A study of well-balanced finite volume methods and refinement indicators for the shallow water equations

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A thesis submitted for the degree of Doctor of Philosophy of The Australian National University
Dedicated to my wife, Asti,
our daughter, Daniella,
and my parents, Wagimin and Sulasmi.
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Declaration

The work in this thesis is my own except where otherwise stated.

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I have presented some parts of this thesis in talks during my PhD candidature. The talks are as follows.

• ”Well-balanced finite volume methods for shallow water flows involving wet/dry interface” in the 53rd Australian Mathematical Society Annual Meeting, The University of South Australia, Adelaide, Australia, 28 Sep – 1 Oct 2009


• ”A new analytical solution for testing debris avalanche numerical models in the standard Cartesian coordinate system” in the 54th Australian Mathematical Society Annual Meeting, The University of Queensland, Brisbane, Australia, 27 – 30 Sep 2010

• ”A new analytical solution for testing debris avalanche numerical models” in Computational Techniques and Applications Conference 2010, The University of New South Wales, Sydney, Australia, 28 Nov – 1 Dec 2010
• "Numerical entropy production for shallow water flows" in Computational Techniques and Applications Conference 2010, The University of New South Wales, Sydney, Australia, 28 Nov – 1 Dec 2010

• "Approximations of the Carrier-Greenspan periodic solution to the shallow water wave equations for flows on a sloping beach" in the 55th Australian Mathematical Society Annual Meeting, The University of Wollongong, Wollongong, Australia, 26 – 29 Sep 2011

• "A finite volume method for shallow water flows on triangular computational grids" in the IEEE International Conference on Advanced Computer Science and Information Systems (ICACISIS) 2011, Jakarta, Indonesia, 17 – 18 Dec 2011

• "Some exact solutions to shallow-water-type models and their use as numerical test problems” in a Seminar at IGPM, RWTH Aachen University, Aachen, Germany, 9 February 2012

I received the following awards during my PhD candidature.

• Best paper award in the IEEE International Conference on Advanced Computer Science and Information Systems (ICACISIS) 2011, Jakarta, Indonesia, 17 – 18 Dec 2011,

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Abstract

This thesis studies solutions to the shallow water equations analytically and numerically. The study is separated into three parts.

The first part is about well-balanced finite volume methods to solve steady and unsteady state problems. A method is said to be well-balanced if it preserves an unperturbed steady state at the discrete level. We implement hydrostatic reconstructions for the well-balanced methods with respect to the steady state of a lake at rest. Four combinations of quantity reconstructions are tested. Our results indicate an appropriate combination of quantity reconstructions for dealing with steady and unsteady state problems.

The second part presents some new analytical solutions to debris avalanche problems and reviews the implicit Carrier-Greenspan periodic solution for flows on a sloping beach. The analytical solutions to debris avalanche problems are derived using characteristics and a variable transformation technique. The analytical solutions are used as benchmarks to test the performance of numerical solutions. For the Carrier-Greenspan periodic solution, we show that the linear approximation of the Carrier-Greenspan periodic solution may result in large errors in some cases. If an explicit approximation of the Carrier-Greenspan periodic solution is needed, higher order approximations should be considered. We propose second order approximations of the Carrier-Greenspan periodic solution and present a way to get higher order approximations.

The third part discusses refinement indicators used in adaptive finite volume methods to detect smooth and nonsmooth regions. In the adaptive finite volume methods, smooth regions are coarsened to reduce the computational costs and nonsmooth regions are refined to get more accurate solutions. We consider the numerical entropy production and weak local residuals as refinement indicators. Regarding the numerical entropy production, our work is the first to implement the numerical entropy production as a refinement indicator into adaptive finite
volume methods used to solve the shallow water equations. Regarding weak local residuals, we propose formulations to compute weak local residuals on nonuniform meshes. Our numerical experiments show that both the numerical entropy production and weak local residuals are successful as refinement indicators.
Abbreviations

1D, 1.5D, 2D  one dimension, one-and-a-half dimensions, two dimensions
CFL  Courant-Friedrichs-Lewy
CG  Carrier-Greenspan
CK  Constantin-Kurganov
FVM  Finite Volume Method(s)
iFEM  innovative Finite Element Method
KKP  Karni-Kurganov-Petrova
MHR  Mangeney-Heinrich-Roche
NEP  Numerical Entropy Production
nNEP  normalised Numerical Entropy Production
SI  Système International (International System)
WLR  Weak Local Residual(s)