THE TRANSITION FROM CONCRETE
TO FORMAL THINKING

A thesis submitted in partial
fulfilment of the requirements
for the degree of Doctor of
Philosophy.

by

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December, 1970.
CHAP TER 3

THE RELATIONSHIP OF THE WORK OF DIENES AND JEEVES TO PIAGET'S THEORY

3.1 Factors Determining the Selection of Experimental Work for Use in the Present Study.

The present investigation combines some of the features of Neimark's (1970) study and some suggestions made by Flavell and Wohlwill (1969) in their theoretical analysis of transition periods, both of which have been discussed in the previous chapter. However, since neither of these publications was available when the study was planned, an account will not be given of the considerations which led to its design. From a review of the earlier experimental literature it was clear that an investigation of the transition from concrete to formal thinking would be largely of an exploratory nature. Nevertheless, it was felt that some ideas could be put forward as to the type of study which would be most informative. Experiments which "test" deductions from Piaget's theory are not easily derived, partly because his own work has not grown out of precise measurements of limited phenomena, but out of a global view of the nature and development of human intelligence. Wohlwill (1970) sees Piaget as providing "systematic descriptive work on the sequential appearance of behaviours in development" and says:-

"Piaget's sequences, to be sure, are embedded in an intricate theoretical superstructure, functioning much as a mathematical model may do in the study of growth or learning curves, the observations serving mainly to confirm the adequacy of the model, rather than to test specific hypotheses in a deductive sense." (Wohlwill, 1970, p. 55).
It was also clear from the literature that many authors felt that his analysis was potentially applicable to any realm of cognitive activity, be it scientific, mathematical, or concerned with the humanities and politics; but equally clear that a method of application was difficult to find.

The present study began when a task was found, in the work of Dienes and Jeeves (1965), for which it seemed possible to predict, in advance, the levels of performance of concrete operational and formal operational subjects. The task was not identical to any used by Inhelder and Piaget (1958) although its structure was that used by Piaget to characterise formal thought, namely the structure of a mathematical group. While Dienes and Jeeves (1965) had found different levels of approach to the task, and a difference between a group of eleven year old children and a group of adults in the level of approach, they had not specifically associated the age difference with Piaget's distinction between concrete and formal thinking.

It is interesting that Easley (1964) has also speculated about the use of a similar problem (embodying the same structure as the INRC group) in a test of formal thinking. After describing a situation in which a snail, on a board, may move to the left or to the right, and in which the board itself may also be moved to the left or right, Easley (1964) demonstrates that the structure of possible movements is the same as the structure of the INRC group of logical operations. He then envisages a set of questions which a subject could be asked about combinations of movements, surmises that quite young children could answer all these
questions correctly after a little practice, and asks "is it sufficient for the child to answer all these questions separately? If the child answers all these questions correctly, is the child then at the level of Formal Operations?" (Easley, 1964, p. 111).

From the work of Dienes and Jeeves (1965) it would appear that the presence or absence of formal operations will be revealed, not by success or failure in learning such a structure, but by the method of approach and the ultimate level of understanding which the subject displays. It was the difference between subjects in these latter aspects, rather than the structure of the problem itself, which recommended the task to the present author. The reasons for this will become clearer when the results of Dienes and Jeeves' (1965) experiments are described. This will be done now, and followed by an interpretation of their findings in terms of Piaget's stages of concrete and formal thought. The present study was designed to investigate this interpretation.

3.2 Background to the experimental work of Dienes and Jeeves.

Dienes has made highly original contributions to the field of mathematics learning (Dienes 1959, 1960, 1963, 1964). This work has received theoretical attention in its own right from developmental psychologists (Bruner and Kenney 1965) and has a place amongst a growing body of experimental evidence in this area (Skemp 1961; Gagné, Mayor and Garstens 1962; Kilpatrick 1964; Gagné and Staff, Univ. of Md. Math. Project 1965; Lovell 1965; Suppes 1965; Law 1969). Dienes has also given an account of the formation of concepts, mathematical concepts in
particular, which he considers has much in common with Piaget's developmental model (Dienes 1959, 1960, 1961). This is not primarily a developmental model, however, but one which he claims is equally applicable to the child and to the adult acquiring a concept for the first time. In his later account (Dienes 1961), he recognises that Piaget might not agree that their accounts describe essentially the same process.

Dienes' point of view is illustrated by the following extracts. He gives an account of Piaget's developmental theory and then says:

"The above is a developmental cycle, ending in adult forms of thinking. If the cognitive process has been kept dynamic, the development does not cease here, but a kind of play activity sets in on a higher level. The experiences collected, the classifications achieved, form the bricks with which new (usually mental) games can be played, and this ushers in a new play stage. Such is the situation when a young mathematician starts out on a new path of research: he plays with his bricks, until he perceives a direction in which he can hopefully turn, he operates his tools in this direction, until he forms a theory in which his bricks will be put together in a certain way. He then proceeds to check his structure logically, i.e., he ascertains the relationships between his pieces and makes sure that the structure does not collapse (freedom from contradiction.) He has then built himself another toy with which he can start out on another game.

It is suggested here that what Piaget perceived as a developmental cycle in the large, as a macrocosmos as it were, also occurs in the formation of every abstract concept as a microcosmos" (Dienes, 1961, p. 283).

Dienes then goes on to suggest that the first stage in the formation of a concept can be described as a "play stage", and an analogy is drawn with Piaget's preoperational stages in the development of thought.

"Play is to be understood as undirected activity, seemingly purposeless, in which there is freedom to experiment; during such play, as it is undirected, we are energised by our own primary sources of energy" (Ibid., p. 284).
Then he asks:—

"What corresponds to the concrete operational stage in the microcosmos of the formation of an individual concept? Surely it is the passing from apparent chaos to structure, from apparent lack of direction to the realization of a direction in our thinking. The playthings, whether concrete objects isomorphic to a structure or structures, or purely mental playthings, are now being put together according to some plan. The plan may not be a very coherent one, it may even be impossible of realization, but the work is no longer random. This is a very much more intense stage, and the energy derived from the play stage may be very necessary to carry us through the vicissitudes of the structured stage. If the structuring has not been too early or too severe, in other words if the cycle is allowed to follow its natural rhythm, the search for the final insight will be exciting and the power will not be reduced or cut off. The endpoint of this stage is the occurrence of the final insight; this is when all the pieces in the game appear to click together as if nothing could be more natural. This moment is a very exciting one, and provides the power for the next stage" (Ibid., p. 284).

The next stage, analogous to Piaget's developmental stage of formal operations, is described as follows:—

"The next stage has really two aspects: one is having a look at what we have done and seeing how it is really put together (logical analysis), the other is making use of what we have done (practice). In the early years this stage consists mainly of practice and of rather little logical dissection; as we grow older and are able to "think about" our structures, we may wax more analytical. There will arise individual variations as to the weighting of this stage towards the logical or towards the practical side, and so variations from the so-called "purist" to the "practical man" will develop over a population" (Ibid., p. 285)

It is plain, from these extracts, that Dienes' primary enthusiasm is for the discovery of mathematical concepts and is it not surprising that much of his work, both theoretical and experimental, is directed at ways of bringing about this discovery, through a sequence of "play", then "structuring" and then "logical analysis and application of the
results" of the kind described above. The mathematical structure forming the basis of Dienes and Jeeves' (1965) work was earlier used by Dienes (1963, 1964) in studies of experimental mathematics learning. This structure, the finite mathematical group (with 2, 3, 4, 6, or 8 elements), was embodied in a number of classroom "games", such as moves around a circle or square, or rotations of an object (such as a book) around various axes. Dienes' theory of the abstraction and generalisation of structures from manipulation of objects, or from playing games, is outlined and applied in detail in these books (Dienes 1963, 1964). The role of active play, both in maintaining motivation and in providing the necessary concrete experiences, is emphasised throughout. The aim is to bring the child to a discovery of the "structure" inherent in a mathematical game, or story, or set of blocks. However, progress to a "formal" stage, of a logical analysis of the structures, will only occur if a number of games, or stories, each embodying the same structure, are compared with one another. When this is done, it is possible for the child to arrive at a formal set of rules, which enable him to find out whether other games have the same structure too, or even to generate games of his own having this structure.

3.3 Experimental work on "Thinking in Structures" by Dienes and Jeeves.

The experiments of Dienes and Jeeves (1965) can be seen as an attempt to reveal the process of the discovery of a mathematical group structure, by subjects to whom it is not a familiar concept. The authors acknowledge the influence of Sir Frederick Bartlett (1958), both in his
orientation towards thought as a process of structuring, or making sense of, experience and in his direct comments on their work. They compare the work of Bartlett with that of Bruner and also with their own work as follows:

"One of the purposes of the study reported here is to propose and use a paradigm by means of which -- putting together of information into wholes in certain ways may be studied. Bartlett has studied the whole process, while Bruner has studied its parts. In this study it is hoped that the whole is being considered through the detailed ways in which its parts are put together. This explains the title; we shall be concerned with the study of the emergence of models, structures in terms of which we think." (Dienes and Jeeves; "Thinking in Structures, 1965, p.12).

In their study, four different structures were used as experimental tasks. Two of these had only two elements and two had four. Both of the four-element structures were mathematical groups (the Klein group and the Cyclic Four Group), but only one of the two-element structures was a group. Each subject learned one two-element and one four-element structure. The variables whose effects were investigated included: age (a comparison of first-year psychology students with a group of children whose average age was eleven and a half years); sex; the order of learning a two and a four-element structure (2-4 order compared with 4-2); the effect of the selection of instances by the subject as compared with the reception of a prearranged sequence; the effect of changing or not changing symbols from the first to the second structure learned; and the effect of the greater symmetry of the Klein group of four elements, as compared with the Cyclic Four Group. These were the major independent variables. The dependent variables will not be meaningful until the apparatus and instructions to subjects have been
described. This is best done by direct quotation (Note that the terms dependent and independent variables have a different meaning, in what follows, from their meaning above):-

"The apparatus consisted of a piece of hardboard with a window in it which could be opened and closed from behind, the subject being situated in front of it. A number of different symbols or cards could be put in the window. The subject was provided with the same cards as those that could appear in the window. An event was the dependent variable. The independent variables in this experiment were:

(a) the card in the window previous to the card appearing for the event.

(b) the card played, that is put on the table, by the subject.

The card that appeared in the window depended therefore on two factors:

(a) the card that was in the window previously

(b) the card that was played by the subject.

These were the two independent variables. This is why the structure was a two-dimensional structure. The two-dimensional dependence of the event on the two independent variables just described was made clear to the subjects before the experiment began.

The instructions given to subjects are described below:

We are going to play a game with you, and your task will be to discover the rules of the game as we play it.

You have these two (four) cards, with . . . on them.

I also have two (four) such cards, with the same pictures (or signs), which I can put in this window.

What I shall want you to do is:

(1) Look in the window, then

(2) Play one of your cards, by putting one on the table so that I can see it.

Then I shall shut the window, then open it again with another picture or the same picture showing.

What you will see in the window next will depend entirely on what was in the window before and on the card you played (repeat).
You are really playing with a mechanism, only I happen to be working it. Your task is to find out how the mechanism works. Do not forget that it is a mechanism. What it does once, it will do again under the same circumstances. If one picture in the window once changed to a certain other picture when you played a certain card, the same will happen again whenever you play the same card when the same picture is in the window. The mechanism does not consist of any sequence (repeat), it could not possibly, since the card you play is your choice and the machine cannot respond until you have played your card.

Lastly, this is not a test, nor an attempt at constructing one. We are not interested in whether you succeed in doing the task or in how rapidly you do so. We are interested in the ways you attack the problem of finding out how the mechanism works.

Any questions?

I now want you to have four (six) preliminary tries, after which I shall ask you to look at the window, play a card, and say what picture you will expect to see in the window next. We shall go on like that until your expectations turn out correct every time" (Ibid., pp. 20-22).

Precautions were taken to ensure that the subject did not think there was any "catch" in the game. Records were kept of all cards played, and the predictions made by the subject. The criterion performance to which subjects were taken is described as follows:-

"The subject went on playing cards until he had made a certain number of consecutive correct predictions. After this he was systematically examined by the experimenter on all the possible combinations of cards in the window and cards played. That is, the experimenter presented him with the table row by row. If the subject knew in each case what the next card in the window would be, this part of the task was at an end. If he did not, he was given a free choice of playing further cards and encouraged to learn all the other possibilities which he had not learnt at this stage. If on examination the subject knew all but a certain number of combinations, he was asked these combinations again. If he then knew them, this part of the task was at an end. If he did not, then the freedom of choice was again restored to him, and the task proceeded until he reached the criterion of the certain number of correct predictions again" (Ibid., p. 22).
When learning and testing of the combinations was complete, the subject was asked a series of questions. The first two of these, asked after completion of each of the games were:

1. "Explain in words how the game works, not just by enumerating all these cases. Have you discovered any principle?" (Ibid., p. 22)

2. "How did you get the rules?" (Ibid., p. 23)

The remaining questions were different, according to whether the structure learned was one with two or with four elements. These questions will only be meaningful in terms of the actual combinations involved in the task, so before quoting them, Table 2 will be presented, showing the structure of combinations for the Cyclic Four Group, and for the Klein Group of four elements. The letters Y, B, O, G stand for the colours of the cards used as elements of the group: Y (yellow), B (blue), O (orange) and G (green).

### Table 2: Examples of the two different groups with four elements, used by Dienes and Jeeves (1965). Letters stand for the colours of cards used as elements of the groups: Y (yellow), B (blue), O (orange) and G (green); it should be pointed out that these were not the only types of symbols used as elements. The letters in the cells of tables show the results of all possible combinations of window (column) and played (row) cards.

#### The Cyclic Group with four Elements

<table>
<thead>
<tr>
<th>Card in window</th>
<th>Card in window</th>
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<tbody>
<tr>
<td>Y</td>
<td>B</td>
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<tr>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>Card</td>
<td>B</td>
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<tr>
<td>played</td>
<td>O</td>
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<tr>
<td>G</td>
<td>G</td>
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</tbody>
</table>

#### The Klein Group of four Elements

<table>
<thead>
<tr>
<th>Card in window</th>
<th>Card in window</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y</td>
<td>B</td>
</tr>
<tr>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>Card</td>
<td>B</td>
</tr>
<tr>
<td>played</td>
<td>O</td>
</tr>
<tr>
<td>G</td>
<td>G</td>
</tr>
</tbody>
</table>
Further questions asked, in Dienes and Jeeves' (1965) experiment, after completion of the learning and testing of the four element games, were as follows:-

"3 (a) If there is a blue in the window, and you want a blue in the window next, what card would you play?
(b) If there is a blue in the window and you want a green in the window next, what colour would you play?
(c) If there is a green in the window and you want an orange in the window next, what colour would you play?
(d) If there is a yellow in the window and you want a yellow in the window next, what card would you play?
(e) If there is an orange in the window and you want a yellow in the window next, what card would you play?
(f) If there is a blue in the window and you want a yellow in the window next, what card would you play?
(g) If there is a green in the window and you want a yellow in the window next, what card would you play?

4. Suppose yellow is in the window; then you play orange then blue.
Could you have obtained the resulting window by playing one card?
If so, which?
Same question for blue then blue, also green then green.

5. Suppose now orange is in the window, then you play orange then blue, could you have obtained the resulting window by playing one card? If so, would it be the same card as when you started on yellow?
Same question for blue then blue, also green then green.

6. Will these "replacement" cards be the same, whatever the initial window, or not?
7. Are the rules for obtaining the "replacement cards" the same or different from the rule followed by the mechanism in showing the next picture after a card is played?

8. Does this game, or any of the rules in it, remind you of anything else you know?" (Ibid., pp. 23-24).

The phrasing of instructions and questions was modified slightly for the group of eleven year old subjects. The dependent variables examined by Dienes and Jeeves (1965) consisted of a number of types of scores. Several different error scores were devised, intended to show how much difficulty was experienced in learning particular combinations, and also to show what sorts of erroneous predictions were commonly made. Of more immediate interest for this discussion are the "strategy scores", devised as an indication of the approach to the problems, and also the different types of evaluation given in response to questions about "how the game works". Discussion of these will be confined to the tasks using four-element groups. However, for these structures, a detailed account will be given, to provide sufficient information for subsequent interpretation and discussion of results.

Firstly, the Strategy Scores devised were as follows:-

(a) **Operator Score** "This is obtained by counting the total number of cards played in runs of three or more of the same card being played in succession, this number being divided by the total number of freely selected instances" (Ibid., p. 25).
(b) **Pattern Score**  "For the purpose of scoring this category, the table of the structure is divided into three sections, which will be referred to as Constant or C section, Squares or S section, and Triangle or T section. The diagram below indicates where these are in the table.

<table>
<thead>
<tr>
<th>Y</th>
<th>B</th>
<th>O</th>
<th>G</th>
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</thead>
<tbody>
<tr>
<td>Y</td>
<td>S</td>
<td>C</td>
<td>C</td>
</tr>
<tr>
<td>B</td>
<td>C</td>
<td>S</td>
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<tr>
<td>G</td>
<td>C</td>
<td>T</td>
<td>T</td>
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</table>

C = "Constant" section  
S = "Squares" section  
T = "Triangle" section

The pattern score is computed in a similar way to the operator score. Runs of three or more combinations from the same section are counted towards the pattern score. Runs of three or more combinations from the same section are counted if interspersed by single correct predictions from other sections. It is not possible to generate any desired instance immediately after any given instance. For example, if a subject has been through all three combinations of the T section he will find himself in the C section; he would need to use an instance from the C section in order to return to the T section. This is not so for the Klein group, in which the triangle properties are entirely symmetrical. But even in this group, all the instances on the S section cannot occur in succession. For these reasons a sectional run is still counted as a run towards the pattern score if it is interspersed by single instances correctly predicted, which do not belong to the section from which the run is taken.

This applies naturally to every section, i.e. C, S, and T".

*(Ibid., p. 23. The table is adapted to conform with Table 2 of this thesis on page 100.)*
In discussing the answers received to questions after the game had been learned, Dienes and Jeeves (1965) say:-

"These answers appeared to sort themselves into easily distinguishable types. It will be recalled that the question was "Explain in words how the game works, not just by enumerating all the cases. Have you discovered any principles?" The types were as follows:

1. **Operational type**

   These subjects appeared to regard the card played as operating on the card in the window, having the power to alter this card.

2. **Pattern type**

   These subjects appeared to regard the game as divided up into a certain number of sub-sections. They regarded the card in the window and the card on the table as on the same level. They described one part of the table at a time as hanging together into a pattern. The whole table appeared to them, by their reports, as a large pattern put together out of certain smaller patterns.

3. **Memory type**

   These subjects stated that they had merely memorized all the different combinations.

There were a great number of subjects who were not of these pure types, particularly in the four-game. There were mixtures of operator and pattern, and even more mixtures of pattern and memory. These were the ones that were only able to verbalize a part of the pattern of the table. It would seem fair to say that the operator strategy is superior to the others, the pattern one being somewhat more superficial.
Correspondingly, a strategy score was invented with five points given for operator, three for pattern, and one for memory, two for partial pattern, and four for a mixture of operator and pattern, and three for the mixture of operator and memory. A list is provided below for reference.

\[
\begin{align*}
\text{Op.} & = 5 \\
\text{Op. Pat} & = 4 \\
\text{Pat.} & = 3 \\
\text{Op. Mem.} & = 3 \\
\text{Pat. Mem.} & = 2 \\
\text{Mem.} & = 1 \\
\text{Fail} & = 0
\end{align*}
\]

(Ibid., p. 36).

Dienes and Jeeves (1965) are able to establish a relationship between a subject’s strategy score and his type of evaluation of the task. Thus it can be said that these two dependent variables reflect, in different but related ways, a subject’s overall approach to the problem. The major distinctions to be made are between an approach characterised by the operator strategy and evaluation, an approach characterised by a pattern strategy and evaluation, and an approach employing rote memory and a nonsystematic investigation strategy. The authors point out that combinations of these types of approach occurred quite commonly, and the "hierarchy" of levels of approach which they suggest (and score on a scale from 0 to 5) has been quoted above. Some examples of evaluations illustrating these different levels are also given, and these are as below:

Operator Evaluations of the Cyclic Four Group.

1. "Arrange the colours in a circle round the spectrum. The yellow
corresponds to "no change". The orange card played will induce a shift of two round this circle. The blue card played would induce a shift of one in the direction yellow-blue-orange-green-yellow. The green card played would induce a shift of one round the circle in the direction yellow-green-orange-blue-yellow" (Ibid., p.42)

2. "Yellow is zero, blue is one, orange is two, and green is three and you simply add them up and when you come to four you go down to zero again," (Ibid., p.42).

A Combined Operator and Pattern Evaluation of the Cyclic Four Group

"If the colours blue and green are present, then these colours dominate. With the colours blue and green, the pairs give an orange and the odd ones give yellow. With the colours yellow and orange the pairs give yellow and the odd ones give an orange. An orange will change a colour and a yellow will maintain it" (Ibid., p.42)

A Combined Operator and Memory Evaluation of the Cyclic Four Group

"A yellow card keeps the card as it is, a card as it is played remains. No change. Orange in the window changes it to the other one of the pair. Yellow into orange, orange into yellow, blue into green, green into blue. The pairs being yellow and orange, blue and green" (Ibid., p.43)

It should be noted that this evaluation, atypically in Dienes and Jeeves' study, regards the window card as the "operator".

A Partial Pattern Evaluation of the Cyclic Four Group.

"When there is yellow in the window it becomes next whatever you play. Green, Blue and orange are related. I couldn't tell you how" (Ibid., pp. 43-44).
A Memory Evaluation of the Cyclic Four Group

"The game seems to be made up of finding pairs of colours that go together. It was learning the pairs. It doesn't matter where they are". (Ibid., p. 44).

The clearcut types of evaluation, illustrated above, do not give a complete description of all the responses made. Dienes and Jeeves remark that they obtained quite a number of "references to mathematics", such as that there were "odds and evens" or that "yellows are ones, orange is a big number, the others are factors. There is no way in which it can be made right" (Ibid., p. 49) It could be suggested that a structuring of the combinations is evident in the response of the subject quoted above as showing purely memory evaluation, where the statements are made:-

"It was learning the pairs. It doesn't matter where they are" (Ibid., p. 44). The meaning of the last sentence appears to be that the combinations obey the Commutative Law, in other words, that changing the positions of the cards, (window and played) does not affect the result. This represents a structuring of the matrix into pairs of combinations and, as such, is not very different from the "pattern" structuring into the so-called "sections" (C, S and T) of the matrix described by Dienes and Jeeves.

Detailed investigations of the effects of a number of variables were made by the authors and these cannot be discussed in full here. The most important aspect, for the present study, is the evidence for a hierarchy of levels of approach to the problem. In this respect, it is the Cyclic Four Group which deserves greatest attention, since it
was in evaluations of this task that the greatest number of operator-type responses were obtained, Dienes and Jeeves (1965) point out that the "roles" of elements in the Klein Group of four elements are much more similar to one another than is true in the Cyclic Four Group. They suggest that this may possibly explain the greater number of high level evaluations of the latter structure - it is noticeable that the examples of types of evaluations, quoted earlier from their book, all apply to the Cyclic Four Group.

The evidence which Dienes and Jeeves (1965) adduce for the hierarchy of levels of evaluation is in a variety of forms. Firstly, the relationship which they were able to establish between operator and pattern strategy scores, and the level of evaluation of the game by the subject (some combination of operator, pattern and memory type evaluation), provides evidence that there are definite categories of approach to the tasks. Secondly, the mean number of instances required to learn the structure is significantly larger for the "memory and pattern-memory" evaluators than for the subjects giving "operator or partial operator" evaluations. The operator-type evaluations thus seem to be associated with a more efficient method of learning the structure. Other measures of the extent to which combinations have been "learned" by the subject (such as the number of questions answered correctly under 3 (a) to (g) at the conclusion of the learning phase) also relate to the type of evaluation of the game. Thirdly, the authors report that the proportion of adult subjects giving high level evaluations of the games is greater than the corresponding proportion of children. These three sources of
information thus build up a picture of the hierarchy of levels of approach to the problem; from rote memory approach, through a structuring of the matrix into subsections as an aid to memory, to an approach which arrives at an operational "role" for each element of the structure. The notion that a developmental study might reveal a progression through these stages is already suggested by Dienes and Jeeves' (1965) comparison of an eleven year old sample with a sample of first year psychology students. It will be argued that such a progression can be predicted, in a reasonably precise way, from Piaget's theory. The most crucial stage will be the point of transition from concrete to formal operational thought.

3.4 Interpretation of Dienes and Jeeves' (1965) results in terms of Piaget's theory.

Dienes and Jeeves' (1965) descriptions of the types of strategies and evaluations, obtained in their experiments with mathematical group structures, lend themselves to interpretation in terms of Piaget's theory. When this is done, it seems plausible to suggest that the difference between a subject whose strategy and evaluation have 'operator' characteristics, and a subject characterised by a "pattern" strategy and evaluation, is equivalent to the distinction between subjects at the stages of formal and concrete operational thought respectively. Not a great deal of reinterpretation of what Dienes and Jeeves have said is involved. One implication of the reinterpretation, however, is that a more prolonged, more clinical, interview of subjects after the completion of the learning would have revealed more of their understanding of the games.
One of the recognised abilities of a child at the stage of concrete operations is the classification and comparison of objects and events. In a Four Group Task, combinations of pairs of elements may be classified and compared with each other. The concrete operational child will tend to classify them in more or less obvious ways, not determined by any understanding of the possible structures which may be found. He may find similarities between some of the combinations and observe a corresponding similarity between their results. Such an approach could be expected to result in a certain amount of structuring of the problem, basically in ways which will enable him to remember that such and such a "group" of combinations leads to such and such a "group" of results.

The pattern evaluations of the tasks described by Dienes and Jeeves (1965) appear to show structurings of this kind. It has been suggested earlier, however, that the division of the matrix which they describe (into C, S and T sections) is not the only structuring of this nature which may be discerned. Specifically, a "grouping together" of each combination, AB, with the combination of the same elements in reverse order, BA, results in six pairs of combinations (excluding the four double ones) and is a structuring of the matrix.

These types of classification of combinations, and their results, can be put forward as the expected achievements of subjects with a concrete operational approach to the problem. The subject is basically "looking to see" what interesting and pleasing features of the task he can discern and describe. He may often be disappointed, at first, that the combinations do not conform to some facts which he already knows,
such as those involved in mixing paints. He will be somewhat more disappointed if he can discover no regularities at all, and the enunciation of a statement such as: "I know the double ones, they are YY, GG, BB, and RR, and the answers go Y, B, Y, B," will give him great satisfaction. These are the kinds of "principles" he expects to find embodied in the task, and he is quite happy to leave it at that. He does not ask himself, and has no answer to the question, if asked: "Why do the answers go like that?"

At the level of formal operational thought, an altogether different approach to the problem can be postulated. The subject should be aware that a number of different types of "formal structure" can be tried out as "models" of the problem. Sometimes, as a preliminary exercise, notions such as colour mixing might be mentioned, and dismissed, with a statement such as: "Well, I suppose it's not just like paints." The trial of one or two combinations should be sufficient to confirm this suspicion. From then on, the subject may proceed in a manner quite similar to that of subjects at the concrete operational stage, in the sense that different combinations and their results are compared with one another. However, his "attitude towards" the regularities that he is able to discover should be quite different. He will not be satisfied unless they explain something to him about the structure of the task. Thus he does not stop at a pattern type of division of the matrix into manageable sections, nor at a statement of "principles" which describe only a subset of the combinations. He is looking for a set of rules which make sense of the results of all sixteen. It may be only after some time
that the symmetries in the situation suggest to him that all elements are of equivalent importance, in the sense that the role of each must be ascertained. Remarks such as the following would indicate this:-

"I see that there are sixteen combinations here; that's four each for each of the elements". The problem then becomes one of "grouping together" combinations appropriately to reveal the roles of the separate elements. This is not an easy task. The pattern type of structuring of the matrix is in fact a hindrance to such an attempt, with the possible exception of the isolation of the C section. What the subject needs to do is to divide the matrix into its four rows (or four columns) and examine the results in each, separately, until he is able to express what happens when a given element is combined with "each in turn", or preferably "any", of the other elements.

A subject who employs an operator strategy and/or gives an operator evaluation of the task, as presented by Dienes and Jeeves (1965) seems to behaving in the above way. As the authors point out, he will be somewhat frustrated in his attempts to investigate the combinations as he wants, because the card in the window is always determined by the previous combination. Nevertheless, it is clear that some of Dienes and Jeeves' (1965) subjects employed such a strategy. The suggestion that it is a formal operational strategy, in Piagetian terms, is made on the basis of the purposes for which combinations are grouped together. At the stage of formal thinking, it is suggested that the subject will experiment with different sorts of structurings of the matrix until he finds one which allows a complete and consistent "explanation" of the
results of combinations to be given. The concrete operational child, by contrast, will be satisfied by whatever structurings, or partial structurings, he is able to achieve. He will not usually substitute new ones for the first ones he discovers, nor even show any signs of further search.

In summary, this reinterpretation of Dienes and Jeeves' (1965) results in the form of Piagetian stage differences is based on notions of the global approaches to the problem which would characterise each stage, and on the types of inference which would be made from observations in each case. It does not seem possible to make the interpretation more specific in terms of particular members of the set of sixteen binary operations. The argument is, rather, that the subject will not conceive of the problem in terms of a variety of different, logically possible, structures, unless he is at the stage of formal thought. Whether or not he is will be shown in the manner in which he groups and regroups the combinations of the matrix. A child at the stage of concrete operations, by contrast, is satisfied with whatever structuring of the matrix he discovers first.

The distinctions made above seem to lead to a comparison of concrete and formal thinking in terms of the mental operations used to conceptualise the task. At the stage of formal operations the subject will strive to arrive at a "role" for each element and will understand and describe the task in terms of these. It can be suggested that these "roles" which he abstracts are what Piaget would mean by formal operations. By comparison, the child at the stage of concrete operations cannot
conceptualise the task except in terms of actions which he may perform on the materials in front of him. Thus he is limited to internalised operations such as "playing a green card against a yellow card" and so on. The "structure" of the task which he perceives can consist only of "groupings together" of similar combinations. The structure perceived by the formal operational child, on the other hand, is derived from interrelating the "roles" of the elements. An attempt is made, in the present study, to bring out the distinction between concrete and formal thinking in these terms, using a variation of Dienes and Jeeves' (1965) Cyclic Four Group Task. The task and testing procedure will be described in the next chapter.

Dienes and Jeeves (1965, 1967) have extended their work on the understanding of mathematical group structures in a number of directions not mentioned in the above account. Jeeves (1968) has given a summary of this work. The central aspect of interest to them appears to be the transfer which occurs between the learning of one group structure and another. In their earlier work (Dienes and Jeeves, 1965) they found evidence that the learning of a smaller structure (with two elements) did not enhance subsequent performance on a larger structure (with four elements) even though the former structure might be included in the latter. Subjects who learned the four-element structure first, performed just as well or better in their evaluation of it, and of a subsequently learned two-element structure, than did subjects who learned them in the reverse, 2-4, order. Transfer effects of this nature have caused great
interest, since they confirm Dienes' view that complex mathematical structures may be taught to children of primary school age. The authors have subsequently extended their study of such transfer effects, using adult subjects and a sample of eleven-year-old children similar to that in their first study, to group tasks with larger numbers of elements, and with a variety of structural characteristics in common (Dienes and Jeeves 1967). Results of this work will not be discussed here. It should be pointed out that they have not directed their enquiries at the mental operations in terms of which a subject attempts to understand the structure. Their comparison of the performance of adults and children is also not designed to throw light on this aspect of performance. Thus the distinction between concrete and formal approaches to the task, attempted in the present study, is not duplicating aspects of their work, although derived directly from it.
CHAPTER 4

THE FOUR GROUP TASK AND QUESTIONING PROCEDURE

4.1 Choice of a Task and its Format.

The work of Dienes and Jeeves (1965) established that children of about eleven years of age were able to memorize the sixteen combinations involved in a four group structure. A structure of this size was considered suitable for the present study, as it could be surmised that a child of twelve or thirteen years, beginning to develop formal operations, would not be daunted by an undue memory load, in his attempt to abstract the group properties. With a group composed of a larger number of elements, this could conceivably be the case. Dienes and Jeeves (1967), in their later work, have deliberately increased the number of group elements to as many as nine, in an attempt to force subjects to abandon a rote memory strategy. This was not considered necessary or advisable in a developmental study, since it would create problems of motivation for the younger children and might actually deter older subjects from the task of structuring the problem.

Since the Cyclic Four Group lends itself more readily to an "operator" evaluation than the Klein Group of Four elements, according to Dienes and Jeeves (1965), it was decided to use the former structure. A number of changes were made in the format in which the problem was presented to subjects. Instead of having an experimenter-operated window, in which the result of the combination of the previous window card and the card played by the subject appeared, three panels with coloured lights were wired to embody the group structure (See Fig. 1). Each panel had, from left to
The Apparatus for the Four Group Task. One light may be switched on, at a time, in each of the "Subject's" and "Experimenter's" panels. If there is a light on in one but not both of these two panels, the same light appears in the "Results" panel. If a light is switched on in each of these panels simultaneously, the result of that particular combination appears in the "Results" panel. The results of all possible combinations are shown in Table 3, Section A.3. The apparatus is drawn to scale and with the panels in the approximate relative positions occupied during testing.
right, a yellow, a green, a blue and a red light. The two smaller panels, with switches, were used to represent the four elements of the group available for combination. The large third panel was used to represent the "state" of the lights in the small panels at any one time. Thus if a single light were turned on in either of the small panels, the same single light would come on in the large panel (only one light was to be turned on at a time in a small panel). If a combination of one light in each panel was turned on, the result of this combination appeared in the large panel. It was thus possible to see both elements of a combination, in the small panels, and the result of their combination, in the large panel, at the same time.

This is different from the situation (as in Dienes and Jeeves, 1965) where a subject has to remember what element was in the window previously, when looking at the new outcome and comparing it with his prediction. With the panels of coloured lights, the structure of the problem may be rather more easily seen. This is the more so because the lights, although arranged in a straight line, are nevertheless in the correct order for a cyclic group. Seen as representing the number of moves around a circle, the lights become 0, 1, 2, 3 in the order Yellow, Green, Blue, Red on the panels. For convenience the letters Y, G, B, R will sometimes be used to refer to the four elements in what follows.

A formal definition of the Axioms of a Group structure will now be given, followed by a table showing the actual combinations and their results in the Cyclic Four Group used in the present study. An indication of the possible "interpretations" of the structure found amongst subjects giving "operator" evaluations by Dienes and Jeeves (1965), is then given, and
finally an outline of the instructions and questioning procedures used in the present study.

4.2 The Axioms of a Mathematical Group.

Lederman (1961) provides a definition of a mathematical group as below. Some examples of infinite groups are all the rational numbers with respect to multiplication and all integers with respect to addition. Examples of finite groups are the four numbers 1, i, -1, -i (where $i = \sqrt{-1}$) with respect to multiplication; and the four clockwise rotations through $90^\circ$, $180^\circ$, $270^\circ$ and $360^\circ$ with respect to sequential performance, or the law of composition "and then".

"The theory of groups deals with certain sets of elements $G = (A, B, C \ldots \ldots \ldots )$ with respect to which a single law of composition is defined. (It is a matter of convention that the notation and nomenclature of multiplication are usually used to express the composition of abstract group elements. Thus we assume that any two elements $A, B$ of $G$, equal or unequal, possess a unique product $C$, and we write $AB = C \ldots \ldots \ldots )$

Definition

A set $G$ of a finite or infinite number of elements, for which a law of composition is defined, forms a group if the following conditions are satisfied:–

1. **Closure.** To every ordered pair of elements, $A, B$ of $G$ there belongs a unique element $C$ of $G$ written:
   
   $C = AB$

2. **Associative Law.** If $A, B, C$ are any 3 elements of $G$, which need not be distinct, then

   $(AB)C = A(BC) = ABC$

3. **Unit Element.** $G$ contains an element $I$, called unit element or identity element, such that for every element $A$ of $G$:

   $AI = IA = A$. 
4. **Inverse or Reciprocal Element.** Corresponding to every element \( A \) of \( G \), there exists in \( G \) an element \( A^{-1} \) such that

\[
AA^{-1} = A^{-1}A = I
\]

5. **Commutative Law.** A group which has the additional property that for every two of its elements, \( AB = BA \) is called an Abelian or Commutative group". (Lederman, 1961, pp. 2-3).

### 4.3 The Structure of the Cyclic Four Group and Some Possible Interpretations.

Entries in Table 3 below show the result (light in the large panel) for each possible pair of lights, one in each of the small panels (panels 1 and 2). The structure is that of a Cyclic Group with four elements. Since one interpretation of this group is in terms of integers "Modulo-4", it is sometimes referred to (e.g. by Dienes and Jeeves 1965, 1967) as the \( M_4 \) group. In the present thesis, the structure itself is referred to as the Cyclic Four Group, and the task based upon it referred to as the Four Group Task.

<table>
<thead>
<tr>
<th></th>
<th>Y</th>
<th>G</th>
<th>B</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Y</strong></td>
<td>Y</td>
<td>G</td>
<td>B</td>
<td>R</td>
</tr>
<tr>
<td><strong>G</strong></td>
<td>G</td>
<td>B</td>
<td>R</td>
<td>Y</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td>B</td>
<td>R</td>
<td>Y</td>
<td>G</td>
</tr>
<tr>
<td><strong>R</strong></td>
<td>R</td>
<td>Y</td>
<td>G</td>
<td>B</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel 1 (Subject)</th>
<th>Panel 2 (Experimenter)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Y</td>
</tr>
<tr>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>G</td>
<td>G</td>
</tr>
<tr>
<td>B</td>
<td>B</td>
</tr>
<tr>
<td>R</td>
<td>R</td>
</tr>
</tbody>
</table>

**Table 43:** The Cyclic Four Group used in the present study. Entries in cells of the matrix are the results of combinations of the corresponding row (subject's panel) and column (experimenter's panel) elements of the structure. The letters Y,G,B,R stand for lights of colour yellow, green, blue and red.
Some possible "interpretations" of the cyclic four group are shown below. The "cyclic" interpretation is readily translatable into a purely numerical system.

<table>
<thead>
<tr>
<th>Cyclic</th>
<th>Numerical</th>
<th>&quot;Fanciful&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y</td>
<td>Y 0</td>
<td>Y is Me</td>
</tr>
<tr>
<td>G</td>
<td>G 1</td>
<td>G is Dad</td>
</tr>
<tr>
<td>B</td>
<td>B 2</td>
<td>B is Love</td>
</tr>
<tr>
<td>R</td>
<td>R 3</td>
<td>R is Mum</td>
</tr>
</tbody>
</table>

The four lights are to be seen as lying on a circle as above. The lights represent the following moves:

- Y - no move
- G - a move of 1 positions clockwise
- B - a move of 2 positions clockwise
- R - a move of 3 positions clockwise

(Note: Equivalent anti-clockwise moves may be substituted for one or more of the clockwise moves).

The lights may be seen as representing a group of integers "Modulo - 4". Y = 0, G = 1, B = 2, R = 3 and any number greater than 3 is equal to the remainder after its division by 4. The numbers combine under addition. (Many other numerical systems are possible, for example:)

- Under Addition
  - (a) 0 5 10 15
  - (b) 0 1 2 -1
  - (c) 4 3 2 1

- Under Multiplication
  - (d) 1 2 4 8

4.4 Presentation of the Task to Subjects; Instructions and Questioning Procedures.

4.4.1 Instructions to subjects (Answers expected of the subject are shown in brackets).

"This is a game with lights. I want you to find out the rules of the game as we play it. This is how it is played.

You have a panel with four different coloured lights on it. Can you tell me what the four colours are?"
I have a panel the same as yours. In these panels the switches underneath turn on the lights above them. In front of us both is a big panel with four lights the same as in our panels, but there are no switches for turning on the lights. As we play the game you will see that lights turned on in your panel and my panel will make lights in the big panel go on. You have to find out the rules which will tell you which light will go in the big panel. In the game you can switch on just ONE light in your panel at a time. Then you can ask me to switch on ONE light in my panel. You can ask me for whichever one you like.

Let's try this out. Suppose that you switch on the R light in your panel. See that the R light comes on in the big panel as well. Now switch it off. Now suppose that you ask me to turn on the B light in my panel. You see that the same thing happens and the B light goes on in the big panel. Now I'll switch it off. That is what happens if we put the lights on separately in our own panels. The game is to find out what happens if we switch on a light in both your panel and my panel together. Let's try this out. Suppose you put on the R light in your panel again. Now let's leave this light ON and you ask me to put on my B light. What happens? You see that the R and B do not both come on in the big panel. Instead just ONE light comes on (G). You can think of this as telling you that:

"Red in your panel then Blue in my panel is Green in the big panel"

(Repeat)

The game is to find out the rules which tell you which light goes on in
the big panel when one in your panel and one in mine are put together like this.

Let me see if you have understood what the game is:

1. If you put on R in your panel, what light will come on in the big panel? (R)

   If you put on B in your panel, what light will come on in the big panel? (B)

   If you put on G in your panel, what light will come on in the big panel? (G)

   If you put on Y in your panel, what light will come on in the big panel? (Y)

2. If you ask me to put on a light in my panel together with a light in your panel how many lights will come on in the big panel? (One)

3. Taking our example of R in your panel and B in my panel, what light did we see come on in the big panel? (G)

   How can you think of this to yourself? ("R in my panel and then B in your panel is G in the big panel").

Now we are going to play the game and this is how I want you to do it.

When we are ready to start you can put on whichever of the lights in your panel you want - just ONE remember. Then you can ask me to switch on one of my lights, but before I do it for you I want you to guess what light will come on in the big panel and to tell me your guess. When you have told me your guess I will put on the light in my panel and you will be able to see whether your guess was right or not.

Then we will switch both the lights off and start again. You can choose
another light in your panel, ask me to turn on another one in my panel and guess what will go on in the big panel. You can choose the same ones again, or different ones - whatever you like. When you have tried out all the lights and you can tell me which light is going to come on in the big panel for all the combinations, the game will be finished.

Any questions? Well now we can start the game. I am going to write down what we do on this paper. (See record sheets, Appendix I).

4.4.2 Criterion of learning.

Before questioning subjects about their understanding of the way the lights combined, it was necessary to be sure that all had achieved a comparable knowledge of the combinations. A suitable criterion proved hard to find. Preliminary work with seven to eleven year olds indicated that a criterion such as ten out of twelve successive trials correct did not ensure that subjects knew the same number of the sixteen combinations. Quite commonly a subject would repeat the same three or four combinations over and over again, giving a correct prediction every time. In this way he could achieve ten out of twelve successive trials correct and perhaps never have tried certain combinations at any stage.

It was therefore decided that learning should proceed until at least twelve out of the sixteen combinations were correctly predicted in a systematic test conducted by the experimenter (with those wrong corrected in not more than three retests). The decision as to when to give such a test was made by the experimenter, taking into account any statements of confidence made by the subject and the general trend of
correct answers (although not in any formalised way). After experience with a number of subjects it became clear that it was expedient to have a "practice" test after no longer than fifteen minutes at the task and so this became part of the procedure. Very often not as many as twelve predicted results of combinations were correct on this test, in which case the subject went back to learning combinations until judged ready to face another test. Rarely were more than two tests required to reach criterion.

The systematic tests for knowledge of the combinations proceeded in the order shown in Table 4 below. While it is clear that such a procedure may structure the problem in a particular way for the subject, this bias seemed preferable to a "random" testing procedure which might make the task of systematising (and remembering) more difficult for the subject.

Panel 1

<table>
<thead>
<tr>
<th>Y</th>
<th>G</th>
<th>B</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

Panel 2

<table>
<thead>
<tr>
<th>Subject</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
</tr>
</tbody>
</table>

| R | 13 | 14 | 15 | 16 |

Table 4: Systematic testing of the subject's knowledge of the combinations in the Four Group Task. Numbers in the cells of the table indicate the order in which the sixteen combinations were tested by the experimenter. In retests, only those previously wrong were retested.

When judged ready for a test, the subject was told "I think it's time to see how many of them you know now. Will you put on your Y light
for me. Now if I put on my Y what will come on in the big panel? And if I put on my G .....? (leave your Y on) etc."

In each case the subject was allowed to see whether his answer was the correct one before going on to the next combination. No retests were done, however, until all sixteen combinations had been tested once. For sample learning records see Appendix I.

It is clear that in further work an attempt should be made to have a more specific learning procedure and criterion. In this study it was felt important to adopt a flexible approach in order to allow each child to develop an understanding of the game in the way which best suited him. It was argued that an accurate measure of the efficiency of learning was of secondary importance to a thorough investigation of the child's approach to, and understanding of, the problem.

4.4.3 Questioning procedures.

4.4.3.1 Initial questioning for level of understanding.

When criterion had been reached, the subject was told that the learning was over, a tape recorder was introduced and shown to him and he was asked:- "Explain how the game works. Tell me any ways you had of remembering which light would go on in the big panel. Did you have any special ideas or rules which helped you to remember them?" Depending on the answer given, further questioning to probe for ideas occurred. For full details of this see Part I of the detailed questioning protocol, in Appendix II.
4.4.3.2. Questioning about the axioms of group theory and "concrete reversibility".

Following the initial questioning for level of understanding of the game, the subject was told: "I want to go on and talk about some other things you know about the game now that we have seen how it works - the sort of thing you might explain to someone who didn't know anything about it". Then followed questions designed to investigate understanding of four of the five axioms of the group. The axioms tested were Nos. 2, 3, 4 and 5 in the definition given in section 4.2. The questions asked were tied as closely as possible to practical examples, so as not to rely too heavily on verbal interchange. The basic questioning method and the examples used are set out below, but for more details for probing questions see Part II of the detailed questioning protocol, in Appendix II. It should be noted that Axiom 1, the Closure Axiom, is, in effect, explained to the subject at the outset of the learning, so that there is no direct questioning on this. Section I of the questioning is not related directly to any group axiom, but probes for the kind of reversibility of action which would be expected at the stage of concrete operations. Essentially this means whether turning ON a light may be reversed by turning it OFF. Questioning about concrete reversibility and the axioms was always done in the order in which they are set out below. Answers expected from the subject appear in brackets.

(i) Concrete Reversibility

Suppose you put on R in your panel and ask me for B in mine. What goes on in the big panel? (G) Good now we have a G in the big panel. Q.1. How could we make R come instead of G in the big panel? . . . . . . . . . . . . (Turn off B in your panel)
Q. 2. How could we make B come instead of G in the big panel?

. . . . . . . . . . . . . . . . . . . . (Turn off R in my panel).

What is happening when we do this?

(ii) Commutative Law

Axiom: A group which has the property that for every two of its elements $AB = BA$ is called an Abelian or Commutative group.

Tests

1. We have seen that R in your panel then B in my panel is G in the big panel.

   If you had put on B in your panel, what would we have to put on in my panel to have G in the big panel?
   $RB = B? = G$ (Answer R)

2. $GY = ?G = Y$ (Answer Y)

3. $BG = ?B = R$ (Answer G)

What can you say about these combinations?

(iii) Unit Element

Axiom: G contains an element I, called the unit element or identity element, such that for every element A of G, $AI = IA = A$.

Tests

a) If you have B in your panel and want B to stay on in the big panel what do you ask me to put on?

   . . . . . . . . . . . . . (Answer Y)
b) If you have Y in your panel and want Y to stay on in the big panel what do you ask me to put on?

. . . . . . . . (Answer Y)

c) If you have R and want R . . . . . . . . ?

. . . . . . . . (Answer Y)

d) If you have G and want G . . . . . . . . ?

. . . . . . . . (Answer Y)

What can you say about the Y light?

(iv) Inverse Element

Axiom: Corresponding to every element A of G there exists in G an element $A^{-1}$ such that $AA^{-1} = A^{-1}A = I$.

Tests

a) If you have G in your panel and want Y in the big panel what do you ask me to switch on?

. . . . . . . . (Answer R)

b) If you have B and want Y . . . . . . ?

. . . . . . . . (Answer B)

c) If you have R and want Y . . . . . . ?

. . . . . . . . (Answer G)

d) If you have Y and want Y . . . . . . ?

. . . . . . . . (Answer Y)

What can you say about the combinations of lights which put yellow on the big panel?
(v) **Associative Law**

**Axiom:** If A, B, C are any 3 elements of G which need not be distinct then \((AB)C = A(BC) = ABC\).

Suppose instead of just two panels with switches, we had three panels and suppose we put on one light in each of the three panels together - let us say:-

B in your panel, B in mine and G in another panel

Could you work out what would come on in the big panel?

(Answer \(RB = G\) and \(BG = R\) and \(GG = B\) or \(RR = B\) ... so Answer B)

Now can you work out what the light in the big panel would be for (usually only one of those given below was chosen):-

- a) \(YBG\) (\(YB = B\) and \(BG = R\) or \(BG = R\) and \(YR = R\) ... so Answer R)
- b) \(YGG\) (\(YG = G\) and \(GG = B\) or \(GG = B\) and \(YB = B\) ... so Answer B)
- c) \(YRR\) (\(YR = R\) and \(RR = B\) or \(RR = B\) and \(YB = B\) ... so Answer B)
- d) \(BBG\) (\(BB = Y\) and \(GG = G\) or \(BG = R\) and \(BR = G\) ... so Answer G)
- e) \(BGG\) (\(BG = R\) and \(RG = Y\) or \(GG = B\) and \(BB = Y\) ... so Answer Y)
- f) \(BRR\) (\(BR = G\) and \(GR = Y\) or \(RR = B\) and \(BB = Y\) ... so Answer Y)

What can you say about the different ways to get the same answer?
4.4.3.3 Final questioning about level of understanding

Following the questions about concrete reversibility and the group axioms, the subject was asked whether the game, or any of the rules in it, reminded him of anything else he knew. Any suggestions made by him were followed to see whether anything of the group structure was embodied in them. If few suggestions were forthcoming, he was asked whether it was like anything he did in school or any games he played at home. If numbers were not mentioned by him, he was finally asked whether it seemed like putting numbers together and why. For more details of questioning at this stage, see Part III of the detailed questioning protocol, in Appendix II.

4.5 Scoring Procedures

No scoring took place during the interview with the subject. From the learning record, and the taperecording of his responses to questioning, a system of scoring, firstly on measures of efficiency in learning the task, and secondly on the level of understanding of the group structure and its axioms, was developed. Descriptions of the scores devised are given in Chapter 7.