be said to imply that this is the case, and what they imply may be open to question, it may not, on reflection, be the
only nor the truest way of viewing the facts in the world. The forms of our common utterance imply the existence or the possibility of an independent array of logical subjects, a, b, c, d, e, etc., each capable of existing or not existing separately without others, and permitting the attribution of characters, the possession of which by one logical subject tells us nothing as to the possession of the same character by another logical subject. They may also imply the presence of relations among subjects which are external and indifferent to their existence and their character. The forms of this type of utterance readily lead to the development of a metaphysical atomism even more drastic than that worked out by Wittgenstein in the Tractatus Logico-philosophicus, an atomism of wholly independent existences, quite contingently characterized and related. But there is nothing to prevent us from holding this metaphysic to be merely an abstract or surface way of regarding the world, completely absurd if regarded as setting forth in completeness what a world conceivably could be, unable to make sense of the rational procedures which enable us comprehensively to understand the world and the beings who share it with us, and yet presupposing these procedures in the comprehensive, soi-disant intelligible view it sets forth of what is."

(Cf (3.3) below)

Findlay might not even accept Goodman's second "syntactical" requirement, which Goodman explains as follows (p.135):

"...if the differentiation [between characters] is not finite, [that is,] if there are two characters such that for some mark no even theoretically workable test could determine that the mark does not belong to both characters, then keeping the
characters separate is not just practically but theoretically impossible." That is, Goodman's second requirement (in addition to disjointness) on a notational scheme is that the characters be finitely differentiated.

In this work we reject Goodman's requirements to the extent that we do not concede that in logic it is ever possible to make a single inscription without thereby automatically committing oneself to making one or more other inscriptions as well. We do, however, impose a somewhat weaker requirement, to the effect that every inscription must be uniquely placed. In (1.3) we explain what this means, and we also explain our refusal to make the above concession.

The unreliability of visual intuition

Our partial rejection of Goodman's requirements implies that we consider the usual visual intuitions concerning characters to be unreliable. In case the reader should wish to counter with the claim that visual intuition is unchallengeable, we hasten to point out that the most outstanding example of the fallaciousness of what have appeared to be irrefutable truths directly ascertainable by means of physical vision and visual intuition has probably occurred in mathematics. Here it was found that logical rigour was unable to countenance any ultimate reliance on the use of geometrical diagrams in mathematical demonstrations. This example seems to be instructive for two reasons. Firstly, one can draw some sort of an analogy between the uncontested acceptance of visual intuition in contemporary formal logic and the uncontested acceptance of the validity and the universal applicability of geometrical demonstrations in eighteenth century mathematics. Secondly, it is instructive to contemplate with what great reluctance mathematicians, in their demonstrations, were
forced to abandon a reliance on and, for that matter, a tenacious confidence in the data of one of the physical senses: this is despite the experience of the initial attempts to prove the fundamental theorem of algebra (the theorem that every polynomial over the integers has complex roots), despite the now legendary protracted quest for an acceptable proof of so "intuitively obvious" a result as the Jordan curve theorem, despite the invention of noneuclidean geometries, and so on.

There are no precisely fixed rules by which one can distinguish two symbols as typographically distinct; rather, it is customary to employ a notation in which every pair of distinct symbols differs typographically to a sufficient extent that any minor inadequacies of printing of the symbols are highly unlikely to cause confusion. Here a little suspicion could perhaps be justified. Why is it that no such precise rules exist? Is it because it is impossible for such rules to generally account for how, e.g., minor misprints are to be taken care of? If so, then it will at least have been established that the data of physical observation of formal symbols are not governed by a sufficient degree of precision as would admit complete formalization, if the formalization required is one by means of which the issue of the validity of all visual intuitions may be discussed. The "visual intuitionist" cannot dispose of this imprecision at all simply by attributing it to the arbitrariness of the choice of the particular notation which is used, since the imprecision is common to every choice of notation.
(1.3) THE SPACES INSCRIPTIONS OCCUPY

On the concept of space

There is something irrevocable about an act of writing down an inscription. Even if the inscription is subsequently erased, or otherwise removed or replaced, it is an eternal and irreversible fact that the act of writing it down was made. In PM there does not appear to be any symbolic acknowledgement or indication of this fact. In this work we attempt to make up for this deficiency by introducing the notion of "space" (or strictly speaking, "spacing") as a fundamental notion for logic, for us just as fundamental as the notion of logical form (strictly, "forming").

We may think of each individual unit of space (that formal expressions may occupy) as being "labelled". After all, in any formal discourse it is impossible to get by without some means of keeping track of all the expressions that are used; otherwise it would not be possible to write them down. And although a metalanguage has labels for (and even names of) all the expressions in any relevant object language as a matter of course, there does not seem to be any effective recognition at all of the labelling that is inseparable from the use of the object language itself (it is inseparable because in the use of the object language different spaces can be distinguished between).

Our notion of "space" is however somewhat more complicated than might at first appear, and is in some ways rather surprisingly counterintuitive.

A common sense view

Common sense suggests that if two inscriptions are distinct then they cannot (simultaneously) occupy exactly the same space. On the other hand, if two inscriptions occupy distinct spaces then they are
Ipsa facto distinct. It follows, then, that common sense suggests two inscriptions are distinct if and only if they necessarily occupy distinct spaces. But in any logic which does not discriminate between the two members of any pair of equiform expressions - such as the logic of PM, for example - this suggestion is somewhat misleading, as we now show.

The indispensibility of spaces

Before making an inscription, or else simultaneously with making an inscription, one must - as we have already mentioned - create or procure a space which that inscription can occupy, and one must allocate that space to that inscription. We refer to the latter activity as placement of an inscription, or, alternatively, as space-allocation. The creation or procurement and allocation of such spaces is normally a tacit process, but it is an unavoidable one whenever one wishes to write something down.

The special use of spaces in logic

It is very common in logic for a space to be allocated to more than one inscription, even though normally only one inscription may occupy such a space at any one time. For example, any space occupied by a propositional variable normally may have any other propositional variable substituted for it without harm, provided that this substitution is uniform for all occurrences of that propositional variable in the theory in question, and provided that all the pairs of spaces which were occupied by distinct propositional variables are still so occupied after the substitution has been carried out. In other words, any space occupied by a propositional variable normally (that is, in nearly all theories) has more than one inscription allocated to it - that is, it makes no difference if it is occupied by 'p' or by
'q' or by ..., as long as the inscription occupying it is of a different shape from the inscriptions which occupy certain other spaces and of the same shape as the inscriptions which occupy certain others.

However, the above common sense assumption amounts to the claim that two inscriptions are distinct if and only if their space-allocations are distinct, since it entails that distinctness between the symbols which are permitted to occupy a given space is entirely determined by their space-allocations. And we have just observed that, wherever certain commonly used logical notations are involved, at least certain spaces are allocated to more than one inscription. Since we wish to use or explicate such notations, the latter fact amounts to a reductio ad absurdum of the above common sense assumption. So we must now concede in this discussion that, in general, two distinct inscriptions can occupy the same space.

**How can one inscription be distinguished from many?**

At this point it may be suggested that two inscriptions are distinct if and only if either they necessarily occupy distinct spaces or they differ in shape. But this new identity criterion breaks no ground in explaining how to distinguish between two inscriptions of the same shape which happen to be occupying the same space. That is, it gives us no basis for knowing that the expression 'A' comprises only one inscription and not (for example) seven, all of the same shape and occupying the same space. If the latter happens to be the case, it will of course be preferable to have seven distinct spaces, each containing only a single inscription; but we cannot guarantee to contrive this, except by convention.

At least, the only way to overcome this problem is to require that different tokens of the same character (that is, of exactly the
same physical shape) must always be allocated to different spaces. This amounts to the introduction of the notion of tokenicity.

**Token space**

We may here distinguish between "token space" and what we perversely choose to call "physical space" (in virtue of the remarks in (1.1), "token space" and "physical space" should of course, strictly speaking, be read as shorthand for "token spacing" and "physical spacing" respectively): physical space (in our sense) is what we have called "space" up till the paragraph before last, and physical space differs from token space in so far as a token space can hold only one inscription and is absolutely unrepeatable, and there is no known general, logical criterion for identifying the number of inscriptions which must exist on the filling of a given collection of physical spaces, other than the requirement that there must be at least as many inscriptions as there are physical spaces. In filling one token space, we may also, unwittingly, be filling others.

If the term "physical" seems odd, something like "blackboard" might be used instead. Our choice of the term "physical" is heavily influenced by late nineteenth century and twentieth century notions of mathematical space as well as by those of contemporary subatomic physics.

Our notion of token space differs from Wittgenstein's notion of "logical space"; in Tractatus 3. 41 he says: "The propositional sign with logical co-ordinates - that is the logical place," whereas for us any symbol (not just a propositional one) gets allocated to a specific piece of territory in token space (= the collection of all individual token spaces) by means of "logical co-ordinates" (i.e., its space allocation).
The elusiveness of token space

We are taking our notion of token space as a primitive notion, with the qualification, however, that individual token spaces are not to be considered as having any reality in isolation from other token spaces but, rather, derive their reality from their relations with all other token spaces. The complexity and importance of our notion of token space and of ones analogous to it seems to be generally overlooked in logic, in a manner which to us is reminiscent of the attitude painters once had to landscapes, in the age when landscape was regarded as "mere background". Let us examine this concept a little.

Firstly, it may be observed that our notion of token space is incommunicable within symbolic pure logic since our conception of what constitutes a symbolic logical notation presupposes the prior existence and the soundness of this notion. To put it another way, for the purposes of this work the general notion of token space cannot be fully or satisfactorily formalized.

Similarly, it is obscure on what basis token spaces are to be identified as distinct or identical, in the sense that our conception of a satisfactory notation presupposes that this question has already been settled prior to the introduction of the notation. We adopt the procedure that any two distinct (names for) individual token spaces are to be assumed (to be names of) distinct spaces.

Thirdly, we have stipulated that a token space is effectively inseparable from its environment, or contexts, or fields of applicability. From this it follows that occupation of or allocation to a particular individual space shall always affect the occupation of or the allocation to other individual spaces.

Some of the bewildering things that Aristotle said about his notion of matter can also be said of token space even though logic is
supposed to be wholly abstracted from matter. The concepts of matter and token space share the qualities of neutrality, of indeterminacy, of standing in an antithetical relationship to the concept of individual shape and the concept of form, and of being all-pervasive and hence, in one sense, somewhat inaccessible.

**Formal languages and mental "languages"**

Token space is also relevant with regard to the distinction between what one writes down and what one thinks. Although one uses spoken and written languages to articulate what one thinks in one's mind, they invariably convey only part of the full content of one's thoughts. A great amount of the distortion involved in the translation of one's thoughts into natural or formal languages may well, moreover, be inevitable. For example, the mind is known to think at a much faster rate than that at which it permits one to put forward one's (spoken) utterances, and it is also known not to be bound to function in the sort of one-dimensional sequential pattern to which the more usual of our natural and artificial languages are invariably constrained. This is explained eloquently by Watts, *In my own way* (pp. 423-424):

"The difficulty is that our waking and attentive consciousness scans the world myopically, one thing, one bit, one fragment after another, so that our impressions of life are strung out in a thin, scrawny thread, lining up small beads of information: whereas nature itself is a stupendously complex pattern where everything is happening altogether everywhere at once. What we know of it is only what we can laboriously line up and review along the thread of this watchfulness."

This circumstance is largely glossed over by formal logic in
its present state of development. Need it be? Surely we may attempt to use special kinds of artificial languages which begin to embody some of the features of our mental "languages", thus formally recognizing that there is a great difference between mental "languages" and the more usual sorts of formal languages. In this work we do, however, confine ourselves to using formal symbols in (effectively) a one-dimensional sequence.

If we are to represent mental "languages" both in a formal way and in such a way as to clearly differentiate them from the formal languages we normally use, either we need to introduce at least one formal operator or functor of some sort which is categorically different from any of the usual logical and metalogical operators and functors, or else we need to blatantly deny formal representations of mental "languages" the status of belonging to symbolic (pure) logic. Otherwise, from a purely formal point of view any talk of mental "languages" as distinct from utterance-languages will surely be irrelevant. In this work we adopt the latter course.

However, mental "languages" are relevant in this work because when one speaks or writes statements (or other linguistic expressions), these take on a definite location in space-time, or a number of different spatiotemporal locations if one speaks or writes them more than once; on the other hand, they cannot be said to have any spatiotemporal location when one thinks them in one's mind. Therefore the act of public utterance of statements (or other expressions) involves an act of physical placement; such a placement constitutes, among other things, a conversion from token spaces to physical spaces. Furthermore, one is highly selective about the particular utterances one chooses to make. The mind does not function so much like, say, a computer output terminal, letting through all the information that
has been fed into it; rather, it is more like the editorial staff of
a popular magazine, carefully reviewing and adapting whatever comes
before it, and divulging only the material likely to be popular. The
decision that a given mental statement deserves to be uttered to others
is by no means an insignificant one. Moreover, often it is basically
irrevocable. This provides one intuitive reason for saying that,
in general, there may be more token spaces than there are physical
spaces involved in a given group of physical placements.

"Token-shape"

In the case of the physical individual spaces to which only
one inscription is allocated, one may say that such spaces are
allocated a unique physical shape. And analogously, since precisely
one inscription is allocated to each token space, it seems tempting
to say that each token space is allocated a unique "token-shape".
But it is not clear that differentiation between different "token-
shapes" is always achieved by physical observation alone (that is, by
using one's eyes), so that it is obscure exactly how such
differentiation is achieved and exactly what counts as a "token-
shape". Because the world is not static, it is not always clear what
other inscriptions will go with a given inscription; there is a
freedom of choice for whoever is writing something down. There is also
the problem that the only way one can keep track of (tokens of)
inscriptions, and indeed even of plain token spaces, is by means of
names for them, and not by dealing directly with them themselves. But
to say that we are unclear about how to differentiate between distinct
"token-shapes" in general is to say that we are not clear about
whether it is in general possible to make an intended inscription
without thereby automatically and unintentionally making some other
inscription as well. Consequently, we refuse to admit the conventionalistic assumption that inscriptions never occur in the above incidental way, even though we concede that this supposition is universally accepted in orthodox logic (though tacitly rather than consciously).

We have required token spaces to be absolutely unrepeatable. On the other hand, the physical spaces we are concerned with in this work are for all practical purposes repeatable, since any book or collection of books containing formal symbols can, in principle, be photocopied or reprinted or otherwise physically reproduced as many times as desired. If we choose, as we well might, to regard the corresponding physical spaces in each reproduction as identical, this provides an illustration of a situation where there definitely are more token spaces than physical spaces (in fact, as many times more as the number of reproductions). The token spaces are not repeatable because they are each unique in an absolute sense - that is, they each differ significantly if one takes extraformal differences into account.
SELECTION AS A PRIMITIVE CONCEPT

Selection and inscription

Every inscription involves acts of selection in its very making - selection of a token space, selection of physical means to refer to that space, and selection of a symbol and a symbol shape to occupy that space. We take the notion of selection as primitive and, moreover, unformalizable.

Selection and abstraction

The concept of selection is a very important one for the foundations of logic. Not only does this concept figure so very obtrusively in the making of an inscription, but the concept of abstraction is very closely linked with it as well.

This is particularly clear if we observe that, in logic, exclusion is always a disguised form of selection since every act of exclusion is an act of selecting to explicitly ignore. It is even more clear if we also use Langer's (p. 33) definition of abstraction, because the latter is then seen to reduce every act of abstraction to a combination of acts of selection:

"The consideration of a form, which several analogous things may have in common, apart from any contents, or 'concrete integuments,' is called abstraction."

That is, Langer identifies every abstraction as the selection of a form combined with the exclusion of any contents. Thus, to give an example, the abstract concept of geometrical shape results from the concept of geometrical figure by abstracting from position and magnitude.
and the concept of abstract patterning. In fact, we shall find it convenient below to formalize the foundations of logic solely in terms of the concept of selection of inscriptions, of definition, and of different kinds of combination.

Selection and exclusion

We might almost equally have taken the concept of exclusion as primitive rather than the concept of selection, since every act of selection is an act of isolating one state/event/activity/object to the exclusion of all others. But we do not choose to do so.

Selection and notation

It should always be remembered that a formal inscription or group of formal inscriptions in itself constitutes a formal representation of one big act of selection. We shall nowhere use any formal apparatus to represent selection per se as separate from acts of inscription. And throughout this work we take the concept of selection itself to be a primitive one.

A single act of selection may involve only one inscription, or it may involve a number of inscriptions; in the latter case, the inscriptions (and the relationships between them) are all being selected simultaneously.

Selection and specificity

In logic, there is often some ambiguity in an act of selection. Such ambiguity is often a direct and necessary result of the abstract character of logic. To quote Weyl (p. 25):

"A science can only determine its domain of investigation up to an isomorphic mapping. In particular it remains
quite indifferent as to the 'essence' of its objects. That which distinguishes the real points in space from number triads or other interpretations of geometry one can only know ... by immediate intuitive perception".

Even in logic, however, there are limits to such ambiguity. For example, Whitehead and Russell's (passim) ambiguous use of the same symbols for the corresponding truth-functional connectives in both object theory and metatheory is unacceptable because of the resulting confusion between object theory and metatheory (this confusion clearly shows up, for example, in the analysis of entailment; see (1.9)).

Part of the problem with ambiguity is that acts of selection are tacit and are not normally formalized; indeed, we fail to see how they satisfactorily could be formalized. On the other hand, illicit ambiguity can normally be dissolved by the use of appropriate selections embedded within selections - that is, in effect, by the introduction of a sufficiently richer, or finer, vocabulary.
(1.5) A VOCABULARY FOR THE FOUNDATIONS OF LOGIC

Symbols

From now on, a formal symbol is always a name of an inscription. Different inscriptions are permitted to share the same name, and in fact customarily must do so if they happen to have the same physical shape.

We assume the availability of an unspecified collection of symbols, which must however include the four "primitive connectors" which we introduce below, as well as any symbols introduced by means of or associated with definitions. Any two occurrences of individual symbols, whether they have the same shape or not, must occupy two distinct token spaces at any given time. (This obviates the need for a distinction between occurrences of relations between token spaces and tokens of such occurrences.)

Place-markers

Although we asserted in (1.3) that token space is fundamentally unformalizable, we do propose to formalize the selection of token spaces. We do so through using

A, B, C, ..., Z; A₁, A₂, ..., which we call place-markers. Each place-marker represents (the selection of) one token space (strictly, act of token enspacing), although in each case it need not be specified which token space this is. Unlike token spaces (which they merely formally represent rather than mimic), place-markers are repeatable any finite number of times. They represent acts of inscription without in any instance specifying what is the identity of the inscription involved. Place-markers are the antithesis of proper names in so far as the uniform interchange
of any pair of them makes no logical difference whatsoever. It of course follows that, for the purposes of logic, individuation between place-markers depends entirely on the relations between place-markers.

**Primitive Connectors**

We also require that the following four symbols must be available:

```
/, /, &, ⊃;
```

these are to be known as the **primitive connectors**. They represent binary relations, the scope of which is clarified, where necessary or helpful, by means of parentheses (either '(', ')' or '[', ']') and, where necessary, empty gaps, in the customary way. There are two different ways in which such parentheses and gaps may be used. The first, more common way is in order when the primitive connector occupies what we call **operational token space**. In this case two parentheses or two empty gaps enclose the scope of the relation, and parentheses also enclose any of the two relata that happens to have a primitive connector occurring inside it and that consists of more than one symbol. Any place-marker or symbol may serve as a relatum. The primitive connector '/' is not permitted to have identical relata (that is, relata of identical physical shape) unless such relata contain symbols (rather than place-markers) which are of the same shape but which have been allocated to distinct token spaces. The purpose of this restriction is to ensure the significance of '/' under its intended interpretation (for which cf next paragraph).

The second way of using parentheses and gaps is to be used when the primitive connector occupies what we call **selective token space** (a more traditional term might be "substantive mode", as distinct from "relational mode"). Here the primitive connector is itself a...
relatum, and its intended interpretation is "the act of ...", the three dots being filled in in accordance with the following interpretation.

Interpretation of the primitive connectors

The primitive connector '/' can be interpreted as representing the act of distinguishing extensionally; the connector 'f' as representing the act of choosing not to distinguish, or identifying, extensionally; the connector '&' as combining or compounding, without in any way altering the identities of what are being combined (although we do admit that sometimes the same compound can be broken up into components in two or more different ways); and the connector '+' as transforming. In view of the intended interpretation of '/', the special nonequiformity restriction we have imposed on the relata of '/' in the preceding paragraph may perhaps make sense if compared with Bradley (p.141), where he says that although the "principle of identity" is often stated in the form of a tautology, "A is A", this is in fact wrong: "For identity without difference is nothing at all. It takes two to make the same, and the least we can have is some change of event in a self-same thing, or the return to that thing from some suggested difference."

In contemporary logic, "relation" actually means the extension of a relation. It is important to have it very clear that we are not making such an identification here; we are not identifying the activities of distinguishing, identifying, compounding, or transforming with the objectifications of these activities.

Thus, on our intended interpretation, '(A/B)&(A&B)' represents the simultaneous acts of both distinguishing and failing to distinguish extensionally between A and B; this does not violate the law of
contradiction because the latter only forbids the simultaneous
distinction between and identification of two entities with respect
to possession of the same specific property, whereas the general
concept of a property has not yet even been formally introduced
and even if it had, we have not claimed and certainly do not by any
means intend to claim that our conception of distinguishing is so
narrow that all cases of distinguishing extensionally are reducible
to cases of distinguishing extensionally with respect to specific
properties; and in any case, it is certainly true that not every
act of nondistinguishing extensionally is made with respect to
all properties. This point is taken up again in (1.6) Part 2.

Unspecified distinctions

As part of our vocabulary we also include

\['/\', '/\', 's', 's', ... , '/\', '/\', \] where \(R, S, ... , Z\) are

unspecified relations.

We refer to the former collectively as "unspecified distinctions".

Place-markers and inscriptions

We assume that it is impossible to formally name or refer to
any inscription other than an inscription of an unspecified distinction
or of a primitive connector or of '='; which is introduced in (1.6),
without first writing down a place-marker and then replacing that
place-marker by the relevant symbol. The notion of replacement
of any place-marker by any symbol or place-marker is taken as
primitive; such replacement must be uniform.

Place-markers, and inscriptions occupying operational token
space, permanently exist in token space but are timeless - normally
they do not ever change their location. On the other hand, all symbols occupying selective token space (that is, all symbols not occupying operational token space) are permitted to change with time from one token space or collection of token spaces to another. What always remains is (effectively) a one-dimensional sequence of symbols, place-markers, parentheses, and empty gaps. In places, this sequence is interrupted, by: change of line of the page (or whatever), change of page or blackboard or book (or whatever), introduction or use of definitions, and informal discussion of the expressions in the one-dimensional sequence.

The concept of replacement is adequately represented on a blackboard when a place-marker is rubbed out and its replacement written in in the very same spot.

Formal ineffability

Our vocabulary, therefore, consists of an unspecified collection of symbols (which must include the primitive connectors and unspecified distinctions), the place-markers, parentheses, and empty gaps (which separate expressions); and also of symbols for, or introduced by, definitions, which we introduce formally in (1.6). Anything in logic which cannot be expressed in terms of the notation we have introduced here or in (1.6) or (1.8) or Chapter 6 shall hereafter be regarded as formally ineffable - ineffable not in any absolute sense but merely within the symbolic discourse to which we confine ourselves in this work.
Collective space

No individual token space can be located in isolation from all other token spaces (because otherwise it would be impossible to identify such a space, as we have explained earlier in this chapter). Rather, each individual token space receives its identity through participating in many groupings or collections of token spaces, from which it is moreover inseparable.

(Some such groupings, however, may be more "relative" than others. For a very concise example of the difference between "more relative" grouping and "less relative" grouping, we turn to Leibniz, De arte combinatoria, para. 5:

"In respect of order, the following situations are different: abcd, bcda, cdab, dabc. In respect of vicinity, however, there is understood to be no variation, but only one situation - namely

\[
\begin{array}{ccc}
  a & b & c \\
  d & & \\
\end{array}
\]

"

Definitions

If the fact that token spaces divide naturally into groupings rather than into isolated units is to have any relevance in our particular symbolic theory, then as far as we can see there must exist at least some different groupings of names of individual logical spaces and of individual symbols which coincide with each other. We call such cases of coincidence definitions, and we use the symbol '=' or '≡' to identify them. Whenever this symbol is used, it signifies that any token of what is on its left can replace (not necessarily uniformly) any token of what is on its right, and vice-versa. Definitions, as we
use this term, always involve place-markers and do not involve any symbols which have been substituted for place-markers. Definitions are therefore timeless and irrevocable. They signify unbreakable fusions of the meanings of the expressions they identify.

On the other hand, the interreplaceability guaranteed by a definition continues to hold if any of the place-markers occurring in the definition are uniformly replaced by any symbols or permissible sequences of symbols. We use the symbol '=' to denote interreplaceability. Moreover, since any symbols occupying operational token space are ambiguous as to which particular inscription they name, and since every definition must contain at least one symbol in operational token space (in order to combine at least some individual token spaces), any given place-markers may be replaced for other place-markers in a definition, provided the replacement is uniform.

Definitions shall also be the means by which we augment our vocabulary over that introduced in (1.5). Thus there are two different categories of definitions: those which introduce new symbols into the vocabulary, and those which do not (e.g. \( A \&(A/B) = A \& B \)).

In (1.8) we will be very much concerned with the former variety, and in Chapter 4 we shall briefly look at the latter. Only the former kind, let us note, is what Whitehead and Russell (p.11) mean by a "definition".

**Equations**

We require that '=' is to occur only in operational token space. We call expressions containing '=' equations. All equations are to be either definitions or else are to have been obtained from definitions by means of legitimate replacement, that is,
from definitions or uniformly by replacement of one or more place-markers.

Equations are a special type of formal proposition. They are the only formal propositions which we consider to be absolutely necessary in the general theory of abstract patterning which we explore in Chapter 4. Whitehead and Russell (p. 11) do not count definitions - and therefore equations generally - as being propositions at all:

"... a definition is ... not true or false, being the expression of a volition, not of a proposition."

We find their sketchy argument for this position (on p.11) virtually incomprehensible and, indeed, unforgiveable (cf our discussion below of their notion of definition).

It is very easy to see why equations are of use if one considers the needs of applied logic. Each symbolic formula can be interpreted as representing an abridged, distorted map of some aspect of reality; and two or more maps of the same territory can be far more useful than one - they might mean all the difference between two dimensions of outlook from which to approach a subject and only one.

Equations and deductive reasoning

According to Pascal, the entire essence of deductive argument is contained in the use of equations. To quote from p. 427:

"Such, then, is the whole art of convincing. It is contained in two principles: to define all notations used, and to prove everything by mentally replacing the identified terms by their definitions."

Leibniz held this view as well. To quote from Parkinson (pp. xv - xvi):
"In the De Arte Combinatoria he remarks briefly that demonstration has one locus or topic, definition. In letters and papers written in 1671 and 1672 he explained that what he meant here was that demonstration was nothing but a combination of definitions: that all a priori propositions simply assert a definition, or part of a definition ..."

And ibid. (p. xxxii):

"Normally he regards a definition as an analysis, that is, as the breaking-up into other concepts of the concept to be defined."

In the version of the propositional calculus given later in this work, the only formal relation figuring in the transformation rules is '='. So we effectively show that the Modus Ponens rule is redundant; a similar conclusion has already been realized in the version of the propositional calculus given in Spencer Brown. We quote from p. 118 of that work:

"If we stand back for a moment to regard the structure of an implicational logic, such as Whitehead and Russell's, we see that it is fully contained in that of an equivalence logic. The difference is in the kind of step used. In one case expressions are detached at the point of implication, in the other they are detached at the point of equivalence.

"If an expression is detached at the point of implication, it of course need not be equivalent to the expression from which it is derived. But if it is a tautology it can be implied only by another tautology, so that, in such cases, the sign of implication can always be replaced by a sign of equivalence. Thus an implicational logic in fact degenerates
into an equivalence logic in respect of the class of [formally] true statements, with which such logics are most intimately concerned."

We find the Modus Ponens rule strange in a number of respects, some of which are explained when we discuss entailment in (1.9); suffice it to say here that we are nonplussed and even somewhat outraged by Whitehead and Russell's failure, in the face of, for example, Pascal's and Leibniz's statements, to explain why they believed themselves licensed to use a formal implication operator in one of their transformation rules instead of an operator intimately linked with every formal definition, such as our '='.

'/', '=' and extensionality

In our understanding of the propositional calculus, the connector '/' plays a role precisely equivalent to that of material equivalence. The rule of substitutivity of material equivalents for the propositional calculus then amounts to the statement that if 'A/B' then 'A = B' where A and B are any pair of wffs. Since the converse holds as well, it follows that the propositional calculus, à la Whitehead and Russell, conflates the distinction between '/' and our '='. We view this conflation in both object theory and metatheory as tragic, and possibly the most scandalous of what we consider are the blunders made by Whitehead and Russell in their formulation of the propositional calculus. Its influence has been so immense that even Marcus evidently has enormous difficulty distinguishing clearly between what is effectively equivalent to our '=' and various equivalence operators which do not at all coincide with '='. We warn the reader now, although we do not take up the issue again explicitly until Chapter 4.
Whitehead and Russell's concept of definition

It seems to us that the conception of definitions presented in *PM* is utterly unrealistic. We quote from p. 11:

"It is to be observed that a definition is, strictly speaking, no part of the subject in which it occurs. For a definition is concerned wholly with the symbols, not with what they symbolise ... Theoretically, it is unnecessary ever to give a definition: we might always use the *definiens* instead, and thus wholly dispense with the *definiendum*. Thus although we employ definitions and do not define "definition," yet "definition" does not appear among our primitive [= undefined] ideas, because the definitions are no part of our subject but are, strictly speaking, mere typographical conveniences....

"In spite of the fact that definitions are theoretically superfluous, it is nevertheless true that they often convey more important information than is contained in the propositions in which they are used."

If definitions really are *theoretically* superfluous, why is it that they can convey any theoretical information at all, not to mention much such information? And why, if they are superfluous, are they so truly important a part of the logical theory of (formal) propositions? And if they really are concerned only with symbols and in no way with the meanings of symbols, then any groups of symbols which are unfortunate enough to constitute a *definiens* or a *definiendum* in some definition will have to have no meaning whatever (since otherwise there will be a definition which is involved with meanings). Moreover, we see here that, according to Whitehead and Russell, the concept of definition is neither defined nor undefined in *PM*. But neither, we
presume, is it meaningless for them. We have to conclude that
Whitehead and Russell were hopelessly confused about the concept
of definition, a concept which figures prominently throughout PM
and especially prominently in the basis on which PM is built
(consider especially the term "undefined", which, incidentally, they
appear to use perhaps with a variety of different meanings.)
PART 2 - DISTINGUISHING AND NONDISTINGUISHING

Support from Leibniz

Leibniz regarded the concepts of distinguishing and nondistinguishing as extremely central to deductive logic, just as we do. He opens his *Mathesis rationis* with the following words:

"The laws of categorical syllogisms may best be proved by reduction to a consideration of the same and the different. For in a proposition or statement what we are doing is to state that two terms are the same as, or different from, each other."

And in *De arte combinatoria* (para. 63), we read:

"Thomas Hobbes, everywhere a profound examiner of principles, rightly stated that everything done by our mind is a computation, by which is to be understood either the addition of a sum or the subtraction of a difference (*De Corpore*, Part 1, Chap. 1, Art. 2). So just as there are two primary signs of algebra and analytics, + and −, in the same way there are as it were two copulas, 'is' and 'is not'; in the former case the mind compounds, in the latter it divides."

---

= and ≠

"To say of two things that they are identical is nonsense, and to say of one thing that it is identical with itself is to say nothing." - *Tractatus* 5. 5303.

Of course, a considerable part of logic is no doubt concerned with saying what Wittgenstein would like to dismiss as "nothing" (since in general one thing can be referred to in various different ways, and to say that these ways are equivalent is to say in different ways that one thing is identical with itself). But what sort of identity may we
suppose Wittgenstein is talking about in this statement? It is, surely an absolute kind of identity, and therefore one which holds timelessly. Equally surely, this cannot be the only sense in which the word "identity" is used in logic, and some at least of the other senses in which it is used are not absolute (for an example, we quote Leibniz, in his *Generales inquisitiones*:

"An entity is either in itself [*per se*] or accidental [*per accidens*]; or, a term is either necessary or mutable. Thus, 'man' is an entity in itself, but 'learned man' or 'king' are accidental entities. For that thing which is called 'a man' cannot cease to be a man except by annihilation; but someone can begin or cease to be a king, or learned, though he himself remains the same.");

not being absolute, they will of necessity be context-dependent, if the word "context" is used in a sufficiently broad sense (cf the next two paragraphs).

These considerations lead us to require the presence of at least two different identity symbols in logic. It may well be better to have three, or more. But we stand amazed at the absence, in mainstream contemporary formal logic, of separate symbols for context-independent identity and context-dependent identity. The only symbol for context-independent identity to be used in this work is the interreplaceability symbol '='; which we have introduced already. (We do not at all claim that all cases of absolute identity are assimilable to '='; those which are not are excluded from consideration in this work.) The symbol '/' shall represent context-dependent identity. Its negation in context shall be represented by '/'.
The intended interpretation of '/' and 'Ã’

Whenever we wish to deliberately omit to draw a distinction between two abstract objects or collections of objects (strictly speaking, spacings or collections of spacings) which are not identified in any equation in our logic, there is no necessary reason why we should in fact have deliberately omitted to make the distinction, any more than we should deliberately have made the distinction.

We call such contingent indentifications ('/') or distinctions ('/') context-dependent, and we do so for a very good reason. This is that although we may distinguish or omit to distinguish between two given abstract entities with respect to one distinction, there is no reason why we could not choose to do the opposite with respect to another, different distinction; hence whether we choose to identify them or not depends on which distinction(s) we happen to consider relevant; and that, very clearly, depends on the context of discourse.

Infinity and so-called contradiction

In (1.8) we shall define an operator '-' which corresponds to logical negation when applied to propositional variables. If P and Q are place-markers which in our logic correspond to the propositional variables of Whitehead and Russell, let '∞P' represent the countably infinite iteration of '-' applied to P. Then, as Spencer Brown points out (p. 58), if

\[ Q = -∞P \]

then

\[ Q = -Q \]

(since '(-∞P) = -∞P' is true).

This equation holds inexorably by virtue of the meaning of 'countable
infinity"; consequently, it must be admitted as a valid equation as long as infinite iteration of '-' is admitted.

Here we come up against the well-known paradoxicality of the mathematical infinite, but now we see it perhaps in its truest, certainly, we consider, in its most blatant colours. It is not just that an infinite whole can be as small as one or more of its parts. Rather, we find here that certain symbolic expressions representing propositions, and equivalent to symbolic expressions of infinite length, are absolutely indistinguishable from their own negations.

It goes against the entire twenty-six hundred year tradition of mathematics to forbid countably infinite iterations of an operator. Nevertheless, because the operator in question happens to be the negation operator (when it is confined to propositional variables), its infinite iteration is normally forbidden; otherwise, "consistency" would be violated. But surely there is another way to look at all this. Surely we can regard '-(A = -A)' as an inadmissible (or invalid) equation in our logic in general, but also as one which can, if desired, be treated as admissible subject to the requirement that A can be replaced only by expressions containing only a finite number of occurrences of any given operator, and containing only a finite number of distinct operators.

In any case, we believe we have demonstrated that the law of contradiction (that it must be true that '-(A & -A)', or '-(A = -A)', where A corresponds to a propositional variable in Whitehead and Russell) cannot be generalized in some directions without the generalization itself being violated within the more general settings involved.

Of course, our demonstration depends on the presupposition that '-•' is well-defined (that is, that it "exists" in the sense of
mathematical "existence"). An adherent of the law of contradiction is compelled to deny that this is so. However, such a denial amounts to a denial that it is permissible to iterate '-' a countably infinite number of times, which, since '-' is a well-defined unary operator, amounts to the rejection of all countably infinite sets, and hence of infinite sets in general. This is equivalent to a strict finitism. But the latter is quite unacceptable and unworkable as a philosophy of mathematics, as is pointed out by Kielkopf (p.182):

"There is no defense for strict finitists who hold that all mathematics should be done in a strict finitistic way since this cannot be done. Nor can there be any defense of a strict finitist who claims that we should eschew all mathematics which cannot be done by strict finitistic means. A philosopher ought not urge a return to barbarism. Similarly, philosophers of mind or philosophers of perception ought not to urge a return to the barbaric stage of talking only of bodies or sensations. Part of the point of Wittgenstein's claim in PI - 124 that philosophy leaves everything as it is, is a rebuke of such primitivism."

**Similarity and difference**

The next point we wish to make is that almost every act of distinguishing involves a simultaneous act of nondistinguishing, and similarly that almost every act of nondistinguishing involves an act of distinguishing. We quote from Alexander, pp. 143-144:

"The term 'identity' may be used for the extreme condition of similarity in which all difference has been excluded. But if this is done, then the characteristic of twoness, which implies some separateness or difference, must be
eliminated. Hence, in this extreme sense no two things can be identical. Complete identity pertains only between a thing and itself. For example, two peas in a pod may look just alike in every respect, but to be two peas they must at least occupy different spaces and in this respect, not all difference has been eliminated. So two peas in a pod cannot be identical. (Of course, we often use the term 'identical' more loosely than this, as when we speak of identical twins.) Complete and total difference is hard to conceive; for if we think of two things as different in all respects, there is at least one respect in which they are still alike: that is, that we are thinking of them both. Now if we eliminate this similarity then we are no longer able to notice any difference since we can no longer think about the two items.

"To generalize, any relation will involve some degree of similarity and some degree of difference between its relata."

Of course, if two things are identical in some respects, they will be different in other respects, and moreover they will never be both identical and different in the same specific respect. However, it is most important to emphasize that it is a fact that we may know two things to be identifiable, or distinguishable, without being able to nominate any specific respect in which this is so. Proof: if A and B are two entities, then they must differ in some respect - we may not know which respect at all - as otherwise they could not be two. And A and B must also have some property in common, since otherwise we would have the reductio ad absurdum that they shared the property of differing from some other entity with respect to every single property. Hence if A and B are two, we can always
validly assert that \((A/B)\&(A\neq B)\)' holds, or, at least, makes sense, 
\[ \begin{array}{c}
R \\
S 
\end{array} \]
for some '/' and '/' .
\[ \begin{array}{c}
R \\
S 
\end{array} \]

This point has significant consequences. It establishes that one may simultaneously make an act of both distinguishing and nondistinguishing in the one same sense, albeit an indeterminate and very abstract sense. Perhaps the mystics who assert (cf Chapter 9) that the universe is simultaneously and in the same sense identical and not identical with their true selves are voicing a similar insight.

Spencer Brown points out (p. 99 and pp. 56-68) that what might be called "the logic of contradictions" - the theory, that is, of the logical properties of expressions such as \((A/B)\&(A\neq B)\)' - has important applications in engineering. He evidently wishes to intimate, on pp. 99-100, that it would be just as ludicrous not to take advantage of "the logic of contradictions" (as we are calling it) on the excuse that it deals with what is meaningless as it would be ludicrous to banish the theory of complex numbers from mathematics on the ground that it involved imaginary numbers and therefore dealt with purely fictional quantities. In fact, he even suggests (pp. 99-100) that he considers the existence of undecidability results to be the inevitable consequence of the virtually total exclusion of anything to do with "the logic of contradictions" from contemporary logic; however, he does not offer more than a very sketchy argument to support his view, even though the latter seems to us to have some intuitive plausibility.
Identity and difference between variables

In (1.8) we define the underlying concepts of quantification theory in terms of the concepts of place-markers, interreplaceability, and the concepts represented by the four primitive connectors. But place-markers, with replacement in each case restricted to the symbols of a given syntactical category, are intended by us to be interpreted as coinciding exactly with "dummy variables" or "arbitrary, representative members of the domain" corresponding to a given syntactical category. The concept of "dummy variables" up till now has been regarded as unrespectable in modern logic, as it cannot be derived from the fundamental concepts of quantificational logic. Since in (1.8) we define the latter in terms of the concepts of place-markers and of impeccably respectable logical concepts, it presumably follows by watertight logic that the basic concepts of quantificational logic cannot after all be any more respectable than that of "dummy variables", contrary to what has been the general supposition up till now!

In Chapter 8 we shall have some ungenerous and more detailed comments to make about the treatment which logicians have given "dummy variables".

We also wish to emphasize here that modern logic does not formally recognize the significance of the question of when two variables (rather than the values of two variables) are identical or different. This will be an important question for us in Chapter 8. Thus in speaking of "distinguishing" and "nondistinguishing" we do not presuppose that it is necessarily constants that are the objects. (By "quantification theory" above, we mean first-order quantification theory.)
The basic units of thought

Because Western logic has always confined its attention to propositions and operations on or within propositions, we find it necessary to turn to psychology for an account of the general (not just the propositional) abstract patterning activities involved in thought. We quote from Communications Research Machines, a psychology textbook written by a panel of 33 and therefore not a particularly biased work. We begin at p. 200:

"... the materials of thought are called cognitive units: [these consist primarily of] the images, symbols, concepts, and rules. The cognitive processes are the operations or routines that the mind performs on these units. The processes include the encoding of information, the storing and retrieving of information from memory, the generation of hypotheses, their evaluation according to criteria, and deductive and inductive reasoning. Out of the interaction of these processes with these units there somehow emerges intelligent thought.

"There are two philosophical attitudes toward thinking. One view assumes there is a psychological executive continually monitoring the cognitive processes, much as an architect supervises the construction of a house. ... An alternative view assumes that the monitoring function is contained within the units and processes themselves...."

"...Mental units are not actual things with substance, size, shape, or a definite location."

The mental units which particularly concern us in this work,
however, can all be in each case allocated to an indefinitely large number of token spaces.

Pp. 201, 203, 260 read:

"The image is probably the first mental unit to develop [in a child's mind]. It is a representation of a specific event ..." (perhaps images are what Wittgenstein considers to be "shown" when in _Tractatus_ 4.1212 he says: "What can be shown, cannot be said."

"Symbols refer to the next level of abstraction in the representation of experience. Symbols are names for things and qualities; the best examples are the names of letters, numbers, objects, or persons .... The major difference between an image and a symbol is that an image represents a specific sight or sound and preserves the relations in that particular experience. The symbol stands for something other than the event....

"All concepts are symbols, but they are much more than that: they represent a set of common attributes among a group of symbols or images. The concept extracts a common meaning from a diverse array of experiences, whereas a symbol is only the name assigned to a particular class of events. Consider a drawing of a cross (+). The eight-month-old represents this event as an image. The three-year-old names it a cross or a plus sign and represents it as a symbol. An adolescent may regard it as a Christian cross and think of it in relation to religion ... In this case, the cross stands as a concept.

"A concept represents a characteristic that belongs to different events..."
Piaget's discovery

The main point of the material we have quoted is to emphasize that thought does not begin with propositions; and indeed, from a developmental (that is, childhood chronological) point of view it apparently begins with images of events rather than of any kind of objects.

Piaget, generally acknowledged as the most influential developmental psychologist of the century, has made the empirical discovery that children of certain ages can grasp certain types of concepts without being able to grasp the abstract concept of a proposition - not, at any rate, to the extent of being able to apply and manipulate truth-functions.

In particular, Piaget has found that children (typically) aged between 8 and 11 years have grasped (in practice, though not in the sense of being able to completely verbalize) the concept of what he calls "groups" of "operations" but have not grasped (in the same sense of "grasped") the concept of truth-functional relations. "Operations" here are actions of certain types, and the paradigm examples Piaget gives of classificatory notions by means of which "groups" of them are formed include those of (sub-) class inclusion within a given class, class intension, seriation (that is, sequential ordering), and juxtaposition. Piaget lists five defining properties for any such "groups"; we are quoting from Thomson, pp. 91-92; the "units" are "operations" of such a kind as to be able to be grouped together ("combined") by some form of classification:

"(1) Composition. Any two units can be combined to produce a new unit.

(2) Reversibility. Two units combined may be separated again."
(3) **Associativity.** The same result may be obtained by combining units in different ways.

(4) **Identity.** Combining an element with its inverse annuls it.

(5) **Tautology.** ...repeating a logical unit only gives repetition or tautology: \( A + A = A \) ('A' is an 'A' and not another thing).

The "grouping" characteristics of the abstract patternings the totality of which in this work we urge ought to be identified with "logic" and which are the focus of attention throughout Chapter 4 and elsewhere seem to conform entirely with Piaget's five defining properties. This conformity is not due to any prior acquaintance of ours with Piaget's or other psychologists' work. Therefore we believe that Piaget's work provides some moderately impressive empirical corroboration of our fundamental contention that the concepts on which we build logic in this work are presupposed in Western logic (it is only too obvious that they are presupposed) without any explication even though in fact it is possible to give an explication of them.

(Specifically, defining properties (1) and (2) are together equivalent to the existence of '$\&$' in our logic; property (3) is equivalent to the associativity of '$\&$'; while (4) is equivalent to '$(A \& (A \lor F)) = F$' (cf Chapter 4), and (5) is equivalent to '$A \& A = A$'.)
(1.8) REDUCTION OF SOME FUNDAMENTAL CONCEPTS OF LOGIC

PM (p.91):
"Since all definitions of terms are effected by means of other terms, every system of definitions which is not circular must start from a certain apparatus of undefined terms. It is to some extent optional what ideas we take as undefined in mathematics... We know no way of proving that such and such a system of undefined ideas contains as few as will give such and such results. Hence we can only say that such and such ideas are undefined in such and such a system, not that they are indefinable."

In this section we attempt to define some of the fundamental concepts of logic in terms of the four primitive connectors, placemakers, and interreplaceability. It should be stressed that in this work we are choosing to explore only one of the different ways in which these fundamental concepts might conceivably be defined in terms of our own fundamental concepts. We are not claiming to do this in necessarily the most concise or the most intuitively appealing way.

**Formal expressions and replacement**

At this stage, let us make clear exactly what we mean by a "formal expression". Any sequence comprised of occurrences of (some or all of) place-markers, primitive connectors, or '=' is a formal expression, provided that it is terminated at each end by empty gaps but does not itself contain empty gaps in the sense of (1.5)(that is, empty gaps used as a special kind of brackets) and provided that it satisfies the scope requirements laid down in (1.5),
with ' 'being treated in exactly the same way as a primitive
connector is treated in the relevant part of (1.5). What is
obtained from any such formal expression by uniformly replacing one
or more place-markers by any formal symbols (any formal symbols
whatever that may be available or accessible) is itself always a
formal expression. Moreover, any place-marker in a given formal
expression may be replaced by any formal expression; that is, the
collection of all formal expressions is closed under (uniform)
replacement of formal expressions for place-markers occurring within
formal expressions. That concludes the specification of what
we mean by "formal expressions".

**Dummy variables and quantification**

As we have already mentioned in (1.6), place-markers are
intended to be read as dummy variables (and therefore, in the case
of the propositional syntactical category, as propositional
variables) - that is, if one wishes to explicitly bring in
universal quantification (we certainly do not), as variables which
are to be read with universal quantification understood over the
entire formal expression in which a dummy variable occurs.
Consequently there can be place-markers appearing on only one side
of an equation. Our practice is as far as feasible the same as
that of Leibniz throughout his logical works; Leibniz managed to
discuss concepts from quantification theory without explicitly
introducing quantifiers.

We wish to avoid using the orthodox concept of quantification
for the following reason. This concept presupposes the concept
of a universe of discourse, and there is no formal or precise
explanation of what is meant by a universe of discourse which is
not built upon orthodox set theory (and possibly other notions in addition). But orthodox set theory presupposes the concept of quantification, so that there is a full circularity in orthodox modern logic. We consider this circularity to be intolerable, and we therefore reject the concept of a universe of discourse as a fundamental one.

**Conceps used in propositional logic**

We introduce the symbol 'T' by

\[ T = A/A \]

Df

'T/A' shall be interpreted to mean that it is true that A (if A represents a proposition) or else that A is a possible entity (if A does not represent a proposition).

We introduce the operator '-', which on propositions corresponds to negation, by

\[ -A = A/T \]

Df

And

\[ F = A/-A \]

Df

introduces 'F'; it may be observed, for later reference, that we have not defined 'F' by the equation 'F = -T', and it may not necessarily be the case in general that this equation will hold (cf Chapter 4).

We define material implication, which we denote by '→', by

\[ A→B = A/(B\&(A/B)) \] if A and B represent propositions;

Df

and material equivalence is to coincide with '↔', and the negation of material equivalence is to coincide with '⊥', on place-markers representing propositions.

We consider it may possibly be desirable to introduce an opposition
operator 'N_A' as follows:
\[ N_A(B) \overset{\text{def}}{=} (B/t)(A/B); \]
in propositional logic, \( N_B(B) \) will be materially equivalent to \(-B\).
We define the disjunction operator 'v' by:
\[ A v B \overset{\text{def}}{=} -(N_A(B) \& N_B(A)). \]
We postpone the discussion of various different entailment operators to (1.9).

**Concepts used in set theory and predicate logic**

'&' represents the comprehension of elements or parts into a whole. The existence of the distinction between elements and parts means that the use of ' & ' can be deceptively complicated. The same elements may sometimes make different wholes; for example, (taken from Langer p. 45) 'RONALD' is a word composed of the same letters (elements) as 'ARNOLD'. On the other hand, the compounding of different parts can sometimes result in the same whole. Thus we happen to require (in Chapter 4) that in our logic, for example, \( A \& (A/B) = A \& B \) should hold (one might interpret this equation as an identification, in the context of 'A&...', of 'A&' with B). We discuss these issues a little more fully in Chapter 5.

The part-whole or subset relation ' \( \subseteq \) ' can be defined quite simply if we make use of the fact that adding a part (that is, a sub-whole) to a whole does not result in anything new being introduced, while adding anything other than a sub-whole does:
\[ B \subseteq A \overset{\text{def}}{=} ((A&B)=A). \]

We introduce our elementhood relation ' \( \epsilon \) ' by first introducing "singletons", '{B}', by the definition
\[ C = \{B\} \overset{\text{def}}{=} ((D/C) \& (C \& (D=B))) \]
we are using '→' throughout this work in such a way that

\((\alpha(D) \rightarrow (D \rightarrow B)) \equiv (\alpha(D) \rightarrow (\alpha(B)))\)

holds, where \(\alpha(D)\) represents any expression and where \(\alpha(B)\) is the same expression as \(\alpha(D)\) except that each occurrence of \(D\) (if any) in \(\alpha(D)\) is replaced by \(B\).

The elementhood relation is then defined as follows:

\[ B \in A \iff [-(C \equiv \{B\}) \lor -(C \subseteq A)] \]

that is (converting from universal to existential quantification), \(B\) is an element of \(A\) just in case there exists a \(C\) identical to '{\(B\}' which is also a subset of \(A\); we introduce existential quantification formally in the next paragraph.

Along with (our version of) the existential quantifier, we now introduce what we call bound variables, namely:

\(\star_A, \star_B, \ldots\), corresponding to place-markers \(A, B, \ldots\), in the following way. Given any admissible expression \(\alpha(A)\) containing the place-marker \(A\) and no bound variables and representing a proposition, we define the existential quantification of \(\alpha(A)\) by:

\[ (E^* A)\alpha(A) = F^\alpha(A) \]

where \(\alpha(A)\) is obtained from \(\alpha(A)\) by uniform replacement of \(\star_A\) for \(A\).

This definition exploits the fact that \(\#\) is intended to correspond to material equivalence and the fact that \(F^\alpha\) is a constant and the standard result of predicate logic that if \(\phi\) is a propositional constant (we are now using orthodox notation) then

\[ (x)(c \equiv \#(x)) \iff c \equiv \#(x) \#(x).) \]

If \(\alpha(A)\) is any admissible expression not containing \(A\), then

\[ (E^* A)\alpha(A) = \alpha(A) \]

And if \(\alpha(A, B, \ldots, A_1, A_2, \ldots, A_n)\) is any admissible expression containing \(A\) and the bound variables \(\star_{A_1}, \star_{A_2}, \ldots, \star_{A_n}\) and no
existential quantifiers, then

\[(E^*A)\alpha(A^*B,A_1^*A_2^*...^*A_n^*A_{n-1}^*) \text{ is obtained from} \]

\[\alpha(A^*B,A_1^*...^*A_{n-1}^*A_n^*) \text{ by uniform replacement of } A_n \text{ for } ^*A_n; \]

and similarly, the latter expression is obtained from

\[\alpha(A,B,A_1^*...^*A_{n-1}^*A_n^*) \text{ by uniform replacement of } ^*A \text{ for } A. \]

The latter definition can be repeatedly applied to reduce the number of bound variables from \(n+1\) to 1.
The concept of change

The concept of change is a very fundamental one for logic and, unlike the concepts of definition and formal distinction and formal identification and abstract compounding and token space, it is a fundamental one in every sphere of human activity. All matter in the universe is continually and unrelentingly changing, and in no instance is it constantly and forever what it appears to be at a given moment. All life, all activity, all manifestations of mind of any sort are forever in a state of flux. Even token space is forever being altered because, as the principle which is traditionally claimed to summarize all of Buddhist metaphysics states, form is inseparable from space, and space is inseparable from form.

In logic we are of course particularly concerned with the concept of (pure) change in abstract. But because the concept of change is such an important one, we are also concerned in logic with some of the most universal species of change as well as with change in abstract.

Transformation rules

In logic, change is generally called "transformation", and we use that term from now on. There are various different types of transformation which are very important in logic: transformation in abstract, substitution (which is almost the same as transformation in abstract and in combinatory logic sometimes amounts to the same thing), the use of calculations, the use of constructions, the use of deducibility or entailment, the use of equivalence relations of different kinds, negation, elimination, the introduction or creation of new terms or concepts, and combinations of various of these.

In modern logic, transformations are normally carried out "within" the object logic yet their statement or description is not part of the
object logic but instead belongs to its metalogic. This is a peculiar situation in the sense that although a demonstration is certainly considered to occur within the object logic, the justifications for each step of the demonstration, even though they are required to be written down along with the demonstration, are not part of the object logic. The result is that it is impossible to use the object logic without making use not just of the object language but also of the metalanguage. The metalanguage is used whenever a substitutional or inferential rule is applied, and whenever a calculation or demonstration is initiated or completed or summarized.

On the other hand, in the metalogic without the metametalogic (that is, in the syntax, without its metatheory and the semantics and so on), the metalanguage is also used. Thus from the fact that the metalanguage is being used one does not know, without further information, whether the object logic is being used or whether the syntax is being used, and for precisely this reason use/mention distinctions become notoriously unmanageable. (Incidentally, we consider the term "object logic" to be a misnomer for this very same reason.) It would seem to be much preferable to have included all transformation rules in the object logic itself, especially since this move would have increased the expressive strength of the object logic. In Chapter 4 the logic we present can be regarded as an intermediate stage towards the achievement of this ideal.

It need hardly be pointed out that logicians still have much to discover about transformations. The simplest sort of transformation is one which involves the interchange of two expressions, or the replacement of a given occurrence of one expression by the occurrence of another. This is normally covered by "rules of substitution" of an orthodox sort. In view of the historical background to the rise of
combinatory logic it need hardly be pointed out, however, that a rigorous account of even these rules was not successfully given by any logician for many years, despite careful attempts by a number of the most well-known logicians.

**Implication and Whitehead-Russell**

Inference, or implication, is a kind of transformation which is very important in logic and one to which the authors of *PM* ascribed absolutely central importance (p.90):

"...the subject to be treated ... is not quite properly described as the theory of propositions. It is in fact the theory of how one proposition can be inferred from another."

Russell attempted to back out of this estimation of the importance of the concept of implication for logic in *The principles of mathematics* (p.vii):

"... the form 'p implies q' is only one of many logical forms that mathematical propositions may take. I was originally led to emphasize this form by the consideration of Geometry.... Such instances led me to lay undue stress on implication, which is only one among truth-functions, and no more important than the others."

But this remark was muddle-headed, for the well-known reason that the entailment relation does not coincide with the material conditional; not to mention the fact that the appearance of the material conditional in the Modus Ponens rule does single it out from all other truth-functions; and it is not the material conditional by itself but the material conditional as it appears in the Modus Ponens rule (and that is something more complex than a mere truth-function) that is under consideration in any case.

It has of course long been recognized that the material conditional
cannot be interpreted as denoting entailment except under very special and restricted circumstances. What alarms us is the dogmatic acceptance accorded to the other aspects of propositional logic as they are presented in PM. This dogmatic acceptance of PM alarms us in other areas of logic and not simply in those connected with the analysis of entailment, as the reader will to some extent have observed from previous subsections. However, this point is relevant in the case of entailment because it is almost always assumed that Modus Ponens is the best, even perhaps the most natural, inferential rule to use in logic; that there is something necessary and unchallengeable in the awkward banishment from the object logic of parts of the metalogic without which it would be impossible, as we explained just above, to use the object logic; and that what needs to be explained in regard to the whole problem of entailment is nothing more than an extensional relation of some sort. None of these assumptions is acceptable to us, and by the end of Chapter 4, if not by the end of this section, it should be clear to the reader why they are not.

Fortunately, Anderson and Belnap and others have at least had sufficient courage to challenge some of the axioms of PM with $\vdash \rightarrow \neg$ read as "entails". (Thus $p \rightarrow pvq$ needs to be rejected if $\vdash \phi$ is read as "entails" and \text `v` as logical disjunction, because logical deduction never leads to an increase in uncertainty.) But we do not consider this move to be by itself anywhere near an adequately radical departure from PM.

What entailment is

Entailment is the preservation of truth through change. Different accounts of entailment differ as to how broad their use of the term "truth" is here (does it mean truth in general, or tautologousness, or axiomatic truth?), and as to how broad a sense of
the term "change" is meant, and as to what sort of conditions preserve truth on what sort of formulas or assertions (hence we have varieties of entailment logics and relevance logics). In the formal logic presented in this work we are interested in making use of (or at any rate we are presupposing) entailment in the broadest possible sense, given the above definition of entailment, that our symbolism will permit. Let us therefore first examine the concept of change, or transformation. Our concept of transformation in abstract, '+', is surely change in the broadest possible sense, and therefore one which, after any appropriate restrictions have been made to it in each case, ought to fit into any of the theories of entailment.

\textit{Transformation in abstract}

As we have already mentioned in (1.8), we are using '+' in such a way that

\[ (*) \, \alpha(A) \& (A \to B) = \alpha(A) \& \alpha(B), \]

where \( \alpha(A) \) is any admissible expression and \( \alpha(B) \) is the expression obtained by replacing \( B \) for \( A \) in \( \alpha(A) \). In the special case where \( A \) and \( B \) are propositional variables and \( \alpha(A) \) is \( 'A_\top' \), the equation (*) might be considered to become equivalent to a statement of the Modus Ponens rule and its converse. So might

\[ A \& (A \to B) = A \& B \]

or

\[ (A \& (A \to B)) \not\to \top = (A \& B) \not\to \top. \]

However, for the moment let us leave aside the confusions which result from the Aristotle-Leibniz-Russell identification of any proposition \( P \) with "\( P \) is true", and let us concentrate on '+'.

The most important point about '+' is probably the fact that if it is treated as if it were an extensional relation then, paradoxically, it is seen to be context-dependent because (from \( (*) \)]
\[ \ldots \&A\&(A+B) = \ldots \&A\&B \]
does not in general hold (at least not from (*), and certainly not on our intended interpretations of '=' and the primitive connectors), nor does
\[ \ldots \& (A/!) \& (A+B) = \ldots \& (A/!) \& (B/!), \]
nor
\[ \ldots \& (A/!) \& (A+B) = \ldots \& (A/!) \& (B/!), \]
where the same expressions have to be substituted for three dots on either side of an equation.
In the current literature, entailment is generally confused with the transformation which is one of the things involved in entailment, and this transformation is treated (without any explanation why, other than the weight of accepted tradition) as an extensional relation. Accordingly, there arises the need to explain why really (so it is supposed) it is not at all context-dependent after all. This amounts to the requirement that A and B need to be in some sense "relevant" to each other. We have already in effect stated what we believe is the correct "relevance" criterion in (*) above - that is, the sense in which A and B need to be "relevant" to each other is that they must each be part of appropriate expressions which are the same but for the fact that one has A in it where the other has B, where by "appropriate" expressions we mean any expressions \( (@(A) \& (@(A+B)) \) such that \( @((A) \& (A+B)) \) is or contains the scope of the relevant occurrence of '+'; this scope differs with different occurrences of '+'; - but in all the current literature the approach is to require either that A and B be "relevant" to each other in a completely unspecified sense of "relevance", or else in an inadequately weak sense.
It is important to mention, in connexion with '+', that the concept of equivalence - deductive equivalence or material equivalence - can lead to the use of disguised forms of '+'. Entailment is the preservation of truth through change, and is therefore the preservation of truth through the application of '+', or of something equivalent.

**Assertion and entailment**

What the exact connexion is between entailment and assertion is obscured by various things. Although entailment is normally spoken of as a relation, it is clearly meant as more than that: it is normally an asserted relation, and moreover one which is correctly asserted, or at least confidently believed to be. And it is a relation asserted either within a logical theory or in circumstances which can be adequately formalized in an appropriate logical theory.

The assertion of an entailment relation is also sometimes confused with its demonstration. Furthermore, as Spencer Brown points out (p.xiv), there is a clear distinction "between the proof of a theorem and the demonstration of a consequence. The concepts of theorem and consequence, and hence of proof and demonstration, are widely confused in current literature, where the words are used interchangeably". That is not all. Entailment is also quite often confused with deductive calculation (whether formal or informal).

We do not propose in this work to attempt to untangle all the intricacies of the relationship between entailment and assertion. We do not in any case consider this to be possible without an exhaustive comparative treatment of the logical properties of any proposition $P$ as distinct from those of the proposition "$P$ is true". Some
small beginnings of such a treatment are found in Chapter 4.

**Why Modus Ponens?**

If A entails B, that means that A is equivalent to B conjoined with something else (where by "conjunction" we mean use of our '&'). It would seem to be more precise and rigorous to demand to know in each case what that something else can be. For this reason, and in view of the difficulties with entailment mentioned just above, it might be wiser, in formal logic, to use something deductively equivalent to the entailment relation but not identical to it. For example, since

\[(p \Rightarrow q) \equiv (p \land q \equiv p)\]

is a thesis of the propositional calculus, it follows that the derivation of B from A in the propositional calculus is deductively equivalent to the derivation of 'A&B' from A. This suggests that formal entailment might without harm be definable by:

\[A \text{ entails } B = A = (A \& B)\]

Df

or

\[A \text{ entails } B = A \neq (A \& B)\]

Df

or even perhaps

\[A \text{ entails } B = A = B .\]

Df

Or one might even (cf Chapter 4) abandon the use of an entailment relation and a Modus Ponens rule altogether, and instead use the definition

\[P \&(P/Q) = P \& Q\]

Df

or

\[P \&(P=Q) = P \& Q .\]

Df

Why not? Is there any reason against it other than the weight of a
dubious tradition which has led to proliferations of logics of entailment, natural deduction, and relevance?

(1.10) THE TOKEN PROBLEMS

The omission of the logic of tokenicity from PM

One of the deficiencies of PM as an account of formal logic is its total failure to provide an account of tokenicity. One reason for this failure is that in PM there is no distinction between what we have termed token space and physical space; at least, there is no recognition of the existence of any such notion as that of token space. Yet our minds do employ clear mental images, or memories, of token spaces as well as of physical spaces. (We do have memories of unique and unrepeatable events and situations and circumstances.)

Every token space is unique and unrepeatable, but physical spaces are repeatable; hence the constituents of token space can all be described as "tokens" and the constituents of physical space can all be described as "types" thereof. Whatever is admissible as a symbolic expression in our formalism must occupy token space, and therefore be a token. In whatever sense it can be considered to occupy physical space, in that sense it is a type as well. We have here a basis for discussing tokenicity of symbolic expressions, which the writers of PM certainly did not.

In this work we are principally concerned with the tokenicity of symbolic expressions and not with tokenicity in general. The latter raises many issues from the theory of reference, and so cannot be properly handled within pure logic.

Place-markers

Our usage of place-markers overcomes another of the problems contemporary
logicians have faced in their attempts at analyzing tokenicity. The way tokenicity has been treated in the literature up till now has been to give different, metalogical names to all tokens of the same type. By contrast, place-markers are part of our object logic. This is true even though place-markers are names of token spaces (strictly speaking, spacings), which are abstract objects (activities). In our view it is high time that the universal prevalence in Western logic of what we call Aristotelian realism was seriously challenged. By Aristotelian realism in logic we mean the doctrine that, for the purposes of logic, reality must be graded into levels of abstraction, with a different level of formal language corresponding to each level of abstraction, and with a strict taboo on mixing up within the same level of language the constituents of one level with entities obtained directly from these by abstraction. A type is obtained directly from its tokens by abstraction. Until modern logic permits the mixing of different levels of abstraction, there never will be an adequate analysis of tokenicity.

**Contexts**

One way to view tokens is to consider them as replicas of one another. This raises two questions, however. The first is: What is to count as the identity criterion by which given entities are to be adjudged as identical replicas? This has already been discussed in (1.6) Part 2.

The second question is whether, when we replicate an occurrence of a given symbol - "a", let us say - we wish to replicate the whole context in which the occurrence of "a" in question happens to be embedded, or whether we wish to replicate it as divorced from its context of occurrence. The latter choice carries difficulties because it is not generally possible to precisely specify the individual token space in question except relative to a context (that is, to use our previous terminology, a grouping or a combination) of certain token spaces.
It seems much more desirable, therefore, to speak of tokens relative to a given context or contexts rather than in an absolute sense. We elaborate on this in Chapters 6 and 8.
CHAPTER 2 - AN INTRODUCTION TO THE LOGIC OF DISTINCTIONS

"Logic is to the philosopher what the telescope is to the astronomer: an instrument of vision." - Langer, p.41.

(2.1) RELATIONS AND DISTINCTIONS

Introduction

The reader may by now be wondering why the foundations for logic dealt with in Chapter 1 are especially relevant to the logic of distinctions rather than to the logic of, say, definitions, or of distinctions and compounding, or whatever. In this chapter we attempt to clarify the motivation for the choice of the title of this work. We also look at some of the general characteristics of distinctions.

In pure logic, relations are equivalent to distinctions!

Every distinction is clearly a relation (because it is something that holds between or connects things).

And using/applying/creating/asserting the distinction between an antecedent (i.e., first term) of a relation R and its corresponding consequent(s) (i.e., second term(s)) is the same as (respectively) using/applying/creating/asserting the relation R.

Hence in pure logic using relations is equivalent to using corresponding distinctions. This is not to say that all relations are indistinguishable from distinctions, by any means. It is just that they are indistinguishable for the purposes of (pure) logic; there certainly are extralogical reasons for distinguishing between them (these include philological reasons at least).

Relations as the basis of logic

In Chapter 1 we derived the primitive concepts of first order predicate logic and the elementhood relation from the concepts of '={', of the four primitive connectors, and of token spaces, and we required that
individual token spaces can only be differentiated between relationally - that is, their participation in various logical relations is in each case inseparable from what constitutes their identity. Thus our formal logic can justifiably be called "the logic of relations". And as we have just seen, "the logic of distinctions" is effectively the same as "the logic of relations". Hence the title of this work.

In this chapter we concentrate mainly on distinctions, however. We reserve the emphasis on relations rather than distinctions for Chapters 3 and 4.
Why propositions? Why always be dogmatic?

The traditional logic confined itself to the analysis of propositions and of the relations between groups of propositions, and it was tacitly presupposed that such an analysis can be successfully applied (even if sometimes in a contrived manner) to account for all instances of what can be counted as deductive argument. Orthodox modern logic adopted this presupposition (appropriately reformulated) as its own, and as a result there remains today a widely prevalent belief which the reader is asked to consider suppressing while he is reading this chapter, and no less than discarding after he has read this chapter. This is the belief that logical reasoning (or, at any rate, that part of logical reasoning which orthodox modern logic is capable of exemplifying) consists solely of two things: the assertion of (at least the formal component of) propositions (or, if one likes, statements, or sentences) which are invariably related to one another in one or another of certain regular, clearly defined ways, and the manipulation or combination of these relations between propositions. To give one example of how the general acceptance of this belief works out in practice, the only strings of symbols normally permitted to exist in an object language are those which represent propositions - that is, the wffs.

Why should the scope of logicality be limited to assertions (or judgments)? Assertions (or judgments) are surely not the only means of expressing rationality, nor the only sort of meaningful sentence, and it is even true that not all illocutionary acts are assertions. Why, then, should propositional logic play as fundamental and dominating a role as it does in many current conceptions of logic? And is it not significantly ironic (if not intolerable) that in symbolic logic the truth or falsity of propositions is irrelevant and only their structure
is relevant while at the very same time the chief identifying
classifier of propositions is that they are either true or false?
Given the right circumstances, it may be eminently rational to ask a
series of questions rather than to utter a series of assertions; or
to express one’s own individual intentions or beliefs rather than
assert any objectively or universally true statements; or to celebrate,
or contemplate, reserving final judgment about what one is doing, rather
than refuse to involve oneself in what one is about to do until one has
first passed precise judgment on it; or to be utterly spontaneous, in
an experience too immediate to be completely reducible to theoretical
concepts; or to attempt to hold one’s mind clear of all conscious
thought whatsoever (so that perhaps one is not thinking about thoughts,
but rather about naked reality); or to act rather than speaking at all.

Is science the sole subject matter of logic?

We quote Bochenski, The logic of religion, pp. 7-8:
"...the historical situation in Western logic has been this: it has
practically always been developed for the sake of science and especially
of mathematics. This was the case with Plato, with Aristotle in his
Posterior Analytics, and finally with Whitehead and Russell. But
science is uniquely interested in propositions. Consequently, the
scope of logic, especially formal logic, has been limited to
propositions and their parts. Moreover, since in science all
propositions which are not logical are factual, logic has been
constructed in such a way as to be able to deal with the factual only.
"But this is by no means a necessary limitation of logic, at least
not where formal logic is concerned. During the twentieth century
a broadening of the scope of logic has been carried on in at least
two directions. One is the ... formal logic of morals, that is,
of a field in which most formulae are not indicative sentences but
imperatives. As to the second ... Austin convincingly showed that certain formulae - called by him "performatives" - can become the object of a formal-logical study. As is known, these performatives can be neither true nor false, but they have two analogous properties, called by Austin "happiness" and "unhappiness", and nothing forbids the construction of a formal logic for them."

Ibid., pp.27-28:

"A remark about the poverty of most semantic analyses of the present day is in order ... It is all too often said that every meaning is either theoretical or emotional; and by "theoretical," propositional meaning is meant. By this (1) all non-propositional complete meanings, such as those of performatives, imperatives, and so on, are reduced to the status of purely emotional meanings, which is simply preposterous; (2) such meanings as those carried by music [e.g. the internal meaningfulness of a Bach fugue] are also declared to be purely emotional, which is - if this is possible - more preposterous still. The reason for this poverty is, however, clear: most semanticians are exclusively interested in science. Science is, however, composed of propositions; it is, consequently, sufficient to divide all meaning into propositional and "other". Yet ... there is many a meaning which is not propositional and yet not emotive at all."

There is no universal sharp dichotomy between that which is true or false and that which is illusory. If there were one, the poetic or aesthetic imagination - perhaps imagination in general - could not exist.

Propositions comprise one end of a spectrum

Some further light is shed on this whole issue by Nuchelmans (pp.2-3):
"... we can ask ourselves whether that which, as the object of an act or attitude of holding something true, is a bearer of truth or falsity can also be the object of other acts or attitudes which have to do with truth-values in a more indirect way. As an extreme case we might take the situation in which one merely entertains a proposition, putting before one's mind the thought that something is the case without either accepting or rejecting that thought. The state of affairs conceived in such a neutral attitude might be regarded as the potential object of acts or attitudes of a more committing kind. And among these latter could also be reckoned acts or attitudes of wishing, regretting, and even commanding, where the connection with truth-values still is a remote one.

"As intermediate cases between the acts or attitudes we have just mentioned and full-blown acts or attitudes of holding something true we have such acts or attitudes as asking, doubting, supposing, assuming. The objects of the acts of supposing and assuming are particularly important in so far as they can be considered as the terms of logical relations, such as the relation of entailment between premisses and conclusion, the relation of implication between antecedent and consequent, or the relations of contradiction and contrariety.

"So there seems to be a continuous transition from the extreme case of merely entertaining a proposition to acts or attitudes of actually holding something true."

**Conclusion**

We must therefore insist that the following root-cutting statement of Russell's (A critical exposition of the philosophy of Leibniz, p.8), if it is not simply quite wrong, is at least in need of cautious
and unusual interpretation:

"That all sound philosophy should begin with an analysis of propositions, is a truth too evident, perhaps, to demand a proof."

(2.3) SENTENCES ARE NOT ENOUGH

Even if contemporary formal logic had advanced to the point where it could fully account for the logical properties of all the different kinds of English sentences, this would still be inadequate, for the following reasons.

Firstly, it would be inadequate because, in at least some sense, the logically fundamental units of linguistic discourse are, at least under some circumstances, phrases rather than whole sentences, and so are worthy of study in themselves and on their own. We have already seen one illustration of this in (1.7). As Quine points out (p.9):

"Not that all or most sentences are learned as wholes. Most sentences are built up rather from learned parts, by analogy with the way in which those parts have previously been seen to occur in other sentences which may or may not have been learned as wholes. What sentences are got by such analogical synthesis, and what ones are got directly, is a question of each individual's own forgotten history."

Secondly, it would be inadequate because abstraction from ordinary (that is, European) language is not the only major source of contemporary formal logic. Another source is pure mathematics and the critique thereof, and then there is also the scientific tradition generally.

But in any case, throughout this work we are concerned ultimately with distinctions or, equivalently, with relations on their own, and
relations can be fully and completely expressed by proper parts of sentences such as "the aura around the moon tonight", and linguistic expressions of such relations only become transformed into sentences by transforming the relations into something generally different, that is, asserted relations, as in "There is an aura around the moon tonight". The significance of this difference is expanded on in the subsequent sections of this chapter, and more fully in Chapter 4. For this reason we cannot be satisfied with just "the logic of ordinary language".

Some of the different sorts of discourse which need not make any use of full sentences are described by Alexander (p.67):

"Expressive-appraisive discourse roughly makes particular uses of signs and symbols which primarily express feelings and emotions. Conative-persuasive-incitive discourse uses signs and symbols which warn and command. Designative-informative-descriptive discourse may use signs as labels and pointers, but mostly uses symbols as idea indicators. Interrogative-explanatory discourse also uses symbols mostly as idea indicators, especially where doubts arise and explanations are necessary from ideas whose referents lie beyond sense experience."
(2.4) WHY A LOGIC OF DISTINCTIONS IS USEFUL

"If a community of fishermen all possess fish-nets with openings three inches in diameter, they run a serious risk of ignoring and even denying the existence of two-inch fish."
- Ornstein, p.xii

Introduction

We now ask the reader to consider again the general question of what is logic. We ask the reader to consider the consequences of what is a broader conception of the scope of logic than that of the traditional logic (cf (2.2) ) and even, as we saw in (2.3), than that of orthodox modern logic. We ask him or her to consider "logical" reasoning to consist of the drawing and use of (conceptual) distinctions (and possibly also of other things as well); and in Chapters 4 and 8 we attempt to begin to trace some of the consequences of this view of logic, using the symbolism introduced in Chapters 1, 3 and 6. Such a conception is broader than the traditional conception for the following reasons. (1) It is not at all necessary to make a statement in order to draw or entertain a distinction in one's mind, even though on the other hand the successful assertion of an English sentence presupposes the drawing of certain distinctions (e.g. in English it always presupposes a subject/predicate distinction and a grammatical/ ungrammatical distinction) and also presupposes (in the case of an asserted sentence) that the sentence must lie on one or another side of certain distinctions (e.g. the true/false and the use/mention distinctions). (2) A proposition is no more and no less than whatever is a significant object of the true/false distinction, so the general logic of distinctions must be in some sense more general than the logic of propositions.

In this work, therefore, we are concerned with penetrating beyond
confinement of logical reasoning simply to the milieu of propositions. But let us be more specific. We now expound some motivations for the study of the formal characteristics of distinctions.

1. Sentences are not always statements

We have already explained in (2.2) that not everything that is a sentence is also a statement. To emphasize this point again, in a slightly different way, we quote Goddard and Routley (p.24):

"To use a sentence to ask a question is to use it interrogatively; to give an order, imperatively; to express a wish, optatively; to make a statement, statementally, or as we ... sometimes say, indicatively. These descriptions are adverbial since they qualify the use of the sentence and not the sentence itself.... The sentence used in an indicative act, for example, need not itself be a grammatical indicative."

2. Peirce's theory of logic

Furthermore, the logic of distinctions may be construed as a significant illustration of the import of Peirce's characterization of logic as the general theory of signs, especially, perhaps, as far as nonlinguistic or "prelinguistic" signs are concerned; since Peirce's characterization of logic is evidently the most inclusive one in existence, any theory which may in any way help to vindicate it is of special interest.

3. Reasoning qua distinction-mongering

The thesis that the logic of distinctions has to do with the core of what is to be counted as "logic" draws intuitive justification from the very widely held view that the drawing of distinctions is a major and indispensable component of all rational thought, - provided
of course, that one is prepared (as one normally is) to view logic as including (or consisting of) the study of the characteristics common to all serious rational thought.

4. The basis of logic in the mathematical tradition

Here we need to make use of the identity of the logic of distinctions with the logic of relations. Modern mathematics is based on relations. As an illustration of this fact we quote Kung (p.27):

"Where was [Russell] to find an acceptable linguistic form for relation sentences? He found it in the logic that had been developed by mathematicians since the middle of the 19th century. As Russell himself reports, his interest in it was aroused in 1900 at the International Congress of Philosophy in Paris, where he became acquainted with the school of G. Peano and was fascinated by the precision of its method: Russell was a mathematician himself but had been disillusioned by the inaccuracy of mathematical arguments... Peano's logic was not restricted to considering the extension of concepts on the subject-predicate model, but was based on monadic and polyadic functions, such as are expressed by 'x is a number', 'x is the successor of y', and so on. After all, most mathematical sentences deal with relations such as "greater than", "less than", "successor of", and so forth. This was precisely what Russell was looking for. It is therefore not surprising that he was enthusiastic and at once began to elaborate further the theory of polyadic functions, i.e., the logic of [the extensions of] relations. (The results of this work are incorporated in The Principles of Mathematics and in Principia Mathematica.)" 

Again, Lyndon (p.8) divides all formal expressions into variables,
functions, relations, and seven logical constants. It would seem significant that the author of a book on logic for mathematicians does not make formal sentences the basic kind of expression, but rather uses a basis for logic rather similar to (though less general than) that given in this work.

5. The existence of ergative natural languages

Logic should at the very least be able to account for the most general or commonly occurring structures in natural languages. But it is a commonplace that there exist natural languages whose grammatical systems are utterly different from that of English. Kung (p.7):

"To mention only one fundamental difference: apart from languages in which sentences are constructed on a bipolar subject-predicate schema, such as the European languages with which we are familiar, there are others, so-called "ergative" languages (for example Basque, classical Tibetan, and Eskimo) which are uni-polar and where a nounlike verb is qualified by the rest of the sentence. For instance, whereas we would say in English: "Buddha has taught the doctrine", or "The doctrine has been taught by Buddha", there is only one construction in classical Tibetan, a more literal rendering of which is: "There has been teaching (nounlike verb) with respect to the doctrine (absolutive) by Buddha (ergative)."

6. The problem of the one and the many

Johnston (p.82) explains the classical problem of the one and the many in dramatic though overstated terms, and at the same time from the point of view of mysticism:

"As the human mind penetrates more deeply into reality, it becomes increasingly aware of unity. It comes to perceive that everything is one. And, at the same time, it knows that everything is not
one. This is the great paradox of mysticism East and West ... But the Greeks ... were much less open to paradox than their cousins east of Suez; and they built up this problem into the central riddle of all philosophical thinking. They called it the problem of the one and the many, and gave it the honored place in all those philosophical disputations that shook and shaped the Western world. From Parmenides and Heraclitus to Plato, and Aristotle to Thomas Aquinas the words rang out in syllogistic disputations: 'How can it be that there is only one thing [sic] and yet there are many things.'"

The logic of distinction is relevant to this problem because the use of distinction in abstract as applied to that which is originally one is precisely what produces many (in abstract).
(2.5) SOME EXAMPLES OF DISTINCTIONS

"Alice knew which was which in a moment, because one of them had 'DUM' embroidered on his collar, and the other 'DEE'. "I suppose they've each got 'TWEEDLE' round at the back of the collar," she said to herself."

- Through the Looking-Glass, Ch.IV.

By a (conceptual) distinction we mean that term in the broadest possible sense - anything whatsoever that constitutes a (conceptual) difference. We use it with the added connotation that we are considering distinctions from the point of view of the formalization of their characteristics. Some perfectly acceptable examples of distinctions include the distinctions between: fact and fantasy; up and down; left and right; early and late; form and content; true and false; free and deterministic; sign type and sign token; practical and theoretical; female consort and (the corresponding) male consort; scientific and emotive; time and timelessness; use and mention; finite and infinite; pinkish and pink; bigger and smaller; subjective and objective; sense and nonsense; being and becoming; premiss and conclusion; mind and body; fact and value; yes and no; I and we; reality and appearance; hotter and colder; red and green; burial and oranges; reason and intuition; tables and chairs; the world and nothing; houses and honey; singing and whistling.

(2.6) DISTINCTIONS AND CONTEMPORARY PHILOSOPHY

Distinctions and contemporary logic

Since, as we attempt to elucidate in this work, much of formal logical reasoning can be reduced to the drawing of distinctions and their application, one might hope that logic, which deals with the
principles common to all analytic reasoning, could give some account
or description of what a distinction is in general and how it can come
into being and how it works. But contemporary logic does not do
this at all. It only accepts distinctions which are ready-made. In
the eyes of many contemporary logicians, it would no doubt be an
enterprise outside their province to attempt in any way at all to reduce
concepts, symbols, or atomic propositions to more fundamental constituents
thereof. And yet no-one will deny that, for example, one cannot have
access to the use of concepts unless one is in the first place able to
draw distinctions.

_The Kafka-Wittgenstein moral_

In Kafka's novels the chief character invariably dies at the end
as a result of some sort of punitive process over which he has had no
conscious control. However, he never considers the possibility of
interrupting his ceaseless attempts to justify himself and to create
"the right impression" with the largely unseen judiciary or other
authorities on whose favour his freedom and continued existence depends.
Therefore he never honestly reveals the person who lies behind the
paranoic and sycophantic facades he perpetually shows, and never acts
without prejudging the reactions of the other characters in the story;
and this, as we choose to interpret Kafka, is precisely what he is guilty
of.

He never tries the simplest and most obvious solution to his dilemma:
that he is after all not consciously guilty of anything, and should
accordingly not try to justify himself. In a somewhat analogous way,
the philosopher should be wary of whatever is so obvious and omnipresent
or omniimplicit in the elaborate ratiocinating that makes up philosophy
that (he may be "alienated" from it, that is:) he may never even think
of stopping to consider how important it might be - unless someone has
pointed it out to him. And language of course is (or, before
Wittgenstein, was destined to be) a prime example of just such a thing.

Has the moral been extended as far as it will comfortably apply
in philosophy? It is surely worth dwelling on this serious question
at least a little longer here. Certain very special distinctions,
such as the subject/predicate distinction, for example, are presupposed
or "given" in all sentences of the English language and, indeed, in all
sentences of orthodox formal languages. This can be most unfortunate,
because, as we have already mentioned, modern formal languages have no
systematic means whatever of providing a critique of nor, more generally,
of studying the general or universal characteristics of (the intensions of)
distinctions, whose use, however, is a ubiquitous aspect of rational
reasoning. This becomes particularly significant, for example, when
one considers the fundamental Eastern critique of Western philosophy
that in mainstream Western philosophy distinctions are often used
uncritically and indeed their implementation is often too hastily
presumed to be completely and irreversibly self-justifying (see the
discussion of relational metaphysics in Chapter 3, and also Chapter 9).

It is not as if this deficiency were irremediable. The majority
of logicians and analytic philosophers will readily acknowledge that,
at least as far as logical things are concerned, what one cannot talk
about clearly in a natural language one must try to invent new signs
and formal systems, or even new formal languages, to express, or else
one must make use of already existing symbol systems if these happen
to do the job. Hence the feasibility of and need for a calculus of
distinctions.

**Distinctions and linguistic analysis**

The so-called "linguistic turn" in philosophy came as a result of
the realization that language, despite its once supposedly quite
innocuous omnipresence in philosophical discourse, in fact conditions
all philosophical discourse. As we have already emphasized, the
position of distinctions with respect to contemporary philosophy is
analogous in some ways to the position language was in before this
"linguistic turn". At present the use of distinctions is completely
taken for granted: although present in all rational reasoning, they
are assumed to be, in themselves, passive, inert, topic-neutral
vehicles of thought.

Now because language normally accepts all its most commonly used
distinctions as "given", it necessarily follows that in enquiring
whether or not distinctions exert only a passive effect on what can
or cannot be thought (or, if one likes, on the existence of philosophical
problems within the terms of a given language), we are largely dealing
with issues which are epistemologically prior to the considerations
involved in the philosophy of language. All the more reason for us,
then, surely, to suspect that perhaps the Kafka-Wittgenstein moral is
relevant here.

Next, consider the following brief and hopefully typical argument
that might be offered in defence of analytic philosophy:

Some of our beliefs, values, attitudes, notions, and judgements are
conditioned, by environment, friends, society, nature, ideologies,
chance, and so on, more than others are. It is very important
to be able to at least begin to separate the latter from the former,
to discern between the more completely conditioned and the relatively
unconditioned. The notions of certainty, truth, meaning, fact,
and so on, are largely dependent on the possibility of such a
discernment, or one very much like it. Consequently it is very
important to scientifically isolate the most reliable criteria
used for such discernment. Moreover, the more obvious and the
more commonly used or presupposed such criteria are, the more
scope there is for forgetting them through overfamiliarity; therefore they should be stated and described and distinguished and analyzed as clearly as possible.

In the light of standard defences of analytic philosophy such as this one, we suggest to the reader that analytic philosophy is hypocritical in the sense that it overlooks and ignores the existence of (the intensions of) distinctions, since an uncritical (or, at any rate, unquestioning) acceptance and use of (the intension of) certain distinctions lies within the very basis of the deductive logic and the common-sense philosophy and the linguistic analysis which are the very corner stones of analytic philosophy. The very criteria of critical rigour which analytic philosophy so often applies with unwavering confidence are therefore, in respect of distinctions, themselves in need of critical and stringent questioning. Here, again, the logic of distinctions can be relevant and useful, as we have already seen, in some ways, in Chapter 1.
Not every distinction is a dichotomy. That is, for some distinctions it is not true that A lies on one side of the distinction if and only if it does not lie on the other, where A represents any entity to which the distinction, or the concept of lying on one side of the distinction, can be meaningfully applied.

The truth/falsity distinction, as applied to propositions, is normally taken to be a dichotomy. This fact is often expressed by the conjunction of the laws of contradiction and excluded middle. Since the truth/falsity distinction is only one example of a distinction, the question naturally arises of what are the analogues of these (so-called) fundamental laws in the general case - that is, if we generalize from the truth/falsity distinction to any arbitrarily selected distinction -, and of whether or not these hold or may reasonably be taken to hold.

We generalize the law of contradiction as follows, for any given distinction:

**Mutual exclusion condition:** Nothing is ever permitted to lie on both sides of the distinction.

We generalize the law of excluded middle as follows, for any given distinction:

**Irrelevance closure condition:** If C does not lie on either side of the distinction which differentiates between A and B, this being the distinction which has been given, then it is impossible to find a distinction such that C and A lie on one side only of it while at the same time B lies on the other.

In (2.8) we give examples of distinctions for which neither of these conditions holds.

From now on we call a distinction exhaustive if it satisfies the
second condition - that is, if wherever it is coherently applicable it is also instantiated -, and exclusive if it satisfies the first condition.

(2.8) THE FORMALIZATION OF DISTINCTIONS: SOME BEGINNINGS

An ultimate distinction

Every act of selection tacitly makes use of the distinction between what is being selected and what is not. Since we are assuming that every formal act of inscription (of symbols or place-markers) involves and represents an act of selection, this distinction is an ultimate one (if not the ultimate one) in the sense that it is used every time we write down any part of our symbolism. Indeed, it is a universal truth that we can only ever know what something is by in some way opposing it to what it is not. This distinction is also closely connected with the distinction between one and none.

This distinction is also worth special attention because some of its properties are the same as those of that distinction which is obtained from all other distinctions by means of abstraction - "the representative distinction". Both are alike in their possession of the following characteristics:

(1) Possession of two sides (There is nothing special about this, however. Every distinction either has precisely two sides or else is equivalent to the combination of a number of distinctions which each have precisely two sides.)

(2) Asymmetry of some sort; in general, on one side of the distinction there is found a comparatively limited group of entities all bearing some sort of mutual resemblance; on the other there is usually a less limited and more diversified collection of entities, of which often at least some will be utterly dissimilar. This is not quite always the case, but it is true in the majority of cases.
(3) Presupposition of the distinction in every conception, every judgment, every piece of formal reasoning.

(4) "Otherness": it is the fact that the two sides of the distinction are in some sense "other", or distinct, from each other, and not that they exist - they may or may not exist -, that makes the distinction coherent and communicable.

On the other hand, the distinction between selection and non-selection is clearly a dichotomy, whereas not every distinction is a dichotomy, as we shall shortly see.

Selection and Spencer Brown's theory of truth

Every formal inscription occurring in a logical theory normally carries with it an intended interpretation. This intended interpretation invariably constitutes some aspect of reality or an abstraction from some aspect of reality. As we noted in (1.4) and again, more explicitly, in the two preceding paragraphs, to concentrate one's attention on it is to concentrate on some aspect of reality to the exclusion of other aspects, and hence to temporarily adopt a distorted view of reality. This is because it is impossible to select one thing without excluding others.

Consequently, every act of symbolic inscription represents a (temporary) distortion of reality.

Now consider the notion of an act of symbolic inscription in abstract; that is, consider the general notion of any act of symbolic inscription. It follows from the above that this a notion of (among other things) distortion of reality, of the substitution of a distorted picture of reality for reality itself. For our purposes of symbolization, as explained in (1.4), this notion is in fact exactly the same as that of selection, as was also explained in (1.4).

Spencer Brown represents this notion by his symbol "\( \neg \)", and he
attempts to represent "reality" itself by the use of empty spaces. Thus "\( \Box \)" by itself is really the same as "\( \Box \)" operating on an empty space (and in particular, on the square piece of empty space of which the symbol forms two sides, and which for Spencer Brown represents one "unit" of reality), and hence it represents, among other things, the general concept of distorting reality. He in fact seems to regard the operation associated with "\( \Box \)" as that which interchanges "reality" and a symbol for "reality" with each other, since he identifies its iteration with one "unit" of empty space (which is presumably for him equivalent to the trivial replacement of "reality" for itself). We sympathize with his motives for using an empty space to represent the non-selection side of the distinction between selection and non-selection: it is, after all, as we interpret him, an imaginative and courageous attempt to communicate the incommunicable (i.e. "reality"). Nevertheless, he still effectively uses empty spaces as symbols, and for the purposes of this work place-markers are at least as good as empty spaces in fulfilling roles analogous to what we understand to be those of his empty spaces (although our concept of token space was in fact obtained completely independently of Spencer Brown's influence, but we were consciously influenced by the Nyāya theory of negation).

We may observe in passing that the concept of selection in abstract is also equivalent, for the purposes of our symbolization, to the concept of "thisness" in its various equivalent forms; for example, as it is used in phrases of the form "the A here (-now)" or in sentences of the form "This is A". This point will become especially relevant later, when we discuss the formalization of the notion of a context of use.

The concept of selection in abstract also has a very great deal in common with such notions as nameability, specifiability, recognition, initiation of communication, and determinateness. We do not comment on these here, but we do comment on what we surmise Spencer Brown intends
as the basis for his theory of truth. This is the idea that, in the context of propositional logic, an empty space represents truth (because truth is "reality" asserted), and its distortion by means of "\( \neg \)" represents falsehood. Hence for Spencer Brown, a proposition is true just in case it is equivalent to (empty space), and false just in case it is equivalent to "\( \neg \)". It is not always clear whether he distinguishes between truth and logical truth, but in spite of this the link between the concept of truth and such concepts as that of selection is in our opinion of very great interest.

We mention these matters here because they would all seem to be involved with the very general problem of how one should go about formalizing the fundamental general concept of distinction in abstract.

**Distinctions and pairs of properties**

With many distinctions we can associate a pair of independent properties, \((F, G)\), where \(F\) and \(G\) are both properties; properties will be formally defined in Chapter 3. The intended interpretation of such pairs covers "properties" in as wide a sense as possible, so that, for example, being either Fred Bloggs or the scorched side of Mercury is for us a perfectly acceptable example of a property, even though it is not a natural property.

We represent the distinction corresponding to \((F, G)\) as applied to \(A\) and \(B\) by

\[
(F, G)(A/B)
\]

and we define this as follows:

\[
(F, G)(A/B) = (FA\&GB)\lor(FB\&GA);
\]

Df

intuitively, this represents \(A\)'s and \(B\)'s being on opposite sides of the distinction between being \(F\) and being \(G\).

Similarly,

\[
(F, G)(A/B) \overset{\text{Df}}{=} (FA\&FB)\lor(GA\&GB);
\]

this signifies \(A\)'s and \(B\)'s being both on the same side of this
distinction. We now look at some examples of distinctions which are thus reducible to pairs of properties.

**Concerning the true/false dichotomy**

One weakness of the true/false distinction is the fact that it is not an example of the most general sort of distinction - it is, after all, a rather limited sort of distinction, namely one which is both exhaustive and exclusive. To see the significance of this, let us consider some other examples of distinctions.

1. The use/mention distinction. Although exhaustive, this is not exclusive (in fact, every instance of mentioning is really also an instance of using).

2. The at least sometimes/always distinction. This has very similar properties to those of the use/mention distinction, except that it is not exhaustive (and is to be compared with the at least sometimes/never distinction, which is exhaustive).

3. The yellowish/bluish distinction, where a colour is "yellowish" if it can be obtained by mixing the pigment yellow with some other pigment(s) (and analogously for "bluish"). This is neither exhaustive nor exclusive. It is not exhaustive because although one can coherently ask whether the colour red is yellowish, or bluish, red is neither; and it is not exclusive because the colour green is both yellowish and bluish.

4. The distinction between numbers divisible by the square of a prime and numbers divisible by $p+1$ for some prime number $p$. This is similar to the yellowish/bluish distinction.

5. The sweet/sour distinction. This is perhaps even more similar to the yellowish/bluish distinction.

One statement which we can make about the yellowish/bluish distinction, for example, is that, given any colour, that colour will either
1. lie on both sides of the distinction, or
2. lie on one side of the distinction only, or
3. lie on neither side of the distinction at all.

Now consider what would happen if instead of assigning one of the two usual truth-values (truth or falsity) to every sentence, we decided to assign a colour to every sentence in some systematic way. In particular, what if we were to assign yellow to every true sentence and blue to every false sentence; would we not then be free to assign other colours to the many sentences which make no claim to having a truth-value (such as sentences which express feelings, likes, fantasies, expectations, nonsense, uncertainties, and so on)? Moreover, would there be any analogue in any of the most familiar logics (such as any of those having the propositional calculus as part of its foundation) to such concepts as "lies on the 'yellowish' side of the yellowish/bluish distinction (though possibly also on the 'bluish')" or "does not lie on the 'bluish' side of the yellowish/bluish distinction", in the same sense that "yellow" would be the direct analogue of "true"? It seems very clear that there is no such analogue, since in such logics every formula must have precisely one truth-value relative to any given interpretation.

From this we can conclude that the use of a more general distinction than the true/false dichotomy but one which may be equally if not (because of its greater universality) more coherent and clear (such as the yellowish/bluish distinction) will achieve all the work the true/false dichotomy can do, and other work besides.
CHAPTER 3 - RELATIONS AND THEIR LOGIC

(3.1) INTRODUCTION

Contents

In this chapter we present somewhat of a jumble of remarks and
details which are intended to throw light on how one may go about the
rather herculean task of developing the formal logic of (the
intensions of) relations. Most of this material was gathered at
a very late stage in the writing of this work, and there may therefore
be many possible lines of development of the ideas presented here
which are not taken up by us.

On the importance of the logic of relations

The significance of the logic of relations - even of that part
which deals with the extensions of relations alone - could hardly
be overestimated. Mathematical logic and the foundations of
mathematics derive ultimately from Peirce's theory of relations and
the ways in which this was developed further by Peano and Frege.
To quote Bochenski, A precis of mathematical logic, p. 66:

"The calculus of [extensions of] relations ... is the newest and
also the most important part of modern logic. Developed
originally for the foundations of mathematics, it has gone beyond
this science to embrace the whole of knowledge. Despite the fact
that it occupies a major place in the treatises of logic, it is still
relatively little developed."

For the early Wittgenstein, the logical properties of relations
were of paramount importance for the understanding of reality. In
Tractatus 4.021 he claims: "A sentence is a picture of reality";
commenting on this claim, Kung (p.52) points out:

"[Here] it is relational sentences ... that Wittgenstein has in
mind: "It is obvious that a sentence of the form $aRb$ strikes us as a picture." (4.012). "That 'a' stands to 'b' in a certain relations says that $aRb$." (3.1432) And he holds that the correspondence between the structure of a sentence and that of a state of affairs is more readily apparent in a formula written in logistic language. In ordinary language matters are much more complicated; for there the outward form of the dress - as Wittgenstein puts it - has been fashioned for quite different purposes than to reveal the form of the body. (4.002)"

The logically fundamental nature of relations

Every proposition whose meaning is determinate contains at least two terms. In order to belong to the same proposition, these terms must be connected by a relation. Thus modern logic is based on the assumption that relations are effectively at least as basic for logic as are propositions. Thus Langer (p.61):

"The simplest logical structures are those expressed by propositions that mention just one relation and its terms. Therefore we ... take such propositions as the most elementary material for logical study."

The reasons for this assumption are explained in Russell, An inquiry into meaning and truth, pp. 32-33:

"... [the essential unity of a sentence], whatever its nature may be, obviously exists in a sentence of atomic form, and should be first investigated in such sentences."

"In every significant sentence, some connexion is essential between what the several words mean - omitting words which merely serve to indicate syntactical structure. We [see] that 'Caesar died' asserts the existence of a common member of two classes, the class of events which was Caesar and the class of events which are deaths.
This is only one of the relations that sentences can assert; syntax shows, in each case, what relation is asserted. Some cases are simpler than 'Caesar died', others are more complex. Suppose I point to a daffodil and say 'this is yellow'; here 'this' may be taken as the proper name of a part of my present visual field, and 'yellow' may be taken as a class-name. This proposition, so interpreted, is simpler than 'Caesar died', since it classifies a given object; it is logically analogous to 'this is a death'. We have to be able to know such propositions before we can know that two classes can have a common member, which is what is asserted by 'Caesar died'. But 'this is yellow' is not so simple as it looks. When a child learns the meaning of the word 'yellow', there is first an object (or rather a set of objects) which is yellow by definition, and then a perception that other objects are similar in colour. Thus when we say to a child 'this is yellow', what (with luck) we convey to him is: 'this resembles in colour the object which is yellow by definition'. Thus classificatory propositions, or such as assign predicates, would seem to be really propositions asserting similarity. If so, the simplest propositions are relational."

The view that relations are the simplest logical structures is not confined to modern logic. In De Morgan, the observation was for the first time (in the West) made that the doctrine of the syllogism can be generalized to the general theory of the composition of relations; this was because for the purposes of the traditional doctrine all propositions are treated as stating a relation (cf e.g., Sinclair, pp.5-6).
Propositions and unasserted relations

At this point we need to introduce the distinction between asserted and unasserted relations. Any relation can be asserted to hold; but not every relation is in fact by itself an asserted relation. This fact can be demonstrated linguistically (Langer, pp.50-51):

"Any symbolic structure, such as a sentence, expresses a proposition, if some symbol in it is understood to represent a relation, and the whole construct is understood to assert that the elements (denoted by the other symbols) are thus related. In ordinary language, the verb usually performs both functions; it names the relation and asserts that it holds among the elements. But if, as is often the case, the relation is named by a preposition or other kind of word, then an extra verb is required to assert the relation.... For example, in 'Brutus killed Caesar,' the verb furnishes both the name of the relation, and the assertion that it holds; but in 'the book upon the table,' the preposition 'upon' merely names a relation, without making any assertion ..." (her italics).

Unasserted relations and the Nyāya logicians

Let us consider the following matrices or open sentences:

(1) x is a man.
(2) x is the father of y.
(3) x opposes y.
(4) x causes y.

It is significant that the Indian Nyāya logicians did not deal directly with sentences or propositions as such. "Rather," (Matilal, pp.32-33): "they would reduce a sentence expressing a cognition to a composite term or terms using the notions of occurrence and locus. Hence,
according to their convention, (1) will be reduced to what I shall call an "ascriptive" expression, e.g.,

(5) Humanity (occurs) in x 
or

(6) Humanity in x.

The difficulty with (6) is that it does not contain an assertion.

The common opinion that making a judgment consists in combining (or separating, as in negative judgments) the two different data of a presentation has not always been maintained by philosophers. The Nyāya probably held that in a declarative sentence what we assert or affirm is a single object.

Thus, we may translate the categorical sentence "John is a man" into the existential sentence "There is a locus, called John, of humanity." But at the next step, following the Nyāya, we get rid of all reference to existence and translate the existential into an "ascriptive" like "Humanity in John."...

"On the analogy of (6), (2) through (4) can also be reduced to ascriptive expressions. That is, locutions like "— is the father of ..." can also be thought to refer to some abstract relations residing in some locus and, at the same time, conditioned or ascertained ... by some entity. Such relations are called 'the relation of fatherhood' ..., 'the relation of opponency' ..., 'the relation of being the cause' ..., 'the relation of concurrence' ..., etc. Although they are called relations, they are nevertheless abstractions from the corresponding relative general terms like 'father'. Accordingly, it [is] convenient to call them 'relational abstracts' ... We translate (2) through (4) as follows:

(7) Fatherhood resident in x and conditioned by y.

(8) Opponency resident in x and conditioned by y.

(9) Causeness resident in x and conditioned by y."
The point of all this is that the Nyāya logicians managed to get by with the use of unasserted relational phrases - such as (6) to (9) - instead of ever using the corresponding asserted relations (such as (1) to (4)).

Once a satisfactory logical theory of (the intensions of) relations has been produced (if ever), therefore, we suggest that all objections to the use of in general unasserted relations plus an assertion operator rather than the use of propositions as the fundamental units of logic will surely have been overcome.

Certain relations, however, are in and by themselves asserted relations.

Two types of assertions

In our logic, assertions are formalized either in the form '...$\mathcal{T}$' or in a form equivalent (in the sense of '$=$') to this form, or else in expressions containing '$=$' (that is, in the form of equations). Both are types of relation.

Every equation holds by definition, analogously to the way that in orthodox logic a wff is always by definition an assertion. On the other hand, not every assertion of the form '...$\mathcal{T}$' holds. If the intended interpretation of '...' in '...$\mathcal{T}$' is that it represents a proposition, then the assertion in question is the assertion that the proposition is true; otherwise, the assertion in question is the assertion that '...' is a possible entity (we do not wish to commit ourselves in this work to any precise explanation of what "possible" should mean here; we do not explore this whole question any further in this work).
Introduction

By a relational metaphysics we mean a metaphysics (or a framework for looking at reality) in which what is considered (ultimately) real is relations only. We are especially interested in that particular species of relational metaphysics in which "relatedness" is the only reality, by which we mean that a relation is considered to be unreal in so far as it is itself a term taking part in other relations but is nevertheless considered to be normally less unreal than any of its terms. In this species of relational metaphysics, the meaning of a term is always entirely relative to the relations in which it participates, and has no separate reality whatsoever in itself. We call this sort of relational metaphysics "Chinese-mystical metaphysics", for reasons to be explained shortly.

Some logical consequences of holding a Chinese-mystical metaphysics

Relational metaphysics of any sort seems extremely strange to the Western mind, because the latter tends to identify reality and existence with each other. However, relations can generally have non-existent terms just as readily as existent ones - it normally makes no difference! The link between existence and reality is no longer a close one at all!

If the reader at this stage wishes to appeal to his or her Western presuppositions and prejudices and claim that, for example, life is more real (or less unreal) than death, we hasten to ask whether such a claim is based on the (in this context) philosophical bias that links reality and existence with each other. If the reader attempts to invoke linguistic evidence of any sort to support such a claim, we hasten to point out that ergative languages provide linguistic evidence in favour of a relational metaphysics rather than a substance/attribute metaphysics, and moreover, that in Hindi and Chinese the ergative and
the subject/predicate sentence-structures exist side by side. The majority of the earth's population speaks Hindi or Chinese. Therefore we find it difficult to see how a true believer in linguistic philosophy can fail to take relational metaphysics almost as seriously, if not exactly as seriously, as a metaphysics based on subjects and predicates.

To continue with the exposition of the startling consequences of Chinese-mystical metaphysics, let us observe that every relation is itself a term of further relations, so that any quest for what is "the ultimate, or the fullest, reality" (this is presumably almost the same as "the universe") in Chinese-mystical metaphysics is doomed to lose itself in an effectively unending maze of relations between relations.

Moreover, given any two things whatsoever, there always exists a relation, perhaps a rather or very artificial one, binding them together (this is the relation corresponding to the distinction between the two things in question). It does not matter how complex, indirect, impractical, or bizarre the relation may be. Hence all things are considered to be interdependent. On the consequences of this, we quote Watts, Time, pp.23-31:

"In reality there are no separate events. Life moves along like water, it's all connected as the source of the river is connected to the mouth and the ocean. All the events or things going on are like whirlpools in a stream. Today you see a whirlpool and tomorrow you see a whirlpool in the same place, but it isn't the same whirlpool because the water is changing every second.

"What is happening is not really what we should call a whirlpool, but rather a whirlpooling. It is an activity, not a thing. And indeed every so-called thing can be called an event. We can call a house, housing, a mat, matting, and we could equally call a cat, a catting. So we could say, "The catting sat on the matting."
And we would thereby have a world in which there were no things but only events. To give another illustration: A flame is something [of which] we say, "There is a flame on the candle." But it would be more correct to say, "There is a flaming on the candle," because a flame is a stream of hot gas. . . .

"So, therefore, we do not need the idea of causality to explain how a prior event influences a following event. Consider it this way: Suppose I'm looking through a narrow slit in a fence, and a snake goes by. I've never seen a snake before, so it is mysterious. Through the fence I see first the snake's head, then I see a long trailing body, and then finally the tail. Then the snake turns around and goes back. Then I see first the head, and then after an interval the tail. Now if I call the head one event and the tail another, it will seem to me that the event head is the cause of the event tail. And the tail is the effect. But if I look at the whole snake I will see a head-tail snake and it would be simply absurd to say that the head of the snake is the cause of the tail, as if the snake came into being as a head first and then a tail. The snake comes into being out of its egg as a head-tail snake. And in exactly the same way all events are really one event. We are looking, when we talk about different events, at different sections or parts of one continuous happening.

"Therefore, the idea of separate events, which have to be linked by a mysterious process called cause and effect, is completely unnecessary. But having thought that way, we think of present events as being caused by past events, and tend to regard ourselves as the puppets of the past, driven along by something that is always behind us."

Perhaps the most unpalatable consequence of Chinese-mystical metaphysics is its rejection of substances (as realities). Thus the concept of the self, or personality, is totally rejected. Watts, The way of Zen, p.67:
"It is fundamental to every school of Buddhism that there is no ego [i.e., personality], no enduring entity which is the constant subject of our changing experiences. For the ego exists in an abstract sense alone, being an abstraction from memory, somewhat like the illusory circle of fire made by a whirling torch. We can, for example, imagine the path of a bird through the sky as a distinct line which it has taken. But this line is as abstract as a line of latitude. In concrete reality, the bird left no line, and, similarly, the past from which our ego is abstracted has entirely disappeared."

And ibid., p.138:

"Submission to fate implies someone who submits, someone who is the helpless puppet of circumstances, and for Zen [and Chinese-mystical metaphysics generally] there is no such person. The duality of subject and object, of the knower and the known, is seen to be just as relative, as mutual, as inseparable as every other. We do not sweat because it is hot; the sweating is the heat. It is just as true to say that the sun is light because of the eyes as to say that the eyes see light because of the sun. The viewpoint is unfamiliar because it is our settled convention to think that heat comes first and then, by causality, the body sweats."

And ibid., pp.75-76:

"...the structure of our language does not permit us to use a transitive verb without a subject and a predicate. When there is 'knowing', grammatical convention requires that there must be someone who knows and something which is known. We are so accustomed to this convention in speaking and thinking that we fail to recognize that it is simply a convention, and that it does not necessarily correspond to the actual experience of knowing. Thus when we say, 'A light flashed,' it is somewhat easier to see through the
grammatical convention and to realize that the flashing is the light ... Our intellectual discomfort in trying to conceive knowing without a distinct 'someone' who knows and a distinct 'something' which is known, is like the discomfort of arriving at a formal dinner in pyjamas. The error is conventional, not existential ...."

So "we see how convention ... populates the world with those ghosts which we call entities and things. So hypnotic, so persuasive is the power of convention that we ... feel these ghosts as realities, and make of them our loves, our ideals, our prized possessions. But the anxiety-laden problem of what will happen to me when I die is, after all, like asking what happens to my fist when I open my hand, or where my lap goes when I stand up."

There are of course very many other interesting things about Chinese-mystical metaphysics, but we do not elaborate them here, because our primary concern is with such a metaphysics confined to the interpretation of formal logic.

(The reader will no doubt recognize some of the imagery used above as originating with Heraclitus; Heraclitus' metaphysics was not Chinese-mystical, however.)

On the interpretation of logic

Let us agree to describe the totality of all the objects and entities of first-order predicate logic and set theory as "the world of logic and mathematics". Then, as was explained in Chapter 1 of this work, we have suggested foundations for logic in Chapter 1 which may be interpreted in such a way that "the world of logic and mathematics" has a Chinese-mystical metaphysics. From now on we explicitly urge such an interpretation as the intended interpretation.

(Thus if one believes there is much truth in the early Wittgenstein's
claim that logic in some sense "mirrors" reality, one is by logical necessity faced with the need to take Chinese-mystical metaphysics quite seriously.)

On the term "Chinese-mystical"

At this point we digress to provide the reader with several references intended to justify our use of the term "Chinese-mystical". Our claim is that Taoism, Mahayana Buddhism, and Zen Buddhism all presuppose a metaphysics of this sort, and since all three are schools of mysticism which at some time or other flourished in China, the term will be accounted for once we have provided evidence for our claim.

As evidence that Taoist metaphysics is of the requisite sort, we cite Yu-Lan, p.68 (throughout the whole chapter from which this quotation is taken the author quite clearly uses the word "things" in a very general sense, in such a way as to refer at least to whatever has being, abstract or concrete, and presumably also - see pp.60-62 - to whatever is "nameable"; hence we may safely take the word "things" below as including all relations):

"From the viewpoint of the Tao [i.e., of Taoism] not only are the distinctions which men make relative. It may also be said that the natures respectively of all things are relative. So also is the difference between the "I" of me and other things. We all equally come from the Tao, and therefore the Tao interpenetrates and makes us one." (For "the Tao" in the last sentence, read "the fundamental reality"). "The Ch' i Wu Lun Chapter [of the works of Chuang Tzu, one of the two most influential philosophers of Taoism] says, [and this should be read as legitimate poetic exaggeration in order to illustrate a philosophical point the more vividly:] "There is nothing larger in the world than the point of a hair, nothing smaller than Mount T'ai, nothing older than a dead child ..."
As evidence that Mahayana Buddhist metaphysics is of the above sort, we first cite Lewis and Slater, p.73:

"The Mahayan teachers ... taught the dependence of things on each other. Nothing exists of itself or by itself. There are no separate entities..."

We also cite Watts, ibid., p.86, in the chapter in which he is giving an exposition of the philosophy of Mahayana Buddhism:

"In the whole universe, within and without, there is nothing whereon to lay any hold, and no one to lay any hold on anything. This [is] discovered through clear awareness of everything that seemed to offer a solution [to man's constraints] or to constitute a reliable reality, through the intuitive wisdom called prajna, [which for Mahayan Buddhism is the source of all knowledge], which sees into the relational character of everything."

Since it is an historical fact that the metaphysical roots of Zen Buddhism are Taoist and Mahayana Buddhist, we do not include any quotations about Zen Buddhism.

Zaidi's account

An attempt to describe a relational metaphysics of a sort different from the Chinese-mystical is made by Zaidi. Zaidi does not appear to have any acquaintance with Eastern philosophy, and his statements are accordingly couched in the framework of modern Western philosophy. We now quote from his paper rather extensively.

On p.412 he points out that the normal dictionary definitions of "relation" "suggest that things, or relata, are logically presupposed by relations - and, as an account of the way we ordinarily conceive of relations, this is unexceptionable. Hardly any philosopher has even considered the alternative. Bradley ... entertained the view ... only long enough to dismiss it out of hand. Nor has any of the
many commentators on Bradley considered this a moot point. So far as I know the alternative view has been seriously entertained by only one philosopher; that is Nietzsche, in a few paragraphs in his notebooks."

We might mention here that the status of relations has always been a burning question throughout the history of Indian philosophy. To illustrate this, we quote from Tripathi (pp. 39-40):

"Looking at the varied and complex nature of Indian systems, it would appear that it is impossible to detect any central or pivotal problem common to all of them. Although we find that they discuss questions regarding the nature of self, God, causation, substance, universals and so on, yet these are particular questions and are neither central nor common... we are trying to discover that basic problem which determines the peculiar logic of every system so that any change in that logic affects the whole metaphysical structure. The problem of relation is, in our view, such a central problem...

And let us point out that the problem raised here is not the same as that raised by Hume. There is no doubt that Hume brought the problem of relation to the forefront in modern philosophy for the first time. He claimed that the central problem of philosophy was the problem of relation in general and of causal relation in particular. The problems regarding substance and quality, etc., were really secondary. All the same, it must be noticed that Hume's formulation of the problem was different, as he was attacking the problem in the context of British empiricism. His problem - whether there is any cognitive basis for our belief in necessary relations - is epistemological, while the problem raised in the Indian systems is metaphysical. The question asked here is whether relations are real."

We now continue with quoting some of Zaidi's more significant remarks. P.414: "The relational perspective is a functional one, but
functional in more than the epistemological sense of holding that one only comes to know of things by what they do; for the relational point of view the doing or activity is metaphysically fundamental, and the status of a 'thing' is derivative, ... even if the 'activity' be no more than the continuing existence of the thing." (Cf (1.1.).)

P.423: "Relational metaphysics does not violate common sense. That might be the case if the ordinary framework had a claim of metaphysical ultimacy written on its face; but it does not. For the logical relations between the categories of the ordinary framework are not self-evident - whether, for example, the categories of quality and relation are equally basic, or whether one or the other is more fundamental... What relational metaphysics does contradict is an interpretation of the ordinary framework which holds quality to be a more fundamental category than or equally as fundamental as relation. Such an interpretation is itself a metaphysics, but an inadequate one. Conversely, relational metaphysics may be judged to be adequate because the purely relational framework provides a simple, unified, non-dualistic perspective through which we can see the underlying logical connection between the categories of the ordinary framework."

Zaidi arrives at the conclusion, on pp.420-421, that:

"we may say that quality is Hume's fundamental category ... The result is that in the Humean conception there is no logical connection between the status of a unity as a quale and as a relatum of independent relation." (By "independent" or "quality-independent" relations he means what we below mean by "external" relations.)

"Since a philosophy committed to quality as the fundamental category must also consider some scheme of independent relation as equally fundamental, it is less simple than a system committed
Zaidi's argument for the above claim about Hume's metaphysics is perhaps best illustrated through reference to another writer's work. We quote Henry, pp.118-119:

"Hume and a Broom. One day Hume, while walking up the steps of his club, received an unusually lively impression of being tumbled back down to the street by the vigorous action of a sweeper with a broom. The sweeper hastened to the bottom of the steps and helped the philosopher to his feet. 'Begging your pardon, sir!' he said, pulling off his cap, 'But I was thinking about my broom, as you might say, and how it's served me these thirty years, so I didn't see you coming, sir.' ... 'Yes, sir!' He said, 'That's on account of the new 'ead. You see, sir, I gives it a new 'ead every summer.' He replaced his cap. 'Nevertheless,' said Hume, 'the handle is likewise in excellent condition. Quite remarkable!' 'Yes sir,' said the sweeper, removing his cap - this time to knock the dust from Hume's hose, 'That'll be on account of the new 'andle I gives it every winter.' 'Come, come, my good man!' smiled Hume. 'We can only attribute identity to this broom provided all the parts continue uninterruptedly and invariably the same.' It was charming to observe the mental confusions of the lower orders.... 'All the parts, sir?' He made as if to remove his cap, but it was already off. 'But not all, sir. After all, if it loses a bristle...'

'If some small or inconsiderable part be added to the mass,' Hume explained kindly, 'or be subtracted from it: though this absolutely destroys the identity of the whole, strictly speaking, yet as we seldom think so accurately, we scruple not to pronounce a mass of matter the same, where we find so trivial an alteration.'

The sweeper replaced his cap, visibly cheered. 'You 'ad me worried for a minute, sir! I likes to think it's the same broom been
'anded down, father to son...! 'Tis the proportion of the change that counts', said Hume, taking a pinch of snuff to soothe his rising irritation. How tiresome these fellows could be! 'A broom cannot undergo a change amounting to one half of its mass and preserve its identity. You would have to supply good reasons indeed to call it the same.'

'But I've always used it for the same job, this broom. Steps and portico. You won't find me using it for the 'all or the street. No sir! This is the steps broom: always 'as been.'... 'And it's always been kept in the same place,' insisted the sweeper, 'under the servants' stairs.' 'Indeed!' said Hume. 'And it's always been used by the same person. No one else touches it, sir; not this broom.'... 'The 'andle wouldn't be no use without the 'ead, would it, sir? And the 'ead wouldn't be no use by itself, if I may put it so bold,' said the sweeper. The philosopher muttered something about 'sympathy of parts' and said 'Pon my soul!', though he knew this to be without significance. Then, to hide his feelings, he tried to take snuff from the sweeper's cap. 'See here, my good fellow!' he said, determined to settle the matter. 'You have a distinct idea of that broom handle which remains invariable until such time as it wears out. Whereupon you change it.' The sweeper nodded. Hume continued, 'You have also a distinct idea of several different objects, namely, the old broom handles and heads and their replacements, existing in succession, and connected together by a close relation.' He paused to hand the sweeper his cap. 'This, my good man, affords as perfect a notion of diversity as if there were no manner of relation among the objects.' He paused again, and the sweeper
nodded. 'Tis certain,' Hume concluded, 'that in your common way of thinking, the former idea of identity or sameness is confounded with the latter idea of diversity.' He turned towards the club door. Dr. Manderville and Mr. Hutcheson would have set up the backgammon board by now, and would not like to be kept waiting. 'Good day to you!' he said, and tossed the sweeper a coin. 'One moment, sir!' said the sweeper. 'If I may be so bold, sir. Tell me then, is the 'andle the broom, or is the 'ead the broom? If the 'ead's the broom or the 'andle's the broom, then this broom is two brooms, and at the same time 'alf a broom.'"

Relations and the sciences

In case the reader should object that it is impossible to reformulate science on the basis of a relational metaphysics, we ask him or her to reflect on the following statement by Popper (p.111): "The empirical basis of objective science has ... nothing 'absolute' about it .... The bold structure of its theories rises, as it were, above a swamp. It is like a building erected on piles. The piles are driven down from above into the swamp but not down to any natural or 'given' base; and if we stop driving the piles deeper, it is not because we have reached firm ground. We simply stop when we are satisfied that the piles are firm enough to carry the structure, at least for the time being."
In formal logic, relations are treated as universals rather than in nominalistic fashion. This is a practical necessity because otherwise one would be committed to maintaining, for example, that there are as many different kinds of negation as there are propositions, since the relation between a proposition and its negation would have to be a different relation with each different proposition. Treating relations as universals loses no generality of interpretation at all in our logic anyway, since the terms of relations are symbolized by place-markers or by symbols which occupy token spaces, so that in our notation each particular instantiation of any given relation-universal will be symbolized differently.

From the point of view of anyone who upholds the logic of relations, therefore, demonstrative terms are the terms in the English language whose characteristics most closely resemble those of the terms of logic (as the latter are always intrinsically dependent on the relations in which they participate). Quine (p. 101) describes some features of demonstratives in the following way:

"A notable trait of 'this', 'this river', 'this water', and similar terms is their transiency of reference, in contrast to tenacious singular terms like 'mama', 'water', 'Nile', 'Nadejda'. Such is the effect not only of the two demonstrative particles, but of the indicator words generally: 'this', 'that', 'I', 'you', 'he', 'now', 'here', 'then', 'there', 'today', 'tomorrow'. The child's learning of 'mama' and 'water' depended on fixity of reference; he was trained, by reinforcement and extinction on multiple occasions of utterance, to adjust to norms or boundaries of reference which were held fast for him. In
learning the indicator words he learns a higher-level technique: how to switch the reference of a term according to systematic cues of context or environment. Demonstrative singular terms thus gained have the convenience of flexibility and the drawback of instability".

(It is interesting to observe that not only are indicator words the terms of English whose characteristics most closely resemble those of the terms of our logic, but they are also those which most closely resemble the terms of Zen. Thus Watts, The way of Zen, p. 108: "Zen communication is always 'direct pointing'"

Quine goes on to describe the relationship between general terms (universals) and demonstratives in English as follows (p. 102):

"... not only are general terms useful for their yield of demonstrative singular terms, but also demonstrative singular terms are useful in getting further general terms. Now this last is an understatement. Demonstrative singular terms figure even in the child's first acquisition of general terms: he has to learn of this apple and that apple, when to identify and when to distinguish .... Demonstrative singular terms, though formed of general terms, are thus needed in getting on to the trick of general terms. The general term and the demonstrative singular are, along with identity ..., interdependent devices that the child of our culture must master all in one mad scramble." However, because our primary concern is with pure logic, we do not study such matters as this in this work.
(3.5) THE FORMALIZATION OF RELATIONS

Notation for relations

Because, for us, (formal) relations are to be universals, each relation must have two "sides", one being the collection of all its antecedents (first terms) and the other the collection of all its consequents (second terms). But lying on the same "side" of a given relation is itself a relation, and moreover at least an equivalence relation if not an identity relation. Similarly, lying on different "sides" is also itself a relation. We represent the former of these special relations, for any given relation $R$ other than '/', by '/', and we shall quite often represent the relation $R$ itself by $R$

'/' if $R$ is not one of the primitive connectors or '=' , or obtained from these by definition. (Thus the assertion of the latter of the two special relations just mentioned - that is, the relation of lying on different "sides" - is equivalent, on $A$ and $B$, to '$(E^*_A)((*^A)^{R} R (*^B)^{C_R})$'.

If $S$ or $T$ or ... is a relation distinct from $R$, a similar notation applies, with $R$ replaced by $S$ or by $T$ or by ...

Some special relations

A special case of note is the one where

$$\frac{A}{B} = \frac{A}{B}$$

holds; we call any such relation $R$ an equivalence relation because membership of either one of its "sides" is equivalent to membership of both "sides".

Properties, of course, can be regarded as one special kind of
equivalence relation, namely, the kind in which antecedent and consequent are identical (in the sense of '!='). Thus we introduce properties into our notation by the definition

\[ F^R(A) \overset{Df}{=} \left( A/A \right) \wedge (A/B \neq (B\cdot A)). \]

Thus 'F^R' is well defined only if R is a property, and 'F^R(A)' means the same as 'A/A'. We may sometimes use F, G, ..., Z, rather than 'F^R', 'F^S', ....

Since a function is commonly taken to be definable as a relation such that each antecedent has only one consequent, we have

\[ R \text{ is a function } \overset{Df}{=} \left( (A/B) \wedge (A/C) \Rightarrow (B=C) \right). \]

One interesting property of equivalence relations is that if '/ = /' holds, then S is an equivalence relation. (A moment's reflection is sufficient to establish this fact.) Accordingly, from now on we interpret '/ to mean "identity (i.e., equivalence) relative to R" while '/' will continue to mean, as we mentioned in (1.6), identity relative to the relation of extensional equivalence (that is, it means identity of extensions). In calling '/ an "identity" relation, we are following the terminology of the traditional rather than the modern logicians. Thus, Barth (pp. 204-205):

"The nature of the relation between 'S' and 'P' in "the (true) judgment", 'S is P', has always been a much debated topic. But all traditional logicians seem to have assumed that predication was closely related to or the same as identification. This assumption is dropped in modern logic. Present-day (modern) logicians hold that the sense of "is" in "Dick is Tom's oldest brother (is the oldest brother of Tom)" is radically different
from that of "is" in "Saul is brown", and that both of these differ from the "is", or "are", in covertly universal propositions like "Bantus are Africans" or "the /a Bantu is an African".

We say that A and B are mutually transparent for the relation R if R is not relevant to distinguishing between nor identifying A and B. Thus we have the following formal definition:

A is transparent with respect to B for R if

\[(A/B) \neq (A/B) \land (B/A) \neq (B/A) \land (A/B) \neq (B/A) \land (B/A) \neq (B/A).\]

In general, expressions of the form 'A/B' may be transparent for R with respect to at least some (meaningful) expressions C; if so, let us call R a functional relation. We use this term because the relations of principal interest in modern logic can be divided into the two categories of functions and predicates, two-place functions being relations which form terms out of pairs of terms. Some relations may be functional without themselves being functions; we fail to see any good natural or heuristic reason why the logic of (intensions of) relations should exclude consideration of such relations, despite the fact that modern logic normally does so.

**Polyadic relations**

If n is greater than 2, any n-adic relation is definable as a two-place relation between an argument and an (n-1)-adic relation; in this way all n-adic relations are ultimately reducible to dyadic relations, and therefore we confine ourselves to the latter in this work.

The n-adic generalization of "property" is "(n-adic) predicate", a property being a 1-adic predicate. If n is greater than 1, "n-adic predicate" means the same as (generally in itself unasserted) "n-adic
Relations and their extensions

We quote from Nidditch, pp. 51-52:

"In 'The Logic of Relatives' an attempt was made by Peirce, using the work of De Morgan, to give a general theory like a mathematics of things having relations; these will be named here 'relation things'. The relation thing 'father' among persons may be said simply to be the class of all the groupings (I:J) where I is a father and J is an offspring of I; so (James Mill:John Stuart Mill) is an element of the class that is the relation thing 'father' but (Roger Bacon:Francis Bacon) is not an element of that class. More generally, any relation thing \( r \) may be said to be the class whose elements are all the groupings (I:J) of the things I and J such that, as one would commonly say, I is an \( r \) of J (or something near this). ... as [asserted] qualities had been turned by Boole into classes of the things that have the qualities, [asserted] relations were turned by Peirce into classes of the groups of things between which there are relations."

Now we do not deny that the logical features of the extensions of asserted relations (or, to use Nidditch's term, "relation things") are no doubt an extremely important aspect of the logic of relations. But the extension of a relation consists of the relata taken in a certain order, and this is something different from the assertion of the relation itself, not to mention the (in general, in itself unasserted) relation itself. That difference is all the difference between content and form. One way in which that difference is crucial is the following.
Because this is not a static universe, many relations have extensions which are continually changing - perhaps even very dynamically so - and hence the identity of these extensions is continually changing and is contingent. But if relations are identified with their extensions, any change of extension means a change of relation, which is absurd (the meaning of the (abstract) motherhood relation does not change with pregnancy rates, or whatever). We illustrate the significance of this point in one field - that of science - by referring the reader back to the quotation from Popper at the end of (3.3).

There is another way in which the contingency of the identity of the extensions of some relations raises problems for those who would wish to identify relations with their extensions. Modern logic treats the extensions of relations as classes. Classes are mathematical objects, the identity of whose elements is analytically fixed.

Then there is the circumstantial linguistic evidence (circumstantial because ordinary language can sometimes be utterly illogical). Relations are typically exemplified by transitive verbs or prepositions or adjectives, and their extensions typically by nouns or noun clauses; and neither verbs nor prepositions nor adjectives are nouns.

For these reasons we cannot admit any theoretical reduction of all abstract relations to (abstract) classes, or to abstract collections of any sort. This opens the door to a new and more general conception of the scope of both mathematics and formal logic: instead of confining their respective subject matters to the study of the characteristics of sets and formal propositions, both disciplines could be extended, in their own respective ways, to the general study of abstract relations.
CONCERNING THE SUBJECT-PREDICATE RELATION

The subject-predicate relation cannot be used to alone give a satisfactory analysis of the structure of propositions. We cite some of the reasons for this from Goddard and Routley (p. 126):

"In admitting multiple subject sentences we are departing from the traditional grammar, enforced by traditional logic, according to which each sentence has just one subject. There are good reasons for this departure. First, the subject required by traditional grammar was never satisfactorily marked out; in fact only conflicting criteria, which failed to survive simple transformations such as the active-passive conversion or translation to other languages, were offered (see Lyons [Introduction to theoretical linguistics, Cambridge U.P., '68]).

We can, however, place an order on the subjects of a sentence in terms of their order of (first) occurrence (in left to right order in English) in the sentence; and frequently the first subject in our sense coincides with the subject of traditional grammar. Secondly, multiple subject or relational sentences are essential for logic, and in order to avoid certain metaphysical impasses, as Russell has explained in detail (in [The principles of mathematics and A critical exposition of the philosophy of Leibniz])."

Traditionally, the true-false relation has been linked up with the subject-predicate and the use-mention relations in a manner that Goddard and Routley (p. 127) explain:

"...the familiar explanation [is] that a subject of a sentence is what the sentence is about, and an associated predicate is what the sentence says about its subject (or subjects). The familiar test has to be handled with caution, and not merely"
because it confuses subjects and predicates, which in the sense concerned should be linguistic items, with what they are about. For one thing, what the sentence is about cannot be identified with what the supposed subject refers to, for the sentence may be about Pegasus and hence refer to nothing. For another, nonsignificant sentences certainly have subject-predicate analyses, but only in a tenuous sense of 'say something about' does the predicate say something about the subject: does 'sleep furiously' say something about colourless green ideas?"

Perhaps more important than the objections raised here, however, is the question of the formal analysis of the true-false and the use-mention distinctions combined with a formal explanation of exactly how they are linked up with the subject-predicate dichotomy. Until this has been done, logic shall remain without an adequate analysis of the structure of propositions.
(3.7) RELATIONS IN NYĀYA LOGIC

The neo-Nyāya ("Navya-nyāya") logicians had made some progress in the study of formal relations. Unfortunately, we were unaware of their work while writing this chapter and Chapter 1. We were also unaware of the work of the traditional logicians on identity theories of the copula, which occupies the bulky seventh chapter in Barth. Both of these sources are undoubtedly worth scrutiny; we regret the omission of it.

Concerning the neo-Nyāya logic of relations, we now include some information from Matilal (pp. 37-40):

"Navya-nyāya concentrates its attention on four types of relations: (1) samyogya (conjunction), (2) samavaya (inherence), (3) svarūpa ..., (4) tādātmya (identity) or abheda (nondifference).

"... Samyoga or conjunction is a well known and the least doubted concept. ... By samyogya the Nyāya means the direct contact of two "embodied" objects of any magnitude from the atomic to the ubiquitous, ...

"The relation of ... inherence has been ... described as a relation connecting the whole ... with the parts ..., ... quality ... and ... action or motion ... with their respective substances, ... and the ... generic property with its locus. It has been conceived as a permanent relation in the sense that it connects the [antecedent] to the [consequent] in such a way that the [antecedent] can never occur or exist as separated from the [consequent] ...."
"There is a very familiar way to criticize the Nyāya concept of relation. Something analogous to Bradley's argument [Appearance and reality, pp. 16-17] can be directed against it. This well-known critique (famous also in the idealistic tradition of India, e.g., in Buddhism and Vedanta) has sometimes been called the paradox of relation. The paradox may be briefly stated in the following manner. When we talk of x as being related to y by the relation r, we have first to relate x to y by r, and then relate r (which is also a property) to x by, say, r', and r to y by, say, r'', another relation. This again may require that x should be related to r' by a further relation r''''. This process can be repeated without end. And thus we find ourselves faced with a regressus ad infinitum.

"The [Nyāya school] averts the difficulty by pointing out that it arises because of the tacit assumption that all relations are essentially different from the relata and hence should be tied by a second relation. There is, however, no a priori necessity, so claims Nyāya, for the relation to be taken as numerically different in all cases from its relata. Nyāya thus postulated a "peculiar" kind of relation, a svarūpa relation, which is not be taken as different from its relata. Thus when we come to a svarūpa relation, the repeating relations, r, r'', etc., are not different from the first relation r, nor are they different from the relata. Through other relations the [antecedent] is tied to the [consequent], but through the svarūpa relation not only the [antecedent] but also the relation itself is tied to the [consequent]."

In this work, however, we avoid "the paradox of relation" in
a different way, namely, by using the distinction (/') corresponding to a relation to represent that relation, rather than using the relation itself (R), except in the case of the primitive connectors and '='. Thus, except in the latter cases, our use of the relation itself is always to be regarded as sheer shorthand for the corresponding distinction.
"Logic, in my view, required to be developed in levels or stages, and at each level new types of logical object and connective might make their appearance, things previously unsayable, but implicit, might become sayable; ... matters of fact might be transformed into necessities and even seeming impossibilities become necessities and so on. What is self-contradictory must indeed always be avoided, but our notion of what is self-contradictory may profoundly alter its face as we adventure further."

Findlay (p. 196)

(4.1) THE SYSTEM PC

To begin this chapter we present one formulation of the propositional calculus in our notation - hereafter this is to be known as "PC". We make no claim that this is necessarily the most intuitively satisfying way to formulate the propositional calculus in our notation. In particular, in this work we do not discuss many of the blatant disadvantages of the conflation of provability and validity in this formulation.

We do not include any formation rules because we are here presupposing those laid down in (1.5) and (1.8), as indeed we are doing throughout this work.

Vocabulary: Place-markers; T, F, -, &, /, =, V; Parentheses.

Primitive equations:

A. Definitions:

\[
\begin{align*}
T & \overset{\text{df}}{=} (A=A) \\
-A & \overset{\text{df}}{=} (A/T) \\
F & \overset{\text{df}}{=} (A/-A) \\
AVB & \overset{\text{df}}{=} -(-B\&-A)\&-(A&B) \\
A\&(A=B) & \overset{\text{df}}{=} A&B \\
\end{align*}
\]
B. Completeness and consistency equation:

\[(\overline{A=\overline{T}})\lor (\overline{-A=\overline{T}})\overline{=T}\]

C. \[A=(A\overline{=T})\overline{.}\]

Meta-"equation": A is a thesis iff A is of the form '...=T', or else is reducible to this form by means of a finite number of applications of the transformation rules to equations of PC.

Transformation rules: 1. Any (admissible, i.e. well-defined) expression may uniformly replace any given place-marker.

2. Let \(\alpha\) be a given equation of PC. In any equation at all in which the right hand side of \(\alpha\) occurs, the left hand side of \(\alpha\) may be replaced for one or more of the occurrences of the right hand side of \(\alpha\) in that equation; and the same holds vice-versa, that is, with "left hand side" and "right hand side" interchanged.

3. If \(\alpha=\beta\) and \(\alpha\) are equations of PC and \(\beta\) is of the form \(\gamma=\delta\), where \(\gamma\) and \(\delta\) are admissible expressions, then \(\beta\) is an admissible expression in PC, and therefore an equation of PC. (The reader will recall from (3.2) and (1.6) Part I that, in our terminology, every equation always holds by definition.)

If place-markers are interpreted as representing propositional variables, and '=' , '/' , and '& ' are respectively interpreted as material equivalence, the negation of material equivalence, and logical conjunction, it follows that then the wffs of PC coincide (as far as interreplaceabiliy is concerned) with those of the propositional calculus plus 'T' and 'F'. This is because: (1) the first two transformation rules are then clearly equivalent to the combination of the rule of substitutivity of material equivalents and the conventions regarding definitions in the propositional calculus; and (2) all truth-functional relations are of course definable in terms of material
equivalence and logical conjunction, and in PC material equivalence is '=?'; and

(3) the completeness and consistency of PC are trivial, as is seen from the appropriately named equation; and

(4) the Modus Ponens rule is not needed in PC in virtue of (1), (2) and (3).

The question we should like the reader to pause and try to think very hard about is, why is the description of PC so simple? Unfortunately, there is not enough space in this work for us to be able to explain our deep-seated doubts about the whole modern concept of formal provability. But leaving that entire question aside, we consider that there are still many things to be learnt from the mathematical simplicity of the formulation of PC. Throughout all of this chapter we attempt to throw some light on these things.
(4.2) THE LOGIC OF PROPOSITION-ABSTRACTS

The logic of asserted and unasserted relations

The distinction between the assertion of a relation and the (in general, though not always, unasserted) relation itself has already been discussed in the preceding chapter. In the logic of relations our primary concern is inevitably with relations in themselves, therefore with relations which in general, though not always, are of themselves unasserted.

In the logic of relations as we understand it propositions are always to be regarded as asserted relations. It is only natural, therefore, to seek an analogue of \( PC \), or something similar, in which asserted relations are replaced by relations in general. Such an analogue will presumably somehow form the nucleus for deductive formal logic from the viewpoint of the formal logic of relations, similarly to the way in which the propositional calculus is the nucleus (or a large part of the nucleus) of contemporary deductive logic. (We emphatically do not claim that we shall fully formulate such an analogue below, only that we shall cast some light on its likely nature).

We wish to emphasize that what we are seeking here is a generalization of contemporary propositional logic (as embodied in the propositional calculus) of which the latter will form a special, restricted case, so that we believe one shall be losing nothing of the advantages of the propositional calculus. That is, we are seeking to see how one might radicalize propositional logic without destroying the tiniest fragment of what is in our opinion useful or desirable within it. Of course, it is true that in doing so we cannot help but expose some of its deficiencies, but, naturally, such exposure of deficiencies will always be for the purpose of
introducing our (putative) improvements over them.

On the logic of unasserted relations

If the reader is prepared to presuppose our strange and unorthodox position on the concept of formal provability (in so far as within propositional logic we utterly conflate provability with validity), he or she will probably find the most natural procedure for seeking the abovementioned analogue of the propositional calculus to be the following. In PC, instead of interpreting the place-markers as propositional variables, interpret them as variables ranging over unasserted relations; then see which equations or rules of PC become unacceptable on the intended interpretation; then also introduce an operator that converts unasserted relations into asserted relations; and then attempt to alter PC accordingly, so to obtain the desired formal calculus.

Let us agree to call unasserted relations proposition-abstracts. Thus every proposition is an asserted proposition-abstract; the term "abstract" refers here to abstraction of the property of assertedness out from a proposition.

Now consider what happens if the place-markers in PC are interpreted as variables for proposition-abstracts. It is very clear that

$$A = (A = !)$$

fails to hold for any proposition-abstract A.

Removal of '$A = (A = ?)$' from the equations of PC results in an incomplete system, however. This is because, for example, the definition of '-' is no longer a thesis. We wish to emphasize that although '$A = (A = ?)$' is unacceptable in the logic of proposition-abstracts, it is certainly acceptable for our above version of the propositional calculus. Despite the practice
of Boole and De Morgan, Russell and Whitehead blithely banished the use of symbols for truth and falsity such as 'T' and 'F' to the metatheory in PM, and then later there came up Tarski's result concerning the undefinability of truth in PM. We have been attempting to suggest earlier, in a number of places, that Whitehead and Russell's work has been overrated generally; and we certainly do not at all support their rejection of the use of the constants 'T' and 'F' in the object theory.

Another point which emerges from the consideration of proposition-abstracts is the difference between '=' and 'f' as applied to propositions. If A represents any proposition, there is a proposition-abstract B for which it is true that A is absolutely identical with "B is true", while on the other hand A merely coincides in truth-value with "A is true" (on most accounts). This can be seen from the following considerations.

**Two senses of "true"**

A proposition-abstract by itself is neither true nor false. Assertion is always assertion of truth. A proposition, however, may be either true or false.

So the sense of "is true" in which "is true" is predicated of a proposition-abstract so as to obtain a proposition is different from the sense in which "is true" is predicated of a proposition. (This is the difference between entertaining a proposition and confirming a proposition.) In one case, what "is true" is predicated of is a proposition-abstract, in the other case it is a proposition. Let us use the symbol 'f' to represent assertion of a proposition-abstract, so that 'fA' always means the proposition which asserts that the proposition-abstract A is true. And in this chapter
we now use the symbols \( \ldots \mathcal{A} \) to mean that \( \ldots \) is a proposition and moreover is true.

We have taken the symbol \( \mathcal{J} \) from Frege, but we use it to represent a rather different sense of "assertion" than Frege's. Although Frege used it as a prefix to theses, it should be remembered that for the formulas of mathematics, as he understood it, truth was no different from thesishood; we, however, are interested in assertion of proposition-abstracts in the general sense, not just assertion of proposition-abstracts in mathematics or in pure syntax. Being a mathematician, Frege seemed to be a little confused about the distinction between validity and truth. It should be particularly stressed that, unlike its normal meaning, for us \( \mathcal{J} \) does not mean thesishood.

A proposition-abstract by itself represents the entertaining of that proposition-abstract. The proposition corresponding to it represents the entertaining of that proposition-abstract as true. It is natural to seek a notation as well for the entertaining of a proposition-abstract as false; for this purpose one may use \( \mathcal{J}^\prime \). Then \( \mathcal{J}^\prime A \) will mean "the 'anti-assertion' of the proposition-abstract A''. (\( \mathcal{J}^\prime \) and \( \mathcal{J}^\prime \) both denote properties, and hence, in our logic, a special sort of relation. Hence our use of them does not conflict with our claim to be basing logic on relations.)

The "system" RPAC

Let us now consider an analogue of PC which embodies the fundamental formal properties of proposition-abstracts, and these only, instead of those of propositions.

With place-markers interpreted as denoting proposition-abstracts,
it is imperative, as we have already mentioned, to remove 'A=(A=T)' (or 'A=(A/\top)', or 'A\neg (A/\top)') from the collection of primitive equations. What about the other primitive equations of PC? If only for the practical expediency of avoiding a notational duplication, let us agree to preserve all definitions intact (except, perhaps, with the alterations to the definitions of 'T' and 'F' that 'T = (A/\top)' and 'F = (A\neg A)').

That only leaves one equation to consider, the completeness and consistency equation of PC. This is clearly unacceptable as it stands (with place-markers being interpreted as denoting proposition-abstracts instead of propositions). Moreover it clearly remains unacceptable unless the three occurrences of '='T' are converted to '/'T' or ']' or equivalents of these. Such a conversion, however, changes the completeness and consistency equation into something which is not an equation. Not only that, but the latter is concerned with the logical properties of asserted proposition-abstracts. For both these reasons, we consider we must exclude it as a primitive assertion or axiom from the analogue of PC for proposition-abstracts without propositions, since the latter must highlight the logical properties of proposition-abstracts in themselves and not those of asserted proposition-abstracts.

This then leaves us with an unusual "system", which we call "PAC". In PAC, there are no axioms, the only primitive equations being definitions. More significantly, however, there are no theses in PAC which do not result directly from substitution in definitions. This is hardly surprising to the extent that all proposition-abstracts are by definition unasserted. But it is in a most unusual, if not even overstrained, sense of "system" that PAC can be said to be a system.

PAC is seen to have various curious properties when it is
regarded as an analogue of the propositional calculus. For one thing, PAC is concerned simply with the manipulation of definitions. Possibly this is more in line with, or vindicates the spirit of, Pascal's and Leibniz's remarks in (1.6) concerning the role of definitions in deductive logic. Or perhaps not, as Weyl (p. 27) intimates:

"... axioms become implicit definitions of the basic concepts occurring in them. The concepts, admittedly, retain a certain range of indeterminacy; but the logical consequences of the axioms are valid, no matter what concrete interpretation may be adopted within this range. Pure mathematics acknowledges but one condition for truth, and that an irremissible one, namely consistency."

In the interests of greater generality, let us agree to concentrate on the "system" which we call "RPAC" rather than on PAC, where RPAC is obtained from PAC by allowing place-markers to denote either proposition-abstracts or relata occurring within any proposition-abstracts which contain no relation symbols other than primitive connectors and '='. Let us also now replace '/' and '/' throughout by '/' and '/', with R unspecified.

Now let us attempt to look for what it is that makes two proposition-abstracts, or relata within proposition-abstracts, extensionally distinct (after all, '/' is part of the vocabulary of RPAC). Perhaps this is best approached by first considering the formal behaviour of the yellowish/bluish distinction, which we looked at at the end of Chapter 2. Consider, for the moment, the special case where place-markers are interpreted to represent colours. Since green lies on both sides of the yellowish/bluish distinction,
it follows that, if $R$ for the moment represents the relation corresponding to the yellowish/bluish distinction,

$$A/A
R$$

has a valid substitution-instance if $A$ ranges over all colours. And since yellow lies on one side only, it follows that

$$A/A
R$$

has a valid substitution-instance; in fact, '$A/A'$ is also valid in the case of the colour green. From our consideration of this particular case, we conclude that in RPAC '$/\!/'$ is not (always) exclusive on pairs of relata of proposition-abstracts. Similarly, it is not (always) exhaustive because red is a colour which does not lie on either side of the yellowish/bluish distinction.

Although '$/\!/'$ cannot be (necessarily) a dichotomy on pairs of relata of proposition-abstracts in RPAC, one might insist that on pairs of proposition-abstracts in RPAC it must be a dichotomy, since the analogue of '$/\!/'$ used in PC (namely '$/\!/'$) is a dichotomy on pairs of wffs of PC. Our only arguments against this are inconclusive and have already been stated in Chapter 2.

However, it is very interesting to observe the effects of denying that '$/\!/'$ is (generally) either exclusive or exhaustive on proposition-abstracts in RPAC, since then the assertion of '$/\!/'$ is neither exclusive nor exhaustive. Such a denial would seem to amount to a claim that it is more logical to use a distinction of this sort rather than the true/false dichotomy as the most basic distinction of deductive logic.
(4.3) CONCERNING UNASSERTED CONTRADICTIONS

From now on we adopt the following rule as self-evident:

If \( \neg \neg B \) is an equation of RPAC containing proposition-abstracts but no propositions, then any corresponding equation with proposition-abstracts replaced by their assertions also holds; and conversely, with "proposition-abstracts" and "propositions" ("assertions") interchanged.

If \( A \) is a proposition and \( A = \neg \neg B \) holds, then from \( \neg \neg \neg B = \neg B \) and the above rule,

\[
\neg \neg A = \neg \neg \neg B \quad \text{and so} \quad \neg \neg \neg A = \neg \neg B \quad \text{holds; whence}
\]

\[
A \neg \neg A \equiv (\neg \neg B \equiv \neg B)
\]

holds, whence, from the first part of the above rule,

\[
A \neg \neg A = \neg \neg (B \equiv \neg B).
\]

Consequently, in the logic of proposition-abstracts it is possible to study the formal properties of self-contradictory forms immaculately abstracted from their assertion (and therefore, one would presume, from any self-invalidation). Admittedly, \( \neg \neg (B \equiv \neg B) \) is necessarily always a false proposition; but then, \( B \equiv \neg B \) is always neither true nor false.

This leads to two different senses of "contradictory": one for propositions and one for their corresponding proposition-abstracts. The latter are by definition free of the "undesirable" quality of being necessarily false, and therefore are presumably worthy of formal enquiry. The Buddhist logicians, and of course in a sense the intuitionists as well, appear to have clearly recognized in the past that there is more to contradictions than necessary nonsense; concerning the Buddhists cf Chi (pp. 156-163).
It is most lamentable that the prejudice in favour of assertions as the sole ultimate constituents of all meaning in modern logic is rather firmly established. Equally clearly, however, the putative development of the logic of relations should uproot it. It is solely on the basis of this prejudice that claims such as the following of Quine's (p.59) make any sense:

"... consider the familiar remark that even the most audacious system-builder is bound by the law of contradiction. How is he really bound? If he were to accept contradiction, he would so readjust his logical laws as to insure distinctions of some sort; for the classical laws yield all sentences as consequences of any contradiction. But then we would proceed to reconstrue his heroically novel logic as a non-contradictory logic, perhaps even as familiar logic, in perverse notation."
(4.4) THE USEFULNESS OF CONTRADICTIONS

In this work a number of possible reasons why one might choose to reject the universal validity of the law of contradiction have been given prior to now. (See (1.5) Part 2, (4.1), (4.2), (4.3), (3.2), and passim in Chapter 2.)

We here add three more possible reasons for rejection of the law of contradiction.

(1) The traditional logicians treated the copula as an identity and not just as similarity (cf Barth), and this has unfortunate consequences. In particular, we quote from Barth (p. 226), where he is concluding a lengthy discussion of some aspects of the identity theories of the copula:

"Summing up: if the copula is understood as an (expression of an) identity or equivalence between concepts, then a logic is obtained which nolens volens will produce an infinite number of contradictions, and in a rather trivial manner at that. One may or may not attempt to turn this vice into a virtue."

(2) Our second reason comes from the analysis of change. To quote R.W. Church (p. 21):

"... it is an old story that change is unintelligible. How can A, which is A, change into Y, which is Y. Let the process of change be a matter of stages as minute as you wish. Still, in the course of the process, however conceived of, there finally would be a moment at which A would no longer be A, and would not yet be Y. Yet to say, in any case whatever, that A may be both A and not A is to utter a self-contradiction. That contradiction
would be implicated in any view on which it were held that one self-identical being may become another self-identical being. What is self-identical may not become: it may only be itself. Any view of change on which what changes is self-identical must, then, be abandoned.

That the phenomenon of change deserves logical investigation is hardly deniable within the logic of relations, since in a relational metaphysics all reality is for ever in a state of change (cf (3.3)). From the viewpoint of orthodox logic it also seems to warrant investigation; we quote Davidson, p. 81:

"Strange goings on! Jones did it slowly, deliberately, in the bathroom, with a knife, at midnight. What he did was butter a piece of toast. We are too familiar with the language of action to notice at first an anomaly: the "it" of "Jones did it slowly, deliberately, ..." seems to refer to some entity, presumably an action, that is then characterized in a number of ways. Asked for the logical form of this sentence, we might volunteer something like "There is an action x such that Jones did x slowly and Jones did x deliberately and Jones did x in the bathroom, ..." and so on. But then we need an appropriate singular term to substitute for 'x'. In fact we know Jones buttered a piece of toast ... The trouble is that we have nothing here we would ordinarily recognize as a singular term."

(3) Our final reason is grounded entirely on our use of Chinese-mystical metaphysics for the interpretation of logic, and more particularly on the doctrine of the interdependence of all
things which such a metaphysics entails. Since all things are taken to be interdependent, it is very misleading to speak of any individual thing or object A as if it possessed any reality independently and of itself. It is also misleading, but definitely less so, to speak of the relations between A and other things or objects as if these had any separate reality. Hence it is less misleading to identify A by means of its relation to \(-A\), its absence or (if A is a proposition or proposition-abstract) negation, than it is to identify A simply as A, separate from the rest of reality. So if A is a proposition or proposition-abstract, in a Chinese-mystical metaphysics the use of \(A \& -A\) is in some sense more in tune with reality and more accurate than the use of A alone.

All of these reasons are reasons to support the claim that it is valid to reject the law of contradiction.

But even if one wishes to adhere to the law of contradiction (or at least to some one formulation of it), all of these reasons apply with at least equal force as support for the claim that it is useful to reject the law of contradiction sometimes.
Let us return now to the question of the difference between the meanings of \( '=' \) and \( '\neq' \) (not to mention \( '\neq' \)).

Even if we confine our attention to propositions only, this introduces the problem of how and where to disentangle \( '\neq' \) and \( '=' \) from each other in propositional logic, since they are wrongly identified with each other there. We shall not offer any attempted solution to this problem here, although we wish to offer two examples of formulas which, according to our own intuitions, definitely should belong to the putative system which would be obtained from PC by sorting out and correctly implementing the distinction between \( '\neq' \) and \( '=' \):

\[
A \land A = A \\
\text{and} \\
((A = B) \land (A \neq B)) = (A = B).
\]

Perhaps

\[
(A \neq T) = (-A \neq F),
\]

or something similar, expresses a satisfactory consistency condition. We fail to see how to express a completeness condition (such as \(((A \neq T) \lor (A \neq F)) \neq T\)', where '\( \lor \)' is defined in terms of '\( \land \)' and '\( ' \)' in the familiar way) as an equation. We also have intuitive preferences for certain definitions, including the following, to be included as part of propositional logic, in addition to those in PC:

\[
A = B = (A \rightarrow B) = (B \rightarrow A) \\
\text{Df} \\
A \land B = B \land A \\
\text{Df}
\]

We do not pursue these questions further in this work.

Lastly, we point out that, firstly, the concept of distinction
in general corresponds to \( \!/ \), with \( R \) unspecified, rather than to \( \! / \).

Similarly we point out that \( \!/ \), with \( R \) of course unspecified, rather than \( \! / \), represents the general concept of context-dependent identification. This can be used as an explanation of why inter-replaceability is in general distinct from extensional equivalence. (The latter is of course a well-known fact (for a comparison of several different ways to account for it, cf Quine, p. 151)).
(4.6) CONCERNING '→'

As mentioned in (1.8) and (1.9), if \( \sigma(A) \) is any (admissible) formal expression then we stipulate that use of '→' is to be such that

\[
\sigma(A) \triangleleft (A \rightarrow B) = \sigma(A) \triangleleft \sigma(B)
\]

holds. One consequence of this is that

\[
(A \rightarrow B) \triangleleft (B \rightarrow A) = (A \rightarrow B) \triangleleft (A \rightarrow A) \quad \text{(interchanging A and B and taking \( \triangleleft \) to be \( 'A \rightarrow B' \))},
\]

\[
= (A \rightarrow A) \triangleleft (B \rightarrow B) \quad \text{(if symmetry of \( \triangleleft \) is assumed)},
\]

\[
= (A \rightarrow A) \triangleleft (B \rightarrow A) \quad \text{(taking \( \sigma(A) \) to be \( 'A \rightarrow A' \))}.
\]

With '→' interpreted as representing transformation, this means that the transformation of A into B combined with the simultaneous, or directly subsequent, transformation of B back into A is equivalent to the transformation of A into itself combined with the transformation of B into itself. (Concerning the reasons for the (vast) difference between \( '(A \rightarrow B) \triangleleft (B \rightarrow A)' \) and \( '(A \rightarrow B) = (B \rightarrow A)' \), cf the discussion in (1.9).)

Although '→' has not figured significantly in this chapter, there are at least two good reasons for desiring the concept of transformation to be formalized in the object theory rather than the metatheory. One of these is that in (1.8) we defined the elementhood relation using '→'. The other comes from the existence of combinatory logic and the desirability of integrating combinatory logic with propositional logic within the one object theory, and from the fact that functional abstraction is interreplaceable with transformation of expressions into the functional images of expressions. That is, the functional abstraction operator corresponding to a function \( F \) (cf (3.5)) is

\[
... \rightarrow F(\ldots).
\]
CHAPTER 5 - THE INADEQUACY OF SET THEORY

Three reasons why the concept of "manyness" is important

The distinction between one and many (and hence also the concept of number) is closely connected with distinction in abstract. Quantity, or the quantum, is that which is divisible - as distinct from that which is non-divisible - into "many" - that is, into two or more constituent parts. For this reason, the analysis of "manyness" will have to comprise a significant part of the logic of distinctions.

Another reason why the concept of a collection or "manyness" is important is its connexion with the concept of compounding, '&'. Whenever the operator '&' joins a pair of terms which are not absolutely identical (or interreplaceable), it collects them together, makes them a "many" or part of a "many".

Then, as well, there is the concept of generalization. This has intimate connexions with the concept of "manyness".

Alexander (p. 213):

"Generalizing ... is the process of grouping together to form a class, or of adding more members to a class. Every class, whether of relatively concrete objects or events or of abstract parts, qualities, relations, or functions, is already a generalization, a product of an act of generalizing."

The inadequacy of existing theories of "manyness"

Of course, not all collections, or "many's", are classes.

Alexander (p. 145):

"... there is quite a difference between an organic whole, such as a whole chicken, and an aggregate or class whole,
such as the whole class of dogs. The organic whole is not composed of a number of similar units (a chicken is not made of a lot of little chickens), whereas a class is composed in this way. So we must remember to distinguish between (1) the organic part-whole relation and (2) the aggregate or class part-whole relation, noting that this latter relation may also be thought of as the relation of species to genus or subclass to class-as-a-whole."

Not only is set theory well known to be resting on shaky ground, but mereology, the only significant existing formal theory of organic wholes, does not by any means account for the logic of all organic wholes. As Rescher points out:

"There exists a well-known axiomatic theory of the part relation, the mereology devised by the Polish logician S. Lesniewski, and subsequently developed by A. Tarski and others. However, ... this axiom system is not applicable in interpretation to many legitimate uses of "is a part of" in scientific and technical discourse... The absence of a formal theory of the part relation able to accommodate a wide portion of the spectrum of scientifically interesting usages of "is a part of" ... would greatly hamper certain investigations in the philosophy of science. It would, for example, wholly block efforts towards a general clarification of such important concepts as "organic whole" or "system" or "Gestalt".... For if we do not possess an exact, formal articulation of the part-whole concept, we are unable fruitfully to subject to precise analysis these other and far more complex concepts in which the notions of part and whole are inextricably involved."
Collections from a relational point of view

From the point of view of the logic of relations as we present it throughout this work, the meanings of any terms that enter into any given element-collection relation or part-whole relation are purely relative - that is, entirely dependent on the meanings of the relations they enter into. This is not so in contemporary set theory (or theory of classes). In particular:

(1) An individual may simultaneously be involved in two conflicting roles, such as that of a director of a business corporation and that of a bearer of public office. In such a situation, it can be meaningfully claimed that that individual, - assuming he is honest and reasonably incorruptible - in so far as he plays one role, is different from that individual in so far as he plays the other role. But of course he plays these roles by virtue of being in each case a member of the class (or collection) of all the individuals who play that role. Thus it can be meaningfully claimed that if x is an object then x-as-a-member-of-y is in general different from x-as-a-member-of-z if y and z are distinct collections of which x is a member. In set theory, however, if sets are taken to be the formal counterparts of collections of real objects then there is no simple way to formalize this claim. There is no simple way to do so because there is currently no well-defined formal theory of "x-as-a-member-of-y"; in Chapter 6 we establish the beginnings of such a theory.

(2) It is blatantly ignored in set theory that the identity of an object x in a collection y may itself be dependent on the identity of y. There are certain collections some or all
of whose members are interdependent in the sense that it is not possible to specify or precisely describe the identities of all such members without describing the collection as a whole. A familiar example of a mathematical collection of this sort is any nondenumerable set. Of course, this fact about nondenumerable sets is conveniently obscured in mathematics by the presumption that it is not possible to precisely describe all of the elements of any nondenumerable set. (It is impossible to precisely describe each element individually, but that is not what is at issue here.)

(3) A collection may have some organization to it over and above the fact that it is a totality of objects. For example, a sentence is more than just a string of words. Without removing or adding any components or elements of a collection, we may change its logical form by changing the relations between them. Thus, as we pointed out in (1.8), the same elements may sometimes make different wholes.

(4) Number concepts are in at least one sense relative rather than absolute, as Saminsky explains (pp. 58, 59):

"Let us lay ten cannon shells close to each other or heap them together into a pyramidal or global mass. This group would leave the impression of 'one' not 'ten', i.e. of one heap. But when the shells are placed pretty close to each other yet clearly separated, the group would not fail to leave the impression of 'ten'." "One cannot say that number is a quality of things, or that it is not: neither makes sense. Number is a quality of experience as a whole, not in parts; for
number's very nature is in that it states the division of experience into objects and the uniting of things in experience."

(5) In view of the inadequacy of set theory for the reasons just given, the contemporary treatment of mathematical objects is also unsound from the point of view of the logic of relations. Consider van Fraassen (p. 8):

"The objects referred to [in this book] (sentences, sets of sentences, formal languages, logical systems) always are or can be construed as mathematical objects. All mathematical objects are sets, and our main tool will be elementary set theory."
CHAPTER 6 - ON THE POSSIBILITY OF NEW FOUNDATIONS FOR MATHEMATICS

Frege (p. 253):

"... I cannot see how arithmetic can be set up on a firm basis, and how numbers can be conceived as logical entities and made objects of thought, if we are not allowed - conditionally at least - to pass from a concept to its extension. Am I always permitted to speak of the extension of a concept, of a class?"

Introduction

In this chapter we suggest two ways in which set theory can be altered to make it, in our eyes, considerably less unacceptable as a formalization of the concept of "manyness" and foundation for mathematics. However, we do not at all believe or claim that these alterations can by themselves produce a theory which overcomes all of what we see to be the inadequacies of set theory. We introduce these two ways together, but what we introduce is not intended to be complete; it is only intended to give insight into the intuitions involved.

Elementhood should be quality-independent

Firstly, we require that the elementhood relation '∈' in any acceptable set theory should be a quality-independent (or "external") relation instead of a quality-dependent (or "internal") relation, which it clearly is in contemporary set theory - in the sense that the meanings of the symbols which may (according to the formation rules) be written on one side of '∈' are in general independent of those that may be written on the other side. (On the other hand,
the meanings of A and B in 'AB' below are stipulated to be necessarily dependent on each other.) This quality-independence requirement is an unavoidable one for us given our definition of 'ε' in (1.8), since this definition makes 'ε' equivalent to a combination of relations all of which are quality-independent. And in any case, our desire to replace extensions of relations by their intensions might itself alone conceivably suffice to secure this requirement, considering the meaning of "external". (Also, the objections given in Chapter 5 to orthodox set theory all seem to support the introduction of this requirement.)

What this requirement amounts to for our purposes here is that, instead of sets, or sets and elements of sets, as its fundamental entities, set theory is now required to have couples, made up of elements of sets and sets respectively, as its fundamental entities. We write these in the form 'AB', to be read "A-qua-element-of-B or A-qua-nonelement-of-B (whichever happens to be the case)"; so 'AB' is meaningful regardless of whether 'AεB' holds or not.

_A collection of tokens must now be a set of tokens_

Our second requirement is that sets should be permitted to contain equiform elements - and from now on we refer to this as "containing the same element more than once" - , and two sets should be identical just in case they both contain the same elements the same number of times. Our reason for making this requirement is as follows.

Within orthodox set theory, two equiform terms are always, despite possible differences in context of occurrence, formally indiscernible from each other. On the other hand, a pair of distinct tokens of a formal symbol are equiform but (in some sense) discernible. It necessarily follows that a collection of two or more equiform
tokens is not, in the orthodox sense of "set", a set of equiform tokens. Set theory as we are conceiving it, however, is concerned with an extension of the notion of set which is such that the last statement is no longer true. It is no longer true because in our type of set theory two equiform terms can be formally discernible; more specifically, they are discernible if they are distinct members of the one, same set. Our type of set theory therefore formally embodies at least one feature of the type/token distinction - namely, the equiformity but nonidentity of distinct tokens of one type - which (barring use/mention conflations) is incommunicable within ordinary set theory. We believe this is a bonus because the naive notion(s) of a collection of abstract entities definitely seem(s) to embrace the possibility that some of the entities in the collection are indistinguishable from others for the purposes for which the collection is used. There are basically at least two types of abstraction, the type which selectively ignores certain properties of things as irrelevant and the type which selects certain properties of things as the only properties which are relevant, in the context in question. And the latter type, at least, sometimes ignores all the properties which distinguish two distinct objects, especially if the objects are very similar to each other. It should be remembered that all sets are abstract objects. But even the concrete examples of "sets of things" which are given by Halmos (p. 1), "a pack of wolves, a bunch of grapes, or a flock of pigeons", are in practice sometimes conceived of as sets in our extended sense rather than as sets in the normal sense. A person being pursued by a pack of wolves may not necessarily be concerned with differentiating between all the individual wolves in the pack; a person eating a bunch of grapes, all of similar size and ripeness, is
unlikely to waste any energy on speculative ratiocination on the subject of what gives each individual grape its uniqueness; a flock of pigeons seen from the ground may appear as so many indistinguishable grey specks.

**Individual sets**

Although we require couples of the form 'AB' to be considered as the fundamental entities of our set theory, we nevertheless introduce single entities by the definition

\[ A \overset{\text{Df}}{=} \hat{A}(A); \]

of course, we require that this definition only makes sense if '{A}' exists, but from our definition of 'e' (see (1.8)) it follows that if A is such that '(\text{E}B)(Ae:B)' holds then '{A}' does exist. In our set theory, "A is a set" shall mean that '(E_B)(Ae:B)' holds, so that the definition above always applies if A is a set.

**Predicate logic with identity**

For the purposes of our set theory, we are presupposing the use of a different form of predicate logic with identity than usual. This predicate logic with identity differs from ordinary predicate logic in the following ways.

We use place-markers rather than variables (with quantification defined as in (1.8)), and we use '=' and the primitive connectors '∈' and '/' exactly as in PC rather than the truth-functional connectives of the PM version of the propositional calculus. We are also presupposing the use of '+' and '∉', and the definition of 'ε' given in (1.8).

Whereas earlier in this work we have permitted single place-markers
to occupy any single selective token spaces at all, for the purposes of our set theory we now permit this only for those place-markers \( A \) for which '{A}' is definable, and which therefore satisfy

\[ A = \hat{A}(A). \]

Wherever a single place-marker \( A \) occurs in or is definable (from the corresponding bound variable '{A}') in our predicate logic with identity, therefore, it is being presupposed that '{A}' is definable and \( A \) already satisfies this equation. Also, we permit uniform replacement of any couplings of place-markers in the form 'BC' for any single place-markers (and so on iteratively - that is, further couplings may replace B or C or both, and so on).

Lastly, we stipulate the following identity condition in addition to the usual identity theory (which requires that '=' be reflexive, symmetric and transitive on all single place-markers occurring in or definable in our theory, and that identicals be interreplaceable):

\[ \hat{A}B\hat{A}C = B=C. \]

\( \text{Df} \)

The question of consistency

Because ordinary predicate logic with identity is consistent, the consistency of our predicate logic with identity is equivalent to the consistency of ordinary predicate logic with identity plus the requirement that there exist at least one function which is definable in it. (This is because in our predicate logic with identity the only requirement on our coupling relation is '(\hat{A}B\hat{A}C)=(B=C)', that is, that it be a function.) Hence our predicate logic with identity must be consistent.

We may now distinguish between it and our set theory proper.
Our set theory proper has only one axiom (actually, it is an
axiom schema), namely, an axiom of comprehension, which we now
introduce. It is:

\[(E^+B)(E(A'B)=A)\] if FA is any admissible expression such
that F is a property or function.

Our set theory proper would therefore seem to be consistent relative
to our predicate logic with identity, for the simple, a priori
reason that a system having only one axiom (schema) cannot be
inconsistent unless that axiom (schema) is self-contradictory.

"Arbitrary elements"

Finally we give something of a small account of the formal
properties of the notion of an "arbitrary element" of a set. If
B is a set, we augment our vocabulary so as to denote "an arbitrary
(or arbitrarily selected) element of B" by 'B'. This, we stipulate,
satisfies the following equations.

\[E(B) \equiv (A \in B) \Rightarrow FA\]

\[(-E)(B) = (A \in B) \Rightarrow -FA\]

\[-(E(B)) \equiv (E(A)(A \in B) \Rightarrow F^*).\]

Hopefully, the extremely brief account we have just given can
go at least a very little way towards placating those who claim that
any talk of arbitrary (abstract) entities is nonsense.

Future work

In this chapter we have not introduced any formal means of
individuating between two equiform elements of a set in our set
theory. Such a device is of course necessary if the advantages of
our set theory are to be fully exploited. However, we postpone
its introduction to a later date than that of this work, as we also do the construction of mathematics from our putative new foundations.
CHAPTER 7 - PROPOSITIONS, SENTENCES, AND UTTERANCES

In this brief chapter we digress to take a look at some of the different theories of propositions which have arisen at some time or other in the history of logic. Our chief interest in these, however, is in their relevance to the analysis of tokenicity, which we discuss in the next chapter.

**Propositions: the medieval kind and the modern kind**

The usages of the term "proposition" in the history of logic can be broken up into two very distinct categories, which we may call the medieval and the modern. A. Church explains this very clearly (pp. 3-6):

"... already by Boethius the word [propositio] has come to be used in a sense which it long retained and which I can attempt to express in other words by speaking of a declarative sentence taken together with its meaning. Basically the same sense of the word as Boethius' is intended when Peter of Spain defines, "Propositio est oratio verum vel falsum significans indicando," and when post-scholastic traditional logicians define a proposition as a judgment expressed in words. ..."

"Though the terminology is by no means uniform among different writers, it seems fair on the whole to take Peter's definition of propositio, just quoted, as representative of the scholastic usage. However, [we add the qualification that] some scholastic logicians use enuntiatio, either as an alternative to propositio, or in order to reserve the word propositio for use in some more special sense. And even Peter of Spain in another passage draws a certain distinction between propositio and
"Contrasted with this scholastic-traditional use of the word proposition is another use of the word which has arisen in more modern times, and which I shall distinguish by speaking of proposition in the traditional sense and proposition in the abstract sense....

"The difference between the two senses may be explained by supposing that we have before us an English declarative sentence, its translation into Latin, and its translation into German. In the traditional sense these are three different propositions. For though the three sentences have the same meaning (each in its own language), the words used are different in each case...

"On the other hand, ... a proposition in the abstract sense, unlike the traditional proposition, may not be said to be of any language; it is not a form of words, and is not a linguistic entity of any kind except in the sense that it may not be obtained by abstraction from language...

"An explicit distinction between proposition in the traditional sense and proposition in the abstract sense first appears in Bolzano's Wissenschaftslehre of 1837. Bolzano's word is Satz, which indeed is the usual German translation of the Latin propositio, and the proposition in the abstract sense is distinguished by calling it Satz an sich.

"In 1892, independently of Bolzano, propositions in the abstract sense were introduced by Frege under the name of Gedanke....

"The abstract notion of proposition appears again in Russell's The Principles of Mathematics in 1903. Russell ... explains that Frege's Gedanke is approximately the same as his own unasserted proposition. Propositions in the abstract sense play an essential role in ... Principia Mathematica ..., as
originally written. And though Russell later repudiated the abstract notion - replacing it in *Introduction to Mathematical Philosophy* by a definition of proposition which closely follows Peter of Spain, and more recently by a psychological notion of proposition - writers such as Eaton, Cohen and Nagel, Lewis and Langford, Carnap, and many others have followed the early Russell in employing the word *proposition* in the abstract sense.

"It should be added that although the use of the particular word *proposition* in this abstract sense is of modern origin, the notion itself is old. In fact the *[lekta]* of the Stoics are, wherever the *[lekton]* of a declarative sentence is in question, propositions in the abstract sense.... And ... the abstract notion appears again in the writings of the later scholastics, beginning with Gregory a Rimini, under the name of *complexe significabile."

In our view of how modern logic ought to be evolving, modern logic ideally needs to incorporate both senses of "proposition". Surely "propositions" have been so important for logic as to deserve the existence of a formal terminology which as far as possible captures the distinctions incorporated in each of the different terminologies throughout history. It seems to us that the full development of the logic of distinctions will not be in sight until the distinctions between the following have been formally clarified (at least from a relational point of view):

- an utterance (of a sentence)
- making an utterance (of a sentence)
- a sentence
- the meaning of a sentence
- a use of a sentence
- a declarative (i.e., indicative) sentence
- a statement
making a statement
a proposition
asserting a proposition
entertaining a proposition.

Propsitions in modern logic

In this subsection, by "proposition" we mean "bearer of truth or falsity". There is quite a diversity of standpoints within the modern, abstract view of propositions. Although they would all concur with Frege's definition (Frege, p. 89n) of a proposition as "the objective content of thought which is capable of being the common property of many", and although they would also all concur in distinguishing a proposition from a sentence, that is about as far as they are prepared to agree.

Following Frege there is a platonistic school of thought (cf Church, Introduction to mathematical logic, p. 25), according to which every proposition is the meaning, or "sense", of some significant sentence, and moreover is an abstract object having objective existence in the real world.

In contrast to the platonists, other modern views of propositions attempt to link propositions, in one way or another, with utterances. Thus Strawson takes "statements" as the bearers of truth or falsity, a "statement" being a declarative sentence "imbedded in the context" (p. 4). They are identified "not only by reference to the words used, but also by reference to the circumstances in which they are used, and, sometimes, by the identity of the person using them." The same view as Strawson's has been adopted, or accepted with slight modifications, by many writers. Nerlich, for example, discusses the distinction between a statement which has been made being true or false, and a statement's having been successfully or unsuccessfully
made. He also lucidly points out that there is a distinction between utterance and assertion (does every utterance of a meaningful sentence involve the successful making of a statement; what about "The king of France is Irish").

Then there is the theory that "eternal sentences" constitute the propositions on which logic should properly be based. An "eternal sentence" is a sentence-token of a particular sort, namely, one which is "eternalized" (cf Quine, pp. 227, 208), and does not vary at all in truth-value with changes in circumstances of utterance.

Another position worth mentioning is that of Cohen. Cohen criticizes Strawson and Quine because they ignore that one way a sentence may change in truth-value is through "meaning-change" (cf his §31). He proposes the replacement of propositions by "sayings", a "saying" being that which "a man repeats to himself, communicates to others, or treats now as a premise and now as a conclusion" (cf his §19).

_On propositions before modern logic_

One medieval logician whose theory of propositions may be of special interest for the analysis of tokenicity is Buridan. Scott (pp. 15, 29):

"... Buridan ... identifies the proposition with a sentence token, a single utterance or inscription. He allows neither classes of these sentences nor what is now known as a sentence type. He gives considerable attention to the rejection of any abstract entity as the significate or "meaning" of a proposition,...

"... [for Buridan] there are no complex entities as significates of propositions. When it is said that 'A is B' signifies
or asserts that A is B, we are not to suppose that A is B (or A-being-B) is something in addition to A and B. To say that 'A is B' signifies A-being-B is to say either that it signifies the mental proposition corresponding to that spoken proposition or that it signifies an A that is, in fact, B".

The other medieval logicians whose theories of the bearers of truth and falsity seem to us particularly relevant to the analysis of tokenicity are Abelard, Ockham, Burleigh, Gregory of Rumini, and Paul of Venice. Concerning Abelard, we quote from Nuchelmans (pp. 156-157):

"According to Abelard the qualifications 'true' and 'false' may be applied in three different ways: to propositiones, to the mental counterparts of propositiones, and to the dicta .... If we call a propositio true or false, we always do so in a derivative sense. The Boethian definition of a propositio as an oratio signifying something true or false is susceptible of a twofold interpretation: 'signifying something true or false' either means the same as 'saying something that is the case in reality or is not the case in reality' or as 'producing a mental counterpart which is true or false' ..... Accordingly, a propositio is true if its dictum is true or if its mental counterpart is true, and false in the contrary cases."

Concerning the others, the reader is enthusiastically referred to Nuchelmans, which is a very scholarly but quite readable study of all aspects of the major Greek and medieval theories of what are the bearers of truth and falsity. Concerning the chaotic confusion of sentence-types and sentence-tokens with each other in post-medieval logic, the reader is referred to Ashworth, pp. 53-55.
What can be concluded from all this?

Nuchelmans (pp. 4-5) claims that the many different theories of propositions examined in his book can all be arranged according to a certain scale:

"At one end of the scale of possible answers we may put the view that it is the utterance-token which determines the sameness and difference of things asserted; in the sense that difference of utterance-token necessarily implies difference of what is asserted. At the other extreme it may be maintained that the thing asserted is the same as long as the factors which determine its truth or falsehood remain the same. In that conception differences of linguistic form, either in the same language or in different languages, are taken to be irrelevant in so far as they do not bring about a change of meaning. And differences of meaning are taken to be irrelevant in so far as they are not accompanied by changes in the factors which determine the truth or falsehood of what is said. The linguistic meanings of the utterances 'He will be hungry when he arrives', 'I am hungry', and 'You were hungry yesterday afternoon' are no doubt different; but as long as the persons who use them do so in order to refer to the same situation there are, according to this view, no corresponding differences in what is asserted. Between these two extremes there are several other possibilities, according to the extent to which variations in form or meaning are taken into consideration."

Surely the different positions on this scale will be capable of formal comparison only after a general formal analysis of tokenicity has been created. For this reason alone, the logic of the type-token distinction is a very important part of the logic of distinctions, and therefore the next chapter is devoted solely to it.
(8.1) Why Tokens Are Important

In this chapter we are confining our attention to those sorts of tokens which are either tokens of symbolic expressions or else are directly involved in the interpretation of symbolic expressions. In this section we give seven reasons why the analysis of tokenicity is important.

1. Importance for the philosophy of mathematics

Quantity is that which is divisible into units. Mathematics can be defined as the science of quantity in abstract.

The paradigm mathematical objects are the whole numbers.

And the learning of mathematics is always first approached by the art of counting, so the very concepts of number and of quantity must rest on counting.

All cases of counting involve and are performed on entities which share a common property, \( F \); and their \( F \)-ness is the only quality that is necessary or relevant for such entities to be countable. Whenever we have completed counting, we say that there are so-and-so many entities which are \( F \). When we count, therefore, in effect we are abstracting all qualities except the relevant \( F \) out of the entities we are counting. The abstract objects which are produced by such abstraction are what we may call "\( F \)-units". And for any given \( F \), \( F \)-units are clearly equisignificant tokens (\( F \)-units are equisignificant by definition, because everything that could have made them otherwise has been abstracted out).

In this way the analysis of tokenicity is crucial to the very basis of the philosophy of mathematics.
We shall be returning to the analysis of counting in (8.4).

2. The inseparability of the concepts of tokenicity and identity from each other

Identity is impossible without tokens, because it takes two to make the same. We do not care if they are exact copies of each other, they still must be two. "For, otherwise," - we quote Bradley (p.141) - "to say, 'It is the same as itself' would be quite unmeaning."

Wittgenstein elaborates on this point in his Remarks on the foundations of mathematics (V-31):

"... if it is impossible for us to recognize an object as different from itself, is it quite possible to recognize two objects as different from one another? I have e.g. two chairs before me and I recognize that they are two. But here I may sometimes believe that they are only one; and in that sense I can also take one for two. - But [- someone will point out -] that doesn't mean that I recognize the chair as different from itself! Very well; but then neither have I recognized the two as different from one another."

And not only is identity impossible without tokens, but conversely, tokens are of course impossible without identity (in some sense of 'identity').
3. The close links between the concepts of tokenicity and context

In a Chinese-mystical metaphysics all meaning is context-dependent, in a rather similar way to the way that words like "this" and "I" and so on in English are context-dependent in meaning. Consequently, from the point of view of the type of interpretation of logic that we are urging, the concept of context is a crucial one.

The concept of context is closely linked with that of tokenicity for the following reason. Because they are by definition equisignificant, two tokens of the same type can only be distinct because they participate in different environments - that is, in different contexts of use. So precisely what it means for two equisignificant tokens to be distinct is that they participate in (at least some) different contexts.

On the other hand, to distinguish between two copies (dare we say "tokens"?) of a given token within what we have agreed to count as one single and irreducible context, we need to use a quite different strategy, unless we wish to get lost in a bottomless swamp built of tokens of tokens of tokens of .... This strategy is connected with the analysis of counting, and we shall return to it in (8.4).

We may point out that the importance of context is played down in the usual sort of interpretation of logic, and is only taken into account explicitly when this is necessary to avoid outright ambiguity. In this respect, the Chinese-mystical sort of interpretation is closer to reality, as we can see from Gurwitsch (p. 1):

"Experience always presents us with objects, things, events, etc., within certain contexts and contextures, and never with isolated and scattered data and facts. Looking at a material thing, e.g., a book, we perceive it in certain surroundings."
We see the table on which the book is lying, we see other books, papers, pencils, pipes, and through the window, a segment of scenery outside the house. Every material thing is perceived amidst other things which form a background for its appearance. Correspondingly, the same is true with regard to thinking. When we are dealing with some theoretical problem, more than the problem alone is given to consciousness."

4. Our rejection of PM as a model of what formal logic should be like

In various places in this work we have offered often angry criticisms of orthodox propositional and predicate logic, and especially of the practice of the authors of PM. However, the only criticism which is directly relevant to the analysis of tokenicity is that PM totally fails to provide any account of, or explicitly recognize the existence of, tokenicity.

5. Our use of the concept of token space

This is one of the primitive concepts of logic as far as we are concerned, and it clearly has close links with the concept of tokenicity, as a reading of (1.3) should disclose.

6. The importance of tokenicity for the question of what are the bearers of truth and falsity

This has already been discussed in the previous chapter.
7. Use/mention confusions are sometimes used to obscure the presence of tokenicity.

Tokens are by definition not repeatable; at least, the same term may function in one sense as a token and in a different sense as a type, and strictly speaking it is only in the sense that it is a type that it can be copied or repeated.

Whereas tokens are not repeatable, all the entities that can be described as names or symbols or referring expressions are repeatable. Since logic uses only entities which can be so described, it might be argued that therefore tokens lie outside the province of logic. This is to make a simple use/mention confusion, however. Although tokens are not repeatable, reference to them is. Names of tokens are every bit as repeatable as are the types of which they are tokens. And when we talk about the logic of tokens, we of course mean logic which uses names of tokens to analyze their special characteristics, and which does not use tokens as such in any special way in which they are not already used in other sorts of logic. We suspect that Whitehead and Russell may have made this simple use/mention confusion, however, because in our estimation PM might well deserve some sort of prize for making the largest number ever of sneaky and misleading use/mention confusions in any book that has been written on logic (cf their notorious use of the same symbol for analogous operators in both object theory and metatheory).
(8.2) GENERAL STRANDS IN THE LITERATURE ON TOKENICITY SO FAR

As far as we are aware, the literature on tokenicity can be divided under five general headings.

1. *Equiformity without identity*

As far as we know, the only attempt (other than our own, below) to provide a notation for tokens which is such that two distinct tokens of the same type are equiform but not identical is that of Reichenbach with his "token-quotes". As Rankin largely explains, this seems to have mainly been quite abortive from the point of view of the logic of tokens, although it did bring to light all sorts of information about the difference between quotation and reference.

All the other work on tokens (except our own) refuses, in one or another of various different ways, to treat distinct tokens of the one type as being absolutely and incorrigibly equiform, and therefore refuses to attack the fundamental problem in any but a roundabout way.

One of the main difficulties, as we understand them, is that any collection of at least two distinct but equiform tokens is not a class or set in the orthodox sense (although of course it is a set in our extended sense of that term, as we explained in Chapter 6), and hence it is impossible to directly apply orthodox set-theoretical methods and structures, on which all of modern mathematical logic is grounded, to any complete formal analysis of equisignificant tokens.

2. *The use of different metatheoretic names for different equiform expressions*

Reichenbach (p. 286) did introduce a theta-symbol such that 'θ' means "this sentence". This is just one example of the various ways
in which the existence of a difference between distinct equiform expressions - and hence the existence of tokens as distinct from types - is ignored in the object theory but not in the metatheory. However, this strategy has the deficiency that it uses continually different names for the one type instead of single, unique names of each of the different tokens of that type. That is, it fails to distinguish in any way between names of a type and tokens of a type.

3. The use of the concept of a "point of reference"

This idea - to construe a token as a pair consisting of a type and a "point of reference" - apparently originated with Bar-Hillel and was developed further by Montague. For Montague, a "point of reference" is any ordered pair consisting of a possible world and the context of utterance (or in place of the latter, some information adequately identifying it); the reader is referred to his almost unreadable paper 'Universal grammar'. Montague's approach to the analysis of tokenicity appears to be the one that is normally adopted in much of the current literature. In 'Pragmatics', he proposes his scheme roughly like this: The syntactical and semantical rules for a language determine an interpreted sentence or clause; this, together with some features of the context of use of the sentence or clause, determines a proposition; this in turn, together with a possible world, determines a truth-value. According to this scheme, both contexts and possible worlds are partial determinants of the truth-value of what is expressed by a given sentence. Montague merges them together, considering a proposition to be a function from contexts-and-possible-worlds (i.e., "points of reference") into truth-values. Pragmatics-and-semantics is then treated as the study of the way in which, not propositions,
but truth-values are dependent on "context", where "context" is now being used in an extended sense (much the same as our own sense) so that part of the context is the possible world in which the sentence is uttered.

We applaud this approach in so far as it abandons all attempts to treat propositions as objects and instead treats them as functions - that is, from the point of view of a relational metaphysics. We would go further and talk of the true/false distinction (or relation) instead of talking of truth-values, and we would reject the whole notion of "possible worlds" because we reject the notion of any sort of "world" as a fundamental notion (because a "world" is a substance), and instead we talk only of "contexts", and we also insist that the identity of a context is not determined in isolation but is dependent on the relations between that context and all other contexts. And although Montague treats propositions as functions, the entities which make up his propositions are treated as things; because in a Chinese-mystical interpretation we are committed to denying the concept of "thing" any fundamental status, we must do the same for subjects, properties and extensions of relations in Montague's syntax and for meanings, denotations, possible worlds, and truth-values in his semantics.

But what particularly worries us about Montague's analysis is the idea that the type is fully present and completely contained in any given one of its tokens. Surely the type is obtained by abstraction from all its tokens, so how can one have the whole type concretely present and not have all its tokens, instead of just one, present as well?
4. Theories of relative identity

Geach's (p. 157) approach to the problem is to regard the relation "being of the same type", defined on pairs of distinct tokens, as one particular instance of an identity relation. For Geach, there is no one relation which is the relation expressed by "is identical with", but there are many different relations, varying with context.

With a Chinese-mystical interpretation of logic, it is trivial that the meaning of identity is context-dependent, because there all meaning is context-dependent. Geach's arguments of course assume a subject/predicate type interpretation of logic, and are therefore not relevant to our concerns.

5. Indexicals

Finally, there is the literature on what Peirce called "indexical expressions" and Goodman calls "indicators" and Reichenbach called "token-reflexives". This whole literature is inconclusive and seems to have little to contribute to the general theory of tokens, so we pass over it.


Substitutivity and identity

Rather than talking of identity, for the moment we find it convenient to talk about substitutivity. There are different kinds of substitutivity. If A and B can be substituted for each other everywhere in our theory, then 'A=B' holds, and the substitutivity in question is what we call interreplaceability. On the other hand, there is the situation where

(*) \( a(A) = a(B) \)

holds, where the only difference between \( a(A) \) and \( a(B) \) is that one has A everywhere where the other has B, and where, in general, the equation (*) holds only for a certain class of expression \( a(...) \). (The first situation can of course be regarded as a special case of the second, namely, the case where the class of expressions \( a(...) \) for which (*) holds comprises all admissible expressions.)

Because (*) is a two-place relation, it may be rewritten in the form

(**) \( A/B \), where \( S \) depends on \( a(...) \).

Moreover, since it is clearly an equivalence relation, it may in fact be written in the form

(***) \( A/\bar{B} \), where \( R \) depends on \( a(...) \).

Hence all cases of substitutivity are reducible to tokenicity in the sense that if A and B satisfy (*) then they are mutual \( R \)-tokens for \( R \) as in (**), where by "mutual \( R \)-tokens" we mean any pair of distinct terms of which the relation \( /' \) is validly predicatable. (In the case where \( /' \) is in fact equivalent to \( /' \), \( R \) may be taken to be \( /' \).)

We observed in (1.6) Part 2 that modern logic formalizes only
one kind of identity. On the other hand, in its so-called
intensional logics it formalizes, or attempts to formalize, various
kinds of restricted substitutivity. We have just seen, however,
that the latter are reducible to \( \neq \)' for some \( R \), or to \( \neq \). Hence
we see that modern logic in effect uses more than one kind of identity,
but uses the term "identity" for only one of these kinds. All we
are effectively doing in using two kinds of identity, therefore, is
to bring this fact out into the open.

**Contexts**

In Chapter 6 we introduced the notion of a new kind of set
theory. We only axiomatized the properties of the relation denoted
by \( \neq \), and we therefore left it open which axioms (if any) \( \epsilon \)
was to satisfy - except to the extent that \( \epsilon \) was defined in (1.8)
in terms of our primitive concepts. But the difference between
orthodox sets and Lesniewski's "heaps" lies in the different
axioms for \( \epsilon \). Hence our notion of set can be taken as a
generalization of both of the other notions.

Now, instead of speaking of sets and elements of sets (in our
extended sense of "set"), we speak of contexts and elements of contexts
respectively. We now choose to interpret all expressions of the
form \( \hat{A}B \) to mean "\( A \) (imbedded) in the context (of) \( B \)." To fix the
notion of context a little, we quote Langer (p. 69) on the subject:

"A formal context involves not only elements, but the relations
which connect such elements."

(This statement seems to accord with our notion of context because
the meaning of \( \hat{A}B \) always involves the meanings of the relations
between \( A \) and \( B \), and not simply the meanings of \( A \) and \( B \) in themselves.)

And p. 79:
"A logical discourse rules out all private and accidental aspects. Its context must be fixed and public. The elements and relations that may enter into its propositions may, therefore, be enumerated in advance. These constitute the formal context of the discourse.

"The total collection of elements in a formal context is called the universe of discourse. This is distinct from the formal context itself."

The formalization of the concept of context is of course hindered by the fact that, to quote Goddard and Routley (p. 40), "the notion of context is inevitably vague". However, talking of the notion of context of a sentence-token they say (p. 41):

"All relevant features of the context [that is, of any given context of a given sentence-token], whether standard or not, may be described by using sentences, so that, from a logical point of view, a context may be represented by a set of sentences [i.e., sentence-tokens], namely those which specify the context."

Thus if A is any given sentence-token and B is the appropriate set of sentence-tokens, with A added if it is not already an element of B, then on their account 'AB' means precisely "A in its context."

On our interpretation of expressions of the form 'AB', therefore, (***) ᾱC = ᾱC represents substitutivity of A and B for each other relative to the context C. Thus we have four different ways - as expressed by (*), (**), (***) and (****) - of expressing the general concept of substitutivity or, equivalently, of an equivalence relation, or, equivalently, of context-dependent identity.
In (8.1) we stipulated that tokens of a given type are to be regarded as distinct if and only if they participate in (at least some) different contexts. We now state this requirement in terms of our formal notation:

\[ A \text{ is the same token as } B \text{ (or, we might also say, is ambiguous with respect to } B) = \hat{AC} = \hat{BC}. \]

Because it is possible to have tokens of tokens, it follows that we are in need of a formal notation for identical copies of a token as well as for tokens imbedded in contexts (that is, we need a notation for the concept of pure numerical difference). For just as in discussing the propositional calculus we do not permit the propositional variables to range over the metalogical propositions which we use in its metalogic (whether formal or informal), similarly in any given context we do not permit the place-markers denoting tokens to range over any tokens of those tokens.

One way to distinguish between equiform tokens is to introduce a special relation \( T \) which we may call a "token-marking relation", and entities which we call "token-markers". Thus, for example, if \( A \) and \( B \) represent equiform but distinct symbols, then we propose that

\[ (E^*_D)(E^*_F)((^{\hat{D}^*}_{G^*} \in C = ^*_G = A) \& (^{\hat{F}^*}_{H^*} \in C = ^*_H = B)) \]

should be required to hold (this formula can be translated into quantification talk as follows: "for all \( C \), there exist \( D \) and \( F \) such that for all \( G \), \( D \) is a token-marker for \( G \)-qua-element-of-\( C \) iff \( G \) is \( A \), and (similarly) such that for all \( H \), \( F \) is a token-marker for \( H \)-qua-element-of-\( C \) iff \( H \) is \( B \)). Of course, there is a need to axiomatize the formal properties of \( T \) in such a way as to accord
with plausible intuitions concerning "place-marking" and to fix
the intended interpretation of $T$. For example, $A$ should presumably
be an element of $B$ iff there exists a token-marker for $\hat{A}B$. We
do not attempt to produce such an axiomatization.

However, we do point out that the concept of "token-marking"
suggests a new analysis of counting. We may now identify the
sequence of natural numbers with the following sequence

$0, 1, \{1,1\}, \{1,1,1\}, \{1,1,1,1\}, \ldots,$

where

$\underbrace{\{1, \ldots, 1\}}_{n \text{ 1's}}$

signifies a set (in our extended sense) such that 1 is the only
element-type of that set and there are $n$ distinct tokens (occurrences)
of the element-type 1 in that set. One might perhaps wish to define
the number 0 by an axiom stipulating that there is a unique set $A$
such that $\hat{A}B = B$ holds (for all sets $B$) and by identifying that
set as the number 0.
The axiom of comprehension in our putative set theory (theory of contexts) guarantees the existence of at least one context (or set), \( U \) say, such that
\[
\hat{\forall}U = A
\]
(take \( F \) in our statement of the axiom of comprehension in Chapter 6 to be the trivial function). Clearly, in virtue of the universality of this equation we may call any such \( U \) a "universal" context.

In Chapter 6 we introduced the notion of "an arbitrary element" of any set \( B \), which we denoted by \( \overset{\circ}{B} \). Now consider \( \overset{\circ}{U} \), where \( U \) is as in the above paragraph; its intended interpretation is of course "an arbitrary element of \( U \)". Hence if \( F \) represents any property and given the notion of "arbitrary element" as primitive, we could, if we wished, define our version of the universal quantification of \( F \)-ness by
\[
F(\overset{\circ}{U}) = FA.
\]
\( \text{Df} \)

However, let us instead agree to use the notation \( \overset{\circ}{A} \), \( \overset{\circ}{B} \), \( \overset{\circ}{C} \), ... to represent distinct examples of "arbitrary elements" of \( U \) (given that the concept of there being at least one "arbitrary element" of \( U \) makes sense, we may use our extended concept of set, combined with the introduction of appropriate axioms concerning appropriate token-markers and \( \overset{\circ}{U} \), to guarantee that the notion of any finite number of essentially significant but distinct tokens of \( \overset{\circ}{U} \) makes sense; and we may take \( \overset{\circ}{A} \), \( \overset{\circ}{B} \), \( \overset{\circ}{C} \), ... to be metatheoretic names of these tokens). Thus we are evidently enabled to replace all occurrences of any given place-marker \( \alpha \) by \( \overset{\circ}{\alpha} \). In this way
the concept of quantification is reducible to the concept of "arbitrary
elementhood". (Cf (1.8).)

As we mentioned in the last paragraph of (1.6), the description
of our version of quantification (cf (1.8)) seems to entail that the
basic concepts of quantificational logic cannot after all be more
respectable than that of "dummy variables". In the present chapter
we have just briefly seen how one may begin to go about expressing
this fact in a formal way.

In orthodox logic, the question: "When are two variables
identical?" is not shown sufficient courtesy to so much as allow it
to be completely formalized, either within a logic or metalogically.
In all orthodox logics all variables in any one syntactical category
have identical ranges, but it clearly will not do there to identify
all variables in the same syntactical category. Consequently two
variables are in fact identified with each other just in case they
are equiform. (A consequence of this is of course that two variables
are equiform just in case they can be identified with each other
(interreplaceably).) But what is the reason for all the eagerness to
formalize and bring out into the open the identity relation between
individuals on the one hand, and to avoid discussion of identity
between variables on the other hand? If it is replied that there
is little significant use for study of the latter identity relation,
we need only turn our attention to the many different varieties of
"arbitrary individuals" that inhabit so many mathematical proofs,
to the unrandom, determinate sets known in statistics as "random
variables", and to the use of the identity sign to denote the
(in general unsymmetric) operation of "replacement" in the "equations"
of computer programming. In all of these cases it can be very
important to know precisely when two "dummy variables" (that is, two
"arbitrary individuals") can and cannot be (absolutely) identified. Unfortunately there is no formal theory of how to decide this question. Our expressions of the form \( \ulcorner U \urcorner \) at least represent the beginnings of the introduction into logic of a notation of a kind without which such a formal theory is quite impossible.
Consider the concept of inseparable union in the sense that A is inseparably united with B if and only if A and B each has no reality when isolated from the other. Then in a subject/predicate metaphysics, A is inseparably united with B if and only if A is identical with B. In a Chinese-mystical metaphysics, on the other hand, this is not so (in general).

As we interpret it, "the problem of the one and the many" referred to in Chapter 2 is precisely equivalent to the problem of how A can be inseparably united with B and yet remain distinct from B. The solution we are giving here is that: (1) Within a subject/predicate metaphysics this problem is indeed insoluble, and to pretend a solution exists within such a conceptual framework is to embrace flat contradiction and absurdity.

(2) Within a Chinese-mystical metaphysics, it is true for any and every A and B that A is inseparably united with B, but it is not generally true that 'A/B' holds.

We now draw the reader's attention to Stace's book. This represents possibly the clearest existing analysis of the statements of different mystics throughout the world and time in such a way as to extract what, if anything, is common to all of them and what are the implications for varying views of reality. Kennick's review gets straight to the heart of Stace's book. He eloquently states (p.388) "the problem of the one and the many" as it is crystallized in Stace's book in the following terms"

"... if we hold that the mystical experience is indeed one of
undifferentiated unity (whatever that might mean), and if the
undifferentiated unity is the unity of the individual self
(as Stace claims), then, as he rightly notes, there is no
principium individuationis on which can be based a distinction
between one self and another, and we are apparently forced to
conclude that "there is therefore a universal cosmic self
with which the mystic makes contact and with which he becomes
identified" (p.152). But I say "apparently" for, by the
argument in question, "the mystic" has vanished, and hence he
cannot experience anything.... It is therefore somewhat
surprising that in Chapter 4 Stace opts for ... the claim that in
the mystical experience the individual self is both identical
with and different from the Universal Self. But where does
or can the difference come in? According to Stace's own
argument there can be no difference where there is nothing to
make the difference, and in the mystic's experience there is
nothing to make a difference, no principium individuationis."

We hope that our explanation of "the problem of the one and the many"
has shown the reader one way to fully and utterly dissolve Stace's
and Kennick's dilemma.

Stace gives various examples of paradoxes which mystics
persistently and eloquently insist on holding (cf p.253); all of
these are reducible to the form "A is F and A is simultaneously and
in the same sense not F"; but in a subject/predicate metaphysics
this is equivalent to "A is inseparably united with something which
is F and also simultaneously with something which is not F"; and
the latter statement is a tautology in any Chinese-mystical
metaphysics! To conclude this triumphantly brief chapter, we quote
Kennick (pp.388-389) on Stace on the paradoxes of mysticism:
"In Chapter 5 ("Mysticism and Logic") Professor Stace confronts the fact that so many of the mystic's statements about his experience seem to be downright paradoxical. He dismisses, rightly in my opinion, several attempts to construe the mystic's statements as only apparently paradoxical. But, wrongly in my opinion, he concludes, from the fact that the mystic's statements are not just nonparadoxical assertions clothed in paradoxical language, that they are "outright logical contradictions" and yet true. What allows him to make this shocking claim is his conviction that "what the paradoxes show is that, although the laws of logic are the laws of our everyday consciousness and experience, they have no application to mystical experience" (p.270); for they are operative only where there is multiplicity, and in the mystical experience there is undifferentiated unity. But this simply will not do, even on Stace's own grounds. One cannot claim that the statements of the mystic are genuinely paradoxical and yet true, and that this shows that there are experiences to which the laws of logic do not apply. For if it is true that the laws of logic do not apply to the mystical experience, then the mystic's statements are not genuinely paradoxical; for we can have a genuine paradox only where the laws of logic do apply. On the other hand, if the mystic's statements are genuinely paradoxical, then they are necessarily false and hence cannot show that there are areas where the laws of logic do not apply. One cannot have it both ways."

(In this passage, simply substitute "paradoxical from the viewpoint of subject/predicate metaphysics" for "paradoxical" in all except the last four sentences.)
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