THE TRANSITION FROM CONCRETE TO FORMAL THINKING

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by

Susan Clare Page (née Somerville)

Australian National University
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CHAPTER 10

CONCRETE AND FORMAL LEVELS OF UNDERSTANDING OF THE FOUR GROUP TASK.

10.1 Introduction

Chapter 3 presented an interpretation of the differences reported by Dienes and Jeeves (1965), between subjects giving "operator" and those giving "pattern" evaluations of a cyclic four group, in terms of the differences between Piaget's stages of Formal and Concrete Thinking. This chapter aims to substantiate that interpretation with results obtained in the present study.

Broadly speaking, the subjects categorised as at the stage of Formal Operations, in the present study, are those who would have been described as "operator" evaluators by Dienes and Jeeves (1965). Subjects regarded as transitional, in the present study, are those who reached an "operator" evaluation only with a certain amount of assistance, at the conclusion of the interview. Since Dienes and Jeeves (1965, 1967) do not conduct extensive interviews at the conclusion of the learning of combinations, they identify a relatively small proportion of their subjects as giving "operator" or "partial operator" evaluations. The proportion of subjects located in the formal operational and transitional stages in the present study is large by comparison.
The difference is greater if the results for Dienes and Jeeves' (1965) sample of 11 year-old subjects, rather than those for their sample of first year University students, are compared with those for subjects in the present investigation. The extensive amount of questioning, probing, and help to express ideas, given in the present study (which may be presumed to be the cause of the discrepancies pointed out above) thus needs some discussion before the interpretation of results is attempted.

10.2 Discussion of the Detailed Interview about the Four Group Task in the Present Study.

10.2.1 General attitude to the role of questioning

It is a tenet of Piaget's theory, evident in his attitude to experimentation, that the suggestion of the correct answer, to a child whose own approach does not lead to that answer, will not alter the child's view. In fact a basic technique employed in the testing of such concepts as conservation, or the understanding of physical laws (such as those involved in the oscillation of a pendulum), is the use of suggestions to the child, on which he is asked to give an opinion. For example, a child who has failed to conserve continuous quantity may be told:-
"But I had a little boy here, just now, who told me that he thought we would still have the same amount to drink - because, he said, although my new glass is thinner, the water has gone higher up - and so it is still the same amount"

and asked:-

"What do you think about that?"

Correspondingly, a typical way to test the confidence of a child who gives a conserving answer, is to suggest to him one showing nonconservation, typical of the earlier stage of thought. Analogously, in testing on the Pendulum Problem, a child who has been experimenting correctly may be shown an example, by the experimenter, in which both the weight and the length of the string are changed simultaneously and asked:-

"Now, do you think what I have just done proves anything about what affects the speed of the tick?"

Experience with the Pendulum Problem, in the present study, confirms the notion that the use of such interviewing techniques does not assist subjects to achieve a better understanding of the problem. On occasions it reveals that a subject is not very sure of the ideas he has put forward, and is willing to "regress" to an explanation characteristic of an earlier developmental stage. Thus such techniques serve to expose more clearly the aspects of the task which cause difficulty to the subject on the one hand, and those upon which he relies for his conclusions on the other.
An example of the former is contained in the interview on the Pendulum Problem, presented as an illustration of the recording techniques, in Appendix I. Here, a 14 year-old subject who has succeeded, with difficulty, in excluding the amplitude of oscillation as a factor, attempts to dismiss the problem altogether in response to a question by the experimenter, at the end of her demonstration. When asked why she is always careful to hold the weight out as far as the red line, in her demonstration, when she has already shown that the distance out makes no difference to the speed of the tick, she replies:

"Oh, I think I would have held them at the red line, even if it wasn't there."

Such a question, rather than constituting a direct suggestion, to the subject, for improvements to her demonstration, acts to reveal an aspect of her reasoning which is not consistent with her stated conclusions. Another example of a Piagetian test in which the direct suggestion of an answer, to a subject, is of no avail is the "resorting task described by Inhelder and Piaget (1964). After a child has sorted a number of geometrical shapes (varying on three dimensions) into two piles using, as a criterion, the dimension of his choice, he is then asked to "do it a new way". The typical preoperational child finds this an impossible task, somehow always arriving at the same dichotomisation of the objects that he made initially. This is so even if the experimenter begins a new classification for him, on the basis of a different dimension - the subject always alters it and returns to the original classification again.
It was with such an attitude to the role of detailed questioning, in the investigation of thought processes, that the long and persistent interview about the Four Group Task was included in the present study. The central aim was to reveal, as clearly as possible, the cognitive structures which the subject brought to bear on the task of memorising, and making sense of, the sixteen combinations of lights. From a pilot study, it was clear that, by the time a child of 9 to 12 years of age had spent the necessary time and effort memorising the combinations, his approach to the task was firmly enough established not to be changed radically by the form of the questions put to him by an experimenter.

10.2.2 Issues concerned with the final general questioning and the classification of subjects in a transitional stage.

Although the attitude to the role of questioning, described above, was held at the outset of the main study, the experimenter was unnecessarily cautious about making specific suggestions to subjects, in the first few weeks of testing. In particular there was some reluctance to mention numbers, specifically, in the final questioning, lest every subject somehow "see" the system immediately. It soon became apparent that this fear of "giving the game away" was unfounded, in that a subject who had not thought of numbers previously would simply answer "no" to the question "Is it like numbers?", without giving the matter much thought. Persistent questioning and
guidance was necessary to lead those who did think that numbers could be used to represent the lights, to find a set of numbers which was suitable.

Thus, after several weeks experience, the experimenter no longer hesitated to ask as many questions as were necessary to make the subject's own approach to the task quite clear, and to examine his ability to think of new (number) ideas. Unfortunately, the subjects tested during the first few weeks were all 12 year-olds, and their tendency to give lower level evaluations of the task and the axioms is due, at least in part, to the experimenter's early reticence in questioning. The overall stage category particularly affected is the transitional stage, ?III, where the improvement from a Stage IIA or Stage IIB performance depends on the subject's response to the final questioning.

The record of the interview with K.D. (F, 11 yrs., School A(P)), in Appendix IV, shows the responses of a subject in the transitional category. She displays an immediate and serious interest in the possibility of using numbers to explain the combinations, and after working out some specific numbers, says emphatically "That's how I could have remembered them".
This response to an invitation to consider whether the lights might be thought of as numbers contrasts with that of other subjects who agree to the general idea, but when asked to say what the numbers should be, either say, without thinking, "Oh, they could be anything we liked", or else are unable to use specific combinations to decide upon, or test out, exact numbers for the lights involved. Quite often a subject may derive the idea that the yellow light must be zero (if adding) or one (if multiplying), but will then assert that the other lights could be "anything at all" (or sometimes that they could all be zero, the same as yellow; or that they must all be the same number, because the same thing happens when each is combined with yellow, i.e. YG=G, YB=B, YR=R).

This last-mentioned type of reasoning, where a regularity observed in combinations of yellow with each of the others is used to explain not the role of yellow itself, but those of all the others, is reminiscent of a type of reasoning, about factors involved in the Pendulum Problem, described by Inhelder and Piaget (1958) and already discussed in Chapter 6. This reasoning, said to characterise children at Stage II B, leads a subject to conclude, for example, that the one factor which has not changed from one situation to the next cannot be having any effect,
The example already quoted is of a child who says "Anyway, it's not the string because it's the same string". A similar difficulty is encountered, on an experimental level, when a subject does not vary the factor which he wishes to investigate, but somehow often changes another factor instead. This difficulty, reported by Inhelder and Piaget (1958) for the Pendulum Problem, was apparent also when subjects were attempting to arrive at the "role" of each element in the Four Group Task. From this discussion it appears that a lack of clarity about which factor to manipulate, and which to leave unchanged (physically or on a mental level), in order to reach conclusions about the one of interest, can be said to characterise an unsuccessful approach to both the Pendulum Problem and the Four Group Task. Whereas it typifies the concrete operational approach to the Pendulum Problem, on the level of the physical manipulation of variables, it seems to extend, on a mental level, to subjects who are trying to disentangle the formal roles of the four elements in the Four Group Task.

On the basis of the points discussed above, it is held that, even when a specific suggestion of a formal (number) system for the combinations is made, by the experimenter, the distinction between a subject who is possibly beginning to acquire formal operations and a subject who is clearly still in the concrete operational stage will be apparent. Some examples of responses to the final questioning, by the two categories of subject, can be given as illustrations.
Concrete operational responses to the final questioning.

T.C. (F, 10 yrs., School A(P))

E: What does the game remind you of?

S: Painting. In Miss B's class we used to get colours in a row and find out what colours make what colours.

E: Anything else?

S: The first time I didn't know the answer and the second time I did.

E: Did you think it was like putting numbers together?

S: No.

E: Could it be, do you think?

S: If you add a number to another number it'll make a new one.

E: Is it like that here?

S: Yes, but then you get the numbers mixed up. If Y is 1, G is 2, .. so the lights go 1, 2, 3, 4 then with YG, G lights up so you would say that number 2 still goes on.

E: Why do you say number 2?

S: Y is the first, G is the second, number 2.

P.S. (M, 11 yrs., School B(P))

E: What does the game remind you of?

S: It reminds me of torches. I have three torches. I used to have a red pencil torch. My brother had a green one. When we turned them on together it turned out red.

E: Anything else?
P.S. (Cont'd):

S: It reminded me a bit of lights in a house or on a car - Y reminds me of this.

E: Is it like putting numbers together?

S: No.

M.H. (F, 13 yrs., School B(S))

E: What does the game remind you of?

S: Painting and traffic lights.

E: Anything else?

S: It reminds me of a stove. When you turn the oven on a red light goes on while it's not heated . . . and when it's heated it goes off.

E: What about the rules?

S: It's like maths in a way . . . Well say this (Y) is ?, then 1 + 1 (puts on YY) is 1 . . . No! 1 + 1 = 2 . . . Oh, 1 x 1 = 1.

E: What would the others be like?

S: Well YR = R so that'd be 1 x 1 = 1.

E: And what would the others be?

S: Ah . . . GB make R so that would have to be 1 also, 1 x 1 = 1.

E: So would they all be 1?

S: Yes.

E: Why do you think they are different colours?

S: To help not get you confused.

E: Why do you think that GR make Y here, and not this one here (E indicates B), if they're all 1's?

S: Well, they're different ones.
B.W. (M, 13 yrs., School B(S))

E: What does the game remind you of?
S: It's just like adding.
E: Can you think of numbers for the lights?
S: Well 1 + 1 are 2 . . . No I don't think that will find them . . . If you say you are multiplying, then 1 \times 10 = 10, 2 \times 10 = 20 (indicates YG = G, YB = B).
E: So for multiplying what would Y be?
S: 10.
E: If you're saying 1 \times 10 = 10 for YG = G, wouldn't Y be 1?
S: No, I'd start from 10 . . . so Y = 10; then YG = G is 10 \times 1 = 10 and YB = B is 10 \times 2 = 20 . . . the ones in the big panel are "tens", the ones in the little panels are units.

J.M. (F, 13 yrs., School A(S))

E: What does the game remind you of?
S: It's not like real colours. Maybe it's like a computer . . . a computer might have the same methods as this.
E: Is it like numbers?
S: Yes . . . you might have colours that stand for numbers . . . G and R . . . G could stand for 6 and R for 7 and together they make a number and that number stands for another colour.
E: What would the numbers have to be?
S: There's 1, 2, 3, 4 . . . so each one . . . there's four colours so each one of these colours would have to stand for a number . . . 1, 2, 3, 4 . . . 1 + 1 = 1 (tries Y + Y = Y) . . . 1 + 2 = 2 (tries Y + G = G) . . . 1 + 3 = 3 (Y + B = B) . . . 1 + 4 = 4 (Y + R = R).
J.M. (Cont'd):

E: Yes, but the only trouble is that in real numbers 1 + 2 doesn't make 2.

S: It doesn't matter really because this is a different thing . . . like numbers, well numbers in maths . . . Not all numbers stand for the same things in life, you know, they sometimes have to be changed.

S.D. (F, 13 yrs., School A(S))

E: What does the game remind you of?

S: Sort of like Algebra, where you have to work out what all the symbols mean before you can find out the answer.

E: What kind of symbols do you mean?

S: I was thinking Y could be "a" because it's at the beginning of the board.

10.2.2.2 Transitional (?III) responses to the final questioning

D.M. (M, 14 yrs., School B(S)) -- The learning record of this subject appears in Appendix I.

E: What does it remind you of?

S: Colours mixed together, but it doesn't always work. Two the same do not always give the same colour.

E: Does it seem more like something else?

S: No.

E: Like adding numbers?

S: It does a bit.

E: Why?
D.M. (Cont'd):

S: Well, with the Y, that would probably equal 0.

E: And the others?

S: Well with the R, RR = B and GG = B, so R and G have to be the same if they're numbers.

E: What about some of the others?

S: G and B = R . . . so that could be 1, 2, 3 . . . 1 + 2 = 3 . . . then GG = B is 1 + 1 = 2.

E: What about G and R?

S: It goes back to 0 again . . . So that numbers above 3 go back to 0 . . . B and R would probably go to G . . . So the numbers are 0, 1, 2, 3 and then up again . . . 4, 5, 6, 7 . . . they keep going around a circle. (S then goes on to explain the Inverse Elements with further questioning.)

R.P. (M, 14 yrs., School A(S)) -- The learning record of this subject appears in Appendix I.

E: What does it remind you of?

S: It's a bit like valencies in science . . . If you have aluminium (Al) and magnesium (Mg) and you put them together they make Al something Mg₂ . . . so they combine.

E: Can you explain about the valencies?

S: Well aluminium has a valency, I don't remember what it is, say 3, and magnesium has a valency of 2 and so together they make a valency of 5, only you don't write it like that, you write Al₂Mg₂.

E: What about Y with the others?

S: With Y and G if Y had a valency of 2 and G had a valency of 3 and you put them together . . . you just get G . . . So you have to take the valency of 2 away . . . or else get an acid with a valency of 0.
R.P. (Cont'd):

E: So what valencies would Y and G have?
S: 0 and 3.
E: What about B and R?
S: They make G... so you could have 1 + 2 = 3.
E: Yes.
S: ... or 2 + 1 = 3 ... or, if we wanted to go along, 0, 1, 2, 3 (in order Y, G, B, R).
E: What happens when you put RB together?
S: ... 3 + 2 = 1 ... I can't have that ... we only go up to four ... oh yes, four would be 0 again.
E: And 5 would be?
S: G, or 1, again and 6 is the same as 2, or B.

P.M. (M, 10 yrs., School A(P))

E: What does it remind you of?
S: Lights going together.
E: Anything else?
S: It reminds me a bit of maths.
E: Why?
S: Well in maths you can turn two figures around and it will be the same.
E: Do you think you could find a number for each light?
S: I don't know.
P.M. (Cont'd):
E: What might Y be?
S: 1, because it's the first light.
E: And the others?
S: G would be 2, B 3 and R 4.
E: Would you like to try it out?
S: YG . . . that's 1 x 2 = 2 . . . GB = R, that's 2 x 3 = . . . 
(tries BB, BR, then a great many different combinations, very 
rapidly . . . then says:-) I think the Y would be 0 because 
0 + 0 = 0 and this would be 2 (G) and B would be 3 and R would 
be 4.
E: Good. You've changed to adding, then?
S: Yes.
E: Try some out.
S: 2 + 0 = 2 (GY = G) . . . 2 + 3 = 4 (G = R). That's not right 
. . . change that (R) to 5 . . . then 2 + 2 = 4 (GG = B), so 
it looks as thought B is 4.
E: Now how will you find out R?
S: By adding these two . . . GB = R, that must be 2 + 4 = 6.
E: So the numbers are?
S: 0, 2, 4, 6 . . . Oh, but 6 + 6 = 4 (RR = B).
E: What could we do? . . . We've got 0, 2, 4, 6 . . . What should 
the next one be?
S: 8 . . . So you could go back . . . 0, 2, 4, 6, 8, 10 . . .
E: How can you get 10?
S: 4 + 6 = 10, that's B + R = G . . . and RR = 12, that'd be 
B equals 12 . . . and then that'd be 14 (R) . . . 8 + 6 = 14 
(YR = R).
The extracts from interviews, quoted above, illustrate the difference between subjects who were ultimately rated as at a transitional stage (?III), and those whose rating did not change from Stage IIA or Stage IIB (both concrete operational) as a result of the final questioning. All were classified as Stage IIA or Stage IIB on the basis of responses to the initial questioning. To be regarded as transitional, the subject had to demonstrate such a ready understanding of a number system (with considerable emphasis on the understanding of its "cyclic" nature), when it was suggested, that it seemed likely that his failure to develop it himself, in the first instance, was purely mischance.

10.2.3 Issues concerned with the initial general questioning

One further aspect of the detailed questioning about the Four Group Task requires discussion before the accounts of concrete and formal levels of understanding are given. This is the degree of assistance given to many subjects, in the initial questioning phase, to achieve a complete understanding of a formal operational system. It is clear from the results in Table 14 of Chapter 8 (and Table 14A of Appendix V) that very few subjects immediately gave a Type (G) response. Most of the subjects categorised in Stage IIIA or Stage IIIB began with a spontaneous Type (E) or Type (F) (or even lower) response and this was developed by questioning (set down in Part I of the Detailed Questioning Protocol in Appendix II) into a Type (G) understanding.
Appendix IV contains interview records of the initial questioning of two subjects who gave a Type (G) response, spontaneously—(M.B., F, 14 yrs., School A(S); V.S., M, 14 yrs., School A(S))—and one interview record in which an initial response of Type (E) is developed into a Type (G) understanding (M.C., M, 14 yrs., School B(S)).

These detailed interviews were considered important as an exploration of the limits of understanding which could be reached by each subject. While it is clear that many ideas were forthcoming during this questioning which would not have been discovered, by the subject, in such a brief contact with the task, the important evidence which these interviews provide is that the ideas can be understood, and subsequently used, by children who are at the appropriate developmental stage. Further, it can be argued that the experimenter could not have succeeded in suggesting the ideas (for example, of cyclic moves) to a subject whose own approach to the problem did not "set the stage" for them. In fact the attempt to do so would be an exhausting and fruitless undertaking for both parties concerned. Before discussing an interview with an eleven year-old subject, which borders on the situation envisaged above, one further point should be made. This is that the degree of development of ideas taking place in the initial questioning does not in any way invalidate the categorisation of subjects into overall stages of concrete and formal thinking.
If subjects were to be categorised merely on the basis of their first, brief, spontaneous responses, instead of taking into account the subsequent development of ideas, only 30 of the 236 categorisations would be different. This was pointed out in Section 8.2.1 of Chapter 8. Thus it can be claimed that the initial general questioning serves merely to explore, to its limits, the level of approach of any subject. The level may be identified, at the outset, as a concrete or formal operational one, without much fear of error.

The interview reported below (with M.P., F, 11 yrs., School A(P)) illustrates the problems, presented by the task, for a subject who is just beginning to glimpse the possibility of a formal system. The subject finds the level of thought and analysis almost impossibly difficult to maintain, even with considerable help from the experimenter. She is one of the "exceptions", marked with an asterisk in Table 14A of Appendix V, and discussed in Section 7.2.1.3 of Chapter 7, who achieved a formal system during Part II of the questioning, relating to the group axioms and concrete reversibility. The interview is reported in a condensed form, since it extended over a period of more than two hours. She is categorised as in a transitional stage (?III) because of the difficulty she has in mastering and using the formal system. Her understanding is continually upset by new facts, or by practical attempts to test the system out.
Parts I and II of the Questioning of Subject M.P. (F, 11 yrs., School A(P))

(Initial response rated Type (B), develops to Type (F) in Part II, questioning about the axioms. It is clear that the Type (G) understanding, to which she is continually led by the experimenter, keeps slipping out of her grasp).

E: How did you remember what the lights would be?

S: Y with another colour becomes the colour turned on second ...GY = G, YR = R ... Y changes to the other colour that you turned on.

E: And when there is no Y?

S: If you turn on G and B it would be Y (tries it and finds GB = R) ... no, I thought it might be R, but then I thought it might be Y because if it was R ... (pauses and appears to think very hard).

E: What are some of the others?

S: B and R = Y.

E: No, G.

S: I was going to say that. Those three colours have been in it twice and Y is out ... Y hasn't turned on yet.

E: Which are the two?

S: GB and it's R and RB = G, so YB = B and RR = Y.

E: No, B. Can you make sense of RR = B?

S: I don't know, I just remembered ... Well, if RR = B, GG must be Y.

E: Why?
S: They're both light colours . . . G and Y . . . If they're both light they couldn't make a dark one.

E: Which is a dark one?

S: R is rather dark. The rest are light.

E: So it's all right for GG to make B?

S: I suppose so. (seems exasperated)

E: Is there anything else that helped you to remember?

S: If two colours are the same and you put them back to front they are still the same.

E: Show me.

S: BG is R and so RG . . . oh no! . . . BG is R so GB would be R!

E: Is that true for any two colours?

S: If two colours make one colour and you turn them back to front they still make the same. It's just like doing a sum; \(6 \times 8 = 48\) and \(8 \times 6 = 48\).

(In the initial questioning, outlined above, the subject is expressing only concrete operational rules for the grouping together of similar pairs of elements, but she also seems considerably agitated and dissatisfied with the account she is able to give. Her account of the Commutative Law is a formal one. She is very half-hearted about the explanation of the "double ones" in terms of light and dark colours. The fact that she sees that some comprehensive system is possible comes out in Part II of the questioning, in the section relating to the Unit Element. A shortened version of this part of the interview is quoted below.)
E: I think you told me about the Y light before.

S: Yes, when you turn on Y and turn on another colour, it becomes the colour that you turned on. Do I have to do G now?

E: What could you do to find out about G?

S: Turn on GB to make R . . . GR to make Y, and . . . GB must make R . . .

E: What else can you turn on with G?

S: Y, and you get G.

E: What else?

S: GR . . . done that! . . . GB . . . done that! . . . oh GG and you get B! The two greens change into blue.

E: How can you say what is happening with G?

S: If you turn on a colour with G . . . oh, I just don't know how the G works!

(A break occurred here, for school recess, The interview resumed half an hour later)

E: Do you remember what you were trying to do before?

S: Trying to work out how, if Y works how it does, then how the G light works.

E: Let's see with YG . . . If we say we have a light here, G, and put Y with it, what happens?

S: The colour we turned on with Y, it becomes that colour.

E: But if we say we already had G?

S: If you already have a colour, and put Y, it stays that colour.

E: Good, now see if you can find what happens when you have a colour and put G . . . If you have B and put G . . . ?

S: It becomes R.

E: . . . G and put G . . . ?
S: It becomes B.
E: ... R and put G ... ?
S: It becomes Y.
E: ... Y and put G ... ?
S: It becomes ... it stays G ... no, it becomes G.

E: So if you have any colour and put G ... ?
S: It changes ... B and put G comes to R ... it changes to the colour next to it.

E: Can you show me?
S: Put G and B and it makes R, because it's the colour next to the B.
E: Do G and R make the colour next to R?
S: Yes, Y, because if they were in a circle they would be next to each other.

E: See if you can explain what the B and R ones do as well.
S: (Tries some out) ... It works in the same way, only it starts with the ... the colour; ... with G it starts at G, with B it starts at B and with R it starts at R.

(Up to this point the progress with understanding the system is quite good, and most of the suggestions have come spontaneously from the subject, after some help in structuring the combinations by the experimenter. From this point on, however, the subject has a great struggle to try to separate the roles of G, B and R and she repeatedly takes refuge in the description in terms of "going round the panel, starting at the colour". Some extracts are shown below.)
E: What can we say for B and any colour?

S: Well, YB stays B.

E: But if we say we start with Y and put B . . . ?

S: It doesn't matter if you turn the colours round, . . . it still makes the same colour.

E: If you have G and put B . . . ?

S: It's R, because that's the colour next to B.

E: If we say we had G and put B . . . ?

S: It's the same as what I said.

E: Well, what about two blues, BB . . . ?

S: . . . That's Y.

E: How do you explain that . . . ?

S: Well BB make Y, YY make Y, RR make B and GG make B . . . two of them make Y and two make B.

E: But how exactly can we explain that B and B make Y, by saying what B does?

S: Because in a circle it's the colour next to it?

E: Next to what? Is Y next to the B?

S: No, but it's next to R . . . because that colour's been left out (the R) and it comes to the next!

E: . . . and for B put with R . . .

S: It's G because that's the colour next to the Y.

E: Do you think you know what would happen if you put R now?

S: Well for G and put R . . . well, you've left two colours out . . . so you must leave two colours out and take the next one on the board.

E: Good. Now can you tell me a rule for each colour?
S: Well when you turn on Y with another colour it becomes the colour you turned on with Y; . . . when you turn on G it becomes the colour next to G; . . . when you turn on B with another colour it becomes the second colour next to it . . . the second one along; when you turn on R it misses out two colours and it's the third.

(Note that there are some errors in this statement of the rules . . . and these were shown up by her attempts to illustrate the rules, using examples. The main difficulty was in knowing which one of the two possible rules to use as an explanation. After some discussion the problem was overcome for BG = R as below).

E: So tell me the two ways to explain GB = R.

S: Which one was turned on first? . . . B? . . . Well from B it's one colour away; from G it's two colours away.

E: So from B it moved one because of . . . ?

S: G . . . or from G it moved two because of B.

(After further discussion of the number of moves they stand for, S was able to answer as below.)

E: So we can forget where we start and just do what?

S: Know how much moves they stand for.

E: And when you put lights together?

S: You add up the number of moves and they make whatever light stands for that many moves.

E: So a one-move light (G) and a two-move light (B) . . . ?

S: . . . must make the three-step colour . . . R.
(When invited to try some out, adding the moves, the subject chose B and R and said that it made 2 + 3 = 5 moves, but had great difficulty finding that 5 moves was the same as ONE move around the circle, and that therefore B and R made G, the one move light.)

She was then asked to explain G and R and said:-

S: R is three moves . . . G is one . . . that's four moves.

E: Which is the same as?

S: R.

E: If you start at R and move four moves . . . ?

S: . . . you get to R.

E: If you start at Y and move four moves . . . ?

S: . . . you get to Y.

E: So, always, if you move four . . . ?

S: You end up at the colour you started with . . .

E: . . . and you might as well . . . ?

S: . . . not move at all!

E: So if two lights add up to four moves, it's the same as . . . ?

S: . . . no moves.

E: Which light is no moves?

S: The Y.

E: So now you can say why G and R make Y?

S: G and R make Y because G is one move and R is three moves . . . and that makes four moves; and so when you move four moves that gets back to the one you started at, so that's no moves, so you go to the one that's no moves . . . the Y.
(This represents a very good understanding of the system, but when the subject was asked what it meant about G and R, in an attempt to arrive at the notion of inverse elements, this again caused a great deal of difficulty. When asked what two lights must be, or do, to each other if together they make no moves, she suggested first that they might be the same. A good deal of help, to translate three moves forward into one move back (for R) was needed before she was able to say "they are the opposite -- they do the opposite thing". However, once she had seen this idea, she responded to questions about the opposite of B as below.)

E: What is the opposite of B?
S: Y.

E: Do BY get you back where you started?
S: Oh, B is the opposite of B!

E: Why?
S: If you took two steps forward and two steps back you would be at the same place and may as well not have moved at all.

E: How can B be two back and two forward?
S: Well two steps forward is one, two (illustrates from G to B to R); and two steps back is one, two (illustrates from G to Y to R).

After this, the subject went on to give a formal explanation of the Associative Law, and then the interview concluded with her saying that the game reminded her most of traffic lights and that she would like to take it apart to see how it worked. Whereas this subject
clearly achieved an excellent understanding of the structure of the problem, this was only possible because of the directive questions put by the experimenter and it was noticeable that the understanding was not very permanent. It was easily upset when she was asked to devise an illustration of a rule or to explain a particular combination. Nevertheless, it is held that the performance of this subject, and others in the ?III category, indicated that they were capable of grasping a formal system, with help.

10.3 An Attempt to Specify Systems of Concrete and Formal Operations in terms of which the Four Group Task may be Conceptualised.

Examination of the effects of the detailed questioning, in the previous section, has indicated that a completely thorough formal level of understanding of the task is rarely achieved, by a subject, without considerable help from the experimenter. It has been argued, however, that the child's spontaneous responses are not an adequate indication of the level of understanding of which he is capable and that detailed exploratory questioning is therefore required. Further, the previous section has shown that such questioning does not prejudice the task of distinguishing a formal operational (or transitional) from a concrete operational subject; rather it serves to make the differences between them, and the aspects of the task causing conceptual problems, more clear.
It was also pointed out, however, that, if the aim were merely to categorise a subject as at the stage of Concrete and Formal Operational Thought, the subject's initial spontaneous response would suffice. The questioning which followed was seen as exploring, in full, the nature and limits of the concrete and formal operational systems. Thus the accounts of the two levels of understanding, which follow, are accounts of the complete systems, which can be uncovered by detailed questioning, but which are certainly not articulated, immediately, by most subjects.

10.3.1 The Four Group Task understood in terms of concrete operations. The operations available to the child at the stage of Concrete Operational Thought are directly derived from actions which he performs. Thus in the case of the Four Group Task, the subject can be said to represent, and be able to combine by reflection, actions of "turning on a R light in this panel and a B light in that panel" etc. Furthermore, he is able to detect similarities between the different actions which are possible, and between the results he observes for different actions. On the basis of these similarities, he may classify types of combinations together and remember their results, as a subgroup of the total combinations.

The most typical groupings of combinations are shown below. They correspond to the sections of the matrix described by Dienes and Jeeves (1965).
10.3.1.1 Combinations grouped together by subjects at the stage of Concrete Operations.

<table>
<thead>
<tr>
<th>&quot;the double ones&quot;</th>
<th>&quot;the ones with Y&quot;</th>
<th>&quot;ones made of R, B, G&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y + Y = Y</td>
<td>Y + G = G</td>
<td>R + B = G</td>
</tr>
<tr>
<td>G + G = B</td>
<td>Y + B = B</td>
<td>G + B = R</td>
</tr>
<tr>
<td>B + B = Y</td>
<td>Y + R = R</td>
<td>G + R = Y</td>
</tr>
<tr>
<td>R + R = B (often omitting Y + Y = Y)</td>
<td>(The last one often seen as &quot;hard&quot; or &quot;the exception&quot;)</td>
<td></td>
</tr>
</tbody>
</table>

Some examples of such groupings, by subjects in Stage IIB of Concrete Operations, are presented in the following extracts from the initial general questioning about the Four Group Task. These groupings appear to serve solely as mnemonic devices, not being related, except perhaps in the case of the combinations with Y, to any ideas about the roles of the lights in the system. Certainly, except for Y, there is no indication that the same light is thought of in the same way, when it occurs in different combinations.

D.B. (F, 14 yrs., School B(S))

E: Did you find out anything that helped you to remember what the lights made?

S: Y helped me a lot, because YG = G, YB = B and YR = R.

E: So Y and any colour . . . ?

S: Is the colour, like YR = R.

E: What about the others, not with Y?
D.B. (Cont'd)
S: I don't know how, I just know RB = G, BB = Y, GG = Y ••• no GG = B, and RR = Y ••• no RR = B ••• those last two were both B. Sometimes I forget which, but two of the double ones make B and two of them make Y. I think it's RR and GG that make B, and BB and YY that make Y. If you go along the panel the answers go Y, B, Y, B.

C.T. (F, 12 yrs., School B(P))
E: Can you tell me how you remembered what the pairs of lights would make in the big panel?
S: Most of them with the Y one came out themselves. Most of G, B and R if you put two of them together it came out the third.
E: Which was the one where it didn't?
S: GR = Y.
E: Was there anything else that helped you?
S: Not that I can think of.

P.G. (M, 12 yrs., School B(P))
E: Can you tell me any ideas that you found that helped you to remember what the answer would be?
S: A colour goes into another colour to make a different colour, but Y and another colour always makes the other colour. When you're doing the other three colours it always goes in turn. One colour is always last. The same colour never comes twice as the answer.
E: Can you give me an example?
S: R and R equals B, that's one colour out. Then B and B could go Y. Then the last colour is R, the one that hasn't had an answer, so that'd have to be it. There's always one colour that hasn't been on when four different ones have been asked and that's the answer.
P.G. (Cont'd):

E: Do you mean that when I ask you to turn on a light, say the G one, and then ask you what all the lights in my panel would make with that one, that when you get to the end of the line there is only one answer left?

S: Yes, there is only one that hasn't been the answer already.

J.S. (F, 11 yrs., School A(P))

E: Can you tell me how you remember what the lights make?

S: YY = Y, GG = B, GB = R, RY = R.

E: You were doing YG = G before and said you had better keep away from them. Why was that?

S: Well YG = G so Y and B will just stick to B.

E: And Y and R . . . ?

S: . . . will be R.

E: Why are those ones easy?

S: Y is a colour which when you put two together they stick to the darkest colour. Y is a colour that doesn't help it in any way.

E: So if you have any colour with Y you get . . . ?

S: . . . the one you're putting with it.

E: Is there anything else?

S: I remembered GB = R from paints.

E: Wasn't there one you said was a trick?

S: Yes, you have to have a guess at it, it makes the oddest colour.

E: Which are you talking about?
J.S. (Cont'd):

S: The machine is put together sort of funny. Can I go through them? BB = Y, that's the funniest; GG = B, that's not so bad.

E: Why, what ought they to be?

S: Just the same colour.

E: Are there any like that?

S: Only Y.

E: I thought you were going to tell me that GR = Y was the funny one.

S: Oh, it was, but I got used to it.

T.G. (M, 11 yrs., School A(P))

E: Can you tell me how you knew what the answers would be?

S: I didn't understand all the time.

E: Were there any that were easy?

S: R and Y, B and Y, G and Y.

E: Why?

S: Because Y is like a white light and it's invisible nearly when you mix it with them.

E: What happens?

S: Well the Y doesn't show, just the other colour like G, B, R.

E: Are there any other ideas?

S: Most of them I guessed and found them quite easy.

E: How did you think about them?

S: I just imagined in my mind the two colours mixed.
T.G. (Cont'd):

E: At the end you said RR would be B when you had never seen it. How did you know that?

S: I said to myself if YY = G and if GG = B and BB = R, well there's been only G left out.

E: Well the answers are really YY = Y, GG = B, BB = Y and so RR will be . . . ?

S: B.

E: Is there a pattern there?

S: Some sort . . . I don't know . . . I just knew by automatic instinct.

B.K. (M, 10 yrs., School A(P))

E: Did you have any way to remember what the lights would make in the big panel?

S: All the Y's with G and B were Y.

E: Can you show me?

S: Well YY = Y, YG = G, YB = B, YR = R.

E: What helps you to remember those?

S: When you say Y, the answer to it is Y and the colour that it's with.

E: The answer is . . . ?

S: The colour that it is with.

E: So when we have Y with any colour the answer is . . . ?

S: Y.

E: No . . .

S: It's the colour!
B.K. (Cont'd):

E: Is there anything else?
S: I can't remember.

E: How do you remember when there isn't a Y? Is there another rule like the Y rule?
S: No.

E: How do you remember GB = R?
S: I think it must be a darker colour than itself. R is a darker colour than G or B.

E: What about RB = G?
S: I remembered the closest colour I could to B.

E: Were there other times that it was the closest colour?
S: In painting it would make the closest colour.

E: So you mean the closest looking colour, not the nearest?
S: Yes the closest looking.

E: Did you have a way to remember RG = Y?
S: Not really.

E: I think the only ones left are the double ones. Do you remember what we get with G and G itself?
S: B. We have to get a darker colour, but not as dark as R.

E: Well what about R and R?
S: We must get a bit lighter colour. If you get two dark colours together you have to get a bit lighter colour.
The following extracts from interviews illustrate the responses of subjects in Stage IIA, who remembered the combinations entirely by rote, not showing any of the mnemonic groupings illustrated in the excerpts above.

J.P. (M, 13 yrs, School A(S))

E: What ideas did you have that helped you to remember what the combinations made?

S: The thing I like most is the light going off and on. You have to get used to it and remember them. I find this easy because I built a switchboard at home. It's got the same colours only it's got white as well. I like electronics. I've got a walkie-talkie and an intercom. I've got a radio and a rectifier to change AC to DC. Later on when I grow up I might be able to build a TV set. So I've got a good memory for lights and switches.

E: Did you find there were any rules or patterns here?

S: Not really, I just remembered them.

E: Was there anything that helped you?

S: I found RY comes back to R and GB comes to R again. A lot of them come to R. Other than that I just memorised them.

G.H. (M, 11 yrs, School A(P))

E: How did you remember what the pairs of lights would make in the big panel?

S: They're easy when you get to know them.

E: What are some of the easy ones?

S: RB = G and RG = Y and RR = B and GY = G and GB = R.

E: Why were they easy?
G.H. (Cont'd):

S: You told me them first. When I couldn't remember them I just guessed.

E: Did you notice anything special about any of the lights?

S: They changed colour.

E: When you put two together do they always change?

S: No sometimes they change, sometimes they stay one colour.

E: When did they change, and when not?

S: The first ones I learned were new, then ... I think I forget them now.

E: So you just learned them until you remembered?

S: Yes.

L.E. (F, 11 yrs., School B(P)) — The learning record of this subject appears in Appendix I.

E: Can you tell me how you remembered what the pairs of lights would make when we put them together?

S: I remembered by when I used to mix colours — they would turn out a different colour. B and G make a colour you wouldn't call B or G but in between.

E: What do they make here?

S: A R.

E: Is it like what you knew before?

S: Yes.

E: Are these colours the same as ordinary colours?

S: They're really not, no. G and B makes an olive kind of colour.
L, E. (Cont'd):

E: So did you stop thinking about them like that?
S: Yes, and I just took a guess. After a while I found I could remember them.
E: How did you do it?
S: It's not really hard because there's only four of them.
E: But there are sixteen different ones you have to remember.
S: I thought there'd only be about eight.
E: Were any easier than others?
S: Yes, BG. I was thinking I had only these two left, R and Y. After I did it a bit I thought the answer was Y, but that was wrong, and so then I always remembered R.
E: How did you know the answer was not G or B itself?
S: I don't know.
E: Was there anything else that helped you?
S: Well with the B and R I took the next one, G. Sometimes I wouldn't remember properly and take Y by mistake.
E: Was the answer always one next to it?
S: No.
E: When was it not?
S: YR = R.
E: How did you remember that?
S: YR = R, there's a R in it already. I just said R because I'd already said R.
10.3.1.2 Explanations of concrete reversibility and the group axioms, by concrete operational subjects.

The other aspect of the responses to questioning, which illustrates the concrete level approach to the Four Group Task, is the explanations given of concrete reversibility and the group axioms. The subject at this stage of development uses as "elements" the operations of "turning on a R light", "turning off a B light", and so on, and understands that they obey the axioms in the following ways:

(i) Concrete Reversibility (not an axiom)

When R and B have been turned on together, to make G (R + B = G), R may be obtained by turning off B (G - B = R) and B by turning off R (G - R = B).

Thus turning R (or B) on is the opposite action of turning it off.

(ii) Commutative Law

When two lights are turned on, say a R and a B, it doesn't matter which is turned on in which of the panels. The same two in the reverse order give the same result. i.e. R + B = B + R = G.
(iii) **Unit Element**

Any light which has Y turned on with it stays the same. Y doesn't change the colour it goes with.

*e.g.* \( R + Y = R \) \( (= Y + R) \)

(Subjects may agree, when it is suggested, that this is true even for Y and Y itself, but this example is not usually quoted spontaneously by them. Some of the explanations seem deliberately to leave this one out.)

(iv) **Inverse Element**

No explanation of inverse elements may be given in terms of lights being turned off and on. Explanations only exist in terms of the "roles" of the lights in some formal system of numbers, or moves around the panel.

(v) **Associative Law**

If a third panel with lights to turn on in it is imagined, the same as panels 1 and 2, and one light turned on in each of the three, then the result will be the same regardless of the order in which the lights are turned on.

*\( \text{e.g.} \) \( R + (B + G) = (R + B) + G = B \)*

*\( \text{i.e.} \) \( R + R = G + G = B \)*

The extracts from interviews which follow serve to illustrate such explanations of each of the axioms and of concrete reversibility.
Explanations of concrete reversibility, by concrete operational subjects.

Subjects were given the example RB = G, and asked to say how they could get back to just B or just R. When they answered correctly that this could be done by turning the appropriate one off, they were asked "What is happening when we do this?". Some typical answers to the question are shown below.

"We're taking the other colour away." (L.E., F, 11 yrs., School A(P))

"G has had the B taken out of it." (W.Bo., M, 10 yrs., School A(P))

"When you turn off this one it stops sending out the B . . . this one is still sending out R. The G has had the B taken away." (M.S., M, 10 yrs., School A(P))

"We've sort of separated the colours and then we've turned it to R because we've taken this one away." (W.Br., M, 13 yrs., School A(S))

"The G vanishes and there's another colour -- half of it is left -- the top half." (J.K., M, 10 yrs., School A(P))

Explanations of the commutative law, by concrete operational subjects

Subjects were given the example RB = G, and asked to answer the question B? = G. If they answered correctly that RB = BR = G, they were asked to state the rule, and whether it was true for any two colours. Some typical answers are shown below.
"If you've got B and I've got R it makes G, and if you give me B and I give you R it still makes G, because we're just swapping the colours round. It's true for any two."

(T.C., F, 10 yrs., School A(P))

"When you mix the R with B it goes the same both ways -- if you do R first in my panel and B second or B first in my panel and R second it's the same. I think it's just for R and B. (Tries GB = R and BG = R and then says:-) I think it might be true for them all."

(J.B., M, 11 yrs., School B(P))

"You just have to have the two colours that make G. You have to have R and B to make G -- if you turn R on this side and B on that it makes G, and on opposite sides it still makes G. It happens with all colours."

(B.D., M, 12 yrs., School B(P))

"It's just the same two back to front and it makes the same colour. Any two can be back to front and still be the same."

(R.W., F, 12 yrs., School A(P))

Explanations of the unit element, by concrete operational subjects

Subjects were asked which light they could put with each of the four lights in turn to keep the light the same. Having seen that the answer was Y in every case, they were asked what they could say about the Y light. Some typical answers appear below:

"If you put Y on to any of the other ones the one you've put it on to will stay on."

(R.W., F, 12 yrs., School A(P))
"The Y doesn't stay on when the others are going to be on. The
other one is normally on and the Y doesn't make much difference."
(J.B., M, 11 yrs., School B(P))

"Whatever colour you put with Y, it stays the colour you put with it."
(M.G., F, 11 yrs., School A(P))

"The Y one if you put it on with G will be G, with B will be B and
with R will be R -- it'll change to whatever colour there is."
(W.B., M., 10 yrs., School A(P))

"Well, when you're going through all the ones with Y, it always has
to be one of them. YG had to be one of those and it was G. Then
with YY there was only one, so it had to be Y. That's how if you
know YY = Y, then YG can't be Y and so it's G. Then YB = B and
YR = R -- there are only two to choose from and you take one of them.
Y is not really making a new colour, just the same colour as the
other one -- but that only applies to Y with the others."
(V.H., F, 13 yrs., School A(S).
The learning record of this subject appears in Appendix I.)

Note: No concrete operational explanation of inverse elements
can be given, so the responses of concrete operational subjects to
questioning about this axiom are not reported in detail here. Such
subjects typically said that the combinations which made Y were hard
to explain, or did not make sense. Some gave "explanations" in
terms of lighter and darker colours, or in terms of the fact that Y
was somehow the only answer left.
Explanations of the associative law, by concrete operational subjects

Subjects were asked to combine the three lights R, B, and G in two different ways to arrive at the answer B. They were then asked whether the answer should be the same, whichever way the lights were combined, and why. Some typical responses are shown below.

"It would be the same because if you put G paint in and R paint in next and then B paint it will make the same colour in the end as if you put the R paint first then the B and then the G."
(T.C., F, 10 yrs., School A(P))

"When you put two colours they mix in and then another colour with them makes the final answer. If the two colours are R and B first it goes G and then another G makes it B. If the colours are R and G first it's Y and then a B makes it B that way too."
(K.D., F, 11 yrs., School A(P))

"Well I turn on R and you turn on B -- that makes G. Then you don't worry about these two any more, and then in the other panel turn on G and it will come out G and G make B . . . Or G and R first make Y then Y and B make B . . . So it's the same whichever two colours you start with."
(K.M., F, 12 yrs., School A(P))

"Well one way it'd be BR which is G -- that'd be G made out of those two -- and then G from another panel would make B. Or you could say GR = Y and then YB = B, so it's the same answer whichever two you do first."
(J.R., F, 13 yrs., School B(S))
Concrete level explanations of these axioms may also be seen, together with details of the questioning leading up to them, in the reports of interviews with K.D. (F, 11 yrs., School A(P)) and with D.C. (F, 13 yrs., School A(S)) in Appendix IV. Part II of the Detailed Questioning Protocol, in Appendix II, shows in detail the questions asked, types of responses made, and the rated levels of understanding.

It should be pointed out that the extracts given above are all from subjects who succeeded in giving a satisfactory explanation of the axiom concerned. There were many subjects, at the stage of Concrete Operations, who failed to give an account of one, or some, of the axioms at all. The types of incorrect responses typically obtained may be seen in Part II of the Detailed Questioning Protocol in Appendix II.

10.3.2. The Four Group Task understood in terms of formal operations

The operations available to the subject at the stage of Formal Thought are abstracted from the combinations of pairs of lights, which are grouped together appropriately for these abstractions to be made. The operations consist of "roles" for each of the lights in some formal system, usually consisting of "numbers of moves around the panel" or just "the numbers 0, 1, 2, 3 (Modulo-4)." Many different sets of numbers will conform to the rules of the Cyclic Four Group - some examples obtained in the present study are: - 0, 2, 4, 6; 0, 5, 10, 15; 0, 1, 2, -1 (all under addition).
Records of the initial general questioning for two subjects who discovered a complete formal system, spontaneously, are contained in Appendix II (the subjects are M.B., F, 14 yrs., School A(S)) and V.S., M, 14 yrs., School B(S)). That appendix also contains the complete record of a subject whose initial response was not a complete system, but merely a structuring of the combinations by rows of the matrix, in a manner suitable for the abstraction of such system of roles (Subject M.C., M, 14 yrs., School B(S)).

10.3.2.1 Combinations grouped together by subjects at the stage of Formal Operations

These groupings of combinations, obtained almost universally from formal operational subjects, are shown below. To enable the inference of the "role" or "action" of each of the lights, combinations are grouped:

<table>
<thead>
<tr>
<th>For Y</th>
<th>For G</th>
<th>For B</th>
<th>For R</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y + Y = Y</td>
<td>G + Y = G</td>
<td>B + Y = B</td>
<td>R + Y = R</td>
</tr>
<tr>
<td>Y + G = G</td>
<td>G + G = B</td>
<td>B + G = R</td>
<td>R + G = Y</td>
</tr>
<tr>
<td>Y + B = B</td>
<td>G + B = A</td>
<td>B + B = Y</td>
<td>R + B = G</td>
</tr>
<tr>
<td>Y + R = A</td>
<td>G + R = Y</td>
<td>B + R = G</td>
<td>R + R = B</td>
</tr>
</tbody>
</table>

Some examples of such groupings taken from records of the initial general questioning on the Four Group Task are shown below. The responses quoted are the subjects' initial spontaneous responses to a request to "explain how the game works".
In most cases, this level of understanding was developed into a complete formal system, by the kind of questioning indicated in Part I of the Detailed Questioning Protocol in Appendix II, and illustrated by the complete record of M.C. (M, 14 yrs., School B(S)) in Appendix IV.

T.G. (M, 13 yrs., School B (S))

E: Can you explain to me now how the game works? How do you remember what the combinations of lights will make?

S: The Y, at the beginning, goes all straight through the same. YY = Y, YG = G, YB = B and YR = R - so they're all the same. If you have Y, it doesn't stay there - it goes down the panel - Y makes the colour on the second panel.

Now, GY = G, GG = B, GB = R and GR = Y, because that (indicates R) was the last one on the board and it just went down the scale on the big board and up to Y again. So going down the board on your panel, with the second one on my panel (G), it goes G, B, R and then Y on the big panel.

Now with BY it starts with the third one down, B; then BG = R, the next one down, BB = Y and BR = G. And with the R one it starts at R, (RY = R), then RG = Y because R was the last one on the board, then RB = G and then RR = B.

R.M. (M, 13 yrs., School A(S))

E: How do you remember what the combinations make?

S: When I was doing the yellows, YY = Y... I counted this as number one, so 1 x 1 = 1; this (G) is 2 and 1 x 2 = 2, YG stays G; the YB stays B; and the YR stays R, because Y is still one. With the greens they were the same. GY = Y, GG went B, GB went R, GR went Y. In the Y it goes 1, 2, 3, 4 like that (indicates Y, G, B, R), in the G it goes 1, 2, 3, 4 from G (indicates G, B, R, Y), in the B it goes from this one (B) 1, 2, 3, 4 (indicates B, R, Y, G) and in the R it goes from this one (R) 1, 2, 3, 4 (indicates R, Y, G, B).
K.P. (M, 11 yrs., School B(P))

E: What did you find out about the combinations of lights?

S: When I found the pattern about YY = Y, YG = G, YB = B, YR = R, I tried GY = Y and so on but I don't think I was very successful. . . . GY = G, GG = B, GB = R and GR = Y . . . Oh good! Can I go through them again? . . . GY = G, then GB went to R, oh yes GG was B . . . so it's GY = G, GG = B, GB = R and GR = Y . . . I was going . . . the first number mentioned wasn't the answer . . . it was the number after . . . the number after the G . . . no, the number after the one you have with G. It's moving it round a circle. It's the number, or the light, after the second colour mentioned (with G). So now I can say for Y, the one with it was the answer, for G it is the one after, for B it will jump 2 spaces and for R it will jump 3.

10.3.2.2 Explanations of concrete reversibility and the group axioms, by formal operational subjects.

The other aspect of the responses to questioning, which illustrates the formal level approach to the Four Group Task, is the explanations given of concrete reversibility and the group axioms. The subject at this stage of development uses as elements the formal "roles" of the lights, such as "a move of 2 to the right", or "no moves"; or just the numbers which are added together; and understands that these elements obey group axioms in the following ways:

(i) Concrete Reversibility

Since, for example, an action such as turning on a B light with a G is seen as "moving the G light two to the right" or "adding on two", the action of turning B off is seen as the opposite, in the sense of "moving it back two" or "subtracting two".
(ii) **Commutative Law**

The combined action or role of two lights can be shown to be the same, irrespective of the order of combination. For example, interpreted either as moves to the right, or as numbers:

\[
R + B = B + R = G \\
3 + 2 = 2 + 3 = 1
\]

(iii) **Unit Element**

The role of the Y light ("no moves" or "zero") is such that Y combined with any other element results in the other element (including YY = Y).

(iv) **Inverse Element**

Considering the action of each light, there is always another light whose action, combined with that of the first, results in the Unit Element. The actions of two lights which combine to make Y must be the "opposite", or inverse of each other. Pairs of inverses in the system are RG, BB and YY, and seen either as "moves around the panel" or as "numbers", the explanations are as below:

\[
R + G = Y \text{ (because } 1 + 3 = 0, \text{ or } 1 + (-1) = 0) \\
B + B = Y \text{ (because } 2 + 2 = 0, \text{ or } 2 + (-2) = 0) \\
Y + Y = Y \text{ (because } 0 + 0 = 0). 
\]

(v) **Associative Law**

If the actions of three lights are to be combined, and these are performed in any order, the same resultant number, or
number of moves, will be obtained. For example, in terms either of "numbers of moves around the panel" or of "numbers":

\[
\begin{align*}
R + (B + G) &= (R + B) + G = B \\
3 + (2 + 1) &= (3 + 2) + 1 \\
i.e. \quad 3 + 3 &= 5 + 1 = 2.
\end{align*}
\]

(Note: Formal operational subjects are also required to give an explanation of the Axiom of Closure, since it is not immediately clear that the combined action of two lights is the same as the action of some single light. This usually involves an explanation of the cyclic nature of the system, so that, for example, \(2 + 3 = 1\).

Although no questioning was specifically included for this, it is obvious that it was understood by subjects.)

The extracts from interviews which follow serve to illustrate formal operational accounts of the axioms, obtained in the present study.

Explanations of concrete reversibility, by formal operational subjects

"Well turning R on is moving one back and turning R off is moving one forward . . . that makes turning G on the same as turning R off."

(Note this combines the concrete form of reversibility with the notion of inverse elements in a formal structure.)

(N.A., F, 11 yrs., School A(P))

"When you turn the B on you go two to the left, so when you turn it off you go two to the right."

(J.C., F, 13 yrs., School B(S))

"Turning the B off is cancelling the combination and moving two steps back. It's the opposite action to turning it on."

(I.B., M, 14 yrs., School B(S))
"You're taking away the B light, or in the system it means going two steps back."  
(G.R., M, 14 yrs., School A(S))

Explanations of the commutative law, by formal operational subjects

"It's the same as just a colour and another colour . . . well, x colour + y colour = y colour + x colour. The colours stand for numbers of moves."  
(I.B., M, 14 yrs., School B(S))

"You can add up 1 + 2 = 3 or 2 + 1 = 3 -- those lights just stand for numbers and if they're added they always make the same number no matter which way they're turned around."  
(A.S., M, 14 yrs., School B(S))

"If you add B to G . . . can I do it in numbers? . . . B = 2, G = 1, add them and you get three -- if you add 1 + 2 it will still equal 3 -- that's the commutative law."  
(C.G., F, 14 yrs., School B(S))

"Well you're sort of going the same number along, it doesn't matter if you go 2 first and then 3, or 3 first and then 2."  
(H.P., F, 14 yrs., School B(S))

Explanations of the unit element, by formal operational subjects

"Any colour with Y ends up the colour you put with Y, because Y stands for nothing, no moves."  
(C.B., F, 13 yrs., School B(S))

"Y is zero."  
(R.M., M, 13 yrs., School A(S))

"The colour that Y goes with . . . Y makes the colour it goes with the same. If you are adding then Y is the number 0."  
(J.A., F, 14 yrs., School A(S).
The learning record of this subject appears in Appendix I.)
"The Y light doesn't move colours ahead any. With Y they all stay the same. Y stands for no moves."  (K.P., M, 11 yrs., School B(P))

Explanations of inverse elements, by formal operational subjects

Subjects were asked what combinations of lights resulted in the Y light and then were questioned to see if they could explain these combinations as pairs of inverse elements. The responses of some subjects who succeeded in doing so are reported below.

"G and R make Y because they stand for one forward and one back and sort of neutralise each other. With B, it stands for two forward or two back and so it neutralises itself."

(H.P., F, 14 yrs., School B(S))

"When two lights add up to Y which is no moves they must cancel each other out. G moves any light one step forward and R moves any light three steps forward, which is one step back, so that is how those two cancel out. With B it is difficult -- I'm trying out the moves forward and back. B and B is two forward and then two back and so the two reactions cancel each other out. B stands for both because round this circle of four, two forward is the same as two back."

(I.B., M, 14 yrs., School B(S))

"The two lights are cancelling each other out. G is 1, R is 3, together they make 4 and that's the nothing move. That's because you've only got three moves, as soon as you make four you are back to nothing. Also 1 back = 3 forward, so for G and R we can say 1 forward and 1 back, so they kill each other. B is the opposite of itself, because B takes anything forward 2, or backward 2, it's the same anyway -- and Y is the opposite of itself."

(A.S., M, 14 yrs., School B(S))
"G and R must be opposites or companions, because turning on the G is the same as turning off the R. Turning on the G makes you go 2 to the right, no 1 to the right and turning on the R makes you go 1 to the left... together they make Y... oh I get it now... if you move 1 to the right, then 1 to the left, you just get back where you were and may as well use Y to stay. B can be the opposite of itself because 2 to the right is the opposite of 2 to the left."

(M.P., M, 10 yrs., School A(P))

Explanations of the associative law, by formal operational subjects

"3 + 2 = 5, then 5 + 1 = 6, so it will turn out to be 6, which is the B light -- or 1 + 2 = 3, then 3 + 3 = 6. It has to be the same, because they're the same numbers." (C.G., F, 14 yrs., School B(S))

"You would just go one back from B to G, when you put the R with the B light, and then if you put another G on to that it would come back again to B. That's because R plus B is 3 plus 2, which is the same as 1 -- G is 1 and then 1 + 1 = 2. To do it another way, G = 1 plus B = 2 which makes 3, the R, and then there's a R left so 3 + 3 = 6 which would be B. It turns out to be so simple once you've found out the numbers."

(M.M., M, 13 yrs., School A(S))

"Well R is 6, B is 4, 6 + 4 = 10 and then plus G which is 2 makes 12 -- and 12 is B. You can add those numbers in any order and it will still come out B."

(J.D., F, 11 yrs., School B(P))

"R, B, and G is 1, 2, and 3... so 3 + 2 = 5, then 5 + 1 = 6... that would equal 1, 2, 3, 4, 5, B! Now I could also have just said B + R = G and then G + G = B, that's the same. If I do G + B first, that's 1 + 2 = 3, which is R, and then from R 1, 2, 3, equals B."

(T.G., M, 13 yrs., School B(S)).
This concludes the description of the formal operational approach to the Four Group Task. It is clear that it is distinct from the concrete operational approach in that the subject understands the structure in terms of a "role" for each element, which is the same for every combination in which that element occurs. The group axioms are also understood in terms of the formal roles of the lights, rather than in terms of the physical manipulations made by the subject.

10.3.3 Summary

The two levels of understanding of the Four Group Task, described in the preceding sections 10.3.1 and 10.3.2, confirm the interpretation, given in Chapter 3, of the differences between Dienes and Jeeves' (1965) "pattern" and "operator" evaluations. The two types of structuring of the matrix, and the two systems of "operations" in terms of which the combinations are understood, are put forward as evidence supporting Piaget's account of the stages of Concrete and Formal Operational Thought. That these two levels of approach to the Four Group Task can be meaningfully related to Piaget's theoretical notions, and to performance on the other measure of operational level (performance on the Pendulum Problem) will be shown in Chapter 11. This next Chapter will also discuss the other results of the study and their theoretical and methodological implications.