Boundaries and equivariant products in unbounded Kasparov theory

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Abstract

The thesis explores two distinct areas of noncommutative geometry: factorisation and boundaries. Both of these topics are concerned with cycles in Kasparov's KK-theory which are defined using unbounded operators, and manipulating these cycles. These unbounded operators generalise the Dirac operators of classical geometry.

The first topic of the thesis is factorisation, which is a process by which one attempts to represent the class of an equivariant spectral triple as a product of two unbounded Kasparov cycles, which, if they exist, are defined using the group action. We provide sufficient conditions for factorisation to be achieved for actions by compact abelian Lie groups. We apply our results to examples from Dirac operators on manifolds and their noncommutative theta-deformations. In particular, we show that the equivariant spectral triple associated to a Dirac operator on the total space of a compact torus principal bundle always factorises.

The second topic of the thesis is relative spectral triples, which can be used to describe (noncommutative) manifolds with boundary. Whereas spectral triples are defined using self-adjoint unbounded operators, relative spectral triples are defined using symmetric unbounded operators. We show that the bounded transform of a relative spectral triple defines a relative Fredholm module, and hence a class in relative K-homology. We use relative spectral triples to investigate the boundary map in the six-term exact sequence of K-homology. We show that the boundary of a relative spectral triple has a simple description in terms of extension theory. With some additional data modelled on the inward normal of a manifold with boundary, we construct a triple which is a candidate for a spectral triple representing the boundary class of a relative spectral triple.

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Contents

Introduction

This thesis is divided into two distinct parts. The first concerns the factorisation of equivariant spectral triples, while the second concerns relative spectral triples. Both parts relate to the field of noncommutative geometry, in particular in Kasparov's KK-theory, [25].

Here we provide an executive summary of these two parts, with details to come later.

Factorisation of equivariant spectral triples

Let A be a separable C^*-algebra carrying an action by a compact Lie group G. The action defines a subalgebra A^G of A, which is the set of elements fixed by G. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a G-equivariant spectral triple for A, which defines a class in the equivariant KK-theory $KK_G^j(A, \mathbb{C})$. A *factorisation* of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a pair of unbounded KK-cycles x and y for $KK_G^{\dim G}(A, A^G)$ and $KK_G^{j+\dim G}(A^G, \mathbb{C})$ respectively, such that the Kasparov product

$$KK_G^{\dim G}(A, A^G) \times KK_G^{j+\dim G}(A^G, \mathbb{C}) \to KK_G^j(A, \mathbb{C}),$$

recovers the class of the spectral triple as $[x]\widehat{\otimes}_{A^G}[y] = [(\mathcal{A}, \mathcal{H}, \mathcal{D})]$. In the first part of the thesis, we provide sufficient conditions for a G-equivariant spectral triple to be factorised in the case that G is compact and abelian.

Provided that the action of the compact abelian Lie group G on A satisfies the "spectral subspace assumption", [11], the cycle for $KK_G^{\dim G}(A, A^G)$ is constructed from the spin Dirac operator on G, and depends only on the action of G on A. In order to construct the cycle for $KK_G^{j+\dim G}(A^G, \mathbb{C})$, we restrict the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ to some character space of G in $\mathcal{H}$, such as the fixed point subspace, and require the existence of a Clifford representation $\eta : \Gamma(\mathbb{C}\mathrm{l}(G))^G \cong \mathbb{C}\mathrm{l}_{\dim G} \to B(\mathcal{H})$ satisfying some compatibility conditions. This Clifford representation provides the appropriate shift in KK-dimension.

Even when these two unbounded KK-cycles can be constructed, it is not automatic that their Kasparov product recovers the class of the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, and indeed in some cases factorisation is impossible (see §6 for such an example). To test

whether factorisation has been realised, we employ Kucerovsky's criteria, [27]. Under the existing assumptions on the action of the compact abelian Lie group G and the Clifford representation, checking Kucerovsky's criteria reduces to checking a positivity condition (Theorem 3.4).

The main class of examples of factorisation studied in this thesis are free, isometric torus actions on compact Riemannian manifolds. We show that given an equivariant Dirac operator on such a manifold, factorisation is always achieved, where the Clifford representation is defined canonically using the fundamental vector field map. We also show that if a torus-equivariant spectral triple is factorised, then so is its θ-deformation, and hence θ-deformations of compact, Riemannian manifolds provide a class of non-commutative examples of factorisation.

Factorisation of circle-equivariant spectral triples has previously been studied in [8] and [15, 16, 53]. In [8], factorisation was studied on the level of the constructive Kasparov product, which attempts to construct an unbounded KK-cycle representing the Kasparov product of two composable unbounded KK-cycles, with the aid of a connection, [8, 24, 32, 33]. In this thesis, we show that the constructive Kasparov product may be applied to the factorisation of an equivariant Dirac operator on a compact Riemannian manifold with a free, isometric torus action. The unbounded KK-cycle thus constructed is defined by a first order, elliptic, self-adjoint differential operator on the manifold, and it represents the same class in equivariant K-homology as the Dirac operator (Theorem 5.10). If the orbits of the torus are embedded isometrically into the manifold, then this constructed differential operator is a bounded perturbation of the Dirac operator (Corollary 5.11). The isometric embedding of the torus orbits is a particular case of the "fibres of constant length" condition of [15, 16, 53], a condition which is also satisfied by the examples studied in [8]. We do not require fibres of constant length in order to achieve factorisation in KK-theory, however.

The bulk of this part of the thesis appears in the preprint [19].

Relative spectral triples

In the second part of the thesis, we define a *relative spectral triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ for an ideal J in a C^*-algebra A, and show that the bounded transform $\mathcal{D}(1+\mathcal{D}^*\mathcal{D})^{-1/2}$ of $\mathcal{D}$ yields a relative Fredholm module and hence a class in relative K-homology. The proof that the bounded transform of a relative spectral triple defines a relative Fredholm module is a refinement of arguments used in [1, 23] to prove that an unbounded Kasparov module (and a generalisation thereof using symmetric operators) defines a class in KK-theory.

The principal difference between a spectral triple and a relative spectral triple is that the operator $\mathcal{D}$ is self-adjoint for a spectral triple, whereas it is merely symmetric for a relative spectral triple. The classical example of a spectral triple is given by a Dirac

operator on a complete Riemannian manifold; the motivating example of a relative spectral triple is given by a Dirac operator on a compact Riemannian manifold with boundary, [3].

We are interested in relative spectral triples and relative K-homology because of the boundary map in K-homology. If J is an ideal in a separable trivially $\mathbb{Z}_2$-graded C^*-algebra A such that A/J is nuclear, then there is a six-term exact sequence:

$$\begin{tikzcd} K^0(A/J) \arrow[r] & K^0(A) \arrow[r] & K^0(J \lhd A) \arrow[d, "\partial"] \\ K^1(J \lhd A) \arrow[u, "\partial"] & K^1(A) \arrow[l] & K^1(A/J) \arrow[l] \end{tikzcd}$$

The maps $\partial : K^*(J \lhd A) \to K^{*+1}(A/J)$ from the relative K-homology of $J \lhd A$ to the K-homology of A/J are the *boundary maps*. Given a relative spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ defining a class in $K^0(J \lhd A)$, there is a simple description of $\partial[(\mathcal{A}, \mathcal{H}, \mathcal{D})]$ in terms of extensions, at least for unital A, using the isomorphism between $K^1(A/J)$ and the extension group $\mathrm{Ext}(A/J)$, [3, 22].

The motivation for relative spectral triples is to be able to describe the boundary map using spectral triples, without having to pass to extensions. The advantage of spectral triples is that they can carry additional geometric information in addition to the classes they define in K-homology.

The boundary map $K^*(J \lhd A) \to K^{*+1}(A/J)$ can also be realised as the Kasparov product with the class in $KK^1(A/J, J)$ of the extension

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

Another motivation for relative spectral triples is to be able to render the Kasparov product with the class of this extension more computable. This would be via some generalisation of the constructive Kasparov product, and would allow the description of the boundary map in terms of unbounded KK-cycles.

We conclude the second part of the thesis by showing that given a relative spectral triple for $J \lhd A$ and a so-called "Clifford normal" (modelled on Clifford multiplication by the inward unit normal on a manifold with boundary), we can construct a boundary Hilbert space with a representation of $A/J \widehat{\otimes} \mathbb{C}\mathrm{l}_1$, as well as a symmetric operator on the boundary Hilbert space. This data is a candidate for a spectral triple representing the boundary class (although we do not show that it is a spectral triple). We also relate the Clifford normal to the pullback algebra $\widetilde{A} = \{(a, b) \in A \oplus A : a - b \in J\}$, and show that, in the case that A is unital and represented non-degenerately, a spectral triple for $\widetilde{A}$ can be constructed from an even relative spectral triple for $J \lhd A$. This is based on the doubling construction for a compact manifold with boundary, [7].

Thesis outline

I Factorisation of equivariant spectral triples

In Chapter 1, we review the theory of Hilbert modules, the equivariant KK-groups, unbounded Kasparov modules and the Kasparov product, which we will need to describe factorisation and some constructions in Part II of this thesis.

Let A be a separable C^*-algebra carrying an action by a compact abelian Lie group G, and let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be an equivariant spectral triple defining a class in $KK_G^j(A, \mathbb{C})$. In Chapter 2, we construct (under some assumptions) the unbounded KK-cycles for $KK_G^{\dim G}(A, A^G)$ and $KK_G^{j+\dim G}(A^G, \mathbb{C})$ which are intended to factorise $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.

Provided the unbounded KK-cycles of Chapter 2 exist, in Chapter 3 we apply Kucerovsky's criteria to prove the main factorisation result, Theorem 3.4, which states that factorisation is achieved when a positivity condition is satisfied.

Given a $\mathbb{T}^n$-equivariant spectral triple, one can use the noncommutative torus to construct a θ-deformed $\mathbb{T}^n$-equivariant spectral triple. In Chapter 4, we prove that if a $\mathbb{T}^n$-equivariant spectral triple factorises, then so does its θ-deformed spectral triple.

In Chapter 5, we examine the example of a equivariant Dirac operator on a compact Riemannian manifold M with a free, isometric action by the n-torus $\mathbb{T}^n$. We show that the spectral triple defined by the Dirac operator factorises. We also show a connection can be used to construct a spectral triple representing the Kasparov product of the classes in $KK_{\mathbb{T}^n}^n(C(M), C(M)^G)$ and $KK_{\mathbb{T}^n}^{j+n}(C(M)^G, \mathbb{C})$.

In Chapter 6, we study in depth the example of the Dirac operator on the 2-sphere, which is rotated about the north-south axis by the circle. The action of the circle on S^2 is not free, and factorisation is not possible for $C(S^2)$. The action on S^2 minus the poles is free however, and we show that factorisation is possible for the continuous functions vanishing at the poles. The factorisation on $S^2 \setminus \{N, S\}$ is an example of factorisation on a non-compact (and non-complete) manifold.

When constructing the unbounded cycles for factorisation, we only use even KK-cycles, even though odd cycles may be used to define classes in KK-theory. This is because if one works with odd cycles, one might obtain an operator on a $\mathbb{Z}_2$-graded Hilbert space which is not odd, which is a problem because it must be odd in order to define a KK-class. In [8, §6], this difficulty is avoided by subtracting the non-odd part of the operator. In Appendix A, we show that this subtraction arises naturally by passing from odd cycles to even cycles.

II Relative spectral triples

In Chapter 7, we review the theory of relative Fredholm modules, extensions and the boundary map in K-homology. This theory provides the motivation for relative spectral

triples.

In Chapter 8, we define a relative spectral triple and show that its bounded transform yields a relative Fredholm module, and hence that a relative spectral triple defines a class in relative K-homology. We also show that the boundary of an even relative spectral triple can be described in terms of extensions.

In Chapter 9, we show that given a relative spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ for an ideal J in a C^*-algebra A and a compatible "Clifford normal", we can construct a candidate for a spectral triple representing the boundary class $\partial[(\mathcal{A}, \mathcal{H}, \mathcal{D})]$, although more data may be needed to show that it is in fact a spectral triple. We also show that in the case that A is unital and represented non-degenerately, a Clifford normal can be used to construct a spectral triple for the pullback C^*-algebra $\widetilde{A} = \{(a, b) \in A \oplus A : a - b \in J\}$.

In Appendix B, we develop the example of the Dirac operator on the unit disc. This is an example of a relative spectral triple, and also provides a counterexample to claims in the literature that the definition of a spectral triple may be weakened and still yield a Fredholm module.

In Appendix C, we show that a naive attempt to construct a boundary spectral triple inspired by Green's formula for a Dirac operator on a manifold with boundary is unsuccessful.

Part I

Factorisation of equivariant spectral triples

Chapter 1

Preliminaries

In this chapter we review the equivariant KK-groups and the Kasparov product, [25]. We will use $\mathbb{Z}_2$-graded C^*-algebras and Hilbert modules throughout. This is not simply generalisation for its own sake: the higher order KK-groups are defined by $KK^n(A,B) := KK(A\widehat{\otimes}\mathbb{C}\mathrm{l}_n, B) \cong KK(A, B\widehat{\otimes}\mathbb{C}\mathrm{l}_n)$, where $\mathbb{C}\mathrm{l}_n$ is the nth Clifford algebra. Even if A is a trivially $\mathbb{Z}_2$-graded C^*-algebra, $A\widehat{\otimes}\mathbb{C}\mathrm{l}_n$ is non-trivially $\mathbb{Z}_2$-graded for $n \geq 1$. While it is true that if A and B are trivially $\mathbb{Z}_2$-graded, one can obtain cycles for $KK^1(A,B)$ from odd Kasparov modules, one can encounter difficulties if one attempts to factorise using odd Kasparov modules. See Appendix A for an illustration of problems that can arise from working with odd spectral triples, and how these issues vanish upon passing to the $\mathbb{Z}_2$-graded setting.

1.1 $\mathbb{Z}_2$-graded C^*-algebras and Hilbert modules

See [5, 25] for more information on the following.

Definition 1.1. Let A be a C^*-algebra. We say that A is **$\mathbb{Z}_2$-graded** if there is a decomposition $A = A^0 \oplus A^1$ into self-adjoint closed linear subspaces, such that $A^j \cdot A^k \subset A^{j+k}$ for $j, k \in \mathbb{Z}_2$. An element $a \in A$ is said to be of **homogeneous degree** $\deg a \in \mathbb{Z}_2$ if $a \in A^{\deg a}$. We say that A is **trivially $\mathbb{Z}_2$-graded** if $A^1 = \{0\}$. A $*$-homomorphism $\phi : A \to B$ between two $\mathbb{Z}_2$-graded C^*-algebras is said to be **$\mathbb{Z}_2$-graded** if $\phi(A^j) \subset B^j$ for $j \in \mathbb{Z}_2$.

Definition 1.2. Let A be a $\mathbb{Z}_2$-graded C^*-algebra. A **right Hilbert module** E over A is a right A-module E equipped with a sesquilinear map, linear in the second variable, $(\cdot|\cdot)_A : E \times E \to A$ such that

a) $(e|f)_A a = (e|fa)_A$ for all $e, f \in E$, $a \in A$,

b) $(e|f)_A^* = (f|e)_A$ for all $e, f \in E$,

c) $(e|e)_A \geq 0$ as an element of A, and $(e|e) = 0$ implies that $e = 0$, and

d) E is complete in the norm $\|e\| = \|(e|e)_A\|^{1/2}$.

If $A = \mathbb{C}$, then Definition 1.2 is just the definition of a Hilbert space. We say that E is **$\mathbb{Z}_2$-graded** if there is a decomposition $E = E^0 \oplus E^1$ into closed linear subspaces, such that $E^j \cdot A^k \subset E^{j+k}$ and $(E^j|E^k)_A \subset A^{j+k}$ for $j, k \in \mathbb{Z}_2$. An element $e \in E^j$ said to be of **homogeneous degree** j. A module map $\Psi : E \to F$ between two $\mathbb{Z}_2$-graded right Hilbert A-modules is **$\mathbb{Z}_2$-graded** if $\Psi(E^j) \subset F^j$ for $j \in \mathbb{Z}_2$.

Left Hilbert modules have an analogous definition, except that the C^*-algebra-valued inner product ${}_A(\cdot|\cdot)$ is linear in the first variable.

Example 1.3. Let A be a C^*-algebra. Then A is a right Hilbert module over itself with inner product $(a|b)_A = a^*b$, and A is a left Hilbert module over itself with ${}_A(a|b) = ab^*$.

Let $p = p^* = p^2 \in \mathrm{End}_A(A^N)$ for some $N \in \mathbb{N}$. Then pA^N is a right Hilbert A-module. The **standard Hilbert module** $\mathcal{H}_A$ over A is the set of sequences $(a_n)_{n\in\mathbb{Z}}$ such that $\sum_n a_n^* a_n$ converges in A, with inner product $(\sum_n a_n | \sum_n b_n)_A = \sum_n a_n^* b_n$.

Example 1.4. Let X be a compact Hausdorff space. The continuous sections of a Hermitian vector bundle V over X form a right Hilbert $C(X)$-module with the pointwise inner product. The Serre-Swan theorem states that up to isomorphism, there is a one-to-one correspondence between Hermitian vector bundles over X and right Hilbert $C(X)$-modules of the form $pC(X)^N$, [45, 48].

Definition 1.5. Let E, F be $\mathbb{Z}_2$-graded right Hilbert modules over a $\mathbb{Z}_2$-graded C^*-algebra A. A map $T : E \to F$ is **adjointable** if there exists $T^* : F \to E$ such that $(Te|f)_A = (e|T^*f)_A$ for all $e \in E$, $f \in F$. If T is adjointable then it is A-linear and bounded, but the converse is not true in general. The set of adjointable operators from E to itself is denoted by $\mathrm{End}_A(E)$, which is a C^*-algebra with the operator norm.

Given $e, f \in E$, define an operator $\Theta_{e,f} \in \mathrm{End}_A(E)$ by $\Theta_{e,f}(g) = e(f|g)_A$, and let $\mathrm{End}_A^0(E) = \overline{\mathrm{span}}\{\Theta_{e,f} : e, f \in E\}$. We call the operators in $\mathrm{End}_A^0(E)$ the **compact endomorphisms** of E. The set $\mathrm{End}_A^0(E)$ is a closed ideal in $\mathrm{End}_A(E)$.

Example 1.6. Let A be a $\mathbb{Z}_2$-graded C^*-algebra, and take A as a right Hilbert module over itself. Then $\mathrm{End}_A(A) = M(A)$, the multiplier algebra of A, and $\mathrm{End}_A^0(A) = A$. Similarly, $\mathrm{End}_A^0(A^N) \cong M_N(A)$.

Definition 1.7. [29, Ch. 9] Let E, F be $\mathbb{Z}_2$-graded right Hilbert modules over a $\mathbb{Z}_2$-graded C^*-algebra A. A (possibly unbounded) **operator** $T : \mathrm{dom}(T) \subset E \to F$ is a densely-defined, A-linear map. We say T is **closed** if its graph $\{(e, Te) : e \in \mathrm{dom}(T)\}$ is closed in $E \oplus F$. We define the adjoint of T by

$$\mathrm{dom}(T^*) = \{f \in F : \exists g \in E \text{ such that } (e|g)_A = (Te|f)_A \ \forall e \in \mathrm{dom}(T)\}$$

and define $T^*f = g$ on $\mathrm{dom}(T^*)$. A closed operator T is **regular** if T^* is densely-defined and $1 + T^*T$ has dense range. Regularity is automatically satisfied on Hilbert spaces. We use the notation $T \subset S$ to denote an extension S of T; i.e. $\mathrm{dom}(T) \subset \mathrm{dom}(S)$ and $S|_{\mathrm{dom}(T)} = T$. An operator $T : \mathrm{dom}(T) \subset E \to E$ is **symmetric** if $T \subset T^*$, and is **self-adjoint** if $T = T^*$.

An operator $T : \mathrm{dom}(T) \subset E \to E$ is said to be of **homogeneous degree** $\deg T \in \mathbb{Z}_2$ if $\mathrm{dom}(T) \cap E^j \subset \mathrm{dom}(T)$ and $T(\mathrm{dom}(T) \cap E^j) \subset E^{j+\deg T}$ for $j \in \mathbb{Z}_2$. There is a decomposition of the adjointable endomorphisms $\mathrm{End}_A(E) = \mathrm{End}_A(E)^0 \oplus \mathrm{End}_A(E)^1$, where the operators in $\mathrm{End}_A(E)^j$ are of homogeneous degree j. Under this decomposition, $\mathrm{End}_A(E)$ is a $\mathbb{Z}_2$-graded C^*-algebra. Operators of degree 0 (resp. 1) are also called **even** (resp. **odd**).

The **$\mathbb{Z}_2$-graded commutator** $[\cdot,\cdot]_\pm$ is defined on operators of homogeneous degree by

$$[T, S]_\pm := TS - (-1)^{\deg T \cdot \deg S} ST,$$

and extends by linearity.

Example 1.8. Let A be a ($\mathbb{Z}_2$-graded) C^*-algebra, let $\mathcal{H}_A$ be the standard Hilbert module as in Example 1.3, and define an operator T on $\mathcal{H}_A$ by

$$\mathrm{dom}(T) = \left\{ (a_n)_{n\in\mathbb{Z}} \in \mathcal{H}_A : \sum_n n^2 a_n^* a_n \in A \right\}, \quad T(a_n)_{n\in\mathbb{Z}} = (na_n)_{n\in\mathbb{Z}}.$$

Then T is self-adjoint and regular, [35, Prop. 4.6].

Definition 1.9. Let E be a right Hilbert module over a C^*-algebra A. Let E^* be the conjugate vector space of E, define a left action of A on E^* by $a \cdot \overline{e} := \overline{e \cdot a^*}$, and define an A-valued left inner product on E^* by ${}_A(\overline{e}|\overline{f}) := (e|f)_A$. Then E^* is a left Hilbert A-module. If A and E are $\mathbb{Z}_2$-graded, then so is E^*, by $\deg \overline{e} = \deg e$. We call E^* the **conjugate module** of E, [38, p. 49].

By a similar construction, if F is a ($\mathbb{Z}_2$-graded) left Hilbert A-module, then the **conjugate module** F^* is a ($\mathbb{Z}_2$-graded) right Hilbert A-module.

Definition 1.10. Let E and F be $\mathbb{Z}_2$-graded right Hilbert modules over $\mathbb{Z}_2$-graded C^*-algebras A and B respectively, and let $\phi : A \to \mathrm{End}_B(F)$ be a $\mathbb{Z}_2$-graded $*$-homomorphism. The algebraic tensor product $E \odot F$ has a $\mathbb{Z}_2$-grading by

$$\deg(e \odot f) = \deg e + \deg f,$$

and a B-valued positive semi-definite inner product

$$(e_1 \odot f_1 | e_2 \odot f_2)_B = (f_1 | \phi((e_1|e_2)_A) f_2)_B.$$

Letting $N = \{z \in E \odot F : (z|z)_B = 0\}$, the quotient $(E \odot F)/N$ is a right pre-Hilbert B-module, which completes to a $\mathbb{Z}_2$-graded right Hilbert B-module which we denote by $E\widehat{\otimes}_A F$, [29, Prop. 4.5]. We call $E\widehat{\otimes}_A F$ the **internal tensor product** of E and F.

Definition 1.11. Let E and F be $\mathbb{Z}_2$-graded right Hilbert modules over $\mathbb{Z}_2$-graded C^*-algebras A and B respectively, and let $\phi : A \to \mathrm{End}_B(F)$ be a $\mathbb{Z}_2$-graded $*$-homomorphism. Suppose $\mathcal{D} : \mathrm{dom}(\mathcal{D}) \subset E \to E$ is a closed operator. We define $\mathcal{D}\widehat{\otimes}1 : \mathrm{dom}(\mathcal{D}\widehat{\otimes}1) \subset E\widehat{\otimes}_A F \to E\widehat{\otimes}_A F$ initially on $\mathrm{span}\{e\widehat{\otimes}f : e \in \mathrm{dom}(\mathcal{D}), f \in F\}$ by

$$(\mathcal{D}\widehat{\otimes}1)(e\widehat{\otimes}f) := \mathcal{D}e\widehat{\otimes}f,$$

and then take the operator closure. If $\mathcal{D}$ is self-adjoint and regular, then so is $\mathcal{D}\widehat{\otimes}1$. The map $\mathrm{End}_A(E) \to \mathrm{End}_B(E\widehat{\otimes}_A F)$, $T \mapsto T\widehat{\otimes}1$ is a $\mathbb{Z}_2$-graded $*$-homomorphism. In particular, if C is a $\mathbb{Z}_2$-graded C^*-algebra and $\psi : C \to \mathrm{End}_A(E)$ is a $\mathbb{Z}_2$-graded $*$-homomorphism, then $c \mapsto \psi(c)\widehat{\otimes}1$ is a $\mathbb{Z}_2$-graded $*$-homomorphism $C \to \mathrm{End}_B(E\widehat{\otimes}_A F)$.

Remark. Note that if T is an operator on F, then one cannot similarly define $1\widehat{\otimes}T$ since if $e \in E$, $f \in \mathrm{dom}(T)$ and $a \in A$ preserves $\mathrm{dom}(T)$, then

$$\begin{aligned}(1\widehat{\otimes}T)(ea\widehat{\otimes}f) &= (-1)^{\deg T\cdot(\deg e+\deg a)}ea\widehat{\otimes}Tf \\ &= (-1)^{\deg T\cdot(\deg e+\deg a)}e\widehat{\otimes}\phi(a)Tf,\end{aligned}$$

whereas

$$\begin{aligned}&(1\widehat{\otimes}T)(e\widehat{\otimes}\phi(a)f) = (-1)^{\deg T\cdot\deg e}e\widehat{\otimes}T(\phi(a)f) \\ &= (-1)^{\deg T\cdot(\deg e+\deg a)}e\widehat{\otimes}\phi(a)Tf + (-1)^{\deg T\cdot\deg e}e\widehat{\otimes}[T,\phi(a)]_{\pm}e.\end{aligned}$$

A solution to the ill-definedness of $1\widehat{\otimes}T$ is a connection; see §5.3.

Definition 1.12. Let A and B be $\mathbb{Z}_2$-graded C^*-algebras. We equip the algebraic tensor product $A \odot B$ with the $\mathbb{Z}_2$-grading $\deg(a \odot b) := \deg a + \deg b$. The product on $A \odot B$ is defined on elements of homogeneous degree by

$$(a_1 \odot b_1)(a_2 \odot b_2) := (-1)^{\deg b_1\cdot\deg a_2}(a_1a_2 \odot b_1b_2),$$

and the involution is defined on elements of homogeneous degree by

$$(a \odot b)^* := (-1)^{\deg a\cdot\deg b}(a^* \odot b^*).$$

The product and involution extend to all of $A \odot B$ by linearity. The completion of $A \odot B$ with respect to a particular C^*-norm (see [25, p. 522] for an exact description) is a $\mathbb{Z}_2$-graded C^*-algebra, which we denote by $A\widehat{\otimes}B$. We call $A\widehat{\otimes}B$ the **$\mathbb{Z}_2$-graded**

tensor product of A and B.

Remark. We will only make use of the $\mathbb{Z}_2$-graded tensor product when at least one of the C^*-algebras is nuclear, in which case there is a unique C^*-completion of the algebraic tensor product.

Definition 1.13. Let E and F be $\mathbb{Z}_2$-graded right Hilbert modules over $\mathbb{Z}_2$-graded C^*-algebras A and B respectively. The algebraic tensor product $E \odot F$ is a right pre-Hilbert $A\widehat{\otimes}B$ module with the grading, right multiplication and inner product defined on elements of homogeneous degree by $\deg(e \odot f) := \deg e + \deg f$,

$$(e \odot f)(a\widehat{\otimes}b) := (-1)^{\deg f \cdot \deg a}(ea \odot fb),$$

and

$$(e_1 \odot f_1 | e_2 \odot f_2)_{A\widehat{\otimes}B} := (-1)^{\deg f_1 \cdot (\deg e_1 + \deg e_2)}(e_1|e_2)_A \widehat{\otimes} (f_1|f_2)_B,$$

and extended to all of $E \odot F$ by linearity. The completion of $E \odot F$ is thus a $\mathbb{Z}_2$-graded right Hilbert $A\widehat{\otimes}B$-module, which we denote by $E\widehat{\otimes}F$. We call $E\widehat{\otimes}F$ the **external tensor product** of E and F. We define a $\mathbb{Z}_2$-graded $*$-homomorphism from $\mathrm{End}_A(E)\widehat{\otimes}\,\mathrm{End}_B(F)$ to $\mathrm{End}_{A\widehat{\otimes}B}(E\widehat{\otimes}F)$ on elements of homogeneous degree by

$$(F_1\widehat{\otimes}F_2)(e\widehat{\otimes}f) := (-1)^{\deg F_2 \cdot \deg e}(F_1 e \widehat{\otimes} F_2 f),$$

which extends by linearity. The restriction of this map to $\mathrm{End}^0_A(E)\widehat{\otimes}\,\mathrm{End}^0_B(F)$ defines an isomorphism between $\mathrm{End}^0_A(E)\widehat{\otimes}\,\mathrm{End}^0_B(F)$ and $\mathrm{End}^0_{A\widehat{\otimes}B}(E\widehat{\otimes}F)$, [25, p. 523].

In particular, if C and D are $\mathbb{Z}_2$-graded C^*-algebras together with $\mathbb{Z}_2$-graded $*$-homomorphisms $\phi_1 : C \to \mathrm{End}_A(E)$ and $\phi_2 : D \to \mathrm{End}_B(F)$, then we can define a $\mathbb{Z}_2$-graded $*$-homomorphism $\phi_1\widehat{\otimes}\phi_2 : C\widehat{\otimes}D \to \mathrm{End}_{A\widehat{\otimes}B}(E\widehat{\otimes}F)$ on elements of homogeneous degree by

$$(\phi_1\widehat{\otimes}\phi_2)(c\widehat{\otimes}d)(e\widehat{\otimes}f) = (-1)^{\deg d \cdot \deg e}(\phi(c)e\widehat{\otimes}\phi(d)f),$$

which extends by linearity.

1.2 The equivariant KK-groups

Definition 1.14. Let A be a $\mathbb{Z}_2$-graded C^*-algebra and let G be a locally compact group. An **action** of G on A is a map α from G into the degree zero $*$-automorphisms of A such that the map $G \times A \to A$, $(g, a) \mapsto \alpha_g(a)$ is continuous. If A carries such an action, we call A a **G-algebra**. If B is another $\mathbb{Z}_2$-graded G-algebra with action β then the $\mathbb{Z}_2$-graded tensor product $A\widehat{\otimes}B$ carries the diagonal action $(\alpha\widehat{\otimes}\beta)_g(a\widehat{\otimes}b) = \alpha_g(a)\widehat{\otimes}\beta_g(b)$.

Remark. We will only consider actions by compact groups in this thesis.

Definition 1.15. Let A and B be $\mathbb{Z}_2$-graded C^*-algebras, with A separable and B σ-unital (i.e. B has a countable approximate identity), carrying respective actions α and β by a compact group G. A **(bounded) equivariant Kasparov A-B-module** (or **KK-cycle**) (ρ, E_B, F) consists of (i) a countably generated $\mathbb{Z}_2$-graded Hilbert B-module E_B, (ii) a $\mathbb{Z}_2$-graded $*$-homomorphism $\rho : A \to \mathrm{End}_B(E)$, (iii) a homomorphism V from G into the bounded invertible (not necessarily adjointable) linear operators on E, which is continuous in the strong operator topology, and (iv) an odd operator $F \in \mathrm{End}_B(E)$ such that:

1) $V_g(\rho(a)eb) = \rho(\alpha_g(a))V_g(e)\beta_g(b)$ and $(V_g e|V_g f)_B = \beta_g((e|f)_B)$ for all $a \in A$, $e \in E$ and $b \in B$;

2) $[F, V_g] = 0$ for all $g \in G$;

3) $[F, \rho(a)]_\pm$, $\rho(a)(F - F^*)$ and $\rho(a)(1 - F^2)$ are compact endomorphisms for all $a \in A$.

We say that (ρ, E_B, F) is **degenerate** if $[F, \rho(a)]_\pm = \rho(a)(F - F^*) = \rho(a)(1 - F^2) = 0$ for all $a \in A$.

Remark. Definition 1.15 and equivariant KK-theory can be generalised to locally compact groups, provided Condition 2) is weakened to $\rho(a)[F, V_g] \in \mathrm{End}^0_B(E)$ for all $a \in A$, $g \in G$.

Let $\mathscr{E}_G(A, B)$ be the set of equivariant Kasparov A-B-modules, which is a semigroup under direct summation. We introduce the following equivalence relations on $\mathscr{E}_G(A, B)$ to make it into a group.

Firstly, we say that (ρ, E_B, F) and (ρ', E'_B, F') are **unitarily equivalent** if there is an invariant degree zero unitary $u \in \mathrm{Hom}_B(E, E')$ such that $\rho'(a) = u\rho(a)u^*$ for all $a \in A$ and $F' = uFu^*$.

Secondly, we say that (ρ, E_B, F) and (ρ, E_B, F') are **operator homotopic** if there is a operator norm continuous map $[0, 1] \to \mathrm{End}_B(E)$, $t \mapsto F_t$ such that (ρ, E_B, F_t) is an equivariant Kasparov A-B-module for all $t \in [0, 1]$ and $F_0 = F$, $F_1 = F'$.

We then say that $(\rho, E_B, F) \sim (\rho', E'_B, F')$ if there are degenerate modules X, X' such that $(\rho, E_B, F) \oplus X$ and $(\rho', E'_B, F') \oplus X'$ are operator homotopic up to unitary equivalence. The quotient of $\mathscr{E}_G(A, B)$ by the equivalence relation generated by $\sim$ is an abelian group, which we denote by $KK_G(A, B)$, [25].

The inverse of the class $[(\rho, E_B, F)] \in KK_G(A, B)$ is the class $[(\rho^{\mathrm{op}}, E_B^{\mathrm{op}}, -F)]$, where $\rho^{\mathrm{op}}(a) = (-1)^{\deg a}\rho(a)$, and E^{op} is the Hilbert module with the opposite grading (i.e. $(E^{\mathrm{op}})^j = E^{j+1}$, $j \in \mathbb{Z}_2$).

Example 1.16. The group $KK_G(\mathbb{C}, B)$ is the equivariant K-theory of B, and the group $KK_G(A, \mathbb{C})$ is the equivariant K-homology of A.

Example 1.17. Let A and B be $\mathbb{Z}_2$-graded G-algebras for a compact group G, with A separable and B σ-unital. If $\rho : A \to B$ is a $\mathbb{Z}_2$-graded equivariant $*$-homomorphism, then $(\rho, B_B, 0)$ is an equivariant Kasparov A-B-module, since $\mathrm{End}^0_B(B) = B$, and hence the map ρ defines a class $[(\rho, B_B, 0)] \in KK_G(A, B)$.

Remark. The definition of $KK_G(A, B)$ given by Kasparov in [25] has an equivalence relation given by a more general homotopy than operator homotopy, which allows the representation ρ of A and Hilbert module E_B to vary as well. However, in the case that A is separable and B is σ-unital, these definitions of $KK_G(A, B)$ are equivalent, [5, Thm. 18.5.3].

The group $KK_G(A, B)$ is homotopy invariant, and the functor KK_G is contravariant in the first variable and covariant in the second. By ignoring the G action (or restricting to the trivial subgroup) we obtain the group $KK(A, B)$. The map $KK_G(A, B) \to KK(A, B)$ is a forgetful functor, which is surjective in the case that A and B carry the trivial action of G.

1.2.1 The higher order KK-groups

For $n \geq 0$, let $\mathbb{C}\mathrm{l}_n$ be the universal unital C^*-algebra generated by n self-adjoint unitaries $c_1, \ldots, c_n$ subject to the commutation relations $c_j c_k + c_k c_j = 2\delta_{j,k} \mathrm{Id}_{\mathbb{C}\mathrm{l}_n}$. The algebra $\mathbb{C}\mathrm{l}_n$ is the nth **Clifford algebra** (where $\mathbb{C}\mathrm{l}_0 = \mathbb{C}$). The Clifford algebra $\mathbb{C}\mathrm{l}_n$ is $\mathbb{Z}_2$-graded by giving the generators $c_1, \ldots, c_n$ degree 1, and $\mathbb{C}\mathrm{l}_n \widehat{\otimes} \mathbb{C}\mathrm{l}_m$ is naturally isomorphic to $\mathbb{C}\mathrm{l}_{n+m}$ as $\mathbb{Z}_2$-graded C^*-algebras, [30, Prop. 1.5]. The higher order KK-groups are defined by setting

$$KK^n_G(A, B) := KK_G(A \widehat{\otimes} \mathbb{C}\mathrm{l}_n, B),$$

where $\mathbb{C}\mathrm{l}_n$ carries the trivial action of G. The groups $KK^n_G(A, B)$ and $KK^{n+2}_G(A, B)$ are canonically isomorphic (via the Morita equivalence between $\mathbb{C}\mathrm{l}_n$ and $\mathbb{C}\mathrm{l}_{n+2}$), and $KK^n_G(A, B \widehat{\otimes} \mathbb{C}\mathrm{l}_m)$ is canonically isomorphic to $KK^{n+m}(A, B)$, [25, Thm. 4 of §5].

1.2.2 Unbounded Kasparov modules

Definition 1.18. Let A and B be $\mathbb{Z}_2$-graded C^*-algebras, with A separable and B σ-unital, carrying respective actions α and β by a compact group G. An **unbounded equivariant Kasparov A-B-module** (or **unbounded KK-cycle**) $(\mathcal{A}, E_B, \mathcal{D})$ consists of (i) an invariant dense sub-$*$-algebra $\mathcal{A} \subset A$, (ii) a countably generated $\mathbb{Z}_2$-graded right Hilbert B-module E, (iii) a homomorphism V from G into the invertible degree

zero bounded linear (not necessarily adjointable) operators on E, where V is continuous in the strong operator topology, (iv) a $\mathbb{Z}_2$-graded $*$-homomorphism $\rho : A \to \mathrm{End}_B(E)$, and (v) an odd, self-adjoint, regular operator $\mathcal{D} : \mathrm{dom}(\mathcal{D}) \subset E \to E$ such that:

1) $V_g(\rho(a)eb) = \rho(\alpha_g(a))V_g(e)\beta_g(b)$ and $(V_g e|V_g f)_B = \beta_g((e|f)_B)$ for all $g \in G$, $a \in A$, $e \in E$ and $b \in B$;

2) $\rho(a) \cdot \mathrm{dom}(\mathcal{D}) \subset \mathrm{dom}(\mathcal{D})$, and the graded commutator $[\mathcal{D}, \rho(a)]_\pm$ is bounded for all $a \in \mathcal{A}$;

3) $\rho(a)(1 + \mathcal{D}^2)^{-1/2}$ is a compact endomorphism for all $a \in \mathcal{A}$;

4) $V_g \cdot \mathrm{dom}(\mathcal{D}) \subset \mathrm{dom}(\mathcal{D})$, and $[\mathcal{D}, V_g] = 0$.

Remark. We normally suppress the notation ρ. We will only employ unbounded equivariant Kasparov A-B-modules for which the action of G on B is trivial, in which case V_g is adjointable with adjoint $V_g^* = V_{g^{-1}}$ for all $g \in G$.

To an unbounded equivariant Kasparov module $(\mathcal{A}, E_B, \mathcal{D})$ we associate a bounded equivariant Kasparov module $(\rho, E_B, \mathcal{D}(1+\mathcal{D}^2)^{-1/2})$, and hence a class in $KK_G(A, B)$. The proof that $(\rho, E_B, \mathcal{D}(1 + \mathcal{D}^2)^{-1/2})$ is a Kasparov module is due to Baaj and Julg, [1]. A refinement of this proof is used in Theorem 8.9 to show that we can associate a relative Fredholm module to a relative spectral triple.

While bounded Kasparov modules are analytically simpler, unbounded Kasparov modules are more geometric and easier to compute with. For example, an unbounded Kasparov module can be defined using a Dirac operator $\mathcal{D}$ on a complete Riemannian manifold, [22], which is local, whereas the associated bounded operator $\mathcal{D}(1 + \mathcal{D}^2)^{-1/2}$ is a non-local pseudodifferential operator. Moreover, unbounded Kasparov modules can carry additional geometric data, as we will see in §5.3.

Definition 1.19. Let A be a separable $\mathbb{Z}_2$-graded C^*-algebra with an action by a compact group G. An **even equivariant spectral triple** $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ for A is an unbounded equivariant Kasparov A-$\mathbb{C}$-module, where $\mathbb{C}$ carries the trivial action of G. If A is trivially $\mathbb{Z}_2$-graded, then one can also define an **odd equivariant spectral triple** $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, which has the same definition, except that $\mathcal{H}^1 = \{0\}$ and $\mathcal{D}$ need not be odd.

Example 1.20. Let M be a compact Riemannian manifold. The **Clifford bundle** $\mathbb{C}\mathrm{l}(M)$ over M is the complex algebra bundle generated by the cotangent bundle and the Clifford relations $v \cdot w + w \cdot v = -2\langle v, w\rangle$ for $v, w \in T_x^*M$. A **Clifford module** over M is a Hermitian vector bundle S together with an $C^\infty(M)$-linear $*$-homomorphism $c : \Gamma^\infty(\mathbb{C}\mathrm{l}(M)) \to \Gamma^\infty(\mathrm{End}((S))$, called **Clifford multiplication**, such that

$$(c(v)s_1|s_2)_{C(M)} + (s_1|c(v)s_2)_{C(M)} = 0, \quad \text{for all } v \in \Gamma^\infty(T^*M),\ s_1, s_2 \in \Gamma^\infty(S).$$

A **Clifford connection** on S is a connection $\nabla^S : \Gamma^\infty(S) \to \Gamma^\infty(T^*M \otimes S)$ which is compatible with the Levi-Civita connection ∇^{LC} on T^*M, in the sense that

$$\nabla^S_X(c(v)s) = c(\nabla^{LC}_X s) + c(v)\nabla^S_X s, \quad \text{for all } v \in \Gamma^\infty(T^*M),\ s \in \Gamma^\infty(S),\ X \in \Gamma^\infty(TM).$$

Given a Clifford connection ∇^S, the **Dirac operator** on S is

$$\mathcal{D} = c \circ \nabla^S : \Gamma^\infty(S) \to \Gamma^\infty(S).$$

The Dirac operator is a first order differential operator, which is essentially self-adjoint with compact resolvent as an operator on $L^2(S)$ (whose inner product is $\langle s_1, s_2\rangle = \int_M (s_1|s_2)\,\mathrm{vol}$), and hence $(C^\infty(M), L^2(S), \mathcal{D})$ is an odd spectral triple, [4, 22, 30].

In particular, the spin Dirac operator on a compact spin (or spin^c) manifold defines a spectral triple, as does the Dirac operator $d + d^*$ on the complexified exterior bundle over a compact Riemannian manifold, [30, p. 121].

The Clifford module S is **$\mathbb{Z}_2$-graded** if S is $\mathbb{Z}_2$-graded as a vector bundle and the Clifford multiplication is $\mathbb{Z}_2$-graded. If S is $\mathbb{Z}_2$-graded, we require that a Clifford connection is even, in which case the Dirac operator $\mathcal{D}$ is odd and hence $(C^\infty(M), L^2(S), \mathcal{D})$ is an *even* spectral triple. For example, the complexified exterior bundle can be $\mathbb{Z}_2$-graded by the degree of differential forms, and if M is even-dimensional and oriented, then the positive/negative eigenspaces of $i^{\dim M/2}c(\mathrm{vol})$ defines a $\mathbb{Z}_2$-grading of any Clifford module, where vol is the Riemannian volume form, [43, p. 142].

Suppose a compact Lie group G acts smoothly on M by isometries. Then the action of G on the cotangent bundle preserves the Clifford relations and hence extends to an action on the Clifford bundle. An **equivariant Clifford module** is a Clifford module S, which carries an action of G which lifts the action on M, such that the Hermitian inner product is preserved by G, and Clifford multiplication is G-invariant. If S is $\mathbb{Z}_2$-graded, then we require that the action of G preserves the $\mathbb{Z}_2$-grading. If a Clifford connection ∇^S is G-invariant, then so is the associated Dirac operator $\mathcal{D}$, [4, Lem. 6.2], and thus $(C^\infty(M), L^2(S), \mathcal{D})$ is an equivariant spectral triple, which is even if and only if S is $\mathbb{Z}_2$-graded.

Definition 1.21. Let A be a trivially $\mathbb{Z}_2$-graded separable C^*-algebra with an action by a compact group G. The K-homology class of an odd equivariant spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ for A is defined by associating to it an even spectral triple for $A\widehat{\otimes}\mathbb{C}\mathrm{l}_1$. Equip $\mathbb{C}^2$ with the $\mathbb{Z}_2$-grading

$$(\mathbb{C}^2)^j = \left\{ v \in \mathbb{C}^2 : \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) v = (-1)^j v \right\}.$$

Let c be the self-adjoint unitary generator of the Clifford algebra $\mathbb{C}\mathrm{l}_1$, and define a $\mathbb{Z}_2$-graded $*$-homomorphism $\mathbb{C}\mathrm{l}_1 \to B(\mathbb{C}^2)$ by $c \mapsto \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, which in turn defines a $\mathbb{Z}_2$-graded

representation of $A\widehat{\otimes}\mathbb{C}\mathrm{l}_1$ on $\mathcal{H}\widehat{\otimes}\mathbb{C}^2$. Equip $\mathcal{H}\widehat{\otimes}\mathbb{C}^2$ with the action $V'_g(\xi\widehat{\otimes}v) = V_g\xi\widehat{\otimes}v$. Let $\omega = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right) \in B(\mathbb{C}^2)$, and define an odd self-adjoint operator $\mathcal{D}\widehat{\otimes}\omega$ on $\mathcal{H}\widehat{\otimes}\mathbb{C}^2$ by

$$\mathrm{dom}(\mathcal{D}\widehat{\otimes}\omega) = \mathrm{span}\{\xi\widehat{\otimes}v : \xi \in \mathrm{dom}(\mathcal{D}), v \in \mathbb{C}^2\}, \quad (\mathcal{D}\widehat{\otimes}\omega)(\xi\widehat{\otimes}v) = \mathcal{D}\xi\widehat{\otimes}\omega v.$$

Then $(\mathcal{A}\widehat{\otimes}\mathbb{C}\mathrm{l}_1, \mathcal{H}\widehat{\otimes}\mathbb{C}^2, \mathcal{D}\widehat{\otimes}\omega)$ is an even G-equivariant spectral triple. The class of the odd spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ in odd K-homology is defined to be the class of the even spectral triple $(\mathcal{A}\widehat{\otimes}\mathbb{C}\mathrm{l}_1, \mathcal{H}\widehat{\otimes}\mathbb{C}^2, \mathcal{D}\widehat{\otimes}\omega)$ in $KK_G(A\widehat{\otimes}\mathbb{C}\mathrm{l}_1, \mathbb{C}) = KK^1_G(A, \mathbb{C})$, [12, Prop. IV.A.13].

1.2.3 The Kasparov product

Let A, B and C be $\mathbb{Z}_2$-graded G-algebras for a compact group G, with A separable and B, C σ-unital. Then there is a map

$$KK_G(A, B) \times KK_G(B, C) \to KK_G(A, C),$$

called the **Kasparov product**, [5, 25]. We use the notation $\alpha\widehat{\otimes}_B\beta$ for the Kasparov product of classes $\alpha \in KK_G(A, B)$ and $\beta \in KK_G(B, C)$. The Kasparov product is distributive, associative and functorial (see [5, §18.7] for an exact description of this functoriality). The Kasparov product makes $KK_G(A, A)$ into a ring, with identity the class of the (unbounded) Kasparov module $(A, A_A, 0)$. If $A = C = \mathbb{C}$ (which carries the trivial action of G) then the Kasparov product is the index pairing between K-theory and K-homology;

$$KK_G(\mathbb{C}, B) \times KK_G(B, \mathbb{C}) \to KK_G(\mathbb{C}, \mathbb{C})$$

The ring $KK_G(\mathbb{C}, \mathbb{C})$ is isomorphic to the representation ring of G, [26], so in particular $KK(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$ and we recover the index as an integer.

The product also respects higher order KK-theory:

$$\begin{aligned} KK^j_G(A, B) \times KK^k_G(B, C) &= KK_G(A\widehat{\otimes}\mathbb{C}\mathrm{l}_j, B) \times KK_G(B\widehat{\otimes}\mathbb{C}\mathrm{l}_k, C) \\ &\cong KK_G(A\widehat{\otimes}\mathbb{C}\mathrm{l}_j, B) \times KK_G(B, C\widehat{\otimes}\mathbb{C}\mathrm{l}_k) \\ &\to KK_G(A\widehat{\otimes}\mathbb{C}\mathrm{l}_j, C\widehat{\otimes}\mathbb{C}\mathrm{l}_k) \cong KK^{j+k}_G(A, C). \end{aligned}$$

The Kasparov product is generally non-constructive. It is often easier to choose a likely-looking candidate representative of the product, and then to check whether it does represent the product. Connes and Skandalis gave conditions to check whether the product is represented in the bounded setting, [14, Appendix A]. Kucerovsky then adapted these conditions to the unbounded setting, [27]. The following is a slightly less general version of [27, Thm. 13] (the domain criterion may be weakened).

Theorem 1.22 (Kucerovsky's criteria for the Kasparov product.)**.** *Let G be a compact group, and let $(\mathcal{A}, E_B, \mathcal{D}_1)$ and $(\mathcal{B}, F_C, \mathcal{D}_2)$ be unbounded equivariant Kasparov A-B- and B-C-modules respectively. An unbounded equivariant Kasparov A-C-module $(\mathcal{A}, (E\widehat{\otimes}_B F)_C, \mathcal{D})$, with $E\widehat{\otimes}_B F$ carrying the diagonal action of G inherited from E and F, represents the Kasparov product of $(\mathcal{A}, E_B, \mathcal{D}_1)$ and $(\mathcal{B}, F_C, \mathcal{D}_2)$ in $KK_G(A, C)$ if the following criteria are satisfied.*

i) ***The connection criterion.*** *For $e \in E$, define $T_e : F \to E\widehat{\otimes}_B F$ by $T_e f = e\widehat{\otimes} f$. The operator T_e is adjointable with adjoint $T_e^*(\widetilde{e\widehat{\otimes}} f) = (e|\widetilde{e})f$. The first criterion is: for all x is some dense subspace of AE, the graded commutator*

$$\left[\begin{pmatrix} \mathcal{D} & 0 \\ 0 & \mathcal{D}_2 \end{pmatrix}, \begin{pmatrix} 0 & T_x \\ T_x^* & 0 \end{pmatrix}\right]_{\pm}$$

is bounded on $\mathrm{dom}(\mathcal{D}) \oplus \mathrm{dom}(\mathcal{D}_2)$.

ii) ***The domain criterion.*** *For all $\mu \in \mathbb{R} \setminus \{0\}$, the resolvent $(i\mu + \mathcal{D})^{-1}$ maps the submodule $C_c^\infty(\mathcal{D}_1\widehat{\otimes}1)(E\widehat{\otimes}_B F)$ into* $\mathrm{dom}(\mathcal{D}_1\widehat{\otimes}1)$.

iii) ***The positivity criterion.*** *There is some $R \in \mathbb{R}$ such that*

$$\big((\mathcal{D}_1\widehat{\otimes}1)x\big|\mathcal{D}x\big)_C + \big(\mathcal{D}x\big|(\mathcal{D}_1\widehat{\otimes}1)x\big)_C \geq R(x|x)_C$$

for all x in a dense subspace of $\mathrm{dom}(\mathcal{D}) \cap \mathrm{dom}(\mathcal{D}_1\widehat{\otimes}1)$.

Remark. Although [27, Thm. 13] is stated for the non-equivariant case, it requires no modification in the equivariant case, [28].

Kucerovsky's criteria are our main tool for testing whether factorisation is achieved.

Chapter 2

The construction of the KK-cycles

Throughout this chapter, G is a compact abelian Lie group, equipped with the normalised Haar measure, and $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is an even G-equivariant spectral triple for a $\mathbb{Z}_2$-graded separable C^*-algebra A carrying an action α of G. (The case that the spectral triple is odd is considered later, in §3.2.)

There are some differences between the cases of G even dimensional and G odd dimensional. We introduce the following notation so that we may handle both cases simultaneously.

Definition 2.1. Let $\mathbb{C}\mathrm{l}_1$ be the Clifford algebra generated by a self-adjoint unitary c. We denote by $\mathfrak{C}$ the $\mathbb{Z}_2$-graded C^*-algebra

$$\mathfrak{C} = \begin{cases} \mathbb{C} & \text{if } G \text{ is even dimensional} \\ \mathbb{C}\mathrm{l}_1 & \text{if } G \text{ is odd dimensional.} \end{cases}$$

We also denote by $\mathfrak{c}$ the generator of $\mathfrak{C}$; i.e.

$$\mathfrak{c} = \begin{cases} 1 & \text{if } G \text{ is even dimensional} \\ c & \text{if } G \text{ is odd dimensional.} \end{cases}$$

We will construct three unbounded KK-cycles. The first cycle (referred to as the left-hand module), is constructed using the spin Dirac operator over G, and defines a class in $KK_G(A, A^G\widehat{\otimes}\mathfrak{C})$. The second cycle, which we call the middle module, represents a class in $KK_G(A^G\widehat{\otimes}\mathfrak{C}, A^G\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(G))^G)$. The module is simply the $\mathbb{Z}_2$-graded Morita equivalence between $A^G\widehat{\otimes}\mathfrak{C}$ and $A^G\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(G))^G \cong A^G\widehat{\otimes}\mathbb{C}\mathrm{l}_{\dim G}$, and so contains no homological information. The third cycle (the right-hand module) is constructed by restricting the spectral triple to a spectral subspace of $\mathcal{H}$, and adding a representation of $\Gamma(\mathbb{C}\mathrm{l}(G))^G$, so that it defines a class in $KK_G(A^G\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(G))^G, \mathbb{C})$.

2.1 The left-hand module

By $\mathrm{Char}(G)$ we denote the characters of G, which are the smooth homomorphisms $\chi : G \to U(1)$. Since G is abelian, the characters form a group under multiplication. For each $\chi \in \mathrm{Char}(G)$, let

$$A_\chi = \{a \in A : \alpha_g(a) = \chi(g)a\}$$

be the **spectral subspace** of A associated with the character χ. Note that the direct sum $\bigoplus_{\chi \in \mathrm{Char}(G)} A_\chi$ is dense in A. For each $\chi \in \mathrm{Char}(G)$, define the **spectral projection** $\Phi_\chi : A \to A$ by

$$\Phi_\chi(a) = \int_G \chi^{-1}(g)\alpha_g(a)\, dg. \tag{2.1}$$

Each Φ_χ is a continuous idempotent with range $\Phi_\chi = A_\chi$.

Definition 2.2. The action of G on A is said to satisfy the **spectral subspace assumption** (SSA) if the norm closure $\overline{A_\chi A_\chi^*}$ is a complemented ideal[1] in the fixed point algebra A^G for each $\chi \in \mathrm{Char}(G)$.

Remark. A particular case of the spectral subspace assumption is if $\overline{A_\chi A_\chi^*} = A^G$ for all $\chi \in \mathrm{Char}(G)$. In this case we say that A has **full spectral subspaces**. This is equivalent to the action of G on A being **free** or **saturated**, [37, 44]. If $A = C_0(X)$ for a locally compact Hausdorff G-space X, then $C_0(X)$ has full spectral subspaces if and only if the action of G on X is free, [37, Prop. 7.1.12 and Thm. 7.2.6].

We define an A^G-valued inner product on A by

$$(a|b)_{A^G} := \Phi_1(a^*b) = \int_G \alpha_g(a^*b)\, dg.$$

With this inner product, A is a right pre-Hilbert A^G-module. Hence the completion of A with respect to $(\cdot|\cdot)_{A^G}$ is a right Hilbert A^G-module, which we denote by X. The $\mathbb{Z}_2$-grading of A defines a $\mathbb{Z}_2$-grading of X, which makes X into a $\mathbb{Z}_2$-graded right Hilbert A^G-module. The action of G on A extends to a unitary action $\alpha : G \to \mathrm{End}_{A^G}(X)$.

Remark. Let $\chi \in \mathrm{Char}(G)$, and let $a, b \in A_\chi$. Then $a^*b \in A^G$, so $(a|b)_{A^G} = a^*b$. Hence A_χ is closed in X, and so

$$X_\chi := \{x \in X : \alpha_g(x) = \chi(g)x\} = A_\chi.$$

[1] An ideal $J \lhd A$ is **complemented** if there is another ideal $I \lhd A$ with $I \cap J = \{0\}$ and $A = I \oplus J$.

The following is a more general version of [35, Lem. 4.2] or [11, Lem. 2.4]. The result there is for the case $G = \mathbb{T}$, but the proof is much the same as in the general case. (The result is stated for n-tori in [36, Lem. 5.2].)

Lemma 2.3. *For each $\chi \in \mathrm{Char}(G)$, the map $\Phi_\chi : A \to A$ extends to an adjointable projection $\Phi_\chi : X \to X$ with range A_χ. Moreover,*

$$(x|y)_{A^G} = \sum_{\chi \in \mathrm{Char}(G)} \Phi_\chi(x)^* \Phi_\chi(y)$$

for all $x, y \in X$, and the sum $\sum_{\chi \in \mathrm{Char}(G)} \Phi_\chi$ converges strictly to the identity on X.

The operator on the left-hand module is constructed from the spin Dirac operator on G, which we review.

Fix an invariant Riemannian metric on G, which is equivalent to fixing an inner product on T_eG. Let $W \cong \mathbb{C}^{\lfloor \dim G/2 \rfloor}$ be an irreducible representation space for $\mathbb{C}\mathrm{l}(T_e^*G)$ and let $\$_G = G \times W$, which since $\mathbb{C}\mathrm{l}(G) = G \times \mathbb{C}\mathrm{l}(T_e^*G)$ is a Clifford module over G (the trivial complex spinor bundle). The Dirac operator on $\$_G$ is

$$\mathcal{D}_G = \sum_{j=1}^{\dim G} c(X_j^\flat) X_j,$$

where $\{X_1, \ldots, X_j\}$ is an invariant orthonormal frame for TG, $X \mapsto X^\flat$ is the Riemannian isomorphism $TG \to T^*G$, and $c : \Gamma(\mathbb{C}\mathrm{l}(G)) \to \Gamma(\mathrm{End}(\$_G))$ is the Clifford representation. Let $V : G \to L^2(\$_G)$ be the unitary representation $V_g s(h) = s(g^{-1}h)$. This representation makes $(C^\infty(G), L^2(\$_G), \mathcal{D}_G)$ into a G-equivariant spectral triple, which is even if and only if $\dim G$ is even, [46]. Then $(C^\infty(G), (L^2(\$_G)\widehat{\otimes}\mathfrak{C})_{\mathfrak{C}}, \mathcal{D}_G\widehat{\otimes}\mathfrak{c})$ is a G-equivariant unbounded Kasparov $C(G)$-$\mathfrak{C}$-module for G either even or odd dimensional.

Definition 2.4. Let $X\widehat{\otimes}(L^2(\$_G)\widehat{\otimes}\mathfrak{C})$ be the external tensor product of the Hilbert modules X and $L^2(\$_G)\widehat{\otimes}\mathfrak{C}$, which is a $\mathbb{Z}_2$-graded right Hilbert $A^G\widehat{\otimes}\mathfrak{C}$-module. Let E_1 be the invariant submodule of $X\widehat{\otimes}(L^2(\$_G)\widehat{\otimes}\mathfrak{C})$ under the diagonal action $g \cdot (x\widehat{\otimes}(s\widehat{\otimes}z)) = \alpha_g(x)\widehat{\otimes}(V_g s\widehat{\otimes}z)$. Let V_1 be the homomorphism from G into the unitaries of E_1 defined by

$$V_{1,g}(x\widehat{\otimes}(s\widehat{\otimes}z)) = \alpha_g(x)\widehat{\otimes}(s\widehat{\otimes}z).$$

For each $\chi \in \mathrm{Char}(G)$, let $p'_\chi \in B(L^2(\$_G))$ be the orthogonal projection onto

$$L^2(\$_G)_\chi = \{s \in L^2(\$_G) : V_g(s) = \chi(g)s\},$$

and define $p_\chi \in \mathrm{End}_{\mathfrak{C}}(L^2(\$_G)\widehat{\otimes}\mathfrak{C})$ by $p_\chi(s\widehat{\otimes}z) = p'_\chi s\widehat{\otimes}z$.

The following result is elementary, but will be quite useful in later calculations.

Lemma 2.5. *For elements of homogeneous degree, the $A^G\widehat{\otimes}\mathfrak{C}$-valued inner product on E_1 can be expressed (for x_1, $x_2 \in X$ and s_1, $s_2 \in L^2(\mathbf{\$}_G)\widehat{\otimes}\mathfrak{C}$) as*

$$\begin{aligned}&(x_1\widehat{\otimes}s_1|x_2\widehat{\otimes}s_2)_{A^G\widehat{\otimes}\mathfrak{C}}\\ &\quad= (-1)^{\deg s_1\cdot(\deg x_1+\deg x_2)}\sum_{\chi\in\mathrm{Char}(G)}\Phi_\chi(x_1)^*\Phi_\chi(x_2)\widehat{\otimes}(p_{\chi^{-1}}s_1|p_{\chi^{-1}}s_2)_{\mathfrak{C}}.\end{aligned}$$

Proposition 2.6. *Define an action of $\bigoplus_{\chi\in\mathrm{Char}(G)}A_\chi$ on E_1 by*

$$\sum_{\chi\in\mathrm{Char}(G)} a_\chi\cdot(x\widehat{\otimes}s) := \sum_{\chi\in\mathrm{Char}(G)} a_\chi x\widehat{\otimes}\chi s. \tag{2.2}$$

This action extends to a $\mathbb{Z}_2$-graded $*$-homomorphism $\rho : A \to \mathrm{End}_{A^G\widehat{\otimes}\mathfrak{C}}(E_1)$ satisfying

$$V_{1,g}(\rho(a)e) = \rho(\alpha_g(a))V_{1,g}(e), \qquad a\in A,\ e\in E_1.$$

Proof. Suppose $a_\chi \in A_\chi$ and $x = \sum_{\nu\in\mathrm{Char}(G)} x_\nu \in X$, where $x_\nu \in A_\nu$ for all characters $\nu \in \mathrm{Char}(G)$. Then

$$\|a_\chi x\|^2 = \sum_{\phi\in\mathrm{Char}(G)}\|a_\chi x_\nu\|^2 \le \|a_\chi\|^2\|x\|^2$$

by Lemma 2.3, so $a_\chi x$ is a well-defined element of x.

Since $\alpha_g(a_\chi^*) = \alpha_g(a_\chi)^* = \overline{\chi(g)}a_\chi^* = \chi^{-1}(g)a_\chi^*$, it follows that $a_\chi^* \in A_{\chi^{-1}}$. Hence if $a_\chi \in A_\chi$ and $x_i\widehat{\otimes}s_i \in E_1$, $i = 1, 2$, each of homogeneous degree, then

$$\begin{aligned}&\big(x_1\widehat{\otimes}s_1\big|a_\chi\cdot(x_2\widehat{\otimes}s_2)\big)_{A^G\widehat{\otimes}\mathfrak{C}} = \big(x_1\widehat{\otimes}s_1\big|a_\chi x_2\widehat{\otimes}\chi s_2\big)_{A^G\widehat{\otimes}\mathfrak{C}}\\ &= (-1)^{\deg s_1\cdot(\deg x_1+\deg a_\chi+\deg x_2)}(x_1|a_\chi x_2)_{A^G}\widehat{\otimes}(s_1|\chi s_2)_{\mathfrak{C}}\\ &= (-1)^{\deg s_1\cdot(\deg x_1+\deg a_\chi+\deg x_2)}(a_\chi^* x_1|x_2)_{A^G}\widehat{\otimes}(\chi^{-1}s_1|s_2)_{\mathfrak{C}}\\ &= \big(a_\chi^* x_1\widehat{\otimes}\chi^{-1}s_1\big|x_2\widehat{\otimes}s_2\big)_{A^G\widehat{\otimes}\mathfrak{C}} = \big(a_\chi^*\cdot(x_1\widehat{\otimes}s_1)\big|x_2\widehat{\otimes}s_2\big)_{A^G\widehat{\otimes}\mathfrak{C}}.\end{aligned}$$

Thus the action (2.2) defines a $*$-homomorphism $\bigoplus_\chi A_\chi \to \mathrm{End}_{A^G\widehat{\otimes}\mathfrak{C}}(E_1)$, which extends to a $*$-homomorphism $\rho : A \to \mathrm{End}_{A^G\widehat{\otimes}\mathfrak{C}}(E_1)$. That ρ is $\mathbb{Z}_2$-graded and equivariant is obvious. □

Definition 2.7. Let $\mathcal{D}_G : \mathrm{dom}(\mathcal{D}_G) \subset L^2(\mathbf{\$}_G) \to L^2(\mathbf{\$}_G)$ be the spin Dirac operator on G, and let $\mathfrak{c}$ be the generator of $\mathfrak{C}$. Define a closed operator $\mathcal{D}_1 : \mathrm{dom}(\mathcal{D}_1) \subset E_1 \to E_1$ initially on the linear span of elements of the form $x\widehat{\otimes}(s\widehat{\otimes}z)$, where $x \in X$, $s \in \mathrm{dom}(\mathcal{D}_G)$

and $z \in \mathfrak{C}$ are of homogeneous degree, by

$$\mathcal{D}_1(x\widehat{\otimes}(s\widehat{\otimes}z)) := (-1)^{\deg x} x\widehat{\otimes}(\mathcal{D}_G s\widehat{\otimes}\mathfrak{c}z),$$

and then take the operator closure. Since $\mathcal{D}_G$ is equivariant, $\mathcal{D}_1$ is well-defined.

Proposition 2.8. *The triple $(\oplus_\chi A_\chi, (E_1)_{A^G\widehat{\otimes}\mathfrak{C}}, \mathcal{D}_1)$ is an unbounded equivariant Kasparov A-$A^G\widehat{\otimes}\mathfrak{C}$-module if and only if the action of G on A satisfies the spectral subspace assumption (Definition 2.2). When the action of G on A satisfies the spectral subspace condition, we call the Kasparov module $(\oplus_\chi A_\chi, (E_1)_{A^G\widehat{\otimes}\mathfrak{C}}, \mathcal{D}_1)$ the* ***left-hand module****.*

Proof. See [11, Prop. 2.9] and the preceding lemmas for a proof when $G = \mathbb{T}$. The general case requires only minor modifications, as in [10, Ch. 5]. □

We henceforth assume that the action of G on A satisfies the spectral subspace assumption.

2.2 The middle module

Recall that G is a compact abelian Lie group, equipped with the trivial spinor bundle $\mathbf{\$}_G$, and $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is an even G-equivariant spectral triple for a $\mathbb{Z}_2$-graded separable C^*-algebra A. We will now construct the middle module, whose job is to correct for the spinor bundle dimensions between the left-hand module and $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.

Let $\Gamma(\mathbf{\$}_G)$ denote the continuous sections of $\mathbf{\$}_G$, which is a right Hilbert $C(G)$-module with the pointwise inner product on $\mathbf{\$}_G$. We also let $\Gamma(\mathbb{C}\mathrm{l}(G))$ denote the C^*-algebra of continuous sections of the Clifford bundle over G. This C^*-algebra is $\mathbb{Z}_2$-graded by

$$\Gamma(\mathbb{C}\mathrm{l}(G))^0 = \{\text{forms of even degree}\}, \quad \Gamma(\mathbb{C}\mathrm{l}(G))^1 = \{\text{forms of odd degree}\}.$$

Let $\rho : \Gamma(\mathbb{C}\mathrm{l}(G)) \to \mathrm{End}_{C(G)}(\Gamma(\mathbf{\$}_G))$ be the Clifford representation, which is a $*$-homomorphism. When G is even dimensional, ρ is a $\mathbb{Z}_2$-graded $*$-homomorphism, but this is not the case when G is odd dimensional. Both $\Gamma(\mathbb{C}\mathrm{l}(G))$ and $\Gamma(\mathbf{\$}_G)$ carry an action of G, where the action on $\mathbb{C}\mathrm{l}(G)$ is generated by the action on T^*G, and we denote their respective fixed point sets by $\Gamma(\mathbb{C}\mathrm{l}(G))^G$ and $\Gamma(\mathbf{\$}_G)^G$. The fixed point algebra $\Gamma(\mathbb{C}\mathrm{l}(G))^G$ is a finite dimensional C^*-algebra, with

$$\Gamma(\mathbb{C}\mathrm{l}(G))^G \cong \mathbb{C}\mathrm{l}_{\dim G} \cong \begin{cases} M_{2^{\dim G/2}}(\mathbb{C}) & \text{if } \dim G \text{ is even} \\ M_{2^{(\dim G-1)/2}}(\mathbb{C}) \oplus M_{2^{(\dim G-1)/2}}(\mathbb{C}) & \text{if } \dim G \text{ is odd.} \end{cases}$$

The fixed sections $\Gamma(\mathbf{\$}_G)^G$ form a finite dimensional vector space, with

$$\Gamma(\mathbf{\$}_G)^G \cong \begin{cases} \mathbb{C}^{2^{\dim G/2}} & \text{if } \dim G \text{ is even} \\ \mathbb{C}^{2^{(\dim G-1)/2}} & \text{if } \dim G \text{ is odd.} \end{cases}$$

Definition 2.9. Let $\mathfrak{c}$ be the generator of the C^*-algebra $\mathfrak{C}$ (Definition 2.1). Define a $\mathbb{Z}_2$-graded $*$-homomorphism $\widetilde{\rho} : \Gamma(\mathbb{C}\mathrm{l}(G))^G \to \mathrm{End}_{\mathfrak{C}}(\Gamma(\mathbf{\$}_G)^G \widehat{\otimes} \mathfrak{C})$ on elements of homogeneous degree by $\widetilde{\rho}(s)(w \widehat{\otimes} z) = \rho(s) w \widehat{\otimes} \mathfrak{c}^{\deg s} z$, which extends to all of $\Gamma(\mathbb{C}\mathrm{l}(G))^G$ by linearity.

Lemma 2.10. *The $\mathbb{Z}_2$-graded $*$-homomorphism $\widetilde{\rho}$ is an isomorphism.*

Proof. It is immediate that $\widetilde{\rho}$ is an isomorphism in the case that G is even dimensional, since in this case the Clifford representation is an isomorphism between $\Gamma(\mathbb{C}\mathrm{l}(G))$ and $\mathrm{End}_{C(G)}(\Gamma(\mathbf{\$}_G))$. So suppose that G is odd dimensional, in which case $\mathfrak{C} = \mathbb{C}\mathrm{l}_1$.

Since the C^*-algebras $\Gamma(\mathbb{C}\mathrm{l}(G))^G$ and $\mathrm{End}_{\mathbb{C}\mathrm{l}_1}(\Gamma(\mathbf{\$}_G)^G \widehat{\otimes} \mathbb{C}\mathrm{l}_1)$ are of finite and equal dimension, we need only show injectivity, and this is equivalent to $\rho(s) = 0 \Longrightarrow s = 0$ for all $s \in \Gamma(\mathbb{C}\mathrm{l}(G))^G$ of homogeneous degree.

Let $\dim G = k$. Under the isomorphisms $\Gamma(\mathbb{C}\mathrm{l}(G))^G \cong M_{2^{(k-1)/2}}(\mathbb{C}) \oplus M_{2^{(k-1)/2}}(\mathbb{C})$ and $\Gamma(\mathbf{\$}_G)^G \cong \mathbb{C}^k$, the representation is $\rho(m_1 \oplus m_2) = m_j$ for $j = 1$ or $j = 2$. Without loss of generality we suppose $\rho(m_1 \oplus m_2) = m_1$. Then $\rho(m_1 \oplus m_2) = 0$ implies that $m_1 = 0$. So suppose that $0 \oplus m_2$ is of homogeneous degree.

The C^*-algebra $\Gamma(\mathbb{C}\mathrm{l}(G))^G = M_{2^{(k-1)/2}}(\mathbb{C}) \oplus M_{2^{(k-1)/2}}(\mathbb{C})$ is $\mathbb{Z}_2$-graded by

$$(\Gamma(\mathbb{C}\mathrm{l}(G))^G)^j = \{A_j \oplus A_j + A_{j+1} \oplus (-A_{j+1}) : A_j \in M_{2^{(k-1)/2}}(\mathbb{C})^j\}.$$

Hence if $0 \oplus m_2$ is of homogeneous degree $j \in \mathbb{Z}_2$, then

$$0 \oplus m_2 = A \oplus A + B \oplus (-B) = (A + B) \oplus (A - B),$$

where $A, B \in M_{2^{(k-1)/2}}(\mathbb{C})$ are of homogeneous degree j and $j+1$ respectively. But this implies that $A = -B$, which means that A and B are of the same homogeneous degree. This can only be true if $A = B = 0$. Hence $0 \oplus m_2 = 0$, and therefore $\widetilde{\rho}$ is injective. $\square$

Since $\widetilde{\rho}$ is an isomorphism onto $\mathrm{End}_{\mathfrak{C}}(\Gamma(\mathbf{\$}_G)^G \widehat{\otimes} \mathfrak{C}) = \mathrm{End}^0_{\mathfrak{C}}(\Gamma(\mathbf{\$}_G)^G \widehat{\otimes} \mathfrak{C})$, $\Gamma(\mathbf{\$}_G)^G \widehat{\otimes} \mathfrak{C}$ is also left Hilbert $\Gamma(\mathbb{C}\mathrm{l}(G))^G$-module, where the left inner product is defined by

$$\widetilde{\rho}\big({}_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(w_1|w_2)\big) w_3 = w_1 (w_2|w_3)_{\mathfrak{C}}.$$

Hence the conjugate module $(\Gamma(\mathbf{\$}_G)^G \widehat{\otimes} \mathfrak{C})^*$ (Definition 1.9) is a right Hilbert $\Gamma(\mathbb{C}\mathrm{l}(G))^G$-module.

The fixed point algebra A^G is a $\mathbb{Z}_2$-graded right Hilbert module over itself, and left multiplication on itself defines a $\mathbb{Z}_2$-graded $*$-homomorphism $A^G \to \mathrm{End}_{A^G}(A^G)$.

The external tensor product $A^G\widehat{\otimes}(\Gamma(\mathbf{\$}_G)^G\widehat{\otimes}\mathfrak{C})^*$ of A^G and $(\Gamma(\mathbf{\$}_G)^G\widehat{\otimes}\mathfrak{C})^*$ is a $\mathbb{Z}_2$-graded right Hilbert $A^G\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(G))^G$-module, which carries a representation $A^G\widehat{\otimes}\mathfrak{C} \to \mathrm{End}_{A^G\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(G))^G}(A^G\widehat{\otimes}(\Gamma(\mathbf{\$}_G)^G\widehat{\otimes}\mathfrak{C})^*)$, as in Definition 1.13. This representation is an isomorphism onto the compact endomorphisms (which is all the endomorphisms in this case), so the triple $(A^G\widehat{\otimes}\mathfrak{C}, (A^G\widehat{\otimes}(\Gamma(\mathbf{\$}_G)^G\widehat{\otimes}\mathfrak{C})^*)_{A^G\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(G))^G}, 0)$ is an (unbounded) equivariant Kasparov $A^G\widehat{\otimes}\mathfrak{C}$-$A^G\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(G))^G$-module, with all actions of G trivial. We call this Kasparov module the **middle module**.

2.3 The right-hand module

To define the right-hand module we require greater compatibility between the action α of G on A and $\mathcal{A} \subset A$ than we have assumed so far. We say that $\mathcal{A}$ is **α-compatible** if

$$\mathcal{A}_\chi := \mathcal{A} \cap A_\chi \text{ is dense in } A_\chi \text{ for all } \chi \in \mathrm{Char}(G).$$

Example 2.11. Suppose $\mathcal{A}$ is complete in some finer topology (for example, if $\mathcal{A}$ is closed in the Lipschitz norm $\|a\|_{\mathcal{D}} = \|a\| + \|[\mathcal{D}, a]\|$, [33], or if $\mathcal{A}$ is a Fréchet algebra), and the action α restricts to a strongly continuous action on $\mathcal{A}$ in this topology. Then the spectral projections Φ_k of (2.1) preserve $\mathcal{A} \subset A \subset X$, which implies that $\mathcal{A}_\chi = \Phi_\chi(\mathcal{A})$ is dense in A_χ for all χ and so $\mathcal{A}$ is α-compatible.

Definition 2.12. For each $\chi \in \mathrm{Char}(G)$, let $\mathcal{H}_\chi = \{\xi \in \mathcal{H} : V_g\xi = \chi(g)\xi\}$ be the spectral subspace corresponding to χ, and define an operator

$$\mathcal{D}_\chi : \mathrm{dom}(\mathcal{D}) \cap \mathcal{H}_\chi \subset \mathcal{H}_\chi \to \mathcal{H}_\chi, \quad \mathcal{D}_\chi\xi := \mathcal{D}\xi.$$

The Hilbert space $\mathcal{H}_\chi$ inherits the $\mathbb{Z}_2$-grading of $\mathcal{H}$.

Lemma 2.13. *Suppose that $\mathcal{A}$ is α-compatible. Let $\mathcal{A}^G$ be the fixed point algebra of $\mathcal{A}$. Then for each $\chi \in \mathrm{Char}(G)$, $(\mathcal{A}^G, \mathcal{H}_\chi, \mathcal{D}_\chi)$ is an even equivariant spectral triple for A^G, where $\mathcal{H}_\chi$ inherits the action of G on $\mathcal{H}$.*

Proof. Since G is represented on $\mathcal{H}$ unitarily, there is an orthogonal decomposition $\mathcal{H} = \bigoplus_{\chi\in\mathrm{Char}(G)} \mathcal{H}_\chi$. The density of $\mathrm{dom}(\mathcal{D})$ in $\mathcal{H}$ thus implies that $\mathrm{dom}(\mathcal{D}_\chi)$ is dense in $\mathcal{H}_\chi$ for all $\chi \in \mathrm{Char}(G)$.

The operator $(1 + \mathcal{D}^2)^{-1/2} \in B(\mathcal{H})$ is self-adjoint, and since $\mathcal{D}$ commutes with the action of G, so too does $(1 + \mathcal{D}^2)^{-1/2}$. Hence $(1 + \mathcal{D}^2)^{-1/2}|_{\mathcal{H}_\chi}$ is a bounded self-adjoint operator on $\mathcal{H}_\chi$, and $(1 + \mathcal{D}^2)^{-1/2}|_{\mathcal{H}_\chi} = (1 + \mathcal{D}_\chi^2)^{-1/2}$ for all $\chi \in \mathrm{Char}(G)$. Hence

$$F_\chi := \mathcal{D}(1 + \mathcal{D}^2)^{-1/2}|_{\mathcal{H}_\chi} = \mathcal{D}_\chi(1 + \mathcal{D}_\chi^2)^{-1/2}$$

is also a bounded self-adjoint operator on $\mathcal{H}_\chi$. Since $\mathcal{D}_\chi = F_\chi(1-F_\chi^2)^{-1/2}$, it follows from [29, Thm. 10.4] that $\mathcal{D}_\chi$ is a self-adjoint operator on $\mathcal{H}_\chi$.

Since $[\mathcal{D}_\chi, a] = [\mathcal{D}, a]|_{\mathcal{H}_\chi}$ and $a(1+\mathcal{D}_\chi^2)^{-1/2} = a(1+\mathcal{D}^2)^{-1/2}|_{\mathcal{H}_\chi}$ for all $a \in \mathcal{A}^G$, it follows that $(\mathcal{A}^G, \mathcal{H}_\chi, \mathcal{D}_\chi)$ satisfies the conditions of Definition 1.18, and hence $(\mathcal{A}^G, \mathcal{H}_\chi, \mathcal{D}_\chi)$ is an even equivariant spectral triple. □

We wish to use the operator $\mathcal{D}_\zeta$ to construct our final Kasparov module, for some fixed $\zeta \in \mathrm{Char}(G)$. We require a representation of the Clifford algebra $\Gamma(\mathbb{C}\mathrm{l}(G))^G$ on $\mathcal{H}_\zeta$, which will define a representation of $A^G \widehat{\otimes} \Gamma(\mathbb{C}\mathrm{l}(G))^G$ on $\mathcal{H}_\zeta$. This Clifford representation allows us to account for the extra spinor dimensions appearing in the left-hand module. The conditions we impose below on the representation and the character ζ ensure that we obtain an even spectral triple $(\mathcal{A}^G \widehat{\otimes} \Gamma(\mathbb{C}\mathrm{l}(G))^G, \mathcal{H}_\zeta, \mathcal{D}_\zeta)$ for $A^G \widehat{\otimes} \Gamma(\mathbb{C}\mathrm{l}(G))^G$, and in addition that Kucerovsky's connection criterion is satisfied (Proposition 3.6).

Simple examples show that $\mathcal{H}_\chi$ may be trivial for any given $\chi \in \mathrm{Char}(G)$, including the trivial character $\chi(g) = 1$.[2] We therefore impose the condition $\overline{A\mathcal{H}_\zeta} = \mathcal{H}$ on the character ζ in order to construct the right-hand module. Choosing ζ in this way allows us to recover the original Hilbert space $\mathcal{H}$ from the three modules.

Remark. Even if $\overline{A\mathcal{H}_\chi} = \mathcal{H}$ for all $\chi \in \mathrm{Char}(G)$, when we come to check whether factorisation has been achieved, the positivity criterion (Theorem 3.4) may be satisfied for some choices of ζ but not for others. For an example see §6.

Definition 2.14. Suppose that $\mathcal{A}$ is α-compatible. Let $\zeta \in \mathrm{Char}(G)$ be such that $\overline{A\mathcal{H}_\zeta} = \mathcal{H}$, and let $\eta : \Gamma(\mathbb{C}\mathrm{l}(G))^G \to B(\mathcal{H})$ be a unital, equivariant $\mathbb{Z}_2$-graded $*$-homomorphism such that

1) $[\eta(s), a]_\pm = 0$ for all $s \in \Gamma(\mathbb{C}\mathrm{l}(G))^G$ and $a \in A^G$, and

2) $a\eta(s) \cdot \mathrm{dom}(\mathcal{D}_\zeta) \subset \mathrm{dom}(\mathcal{D})$ and $[\mathcal{D}, \eta(s)]_\pm a P_\zeta$ is bounded on $\mathcal{H}$ for all $a \in \oplus_\chi \mathcal{A}_\chi$ and $s \in \Gamma(\mathbb{C}\mathrm{l}(G))^G$, where $P_\zeta \in B(\mathcal{H})$ is the orthogonal projection onto $\mathcal{H}_\zeta$. We call η the **Clifford representation**.

Given such a character ζ and a Clifford representation η, we define a $\mathbb{Z}_2$-graded $*$-homomorphism $A^G \widehat{\otimes} \Gamma(\mathbb{C}\mathrm{l}(G))^G \to B(\mathcal{H}_\zeta)$ by $(a \widehat{\otimes} s) \cdot \xi := a\eta(s)\xi$. If $\mathcal{A}$ is α-compatible, the conditions on η and Lemma 2.13 ensure that $(\mathcal{A}^G \widehat{\otimes} \Gamma(\mathbb{C}\mathrm{l}(G))^G, \mathcal{H}_\zeta, \mathcal{D}_\zeta)$ is an even equivariant spectral triple for A^G, which we call the **right-hand module**.

[2] A very basic example is the representation $g \mapsto \chi(g)$, for which all spectral subspaces besides $\mathcal{H}_\chi$ are trivial. A more complicated example is the lift to the spinor bundle $\mathbf{\$}_{S^2}$ of the "double" action $t \cdot (\theta, \phi) = (\theta, \phi + 4\pi t)$ of the circle on the 2-sphere, which is mentioned in §6. The representation $V : \mathbb{T} \to U(L^2(\mathbf{\$}_{S^2}))$ is (in one trivialisation of $\mathbf{\$}_{S^2}$)

$$V_t \begin{pmatrix} f(\theta,\phi) \\ g(\theta,\phi) \end{pmatrix} = \begin{pmatrix} e^{2\pi i t} f(\theta, \phi - 4\pi t) \\ e^{-2\pi i t} g(\theta, \phi - 4\pi t) \end{pmatrix}.$$

One can see that the spectral subspace $L^2(\mathbf{\$}_{S^2})_\ell$ corresponding to $\chi_\ell(t) = e^{2\pi i t \ell}$ is non-trivial if and only if ℓ is odd, so in particular the fixed point subspace is trivial.

Remark. Condition 2) of Definition 2.14 is stronger than necessary to ensure that we obtain an equivariant spectral triple for $A^G \widehat{\otimes} \Gamma(\mathbb{C}\mathrm{l}(G))^G$, but this stronger condition is sufficient to prove that Kucerovsky's connection criteria is satisfied.

Chapter 3

The Kasparov product of the KK-cycles

Recall that G is a compact abelian Lie group, equipped with the normalised Haar measure and a trivial spinor bundle $\mathbf{\$}_G$, and $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is an even G-equivariant spectral triple for a $\mathbb{Z}_2$-graded separable G-algebra A. Let $\zeta \in \mathrm{Char}(G)$ and $\eta : \Gamma(\mathbb{C}\mathrm{l}(G))^G \to B(\mathcal{H})$ satisfy the conditions of Definition 2.14, so in particular $\mathcal{A}$ is α-compatible.

3.1 Sufficient criteria for factorisation

The next result can be proved with a straightforward application of Theorem 1.22.

Proposition 3.1. *The Kasparov product of the left-hand and middle modules is the class in $KK_G(A, A^G\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(G))^G)$ of the unbounded Kasparov module*

$$\Big(\oplus_\chi A_\chi, \big(E_1 \widehat{\otimes}_{A^G\widehat{\otimes}\mathfrak{C}} (A^G\widehat{\otimes}(\Gamma(\mathbf{\$}_G)^G\widehat{\otimes}\mathfrak{C})^*)\big)_{A^G\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(G))^G}, \mathcal{D}_1\widehat{\otimes}1\Big).$$

To determine whether the Kasparov product of the left-hand, middle and right-hand modules (which is some class in $KK_G(A, \mathbb{C})$) is represented by $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, we first construct an isomorphism

$$\Psi : \big(E_1 \widehat{\otimes}_{A^G\widehat{\otimes}\mathfrak{C}} (A^G\widehat{\otimes}(\Gamma(\mathbf{\$}_G)^G\widehat{\otimes}\mathfrak{C})^*)\big)\widehat{\otimes}_{A^G\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(G))^G}\mathcal{H}_\zeta \to \mathcal{H},$$

which will allow us to use Kucerovsky's criteria (Theorem 1.22). We would like to define the map Ψ on elements of homogeneous degree by

$$\Psi\Big(\big((y\widehat{\otimes}u)\widehat{\otimes}(a\widehat{\otimes}\overline{w})\big)\widehat{\otimes}\xi\Big) := (-1)^{\deg u\cdot\deg a} \sum_{\chi\in\mathrm{Char}(G)} \Phi_\chi(y)a\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(\chi^{-1}p_{\chi^{-1}}u|w)\big)\xi, \tag{3.1}$$

where $p_\chi \in \mathrm{End}_{\mathfrak{C}}(L^2(\$_G)\widehat{\otimes}\mathfrak{C})$ and $\Phi_\chi \in \mathrm{End}_{A^G}(X)$ are the spectral subspace projections of Definition 2.4 and Lemma 2.3 respectively.

To see that Ψ is well-defined, even on homogeneous elements, we need to know that the sum over characters converges. This is established by the following lemma.

Lemma 3.2. *The map Ψ is a well-defined isometry.*

Proof. For $i = 1, 2$, let

$$\big((y_i\widehat{\otimes}u_i)\widehat{\otimes}(a_i\widehat{\otimes}\overline{w_i})\big)\widehat{\otimes}\xi_i \in \big(E_1\widehat{\otimes}_{A^G\widehat{\otimes}\mathfrak{C}}(A^G\widehat{\otimes}(\Gamma(\$_G)^G\widehat{\otimes}\mathfrak{C})^*)\big)\widehat{\otimes}_{A^G\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(G))^G}\mathcal{H}_\zeta$$

be elements of homogeneous degree. Then using Lemma 2.5,

$$\begin{aligned}
&\big\langle\big((y_1\widehat{\otimes}u_1)\widehat{\otimes}(a_1\widehat{\otimes}\overline{w_1})\big)\widehat{\otimes}\xi_1, \big((y_2\widehat{\otimes}u_2)\widehat{\otimes}(a_2\widehat{\otimes}\overline{w_2})\big)\widehat{\otimes}\xi_2\big\rangle \\
&= (-1)^{\deg u_1(\deg y_1+\deg y_2)+(\deg u_1+\deg u_2)\deg a_2+\deg w_1(\deg a_1+\deg y_1+\deg y_2+\deg a_2)} \\
&\times \sum_{\chi\in\mathrm{Char}(G)} \Big\langle \xi_1, a_1^*\Phi_\chi(y_1)^*\Phi_\chi(y_2)a_2\eta\big({}_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(w_1|w_2(p_{\chi^{-1}}u_2|p_{\chi^{-1}}u_1)_{\mathfrak{C}})\big)\xi_2\Big\rangle \\
&= (-1)^{\deg u_1\cdot\deg a_1+\deg u_2\cdot\deg a_2}\sum\Big\langle \Phi_\chi(y_1)a_1\eta\big({}_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(\chi^{-1}p_{\chi^{-1}}u_1|w_1)\big)\xi_1, \\
&\qquad\qquad \Phi_\chi(y_2)a_2\eta\big({}_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(\chi^{-1}p_{\chi^{-1}}u_2|w_2)\big)\xi_2\Big\rangle \\
&= \Big\langle \Psi\Big(\big((y_1\widehat{\otimes}u_1)\widehat{\otimes}(a_1\widehat{\otimes}\overline{w_1})\big)\widehat{\otimes}\xi_1\Big), \Psi\Big(\big((y_2\widehat{\otimes}u_2)\widehat{\otimes}(a_2\widehat{\otimes}\overline{w_2})\big)\widehat{\otimes}\xi_2\Big)\Big\rangle .
\end{aligned}$$

The penultimate line follows from

$$\begin{aligned}
&{}_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(w_1|\chi^{-1}p_{\chi^{-1}}u_1)_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(\chi^{-1}p_{\chi^{-1}}u_2|w_2) \\
&= {}_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(w_1|w_2(p_{\chi^{-1}}u_2|p_{\chi^{-1}}u_1)_{\mathfrak{C}}),
\end{aligned} \tag{3.2}$$

which in turn follows from $(\chi^{-1}p_{\chi^{-1}}u_2|\chi^{-1}p_{\chi^{-1}}u_1)_{\mathfrak{C}} = (p_{\chi^{-1}}u_2|p_{\chi^{-1}}u_1)_{\mathfrak{C}}$.

We have established that the sum $\sum\Phi_\chi(y)a\eta\big({}_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(\chi^{-1}p_{\chi^{-1}}u|w)\big)\xi$ converges. It only remains to check that Ψ is well-defined with respect to the balanced tensor products, which is a straightforward exercise. □

Proposition 3.3. *The map Ψ is a unitary, equivariant, $\mathbb{Z}_2$-graded, A-linear isomorphism. The inverse*

$$\Psi^{-1} : \mathcal{H} \to \big(E_1\widehat{\otimes}_{A^G\widehat{\otimes}\mathfrak{C}}(A^G\widehat{\otimes}(\Gamma(\$_G)^G\widehat{\otimes}\mathfrak{C})^*)\big)\widehat{\otimes}_{A^G\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(G))^G}\mathcal{H}_\zeta$$

is defined as follows. Let $(x_j)_{j=1}^n$ be a G-invariant global orthonormal frame for $\$_G$, and let $(\phi_\ell)_{\ell=1}^\infty$ be an approximate identity for A^G of homogeneous degree zero. For $\xi \in \mathcal{H}$,

choose sequences $(a_k)_{k=1}^{\infty} \subset A$ *and* $(\xi_k)_{k=1}^{\infty} \subset \mathcal{H}_\zeta$ *such that* $a_k\xi_k \to \xi$ *as* $k \to \infty$. *Then*

$$\Psi^{-1}(\xi) := \sum_{\chi\in\mathrm{Char}(G)} \sum_{j=1}^{n} \lim_{k\to\infty} \lim_{\ell\to\infty} \big((\Phi_\chi(a_k)\widehat{\otimes}(\chi x_j\widehat{\otimes}1))\widehat{\otimes}(\phi_\ell\widehat{\otimes}\overline{x_j\widehat{\otimes}1})\big)\widehat{\otimes}\xi_k.$$

Proof. It is immediate that Ψ is equivariant and $\mathbb{Z}_2$-graded, and Ψ is an isometry by Lemma 3.2. So it remains to show that (i) Ψ is A-linear, and (ii) Ψ^{-1} is an inverse for Ψ.

(i) Let $b \in A$. Then

$$\begin{aligned}
\Psi\Big(b\cdot\big((y\widehat{\otimes}u)\widehat{\otimes}(a\widehat{\otimes}\overline{w})\big)\widehat{\otimes}\xi\Big) &= \sum_{\mu\in\mathrm{Char}(G)} \Psi\Big(\big((\Phi_\mu(b)y\widehat{\otimes}\mu u)\widehat{\otimes}(a\widehat{\otimes}\overline{w})\big)\widehat{\otimes}\xi\Big)\\
&= (-1)^{\deg u\cdot\deg a} \sum_{\chi,\mu\in\mathrm{Char}(G)} \Phi_\chi(\Phi_\mu(b)y)a\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(\chi^{-1}p_{\chi^{-1}}\mu u|w)\big)\xi\\
&= (-1)^{\deg u\cdot\deg a} \sum_{\chi,\mu} \Phi_\mu(b)\Phi_\chi(y)a\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(\chi^{-1}p_{\chi^{-1}}u|w)\big)\xi\\
&= b\Psi\Big(\big((y\widehat{\otimes}u)\widehat{\otimes}(a\widehat{\otimes}\overline{w})\big)\widehat{\otimes}\xi\Big),
\end{aligned}$$

so Ψ is A-linear.

(ii) We first check that Ψ^{-1} is well-defined, which means checking that the limits exist and that the sum converges. Suppose $\xi \in \mathcal{H}$, and choose sequences $(a_k)_{k=1}^{\infty} \subset A$ and $(\xi_k)_{k=1}^{\infty} \subset \mathcal{H}_\zeta$ such that $a_k\xi_k \to \xi$ as $k \to \infty$, which exist since $\overline{A\mathcal{H}_\zeta} = \mathcal{H}$. Since $\sum_{j=1}^{n} {}_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(x_j\widehat{\otimes}1|x_j\widehat{\otimes}1) = 1$,

$$\begin{aligned}
&\Psi\left(\sum_{j=1}^{n}\big((\Phi_\chi(a_k)\widehat{\otimes}(\chi x_j\widehat{\otimes}1))\widehat{\otimes}(\phi_\ell\widehat{\otimes}\overline{x_j\widehat{\otimes}1})\big)\widehat{\otimes}\xi_k\right)\\
&= \sum_{j=1}^{n}\Phi_\chi(a_k)\phi_\ell\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(x_j\widehat{\otimes}1|x_j\widehat{\otimes}1)\big)\xi_k = \Phi_\chi(a_k)\phi_\ell\xi_k = P_{\chi\zeta}(a_k\phi_\ell\xi_k),
\end{aligned}$$

where $P_{\chi\zeta} \in B(\mathcal{H})$ is the orthogonal projection onto $\mathcal{H}_{\chi\zeta}$, and

$$\lim_{k\to\infty}\lim_{\ell\to\infty} P_{\chi\zeta}(a_k\phi_\ell\xi_k) = \lim_{k\to\infty} P_{\chi\zeta}(a_k\xi_k) = P_{\chi\zeta}\xi.$$

Since Ψ is an isometry, this establishes that the limits exist. Moreover,

$$\sum_{\chi\in\mathrm{Char}(G)} P_{\chi\zeta}\xi = \sum_{\chi\in\mathrm{Char}(G)} P_\chi\xi = \xi,$$

so the sum converges. This calculation also shows that Ψ^{-1} is a right inverse for Ψ, so that Ψ is surjective. Since Ψ is injective, it follows that Ψ is invertible with inverse Ψ^{-1}. □

Now that we have the isomorphism Ψ, we can use Kucerovsky's criteria (Theorem 1.22), to determine if $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ represents the Kasparov product of the left-hand, middle and right-hand modules. More precisely, $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is unitarily equivalent as an unbounded equivariant Kasparov module to

$$\Big(\mathcal{A}, \big(E_1 \widehat{\otimes}_{A^G \widehat{\otimes} \mathfrak{C}} (A^G \widehat{\otimes} (\Gamma(\mathbf{\$}_G)^G \widehat{\otimes} \mathfrak{C})^*)\big) \widehat{\otimes}_{A^G \widehat{\otimes} \Gamma(\mathbb{C}\mathrm{l}(G))^G} \mathcal{H}_\zeta, \Psi^{-1} \circ \mathcal{D} \circ \Psi\Big),$$

and now Kucerovsky's criteria may be applied to determine whether factorisation has been achieved.

Theorem 3.4 (The criterion for factorisation)**.** *Let $\zeta \in \mathrm{Char}(G)$ and the Clifford representation $\eta : \Gamma(\mathbb{C}\mathrm{l}(G))^G \to B(\mathcal{H})$ satisfy the conditions of Definition 2.14, so in particular $\mathcal{A}$ is α-compatible. Let $(x_j)_{j=1}^n$ be a G-invariant global orthonormal frame for $\mathbf{\$}_G$, and for each $\chi \in \mathrm{Char}(G)$, let $P_\chi \in B(\mathcal{H})$ be the orthogonal projection onto $\mathcal{H}_\chi$. If there is some $R \in \mathbb{R}$ such that*

$$\begin{aligned}\sum_{j=1}^n \Big(&\Big\langle \mathcal{D}\xi, \eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(\chi^{-1}\mathcal{D}_G(\chi x_j)\widehat{\otimes}\mathfrak{c}|x_j\widehat{\otimes}1)\big)P_{\chi\zeta}\xi\Big\rangle \\ &+ \Big\langle \eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(\chi^{-1}\mathcal{D}_G(\chi x_j)\widehat{\otimes}\mathfrak{c}|x_j\widehat{\otimes}1)\big)P_{\chi\zeta}\xi, \mathcal{D}\xi\Big\rangle\Big) \geq R\|\xi\|^2 \end{aligned} \tag{3.3}$$

for all $\chi \in \mathrm{Char}(G)$, $\xi \in \mathrm{dom}(\mathcal{D})$, then $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ represents the Kasparov product of left-hand, middle and right-hand modules.

Theorem 3.4 is proved by showing that Kucerovsky's domain and connection conditions hold under the existing assumptions. The remaining positivity condition is precisely the condition (3.3).

We recall the operators T_e appearing in the connection criterion of Theorem 1.22, adapted to this setting.

Definition 3.5. For $e \in E_1 \widehat{\otimes}_{A^G \widehat{\otimes} \mathfrak{C}} (A^G \widehat{\otimes} (\Gamma(\mathbf{\$}_G)^G \widehat{\otimes} \mathfrak{C})^*)$, define

$$T_e : \mathcal{H}_\zeta \to \big(E_1 \widehat{\otimes}_{A^G \widehat{\otimes} \mathfrak{C}} (A^G \widehat{\otimes} (\Gamma(\mathbf{\$}_G)^G \widehat{\otimes} \mathfrak{C})^*)\big) \widehat{\otimes}_{A^G \widehat{\otimes} \Gamma(\mathbb{C}\mathrm{l}(G))^G} \mathcal{H}_\zeta$$

by $T_e f = e \widehat{\otimes} \xi$. The operator T_e is adjointable, and $T_e^*(\widetilde{e \widehat{\otimes}} \xi) = (e|\widetilde{e})_{A^G \widehat{\otimes} \Gamma(\mathbb{C}\mathrm{l}(G))^G} \xi$.

Proposition 3.6 (The connection criterion)**.** *The graded commutators*

$$\left[\begin{pmatrix} \Psi^{-1} \circ \mathcal{D} \circ \Psi & 0 \\ 0 & \mathcal{D}_\zeta \end{pmatrix}, \begin{pmatrix} 0 & T_e \\ T_e^* & 0 \end{pmatrix}\right]_\pm$$

are bounded for all $e \in Y$, where $Y \subset E_1 \widehat{\otimes}_{A^G \widehat{\otimes} \mathfrak{C}} (A^G \widehat{\otimes} (\Gamma(\mathbf{\$}_G)^G \widehat{\otimes} \mathfrak{C})^*)$ is the dense

subspace

$$Y := \operatorname{span}\left\{(z\widehat{\otimes}s)\widehat{\otimes}(a\widehat{\otimes}\overline{w}) \in E_1\widehat{\otimes}_{A^G\widehat{\otimes}\mathfrak{C}}(A^G\widehat{\otimes}(\Gamma(\mathbf{\$}_G)^G\widehat{\otimes}\mathfrak{C})^*) : z, a \in \oplus_\chi \mathcal{A}_\chi\right\}.$$

Proof. Let $e = (z\widehat{\otimes}s)\widehat{\otimes}(a\widehat{\otimes}\overline{w}) \in Y$, let $\big((y\widehat{\otimes}t)\widehat{\otimes}(b\widehat{\otimes}\overline{v})\big)\widehat{\otimes}\xi \in \operatorname{dom}(\Psi^{-1}\circ\mathcal{D}\circ\Psi)$ and let $\psi \in \operatorname{dom}(\mathcal{D}_\zeta)$, each of homogeneous degree. Then the upper entry of the column vector

$$\left[\begin{pmatrix}\Psi^{-1}\circ\mathcal{D}\circ\Psi & 0\\ 0 & \mathcal{D}_\zeta\end{pmatrix}, \begin{pmatrix}0 & T_e\\ T_e^* & 0\end{pmatrix}\right]_\pm \begin{pmatrix}\big((y\widehat{\otimes}t)\widehat{\otimes}(b\widehat{\otimes}\overline{v})\big)\widehat{\otimes}\xi\\ \psi\end{pmatrix}$$

is

$$\begin{aligned}
&\Psi^{-1}\circ\mathcal{D}\circ\Psi\circ T_e\psi - (-1)^{\deg z+\deg s+\deg a+\deg w}T_e\circ\mathcal{D}_\zeta\psi\\
&= \Psi^{-1}\circ\mathcal{D}\circ\Psi\Big(\big((z\widehat{\otimes}s)\widehat{\otimes}(a\widehat{\otimes}\overline{w})\big)\widehat{\otimes}\psi\Big)\\
&\quad - (-1)^{\deg z+\deg s+\deg a+\deg w}\big((z\widehat{\otimes}s)\widehat{\otimes}(a\widehat{\otimes}\overline{w})\big)\widehat{\otimes}\mathcal{D}_\zeta\psi\\
&= (-1)^{\deg s\cdot\deg a}\Psi^{-1}\circ\mathcal{D}\sum_{\chi\in\operatorname{Char}(G)}\Phi_\chi(z)a\eta\big(_{\Gamma(\mathbb{Cl}(G))^G}(\chi^{-1}p_{\chi^{-1}}s|w)\big)\psi\\
&\quad - (-1)^{\deg z+\deg s+\deg a+\deg w+\deg s\cdot\deg a}\\
&\qquad\qquad\times\Psi^{-1}\sum_{\chi\in\operatorname{Char}(G)}\Phi_\chi(y)a\eta\big(_{\Gamma(\mathbb{Cl}(G))^G}(\chi^{-1}p_{\chi^{-1}}s|w)\big)\mathcal{D}_\zeta\psi\\
&= (-1)^{\deg s\cdot\deg a}\Psi^{-1}\sum_{\chi\in\operatorname{Char}(G)}[\mathcal{D},\Phi_\chi(z)a\eta\big(_{\Gamma(\mathbb{Cl}(G))^G}(\chi^{-1}p_{\chi^{-1}}s|w)\big)]_\pm\psi,
\end{aligned}$$

and we estimate

$$\begin{aligned}
&\left\|(-1)^{\deg s\cdot\deg a}\Psi^{-1}\sum_{\chi\in\operatorname{Char}(G)}[\mathcal{D},\Phi_\chi(z)a\eta\big(_{\Gamma(\mathbb{Cl}(G))^G}(\chi^{-1}p_{\chi^{-1}}s|w)\big)]_\pm\psi\right\|^2\\
&= \sum_{\chi\in\operatorname{Char}(G)}\left\|[\mathcal{D},\Phi_\chi(z)a\eta\big(_{\Gamma(\mathbb{Cl}(G))^G}(\chi^{-1}p_{\chi^{-1}}s|w)\big)]_\pm P_\zeta\psi\right\|^2\\
&\le \|\psi\|^2\sum_{\chi\in\operatorname{Char}(G)}\left\|[\mathcal{D},\Phi_\chi(z)a\eta\big(_{\Gamma(\mathbb{Cl}(G))^G}(\chi^{-1}p_{\chi^{-1}}s|w)\big)]_\pm P_\zeta\right\|^2,
\end{aligned}$$

where the sum converges since $z \in \oplus_\chi\mathcal{A}_\chi$. Hence the upper entry is a bounded function of ψ. For the lower entry we have

$$\begin{aligned}
&\mathcal{D}_\zeta\circ T_e^*\big(\big((y\widehat{\otimes}t)\widehat{\otimes}(b\widehat{\otimes}v)\big)\widehat{\otimes}\xi\big) = \mathcal{D}_\zeta\big(\big((z\widehat{\otimes}s)\widehat{\otimes}(a\widehat{\otimes}\overline{w})\big|(y\widehat{\otimes}t)\widehat{\otimes}(b\widehat{\otimes}v)\big)_{A^G\widehat{\otimes}\Gamma(\mathbb{Cl}(G))^G}\xi\big)\\
&= (-1)^{\deg s\cdot(\deg z+\deg y)+\deg w\cdot(\deg a+\deg z+\deg y+\deg b)+\deg b\cdot(\deg s+\deg t)}\\
&\times\sum_\chi\mathcal{D}_\zeta\Big(a^*\Phi_\chi(z)^*\Phi_\chi(y)b\eta\big(_{\Gamma(\mathbb{Cl}(G))^G}(\chi^{-1}p_{\chi^{-1}}s|w)_{\Gamma(\mathbb{Cl}(G))^G}(\chi^{-1}p_{\chi^{-1}}t|v)\big)\xi\Big)
\end{aligned}$$

using Lemma 2.5 and Equation (3.2). Let $(x_j)_{j=1}^n$ be a G-invariant, global orthonormal frame for $\mathbf{\$}_G$, and let $(\phi_\ell)_{\ell=1}^\infty$ be an approximate identity for A^G of homogeneous degree zero. For each $\chi \in \mathrm{Char}(G)$, let $(c_k^\chi)_{k=1}^\infty \subset A$ and $(\sigma_k^\chi)_{k=1}^\infty \subset \mathcal{H}_\zeta$ be sequences such that $\lim_{k\to\infty} c_k^\chi \sigma_k^\chi = \mathcal{D}\big(\Phi_\chi(y) b\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(\chi^{-1} p_{\chi^{-1}} t|v)\big)\xi\big)$. Then

$$\begin{aligned}
T_e^* \circ \Psi^{-1} \circ \mathcal{D} \circ \Psi\big(\big((y\widehat{\otimes}t)\widehat{\otimes}(b\widehat{\otimes}v)\big)\widehat{\otimes}\xi\big) &= (-1)^{\deg t\cdot\deg b+\deg s\cdot\deg z+\deg w\cdot(\deg a+\deg z)} \\
&\quad\times \sum_{\chi,\nu}\sum_{j=1}^n \lim_{k\to\infty} a^*\Phi_\nu(z)^*\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(w|(x_j\widehat{\otimes}1)\cdot(\nu x_j\widehat{\otimes}1|p_{\nu^{-1}}s)_{\mathfrak{C}})\big)\Phi_\nu(c_k^\chi)\sigma_k^\chi \\
&= (-1)^{\deg t\cdot\deg b+\deg s\cdot\deg z+\deg w\cdot(\deg a+\deg z)}\sum_\chi\sum_{j=1}^n a^*\Phi_\chi(z)^* \\
&\quad\times \eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(w|(x_j\widehat{\otimes}1)\cdot(\chi x_j\widehat{\otimes}1|p_{\chi^{-1}}s)_{\mathfrak{C}})\big)\mathcal{D}\Big(\Phi_\chi(y)b\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(\chi^{-1}p_{\chi^{-1}}t|v)\big)\xi\Big)
\end{aligned}$$

where we have used

$$\begin{aligned}
\lim_{k\to\infty}\Phi_\nu(c_k^\chi)\sigma_k^\chi = \lim_{k\to\infty} P_{\nu\zeta}c_k^\chi\sigma_k^\chi &= P_{\nu\zeta}\mathcal{D}\big(\Phi_\chi(y)b\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(\chi^{-1}p_{\chi^{-1}}t|v)\big)\xi\big) \\
&= \delta_{\nu,\chi}\mathcal{D}\big(\Phi_\chi(y)b\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(\chi^{-1}p_{\chi^{-1}}t|v)\big)\xi\big).
\end{aligned}$$

Since $\chi^{-1}p_{\chi^{-1}}s = \sum_{j=1}^n (x_j\widehat{\otimes}1)\cdot(\chi x_j\widehat{\otimes}1|p_{\chi^{-1}}s)_{\mathfrak{C}}$,

$$\begin{aligned}
&T_e^* \circ \Psi^{-1} \circ \mathcal{D} \circ \Psi\big(\big((y\widehat{\otimes}t)\widehat{\otimes}(b\widehat{\otimes}v)\big)\widehat{\otimes}\xi\big) = (-1)^{\deg t\cdot\deg b+\deg s\cdot\deg z+\deg w\cdot(\deg a+\deg z)} \\
&\times\sum_\chi a^*\Phi_\chi(z)^*\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(w|\chi^{-1}p_{\chi^{-1}}s)\big)\mathcal{D}\Big(\Phi_\chi(y)b\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(\chi^{-1}p_{\chi^{-1}}t|v)\big)\xi\Big).
\end{aligned}$$

Hence the lower entry is

$$\begin{aligned}
&\mathcal{D}_\zeta \circ T_e^*\big(\big((y\widehat{\otimes}t)\widehat{\otimes}(b\widehat{\otimes}v)\big)\widehat{\otimes}\xi\big) \\
&\quad - (-1)^{\deg z+\deg s+\deg a+\deg w}T_e^* \circ \Psi^{-1} \circ \mathcal{D} \circ \Psi\big(\big((y\widehat{\otimes}t)\widehat{\otimes}(b\widehat{\otimes}v)\big)\widehat{\otimes}\xi\big) \\
&= (-1)^{\deg s\cdot(\deg z+\deg y)+\deg w\cdot(\deg a+\deg z+\deg y+\deg b)+\deg b\cdot(\deg s+\deg t)} \\
&\times\sum_\chi \mathcal{D}_\zeta\Big(a^*\Phi_\chi(z)^*\Phi_\chi(y)b\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(\chi^{-1}p_{\chi^{-1}}s|w)_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(\chi^{-1}p_{\chi^{-1}}t|v)\big)\xi\Big) \\
&- (-1)^{\deg z+\deg s+\deg a+\deg w+\deg t\cdot\deg b+\deg s\cdot\deg z+\deg w\cdot(\deg a+\deg z)}\times \\
&\sum_\chi a^*\Phi_\chi(z)^*\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(w|\chi^{-1}p_{\chi^{-1}}s)\big)\mathcal{D}\Big(\Phi_\chi(y)b\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(\chi^{-1}p_{\chi^{-1}}t|v)\big)\xi\Big) \\
&= (-1)^{\deg t\cdot\deg b+\deg s\cdot\deg z+\deg w\cdot(\deg a+\deg z)} \\
&\qquad\times\sum_\chi[\mathcal{D}, a^*\Phi_\chi(z)^*\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(w|\chi p_\chi s)\big)]_\pm\Phi_\chi(y)b\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(\chi p_\chi t|v)\big)\xi.
\end{aligned}$$

Since Ψ is unitary, the sum $\sum_{\chi\in\mathrm{Char}(G)}\Phi_\chi(y)a\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(\chi^{-1}p_{\chi^{-1}}t|w)\big)\xi$ converges, so

$$\begin{aligned}
&\sum_{\chi\in\mathrm{Char}(G)}[\mathcal{D},a^*\Phi_\chi(z)^*\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(w|\chi^{-1}p_{\chi^{-1}}s)\big)]_\pm\Phi_\chi(y)b\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(\chi^{-1}p_{\chi^{-1}}t|v)\big)\xi\\
&=\left(\sum_\nu P_\zeta[\mathcal{D},a^*\Phi_\nu(z)^*\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(w|\nu^{-1}p_{\nu^{-1}}s)\big)]_\pm\right)\\
&\qquad\qquad\times\left(\sum_\chi\Phi_\chi(y)b\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(\chi^{-1}p_{\chi^{-1}}t|v)\big)\xi\right).
\end{aligned}$$

Thus the norm of the lower entry is bounded by

$$\begin{aligned}
&\left\|\sum_\nu P_\zeta[\mathcal{D},a^*\Phi_\nu(z)^*\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(w|\nu^{-1}p_{\nu^{-1}}s)\big)]_\pm\right\|\\
&\qquad\qquad\times\sum_\chi\left\|\Phi_\chi(y)b\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(\chi^{-1}p_{\chi^{-1}}t|v)\big)\xi\right\|\\
&=\left\|\sum_\nu P_\zeta[\mathcal{D},a^*\Phi_\nu(z)^*\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(w|\nu^{-1}p_{\nu^{-1}}s)\big)]_\pm\right\|\left\|\big((y\widehat{\otimes}t)\widehat{\otimes}(b\widehat{\otimes}\overline{v})\big)\widehat{\otimes}\xi\right\|,
\end{aligned}$$

since Ψ is unitary. Since $\sum_\nu P_\zeta[\mathcal{D},a^*\Phi_\nu(z)^*\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(w|\nu^{-1}p_{\nu^{-1}}s)\big)]_\pm$ is a finite sum of bounded operators, it is bounded. Therefore the lower entry is a bounded function of $\big((y\widehat{\otimes}t)\widehat{\otimes}(b\widehat{\otimes}\overline{v})\big)\widehat{\otimes}\xi$. □

Lemma 3.7. *Let $(x_j)_{j=1}^n$ be a G-invariant global orthonormal frame for $\mathbf{\$}_G$, let $\mathcal{D}_G$ be the Dirac operator on $\mathbf{\$}_G$, and let $P_\chi\in B(\mathcal{H})$ be the projection onto $\mathcal{H}_\chi$ for $\chi\in$ $\mathrm{Char}(G)$. Then*

$$\Psi\circ(\mathcal{D}_1\widehat{\otimes}1)\widehat{\otimes}1\circ\Psi^{-1}=\sum_{\chi\in\mathrm{Char}(G)}\sum_{j=1}^n\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(\chi^{-1}\mathcal{D}_G(\chi x_j)\widehat{\otimes}\mathfrak{c}|x_j\widehat{\otimes}1)\big)P_{\chi\zeta}.$$

Proof. Let $\mathfrak{c}$ be the generator of $\mathfrak{C}$, let $\xi\in\mathrm{dom}(\Psi\circ(\mathcal{D}_1\widehat{\otimes}1)\widehat{\otimes}1\circ\Psi^{-1})$, and choose sequences $(a_k)_{k=1}^\infty\subset A$ and $(\xi_k)_{k=1}^\infty\subset\mathcal{H}_\zeta$ such that $a_k\xi_k\to\xi$ as $k\to\infty$. Then

$$\begin{aligned}
&\Psi\circ(\mathcal{D}_1\widehat{\otimes}1)\widehat{\otimes}1\circ\Psi^{-1}\xi\\
&=\Psi\sum_\chi\sum_{j=1}^n\lim_{k\to\infty}\lim_{\ell\to\infty}(-1)^{\deg a_k}\big((\Phi_\chi(a_k)\widehat{\otimes}(\mathcal{D}_G(\chi x_j)\widehat{\otimes}\mathfrak{c})\big)\widehat{\otimes}(\phi_\ell\widehat{\otimes}\overline{x_j\widehat{\otimes}1})\big)\widehat{\otimes}\xi_k\\
&=\sum_\chi\sum_{j=1}^n\lim_{k\to\infty}\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(\chi^{-1}\mathcal{D}_G(\chi x_j)\widehat{\otimes}\mathfrak{c}|x_j\widehat{\otimes}1)\big)\Phi_\chi(a_k)\xi_k\\
&=\sum_\chi\sum_{j=1}^n\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(\chi^{-1}\mathcal{D}_G(\chi^{-1}x_j)\widehat{\otimes}\mathfrak{c}|x_j\widehat{\otimes}1)\big)P_{\chi\zeta}\xi.
\end{aligned}$$

□

Proposition 3.8 (The domain criterion)**.** *For all $\mu \in \mathbb{R} \setminus \{0\}$, the resolvent $(i\mu + \mathcal{D})^{-1}$ maps the submodule $C_c^\infty(\Psi \circ (\mathcal{D}_1 \widehat{\otimes} 1) \widehat{\otimes} 1 \circ \Psi^{-1})\mathcal{H}$ into* $\mathrm{dom}(\Psi \circ (\mathcal{D}_1 \widehat{\otimes} 1) \widehat{\otimes} 1 \circ \Psi^{-1})$.

Proof. Using Lemma 3.7 and the fact that $(1 + \mathcal{D}_G)^{-1/2}$ is a compact operator, if $\xi \in C_c^\infty(\Psi \circ (\mathcal{D}_1 \widehat{\otimes} 1) \widehat{\otimes} 1 \circ \Psi^{-1})\mathcal{H}$, then $P_\chi \xi = 0$ for all but finitely many $\chi \in \mathrm{Char}(G)$. Since $(i\mu + \mathcal{D})^{-1}$ commutes with the action of G, it preserves $\mathcal{H}_\chi$ for all $\chi \in \mathrm{Char}(G)$. Hence if $\xi \in C_c^\infty(\Psi \circ (\mathcal{D}_1 \widehat{\otimes} 1) \widehat{\otimes} 1 \circ \Psi^{-1})\mathcal{H}$, then $P_\chi (i\mu + \mathcal{D})^{-1} \xi = 0$ for all but finitely many $\chi \in \mathrm{Char}(G)$. Lemma 3.7 then implies that $(i\mu + \mathcal{D})^{-1} \xi \in \mathrm{dom}(\Psi \circ (\mathcal{D}_1 \widehat{\otimes} 1) \widehat{\otimes} 1 \circ \Psi^{-1})$. □

Since the connection and domain criteria of [27, Thm. 13] are satisfied (Propositions 3.6 and 3.8 respectively), Theorem 3.4 is proved by combining the remaining positivity criterion with Lemma 3.7.

3.2 Factorisation for an odd spectral triple

Recall that G is a compact abelian Lie group, equipped with the normalised Haar measure and a trivial spinor bundle $\mathbf{\$}_G$. However, suppose that rather than an *even* G-equivariant spectral triple, we instead have an *odd* G-equivariant spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. We make the following definition analogously to Definition 2.14.

Definition 3.9. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be an odd, G-equivariant spectral triple for a trivially $\mathbb{Z}_2$-graded separable G-algebra A, and suppose that $\mathcal{A}$ is α-compatible. Let $\zeta \in \mathrm{Char}(G)$ satisfy $\overline{A\mathcal{H}_\zeta} = \mathcal{H}$, and let $\eta : \Gamma(\mathbb{C}\mathrm{l}(G))^G \to B(\mathcal{H})$ be a unital, equivariant $*$-homomorphism such that

1) $[\eta(s), a] = 0$ for all $s \in \Gamma(\mathbb{C}\mathrm{l}(G))^G$ and $a \in A^G$, and

2) $a\eta(s) \cdot \mathrm{dom}(\mathcal{D}_\zeta) \subset \mathrm{dom}(\mathcal{D})$ and $(\mathcal{D}\eta(s) - (-1)^{\deg s}\eta(s)\mathcal{D})aP_\zeta$ is bounded on $\mathcal{H}$ for all $a \in \oplus_\chi \mathcal{A}_\chi$, $s \in \Gamma(\mathbb{C}\mathrm{l}(G))^G$, where $P_\zeta \in B(\mathcal{H})$ is the orthogonal projection onto $\mathcal{H}_\zeta$.

Define a Clifford representation $\widetilde{\eta} : \Gamma(\mathbb{C}\mathrm{l}(G))^G \to B(\mathcal{H} \widehat{\otimes} \mathbb{C}^2)$ by $\widetilde{\eta}(s) = \eta(s) \widehat{\otimes} \omega^{\deg s}$, where $(\eta(s) \widehat{\otimes} \omega^{\deg s})(\xi \widehat{\otimes} v) = \eta(s)\xi \widehat{\otimes} \omega^{\deg s} v$.

It is easy to see that the pair $(\zeta, \widetilde{\eta})$ satisfy the conditions of Definition 2.14 for the associated even G-equivariant spectral triple $(\mathcal{A} \widehat{\otimes} \mathbb{C}\mathrm{l}_1, \mathcal{H} \widehat{\otimes} \mathbb{C}^2, \mathcal{D} \widehat{\otimes} \omega)$ (Definition 1.21). The next result follows easily from Theorem 3.4 applied to $(\mathcal{A} \widehat{\otimes} \mathbb{C}\mathrm{l}_1, \mathcal{H} \widehat{\otimes} \mathbb{C}^2, \mathcal{D} \widehat{\otimes} \omega)$.

Theorem 3.10. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be an odd, G-equivariant spectral triple for a trivially $\mathbb{Z}_2$-graded G-algebra A, and let $\zeta \in \mathrm{Char}(G)$ and $\eta : \Gamma(\mathbb{C}\mathrm{l}(G))^G \to B(\mathcal{H})$ be as in Definition 3.9, so in particular $\mathcal{A}$ is α-compatible. Let $(x_j)_{j=1}^n$ be a G-invariant global*

orthonormal frame for $\mathbf{\$}_G$*. If there is some* $R \in \mathbb{R}$ *such that*

$$\sum_{j=1}^{n} \Big(\Big\langle \mathcal{D}\xi, \eta\big({}_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(\chi^{-1}\mathcal{D}_G(\chi x_j)\widehat{\otimes}\mathfrak{c}|x_j\widehat{\otimes}1)\big)P_{\chi\zeta}\xi \Big\rangle \\ + \Big\langle \eta\big({}_{\Gamma(\mathbb{C}\mathrm{l}(G))^G}(\chi^{-1}\mathcal{D}_G(\chi x_j)\widehat{\otimes}\mathfrak{c}|x_j\widehat{\otimes}1)\big)P_{\chi\zeta}\xi, \mathcal{D}\xi \Big\rangle \Big) \geq R\|\xi\|^2$$

for all $\chi \in \mathrm{Char}(G)$*,* $\xi \in \mathrm{dom}(\mathcal{D})$*, then the odd spectral triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *represents the Kasparov product of the left-hand, middle and right-hand modules associated to* $(\mathcal{A}\widehat{\otimes}\mathbb{C}\mathrm{l}_1, \mathcal{H}\widehat{\otimes}\mathbb{C}^2, \mathcal{D}\widehat{\otimes}\omega)$*.*

3.3 Circle factorisation

The factorisation procedure is much simpler for circle-equivariant spectral triples, and this is the setting in [8, 15, 16], so we briefly discuss this particular case. Index the characters of $\mathbb{T}$ by $\mathbb{Z}$, where $\chi_k(z) = z^k$ for $k \in \mathbb{Z}$. We will often use k to stand for the character χ_k; e.g. A_k instead of A_{χ_k}. We equip the circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ with the coordinate $t \in [0, 1)$, where $z = e^{2\pi i t}$.

Fix a $\mathbb{T}$-equivariant spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ for a $\mathbb{Z}_2$-graded $\mathbb{T}$-algebra A. Recall that X is the completion of A in the $A^{\mathbb{T}}$-valued inner product. Since $L^2(\mathbf{\$}_{\mathbb{T}})_k \cong \mathbb{C}$ for every $k \in \mathbb{Z}$, we see that the left-hand module is

$$E_1 \cong X\widehat{\otimes}\mathbb{C}\mathrm{l}_1,$$

as $\mathbb{Z}_2$-graded right Hilbert $A^{\mathbb{T}}\widehat{\otimes}\mathbb{C}\mathrm{l}_1$-modules.

The Dirac operator on the spinor bundle $\mathbf{\$}_{\mathbb{T}} \cong \mathbb{T} \times \mathbb{C}$ is $\mathcal{D}_{\mathbb{T}} = \frac{1}{2\pi i}\frac{d}{dt}$, so $\mathcal{D}_{\mathbb{T}} f = -kf$ for $f \in L^2(\mathbf{\$}_{\mathbb{T}})_k = \mathrm{span}\{e^{-2\pi i k t}\}$. So with respect to the orthogonal decomposition $X\widehat{\otimes}\mathbb{C}\mathrm{l}_1 = \bigoplus_{k\in\mathbb{Z}} A_k\widehat{\otimes}\mathbb{C}\mathrm{l}_1$, the operator $\mathcal{D}_1$ acts on elements of homogeneous degree by

$$\mathcal{D}_1 \sum_{k\in\mathbb{Z}} a_k\widehat{\otimes}z = -\sum_{k\in\mathbb{Z}} (-1)^{\deg a_k} k a_k\widehat{\otimes}cz.$$

The left-hand module $(A, (X\widehat{\otimes}\mathbb{C}\mathrm{l}_1)_{A^{\mathbb{T}}\widehat{\otimes}\mathbb{C}\mathrm{l}_1}, \mathcal{D}_1)$ is the even unbounded Kasparov A-$A^{\mathbb{T}}\widehat{\otimes}\mathbb{C}\mathrm{l}_1$-module corresponding to the odd cycle constructed in [35] (up to a sign).

Let (χ_ℓ, η) be as in Definition 2.14. Recalling the isomorphism

$$\widetilde{\rho} : \Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))^{\mathbb{T}} \to \mathbb{C}\mathrm{l}_1, \quad \widetilde{\rho}(ic(dt)) = c,$$

we have

$${}_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))^{\mathbb{T}}}(\chi_k^{-1}\mathcal{D}_{\mathbb{T}}(\chi_k)\widehat{\otimes}c|1\widehat{\otimes}1) = ikc(dt).$$

Hence the factorisation criterion is then formulated as follows: If there is some $R \in \mathbb{R}$

such that

$$\langle \mathcal{D}\xi, ik\eta(c(dt))P_{k+\ell}\xi\rangle + \langle ik\eta(c(dt))P_{k+\ell}\xi, \mathcal{D}\xi\rangle \Big) \geq R\|\xi\|^2$$

for all $k \in \mathbb{Z}$ and $\xi \in \mathrm{dom}(\mathcal{D})$, then $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ represents the Kasparov product of left-hand and right-hand modules.

Chapter 4

Factorisation of θ-deformations

Given a $\mathbb{T}^n$-equivariant spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ for a $\mathbb{Z}_2$-graded $\mathbb{T}^n$-algebra A and a skew-symmetric matrix $\theta \in M_n(\mathbb{R})$, one can construct the θ-deformed $\mathbb{T}^n$-equivariant spectral triple $(\mathcal{A}_\theta, \mathcal{H}_\theta, \mathcal{D}_\theta)$ for the θ-deformed algebra A_θ. We show that if factorisation is achieved for $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, then it is also achieved for $(\mathcal{A}_\theta, \mathcal{H}_\theta, \mathcal{D}_\theta)$.

We first recall the construction of a θ-deformed $\mathbb{T}^n$-equivariant spectral triple, [13, 42, 52].

Definition 4.1. Let $\theta \in M_n(\mathbb{R})$ be a skew-symmetric matrix. The **noncommutative torus** $C(\mathbb{T}^n)_\theta$ is the universal C^*-algebra generated by n unitaries $U_1, \ldots, U_n$ subject to the commutation relations $U_j U_k = e^{2\pi i \theta_{jk}} U_k U_j$ for $j, k = 1, \ldots, n$.

The noncommutative torus $C(\mathbb{T}^n)_\theta$ carries an action by the n-torus $\mathbb{T}^n$, which is given by $t \cdot U_j = e^{2\pi i t^j} U_j$, where $t = (t^1, \ldots, t^n) \in \mathbb{T}^n$ are the standard torus coordinates.

Definition 4.2. Let A be a $\mathbb{Z}_2$-graded C^*-algebra with an action α of $\mathbb{T}^n$, and let $\theta \in M_n(\mathbb{R})$ be a skew-symmetric matrix. Equip the tensor product $A \widehat{\otimes} C(\mathbb{T}^n)_\theta$ with the diagonal action $t \cdot (a \widehat{\otimes} b) = \alpha_t(a) \widehat{\otimes} (t \cdot b)$ by $\mathbb{T}^n$. The **θ-deformation** of A is the invariant sub-C^*-algebra $A_\theta := (A \widehat{\otimes} C(\mathbb{T}^n)_\theta)^{\mathbb{T}^n}$.

The θ-deformation A_θ carries an action $\alpha^{(\theta)}$ of $\mathbb{T}^n$, given by $\alpha_t^{(\theta)}(a \widehat{\otimes} b) = \alpha_t(a) \widehat{\otimes} b$.

Definition 4.3. Let $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ be a $\mathbb{Z}_2$-graded Hilbert space with a strongly continuous unitary representation $V : \mathbb{T}^n \to U(\mathcal{H})$ such that $V_t \cdot \mathcal{H}^j \subset \mathcal{H}^j$ for $t \in \mathbb{T}^n$, $j \in \mathbb{Z}_2$. Let $\theta \in M_n(\mathbb{R})$ be a skew-symmetric matrix. Viewing $C(\mathbb{T}^n)_\theta$ as a right Hilbert module over itself, form the $\mathbb{Z}_2$-graded right Hilbert $C(\mathbb{T}^n)_\theta$-module $\mathcal{H} \widehat{\otimes} C(\mathbb{T}^n)_\theta$. This module carries an action by $\mathbb{T}^n$, given by $t \cdot (\xi \widehat{\otimes} b) = V_t \xi \widehat{\otimes} (t \cdot b)$. The **$\theta$-deformation** of $\mathcal{H}$ is the $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}_\theta := (\mathcal{H} \widehat{\otimes} C(\mathbb{T}^n)_\theta)^{\mathbb{T}^n}$. We define a unitary representation $V^{(\theta)} : \mathbb{T}^n \to U(\mathcal{H}_\theta)$ by $V_t^{(\theta)}(\xi \widehat{\otimes} b) = V_t \xi \widehat{\otimes} b$.

We will now define the θ-deformation of a $\mathbb{T}^n$-equivariant spectral triple.

Definition 4.4. Suppose that $\mathcal{A}$ is α-compatible. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a $\mathbb{T}^n$-equivariant spectral triple, and let $\theta \in M_n(\mathbb{R})$ be skew-symmetric. Represent the θ-deformed algebra A_θ on $\mathcal{H}_\theta$ by $(a\widehat{\otimes}b)(\xi\widehat{\otimes}c) = a\xi\widehat{\otimes}bc$ (for $a \in A$, $b \in C(\mathbb{T}^n)_\theta$), and setting $U^k := U_1^{k_1} \cdots U_n^{k_n}$ for $k \in \mathbb{Z}^n$, let

$$\mathcal{A}_\theta = \operatorname{span}\{a_k\widehat{\otimes}U^{-k} \in A_\theta : a_k \in \mathcal{A} \cap A_k,\ k \in \mathbb{Z}^n\}$$

which is a dense sub-$*$-algebra of A_θ compatible with $\alpha^{(\theta)}$, and define an operator $\mathcal{D}_\theta$ on $\mathcal{H}_\theta$ by $\mathcal{D}_\theta(\xi\widehat{\otimes}b) = \mathcal{D}\xi\widehat{\otimes}b$ for $\xi \in \operatorname{dom}(\mathcal{D})$.

Proposition 4.5. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a $\mathbb{T}^n$-equivariant spectral triple, and let $\theta \in M_n(\mathbb{R})$ be skew-symmetric. Then the triple $(\mathcal{A}_\theta, \mathcal{H}_\theta, \mathcal{D}_\theta)$ is a $\mathbb{T}^n$-equivariant spectral triple, which we call the θ-**deformation** of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.*

Proof. Let $u : \mathcal{H} \to \mathcal{H}_\theta$ be the unitary isomorphism

$$u\left(\sum_{k\in\mathbb{Z}^n} \xi_k\right) = \sum_{k\in\mathbb{Z}^n} \xi_k\widehat{\otimes}U^{-k},$$

where $U^k = U_1^{k_1} \cdots U_n^{k_n}$. Let $a\widehat{\otimes}U^{-j} \in \mathcal{A}_\theta$. Then

$$\begin{aligned} u^{-1} \circ \rho(a\widehat{\otimes}U^{-j}) \circ u \sum_{k\in\mathbb{Z}^n} \xi_k &= u^{-1} \sum_{k\in\mathbb{Z}^n} a\xi_k\widehat{\otimes}U^{-j}U^{-k} \\ = u^{-1} \sum_{k\in\mathbb{Z}^n} a\xi_k\widehat{\otimes}e^{2\pi i \sum_{\ell<p} j_p k_\ell \theta_{\ell p}} U^{-j-k} &= \sum_{k\in\mathbb{Z}^n} e^{2\pi i \sum_{\ell<p} j_p k_\ell \theta_{\ell p}} a\xi_k. \end{aligned}$$

Now, $\xi = \sum_{k\in\mathbb{Z}^n} \xi_k \in \operatorname{dom}(\mathcal{D})$ if and only if $\xi_k \in \operatorname{dom}(\mathcal{D})$ for each $k \in \mathbb{Z}^n$, and $\sum_{k\in\mathbb{Z}^n} \|D\xi_k\|^2 < \infty$. So if $\xi = \sum_{k\in\mathbb{Z}^n} \xi_k \in \operatorname{dom}(\mathcal{D})$, then we see that each summand in the $\mathbb{Z}^n$ expansion of $u^{-1} \circ \rho(a\widehat{\otimes}U^{-j}) \circ u$ is in $\operatorname{dom}(\mathcal{D})$, and

$$\sum_{k\in\mathbb{Z}^n} \|\mathcal{D}e^{2\pi i \sum_{\ell<p} j_p k_\ell \theta_{\ell p}} a\xi_k\|^2 = \sum_{k\in\mathbb{Z}} \|\mathcal{D}a\xi_k\|^2 < \infty,$$

since $a\xi \in \operatorname{dom}(\mathcal{D})$. Thus $\mathcal{A}_\theta$ preserves $\operatorname{dom}(\mathcal{D}_\theta)$. Under the unitary u, the commutator $[\mathcal{D}_\theta, \rho(a\widehat{\otimes}U^{-j})]$ is

$$u^{-1} \circ [\mathcal{D}_\theta, \rho(a\widehat{\otimes}U^{-j})] \circ u \sum_{k\in\mathbb{Z}} \xi_k = \sum_{k\in\mathbb{Z}} [\mathcal{D}, e^{2\pi i \sum_{\ell<p} j_p k_\ell \theta_{\ell p}} a]\xi_k,$$

and thus if P_k denotes the orthogonal projection onto $\mathcal{H}_k$, then

$$\|\mathcal{D}_\theta, \rho(a\widehat{\otimes}U^{-j})]\| = \sup_{k\in\mathbb{Z}^n} \|[\mathcal{D}, e^{2\pi i \sum_{\ell<p} j_p k_\ell \theta_{\ell p}} a]P_k\| \leq \|[\mathcal{D}, a]\| < \infty,$$

so $\mathcal{D}_\theta$ has bounded commutators with $\mathcal{A}_\theta$. It remains to check that for all $a \in A_\theta$,

$a(1+\mathcal{D}_\theta^2)^{-1/2}$ is compact. We have

$$u^{-1}\circ\rho(a\widehat{\otimes}U^{-j})(1+\mathcal{D}_\theta^2)^{-1/2}\circ u=\sum_k e^{2\pi i\sum_{\ell<p}j_pk_\ell\theta_{\ell p}}aP_k(1+\mathcal{D}^2)^{-1/2}P_k.$$

Since

$$a(1+\mathcal{D}^2)^{-1/2}=\sum_{k\in\mathbb{Z}^n}aP_k(1+\mathcal{D}^2)^{-1/2}P_k$$

is compact, and the summands have mutually orthogonal ranges and supports, we see that each summand in $\sum_k e^{2\pi i\sum_{\ell<p}j_pk_\ell\theta_{\ell p}}aP_k(1+\mathcal{D}^2)^{-1/2}P_k$ is compact, and that the sum converges in operator norm to a compact operator. Therefore $a(1+\mathcal{D}_\theta^2)^{-1/2}$ is compact for all $a\in\mathcal{A}_\theta$.

It remains to show equivariance. Clearly $\mathbb{T}^n\cdot\mathcal{A}_\theta\subset\mathcal{A}_\theta$, and we have already established that the representation of A_θ on $\mathcal{H}_\theta$ intertwines the actions of $\mathbb{T}^n$. Since the unitary isomorphism $u:\mathcal{H}\to\mathcal{H}_\theta$ intertwines the actions of $\mathbb{T}^n$ on these Hilbert spaces, the remaining conditions for a $\mathbb{T}^n$-equivariant spectral triple are satisfied. □

Proposition 4.6. *Let A be a $\mathbb{T}^n$-algebra, and let $\theta\in M_n(\mathbb{R})$ be skew-symmetric. Then A_θ satisfies the spectral subspace assumption if and only if A does.*

Proof. Let $\psi:A^{\mathbb{T}^n}\to A_\theta^{\mathbb{T}^n}$ be the $*$-isomorphism $\psi(a)=a\widehat{\otimes}1$. Then $\psi(\overline{A_kA_k^*})=\overline{(A_\theta)_k(A_\theta)_k^*}$ for all $k\in\mathbb{Z}^n$. □

Definition 4.7. Recall the equivariant unitary $u:\mathcal{H}\to\mathcal{H}_\theta$ defined by $u\left(\sum_{k\in\mathbb{Z}^n}\xi_k\right)=\sum_{k\in\mathbb{Z}^n}\xi_k\widehat{\otimes}U^{-k}$. Given $\eta:\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}\to B(\mathcal{H})$, define $\eta_\theta:\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}\to B(\mathcal{H}_\theta)$ by $\eta_\theta(s)=u\circ\eta(s)\circ u^*$.

Proposition 4.8. *The pair (ℓ,η_θ) satisfies the conditions of Definition 2.14 for the θ-deformed spectral triple $(\mathcal{A}_\theta,\mathcal{H}_\theta,\mathcal{D}_\theta)$ if and only if (ℓ,η) satisfies those conditions for $(\mathcal{A},\mathcal{H},\mathcal{D})$. Consequently if $(\mathcal{A},\mathcal{H},\mathcal{D})$ factorises then so does $(\mathcal{A}_\theta,\mathcal{H}_\theta,\mathcal{D}_\theta)$.*

Proof. If $\xi\widehat{\otimes}U^{-\ell}\in(\mathcal{H}_\theta)_\ell$ and $a\widehat{\otimes}U^{-k}\in(A_\theta)_k$, then $(a\widehat{\otimes}U^{-k})(\xi\widehat{\otimes}U^{-\ell})=\lambda a\xi\widehat{\otimes}U^{-k-\ell}$ for some $\lambda\in U(1)$. Hence $\overline{A_\theta(\mathcal{H}_\theta)_\ell}=\mathcal{H}$ if and only if $\overline{A\mathcal{H}_\ell}=\mathcal{H}$.

Recall the $*$-isomorphism $\psi:A^{\mathbb{T}^n}\to A_\theta^{\mathbb{T}^n}$, $\psi(a)=a\widehat{\otimes}1$. Then $u(a\xi)=\psi(a)u(\xi)$ for all $a\in A^{\mathbb{T}^n}$, $\xi\in\mathcal{H}$. Hence $u\circ[\eta(s),a]_\pm\circ u^*=[\eta_\theta(s),\psi(a)]_\pm$ for all $s\in\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}$, $a\in A^{\mathbb{T}^n}$, so Condition (1) is satisfied for the θ-deformation if and only if it is satisfied for the original spectral triple.

By construction, $\oplus_k(\mathcal{A}_\theta)_k=\mathcal{A}_\theta$. Let $a\widehat{\otimes}U^{-k}\in(\mathcal{A}_\theta)_k$ and let $s\in\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}$. If $\xi\widehat{\otimes}U^{-\ell}\in(\mathcal{H}_\theta)_\ell$ then $u^*\big((a\widehat{\otimes}U^{-k})\eta_\theta(s)(\xi\widehat{\otimes}U^{-\ell})\big)=\lambda a\eta(s)\xi$ for some $\lambda\in U(1)$. Since $\mathcal{D}_\theta=u\circ\mathcal{D}\circ u^*$, it follows that $a\eta(s)\cdot\mathrm{dom}(\mathcal{D}_\ell)\subset\mathrm{dom}(\mathcal{D})$ for all $a\in\oplus_k\mathcal{A}_k$, $s\in\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}$ if and only if $b\eta_\theta(s)\cdot\mathrm{dom}((\mathcal{D}_\theta)_\ell)\subset\mathrm{dom}(\mathcal{D}_\theta)$ for all $b\in\mathcal{A}_\theta$, $s\in\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}$.

Let $a\widehat{\otimes}U^{-k} \in (\mathcal{A}_\theta)_k$, and let $s \in \Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}$. Then

$$u^* \circ [\mathcal{D}_\theta, \eta_\theta(s)]_\pm (a\widehat{\otimes}U^{-k})P_\ell \circ u = \lambda[\mathcal{D}, \eta(s)]_\pm a P_\ell$$

for some $\lambda \in U(1)$ depending on k, ℓ and θ. Therefore (ℓ, η) satisfies Condition 2) if and only if (ℓ, η_θ) satisfies Condition 2).

Since $\mathcal{D}_\theta = u \circ \mathcal{D} \circ u^*$ and $\eta_\theta = u \circ \eta \circ u^*$, clearly the factorisation criterion (Theorems 3.4, 3.10) is satisfied for (ℓ, η_θ) and the θ-deformed spectral triple if and only if it is satisfied for (ℓ, η) and the original spectral triple. $\square$

Chapter 5

Factorisation for compact manifolds

Throughout this chapter, let (M, g) be a compact Riemannian manifold with a smooth, free, isometric left action by the n-torus $\mathbb{T}^n$, and let S be a (possibly $\mathbb{Z}_2$-graded) equivariant Clifford module over M equipped with an invariant Clifford connection ∇^S, as in Example 1.20. Then $(C^\infty(M), L^2(S), \mathcal{D})$ is a $\mathbb{T}^n$-equivariant spectral triple, where $\mathcal{D}$ is the associated Dirac operator on S. The spectral triple is even if S is $\mathbb{Z}_2$-graded; otherwise it is odd.

We will show that $(C^\infty(M), L^2(S), \mathcal{D})$ can always be factorised. Since the torus action is free, $C(M)$ has full spectral subspaces (a special case of the spectral subspace assumption) by [37, Thm. 7.2.6]. We show that the remaining two conditions for factorisation (the existence of the Clifford representation $\eta : \Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n} \to B(L^2(S))$ and the positivity criterion) are satisfied in turn. Compatibility of $C^\infty(M)$ with the action is satisfied since we assume the action to be smooth.

We will conclude this chapter by showing that a connection may be used to construct an unbounded KK-cycle which represents the Kasparov product of the factorisation. This cycle is unitarily equivalent to $(C^\infty(M), L^2(S), T)$, where T is a first order, self-adjoint elliptic differential operator on the vector bundle S. If the orbits of the torus are embedded isometrically in M, then T is a perturbation of the Dirac operator $\mathcal{D}$ by a smooth bundle endomorphism.

5.1 The Clifford representation

We require a character $\ell \in \mathbb{Z}^n$ and a map $\eta : \Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n} \to B(L^2(S))$ satisfying the conditions of Definition 2.14 (or Definition 3.9 if S is trivially graded). The following lemma shows that any $\ell \in \mathbb{Z}^n$ satisfies the condition (and indeed factorisation is achieved for any choice of ℓ).

Lemma 5.1. *Let N be a Riemannian manifold with a smooth free left action by the n-torus $\mathbb{T}^n$, and let F be an equivariant Hermitian vector bundle over N. Then $\overline{C_0(N)L^2(F)_\ell} = L^2(F)$ for all $\ell \in \mathbb{Z}^n$.*

Proof. Since $L^2(F) = \bigoplus_{k\in\mathbb{Z}^n} L^2(F)_k$, it is enough for $C_0(N)_{k-\ell}L^2(F)_\ell$ to be dense in $L^2(F)_k$ for all $k \in \mathbb{Z}^n$. We show that $C_0(N)_{k-\ell}\Gamma_c(F)_\ell = \Gamma_c(F)_k$ for all $k \in \mathbb{Z}$, which since $\Gamma_c(F)$ is dense in $L^2(F)$ proves the result.

Let $\xi \in \Gamma_c(F)_k$. Since ξ has compact support, there is a finite collection of open sets $(U_i)_{i=1}^N$ which cover the support of ξ, such that $U_i \cong \pi(U_i) \times \mathbb{T}^n$ as $\mathbb{T}^n$-spaces, recalling the quotient map $\pi : N \to N/\mathbb{T}^n$. Let $(\phi_n)_{n=1}^N$ be an invariant partition of unity for $\bigcup_{i=1}^N U_i$ subordinate to $(U_i)_{i=1}^N$. For each $i = 1, \ldots, N$, let $f_i \in C_0(\pi(U_i))$ be a function such that $(f_i \circ \pi)\phi_i = f_i \circ \pi$, and let $a_i, b_i \in C_0(U_i)$ be the functions corresponding to $f_i \otimes \chi_{k-\ell}$ and $f_i \otimes \chi_{\ell-k}$ respectively under the equivariant $*$-isomorphism $C_0(U_i) \cong C_0(\pi(U_i)) \otimes C(\mathbb{T}^n)$. Note that $b_i a_i \phi_i = \phi_i$ and $a_i\xi \in \Gamma_c(F)_\ell$, so

$$\xi = \sum_{i=1}^N \phi_i\xi = \sum_{i=1}^N b_i a_i \phi_i \xi_i \in C_0(N)_{k-\ell}\Gamma_c(F)_\ell. \qquad \square$$

We will assume that $\ell \in \mathbb{Z}^n$ is fixed from now on. The choice of ℓ does not affect the factorisation. This means we could choose $\ell = 0$ for convenience, but we will leave ℓ arbitrary in order to show that factorisation is achieved for all choices of ℓ.

Next we define the map $\eta : \Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n} \to B(L^2(S))$. First recall that the fundamental vector field $X^{(v)} \in \Gamma^\infty(TM)$ associated to $v \in T_e\mathbb{T}^n$ is $X_x^{(v)} = \frac{d}{dt}\exp(tv)\cdot x\big|_{t=0}$. Since the action of the n-torus $\mathbb{T}^n$ on M is free, the fundamental vector field of a non-zero vector in $T_e\mathbb{T}^n$ is non-vanishing. The canonical isomorphisms $T_e\mathbb{T}^n \cong \Gamma(T^*\mathbb{T}^n)^{\mathbb{T}^n}$ and $TM \cong T^*M$, along with the fundamental vector field map, give us an equivariant, $\mathbb{Z}_2$-graded map $\Gamma(T^*\mathbb{T}^n)^{\mathbb{T}^n} \to \Gamma^\infty(T^*M)$. However, this map need not be an isometry and hence need not extend to a $*$-homomorphism $\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n} \to \Gamma^\infty(\mathbb{C}\mathrm{l}(M))$. We will modify this map to obtain a $*$-homomorphism.

For $j = 1, \ldots, n$, let $X_j \in \Gamma^\infty(TM)^{\mathbb{T}^n}$ be the fundamental vector field associated to $\frac{\partial}{\partial t^j} \in T_e\mathbb{T}^n$. Observe that $\{X_1(x), \ldots, X_n(x)\}$ is a linearly independent set for every $x \in M$. For each $x \in M$, let $W(x) = (W^{jk}(x))_{j,k=1}^n \in M_n(\mathbb{R})$ be the inverse square root of the positive-definite matrix $(g(X_j(x), X_k(x)))_{j,k=1}^n$, where g is the metric on M. Letting x vary, we obtain functions $W^{jk} \in C^\infty(M)^{\mathbb{T}^n}$ for $j, k = 1, \ldots, n$. Let

$$v^k = \sum_{j=1}^n X_j^\flat W^{jk} \in \Gamma^\infty(T^*M)^{\mathbb{T}^n}, \quad k = 1, \ldots, n, \tag{5.1}$$

where $TM \to T^*M$, $X \mapsto X^\flat$ is the canonical isomorphism. Then $\{v^1(x), \ldots, v^n(x)\}$ is an orthonormal set for all $x \in M$. We call the functions $W^{jk} \in C^\infty(M)^{\mathbb{T}^n}$, $j, k =$

$1, \ldots, n$ the **normalisation functions**.

Definition 5.2. The map $\Gamma(T^*\mathbb{T}^n)^{\mathbb{T}^n} \ni dt^k \mapsto -v^k = -\sum_{j=1}^n X_j^\flat W^{jk} \in \Gamma^\infty(T^*M)^{\mathbb{T}^n}$ is now not only equivariant and $\mathbb{Z}_2$-graded (when S is $\mathbb{Z}_2$-graded), but is also an isometry. It thus extends to a unital $*$-homomorphism $\eta : \Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n} \to \Gamma^\infty(\mathbb{C}\mathrm{l}(M)) \subset B(L^2(S))$.

Remark. The appearance of a minus sign in the definition of the Clifford representation η arises as follows. The action of $\mathbb{T}^n$ on sections of the Clifford module S is given by $V_{\exp(tv)}u(x) = \exp(tv) \cdot u(\exp(-tv) \cdot x)$. So the more natural convention to define η is to use the vector field $Y_x^{(v)} = \frac{d}{dt} \exp(-tv) \cdot x\big|_{t=0} = -X_x^{(v)}$.

As functions are central in the endomorphisms, η satisfies Condition 1) of Definition 2.14, so it remains to check Condition 2). Since the image of η consists of smooth sections of $\mathbb{C}\mathrm{l}(M)$, $\eta(s) \cdot \mathrm{dom}(\mathcal{D}) \cap L^2(S)_\ell \subset \mathrm{dom}(\mathcal{D})$ for all $s \in \Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}$. Before showing that $[\mathcal{D}, \eta(s)]_\pm P_\ell$ is bounded for all $s \in \Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}$, we prove a lemma.

Lemma 5.3. *Let N be a Riemannian manifold, and let G be a Lie group acting smoothly on N. Let F be an equivariant Hermitian vector bundle over N. The equivariance of F defines a unitary representation $V : G \to U(L^2(F))$.*

Let $v \in \mathfrak{g}$, and let $X^{(v)} \in \Gamma^\infty(TN)$ be the fundamental vector field associated to v. Define a one-parameter unitary group on $L^2(F)$ by $\gamma_v(t) = V_{\exp(tv)}$. Let A be the infinitesimal generator of γ_v, characterised by $\gamma_v(t) = e^{itA}$. Then

1) $A : \Gamma^\infty(F) \to \Gamma^\infty(F)$, and

2) $iA + \nabla_{X^{(v)}} \in \Gamma^\infty(\mathrm{End}(F))$ for any connection ∇ on F.

In particular, if N is compact, then $iA + \nabla_{X^{(v)}} \in B(L^2(F))$ for any connection ∇.

Proof. Let $u \in \Gamma^\infty(F)$. Working on a local trivialisation of F, we can view u as a $\mathbb{C}^k$-valued function on N. Since $\gamma_v(t)u(x) = \exp(tv) \cdot u(\exp(-tv) \cdot x)$, in this trivialisation,

$$iAu(x) = \frac{d}{dt}\gamma_v(t)u(x)\Big|_{t=0} = Bu(x) - X_x^{(v)}(u), \tag{5.2}$$

where $B \in M_k(\mathbb{C})$ is the derivative at $t = 0$ of the curve $t \mapsto \exp(tv) \in M_k(\mathbb{C})$. Equation (5.2) shows 1) and 2), since if ∇ is a connection then locally $\nabla_{X^{(v)}} = X^{(v)} + \omega$, where ω is a locally-defined $M_k(\mathbb{C})$-valued function on N. □

The next result shows that the pair (ℓ, η) satisfy the remaining Condition 2) of Definition 2.14.

Proposition 5.4. *Let η be as in Definition 5.2 and $\ell \in \mathbb{Z}^n$. Then the graded commutator $[\mathcal{D}, \eta(s)]_\pm P_\ell$ is bounded for all $s \in \Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}$.*

Proof. For $j = 1, \ldots, n$, let X_j be the fundamental vector field associated to $\frac{\partial}{\partial t^j}$, and let $v_j = \sum_{k=1}^n X_k W^{kj}$ be the normalised vector field as in Equation (5.1). Let $U \subset M$ be an open set such that $M|_U$ is parallelisable, and choose vector fields $(w_1, \ldots, w_{m-n}) \subset \Gamma^\infty(TU)$ (where $m := \dim M$) such that $(v_1, \ldots, v_n, w_1, \ldots, w_{m-n})$ is an orthonormal frame for TU. We can locally express the Dirac operator $\mathcal{D}$ as

$$\mathcal{D}|_U = \sum_j c(v_j^\flat)\nabla^S_{v_j} + \sum_i c(w_i^\flat)\nabla^S_{w_i},$$

where $v \mapsto v^\flat$ is the isomorphism $TM \to T^*M$ determined by the Riemannian metric, and c denotes Clifford multiplication.

Since $\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}$ is generated by $(c(dt^k))_{k=1}^n$, we need only show that the anti-commutator $\{\mathcal{D}, c(v_j^\flat)\}P_\ell$ is bounded for $j = 1, \ldots, n$. Letting ∇^{LC} be the Levi-Civita connection on T^*M and using the compatibility between ∇^S and ∇^{LC}, we have

$$\begin{aligned}
\{\mathcal{D}, c(v_j^\flat)\}|_U &= \sum_i c(v_i^\flat)c(v_j^\flat)\nabla^S_{v_i} + \sum_i c(w_i^\flat)c(v_j^\flat)\nabla^S_{w_i} + \sum_i c(v_i^\flat)c(\nabla^{LC}_{v_i} v_j^\flat) \\
&\quad + \sum_i c(w_i^\flat)c(\nabla^{LC}_{w_i} v_j^\flat) + \sum_i c(v_j^\flat)c(v_i^\flat)\nabla^S_{v_i} + \sum_i c(v_j^\flat)c(w_i^\flat)\nabla^S_{w_i} \\
&= -2\nabla^S_{v_j} + \sum_i c(v_i^\flat)c(\nabla^{LC}_{v_i} v_j^\flat) + \sum_i c(w_i^\flat)c(\nabla^{LC}_{w_i} v_j^\flat).
\end{aligned}$$

The second and third terms are smooth bundle endomorphisms which are independent of the choice of $(f_1, \ldots, f_{m-n})$, and so globally

$$\begin{aligned}
\{\mathcal{D}, c(v_j^\flat)\} &= -2\nabla^S_{v_j} + \text{ bundle endomorphism} \\
&= -2\sum_k W^{kj}\nabla^S_{X_k} + \text{bundle endomorphism.}
\end{aligned}$$

Since M is compact, every endomorphism is bounded, and so it is enough to show that $\nabla^S_{X_j} P_\ell$ is bounded. By Lemma 5.3, $\nabla^S_{X_j} = -iA_j + \omega$ for some $\omega \in \Gamma^\infty(\mathrm{End}(S))$, where A_j is the infinitesimal generator of the one-parameter group $s \mapsto V_{\exp(s\frac{\partial}{\partial t^j})} \in U(L^2(S))$. Since $V_{\exp(s\frac{\partial}{\partial t^j})} = \sum_{k\in\mathbb{Z}^n} e^{2\pi i s k_j} P_k$, $A_j = \sum_{k\in\mathbb{Z}^n} 2\pi k_j P_k$, and thus

$$\nabla^S_{X_j} P_\ell = -iA_j P_\ell + \omega P_\ell = -2\pi i \ell_j P_\ell + \omega P_\ell$$

is bounded, and so we have shown that $\{\mathcal{D}, c(v_j^\flat)\}P_\ell$ is bounded. □

5.2 The positivity criterion

Now that we have a pair (ℓ, η) satisfying the conditions of Definition 2.14, it remains to check the positivity criterion. To this end we derive a formula for $\Psi \circ (\mathcal{D}_1 \widehat{\otimes} 1)\widehat{\otimes} 1 \circ \Psi^{-1}$,

recalling from Equation (3.1) the isomorphism

$$\Psi : \Big(E_1 \widehat{\otimes}_{C(M)^{\mathbb{T}^n} \widehat{\otimes} \mathfrak{C}} \big(C(M)^{\mathbb{T}^n} \widehat{\otimes} (\Gamma(\mathbf{\$}_{\mathbb{T}^n})^{\mathbb{T}^n} \widehat{\otimes} \mathfrak{C})^*\big)\Big) \widehat{\otimes}_{C(M)^{\mathbb{T}^n} \widehat{\otimes} \Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}} L^2(S)_\ell \to L^2(S).$$

Lemma 5.5. *For $j = 1, \ldots, n$, let $X_j \in \Gamma^\infty(TM)$ be the fundamental vector field associated to $\frac{\partial}{\partial t^j} \in T_e\mathbb{T}^n$, with corresponding covector field $X_j^\flat$, and let A_j be the infinitesimal generator of the one-parameter unitary group $t \mapsto V_{\exp(t\frac{\partial}{\partial t^j})} \in U(L^2(S))$. Let $W^{jk} \in C^\infty(M)^{\mathbb{T}^n}$ be the normalisation functions. Then*

$$\Psi \circ (\mathcal{D}_1 \widehat{\otimes} 1) \widehat{\otimes} 1 \circ \Psi^{-1} = -i \sum_{j,q=1}^{n} W^{qj} c(X_q^\flat)(A_j - 2\pi\ell_j).$$

Proof. Let $(x_r)_{r=1}^{2^{\lfloor n/2 \rfloor}}$ be an invariant, global orthonormal frame for $\mathbf{\$}_{\mathbb{T}^n}$. By Lemma 3.7,

$$\Psi \circ (\mathcal{D}_1 \widehat{\otimes} 1) \widehat{\otimes} 1 \circ \Psi^{-1} = \sum_{k \in \mathbb{Z}^n} \sum_{r=1}^{2^{\lfloor n/2 \rfloor}} \eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}}(\chi_k^{-1} \mathcal{D}_{\mathbb{T}^n}(\chi_k x_r) \widehat{\otimes} \mathfrak{c} | x_r \widehat{\otimes} 1)\big) P_{k+\ell}.$$

Since we are using the trivial flat spinor bundle over $\mathbb{T}^n$, $\mathcal{D}_{\mathbb{T}^n} x_r = 0$ for all r, and $[\mathcal{D}_{\mathbb{T}^n}, \chi_k] = 2\pi i \sum_j k_j \chi_k c(dt^j)$.

Recall that the Clifford representation $\eta : \Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n} \to B(L^2(S))$ is defined by $c(dt^j) \mapsto -\sum_{q=1}^n c(X_q^\flat) W^{qj}$. Hence

$$\begin{aligned}
\Psi \circ (\mathcal{D}_1 \widehat{\otimes} 1) \widehat{\otimes} 1 \circ \Psi^{-1} &= 2\pi i \sum_{k \in \mathbb{Z}^n} \sum_{r=1}^{2^{\lfloor n/2 \rfloor}} \sum_{j=1}^{n} k_j \eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}}(c(dt^j) x_r \widehat{\otimes} \mathfrak{c} | x_r \widehat{\otimes} 1)\big) P_{k+\ell} \\
&= 2\pi i \sum_{k \in \mathbb{Z}^n} \sum_{r=1}^{2^{\lfloor n/2 \rfloor}} \sum_{j=1}^{n} k_j \eta(c(dt^j)) \eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_r \widehat{\otimes} 1 | x_r \widehat{\otimes} 1)\big) P_{k+\ell} \\
&= -2\pi i \sum_{k \in \mathbb{Z}^n} \sum_{j,p=1}^{n} k_j W^{pj} c(X_p^\flat) P_{k+\ell} = -i \sum_{j,q=1}^{n} W^{qj} c(X_q^\flat)(A_j - 2\pi\ell_j).
\end{aligned}$$

□

Theorem 5.6. *The positivity criterion is satisfied; that is there is some $R \in \mathbb{R}$ such that*

$$\big\langle \mathcal{D}\xi, \Psi \circ (\mathcal{D}_1 \widehat{\otimes} 1) \widehat{\otimes} 1 \circ \Psi^{-1} \xi \big\rangle + \big\langle \Psi \circ (\mathcal{D}_1 \widehat{\otimes} 1) \widehat{\otimes} 1 \circ \Psi^{-1} \xi, \mathcal{D}\xi \big\rangle \geq R \|\xi\|^2$$

for all $\xi \in \mathrm{dom}(\mathcal{D}) \cap \Psi(\mathrm{dom}((\mathcal{D}_1 \widehat{\otimes} 1) \widehat{\otimes} 1))$. Thus $(C^\infty(M), L^2(S), \mathcal{D})$ factorises.

Proof. For $j = 1, \ldots, n$, let $X_j \in \Gamma^\infty(TM)$ be the fundamental vector field corresponding to $\frac{\partial}{\partial t^j} \in T_e\mathbb{T}^n$, and let $v_j = \sum_{p=1}^n X_p W^{pj}$ be the normalised vector field as in Equation (5.1). Let $U \subset M$ be an open set such that $M|_U$ is parallelisable, and choose vector fields $(w_1, \ldots, w_{m-n}) \subset \Gamma^\infty(TU)$ (where $m := \dim M$) such that $(v_1, \ldots, v_n, w_1, \ldots, w_{m-n})$ is an orthonormal frame for TU.

Recall that we can locally express the Dirac operator $\mathcal{D}$ as

$$\mathcal{D}|_U = \sum_{j=1}^{n} c(v_j^\flat)\nabla^S_{v_j} + \sum_{i=1}^{m-n} c(w_i^\flat)\nabla^S_{w_i}.$$

Since M is compact, by using a partition of unity it is enough to prove the positivity for sections with support in an open set V with $\overline{V} \subset U$.

For each $j = 1, \ldots, n$, let A_j be the generator of the one-parameter unitary group $s \mapsto V_{\exp(s\frac{\partial}{\partial t^j})} \in U(L^2(S))$. Then for $\xi \in \Gamma^\infty(S)$ with support in V,

$$\begin{aligned}
&\left\langle \mathcal{D}\xi, \Psi \circ (\mathcal{D}_1 \widehat{\otimes} 1)\widehat{\otimes} 1 \circ \Psi^{-1}\xi \right\rangle + \left\langle \Psi \circ (\mathcal{D}_1 \widehat{\otimes} 1)\widehat{\otimes} 1 \circ \Psi^{-1}\xi, \mathcal{D}\xi \right\rangle \\
&= \sum_{j,p} \left\langle c(v_j^\flat)\nabla^S_{v_j}\xi, -ic(v_p^\flat)(A_p - 2\pi\ell_p)\xi \right\rangle + \sum_{j,p} \left\langle c(w_j^\flat)\nabla^S_{w_j}\xi, -ic(v_p^\flat)(A_p - 2\pi\ell_p)\xi \right\rangle \\
&\quad + \sum_{j,p} \left\langle -ic(v_p^\flat)(A_p - 2\pi\ell_p)\xi, c(v_j^\flat)\nabla^S_{v_j}\xi \right\rangle + \sum_{j,p} \left\langle -ic(v_p^\flat)(A_p - 2\pi\ell_p)\xi, c(w_j^\flat)\nabla^S_{w_j}\xi \right\rangle.
\end{aligned}$$

Given $X \in \Gamma^\infty(TM)$, the (formal) adjoint of ∇_X is $(\nabla^S_X)^* = -\nabla^S_X - \operatorname{div} X$. Using the compatibility between ∇^S and the Levi-Civita connection ∇^{LC} on T^*M, we compute

$$\begin{aligned}
&\left\langle \mathcal{D}\xi, \Psi \circ (\mathcal{D}_1 \widehat{\otimes} 1)\widehat{\otimes} 1 \circ \Psi^{-1}\xi \right\rangle + \left\langle \Psi \circ (\mathcal{D}_1 \widehat{\otimes} 1)\widehat{\otimes} 1 \circ \Psi^{-1}\xi, \mathcal{D}\xi \right\rangle \\
&= 4\pi i \sum_{j,k} (k_j - \ell_j) \left\langle \xi, \nabla^S_{v_j} P_k \xi \right\rangle - 2\pi i \sum_{j,p,k} (k_p - \ell_p) \left\langle \xi, c(\nabla^{LC}_{v_j} v_j^\flat) c(v_p^\flat) P_k \xi \right\rangle \\
&\quad - 2\pi i \sum_{j,p,k} (k_p - \ell_p) \left\langle \xi, \big(c(v_j^\flat)c(\nabla^{LC}_{v_j} v_p^\flat) + (\operatorname{div} v_j) c(v_j^\flat) c(v_p^\flat)\big) P_k \xi \right\rangle \\
&\quad - 2\pi i \sum_{j,p,k} (k_p - \ell_p) \left\langle \xi, c(\nabla^{LC}_{w_j} w_j^\flat) c(v_p^\flat) \right\rangle \\
&\quad - 2\pi i \sum_{j,p,k} (k_p - \ell_p) \left\langle \xi, \big(c(w_j^\flat)c(\nabla^{LC}_{w_j} v_p^\flat) + (\operatorname{div} w_j) c(w_j^\flat) c(v_p^\flat)\big) P_k \xi \right\rangle.
\end{aligned}$$

Let $\omega_j = \nabla^S_{X_j} + iA_j \in \Gamma^\infty(\operatorname{End}(S))$, as in Lemma 5.3. Since $A_j P_k = 2\pi k_j P_k$,

$$\begin{aligned}
&\left\langle \mathcal{D}\xi, \Psi \circ (\mathcal{D}_1 \widehat{\otimes} 1)\widehat{\otimes} 1 \circ \Psi^{-1}\xi \right\rangle + \left\langle \Psi \circ (\mathcal{D}_1 \widehat{\otimes} 1)\widehat{\otimes} 1 \circ \Psi^{-1}\xi, \mathcal{D}\xi \right\rangle \\
&= 8\pi^2 \sum k_p (k_j - \ell_j) \left\langle \xi, W^{jp} P_k \xi \right\rangle + 4\pi i \sum (k_j - \ell_j) \left\langle \xi, W^{jp} \omega_p P_k \xi \right\rangle \\
&\quad - 2\pi i \sum (k_p - \ell_p) \left\langle \xi, c(\nabla^{LC}_{v_j} v_j^\flat) c(v_p^\flat) P_k \xi \right\rangle \\
&\quad - 2\pi i \sum (k_p - \ell_p) \left\langle \xi, c(\nabla^{LC}_{w_j} w_j^\flat) c(v_p^\flat) P_k \xi \right\rangle \\
&\quad - 2\pi i \sum (k_p - \ell_p) \left\langle \xi, \big(c(v_j^\flat)c(\nabla^{LC}_{v_j} v_p^\flat) + (\operatorname{div} v_j) c(v_j^\flat) c(v_p^\flat)\big) P_k \xi \right\rangle \\
&\quad - 2\pi i \sum (k_p - \ell_p) \left\langle \xi, \big(c(w_j^\flat)c(\nabla^{LC}_{w_j} v_p^\flat) + (\operatorname{div} w_j) c(w_j^\flat) c(v_p^\flat)\big) P_k \xi \right\rangle.
\end{aligned}$$

We estimate:

$$\begin{aligned}
&\left\langle \mathcal{D}\xi, \Psi\circ(\mathcal{D}_1\widehat{\otimes}1)\widehat{\otimes}1\circ\Psi^{-1}\xi\right\rangle + \left\langle \Psi\circ(\mathcal{D}_1\widehat{\otimes}1)\widehat{\otimes}1\circ\Psi^{-1}\xi, \mathcal{D}\xi\right\rangle \\
&\geq 8\pi^2 \sum_{k\in\mathbb{Z}^n}\sum_{j,p=1}^{n} k_p(k_j-\ell_j)\left\langle P_k\xi, W^{jp}P_k\xi\right\rangle - \sum_{k\in\mathbb{Z}^n}\sum_{p=1}^{n}|k_p-\ell_p|C_p\left\langle P_k\xi, P_k\xi\right\rangle,
\end{aligned}$$

for some constants $C_p \in [0,\infty)$, $p=1,\ldots,n$, which are based on the norms of the endomorphisms such as $W^{jp}\omega_p$ and $(\operatorname{div} w_j)c(w_j^\flat)c(v_p^\flat)$ on the compact set $\overline{V}$.

For each $x\in M$, let $\lambda(x)>0$ be the smallest eigenvalue of the positive-definite real matrix $(W^{jp}(x))_{p,q=1}^n$. Then $\sum_{j,p}k_jk_pW^{jp}(x)\geq\lambda(x)\sum_{j=1}^n k_j^2$, and so we can estimate

$$\begin{aligned}
&(8\pi^2)^{-1}\left\langle \mathcal{D}\xi, \Psi\circ(\mathcal{D}_1\widehat{\otimes}1)\widehat{\otimes}1\circ\Psi^{-1}\xi\right\rangle + (8\pi^2)^{-1}\left\langle \Psi\circ(\mathcal{D}_1\widehat{\otimes}1)\widehat{\otimes}1\circ\Psi^{-1}\xi, \mathcal{D}\xi\right\rangle \\
&\geq \inf_{x\in M}\{\lambda(x)\}\sum_{k\in\mathbb{Z}^n}\sum_{j=1}^{n}k_j^2\|P_k\xi\|^2 - n\sup_{j,p}\Big\{|\ell_p|\sup_{x\in M}\{|W^{jp}(x)|\}\Big\}\sum_{k\in\mathbb{Z}^n}\sum_{r=1}^{n}|k_r|\|P_k\xi\|^2 \\
&\quad - \sum_{k\in\mathbb{Z}^n}\sum_{p=1}^{n}|k_p-\ell_p|C_p\|P_k\xi\|^2 \\
&\geq (8\pi)^{-1}\sum_{k\in\mathbb{Z}^n}\left(a\sum_{j=1}^{n}k_j^2 - b\sum_{j=1}^{n}|k_j| - d\sum_{j=1}^{n}|k_j-\ell_j|\right)\|P_k\xi\|^2,
\end{aligned}$$

where we have relabelled some constants and set $d:=\sup_p\{C_p\}$. Since M is compact, the constant $a=8\pi^2\inf_{x\in M}\{\lambda(x)\}$ is strictly positive, and so the function

$$Q:\mathbb{Z}^n\to\mathbb{R},\qquad Q(k)=a\sum_{j=1}^{n}k_j^2 - b\sum_{j=1}^{n}|k_j| - d\sum_{j=1}^{n}|k_j-\ell_j|$$

is bounded from below by some $R\in\mathbb{R}$. Hence

$$\begin{aligned}
\left\langle \mathcal{D}\xi, \Psi\circ(\mathcal{D}_1\widehat{\otimes}1)\widehat{\otimes}1\circ\Psi^{-1}\xi\right\rangle + \left\langle \Psi\circ(\mathcal{D}_1\widehat{\otimes}1)\widehat{\otimes}1\circ\Psi^{-1}\xi, \mathcal{D}\xi\right\rangle &\geq R\sum_{k\in\mathbb{Z}^n}\|P_k\xi\|^2 \\
&= R\|\xi\|^2. \qquad \square
\end{aligned}$$

Since the positivity criterion is satisfied, it follows from Theorem 3.4 (or Theorem 3.10 if the Clifford module S is trivially $\mathbb{Z}_2$-graded) that the $\mathbb{T}^n$-equivariant spectral triple $(C^\infty(M), L^2(S), \mathcal{D})$ factorises; that is we can construct unbounded representatives for classes $\alpha\in KK^n(C(M), C(M)^{\mathbb{T}^n})$ and $\beta\in KK^{j+n}(C(M)^{\mathbb{T}^n},\mathbb{C})$ such that

$$\alpha\widehat{\otimes}_{C(M)^{\mathbb{T}^n}}\beta = [(C^\infty(M), L^2(S), \mathcal{D})]\in KK^j(C(M),\mathbb{C}),$$

where $j=0$ (resp. $j=1$) if $(C^\infty(M), L^2(S), \mathcal{D})$ is even (resp. odd).

5.3 The constructive Kasparov product

Recall that (M, g) is a compact Riemannian manifold with a free, isometric left action of $\mathbb{T}^n$, (S, ∇^S) is an equivariant Clifford module over M with Dirac operator $\mathcal{D}$, and $\ell \in \mathbb{Z}^n$ is fixed.

We have seen that $(C^\infty(M), L^2(S), \mathcal{D})$ represents the class in $KK^*_{\mathbb{T}^n}(C(M), \mathbb{C})$ of the product of the unbounded equivariant Kasparov module

$$\Big(\oplus_k A_k, \big(E_1 \widehat{\otimes}_{A^{\mathbb{T}^n} \widehat{\otimes} \mathfrak{C}} (A^{\mathbb{T}^n} \widehat{\otimes} (\Gamma(\$_{\mathbb{T}^n})^{\mathbb{T}^n} \widehat{\otimes} \mathfrak{C})^*)\big)_{A^{\mathbb{T}^n} \widehat{\otimes} \Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}}, \mathcal{D}_1 \widehat{\otimes} 1\Big)$$

(the product of the left-hand and middle modules, where $A = C(M)$) and the equivariant spectral triple $(C^\infty(M)^{\mathbb{T}^n} \widehat{\otimes} \Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}, \mathcal{H}_\ell, \mathcal{D}_\ell)$ (the right-hand module). We now show that the constructive Kasparov product [8, 24, 32, 33] can be used to produce a representative of the product of these two cycles. The representative thus obtained is unitarily equivalent to $(C^\infty(M), L^2(S), T)$ for some self-adjoint, first order elliptic differential operator T on S. If the orbits of $\mathbb{T}^n$ are embedded isometrically into M, then T is a perturbation of the original operator $\mathcal{D}$ by a smooth bundle endomorphism.

Definition 5.7. Let G be a compact group, and let A and B be $\mathbb{Z}_2$-graded C^*-algebras carrying respective actions α and β of G, with A separable and B σ-unital. Let E be a $\mathbb{Z}_2$-graded right Hilbert A-module with a homomorphism V from G into the invertible degree zero bounded operators on E which is compatible with α, and let $(\mathcal{A}, F_B, T)$ be an unbounded equivariant Kasparov A-B-module. There is a natural action of G on $E \widehat{\otimes}_A \mathrm{End}_B(F_B)$ given by $g \cdot (e \widehat{\otimes} B) = V_g(e) \widehat{\otimes} U_g B U_g^{-1}$, where U is the action of G on F_B. A T-**connection** on E is a linear map ∇ from an invariant dense subspace $\mathcal{E} \subset E$ which is a right $\mathcal{A}$-module into $E \widehat{\otimes}_A \mathrm{End}_B(F_B)$, such that $g \cdot \nabla(e) = \nabla(V_g(e))$ for all $g \in G$, $e \in \mathcal{E}$, and such that

$$\nabla(ea) = \nabla(e)a + (-1)^{\deg e} e \widehat{\otimes} [T, a]_\pm, \qquad \text{for all } e \in \mathcal{E},\ a \in \mathcal{A}. \tag{5.3}$$

We define an operator $1 \widehat{\otimes}_\nabla T$ on the dense subspace

$$\mathrm{span}\{e \widehat{\otimes} f : e \in \mathcal{E}, f \in \mathrm{dom}(T)\} \subset E \widehat{\otimes}_A F$$

by

$$(1 \widehat{\otimes}_\nabla T)(e \widehat{\otimes} f) = (-1)^{\deg e} e \widehat{\otimes} T f + \nabla(e) f.$$

The equivariance of ∇ ensures that $1 \widehat{\otimes}_\nabla T$ is equivariant. We say that ∇ is **Hermitian** if for all $e_1, e_2 \in \mathcal{E}$ and $f_1, f_2 \in \mathrm{dom}(T)$,

$$\big(e_1 \widehat{\otimes} f_1 \big| (\nabla e_2) f_2\big)_B - \big((\nabla e_1) f_1 \big| e_2 \widehat{\otimes} f_2\big)_B = (-1)^{\deg e_1} \big(f_1 \big| [T, (e_1|e_2)_A]_\pm f_2\big)_B.$$

Lemma 5.8. *Let ∇ be a T-connection. If ∇ is Hermitian, then $1\widehat{\otimes}_\nabla T$ is symmetric.*

Proof. Suppose ∇ is Hermitian, and let $e_1\widehat{\otimes}f_1$, $e_2\widehat{\otimes}f_2 \in \operatorname{dom}(1\widehat{\otimes}_\nabla T)$. Then

$$\begin{aligned}
&\big(e_1\widehat{\otimes}f_1\big|(1\widehat{\otimes}_\nabla T)(e_2\widehat{\otimes}f_2)\big)_B - \big((1\widehat{\otimes}_\nabla T)e_1\widehat{\otimes}f_1\big|(e_2\widehat{\otimes}f_2\big)_B \\
&= (-1)^{\deg e_2}\big(f_1\big|(e_1|e_2)_A T f_2\big)_B + \big(e_1\widehat{\otimes}f_1\big|(\nabla e_2)f_2\big)_B \\
&\quad - (-1)^{\deg e_1}\big(Tf_1\big|(e_1|e_2)_A f_2\big)_B - \big((\nabla e_1)f_1\big|e_2\widehat{\otimes}f_2\big)_B \\
&= -(-1)^{\deg e_1}\big(f_1\big|[T,(e_1|e_2)_A]_\pm f_2\big)_B + \big(e_1\widehat{\otimes}f_1\big|(\nabla e_2)f_2\big)_B - \big((\nabla e_1)f_1\big|e_2\widehat{\otimes}f_2\big)_B = 0,
\end{aligned}$$

using the fact that T is symmetric. □

We wish to construct a $\mathcal{D}_\ell$-connection which will allow us to take the constructive Kasparov product. We will need suitable coordinates to define the connection, which we now describe.

Let $x \in M$, and choose tangent vectors $(v_1, \dots, v_{m-n})$ which span the subspace $\operatorname{span}\{X_1(x), \dots, X_n(x)\}^\perp \subset T_xM$, where we recall that X_j is the fundamental vector field associated to $\frac{\partial}{\partial t^j} \in T_e\mathbb{T}^n$. Let $(x^1, \dots, x^n, y^1, \dots, y^{m-n})$ be the geodesic normal coordinates around x corresponding to $(X_1(x), \dots, X_n(x), v_1, \dots, v_{m-n})$. There is a neighbourhood U of x such that $U \cong \pi(U) \times \mathbb{T}^n$ as $\mathbb{T}^n$-spaces, where $\pi : M \to M/\mathbb{T}^n$ is the quotient map, so the standard coordinates $(t^1, \dots, t^n) \in (0,1)^n$ on $\mathbb{T}^n$ give us coordinates $(t^1, \dots, t^n, y^1, \dots, y^{m-n})$ in a neighbourhood of x. Since $g(X_j(x), v_p) = 0$ and $X_j = \frac{\partial}{\partial t^j}$, it follows from the fact that a geodesic is orthogonal to one orbit of $\mathbb{T}^n$ if and only if it is orthogonal to every orbit of $\mathbb{T}^n$ that it intersects, [41, Prop. 2], that $g(\frac{\partial}{\partial t^j}, \frac{\partial}{\partial y^p}) = 0$ on the coordinate chart for $j = 1, \dots, n$, $p = 1, \dots, m-n$.

Let $(U_i)_{i=1}^N$ be a finite cover of M by such coordinate neighbourhoods, and for each $k \in \mathbb{Z}^n$ and $i = 1, \dots, N$, define smooth functions $\chi_{i,k}$ on U_i by

$$\chi_{i,k}(t^1, \dots, t^n, y^1, \dots, y^{m-n}) = e^{-2\pi i \sum_{j=1}^n k_j t^j}.$$

Observe that if $h \in C(M)_k$ has support in U_i, then $h\chi_{i,k}^{-1} \in C(M)^{\mathbb{T}^n}$. Let $(\phi_i)_{i=1}^N$ be an invariant partition of unity subordinate to $(U_i)_{i=1}^N$, and for each $i = 1, \dots, N$, let $\psi_i \in C^\infty(M)$ be an invariant function with support in U_i, such that ψ_i is 1 in a neighbourhood of $\operatorname{supp}\phi_i$. For $f \in C(M)$,

$$\Phi_k(f) = \sum_i \phi_i\psi_i\Phi_k(f) = \sum_i \phi_i\chi_{i,k}(\Phi_k(f)\psi_i\chi_{i,k}^{-1}).$$

Let $(x_r)_{r=1}^{2\lfloor n/2\rfloor}$ be an invariant orthonormal frame for $\mathbf{\$}_{\mathbb{T}^n}$ of homogeneous degree, such that x_1 is of even degree (in the case that $\mathbf{\$}_{\mathbb{T}^n}$ is $\mathbb{Z}_2$-graded). For $i = 1, \dots, N$, $k \in \mathbb{Z}^n$

and $r = 1, \dots, 2^{\lfloor n/2 \rfloor}$, we define

$$y_{i,k,r} := \phi_i \chi_{i,k} \widehat{\otimes} (\chi_k x_r \widehat{\otimes} 1)) \widehat{\otimes} (1 \widehat{\otimes} \overline{x_1 \widehat{\otimes} 1}) \in E_1 \widehat{\otimes}_{C(M)^{\mathbb{T}^n} \widehat{\otimes} \mathfrak{C}} \big(C(M)^{\mathbb{T}^n} \widehat{\otimes} (\Gamma(\mathbf{\$}_{\mathbb{T}^n})^{\mathbb{T}^n} \widehat{\otimes} \mathfrak{C})^* \big).$$

We also make the abbreviations

$$f_{i,k} := \Phi_k(f) \psi_i \chi_{i,k}^{-1}, \quad u_{k,r} := x_1 \widehat{\otimes} (x_r \widehat{\otimes} 1 | \chi_k^{-1} p_{\chi_k^{-1}} u)_{\mathfrak{C}}$$

for $f \in C(M)$, $u \in L^2(\mathbf{\$}_{\mathbb{T}^n})$, $i = 1, \dots, N$, $k \in \mathbb{Z}^n$ and $r = 1, \dots, 2^{\lfloor n/2 \rfloor}$. Then given

$$\begin{aligned} &\big((f \widehat{\otimes} u) \widehat{\otimes} (h \widehat{\otimes} \overline{w}) \big) \widehat{\otimes} \xi \\ &\quad \in \Big(E_1 \widehat{\otimes}_{C(M)^{\mathbb{T}^n} \widehat{\otimes} \mathfrak{C}} \big(C(M)^{\mathbb{T}^n} \widehat{\otimes} (\Gamma(\mathbf{\$}_{\mathbb{T}^n})^{\mathbb{T}^n} \widehat{\otimes} \mathfrak{C})^* \big) \Big) \widehat{\otimes}_{C(M)^{\mathbb{T}^n} \widehat{\otimes} \Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}} L^2(S)_\ell, \end{aligned}$$

we may write

$$\big((f \widehat{\otimes} u) \widehat{\otimes} (h \widehat{\otimes} \overline{w}) \big) \widehat{\otimes} \xi = \sum_{k,r,i} y_{i,k,r} \widehat{\otimes} f_{i,k} h \eta\big({}_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}} (u_{k,r} | w) \big) \xi. \tag{5.4}$$

Define a $\mathcal{D}_\ell$-connection on $E_1 \widehat{\otimes}_{C(M)^{\mathbb{T}^n} \widehat{\otimes} \mathfrak{C}} \big(C(M)^{\mathbb{T}^n} \widehat{\otimes} (\Gamma(\mathbf{\$}_{\mathbb{T}^n}) \widehat{\otimes} \mathfrak{C})^* \big)$ by

$$\nabla\big((f \widehat{\otimes} u) \widehat{\otimes} (h \widehat{\otimes} \overline{w}) \big) = \sum_{k \in \mathbb{Z}^n} \sum_{r=1}^{2^{\lfloor n/2 \rfloor}} \sum_{i=1}^{N} (-1)^{\deg x_r} y_{i,k,r} \widehat{\otimes} \big[\mathcal{D}_\ell, f_{i,r} h \eta\big({}_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}} (u_{k,r} | w) \big) \big]_\pm.$$

Equation (5.4) implies that ∇ is equivariant and satisfies (5.3).

Lemma 5.9. *The $\mathcal{D}_\ell$-connection ∇ is Hermitian.*

Proof. Take $\big((f^1 \widehat{\otimes} u^1) \widehat{\otimes} (h^1 \widehat{\otimes} \overline{w^1}) \big) \widehat{\otimes} \xi^1$ and $\big((f^2 \widehat{\otimes} u^2) \widehat{\otimes} (h^2 \widehat{\otimes} \overline{w^2}) \big) \widehat{\otimes} \xi^2$, each of homogeneous degree. Then

$$\begin{aligned} &\Big\langle \big((f^1 \widehat{\otimes} u^1) \widehat{\otimes} (h^1 \widehat{\otimes} \overline{w^1}) \big) \widehat{\otimes} \xi^1, \nabla\big((f^2 \widehat{\otimes} u^2) \widehat{\otimes} (h^2 \widehat{\otimes} \overline{w^2}) \big) \xi^2 \Big\rangle \\ &\quad - \Big\langle \nabla\big((f^1 \widehat{\otimes} u^1) \widehat{\otimes} (h^1 \widehat{\otimes} \overline{w^1}) \big) \xi^1, \big((f^2 \widehat{\otimes} u^2) \widehat{\otimes} (h^2 \widehat{\otimes} \overline{w^2}) \big) \widehat{\otimes} \xi^2 \Big\rangle \\ &= \sum_{i,k,r} (-1)^{\deg x_r} \\ &\Big\langle \phi_i f_{i,k}^1 h^1 \eta\big({}_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}} (u_{k,r}^1 | w^1) \big) \xi^1, \big[\mathcal{D}_\ell, f_{i,k}^2 h^2 \eta\big({}_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}} (u_{k,r}^2 | w^2) \big) \big]_\pm \xi_2 \Big\rangle \\ &\quad - \Big\langle \big[\mathcal{D}_\ell, f_{i,k}^1 h^1 \eta\big({}_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}} (u_{k,r}^1 | w^1) \big) \big]_\pm \xi_1, \phi_i f_{i,k}^2 h^2 \eta\big({}_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}} (u_{k,r}^2 | w^2) \big) \xi^2 \Big\rangle \end{aligned}$$

using the fact that

$$\begin{aligned} &\big(y_{i,k,r} \big| (f \widehat{\otimes} u) \widehat{\otimes} (h \widehat{\otimes} \overline{w}) \big)_{C(M)^{\mathbb{T}^n} \widehat{\otimes} \Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}} \\ &= \phi_i \chi_{i,k}^{-1} \Phi_k(f) h \widehat{\otimes}_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}} (x_1 \widehat{\otimes} (x_r \widehat{\otimes} 1 | \chi_k^{-1} p_{\chi_k^{-1}} u)_{\mathfrak{C}} | w) = \phi_i f_{i,k} h \widehat{\otimes}_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}} (u_{k,r} | w). \end{aligned}$$

Using the identity $[\mathcal{D}_\ell, a]_\pm^* = -(-1)^{\deg a}[\mathcal{D}_\ell, a^*]_\pm$ and the commutator relation

$$[\mathcal{D}_\ell, ab]_\pm = (-1)^{\deg a} a[\mathcal{D}_\ell, b]_\pm + [\mathcal{D}_\ell, a]_\pm b,$$

we find that

$$\begin{aligned}
&\Big\langle \big((f^1\widehat{\otimes}u^1)\widehat{\otimes}(h^1\widehat{\otimes}\overline{w^1})\big)\widehat{\otimes}\xi^1, \nabla\big((f^2\widehat{\otimes}u^2)\widehat{\otimes}(h^2\widehat{\otimes}\overline{w^2})\big)\xi^2\Big\rangle \\
&- \Big\langle \nabla\big((f^1\widehat{\otimes}u^1)\widehat{\otimes}(h^1\widehat{\otimes}\overline{w^1})\big)\xi^1, \big((f^2\widehat{\otimes}u^2)\widehat{\otimes}(h^2\widehat{\otimes}\overline{w^2})\big)\widehat{\otimes}\xi^2\Big\rangle \\
&= \sum_{i,k,r}(-1)^{\deg u^1+\deg w^1} \\
&\Big\langle \xi^1, \phi_i\big[\mathcal{D}_\ell, (f^1_{i,k})^* f^2_{i,k}(h^1)^* h^2 \eta\big(_{\Gamma(\mathrm{Cl}(\mathbb{T}^n))^{\mathbb{T}^n}}(w^1|w^2(u^2_{k,r}|u^1_{k,r})_{\mathfrak{C}})\big)\big]_\pm \xi_2\Big\rangle .
\end{aligned}$$

Since $\mathcal{D}_\ell$ is local and ψ_i is 1 in a neighbourhood of supp ϕ_i, we may replace $(f^1_{i,k})^* f^2_{i,k}$ by $\Phi_k(f^1)^*\Phi_k(f^2)$ in the above equation. Using $\sum_r (u^2_{k,r}|u^1_{k,r})_{\mathfrak{C}} = (\chi_k^{-1} p_{\chi_k^{-1}} u^2|\chi_k^{-1} p_{\chi_k^{-1}} u^1)_{\mathfrak{C}}$ and $\sum_i \phi_i = 1$ then allows us to resolve the sums over i and r, and then substituting the identity

$$\begin{aligned}
&\big((f^1\widehat{\otimes}u^1)\widehat{\otimes}(h^1\widehat{\otimes}\overline{w^1})\big|\big((f^1\widehat{\otimes}u^1)\widehat{\otimes}(h^1\widehat{\otimes}\overline{w^1})\big)_{C(M)^{\mathbb{T}^n}\widehat{\otimes}\Gamma(\mathrm{Cl}(\mathbb{T}^n))^{\mathbb{T}^n}} \\
&= \sum_{k\in\mathbb{Z}^n} \Phi_k(f^1)^*\Phi_k(f^2)(h^1)^* h^2 \widehat{\otimes}_{\Gamma(\mathrm{Cl}(\mathbb{T}^n))^{\mathbb{T}^n}} \big(w^1\big|w^2(\chi_k^{-1} p_{\chi_k^{-1}} u^2|\chi_k^{-1} p_{\chi_k^{-1}} u^1)_{\mathfrak{C}}\big),
\end{aligned}$$

completes the proof that ∇ is Hermitian. □

Writing $1\widehat{\otimes}_\nabla \mathcal{D}_\ell = (1\widehat{\otimes}1)\widehat{\otimes}_\nabla \mathcal{D}_\ell$ and $\mathcal{D}_1\widehat{\otimes}1 = (\mathcal{D}_1\widehat{\otimes}1)\widehat{\otimes}1$ for short, the following result shows that the constructive Kasparov product yields a spectral triple.

Theorem 5.10. *For $j = 1, \ldots, n$, let $X_j \in \Gamma^\infty(M)$ be the fundamental vector field associated to $\frac{\partial}{\partial t^j} \in T_e\mathbb{T}^n$. Let $(h_{jk})_{j,k=1}^n = (g(X_j, X_k))_{j,k=1}^n$, $(h^{jk}) = (h_{jk})^{-1}$, and let $(W^{jk})_{j,k=1}^n$ be the normalisation functions; i.e. $(W^{jk}) = \sqrt{(h^{jk})}$. Then*

$$\Psi \circ \Big(1\widehat{\otimes}_\nabla \mathcal{D}_\ell + \mathcal{D}_1\widehat{\otimes}1\Big) \circ \Psi^{-1} = \mathcal{D} + \sum_{j,p=1}^n (W^{pj} - h^{pj}) c(X_p^\flat)\nabla^S_{X_j} + B,$$

where $B \in \Gamma^\infty(\mathrm{End}(S))$. Thus $\Psi \circ (1\widehat{\otimes}_\nabla \mathcal{D}_\ell + \mathcal{D}_1\widehat{\otimes}1) \circ \Psi^{-1}$ is a first order, self-adjoint, equivariant, elliptic differential operator. Hence

$$(C^\infty(M), L^2(S), \Psi \circ (1\widehat{\otimes}_\nabla \mathcal{D}_\ell + \mathcal{D}_1\widehat{\otimes}1) \circ \Psi^{-1})$$

is an equivariant spectral triple representing the Kasparov product (which is also represented by $(C^\infty(M), L^2(S), \mathcal{D})$).

Proof. First we note that if S is $\mathbb{Z}_2$-graded, then ∇ is odd, and so in that case the

operator $1\widehat{\otimes}_\nabla \mathcal{D}_\ell + \mathcal{D}_1\widehat{\otimes}1$ is odd. Given $\xi \in L^2(S)$,

$$\Psi^{-1}(\xi) = \sum_{i=1}^{N}\sum_{k\in\mathbb{Z}^n}\sum_{r=1}^{2^{\lfloor n/2\rfloor}} \big((\psi_i\chi_{i,k-\ell}\widehat{\otimes}(\chi_{k-\ell}^{-1}x_r\widehat{\otimes}1))\widehat{\otimes}(1\widehat{\otimes}\overline{x_r\widehat{\otimes}1})\big)\widehat{\otimes}\chi_{i,\ell-k}\phi_i P_k\xi.$$

Using this expression for Ψ^{-1} we can compute

$$\begin{aligned}\Psi\circ 1\widehat{\otimes}_\nabla\mathcal{D}_\ell\circ\Psi^{-1} = &\sum_{k,r,i,q}(-1)^{\deg x_r}\phi_i\chi_{i,k-\ell}\eta\big({}_{\Gamma(\mathbb{Cl}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_r\widehat{\otimes}1|x_1\widehat{\otimes}1)\big)\\ &\times\big[\mathcal{D},\psi_q\chi_{q,k-\ell}\psi_i\chi_{i,k-\ell}^{-1}\eta\big({}_{\Gamma(\mathbb{Cl}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_1\widehat{\otimes}1|x_r\widehat{\otimes}1)\big)\big]_\pm\chi_{q,\ell-k}\phi_q P_k\\ &+\sum_{i,k,r}\psi_i\chi_{i,k-\ell}\eta\big({}_{\Gamma(\mathbb{Cl}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_r\widehat{\otimes}1|x_r\widehat{\otimes}1)\big)\mathcal{D}\chi_{i,\ell-k}\phi_i P_k. \qquad (5.5)\end{aligned}$$

Since $\sum_{r=1}^{2^{\lfloor n/2\rfloor}}{}_{\Gamma(\mathbb{Cl}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_r\widehat{\otimes}1|x_1\widehat{\otimes}1) = 1$ and $\sum_{i=1}^N\phi_i = 1$, the second term of Equation (5.5) simplifies to

$$\begin{aligned}&\sum_{i=1}^{N}\sum_{k\in\mathbb{Z}^n}\sum_{r=1}^{2^{\lfloor n/2\rfloor}}\psi_i\chi_{i,k-\ell}\eta\big({}_{\Gamma(\mathbb{Cl}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_r\widehat{\otimes}1|x_r\widehat{\otimes}1)\big)\mathcal{D}\chi_{i,\ell-k}\phi_i P_k\\ &=\sum_{i=1}^{N}\sum_{k\in\mathbb{Z}^n}\chi_{i,k-\ell}[\mathcal{D},\psi_i\chi_{i,\ell-k}]\phi_i P_k + \mathcal{D}.\end{aligned}$$

Let I denote the first term of Equation (5.5). By several applications of the graded commutator relation $[a,bc]_\pm = (-1)^{\deg b}b[a,c]_\pm + [a,b]_\pm c$, we can simplify I to

$$\begin{aligned}I = &\sum_{k\in\mathbb{Z}^n}\sum_{q=1}^{N}[\mathcal{D},\psi_q\chi_{q,k-\ell}]\chi_{q,\ell-k}\phi_q P_k + \sum_{k\in\mathbb{Z}^n}\sum_{r=1}^{2^{\lfloor n/2\rfloor}}\sum_{i=1}^{N}(-1)^{\deg x_r}\phi_i\chi_{i,k-\ell}\\ &\times\eta\big({}_{\Gamma(\mathbb{Cl}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_r\widehat{\otimes}1|x_1\widehat{\otimes}1)\big)[\mathcal{D},\psi_i\chi_{i,k-\ell}^{-1}]\eta\big({}_{\Gamma(\mathbb{Cl}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_1\widehat{\otimes}1|x_r\widehat{\otimes}1)\big)P_k\\ &+\sum_{r=1}^{2^{\lfloor n/2\rfloor}}(-1)^{\deg x_r}\eta\big({}_{\Gamma(\mathbb{Cl}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_r\widehat{\otimes}1|x_1\widehat{\otimes}1)\big)\big[\mathcal{D},\eta\big({}_{\Gamma(\mathbb{Cl}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_1\widehat{\otimes}1|x_r\widehat{\otimes}1)\big)\big]_\pm.\end{aligned}$$

Recall that $\chi_{i,k} = e^{-2\pi i\sum_{j=1}^n t^j k_j}$ in the coordinates $(t^1,\dots,t^n,y^1,\dots,y^{m-n})$ on U_i, and so $\chi_{i,k}^{-1}[\mathcal{D},\psi_i\chi_{i,k}] = \chi_{i,k}^{-1}c(d\chi_{i,k}) = -2\pi i\sum_j k_j c(dt^j)$.

Write

$$\mathcal{D} = \sum_{j=1}^{n}c(dt^j)\nabla^S_{X_j} + \sum_{s=1}^{m-n}c(dy^s)\nabla^S_{\partial_{y^s}}.$$

Since $g(\partial_{t^j},\partial_{y^p}) = 0$ and $X_j^\flat = \sum_{p=1}^n h_{jk}dt^k$, the Clifford vector $c(dy^p)$ anticommutes with $c(X_j^\flat)$ and hence graded commutes with the image of $\Gamma(\mathbb{Cl}(\mathbb{T}^n))^{\mathbb{T}^n}$ under η for each $p = 1,\dots,m-n$. Using this commutativity as well as the compatibility of ∇^S

with the Levi-Civita connection, the first term of Equation (5.5) is locally

$$
\begin{aligned}
I = &-2\pi i \sum_{k,j} c(dt^j)(k_j - \ell_j)P_k + 2\pi i \sum_{k,r,j} (-1)^{\deg x_r} \\
&\qquad \times \eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_r \widehat{\otimes} 1 | x_1 \widehat{\otimes} 1)\big) c(dt^j) \eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_1 \widehat{\otimes} 1 | x_r \widehat{\otimes} 1)\big)(k_j - \ell_j)P_k \\
&+ \sum_{r,j} (-1)^{\deg x_r} \eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_r \widehat{\otimes} 1 | x_1 \widehat{\otimes} 1)\big) \big[c(dt^j)\nabla^S_{X_j}, \eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_1 \widehat{\otimes} 1 | x_r \widehat{\otimes} 1)\big)\big]_\pm \\
&+ \sum_{r,p} (-1)^{\deg x_r} \eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_r \widehat{\otimes} 1 | x_1 \widehat{\otimes} 1)\big) \big[c(dy^p)\nabla^S_{\partial_{y^p}}, \eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_1 \widehat{\otimes} 1 | x_r \widehat{\otimes} 1)\big)\big]_\pm \\
= &\sum_j c(dt^j)(\nabla^S_{X_j} + \omega_j - 2\pi\ell_j) - \sum_{r,j} (-1)^{\deg x_r} \eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_r \widehat{\otimes} 1 | x_1 \widehat{\otimes} 1)\big) \\
&\qquad \times c(dt^j) \eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_1 \widehat{\otimes} 1 | x_r \widehat{\otimes} 1)\big)(\nabla^S_{X_j} + \omega_j - 2\pi\ell_j) \\
&+ \sum_{r,j} (-1)^{\deg x_r} \eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_r \widehat{\otimes} 1 | x_1 \widehat{\otimes} 1)\big) c(dt^j) \nabla^{LC}_{X_j}\big(\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_1 \widehat{\otimes} 1 | x_r \widehat{\otimes} 1)\big)\big) \\
&+ \sum_{r,j} (-1)^{\deg x_r} \eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_r \widehat{\otimes} 1 | x_1 \widehat{\otimes} 1)\big) \big[c(dt^j), \eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_1 \widehat{\otimes} 1 | x_r \widehat{\otimes} 1)\big)\big]_\pm \nabla^S_{X_j} \\
&+ \sum_{r,p} (-1)^{\deg x_r} \eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_r \widehat{\otimes} 1 | x_1 \widehat{\otimes} 1)\big) c(dy^p) \nabla^{LC}_{\partial_{y^p}}\big(\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_1 \widehat{\otimes} 1 | x_r \widehat{\otimes} 1)\big)\big)
\end{aligned}
$$

for $\omega_j \in \Gamma^\infty(\mathrm{End}(S))$ for $j = 1, \ldots, n$, using $A_j = 2\pi \sum_{k\in\mathbb{Z}^n} k_j P_k$ and Lemma 5.3. Here ∇^{LC} denotes the extension of the Levi-Civita connection on the cotangent bundle to the Clifford bundle. Using $\sum_{r=1}^{2\lfloor n/2\rfloor} {}_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_r \widehat{\otimes} 1 | x_1 \widehat{\otimes} 1)_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_1 \widehat{\otimes} 1 | x_r \widehat{\otimes} 1) = 1$ and the fact that $c(dy^p)$ graded commutes with the image of η, we can make some cancellations and, working locally, simplify the first term of Equation (5.5) to

$$
\begin{aligned}
I = &\sum c(dt^j)(\omega_j - 2\pi\ell_j) - \sum (-1)^{\deg x_r} \eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_r \widehat{\otimes} 1 | x_1 \widehat{\otimes} 1)\big) c(dt^j) \\
&\qquad \times \eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_1 \widehat{\otimes} 1 | x_r \widehat{\otimes} 1)\big)(\omega_j - 2\pi\ell_j) \\
&+ \sum (-1)^{\deg x_r} \eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_r \widehat{\otimes} 1 | x_1 \widehat{\otimes} 1)\big) c(dt^j) \nabla^{LC}_{X_j}\big(\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_1 \widehat{\otimes} 1 | x_r \widehat{\otimes} 1)\big)\big) \\
&+ \sum (-1)^{\deg x_r} \eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_r \widehat{\otimes} 1 | x_1 \widehat{\otimes} 1)\big) c(dy^p) \nabla^{LC}_{\partial_{y^p}}\big(\eta\big(_{\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}^n))^{\mathbb{T}^n}}(x_1 \widehat{\otimes} 1 | x_r \widehat{\otimes} 1)\big)\big)
\end{aligned}
$$

which is a smooth bundle endomorphism. The second term of Equation (5.5) is

$$
\begin{aligned}
&\sum_{i=1}^N \sum_{k\in\mathbb{Z}^n} \chi_{i,k-\ell}[\mathcal{D}, \psi_i \chi_{i,\ell-k}]\phi_i P_k = 2\pi i \sum_{k\in\mathbb{Z}^n} \sum_{j=1}^n c(dt^j)(k_j - \ell_j)P_k \\
&= -\sum_j c(dt^j)(\nabla^S_{X_j} + \omega_j - 2\pi\ell_j) = -\sum_{j,q} h^{jq} c(X_q^\flat)(\nabla^S_{X_j} + \omega_j - 2\pi\ell_j)
\end{aligned}
$$

for some $\omega_j \in \Gamma^\infty(\mathrm{End}(S))$ by Lemma 5.3. Putting the expressions for Equation (5.5)

together with Lemma 5.5 and Lemma 5.3 yields

$$\begin{aligned}\Psi \circ (1\widehat{\otimes}_\nabla \mathcal{D}_\ell + \mathcal{D}_1\widehat{\otimes}1) \circ \Psi^{-1} &= \mathcal{D} + \sum_{j,p}(W^{pj} - h^{pj})c(X_p^\flat)\nabla^S_{X_j} + B \\ &= \sum_p c(dy^p)\nabla^S_{\partial_{y^p}} + \sum_{j,p,q} W^{pj}h_{pq}c(dt^q)\nabla^S_{X_j} + B \end{aligned} \tag{5.6}$$

for some $B \in \Gamma^\infty(\mathrm{End}(S))$, which establishes that $\Psi \circ (1\widehat{\otimes}_\nabla \mathcal{D}_\ell + \mathcal{D}_1\widehat{\otimes}1) \circ \Psi^{-1}$ is a first order differential operator. Since $(W^{rj})_{rj=1}^n$ and $(h_{rq})_{r,q=1}^n$ are invertible, Equation (5.6) also shows that the operator $\Psi \circ (1\widehat{\otimes}_\nabla \mathcal{D}_\ell + \mathcal{D}_1\widehat{\otimes}1) \circ \Psi^{-1}$ is elliptic. Since ∇ is Hermitian, $1\widehat{\otimes}_\nabla \mathcal{D}_\ell$ is symmetric, and so $1\widehat{\otimes}_\nabla \mathcal{D}_\ell + \mathcal{D}_1\widehat{\otimes}1$ is the sum of a symmetric operator with a self-adjoint operator, which is symmetric. Elliptic operator theory, [22, 30], implies that $\Psi \circ (1\widehat{\otimes}_\nabla \mathcal{D}_\ell + \mathcal{D}_1\widehat{\otimes}1) \circ \Psi^{-1}$ is essentially self-adjoint with compact resolvent, and hence $(C^\infty(M), L^2(S), \Psi \circ (1\widehat{\otimes}_\nabla \mathcal{D}_\ell + \mathcal{D}_1\widehat{\otimes}1) \circ \Psi^{-1})$ is an equivariant spectral triple. That $(C^\infty(M), L^2(S), \Psi \circ (1\widehat{\otimes}_\nabla \mathcal{D}_\ell + \mathcal{D}_1\widehat{\otimes}1) \circ \Psi^{-1})$ represents the product is now a straightforward application of Kucerovsky's criteria. □

Corollary 5.11. *Suppose that each orbit is an isometric embedding of $\mathbb{T}^n$ in M. Equivalently, the fundamental vector fields $T_e\mathbb{T}^n \ni v \mapsto X^{(v)} \in \Gamma^\infty(TM)$ satisfy $(X^{(v)}|X^{(v)})_{C(M)} = \|v\|^2$. Then*

$$\mathcal{D} - \Psi \circ (1\widehat{\otimes}_\nabla \mathcal{D}_\ell + \mathcal{D}_1\widehat{\otimes}1) \circ \Psi^{-1} \in \Gamma^\infty(\mathrm{End}(S)).$$

Proof. In this case, $h^{jk} = W^{jk} = \delta^{jk}$, and so the identity in Theorem 5.10 becomes $\Psi \circ (1\widehat{\otimes}_\nabla \mathcal{D}_\ell + \mathcal{D}_1\widehat{\otimes}1) \circ \Psi^{-1} = \mathcal{D} + B$ where $B \in \Gamma^\infty(\mathrm{End}(S))$. □

Theorem 5.10 implies that the constructive Kasparov product is sensitive to metric data. Factorisation on the level of the constructive Kasparov product (and not just the level of KK-classes) has applications to gauge theory, as in [8], but Theorem 5.10 shows that in many cases this finer factorisation may not be achieved. An example of when the hypothesis of Corollary 5.11 is satisfied is when $\mathbb{T}^n$ is a subgroup of a compact Lie group, acting on the Lie group by left multiplication. This case includes the 2-torus and Hopf fibration examples of [8].

Chapter 6

Example: the Dirac operator on the 2-sphere

The spin Dirac operator $\mathcal{D}$ on the spinor bundle $\$_{S^2}$ over the 2-sphere S^2 defines an even spectral triple $(C^\infty(S^2), L^2(\$_{S^2}), \mathcal{D})$. The circle acts on S^2 by rotation about the north-south axis, and there are countably infinitely many lifts of this action to $L^2(\$_{S^2})$, such that $(C^\infty(S^2), L^2(\$_{S^2}), \mathcal{D})$ is an equivariant spectral triple. One can then ask whether any of these spectral triples can be factorised, but since the action of $\mathbb{T}$ on S^2 is not free we cannot apply the earlier theory.

In fact, we cannot factorise $(C^\infty(S^2), L^2(\$_{S^2}), \mathcal{D})$, since the spectral subspace assumption is not satisfied, and, more seriously, $K^1(C(S^2)^{\mathbb{T}}) = K^1([0,1]) = \{0\}$. Since the class of the spectral triple $(C^\infty(S^2), L^2(\$_{S^2}), \mathcal{D})$ in $K^0(C(S^2))$ is non-zero, it is impossible to recover this class under the Kasparov product between $KK^1(C(S^2), C(S^2)^{\mathbb{T}})$ and $KK^1(C(S^2)^{\mathbb{T}}, \mathbb{C}) = \{0\}$. It follows that factorisation is also impossible in equivariant KK-theory, since if factorisation were possible in $KK_{\mathbb{T}}$ then the forgetful functor $KK_{\mathbb{T}} \to KK$ would imply it would also work in non-equivariant KK-theory.

Instead, we remove the poles, and instead consider the equivariant spectral triple $(C_c^\infty(S^2 \setminus \{N, S\}), L^2(\$_{S^2}), \mathcal{D})$ and ask whether this equivariant spectral triple can be factorised. The circle now acts freely, and hence the spectral subspace assumption is satisfied.

We show that factorisation is achieved for $(C_c^\infty(S^2 \setminus \{N, S\}), L^2(\$_{S^2}), \mathcal{D})$ for every possible lift of the circle action, thus providing a non-compact example of factorisation. Unlike for a free action on a compact manifold, the positivity criterion is satisfied for precisely two choices of the character $\ell \in \mathbb{Z}$ of Definition 2.14 used to define the right-hand module.

We will describe the Dirac operator $\mathcal{D}$ on the spinor bundle $\$_{S^2}$ over S^2, [20, 50].

Let N be the North pole of S^2, and let U_N be $S^2 \setminus \{N\}$. A chart for U_N is given by stereographic projection onto $\mathbb{C}$. This chart defines a trivialisation of the spinor bundle

$\$_{S^2}$. All work will be done in the U_N trivialisation unless explicitly stated otherwise. We will work in the standard polar coordinates $(\theta, \phi) \in (0, \pi) \times (0, 2\pi)$. These coordinates are not valid on all of S^2, but since they are defined on a dense subset of S^2 it is enough to work exclusively in these coordinates in order the check the positivity criterion of Theorem 3.4.

The spinor Dirac operator is given by

$$\mathcal{D} = \begin{pmatrix} 0 & e^{i\phi}\big(i\partial_\theta + \csc(\theta)\partial_\phi + i\cot(\theta/2)/2\big) \\ e^{-i\phi}\big(i\partial_\theta - \csc(\theta)\partial_\phi + i\cot(\theta/2)/2\big) & 0 \end{pmatrix}. \tag{6.1}$$

The Hilbert space $L^2(\$_{S^2})$ is graded by $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$; i.e.

$$L^2(\$_{S^2})^j = \{\xi \in L^2(\$_{S^2}) : \gamma\xi = (-1)^j\xi\}.$$

The action of the circle $\mathbb{T}$ on S^2 is $t \cdot (\theta, \phi) = (\theta, \phi + 2\pi t)$. There are countably infinitely many lifts of this action which make $(C^\infty(S^2), L^2(\$_{S^2}), \mathcal{D})$ into a $\mathbb{T}$-equivariant spectral triple.

Proposition 6.1. *Any representation of the circle on $L^2(\$_{S^2})$ which makes the spectral triple $(C^\infty(S^2), L^2(\$_{S^2}), \mathcal{D})$ into a $\mathbb{T}$-equivariant spectral triple is equal to V_k for some $k \in \mathbb{Z}$, where*

$$V_{k,t}\begin{pmatrix} f(\theta,\phi) \\ g(\theta,\phi) \end{pmatrix} := \begin{pmatrix} e^{2\pi ikt} f(\theta, \phi - 2\pi t) \\ e^{2\pi i(k-1)t} g(\theta, \phi - 2\pi t) \end{pmatrix}.$$

Proof. We require the action of $\mathbb{T}$ on $L^2(\$_{S^2})$ to be compatible with the action α of $\mathbb{T}$ on $C(S^2)$, which is $\alpha_t(f)(\theta, \phi) = f(\theta, \phi - 2\pi t)$. Hence the action on spinors is of the form

$$V_t\begin{pmatrix} f(\theta,\phi) \\ g(\theta,\phi) \end{pmatrix} = \begin{pmatrix} a & b \\ d & h \end{pmatrix}\begin{pmatrix} f(\theta, \phi - 2\pi t) \\ g(\theta, \phi - 2\pi t) \end{pmatrix},$$

where a, b, d and h can *a priori* depend on θ, ϕ and t. Since the action of $\mathbb{T}$ should commute with the grading, we require $b = d = 0$. Requiring that the action is unitary, that it commutes with $\mathcal{D}$ and that it is a group homomorphism determines that $a = e^{2\pi ikt}$ and $h = e^{2\pi i(k-1)t}$ for some $k \in \mathbb{Z}$. □

Remark. The spinor bundle has a real structure provided by a conjugation operator

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \circ C,$$

where C is complex conjugation. None of these actions preserve the real structure (i.e. commute with J) , so they are spin^c but not spin actions. There is however a unique

lift of the "double" action of $\mathbb{T}$, $t \cdot (\theta, \phi) = (\theta, \phi + 4\pi t)$, to a spin action given by setting $k = 1/2$ and replacing t by $2t$ in Proposition 6.1.

We fix $k \in \mathbb{Z}$ for the remainder of this section, which fixes a unitary representation $V_k : \mathbb{T} \to U(L^2(\$_{S^2}))$.

The spectral subspaces of $C(S^2)$ are

$$C(S^2)_j = \begin{cases} \{f(\theta) : f \in C([0,1])\} & \text{if } j = 0 \\ \{f(\theta)e^{-ij\phi} : f \in C_0((0,1))\} & \text{if } j \neq 0. \end{cases}$$

Hence

$$\overline{C(S^2)_j C(S^2)_j^*} \cong \begin{cases} C([0,1]) & \text{if } j = 0 \\ C_0((0,1)) & \text{if } j \neq 0. \end{cases}$$

Since $C_0((0,1))$ is not a complemented ideal in $C(S^2)^{\mathbb{T}} \cong C([0,1])$, $C(S^2)$ does not satisfy the spectral subspace assumption, and so we cannot define the left-hand module if we use the C^*-algebra $C(S^2)$. However, the SSA is satisfied for $C_0(S^2 \setminus \{N, S\})$, since the action on $S^2 \setminus \{N, S\}$ is free, [37, Thm. 7.2.6].

By taking the fundamental vector field map and normalising as in §5, we define the map $\eta : \Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))^{\mathbb{T}} \to B(L^2(\$_{S^2}))$ by

$$\eta(c(dt)) = -\frac{1}{\sqrt{g(d\phi, d\phi)}} c(d\phi) = \begin{pmatrix} 0 & -e^{i\phi} \\ e^{-i\phi} & 0 \end{pmatrix}$$

We check that η satisfies the conditions of Definition 2.14. Clearly $\eta(c(dt))$ commutes with the algebra, so Condition (1) is satisfied. Since $a\eta(c(dt))$ is a smooth bundle endomorphism for all $a \in C_c^\infty(S^2 \setminus \{S, N\})$, $a\eta(c(dt))$ preserves $\mathrm{dom}(\mathcal{D})$. It remains to check the commutation condition. We compute:

$$\{\mathcal{D}, \eta(c(dt))\} = \begin{pmatrix} 2\csc(\theta)\partial_\phi - i\csc(\theta) & 0 \\ 0 & 2\csc(\theta)\partial_\phi + i\csc(\theta) \end{pmatrix}.$$

Hence if $f(\theta)e^{-ij\phi} \in C_c^\infty(S^2 \setminus \{S, N\})_j$, then

$$\begin{aligned} &\{\mathcal{D}, \eta(c(dt))\} f(\theta)e^{-ij\phi} P_\ell \\ &= \begin{pmatrix} 2i\csc(\theta)(k - \ell - j) - i\csc(\theta) & 0 \\ 0 & 2i\csc(\theta)(k - \ell - j - 1) + i\csc(\theta) \end{pmatrix} \\ &\qquad \times f(\theta)e^{-ij\phi} P_\ell \\ &= -i\csc(\theta)(2j + 2\ell - 2k + 1) f(\theta)e^{-ij\phi} P_\ell. \end{aligned}$$

Since $f \in C_c((0,\pi))$, $\{\mathcal{D}, \eta(c(dt))\} f(\theta)e^{-ij\phi} P_\ell$ is bounded, and so Condition 2) of Definition 2.14 is satisfied. Therefore (ℓ, η) satisfy the conditions of Definition 2.14 for any

$\ell \in \mathbb{Z}$.

Let $n, \ell \in \mathbb{Z}$, and let $\xi = \begin{pmatrix} f(\theta)e^{i(k-n-\ell)\phi} \\ g(\theta)e^{i(k-n-\ell-1)\phi} \end{pmatrix} \in \mathrm{dom}(\mathcal{D}) \cap L^2(\$_{S^2})_{n+\ell}$. Then as in §3.3 the positivity criterion reduces to (after integrating over the ϕ coordinates)

$$\begin{aligned}
&\langle \mathcal{D}\xi, in\eta(c(dt))P_{n+\ell}\xi\rangle + \langle in\eta(c(dt))P_{n+\ell}\xi, \mathcal{D}\xi\rangle \\
&= \int_0^\pi d\theta \sin(\theta) \times \\
&\quad \Big(\overline{(ig'(\theta) + i(k-n-\ell-1)\csc(\theta)g(\theta) + i\cot(\theta/2)g(\theta)/2)}(-ing(\theta)) \\
&\quad + \overline{(if'(\theta) - i(k-n-\ell)\csc(\theta)f(\theta) + i\cot(\theta/2)f(\theta)/2)}(inf(\theta)) \\
&\quad + \overline{g(\theta)}(ig'(\theta) + i(k-n-\ell-1)\csc(\theta)g(\theta) + i\cot(\theta/2)g(\theta)/2) \\
&\quad + \overline{inf(\theta)}(if'(\theta) - i(k-n-\ell)\csc(\theta)f(\theta) + i\cot(\theta/2)f(\theta)/2)\Big) \\
&= 4\pi n(n-k+\ell+1/2)\int_0^\pi d\theta\Big(|f(\theta)|^2 + |g(\theta)|^2\Big).
\end{aligned}$$

If $p(n) = 2n(n-k+\ell+1/2)$ is non-negative for all $n \in \mathbb{Z}$, then the positivity criterion is satisfied. Conversely, since $\int_0^\pi d\theta\Big(|f(\theta)|^2 + |g(\theta)|^2\Big)$ is not bounded by $\|\xi\|^2$, if $p(n) < 0$ for some $n \in \mathbb{Z}$, then $\langle \mathcal{D}\xi, -in\eta(c(dt))P_{n+\ell}\xi\rangle + \langle -in\eta(c(dt))P_{n+\ell}\xi, \mathcal{D}\xi\rangle$ is not bounded from below and the positivity criterion is not satisfied.

Since $\ell \in \mathbb{Z}$ has thus far not been fixed, we will determine for which values of ℓ the polynomial $p : \mathbb{Z} \to \mathbb{R}$ is non-negative. As a real-valued polynomial, p has a minimum at $x = (k-\ell)/2 - 1/4$.

Suppose $k-\ell$ is even. Then the integer values of n either side of this minimum are $n = (k-\ell)/2-1$ and $n = (k-\ell)/2$, at which $p(n)$ has values $-(\ell-k+2)(\ell-k-1)/2$ and $-(\ell-k+1)(\ell-k)/2$ respectively. The smaller of these two values is $(\ell-k+1)(\ell-k)/2$. As a function of ℓ, $q(\ell) = -(\ell-k+1)(\ell-k)/2$ has a maximum at $\ell = k-1/2$. The integer values on either side of this maximum with $k-\ell$ even are $\ell = k$ and $\ell = k-2$, at which $q(\ell)$ has respective values 0 and -1. Therefore if $k-\ell$ is even, then $p(n)$ is non-negative if and only if $\ell = k$.

Suppose now that $k-\ell$ is odd. Then the integer values of n either side of the minimum $n = (k-\ell)/2 - 1/4$ are $n = (k-\ell)/2 - 1/2$ and $n = (k-\ell)/2 + 1/2$, at which $p(n)$ has respective values $-(\ell-k+1)(\ell-k)/2$ and $-(\ell-k+2)(\ell-k-1)/2$, the smallest of which is $p((k-\ell)/2 - 1/2) = -(\ell-k+1)(\ell-k)/2$. As a function of ℓ, $r(\ell) = -(\ell-k+1)(\ell-k)/2$ has a maximum at $\ell = k-1/2$. The values on either side such that $k-\ell$ is odd are $\ell = k-1$ and $\ell = k+1$, at which $r(\ell)$ has respective values 0 and -1. Therefore if $k-\ell$ is odd, then $p(n)$ is non-negative if and only if $\ell = k-1$.

Thus factorisation is achieved for $(C_c^\infty(S^2 \setminus \{N, S\}), L^2(\$_{S^2}), \mathcal{D})$ for any lift V_k of the circle action to $L^2(\$_{S^2})$, by choosing the characters $\ell = k$ or $\ell = k-1$ when constructing the right-hand module.

We conclude the 2-sphere example by examining the operator on the right-hand module, which, upon identifying $C_0(S^2 \setminus \{N, S\})^{\mathbb{T}}$ with $C_0((0,\pi))$ and $\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))^{\mathbb{T}}$ with $\mathbb{C}\mathrm{l}_1$, defines a spectral triple for $C_0((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1$. One might wonder whether it can be obtained from an odd spectral triple for $C_0((0,\pi))$, such as that defined by (some self-adjoint extension of) the Dirac operator on $(0,\pi)$. We show that this is not the case; for each $\ell \in \mathbb{Z}$ there is no odd spectral triple $(C_c^\infty((0,\pi)), \mathcal{H}', \mathcal{D}')$ such that the right-hand module is the even spectral triple corresponding to $(C_c^\infty((0,\pi)), \mathcal{H}', \mathcal{D}')$. We also show that the right-hand module nevertheless represents the same class in K-homology as the Dirac operator on $(0,\pi)$.

Let $k, \ell \in \mathbb{Z}$ be fixed, where $V_k : \mathbb{T} \to U(L^2(\mathbf{\$}_{S^2}))$ is the representation and (ℓ, η) is the pair of Definition 2.14. Define $F : \mathcal{H}_\ell \to L^2([0,\pi])\widehat{\otimes}\mathbb{C}^2$ by

$$F\left(\begin{pmatrix} f(\theta)e^{i(k-\ell)\phi} \\ g(\theta)e^{i(k-\ell-1)\phi} \end{pmatrix}\right) = \sqrt{\sin\theta}\begin{pmatrix} if(\theta) \\ g(\theta) \end{pmatrix}.$$

The map F is a $C_0((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1$-linear $\mathbb{Z}_2$-graded unitary isomorphism between $L^2(\mathbf{\$}_{S^2})_\ell$ and $L^2([0,\pi])\widehat{\otimes}\mathbb{C}^2$, where the latter space is graded by $1\widehat{\otimes}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and the action of $\mathbb{C}\mathrm{l}_1$ is given by $c \mapsto 1\widehat{\otimes}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. We can compute

$$F \circ \mathcal{D}_\ell \circ F^{-1} = -i\partial_\theta\widehat{\otimes}\omega - (k - \ell - 1/2)\csc(\theta)\widehat{\otimes}c,$$

where $\omega = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$. Hence the right-hand module is unitarily equivalent to the spectral triple

$$\Big(C_c^\infty((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1, L^2([0,\pi])\widehat{\otimes}\mathbb{C}^2,\ -i\partial_\theta\widehat{\otimes}\omega - (k - \ell - 1/2)\csc(\theta)\widehat{\otimes}c\Big).$$

If $(C_c^\infty((0,\pi)), L^2([0,\pi]), \mathcal{D}')$ is an odd spectral triple, then the corresponding even spectral triple is $(C_c^\infty((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1, L^2([0,\pi])\widehat{\otimes}\mathbb{C}^2, \mathcal{D}'\widehat{\otimes}\omega)$. The presence of the $\csc(\theta)\widehat{\otimes}c$ factor means that the right-hand module is not the even spectral triple corresponding to any odd spectral triple.

Let $x \in KK(C_0((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1, \mathbb{C})$ denote the K-homology class of the spectral triple $\big(C_c^\infty((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1, L^2([0,\pi])\widehat{\otimes}\mathbb{C}^2,\ -i\partial_\theta\widehat{\otimes}\omega - (k - \ell - 1/2)\csc(\theta)\widehat{\otimes}c\big)$, where $k, \ell \in \mathbb{Z}$ are fixed. We will now show that x is represented by the Dirac operator on $(0,\pi)$.

The Dirac operator on $(0,\pi)$ is $-i\partial_\theta$, which we give the self-adjoint boundary conditions

$$\mathrm{dom}(-i\partial_\theta) = \{f \in AC([0,\pi]) : f' \in L^2([0,\pi]) \text{ and } f(0) = f(1)\},$$

as in [39, p. 259], where AC denotes the absolutely continuous functions, which are functions of the form $f(x) = f(0) + \int_0^x h(y)\,dy$ for some $h \in L^1([0,\pi])$. To this Dirac op-

erator we associate the even spectral triple $(C_c^\infty((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1, L^2([0,\pi])\widehat{\otimes}\mathbb{C}^2, -i\partial_\theta\widehat{\otimes}\omega)$, and hence a class $y \in KK(C_0((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1, \mathbb{C})$. In order to show $x = y$, we take the index pairing with a K-theory class.

Lemma 6.2. *The triple* $(\mathbb{C}, C_0((0,\pi))_{C_0((0,\pi))}, -\cot(\theta))$ *(where* $-\cot(\theta)$ *acts by multiplication) is an odd unbounded Kasparov* $\mathbb{C}$*-*$C_0((0,\pi))$*-module, and hence it defines a class* $z \in KK(\mathbb{C}, C_0((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1)$.

Proof. Since $-\cot(\theta)$ is real-valued, it is self-adjoint and regular as an unbounded multiplier on $C_0((0,\pi))$, [29, p. 117]. The bounded commutators condition is trivially satisfied, so it remains to check the compactness of the resolvent. Since $(1+\cot^2(\theta))^{-1/2} = \sin(\theta) \in C_0((0,\pi))$, the fact that $\mathrm{End}^0_{C_0((0,\pi))}(C_0((0,\pi))) = C_0((0,\pi))$ proves that $(1+\cot^2(\theta))^{-1/2}$ is a compact endomorphism. Therefore $(\mathbb{C}, C_0((0,\pi))_{C_0((0,\pi))}, -\cot(\theta))$ is an odd unbounded Kasparov module. To this odd module we associate the even unbounded Kasparov module $\big(\mathbb{C}, (C_0((0,\pi)\widehat{\otimes}\mathbb{C}\mathrm{l}_1)_{C_0((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1}, -\cot(\theta)\widehat{\otimes}c\big)$ and hence a class $z \in KK(\mathbb{C}, C_0((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1)$. □

Proposition 6.3. *The triple* $(\mathbb{C}, L^2([0,\pi])\widehat{\otimes}\mathbb{C}^2, -i\partial_\theta\widehat{\otimes}\omega - \cot(\theta)\widehat{\otimes}c)$ *is an even spectral triple, which represents the Kasparov product* $z\widehat{\otimes}_{C_0((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1}x \in KK(\mathbb{C},\mathbb{C})$ *as well as the Kasparov product* $z\widehat{\otimes}_{C_0((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1}y$*, and hence* $z\widehat{\otimes}_{C_0((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1}x = z\widehat{\otimes}_{C_0((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1}y$.

Proof. The square of $-i\partial_\theta\widehat{\otimes}\omega - \cot(\theta)\widehat{\otimes}c$ is

$$\big(-i\partial_\theta\widehat{\otimes}\omega - \cot(\theta)\widehat{\otimes}c\big)^2 = \begin{pmatrix} -\frac{d^2}{d\theta^2} - 1 & 0 \\ 0 & -\frac{d^2}{d\theta^2} + \cot^2(\theta) + \csc^2(\theta) \end{pmatrix}.$$

A complete eigenbasis for $-\frac{d^2}{d\theta^2}-1$ is $\{\sin(k\theta) : k \in \mathbb{N}\}$. Since $-\frac{d^2}{d\theta^2}+\cot^2(\theta)+\csc^2(\theta)$ has trivial kernel, it follows that $\{k\cos(k\theta)-\cot(\theta)\sin(k\theta) : k \geq 2\}$ is a complete eigenbasis for $-\frac{d^2}{d\theta^2} + \cot^2(\theta) + \csc^2(\theta)$. (We arrived at this eigenbasis by $(\partial_\theta - \cot(\theta))\sin(k\theta) = k\cos(k\theta) - \cot(\theta)\sin(k\theta)$.) Putting these two eigenbases together, we obtain the eigenbasis

$$\left\{\begin{pmatrix}\sin(\theta)\\ 0\end{pmatrix}\right\} \bigcup \left\{\begin{pmatrix}\sqrt{k^2-1}\sin(k\theta)\\ k\cos(k\theta)-\cot(\theta)\sin(k\theta)\end{pmatrix}, \begin{pmatrix}-\sqrt{k^2-1}\sin(k\theta)\\ k\cos(k\theta)-\cot(\theta)\sin(k\theta)\end{pmatrix} : k \geq 2\right\}$$

for $-i\partial_\theta\widehat{\otimes}\omega - \cot(\theta)\widehat{\otimes}c$. The corresponding eigenvalues of $-i\partial_\theta\widehat{\otimes}\omega - \cot(\theta)\widehat{\otimes}c$ are therefore $\{0\}\cup\{\sqrt{k^2-1}, -\sqrt{k^2-1} : k \geq 2\}$, each with multiplicity 1. Since these eigenvalues are real, $-i\partial_\theta\widehat{\otimes}\omega - \cot(\theta)\widehat{\otimes}c$ is essentially self-adjoint on the linear span of the eigenbasis, and since the eigenvalues go to $\pm\infty$, $-i\partial_\theta\widehat{\otimes}\omega - \cot(\theta)\widehat{\otimes}c$ has compact resolvent. Therefore $(\mathbb{C}, L^2([0,\pi])\widehat{\otimes}\mathbb{C}^2, -i\partial_\theta\widehat{\otimes}\omega - \cot(\theta)\widehat{\otimes}c)$ is an even spectral triple.

To show that $(\mathbb{C}, L^2([0,\pi])\widehat{\otimes}\mathbb{C}^2, -i\partial_\theta\widehat{\otimes}\omega - \cot(\theta)\widehat{\otimes}c)$ represents the Kasparov products $z\widehat{\otimes}_{C_0((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1}x$ and $z\widehat{\otimes}_{C_0((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1}y$, we use Kucerovsky's criteria, Theorem 1.22,

noting that since $C_0((0,\pi))$ is represented non-degenerately on $L^2([0,\pi])$, we can naturally identify $(C_0((0,\pi))\widehat{\otimes}\mathrm{Cl}_1)\widehat{\otimes}_{C_0((0,\pi))\widehat{\otimes}\mathrm{Cl}_1}(L^2([0,\pi])\widehat{\otimes}\mathbb{C}^2)$ with $L^2([0,\pi])\widehat{\otimes}\mathbb{C}^2$ via

$$(f\widehat{\otimes}z)\widehat{\otimes}(\xi\widehat{\otimes}v)\mapsto f\xi\widehat{\otimes}zv.$$

The connection criterion. Let $e=f\widehat{\otimes}a\in C_c^\infty((0,\pi))\widehat{\otimes}\mathrm{Cl}_1$ be of homogeneous degree. Under the identification of $(C_0((0,\pi))\widehat{\otimes}\mathrm{Cl}_1)\widehat{\otimes}_{C_0((0,\pi))\widehat{\otimes}\mathrm{Cl}_1}(L^2([0,\pi])\widehat{\otimes}\mathbb{C}^2)$ with $L^2([0,\pi])\widehat{\otimes}\mathbb{C}^2$, $T_e(\zeta\widehat{\otimes}w)=f\zeta\widehat{\otimes}aw$, and $T_e^*(\xi\widehat{\otimes}v)=f^*\xi\widehat{\otimes}a^*v$. Let $A\in\mathbb{R}$. Then

$$\begin{aligned}
&\left[\begin{pmatrix}-i\partial_\theta\widehat{\otimes}\omega-\cot(\theta)\widehat{\otimes}c & 0\\ 0 & -i\partial_\theta+A\csc(\theta)\widehat{\otimes}c\end{pmatrix},\begin{pmatrix}0 & T_e\\ T_e^* & 0\end{pmatrix}\right]_\pm\begin{pmatrix}\xi\widehat{\otimes}v\\ \zeta\widehat{\otimes}w\end{pmatrix}\\
&=\begin{pmatrix}\big(-\cot(\theta)\widehat{\otimes}c-i\partial_\theta\widehat{\otimes}\omega\big)(f\zeta\widehat{\otimes}aw)\\ \big(-i\partial_\theta\widehat{\otimes}\omega+A\csc(\theta)\widehat{\otimes}c\big)(f^*\xi\widehat{\otimes}a^*v)\end{pmatrix}\\
&\quad-(-1)^{\deg a}\begin{pmatrix}-if\partial_\theta\zeta\widehat{\otimes}a\omega w+Af\csc(\theta)\zeta\widehat{\otimes}acw\\ -f^*\cot(\theta)\xi\widehat{\otimes}a^*cv-if^*\partial_\theta\xi\widehat{\otimes}a^*\omega v\end{pmatrix}\\
&=\begin{pmatrix}-i\partial_\theta(f)\zeta\widehat{\otimes}\omega aw-f(\cot(\theta)+(-1)^{\deg a}A\csc(\theta))\zeta\widehat{\otimes}caw\\ -i\partial_\theta(f^*)\xi\widehat{\otimes}\omega a^*v+f^*(A\csc(\theta)+(-1)^{\deg a}\cot(\theta))\xi\widehat{\otimes}ca^*v\end{pmatrix},
\end{aligned}$$

which is a bounded function of $\left(\begin{smallmatrix}\xi\widehat{\otimes}v\\ \zeta\widehat{\otimes}w\end{smallmatrix}\right)$ for any value of A since f has compact support in $(0,\pi)$. Therefore the connection criterion is satisfied in both cases.

The domain criterion. Observe that $C_c^\infty\big((-\cot(\theta)\widehat{\otimes}c)(L^2([0,\pi])\widehat{\otimes}\mathbb{C}^2)\big)$ is contained in $L_c^2((0,\pi))\widehat{\otimes}\mathbb{C}^2$, where $L_c^2((0,\pi))$ denotes the square integrable functions with essential support in $(0,\pi)$. Since $-i\partial_\theta\widehat{\otimes}\omega-\cot(\theta)\widehat{\otimes}c$ is a local operator, its resolvent preserves $L_c^2((0,\pi))$. If $f\in L_c^2((0,\pi))$, then $\|\cot(\theta)f\|^2<\infty$, and hence $L_c^2((0,\pi))\widehat{\otimes}\mathbb{C}^2\subset\mathrm{dom}(-\cot(\theta)\widehat{\otimes}c)$. Therefore the domain criterion is satisfied (in both cases).

The positivity criterion. Using integration by parts, we compute:

$$\begin{aligned}
&\left\langle(-\cot(\theta)\widehat{\otimes}c)\begin{pmatrix}\xi\\ \zeta\end{pmatrix},(-i\partial_\theta\widehat{\otimes}\omega-\cot(\theta)\widehat{\otimes}c)\begin{pmatrix}\xi\\ \zeta\end{pmatrix}\right\rangle\\
&+\left\langle(-i\partial_\theta\widehat{\otimes}\omega-\cot(\theta)\widehat{\otimes}c)\begin{pmatrix}\xi\\ \zeta\end{pmatrix},(-\cot(\theta)\widehat{\otimes}c)\begin{pmatrix}\xi\\ \zeta\end{pmatrix}\right\rangle\\
&=2\|\cot(\theta)\xi\|^2+2\|\cot(\theta)\zeta\|^2+\langle\cot(\theta)\xi,\partial_\theta\xi\rangle-\langle\cot(\theta)\zeta,\partial_\theta\zeta\rangle+\langle\partial_\theta\xi,\cot(\theta)\xi\rangle\\
&\quad-\langle\cot(\theta)\zeta,\partial_\theta\zeta\rangle\\
&=2\|\cot(\theta)\xi\|^2+2\|\cot(\theta)\zeta\|^2+\big\langle\csc^2(\theta)\xi,\xi\big\rangle-\big\langle\csc^2(\theta)\zeta,\zeta\big\rangle\\
&=3\|\cot(\theta)\xi\|^2+\|\cot(\theta)\zeta\|^2+\|\xi\|^2-\|\zeta\|^2\geq-(\|\xi\|^2+\|\zeta\|^2),
\end{aligned}$$

using $\cot^2(\theta)-\csc^2(\theta)=-1$. Hence the positivity criterion is satisfied in both cases, and therefore $(\mathbb{C},L^2([0,\pi])\widehat{\otimes}\mathbb{C}^2,-i\partial_\theta\widehat{\otimes}\omega-\cot(\theta)\widehat{\otimes}c)$ represents both of the Kasparov

products $z\widehat{\otimes}_{C_0((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1}x$ and $z\widehat{\otimes}_{C_0((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1}y$. □

Corollary 6.4. *The even spectral triples* $\big(C_c^\infty((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1, L^2([0,\pi])\widehat{\otimes}\mathbb{C}^2,\ -i\partial_\theta\widehat{\otimes}\omega\big)$ *and* $\big(C_c^\infty((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1, L^2([0,\pi])\widehat{\otimes}\mathbb{C}^2,\ -i\partial_\theta\widehat{\otimes}\omega-(k-\ell-1/2)\csc(\theta)\widehat{\otimes}c\big)$ *represent the same class in* $KK(C_0((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1,\mathbb{C})$.

Proof. Under the isomorphisms

$$KK(C_0((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1,\mathbb{C})\cong KK(\mathbb{C},C_0((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1)\cong KK(\mathbb{C},\mathbb{C})\cong\mathbb{Z},$$

the index pairing $KK(C_0((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1,\mathbb{C})\times KK(\mathbb{C},C_0((0,\pi))\to KK(\mathbb{C},\mathbb{C})$ is the multiplication map $\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}$. The isomorphism $KK(\mathbb{C},\mathbb{C})\to\mathbb{Z}$ is the index map. The **index** of an even spectral triple $\big(\mathbb{C},\mathcal{H}^0\oplus\mathcal{H}^1,\left(\begin{smallmatrix}0&\mathcal{D}^1\\ \mathcal{D}^0&0\end{smallmatrix}\right)\big)$, where $\mathbb{C}$ is represented non-degenerately, is $\dim\ker(\mathcal{D}^0)-\dim\ker(\mathcal{D}^1)$. The eigenvalue calculation in the proof of Proposition 6.3 showed that the spectral triple $(\mathbb{C},L^2([0,\pi])\widehat{\otimes}\mathbb{C}^2,-i\partial_\theta\widehat{\otimes}\omega-\cot(\theta)\widehat{\otimes}c)$ has index 1, and hence it represents the identity of the ring $KK(\mathbb{C},\mathbb{C})\cong\mathbb{Z}$. Since $\mathbb{Z}$ is an integral domain, if

$$z\widehat{\otimes}_{C_0((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1}x=z\widehat{\otimes}_{C_0((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1}y=1\in KK(\mathbb{C},\mathbb{C})\cong\mathbb{Z},$$

then $x=y\in KK(C_0((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1,\mathbb{C})$, and so Proposition 6.3 proves the result. □

Part II

Relative spectral triples and the boundary map in K-homology

Chapter 7

Preliminaries

7.1 The boundary map in K-homology

Let A be a separable $\mathbb{Z}_2$-graded C^*-algebra. We define the K-homology groups $K^j(A)$ by $K^j(A) = KK^j(A, \mathbb{C})$ for $j = 0, 1$. Suppose that A is trivially $\mathbb{Z}_2$-graded. Let J be an ideal in A, which we denote by $J \lhd A$. (By ideal we mean a closed, two-sided $*$-ideal.) Then there is a short exact sequence

$$0 \longrightarrow J \xrightarrow{\iota} A \xrightarrow{\pi} A/J \longrightarrow 0 \tag{7.1}$$

where ι and π are the inclusion and quotient maps respectively. Suppose that A/J is nuclear.[1] Then the short exact sequence (7.1) gives us a six-term exact sequence in K-homology [22, Thm. 5.3.10]:

$$\begin{tikzcd}
K^0(A/J) \arrow[r, "\pi^*"] & K^0(A) \arrow[r, "\iota^*"] & K^0(J) \arrow[d, "\partial"] \\
K^1(J) \arrow[u, "\partial"] & K^1(A) \arrow[l, "\iota^*"] & K^1(A/J) \arrow[l, "\pi^*"]
\end{tikzcd}$$

The maps π^* and ι^* are the pullbacks of quotient map π and inclusion map ι respectively. (The pullback $f^* : K^j(B) \to K^j(A)$ of a $*$-homomorphism $f : A \to B$ is $f^*([(\rho, \mathcal{H}, F)]) = [(\rho \circ f, \mathcal{H}, F)]$, where $(\rho, \mathcal{H}, F)$ is a Kasparov B-$\mathbb{C}$-module.) The maps $\partial : K^j(J) \to K^{j+1}(A/J)$ are the **boundary maps**. It is these boundary maps that concern us in this second part of the thesis, in particular the even-to-odd boundary map $\partial : K^0(J) \to K^1(A/J)$.

[1] For the six-term exact sequence, this assumption may be weakened to: the short exact sequence $0 \to J \to A \to A/J \to 0$ is **semisplit**, [22, Defn. 5.3.6]. We will restrict ourselves to the case that A/J is nuclear, since in this case the extension group $\mathrm{Ext}(A/J)$ exists and is isomorphic to the odd K-homology group $K^1(A/J)$.

Example 7.1. Let $\overline{M}$ be a compact manifold with boundary, with interior M. We have the ideal $C_0(M)$ in the C^*-algebra $C(\overline{M})$, and hence a six-term exact sequence:

$$\begin{tikzcd} K^0(C(\partial M)) \arrow[r, "\pi^*"] & K^0(C(\overline{M})) \arrow[r, "\iota^*"] & K^0(C_0(M)) \arrow[d, "\partial"] \\ K^1(C_0(M)) \arrow[u, "\partial"] & K^1(C(\overline{M})) \arrow[l, "\iota^*"] & K^1(C(\partial M)) \arrow[l, "\pi^*"] \end{tikzcd}$$

Suppose $\overline{M}$ is spin^c with spinor bundle $\$_M$ (a type of Clifford module; see [20, 30]), which is non-trivially $\mathbb{Z}_2$-graded if and only if $\dim M$ is even, with a Clifford connection. Then given a Hermitian vector bundle S on $\overline{M}$, there is a twisted Dirac operator $\mathcal{D}^S$ on the Clifford module $\$_M \otimes S$. Let $\mathcal{D}^S_{\min}$ be the closure of $\mathcal{D}^S$ initially defined on smooth sections of $\$_M \otimes S$ with compact support in the interior M, which is a symmetric operator. Then any self-adjoint extension $\mathcal{D}^S_e$ of $\mathcal{D}^S_{\min}$ defines an even (resp. odd) spectral triple $(C_c^\infty(M), L^2(\$_M \otimes S), \mathcal{D}^S_e)$ if M is even (resp. odd) dimensional, and hence a class in $K^{\dim M}(C_0(M))$.

Restricting the spin^c structure and the vector bundle S to the boundary ∂M and choosing a Clifford connection defines an essentially self-adjoint twisted Dirac operator $\mathcal{D}^{S|_{\partial M}}$ on the boundary, and hence a spectral triple $(C^\infty(\partial M), L^2(\$_{\partial M} \otimes S|_{\partial M}), \mathcal{D}^{S|_{\partial M}})$ defining a class in $K^{\dim M+1}(C(\partial M))$. Propositions 4.4 and 5.1 of [3] show that under the boundary map

$$\partial : K^{\dim M}(C_0(M)) \to K^{\dim M+1}(C(\partial M))$$

of the six-term exact sequence,

$$\partial[(C_c^\infty(M), L^2(\$_M \otimes S), \mathcal{D}^S_e)] = [(C^\infty(\partial M), L^2(\$_{\partial M} \otimes S|_{\partial M}), \mathcal{D}^{S|_{\partial M}})].$$

7.2 Relative Fredholm modules

To compute the boundary map, it is more useful to use relative Fredholm modules, which define classes in relative K-homology.

Definition 7.2. Let A be a separable $\mathbb{Z}_2$-graded C^*-algebra, and let $J \lhd A$ be an ideal. An **even relative Fredholm module** $(\rho, \mathcal{H}, F)$ for $J \lhd A$ consists of an $\mathbb{Z}_2$-graded representation $\rho : A \to B(\mathcal{H})$ on a separable $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}$, and an odd operator $F \in B(\mathcal{H})$ such that $[F, \rho(a)]_\pm$, $\rho(j)(F - F^*)$ and $\rho(j)(1 - F^2)$ are compact for all $a \in A$, $j \in J$. A relative Fredholm module is **degenerate** if

$$[F, \rho(a)]_\pm = \rho(j)(F - F^*) = \rho(j)(1 - F^2) = 0, \quad \text{for all } a \in A,\ j \in J.$$

If A is trivially $\mathbb{Z}_2$-graded, then we can define **odd relative Fredholm modules** for $J \lhd A$, which have the same definition except that $\mathcal{H}$ is trivially $\mathbb{Z}_2$-graded and F need not be odd. (An even/odd **Fredholm module** for A is an even/odd relative Fredholm module for $A \lhd A$.)

As in §1.2, we define the **relative K-homology group** $K^0(J \lhd A)$ to be equivalence classes of even relative Fredholm modules under the equivalence relation generated by unitary equivalence, operator homotopy and the addition of degenerate Fredholm modules, and set $K^1(J \lhd A) := K^0(J \widehat{\otimes} \mathbb{C}\mathrm{l}_1 \lhd A \widehat{\otimes} \mathbb{C}\mathrm{l}_1)$. If A is trivially $\mathbb{Z}_2$-graded, we can also define $K^1(J \lhd A)$ to be equivalence classes of odd relative Fredholm modules under the same equivalence relations.

If $(\rho, \mathcal{H}, F)$ is a relative Fredholm module for $J \lhd A$, then restricting the representation ρ to J gives a Fredholm module for J. Hence there is a natural map $K^j(J \lhd A) \to K^j(J)$. The excision theorem says that this map is in fact an isomorphism, [22, Thm. 5.4.5]. So we could also write the six-term exact sequence as (for A trivially $\mathbb{Z}_2$-graded and A/J nuclear):

$$\begin{tikzcd}
K^0(A/J) \arrow[r, "\pi^*"] & K^0(A) \arrow[r, "\iota^*"] & K^0(J \lhd A) \arrow[d, "\partial"] \\
K^1(J \lhd A) \arrow[u, "\partial"] & K^1(A) \arrow[l, "\iota^*"] & K^1(A/J) \arrow[l, "\pi^*"]
\end{tikzcd}$$

The advantage of using the relative K-homology groups $K^j(J \lhd A)$ instead of $K^j(J)$ is that the boundary map is more "computable". In [21], Higson writes down Fredholm modules representing $\partial[(\rho, \mathcal{H}, F)]$ for a relative Fredholm module $(\rho, \mathcal{H}, F)$ (see also [22, Prop. 8.5.6]). However, Higson's expression for the boundary map is not entirely constructive. It is assumed that one has a completely positive splitting (see [22, Defn. 5.3.6]) $\sigma : \widetilde{A/J} \to \widetilde{A}$ (where $\widetilde{A}$ denotes the unitisation of A in the case that A is not unital) and a Stinespring dilation (see [22, Defn. 8.5.5]), which are known to exist by the nuclearity of A/J, but which are very difficult to construct in examples.

7.3 Extensions and the boundary map

The even-to-odd boundary map $\partial : K^0(J \lhd A) \to K^1(A/J)$ can also be described using extensions. The following can be found in [22, p. 39 ff] or [25, §7]. In [25], Kasparov defines the extension groups in greater generality than we do here. We are only interested in extensions for non-equivariant K-homology, but extension theory can be extended to equivariant KK-theory as well.

Definition 7.3. Let A be a separable, nuclear (trivially $\mathbb{Z}_2$-graded) C^*-algebra. An **extension** of A is a $*$-homomorphism $\alpha : A \to Q(\mathcal{H})$, where $Q(\mathcal{H}) = B(\mathcal{H})/K(\mathcal{H})$

is the Calkin algebra of a separable infinite dimensional (trivially $\mathbb{Z}_2$-graded) Hilbert space $\mathcal{H}$. Two extensions $\alpha : A \to Q(\mathcal{H})$ and $\alpha' : A \to Q(\mathcal{H}')$ are **unitarily equivalent** if there is a unitary $u : \mathcal{H} \to \mathcal{H}'$ such that $\pi(u)\alpha(a)\pi(u^*) = \alpha'(a)$ for all $a \in A$, where $\pi : B(\mathcal{H}) \to Q(\mathcal{H})$ is the quotient map. We say that an extension $\alpha : A \to Q(\mathcal{H})$ is **split** if there is a lifting $\widetilde{\alpha} : A \to B(\mathcal{H})$ of α.

The **extension group** $\mathrm{Ext}(A)$ is the set of unitary equivalence classes of extensions of A modulo the split extensions. If $\alpha : A \to Q(\mathcal{H})$ and $\alpha' : A \to Q(\mathcal{H}')$ are two extensions then the sum is defined to be

$$\alpha \oplus \alpha' : A \to Q(\mathcal{H}) \oplus Q(\mathcal{H}') \subset Q(\mathcal{H} \oplus \mathcal{H}').$$

Remark. It is not obvious that $\mathrm{Ext}(A)$ is a group and not just a semigroup, and it need not be if we drop the assumption that A is nuclear, [5, p. 128].

Remark. Extensions can be equivalently described in terms of short exact sequences

$$0 \longrightarrow K(\mathcal{H}) \longrightarrow E \longrightarrow A \longrightarrow 0$$

of C^*-algebras. Given $\alpha : A \to Q(\mathcal{H})$, set

$$E = \{(a, T) \in A \oplus B(\mathcal{H}) : \alpha(a) = \pi(T)\},$$

with the obvious maps $K(\mathcal{H}) \to E$ and $E \to A$.

Theorem 7.4 ([25, Thm. 1 of §7])**.** *Let A be a separable, nuclear, trivially $\mathbb{Z}_2$-graded C^*-algebra. Then* $\mathrm{Ext}(A) \cong K^1(A)$.

Given an odd Fredholm module $(\rho, \mathcal{H}, F)$ for A, the corresponding extension is $\alpha : A \to Q(\mathcal{H})$, where

$$\alpha(a) = \pi\left(\frac{F+1}{2}\rho(a)\right),$$

where $\pi : B(\mathcal{H}) \to Q(\mathcal{H})$ is the quotient map. The conditions $[F, \rho(a)]$, $\rho(a)(F - F^*)$, $\rho(a)(F^2 - 1) \in K(\mathcal{H})$ ensure the α is a $*$-homomorphism. The map $(\rho, \mathcal{H}, F) \mapsto \alpha$ gives us the direction $K^1(A) \to \mathrm{Ext}(A)$; unfortunately the direction $\mathrm{Ext}(A) \to K^1(A)$ is not so easy, involving a choice of completely positive splitting and a Stinespring dilation.

One advantage of the extension picture of odd K-homology is that the boundary map $K^0(J \lhd A) \to K^1(A/J)$ becomes easier to describe. Let $(\rho, \mathcal{H}, F)$ be an even relative Fredholm module for $J \lhd A$, where A is trivially $\mathbb{Z}_2$-graded and separable and A/J is nuclear. Suppose that $F = F^*$, and that F^2 is a projection; any even relative Fredholm module for $J \lhd A$ is equivalent to a Fredholm module of this form, [22, Lem. 8.5.4]. Let $F^j : \mathcal{H}^j \to \mathcal{H}^{j+1}$, $j = 1, 2$ be the even-to-odd and odd-to-even parts of F.

Define extensions

$$\alpha^0 : A/J \to Q(\ker(F^0)), \quad \alpha^0(a) = \pi(P_{\ker(F^0)}\widetilde{a}P_{\ker(F^0)}),$$
$$\alpha^1 : A/J \to Q(\ker(F^1)), \quad \alpha^1(a) = \pi(P_{\ker(F^1)}\widetilde{a}P_{\ker(F^1)}),$$

where $\widetilde{a} \in A$ is any lift of A/J, $P_{\ker(F^j)}$ is the orthogonal projection onto $\ker(F^j)$ and $\pi : B(\ker(F^j)) \to Q(\ker(F^j))$ is the quotient map. Then [3, p. 784], [22, Rmk. 8.5.7]

$$\partial[(\rho, \mathcal{H}, F)] = [\alpha_0] - [\alpha_1] \in \mathrm{Ext}(A/J) \cong K^1(A/J). \tag{7.2}$$

In practice we may not be given an even relative Fredholm module where $F = F^*$ and F^2 is a projection. Fortunately we can still describe the boundary of an even relative Fredholm module using extensions in a wider variety of cases, as the next two results show.

Lemma 7.5. *Let $(\rho, \mathcal{H}, F)$ be an even relative Fredholm module for $J \lhd A$, where A is a $\mathbb{Z}_2$-graded, separable C^*-algebra. Let $T : \mathcal{H}^0 \to \mathcal{H}^1$ be the even-to-odd part of F, and suppose that*

1) $\|T\| \leq 1$,

2) $\rho(a)^0(1 - T^*T)^{1/2}(1 - P_{\ker T})$ and $\rho(a)^1(1 - TT^*)^{1/2}(1 - P_{\ker T^*})$ are compact for all $a \in A$, where $\rho(a)^j : \mathcal{H}^j \to \mathcal{H}$ is the decomposition of $\rho(a)$ with respect to the $\mathbb{Z}_2$-grading of $\mathcal{H}$.

Let $\tilde{F} = \left(\begin{smallmatrix} 0 & T^* \\ T & 0 \end{smallmatrix}\right) \in B(\mathcal{H})$, and let

$$G = \begin{pmatrix} \tilde{F} & (1 - \tilde{F}^2)^{1/2}(1 - P_{\ker(\tilde{F})}) \\ (1 - \tilde{F}^2)^{1/2}(1 - P_{\ker(\tilde{F})}) & -\tilde{F} \end{pmatrix}.$$

Then $(\rho \oplus 0, \mathcal{H} \oplus \mathcal{H}^{op}, G)$ is an even relative Fredholm module for $J \lhd A$, representing the same class as $(\rho, \mathcal{H}, F)$.

Proof. The idea is similar to that of [5, §17.6]. We note first that $[0, 1] \ni t \mapsto tF + (1-t)\tilde{F}$ is an operator homotopy of relative Fredholm modules, and thus $(\rho, \mathcal{H}, F)$ and $(\rho, \mathcal{H}, \tilde{F})$ are relative Fredholm modules representing the same class. We then observe that the relative Fredholm module $(0, \mathcal{H}^{\mathrm{op}}, -\tilde{F})$ is degenerate and so $\left(\rho \oplus 0, \mathcal{H} \oplus \mathcal{H}^{\mathrm{op}}, \left(\begin{smallmatrix} \tilde{F} & 0 \\ 0 & -\tilde{F} \end{smallmatrix}\right)\right)$ is a relative Fredholm module representing the same class as $(\rho, \mathcal{H}, \tilde{F})$. Since $\|\tilde{F}\| \leq 1$, the operator G is well-defined, and we see that

$$(\rho \oplus 0)(a)\left(G - \begin{pmatrix} \tilde{F} & 0 \\ 0 & -\tilde{F} \end{pmatrix}\right) = \begin{pmatrix} 0 & \rho(a)(1 - \tilde{F}^2)^{1/2}(1 - P_{\ker(\tilde{F})}) \\ 0 & 0 \end{pmatrix},$$

which is compact by 2), and thus G is a "relatively compact" perturbation of $\begin{pmatrix} \tilde{F} & 0 \\ 0 & -\tilde{F} \end{pmatrix}$, and so the straight line homotopy between G and $\begin{pmatrix} \tilde{F} & 0 \\ 0 & -\tilde{F} \end{pmatrix}$ is a homotopy of relative Fredholm modules. □

Proposition 7.6. *Let $J \lhd A$ and $(\rho, \mathcal{H}, F)$ be as in Lemma 7.5, and assume in addition that A is trivially $\mathbb{Z}_2$-graded and A/J is nuclear. Let $F^0 : \mathcal{H}^0 \to \mathcal{H}^1$ be the even-to-odd part of F, and define extensions β^j, $j = 0, 1$ of A/J by*

$$\beta^0 : A/J \to Q(\ker(F^0)), \quad \beta^0(a) = \pi(P_{\ker(F^0)} \widetilde{a} P_{\ker(F^0)}),$$
$$\beta^1 : A/J \to Q(\ker((F^0)^*)), \quad \beta^1(a) = \pi(P_{\ker((F^0)^*)} \widetilde{a} P_{\ker((F^0)^*)}),$$

where $\widetilde{a} \in A$ is any lift of A/J and $\pi : B(\ker(F^j)) \to Q(\ker(F^j))$ is the quotient map. Then

$$\partial[(\rho, \mathcal{H}, F)] = [\beta_0] - [\beta_1] \in \mathrm{Ext}(A/J) \cong K^1(A/J).$$

Proof. The relative Fredholm module $(\rho, \mathcal{H}, F)$ is equivalent to $(\rho \oplus 0, \mathcal{H} \oplus \mathcal{H}^{\mathrm{op}}, G)$, where G is as in Lemma 7.5. Since $G = G^*$ and $G^2 = \begin{pmatrix} 1 - P_{\ker(\tilde{F})} & 0 \\ 0 & 1 - P_{\ker(\tilde{F})} \end{pmatrix}$ is a projection, we can apply Equation (7.2). Since $\ker(G) = \ker(\tilde{F}) \oplus \ker(\tilde{F})$, $\tilde{F}^0 = F^0$, $\tilde{F}^1 = (F^0)^*$, and the representation of A is $\rho \oplus 0$, the extensions $\alpha^0 : A/J \to Q(\ker(G^0))$ and $\alpha^1 : A/J \to Q(\ker(G^1))$ are equivalent to β^0 and β^1 respectively. □

Although the boundary map has a nice description in terms of extensions, getting from an extension back to a Fredholm module is not straightforward. Ideally we would want not just a Fredholm module representing the extension class, but a spectral triple, as in Example 7.1. Spectral triples carry geometric information as well K-homological information, so one would like to be able to compute the boundary map in the unbounded setting.

Since we know that the boundary map is (in principle) computed from relative Fredholm modules, the first step of computing the boundary map in the unbounded setting is to find unbounded representatives of relative Fredholm modules, just as spectral triples are unbounded representatives of Fredholm modules. These unbounded representatives are relative spectral triples.

Chapter 8

Relative spectral triples

In this chapter we introduce relative spectral triples, and show that the bounded transform of a relative spectral triple is a relative Fredholm module, and hence that a relative spectral triple defines a class in relative K-homology. Relative spectral triples are defined using symmetric operators, and are related to the "half-closed cycles" of [23], and also studied in [17], which define classes in non-relative K-homology: if $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a relative spectral triple for an ideal J in a separable $\mathbb{Z}_2$-graded C^*-algebra A, then $(\mathcal{A} \cap J, \mathcal{H}, D)$ is a half-closed cycle for J.

Definition 8.1. Let A be a $\mathbb{Z}_2$-graded separable C^*-algebra and let $J \lhd A$ be an ideal. An **even relative spectral triple** $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ for $J \lhd A$ consists of an even representation $\rho : A \to B(\mathcal{H})$ on a $\mathbb{Z}_2$-graded separable Hilbert space $\mathcal{H}$, an odd closed symmetric operator $\mathcal{D} : \mathrm{dom}(\mathcal{D}) \subset \mathcal{H} \to \mathcal{H}$, and a dense sub-$*$-algebra $\mathcal{A} \subset A$, such that

1) $\rho(a) \cdot \mathrm{dom}(\mathcal{D}) \subset \mathrm{dom}(\mathcal{D})$ and the graded commutator $[\mathcal{D}, \rho(a)]_\pm$ is bounded for all $a \in \mathcal{A}$,

2) $\rho(j) \cdot \mathrm{dom}(\mathcal{D}^*) \subset \mathrm{dom}(\mathcal{D})$ for all $j \in \mathcal{J} := J \cap \mathcal{A}$,

3) $\rho(a)(1 + \mathcal{D}^*\mathcal{D})^{-1/2}$ is compact for all $a \in \mathcal{A}$, and

4) there exists an approximate identity $(\phi_k)_{k=1}^\infty \subset \rho(\mathcal{A})$ for $C_{\mathcal{D}}(\mathcal{A})$, where $C_{\mathcal{D}}(\mathcal{A}) \subset B(\mathcal{H})$ is the C^*-algebra generated by $\{\rho(a), [\mathcal{D}, \rho(a)]_\pm : a \in \mathcal{A}\}$.

If A is trivially $\mathbb{Z}_2$-graded, an **odd relative spectral triple** $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ for $J \lhd A$ has the same definition except that $\mathcal{H}$ is trivially $\mathbb{Z}_2$-graded and $\mathcal{D}$ need not be odd.

We will usually omit the notation ρ.

Remark. The assumption 4) is used to show that $[\mathcal{D}(1+\mathcal{D}^*\mathcal{D})^{-1/2}, a]_\pm$ is compact for all $a \in \mathcal{A}$. It is not needed if $a(1 + \mathcal{D}\mathcal{D}^*)^{-1/2}$ is compact for all $a \in \mathcal{A}$ (which is not generally the case), which is why it does not appear in the definition of a spectral triple. If $\mathcal{A}$ is unital and the representation ρ is non-degenerate, then clearly 4) is satisfied. We note that 4) is equivalent to $\rho(\mathcal{A})C_{\mathcal{D}}(\mathcal{A})$ being dense in $C_{\mathcal{D}}(\mathcal{A})$.

Remark. In [22, Exercises 10.9.18-19], the notion of an "unbounded relative Fredholm module" is introduced (or rather, it is left to the reader to introduce). An unbounded relative Fredholm module $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ for an ideal J in a C^*-algebra A consists of (1) a representation ρ of A on a separable Hilbert space $\mathcal{H}$, (2) a self-adjoint operator $\mathcal{D}$ on $\mathcal{H}$ such that $j(1+\mathcal{D}^2)^{-1/2}$ is compact for all $j \in J$, and (3) a dense sub-$*$-algebra $\mathcal{A} \subset A$ such that $[\mathcal{D}, a]_\pm$ is bounded on $\mathrm{dom}(\mathcal{D})$, and such that (4) for all $a \in \mathcal{A}$, there exists an approximate identity $(\phi_k)_{k=1}^\infty$ for J such that $\phi_k[\mathcal{D}, a]_\pm$ and $[\mathcal{D}, a]_\pm \phi_k$ converge in operator norm to $[\mathcal{D}, a]_\pm$ as $k \to \infty$. It can be shown that if $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is an unbounded relative Fredholm module for $J \lhd A$, then the bounded transform $(\rho, \mathcal{H}, \mathcal{D}(1+\mathcal{D}^2)^{-1/2})$ is a relative Fredholm module for $J \lhd A$.

However, Condition (4) is quite restrictive. For example, it is not satisfied in simple examples, such as for $\mathcal{D}$ a Dirac operator on a compact manifold with boundary $\overline{M}$, $J = C_0(M)$ and $A = C(\overline{M})$. In this case, we can choose $a \in C^\infty(\overline{M})$ such that $[\mathcal{D}, a]$ is a section of the Clifford bundle not vanishing at the boundary, but $\overline{C_0(M)\Gamma(\mathbb{C}\mathrm{l}(\overline{M}))}$ consists only of sections vanishing at the boundary.

Example 8.2. Let M be an open submanifold of a complete Riemannian manifold $\tilde{M}$, and let S be a (possibly $\mathbb{Z}_2$-graded) Clifford module over M with Clifford connection ∇, which we assume extend to $\tilde{M}$. Let $\mathcal{D}$ be the closure of the Dirac operator initially defined on smooth sections of S with compact support in M. Then $(C_c^\infty(\overline{M}), L^2(S), \mathcal{D})$ is a relative spectral triple for $C_0(M) \lhd C_0(\overline{M})$, which is even if and only if S is $\mathbb{Z}_2$-graded. (Here $\overline{M}$ is the closure of M in $\tilde{M}$, and $C_c^\infty(\overline{M})$ is the space of restrictions of $C_c^\infty(\tilde{M})$ to $\overline{M}$.) It is not hard to see that Conditions 1), 2) and 4) of Definition 8.1 are satisfied, and $f(1+\mathcal{D}^*\mathcal{D})^{-1/2}$ is compact for all $f \in C_0(\overline{M})$ by elliptic operator theory, in particular the Rellich Lemma and the identification of $\mathrm{dom}(\mathcal{D})$ with the closure of $\Gamma_c^\infty(M; S)$ in the first Sobolev space, [3, Prop. 3.1], [22, 10.4.3].

In particular we obtain a relative spectral triple when $\overline{M}$ is a complete Riemannian manifold with boundary, although the case when M is an open submanifold of a complete manifold is much more general. For a concrete example of a relative spectral triple for a manifold with boundary, see Appendix B.

The main result of this second part of the thesis is that the bounded transform $(\rho, \mathcal{H}, \mathcal{D}(1+\mathcal{D}^*\mathcal{D})^{-1/2})$ of a relative spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a relative Fredholm module, and hence a relative spectral triple defines a class in relative K-homology. The proof follows the same ideas as the proof that the bounded transform of an unbounded Kasparov module is a bounded Kasparov module, [1]. A similar method is also used in [23, §3] to show that the bounded transform of a "half-closed operator" [23, p. 77] defines a bounded Kasparov module.

The results of this chapter also hold in the equivariant setting, as well as in the Hilbert module setting (under an additional regularity assumption on the operator),

but since we are focussing on the boundary map for non-equivariant K-homology we have no need to work in such generality.

Remark. One can also define relative spectral triples "of the second kind". Given an ideal J in a separable, $\mathbb{Z}_2$-graded C^*-algebra A, such a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ consists of a dense sub-$*$-algebra $\mathcal{A} \subset A$, a representation $\rho : A \to B(\mathcal{H})$ on a separable Hilbert space $\mathcal{H}$, and a self-adjoint operator $\mathcal{D}$ on $\mathcal{H}$ such that

1) $\rho(a) \cdot \mathrm{dom}(\mathcal{D}) \subset \mathrm{dom}(\mathcal{D})$ and $[\mathcal{D}, \rho(a)]_\pm$ is bounded for all $a \in \mathcal{A}$, and

2) $\rho(j)(1 + \mathcal{D}^2)^{-1/2}$ is compact for all $j \in J \cap \mathcal{A}$.

By restricting to the ideal J, a relative spectral triple of the second kind defines a spectral triple for J. However, unlike relative spectral triples, they do not necessarily define relative Fredholm modules, so it is unclear how useful they are in computing the boundary map. Relative spectral triples of the second kind are of interest because they arise naturally in examples, such as [36].

Relative spectral triples of the second kind can also be constructed from relative spectral triples. If $\left(\mathcal{A}, \mathcal{H}^0 \oplus \mathcal{H}^1, \mathcal{D} = \left(\begin{smallmatrix} 0 & \mathcal{D}^1 \\ \mathcal{D}^0 & 0 \end{smallmatrix}\right)\right)$ is an even relative spectral triple for $J \lhd A$, then $(\mathcal{A}, \mathcal{H}^0 \oplus \mathcal{H}^1, \left(\begin{smallmatrix} 0 & (\mathcal{D}^0)^* \\ \mathcal{D}^0 & 0 \end{smallmatrix}\right)$ is a triple of the second kind for $J \lhd A$. Alternatively, let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a relative spectral triple for $J \lhd A$ (even or odd), let $\mathcal{D}_e$ be a self-adjoint extension of $\mathcal{D}$, and let $\mathcal{A}_e := \{a \in \mathcal{A} : a \cdot \mathrm{dom}(\mathcal{D}_e) \subset \mathrm{dom}(\mathcal{D}_e)\}$, which is a sub-$*$-algebra of $\mathcal{A}$ containing $\mathcal{J} = \mathcal{A} \cap J$ as an ideal. Since $j(1 + \mathcal{D}_e^2)^{-1/2}$ is compact for all $j \in J$ (see Theorem 8.11 below), $(\mathcal{A}_e, \mathcal{H}, \mathcal{D}_e)$ is a spectral triple of the second kind for $J \lhd \overline{\mathcal{A}_e}$, where $\overline{\mathcal{A}_e}$ is the norm closure of $\mathcal{A}_e$ in A.

We begin with some results concerning non-self-adjoint operators.

Lemma 8.3. *Let $\mathcal{D} : \mathrm{dom}(\mathcal{D}) \subset \mathcal{H} \to \mathcal{H}$ be a closed (unbounded) operator on a separable Hilbert space $\mathcal{H}$. Then* $\mathrm{dom}(\mathcal{D}) = (1 + \mathcal{D}^*\mathcal{D})^{-1/2}\mathcal{H}$.

Proof. See [29, pp. 97–98]. □

Lemma 8.4. *Let $\mathcal{D} : \mathrm{dom}(\mathcal{D}) \subset \mathcal{H} \to \mathcal{H}$ be a closed operator on a separable Hilbert space $\mathcal{H}$. Then*

$$\mathcal{D}(1 + \mathcal{D}^*\mathcal{D})^{-1/2} = \overline{(1 + \mathcal{D}\mathcal{D}^*)^{-1/2}\mathcal{D}}$$

and

$$(\mathcal{D}(1 + \mathcal{D}^*\mathcal{D})^{-1/2})^* = \overline{(1 + \mathcal{D}^*\mathcal{D})^{-1/2}\mathcal{D}^*} = \mathcal{D}^*(1 + \mathcal{D}\mathcal{D}^*)^{-1/2}.$$

Remark. The notation $\overline{A}$ denotes the operator closure of a closeable operator A (i.e. $\overline{A}$ is the operator whose graph is the closure of the graph of A).

Proof. Suppose $\xi \in \mathrm{dom}(\mathcal{D}^*)$ and $\eta \in \mathcal{H}$. Then

$$\langle (1 + \mathcal{D}^*\mathcal{D})^{-1/2}\mathcal{D}^*\xi, \eta\rangle = \langle \mathcal{D}^*\xi, (1 + \mathcal{D}^*\mathcal{D})^{-1/2}\eta\rangle = \langle \xi, \mathcal{D}(1 + \mathcal{D}^*\mathcal{D})^{-1/2}\eta\rangle$$

where the last equality is justified by Lemma 8.3. Since $\mathrm{Dom}(\mathcal{D}^*)$ is dense in $\mathcal{H}$, it follows that $(\mathcal{D}(1+\mathcal{D}^*\mathcal{D})^{-1/2})^* = \overline{(1+\mathcal{D}^*\mathcal{D})^{-1/2}\mathcal{D}^*}$.

We now prove $\mathcal{D}(1+\mathcal{D}^*\mathcal{D})^{-1/2} = \overline{(1+\mathcal{D}\mathcal{D}^*)^{-1/2}\mathcal{D}}$. By [29, Thm. 10.4],

$$(\mathcal{D}^*(1+\mathcal{D}\mathcal{D}^*)^{-1/2})^* = \mathcal{D}(1+\mathcal{D}^*\mathcal{D})^{-1/2}.$$

We already proved that $(\mathcal{D}^*(1+\mathcal{D}\mathcal{D}^*)^{-1/2})^* = \overline{(1+\mathcal{D}\mathcal{D}^*)^{-1/2}\mathcal{D}}$, so

$$\mathcal{D}(1+\mathcal{D}^*\mathcal{D})^{-1/2} = \overline{(1+\mathcal{D}\mathcal{D}^*)^{-1/2}\mathcal{D}}. \qquad \square$$

Lemma 8.5. *Let $\mathcal{D} : \mathrm{dom}(\mathcal{D}) \subset \mathcal{H} \to \mathcal{H}$ be an odd closed symmetric operator on a $\mathbb{Z}_2$-graded separable Hilbert space $\mathcal{H}$, and let $\mathcal{A} \subset B(\mathcal{H})$ be a sub-$*$-algebra such that for all $a \in \mathcal{A}$, $a \cdot \mathrm{dom}(\mathcal{D}) \subset \mathrm{dom}(\mathcal{D})$ and $[\mathcal{D}, a]_\pm$ is bounded. Then*

1) $a \cdot \mathrm{dom}(\mathcal{D}^*) \subset \mathrm{dom}(\mathcal{D}^*)$, so that $[\mathcal{D}^*, a]_\pm$ is defined on $\mathrm{dom}(\mathcal{D}^*)$ for all $a \in \mathcal{A}$,

2) $[\mathcal{D}^*, a]_\pm$ is bounded and extends to $\overline{[\mathcal{D}, a]_\pm}$ for all $a \in \mathcal{A}$, and

3) for all $a \in \mathcal{A}$ of homogeneous degree,

$$\begin{aligned}[(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}, a] = &-\mathcal{D}^*(1+\lambda+\mathcal{D}\mathcal{D}^*)^{-1}[\mathcal{D}, a]_\pm(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1} \\ &- (-1)^{\deg a}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}[\mathcal{D}^*, a]_\pm\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}.\end{aligned}$$

Proof. 1) Let $\xi \in \mathrm{dom}(\mathcal{D}^*)$, and let $a \in \mathcal{A}$ be of homogeneous degree. By the definition of the adjoint, $a\xi \in \mathrm{dom}(\mathcal{D}^*)$ if and only if there exists $\zeta \in \mathcal{H}$ such that $\langle \mathcal{D}\eta, a\xi\rangle = \langle \eta, \zeta\rangle$ for all $\eta \in \mathrm{dom}(\mathcal{D})$. We claim that this equality holds for $\zeta = (-1)^{\deg a} a\mathcal{D}^*\xi + \overline{[\mathcal{D}, a]_\pm}\xi$. Since $\mathcal{D}$ is symmetric, $[\mathcal{D}, a]_\pm^* = -(-1)^{\deg a}\overline{[\mathcal{D}, a^*]_\pm}$, and so for $\eta \in \mathrm{dom}(\mathcal{D})$,

$$\begin{aligned}\left\langle \eta, (-1)^{\deg a} a\mathcal{D}^*\xi \right\rangle + \left\langle \eta, \overline{[\mathcal{D}, a]_\pm}\xi \right\rangle &= (-1)^{\det a}\langle \mathcal{D}a^*\eta, \xi\rangle - (-1)^{\deg a}\langle [\mathcal{D}, a^*]_\pm\eta, \xi\rangle \\ &= \langle a^*\mathcal{D}\eta, \xi\rangle = \langle \mathcal{D}\eta, a\xi\rangle,\end{aligned}$$

and so the claim is proved.

2) Let $\xi \in \mathrm{dom}(\mathcal{D}^*)$, $\eta \in \mathrm{dom}(\mathcal{D})$ and let $a \in \mathcal{A}$ be of homogeneous degree. Then

$$\begin{aligned}\langle [\mathcal{D}^*, a]_\pm\xi, \eta\rangle &= \langle \mathcal{D}^* a\xi, \eta\rangle - (-1)^{\deg a}\langle a\mathcal{D}^*\xi, \eta\rangle \\ &= \langle \xi, a^*\mathcal{D}\eta\rangle - (-1)^{\deg a}\langle \xi, \mathcal{D}a^*\eta\rangle = -(-1)^{\deg a}\langle \xi, [\mathcal{D}, a^*]_\pm\eta\rangle.\end{aligned}$$

Hence $-(-1)^{\deg a}[\mathcal{D}, a^*]_\pm \subset ([\mathcal{D}^*, a]_\pm)^*$. Since adjoints are necessarily closed, we have $-(-1)^{\deg a}\overline{[\mathcal{D}, a^*]_\pm} \subset ([\mathcal{D}^*, a]_\pm)^*$. Since the operator $\overline{[\mathcal{D}, a^*]_\pm}$ is everywhere defined and bounded, it follows that $([\mathcal{D}^*, a]_\pm)^*$ is also everywhere defined and bounded, and so $[\mathcal{D}^*, a]_\pm$ is also bounded. Since $\mathcal{D}$ is symmetric, $[\mathcal{D}, a]_\pm$ is a restriction of $[\mathcal{D}^*, a]_\pm$ and so they both have to same bounded extension.

3) We prove this using a refinement of the argument in the proof of [9, Lem. 2.3]. Ob-

serve that $\mathrm{range}((1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}) = \mathrm{dom}(\mathcal{D}^*\mathcal{D})$, since $\mathrm{dom}(1+\lambda+\mathcal{D}^*\mathcal{D}) = \mathrm{dom}(\mathcal{D}^*\mathcal{D})$. We have

$$\begin{aligned}
&[(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}, a] = (1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}a - a(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1} \\
&= (1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}a(1+\lambda+\mathcal{D}^*\mathcal{D})(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1} - a(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1} \\
&= \big(-\mathcal{D}^*\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}a + (1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}a\mathcal{D}^*\mathcal{D}\big)(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1},
\end{aligned}$$

since $(1+\lambda)(1+\lambda+x)^{-1}-1 = -x(1+\lambda+x)^{-1}$. Using Lemma 8.4 applied to $(1+\lambda)^{-1/2}\mathcal{D}$, we find

$$\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1} = \overline{(1+\lambda+\mathcal{D}\mathcal{D}^*)^{-1}\mathcal{D}}.$$

Similarly,

$$\mathcal{D}^*(1+\lambda+\mathcal{D}\mathcal{D}^*)^{-1} = \overline{(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}\mathcal{D}^*}.$$

Recall that $(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}\mathcal{H} = \mathrm{dom}(\mathcal{D}^*\mathcal{D}) \subset \mathrm{dom}(\mathcal{D})$, from which it follows that $a(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}\mathcal{H} \subset \mathrm{dom}(\mathcal{D})$ and $a\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}\mathcal{H} \subset \mathrm{dom}(\mathcal{D}^*)$. The following is then well-defined.

$$\begin{aligned}
&[(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}, a] = \\
&\big(-\mathcal{D}^*\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}a + (1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}a\mathcal{D}^*\mathcal{D}\big)(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1} \\
&= \big(-\mathcal{D}^*(1+\lambda+\mathcal{D}\mathcal{D}^*)^{-1}\mathcal{D}a + (1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}a\mathcal{D}^*\mathcal{D}\big)(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1} \\
&= \big(-\mathcal{D}^*(1+\lambda+\mathcal{D}\mathcal{D}^*)^{-1}\mathcal{D}a + (-1)^{\deg a}\mathcal{D}^*(1+\lambda+\mathcal{D}\mathcal{D}^*)^{-1}a\mathcal{D} \\
&\quad - (-1)^{\deg a}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}\mathcal{D}^*a\mathcal{D} + (1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}a\mathcal{D}^*\mathcal{D}\big) \\
&\qquad\qquad\qquad\qquad \times (1+\lambda+\mathcal{D}^*\mathcal{D})^{-1} \\
&= -\mathcal{D}^*(1+\lambda+\mathcal{D}\mathcal{D}^*)^{-1}[\mathcal{D}, a]_\pm(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1} \\
&\quad - (-1)^{\deg a}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}[\mathcal{D}^*, a]_\pm\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}
\end{aligned}$$

Note that 3) is true even if $\mathcal{D}$ is not symmetric, provided that 1) is assumed. □

Lemma 8.6. *Let $\mathcal{D} : \mathrm{dom}(\mathcal{D}) \subset \mathcal{H} \to \mathcal{H}$ be a closed operator and $\mathcal{A} \subset B(\mathcal{H})$ be a $*$-algebra such that $(1+\mathcal{D}^*\mathcal{D})^{-1/2}a$ is compact for all $a \in \mathcal{A}$. Then $(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1/2}a$ is compact for all $a \in \mathcal{A}$.*

Proof. It follows from the first resolvent formula that

$$\begin{aligned}
&(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1/2}a \\
&= (1+\lambda+\mathcal{D}^*\mathcal{D})^{1/2}(1+\mathcal{D}^*\mathcal{D})^{-1}a - \lambda(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1/2}(1+\mathcal{D}^*\mathcal{D})^{-1}a.
\end{aligned}$$

By the functional calculus, $(1+\lambda+\mathcal{D}^*\mathcal{D})^{1/2}(1+\mathcal{D}^*\mathcal{D})^{-1/2}$ is bounded by $\sqrt{1+\lambda}$, and

hence

$$(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1/2}a = \underbrace{(1+\lambda+\mathcal{D}^*\mathcal{D})^{1/2}(1+\mathcal{D}^*\mathcal{D})^{-1/2}}_{\text{bounded}}\underbrace{(1+\mathcal{D}^*\mathcal{D})^{-1/2}a}_{\text{compact}}$$
$$-\underbrace{\lambda(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1/2}(1+\mathcal{D}^*\mathcal{D})^{-1/2}}_{\text{bounded}}\underbrace{(1+\mathcal{D}^*\mathcal{D})^{-1/2}a}_{\text{compact}}$$

is compact for all $a \in \mathcal{A}$. □

The main tool used to prove that the bounded transform of a relative spectral triple is a Fredholm module is the integral formula for fractional powers, [34, p. 8], which states that for $0 < r < 1$ and B a positive bounded operator,

$$B^r = \frac{\sin(r\pi)}{\pi}\int_0^\infty \lambda^{-r}(1+\lambda B)^{-1}B\,d\lambda,$$

where convergence is in operator norm as a Riemann integral. Setting $B = (1+\mathcal{D}^*\mathcal{D})^{-1}$ and $r = 1/2$ gives us (after some rearranging)

$$(1+\mathcal{D}^*\mathcal{D})^{-1/2} = \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}\,d\lambda. \tag{8.1}$$

We would like to be able to take terms such as $\mathcal{D}[(1+\mathcal{D}^*\mathcal{D})^{-1/2}, a]_\pm$ and use (8.1) to write

$$\mathcal{D}[(1+\mathcal{D}^*\mathcal{D})^{-1/2}, a]_\pm = \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}\mathcal{D}[(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}, a]\,d\lambda.$$

For this expression to be well-defined, we require that the integral converges in operator norm. Thus we will now prove some estimates which will be used that various integrals converge.

Lemma 8.7. *Let $\mathcal{D} : \mathrm{dom}(\mathcal{D}) \subset \mathcal{H} \to \mathcal{H}$ be a closed operator on a separable Hilbert space $\mathcal{H}$. Then*

a) $\|\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1/2}\| \leq 1$, *and*

b) $\|(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1/2}\| \leq \frac{1}{\sqrt{1+\lambda}}$

for all $\lambda \in [0,\infty)$.

Proof. a) Letting $T = (1+\lambda)^{-1/2}\mathcal{D}$, we have $T(1+T^*T)^{-1/2} = \mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1/2}$, and $\|T(1+T^*T)^{-1/2}\| \leq 1$ by [29, Thm. 10.4].

b) If $x \in [0,\infty)$, then $(1+\lambda+x)^{-1/2} \leq (1+\lambda)^{-1/2}$. Since $\mathcal{D}^*\mathcal{D} \geq 0$, the functional calculus then implies that $\|(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1/2}\| \leq (1+\lambda)^{-1/2}$. □

Lemma 8.8. *Let A be a separable $\mathbb{Z}_2$-graded C^*-algebra, and let $J \lhd A$ be an ideal. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a relative spectral triple for $J \lhd A$, and let $\mathcal{D}_e \subset \mathcal{D}^*$ be a closed extension of $\mathcal{D}$. Then*

$$\begin{aligned}
&j\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1} - (1+\lambda+\mathcal{D}_e^*\mathcal{D}_e)^{-1}j\mathcal{D} \\
&= \mathcal{D}_e^*(1+\lambda+\mathcal{D}_e\mathcal{D}_e^*)^{-1}[\mathcal{D}^*, j]_\pm \mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1} \\
&\quad + (-1)^{\deg j}(1+\lambda+\mathcal{D}_e^*\mathcal{D}_e)^{-1}[\mathcal{D}, j]_\pm \mathcal{D}^*\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}
\end{aligned}$$

for all $j \in \mathcal{J} = \mathcal{A} \cap J$ of homogeneous degree and $\lambda \in [0, \infty)$, where both sides of the equation are defined on $\mathrm{dom}(\mathcal{D})$*, and hence*

$$\|\overline{j\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1} - (1+\lambda+\mathcal{D}_e^*\mathcal{D}_e)^{-1}j\mathcal{D}}\| \leq \frac{2\|\overline{[\mathcal{D}, j]_\pm}\|}{1+\lambda}.$$

Proof. We first address some domain issues. Since $(1+\lambda+\mathcal{D}^*\mathcal{D})(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1} = 1$, it follows that

$$(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1} : \mathrm{dom}(\mathcal{D}) \to \{\zeta \in \mathrm{dom}(\mathcal{D}^*\mathcal{D}) : (1+\lambda+\mathcal{D}^*\mathcal{D})\zeta \in \mathrm{dom}(\mathcal{D})\}.$$

Let

$$\mu \in \{\zeta \in \mathrm{dom}(\mathcal{D}^*\mathcal{D}) : (1+\lambda+\mathcal{D}^*\mathcal{D})\zeta \in \mathrm{dom}(\mathcal{D})\},$$

and let $\eta = (1+\lambda+\mathcal{D}^*\mathcal{D})\mu$. Since $\mathrm{dom}(\mathcal{D}^*\mathcal{D}) \subset \mathrm{dom}(\mathcal{D})$, $(1+\lambda)\mu \in \mathrm{dom}(\mathcal{D})$, and hence

$$\mathcal{D}^*\mathcal{D}\mu = \eta - (1+\lambda)\mu \in \mathrm{dom}(\mathcal{D}).$$

That is,

$$\mathcal{D}^*\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1} \cdot \mathrm{dom}(\mathcal{D}) \subset \mathrm{dom}(\mathcal{D}). \tag{8.2}$$

Hence

$$\begin{aligned}
&j\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1} - (1+\lambda+\mathcal{D}_e^*\mathcal{D}_e)^{-1}j\mathcal{D} \\
&= \Big(j\mathcal{D} - (1+\lambda+\mathcal{D}_e^*\mathcal{D}_e)^{-1}j\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})\Big)(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1} \\
&= \Big(j\mathcal{D} - (1+\lambda)(1+\lambda+\mathcal{D}_e^*\mathcal{D}_e)^{-1}j\mathcal{D} - (1+\lambda+\mathcal{D}_e^*\mathcal{D}_e)^{-1}j\mathcal{D}\mathcal{D}^*\mathcal{D}\Big) \\
&\qquad \times (1+\lambda+\mathcal{D}^*\mathcal{D})^{-1} \qquad \text{(this is well-defined by Equation (8.2))} \\
&= \Big(\mathcal{D}_e^*\mathcal{D}_e(1+\lambda+\mathcal{D}_e^*\mathcal{D}_e)^{-1}j\mathcal{D} - (1+\lambda+\mathcal{D}_e^*\mathcal{D}_e)^{-1}j\mathcal{D}\mathcal{D}^*\mathcal{D}\Big)(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1},
\end{aligned}$$

since $1 - (1+\lambda)(1+\lambda+x)^{-1} = x(1+\lambda+x)^{-1}$. By Lemma 8.4,

$$\mathcal{D}_e(1+\lambda+\mathcal{D}_e^*\mathcal{D}_e)^{-1}|_{\mathrm{dom}(\mathcal{D})} = (1+\lambda+\mathcal{D}_e\mathcal{D}_e^*)^{-1}\mathcal{D}^*|_{\mathrm{dom}(\mathcal{D})}$$

since $\mathcal{D}_e \subset \mathcal{D}^*$. Since $j \cdot \mathrm{dom}(\mathcal{D}^*) \subset \mathrm{dom}(\mathcal{D})$,

$$\begin{aligned}&j\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1} - (1+\lambda+\mathcal{D}_e^*\mathcal{D}_e)^{-1}j\mathcal{D} = \\ &\big(\mathcal{D}_e^*(1+\lambda+\mathcal{D}_e\mathcal{D}_e^*)^{-1}\mathcal{D}^*j\mathcal{D} - (1+\lambda+\mathcal{D}_e^*\mathcal{D}_e)^{-1}j\mathcal{D}\mathcal{D}^*\mathcal{D}\big)(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}.\end{aligned}$$

Similarly, Lemma 8.4 and the fact that $\mathcal{D} \subset \mathcal{D}_e^*$ imply that

$$\mathcal{D}_e^*(1+\lambda+\mathcal{D}_e\mathcal{D}_e^*)^{-1}|_{\mathrm{dom}(\mathcal{D})} = (1+\lambda+\mathcal{D}_e^*\mathcal{D}_e)^{-1}\mathcal{D}$$

Since $j \cdot \mathrm{dom}(\mathcal{D}) \subset \mathrm{dom}(\mathcal{D})$, it follows that

$$\mathcal{D}_e^*(1+\lambda+\mathcal{D}_e\mathcal{D}_e^*)^{-1}j|_{\mathrm{dom}(\mathcal{D})} = (1+\lambda+\mathcal{D}_e^*\mathcal{D}_e)^{-1}\mathcal{D}j|_{\mathrm{dom}(\mathcal{D})}.$$

Hence

$$\begin{aligned}&j\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1} - (1+\lambda+\mathcal{D}_e^*\mathcal{D}_e)^{-1}j\mathcal{D} \\ &= \big(\mathcal{D}_e^*(1+\lambda+\mathcal{D}_e\mathcal{D}_e^*)^{-1}\mathcal{D}^*j\mathcal{D} - (-1)^{\deg j}\mathcal{D}_e^*(1+\lambda+\mathcal{D}_e\mathcal{D}_e^*)^{-1}j\mathcal{D}^*\mathcal{D} \\ &\quad + (-1)^{\deg j}(1+\lambda+\mathcal{D}_e^*\mathcal{D}_e)^{-1}\mathcal{D}j\mathcal{D}^*\mathcal{D} - (1+\lambda+\mathcal{D}_e^*\mathcal{D}_e)^{-1}j\mathcal{D}\mathcal{D}^*\mathcal{D}\big) \\ &\qquad \times (1+\lambda+\mathcal{D}^*\mathcal{D})^{-1} \\ &= \mathcal{D}_e^*(1+\lambda+\mathcal{D}_e\mathcal{D}_e^*)^{-1}[\mathcal{D}^*, j]_\pm\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1} \\ &\quad + (-1)^{\deg j}(1+\lambda+\mathcal{D}_e^*\mathcal{D}_e)^{-1}[\mathcal{D}, j]_\pm\mathcal{D}^*\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}.\end{aligned}$$

Thus

$$\begin{aligned}&\overline{j\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1} - (1+\lambda+\mathcal{D}_e^*\mathcal{D}_e)^{-1}j\mathcal{D}} \\ &= \mathcal{D}_e^*(1+\lambda+\mathcal{D}_e\mathcal{D}_e^*)^{-1}\overline{[\mathcal{D}, j]_\pm}\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1} \\ &\quad + (-1)^{\deg j}(1+\lambda+\mathcal{D}_e^*\mathcal{D}_e)^{-1}\overline{[\mathcal{D}, j]_\pm}\mathcal{D}^*\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1},\end{aligned}$$

where we used Lemma 8.5 2) for $\overline{[\mathcal{D}^*, j]_\pm} = \overline{[\mathcal{D}, j]_\pm}$, and so

$$\begin{aligned}&\|\overline{j\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1} - (1+\lambda+\mathcal{D}_e^*\mathcal{D}_e)^{-1}j\mathcal{D}}\| \\ &\leq \|\mathcal{D}_e^*(1+\lambda+\mathcal{D}_e\mathcal{D}_e^*)^{-1}\|\|\overline{[\mathcal{D}, j]_\pm}\|\|\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}\| \\ &\quad + \|(1+\lambda+\mathcal{D}_e^*\mathcal{D}_e)^{-1}\|\|\overline{[\mathcal{D}, j]_\pm}\|\|\mathcal{D}^*\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}\| \\ &\leq \frac{1}{\sqrt{1+\lambda}}\|\overline{[\mathcal{D}, j]_\pm}\|\frac{1}{\sqrt{1+\lambda}} + \frac{1}{1+\lambda}\|\overline{[\mathcal{D}, j]_\pm}\| = \frac{2\|\overline{[\mathcal{D}, j]_\pm}\|}{1+\lambda},\end{aligned}$$

using Lemma 8.7 and the fact that $\|\mathcal{D}^*\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}\| \leq 1$ by the functional calculus. □

Theorem 8.9. *Let A be a separable $\mathbb{Z}_2$-graded C^*-algebra, and let $J \lhd A$ be an ideal.*

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a relative spectral triple for $J \lhd A$, and let $F = \mathcal{D}(1+\mathcal{D}^*\mathcal{D})^{-1/2}$ be the bounded transform of $\mathcal{D}$. Then $(\rho, \mathcal{H}, F)$ is a relative Fredholm module for $J \lhd A$, where $\rho : A \to B(\mathcal{H})$ is the representation.

Proof. We prove the compactness of $[F, a]_\pm$, $j(F - F^*)$ and $j(1 - F^2)$ for $a \in A$, $j \in J$ in turn. The method is to use the integral formula for fractional powers (8.1) in conjunction with various estimates we have proved.

We first prove that $[F, a]_\pm$ is compact for all $a \in A$. Since $\mathcal{A}$ is dense in A it is enough to show $[F, a]_\pm$ is compact for all $a \in \mathcal{A}$. Let $a \in \mathcal{A}$ be of homogeneous degree. We can write

$$[F, a]_\pm = [\mathcal{D}, a]_\pm (1+\mathcal{D}^*\mathcal{D})^{-1/2} + \mathcal{D}[(1+\mathcal{D}^*\mathcal{D})^{-1/2}, a]. \tag{8.3}$$

Let $(\phi_k)_{k=1}^\infty \subset \rho(\mathcal{A})$ be an approximate identity for $C_{\mathcal{D}}(\mathcal{A})$. Then

$$[\mathcal{D}, a]_\pm (1+\mathcal{D}^*\mathcal{D})^{-1/2} = \lim_{k\to\infty} \overline{[\mathcal{D}, a]_\pm} \phi_k (1+\mathcal{D}^*\mathcal{D})^{-1/2}$$

and so the first term of (8.3) is compact. By Lemma 8.5 3) and the integral formula for fractional powers (8.1), the second term of (8.3) is

$$\begin{aligned}
&\mathcal{D}[(1+\mathcal{D}^*\mathcal{D})^{-1/2}, a] \\
&= -\frac{1}{\pi}\int_0^\infty \lambda^{-1/2}\Big(\mathcal{D}\mathcal{D}^*(1+\lambda+\mathcal{D}\mathcal{D}^*)^{-1}[\mathcal{D}, a]_\pm(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1} \\
&\qquad + (-1)^{\deg a}\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}[\mathcal{D}^*, a]_\pm \mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}\Big)\, d\lambda.
\end{aligned}$$

We note that

$$\|\mathcal{D}\mathcal{D}^*(1+\lambda+\mathcal{D}\mathcal{D}^*)^{-1}[\mathcal{D}, a]_\pm(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}\| \le \frac{\|\overline{[\mathcal{D}, a]_\pm}\|}{1+\lambda},$$

and

$$\|\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}[\mathcal{D}^*, a]_\pm \mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}\| \le \frac{\|\overline{[\mathcal{D}, a]_\pm}\|}{1+\lambda},$$

using Lemma 8.5 2) and the bounds in Lemma 8.7. Hence the integral converges in operator norm. The integrand is compact, since we have the operator norm limits of compact operators

$$\begin{aligned}
&\mathcal{D}\mathcal{D}^*(1+\lambda+\mathcal{D}\mathcal{D}^*)^{-1}[\mathcal{D}, a]_\pm(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1} \\
&= \lim_{k\to\infty} \mathcal{D}\mathcal{D}^*(1+\lambda+\mathcal{D}\mathcal{D}^*)^{-1}\overline{[\mathcal{D}, a]_\pm}\phi_k(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1},
\end{aligned}$$

and

$$\begin{aligned}&\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}[\mathcal{D}^*,a]_\pm\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}\\&=\lim_{k\to\infty}\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}\phi_k\overline{[\mathcal{D},a]_\pm}\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}.\end{aligned}$$

Since the integral converges in operator norm, the integral is compact as well. Hence $[F,a]_\pm$ is compact.

We now prove that $j(F-F^*)$ is compact for all $j\in\mathcal{J}$. Let $j\in\mathcal{J}$ be of homogeneous degree. Then since $j\phi_k,\phi_k j\in\mathcal{J}$ and $\mathcal{J}\cdot\mathrm{dom}(\mathcal{D})\subset\mathrm{dom}(\mathcal{D})$,

$$\begin{aligned}&\big(\phi_n jF\phi_k-(-1)^{\deg j}\phi_nF^*j\phi_k\big)|_{\mathrm{dom}(\mathcal{D})}\\&=\big(\phi_nj\mathcal{D}(1+\mathcal{D}^*\mathcal{D})^{-1/2}\phi_k-(-1)^{\deg j}\phi_n(1+\mathcal{D}^*\mathcal{D})^{-1/2}\mathcal{D}j\phi_k\big)\Big|_{\mathrm{dom}(\mathcal{D})}\\&=\big(\phi_nj\mathcal{D}(1+\mathcal{D}^*\mathcal{D})^{-1/2}\phi_k-\phi_n(1+\mathcal{D}^*\mathcal{D})^{-1/2}j\mathcal{D}\phi_k\\&\qquad-(-1)^{\deg j}\phi_n(1+\mathcal{D}^*\mathcal{D})^{-1/2}[\mathcal{D},j]_\pm\phi_k\big)\Big|_{\mathrm{dom}(\mathcal{D})}.\end{aligned}$$

Since $\phi_n(1+\mathcal{D}^*\mathcal{D})^{-1/2}\overline{[\mathcal{D},j]}\phi_k$ is compact, it follows from the above calculation that

$$\phi_n(jF-F^*j)\phi_k\sim\phi_nj\mathcal{D}(1+\mathcal{D}^*\mathcal{D})^{-1/2}\phi_k-\overline{\phi_n(1+\mathcal{D}^*\mathcal{D})^{-1/2}j\mathcal{D}\phi_k}$$

where $\sim$ denotes equality modulo compact operators. By Lemma 8.8 and the integral formula for fractional powers (8.1),

$$\begin{aligned}&\phi_nj\mathcal{D}(1+\mathcal{D}^*\mathcal{D})^{-1/2}\phi_k-\overline{\phi_n(1+\mathcal{D}^*\mathcal{D})^{-1/2}j\mathcal{D}\phi_k}\\&=\frac{1}{\pi}\int_0^\infty\lambda^{-1/2}\Big(\phi_n\mathcal{D}^*(1+\lambda+\mathcal{D}\mathcal{D}^*)^{-1}\overline{[\mathcal{D},j]_\pm}\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}\phi_k\\&\qquad+(-1)^{\deg j}\phi_n(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}\overline{[\mathcal{D},j]_\pm}\mathcal{D}^*\mathcal{D}(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1}\phi_k\Big)\,d\lambda.\end{aligned}$$

The integral converges in operator norm by the estimates in Lemma 8.8. By Lemma 8.6, $(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1/2}\phi_k$ and $\phi_n(1+\lambda+\mathcal{D}^*\mathcal{D})^{-1/2}$ are compact for all $n,k\in\mathbb{N}$. Hence both terms in the integrand are compact, and so the integral is compact. Hence $\phi_n(jF-(-1)^{\deg j}F^*j)\phi_k$ is compact.

Now, $[F,a]_\pm\sim 0$ for all $a\in\mathcal{A}$, and taking adjoints shows that $[F^*,a]_\pm\sim 0$ for all $a\in\mathcal{A}$. Hence

$$0\sim\phi_n(jF-(-1)^{\deg j}F^*j)\phi_k\sim\phi_nj\phi_k(F-F^*).$$

Since

$$\lim_{n,k\to\infty}\phi_nj\phi_k(F-F^*)=j(F-F^*)$$

in operator norm, it follows that $j(F-F^*)$ is compact.

Finally, we show that $j(1-F^2)$ is compact for each $j \in \mathcal{J}$. Since $jF = jF^* + K$ for some $K \in \mathcal{K}(\mathcal{H})$, it follows that

$$j(1-F^2) = j1 - (jF^* + K)F = j(1-F^*F) - KF,$$

so it is enough to show that $j(1-F^*F)$ is compact for all $j \in \mathcal{J}$. We find

$$\begin{aligned} j(1-F^*F) &= j(1-\mathcal{D}^*(1+\mathcal{D}\mathcal{D}^*)^{-1/2}\mathcal{D}(1+\mathcal{D}^*\mathcal{D})^{-1/2}) \\ &= j(1-\mathcal{D}^*\mathcal{D}(1+\mathcal{D}^*\mathcal{D})^{-1}) \qquad \text{(Lemma 8.4)} \\ &= j(1+\mathcal{D}^*\mathcal{D})^{-1}. \end{aligned}$$

Since $j(1+\mathcal{D}^*\mathcal{D})^{-1/2}$ is compact by assumption, it follows that $j(1-F^*F)$ is compact. □

Remark. If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a relative spectral triple for an ideal J in a separable, $\mathbb{Z}_2$-graded C^*-algebra A, then the bounded transform $F_{\mathcal{D}^*} = \mathcal{D}^*(1+\mathcal{D}\mathcal{D}^*)^{-1/2}$ of $\mathcal{D}^*$ also defines a relative Fredholm module with the same class as $F_{\mathcal{D}} = \mathcal{D}(1+\mathcal{D}^*\mathcal{D})^{-1/2}$ in relative K-homology, even though $(\mathcal{A}, \mathcal{H}, \mathcal{D}^*)$ is not necessarily a relative spectral triple. This is because the path $[0,1] \ni t \mapsto tF_{\mathcal{D}} + (1-t)F_{\mathcal{D}^*}$ is an operator homotopy of relative Fredholm modules, using the fact that $F_{\mathcal{D}}^* = F_{\mathcal{D}^*}$, [29, Thm. 10.4].

The next result shows that if $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a relative spectral triple for an ideal J in a separable $\mathbb{Z}_2$-graded C^*-algebra A, then the relative K-homology class can also be represented by the phase of $\mathcal{D}$, and hence by a partial isometry, at least in the case that A is unital and represented non-degenerately.

Proposition 8.10. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a relative spectral triple for an ideal J in a separable $\mathbb{Z}_2$-graded* unital *C^*-algebra A represented non-degenerately on $\mathcal{H}$. Let V be the phase of $\mathcal{D}$, which is the partial isometry with initial space* $\ker(\mathcal{D})^\perp$ *and final space* $\ker(\mathcal{D}^*)^\perp$ *defined by $\mathcal{D} = V|\mathcal{D}|$,* [39, Thm. VIII.32]. *Then $(\rho, \mathcal{H}, V)$ is a relative Fredholm module with the same class as the bounded transform $(\rho, \mathcal{H}, \mathcal{D}(1+\mathcal{D}^*\mathcal{D})^{-1/2})$, where $\rho : A \to B(\mathcal{H})$ is the representation.*

Proof. We claim that $V - \mathcal{D}(1+\mathcal{D}^*\mathcal{D})^{-1/2}$ is compact, from which it follows that $[0,1] \ni t \mapsto tV + (1-t)\mathcal{D}(1+\mathcal{D}^*\mathcal{D})^{-1/2}$ is an operator homotopy of relative Fredholm modules. Since $\mathcal{D} = V|\mathcal{D}| = V(\mathcal{D}^*\mathcal{D})^{1/2}$,

$$\mathcal{D}(1+\mathcal{D}^*\mathcal{D})^{-1/2} - V = V\left((\mathcal{D}^*\mathcal{D})^{1/2}(1+\mathcal{D}^*\mathcal{D})^{-1/2} - 1\right).$$

Since $(1+x)^{1/2}(x^{1/2}(1+x)^{-1/2} - 1)$ is a bounded continuous function on $[0,\infty)$, the continuous functional calculus implies that $(1+\mathcal{D}^*\mathcal{D})^{1/2}\left((\mathcal{D}^*\mathcal{D})^{1/2}(1+\mathcal{D}^*\mathcal{D})^{-1/2} - 1\right)$ is a bounded operator, [39, Thm. VIII.5], which we denote by T. Since A is a unital

C^*-algebra represented non-degenerately on $\mathcal{H}$, $(1+\mathcal{D}^*\mathcal{D})^{-1/2}$ is compact and thus $\mathcal{D}(1+\mathcal{D}^*\mathcal{D})^{-1/2} - V = VT(1+\mathcal{D}^*\mathcal{D})^{-1/2}$ is compact, proving the claim. □

Remark. An open question is whether Proposition 8.10 is true when the algebra is non-unital. It is true that $(\mathcal{D}(1+\mathcal{D}^*\mathcal{D})^{-1/2} - V)a$ is compact for all $a \in \mathcal{A}$ in the non-unital case, but we would also need to show that $a(\mathcal{D}(1+\mathcal{D}^*\mathcal{D})^{-1/2} - V)$ is compact for all $a \in \mathcal{A}$. If we assumed then $a(1 - P_{\ker(\mathcal{D}^*)}(1+\mathcal{D}\mathcal{D}^*)^{-1/2}$ is compact for all $a \in \mathcal{A}$, then since

$$\begin{aligned} &a(\mathcal{D}(1+\mathcal{D}^*\mathcal{D})^{-1/2} - V) = aV\left((\mathcal{D}^*\mathcal{D})^{1/2}(1+\mathcal{D}^*\mathcal{D})^{-1/2} - 1\right) \\ &= a(1 - P_{\ker(\mathcal{D}^*)})(1+\mathcal{D}\mathcal{D}^*)^{-1/2}(1+\mathcal{D}\mathcal{D}^*)^{1/2}\left((\mathcal{D}\mathcal{D}^*)^{1/2}(1+\mathcal{D}\mathcal{D}^*)^{-1/2} - 1\right)V \end{aligned}$$

it would follow that $a(\mathcal{D}(1+\mathcal{D}^*\mathcal{D})^{-1/2} - V)$ is compact.

The following result is a specialisation of [23, Thm. 3.2]. It is proved by using Lemma 8.8 and the integral formula of fractional powers (8.1) to show that $j(F_{\mathcal{D}} - F_{\mathcal{D}_e})$ is compact for all $j \in \mathcal{J}$.

Theorem 8.11. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a relative spectral triple for an ideal J in a separable $\mathbb{Z}_2$-graded C^*-algebra A, and let $\mathcal{D} \subset \mathcal{D}_e \subset \mathcal{D}^*$ be a closed extension of $\mathcal{D}$. Then*

1) $(\rho, \mathcal{H}, F_{\mathcal{D}_e} = \mathcal{D}_e(1+\mathcal{D}_e^*\mathcal{D}_e)^{-1/2})$ is a Fredholm module for J, where $\rho : A \to B(\mathcal{H})$ is the representation, and so $\mathcal{D}_e$ defines a class $[\mathcal{D}_e] \in K^*(J)$, and

2) $[\mathcal{D}_e] = [\mathcal{D}] \in K^*(J)$; i.e. the K-homology class is independent of the choice of extension.

Remark. It should be emphasised that $F_{\mathcal{D}_e}$ does not generally define a Fredholm module for A, even if $\mathcal{D}_e$ is self-adjoint with compact resolvent. If $\mathcal{D}_e$ is self-adjoint with compact resolvent, the triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}_e)$ appears to satisfy the conditions of a spectral triple since $[\mathcal{D}_e, a]_\pm$ is well-defined and bounded on $\mathrm{dom}(\mathcal{D})$, which is dense in $\mathcal{H}$. However, if $[\mathcal{D}_e, a]_\pm$ is not well-defined on $\mathrm{dom}(\mathcal{D}_e)$ the bounded transform need not define a Fredholm module, [18]. We amplify on this issue regarding domains in Appendix B, and present a counterexample to the claim that it is enough for $[\mathcal{D}_e, a]_\pm$ to be defined and bounded on some dense subspace in order to obtain a Fredholm module for A.

If $\mathcal{D}$ does admit a self-adjoint extension $\mathcal{D}_e$ such that $a \cdot \mathrm{dom}(\mathcal{D}_e) \subset \mathrm{dom}(\mathcal{D}_e)$ and $a(1+\mathcal{D}_e^2)^{-1/2}$ is compact for all $a \in \mathcal{A}$, then $(\mathcal{A}, \mathcal{H}, \mathcal{D}_e)$ is a spectral triple for A and hence defines a class in $K^*(A)$. It follows from the exactness of the six-term exact sequence in K-homology that $\partial([\mathcal{D}]) = 0 \in K^{*+1}(A/J)$ (for A trivially $\mathbb{Z}_2$-graded and A/J nuclear), since $[\mathcal{D}] = \iota^*([\mathcal{A}, \mathcal{H}, \mathcal{D}_e)] \in K^*(J)$, where $\iota : J \to A$ is the inclusion map. So the non-vanishing of $\partial([\mathcal{D}])$ is an obstruction to the existence of such extensions. For

a Dirac operator on a compact manifold with boundary, this obstruction is expressed in [3, Cor. 4.2].

We return to the boundary map in K-homology and extensions. Suppose J is an ideal in a C^*-algebra A, which is represented on a Hilbert space $\mathcal{H}$. In an abuse of notation, given $V \subset \mathcal{H}$ a closed subspace such that $P_V j P_V$ is compact for all $j \in J$ and $[P_V, a]$ is compact for all $a \in A$, we also denote by V the extension $A/J \to Q(V)$ given by $a \mapsto \pi(P_V \widetilde{a} P_V)$, where $\widetilde{a} \in A$ is any lift of $a \in A/J$ and $\pi : B(V) \to Q(V)$ is the quotient map.

Proposition 8.12. *Suppose A is a trivially $\mathbb{Z}_2$-graded unital C^*-algebra, $J \lhd A$ is an ideal such that A/J is nuclear, and $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is an even relative spectral triple for $J \lhd A$. Write $\mathcal{D} = \left(\begin{smallmatrix} 0 & \mathcal{D}^1 \\ \mathcal{D}^0 & 0 \end{smallmatrix}\right)$ with respect to the $\mathbb{Z}_2$-grading of $\mathcal{H}$. Then*

$$\partial[(\mathcal{A}, \mathcal{H}, \mathcal{D})] = [\ker((\mathcal{D}^*)^0)] = -[\ker((\mathcal{D}^*)^1)] \in \mathrm{Ext}(A/J) \cong K^1(A/J).$$

Proof. Suppose first that A is represented non-degenerately on $\mathcal{H}$. Then $(1+\mathcal{D}^*\mathcal{D})^{-1/2}$ is compact. Let V be the phase of $\mathcal{D}$, which is the partial isometry with initial space $\ker(\mathcal{D})^\perp$ and final space $\ker(\mathcal{D}^*)^\perp$ defined by $\mathcal{D} = V|\mathcal{D}|$, [39, Thm. VIII.32]. Then $\mathcal{D}\mathcal{D}^* = V|\mathcal{D}|^2 V^* = V\mathcal{D}^*\mathcal{D}V^*$, which implies that

$$(1 - P_{\ker(\mathcal{D}^*)})(1 + \mathcal{D}\mathcal{D}^*)^{-1/2} = VV^*(1 + \mathcal{D}\mathcal{D}^*)^{-1/2} = V(1 + \mathcal{D}^*\mathcal{D})^{-1/2}V^*,$$

and hence $(1 - P_{\ker(\mathcal{D}^*)})(1 + \mathcal{D}\mathcal{D}^*)^{-1/2}$ is compact.

Let $F = \mathcal{D}(1 + \mathcal{D}^*\mathcal{D})^{-1/2}$ be the bounded transform of $\mathcal{D}$. Since $(1 - F^*F)^{1/2} = (1 + \mathcal{D}^*\mathcal{D})^{-1/2}$ and $(1 - FF^*)^{1/2} = (1 + \mathcal{D}\mathcal{D}^*)^{-1/2}$, we see that the assumptions of Lemma 7.5 are satisfied. Since $\ker(F^0) = \ker(\mathcal{D}^0)$, and $\ker((F^0)^*) = \ker((\mathcal{D}^*)^1)$, it follows from Proposition 7.6 that

$$\partial[(\mathcal{A}, \mathcal{H}, \mathcal{D})] = [\ker(\mathcal{D}^0)] - [\ker((\mathcal{D}^*)^1)].$$

Since $(1 + \mathcal{D}^*\mathcal{D})^{-1/2}$ is compact, the space $\ker(\mathcal{D}^0)$ is finite dimensional and hence the extension $A/J \to Q(\ker(\mathcal{D}^0))$ is trivial, and so $\partial[(\mathcal{A}, \mathcal{H}, \mathcal{D})] = -[\ker((\mathcal{D}^*)^1)]$.

If $(\rho, \mathcal{H}, F)$ is a relative Fredholm module, then $(\rho, \mathcal{H}, F^*)$ is also a relative Fredholm module representing the same class in relative K-homology, since the map $[0,1] \ni t \mapsto tF + (1-t)F^*$ is an operator homotopy of relative Fredholm modules. Hence Proposition 7.6 applied to $(\rho, \mathcal{H}, F^* = \mathcal{D}^*(1 + \mathcal{D}\mathcal{D}^*)^{-1/2})$ yields

$$\partial[(\mathcal{A}, \mathcal{H}, \mathcal{D})] = [\ker((\mathcal{D}^*)^0)] - [\ker(\mathcal{D}^1)],$$

where $\ker(\mathcal{D}^1)$ is finite dimensional and thus defines a trivial extension.

If A is not represented non-degenerately, we can first "cut-down" the relative spectral triple to $(\mathcal{A}, \rho(1)\mathcal{H}, \rho(1)\mathcal{D}\rho(1))$ without altering the class in relative K-homology. The algebra A is represented non-degenerately on $\rho(1)\mathcal{H}$, and hence the above argument shows that

$$\partial[(\mathcal{A}, \mathcal{H}, \mathcal{D})] = [\rho(1)\ker((\mathcal{D}^*)^0)] = -[\rho(1)\ker((\mathcal{D}^*)^1)].$$

For $j = 0, 1$, $\ker((\mathcal{D}^*)^j)) = \rho(1)\ker((\mathcal{D}^*)^j) \oplus (1-\rho(1))\ker((\mathcal{D}^*)^j)$ as both subspaces and extensions. Since the extension defined by $(1-\rho(1))\ker((\mathcal{D}^*)^j)$ is trivial it follows that $[\ker((\mathcal{D}^*)^j)] = [\rho(1)\ker((\mathcal{D}^*)^j)]$. □

Remark. Proposition 8.12 also holds for non-unital algebras A, provided one assumes that $\rho(a)(1 - P_{\ker(\mathcal{D}^*)})(1 + \mathcal{D}\mathcal{D}^*)^{-1/2}$ is compact for all $a \in A$. Although $\ker(\mathcal{D})$ is not necessarily finite-dimensional in this case, it is nevertheless true that $aP_{\ker(\mathcal{D})}$ is compact for all $a \in A$ and hence $\ker(\mathcal{D}^0)$ and $\ker(\mathcal{D}^1)$ define trivial extensions of A/J.

Chapter 9

The Clifford normal

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a relative spectral triple for an ideal J in a $\mathbb{Z}_2$-graded, separable C^*-algebra A. We would like to be able to construct a "boundary spectral triple" $(\mathcal{A}/\mathcal{J}\widehat{\otimes}\mathbb{C}\mathrm{l}_1, \mathcal{H}_\partial, \mathcal{D}_\partial)$ for $A/J\widehat{\otimes}\mathbb{C}\mathrm{l}_1$ such that $\partial([\mathcal{A}, \mathcal{H}, \mathcal{D})] = [(\mathcal{A}/\mathcal{J}\widehat{\otimes}\mathbb{C}\mathrm{l}_1, \mathcal{H}_\partial, \mathcal{D}_\partial)]$, where $\partial : K^*(J \lhd A) \to K^{*+1}(A/J)$ is the boundary map in K-homology (when the boundary map makes sense). In this chapter we show that what we call a *Clifford normal* satisfying some assumptions can be used to construct a Hilbert space $\mathcal{H}_n$ carrying a representation of $A/J\widehat{\otimes}\mathbb{C}\mathrm{l}_1$ (or a $\mathbb{Z}_2$-graded Hilbert space carrying a representation of A/J in the case that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is odd). Under an additional assumption we can construct a symmetric operator $\mathcal{D}_n$ on $\mathcal{H}_n$. It is unknown however whether $(\mathcal{A}/\mathcal{J}\widehat{\otimes}\mathbb{C}\mathrm{l}_1, \mathcal{H}_n, \mathcal{D}_n)$ defines a spectral triple, or whether addition data or assumptions are required. Motivated by the doubling construction on a manifold with boundary, [7, Ch. 9], we also use the Clifford normal to construct a spectral triple for the pullback algebra $\widetilde{A} = \{(a, b) \in A \oplus A : a - b \in J\}$ in the case that A is unital and represented non-degenerately.

The motivation for the Clifford normal comes from the classical example of a manifold with boundary. Let $\mathcal{D}$ be a Dirac operator on a Clifford module S over a compact Riemannian manifold with boundary $\overline{M}$. Let $(\cdot|\cdot)$ denote the pointwise inner product on S. For sections $\xi, \eta \in \Gamma^\infty(S)$, we have Green's formula [7, Prop. 3.4]

$$\langle \mathcal{D}\xi, \eta\rangle_{L^2(S)} - \langle \xi, \mathcal{D}\eta\rangle_{L^2(S)} = \int_{\partial M} (\xi|c(n)\eta)\ \mathrm{vol}_{\partial M} = \langle \xi, c(n)\eta\rangle_{L^2(S|_{\partial M})} \tag{9.1}$$

where $n \in \Gamma^\infty(T^*\overline{M}|_{\partial M})$ is the inward unit normal, and c denotes Clifford multiplication. If $\widetilde{n}$ is some smooth extension of n to the whole manifold, then the inner product on the boundary can be expressed as

$$\langle \xi, \eta\rangle_{L^2(S|_{\partial M})} = \langle \xi, \mathcal{D}c(\widetilde{n})\eta\rangle_{L^2(S)} - \langle \mathcal{D}\xi, c(\widetilde{n})\eta\rangle_{L^2(S)}\,.$$

The operator $\widetilde{n}$ is the model for the Clifford normal.

Remark. In Appendix C we show that the naive application of Green's formula alone (i.e. without a Clifford normal) fails to produce a boundary Hilbert space carrying a representation of the quotient C^*-algebra A/J.

Definition 9.1. Let A be a separable $\mathbb{Z}_2$-graded C^*-algebra, let $J \lhd A$ be an ideal, and let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a relative spectral triple for $J \lhd A$. A **Clifford normal** for $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is an odd (in the case that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is even) operator $n \in B(\mathcal{H})$ such that

1) $n \cdot \mathrm{dom}(\mathcal{D}) \subset \mathrm{dom}(\mathcal{D})$ and $\mathrm{dom}(n) := \mathrm{dom}(\mathcal{D}^*) \cap n\,\mathrm{dom}(\mathcal{D}^*)$ is a core for $\mathcal{D}^*$;

2) $n^* = -n$;

3) $[\mathcal{D}^*, n]$ is a symmetric operator (on $\mathrm{dom}(n)$);

4) $[n, a]_\pm \cdot \mathrm{dom}(n) \subset \mathrm{dom}(\mathcal{D})$ for all $a \in \mathcal{A}$;

5) $(n^2 + 1) \cdot \mathrm{dom}(n) \subset \mathrm{dom}(\mathcal{D})$;

6) $\langle \xi, \mathcal{D}^* n\xi \rangle - \langle \mathcal{D}^*\xi, n\xi \rangle \geq 0$ for all $\xi \in \mathrm{dom}(n)$;

7) For $w, z \in \mathrm{dom}(\mathcal{D}^*)$, if $\langle w, \mathcal{D}^* n\xi \rangle - \langle \mathcal{D}^* w, n\xi \rangle = -\langle z, \mathcal{D}^*\xi \rangle + \langle \mathcal{D}^* z, \xi \rangle$ for all $\xi \in \mathrm{dom}(n)$, then $w + nz \in \mathrm{dom}(\mathcal{D})$.

Remark. Condition 2) of the Definition 9.1 can be weakened to $(n + n^*) \cdot \mathrm{dom}(n) \subset \mathrm{dom}(\mathcal{D})$ instead of $n = -n^*$, and in the case that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is even, the condition that n is odd can be weakened to $(n\gamma + \gamma n) \cdot \mathrm{dom}(n) \subset \mathrm{dom}(\mathcal{D})$, where γ is the grading operator on $\mathcal{H}$, but in practice we shouldn't need this level of generality.

Note that Conditions 1) and 5) together imply that n preserves $\mathrm{dom}(n)$.

The non-degeneracy condition 7) of Definition 9.1 is used to show that the Hermitian form of Definition 9.3 is non-degenerate. Condition 7) is also used to prove the self-adjointness of the operator built in the "doubling construction" of §9.3.

Example 9.2. Let $\mathcal{D}$ be (the minimal extension of) a Dirac operator on a Clifford module S over a compact Riemannian manifold with boundary $\overline{M}$. Then as in Example 8.2, $(C^\infty(\overline{M}), L^2(S), \mathcal{D})$ is a relative spectral triple for $C_0(M) \lhd C(\overline{M})$. We can extend the inward unit normal on the boundary to a unitary endomorphism defined on a collar neighbourhood of the boundary, and use a cut-off function to define an anti-self-adjoint endomorphism n over the whole manifold.

It follows from [2, Thm. 6.7] that $\mathrm{dom}(n) = H^1(M, S)$, the first Sobolev space, which is densely contained in $\mathrm{dom}(D^*)$ with respect to the graph norm. The domain of $\mathcal{D}$ is $\{\xi \in H^1(M, S) : \xi|_{\partial M} = 0\}$, [2, Cor. 6.6]. So we see that Condition 1) of Definition 9.1 is satisfied. Condition 2) is true by construction, 4) holds since functions commute

with endomorphisms, and 5) is true since $n^2 = -1$ is a neighbourhood of the boundary. Green's formula (9.1) implies that Conditions 6) and 7) are satisfied.[1]

To address Condition 3), we examine the behaviour of the Dirac operator near the boundary. In a collar neighbourhood of the boundary, $\mathcal{D}^*$ has the form

$$\mathcal{D}^* = n\left(\frac{\partial}{\partial u} + B_u\right)$$

where u is the inward normal coordinate and B_u is a family of Dirac operators over the boundary, [7, p. 50]. Near the boundary,

$$[\mathcal{D}^*, n] = n\left(\frac{\partial}{\partial u} + B_u\right)n - n^2\left(\frac{\partial}{\partial u} + B_u\right) = n\frac{\partial n}{\partial u} + nB_u n + B_u.$$

The second and third terms are symmetric, and since n commutes with $\frac{\partial n}{\partial u}$, it is easy to check that $n\frac{\partial n}{\partial u}$ is symmetric. Hence $[\mathcal{D}^*, n]$ is symmetric, and thus n is a Clifford normal for the relative spectral triple $(C^\infty(\overline{M}), L^2(S), \mathcal{D})$.

If M is merely an open submanifold of a complete manifold, then we still obtain a relative spectral triple, as in Example 8.2. However, in this case $\overline{M}$ need not be a manifold with boundary and there need not be a normal. So the Clifford normal n is additional structure that is imposed on the geometry in order to obtain a reasonable boundary.

We fix the following data for the rest of this section: let A be a separable $\mathbb{Z}_2$-graded C^*-algebra, let $J \lhd A$ be an ideal, and let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a relative spectral triple for $J \lhd A$.

9.1 The boundary Hilbert space

Definition 9.3. Let n be a Clifford normal for $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. Define a sesquilinear form $\langle \cdot, \cdot \rangle_n$ on $\operatorname{dom}(n)/\operatorname{dom}(\mathcal{D})$ by

$$\langle [\xi], [\eta] \rangle_n = \langle \xi, \mathcal{D}^* n\eta \rangle - \langle \mathcal{D}^*\xi, n\eta \rangle .$$

Lemma 9.4. *Let n be a Clifford normal for the relative spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. The form $\langle \cdot, \cdot \rangle_n$ is a Hermitian inner product.*

Proof. We first establish that $\langle \cdot, \cdot \rangle_n$ is well-defined. If $\xi \in \operatorname{dom}(\mathcal{D})$ and $\eta \in \operatorname{dom}(n)$,

[1] Although Green's formula is stated for smooth sections, it extends to functions in $\operatorname{dom}(n) = H^1(M, S)$ since smooth sections are dense in $H^1(M, S)$, [2, Lemma 6.4(i)], and since the trace map $H^1(M, S) \ni \xi \mapsto \xi|_{\partial M} \in H^{1/2}(\partial M, S|_{\partial M}) \subset L^2(\partial M, S|_{\partial M})$ is continuous, [7, Cor. 11.6].

then

$$\langle \xi, \mathcal{D}^* n\eta\rangle - \langle \mathcal{D}^*\xi, n\eta\rangle = \langle \xi, \mathcal{D}^* n\eta\rangle - \langle \mathcal{D}\xi, n\eta\rangle = \langle \xi, \mathcal{D}^* n\eta\rangle - \langle \xi, \mathcal{D}^* n\eta\rangle = 0,$$

On the other hand, if $\eta \in \operatorname{dom}(\mathcal{D})$ and $\xi \in \operatorname{dom}(n)$, then

$$\langle \xi, \mathcal{D}^* n\eta\rangle - \langle \mathcal{D}^*\xi, n\eta\rangle = \langle \xi, \mathcal{D} n\eta\rangle - \langle \mathcal{D}^*\xi, n\eta\rangle = \langle \mathcal{D}^*\xi, n\eta\rangle - \langle \mathcal{D}^*\xi, n\eta\rangle = 0.$$

These calculations show that $\langle [\xi], [\eta]\rangle_n$ does not depend on the choice of representatives $\xi, \eta \in \operatorname{dom}(n)$ of $[\xi], [\eta] \in \operatorname{dom}(n)/\operatorname{dom}(D)$, and hence that $\langle \cdot, \cdot\rangle_n$ is well-defined.

To show that the form is Hermitian, we compute

$$\begin{aligned}
&\overline{\langle [\xi], [\eta]\rangle_n} = \overline{\langle \xi, \mathcal{D}^* n\eta\rangle} - \overline{\langle \mathcal{D}^*\xi, n\eta\rangle} = \langle \mathcal{D}^* n\eta, \xi\rangle - \langle n\eta, \mathcal{D}^*\xi\rangle \\
&= \langle n\mathcal{D}^*\eta, \xi\rangle + \langle [\mathcal{D}^*, n]\eta, \xi\rangle + \langle \eta, n\mathcal{D}^*\xi\rangle = -\langle \mathcal{D}^*\eta, n\xi\rangle + \langle \eta, [\mathcal{D}^*, n]\xi\rangle + \langle \eta, n\mathcal{D}^*\xi\rangle \\
&= \langle \eta, \mathcal{D}^* n\xi\rangle - \langle \mathcal{D}^*\eta, n\xi\rangle = \langle [\eta], [\xi]\rangle_n .
\end{aligned}$$

Conditions 6) and 7) of Definition 9.1 ensure that $\langle \cdot, \cdot\rangle_n$ is positive-definite, and thus $\langle \cdot, \cdot\rangle_n$ is an inner product. □

The completion of $\operatorname{dom}(n)/\operatorname{dom}(\mathcal{D})$ with respect to the norm coming from $\langle \cdot, \cdot\rangle_n$ is a Hilbert space, which we call the **boundary Hilbert space** and denote by $\mathcal{H}_n$.

Lemma 9.5. *Let n be a Clifford normal for the relative spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. Given $[a] \in \mathcal{A}/\mathcal{J}$, define*

$$\rho_\partial([a]) : \operatorname{dom}(n)/\operatorname{dom}(\mathcal{D}) \to \operatorname{dom}(n)/\operatorname{dom}(\mathcal{D}), \quad \rho_\partial([a])[\xi] = [a\xi].$$

The map $[a] \mapsto \rho_\partial([a])$ is multiplicative, and satisfies

$$\langle \rho_\partial([a])[\xi], [\eta]\rangle_n = \langle [\xi], \rho_\partial([a]^*)[\eta]\rangle_n , \qquad [\xi], [\eta] \in \operatorname{dom}(n)/\operatorname{dom}(\mathcal{D}).$$

Proof. That $[a] \mapsto \rho_\partial([a])$ is multiplicative is immediate. Let $\xi, \eta \in \operatorname{dom}(n)$, and let $a \in \mathcal{A}$ be of homogeneous degree. Then

$$\begin{aligned}
&\langle \rho_\partial([a])[\xi], [\eta]\rangle_n - \langle [\xi], \rho_\partial([a]^*)[\eta]\rangle_n \\
&= \langle a\xi, \mathcal{D}^* n\eta\rangle - \langle \mathcal{D}^* a\xi, n\eta\rangle - \langle \xi, \mathcal{D}^* n a^*\eta\rangle + \langle \mathcal{D}^*\xi, n a^*\eta\rangle \\
&= \langle \xi, a^*\mathcal{D}^* n\eta\rangle - \langle \mathcal{D}^* a\xi, n\eta\rangle - (-1)^{\deg a}\langle \xi, \mathcal{D}^* a^* n\eta\rangle - \langle \xi, \mathcal{D}[n, a^*]_\pm \eta\rangle \\
&\quad + (-1)^{\deg a}\langle \mathcal{D}^*\xi, a^* n\eta\rangle + \langle \mathcal{D}^*\xi, [n, a^*]_\pm \eta\rangle \\
&= -(-1)^{\deg a}\langle \xi, [\mathcal{D}^*, a^*]_\pm n\eta\rangle - \langle [\mathcal{D}^*, a]_\pm \xi, n\eta\rangle = 0,
\end{aligned}$$

since $([\mathcal{D}^*, a]_\pm)^* \supset -(-1)^{\deg a}[\mathcal{D}^*, a^*]_\pm$. □

Remark. It is not clear whether $\rho_\partial(a)$ is bounded as an operator on $\mathcal{H}_n$ for all $a \in \mathcal{A}/\mathcal{J}$.

Definition 9.6. Let n be a Clifford normal for the relative spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. We define an operator $n_\partial : \operatorname{dom}(n)/\operatorname{dom}(\mathcal{D}) \to \operatorname{dom}(n)/\operatorname{dom}(\mathcal{D})$ by $n_\partial[\xi] := [n\xi]$.

Lemma 9.7. *Let n be a Clifford normal for the relative spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. The operator n_∂ has the properties $n_\partial^2 = -1$ and $\langle n_\partial[\xi], n_\partial[\eta]\rangle_n = \langle [\xi], [\eta]\rangle_n$ for all $[\xi], [\eta] \in \operatorname{dom}(n)/\operatorname{dom}(\mathcal{D})$.*

Proof. The first claim follows from $(n^2+1)\cdot \operatorname{dom}(n) \subset \operatorname{dom}(\mathcal{D})$. For the second claim, we have

$$\begin{aligned}
&\langle n_\partial[\xi], n_\partial[\eta]\rangle_n = \left\langle n\xi, \mathcal{D}^* n^2\eta\right\rangle - \left\langle \mathcal{D}^* n\xi, n^2\eta\right\rangle \\
&= -\langle n\xi, \mathcal{D}^*\eta\rangle + \langle \mathcal{D}^* n\xi, \eta\rangle + \left\langle n\xi, \mathcal{D}(n^2+1)\xi\right\rangle - \left\langle \mathcal{D}^* n\xi, (n^2+1)\xi\right\rangle \\
&= \langle \xi, n\mathcal{D}^*\eta\rangle - \langle \mathcal{D}^*\xi, n\eta\rangle + \langle [\mathcal{D}^*, n]\xi, \eta\rangle \\
&= \langle \xi, \mathcal{D}^* n\eta\rangle - \langle \xi, [\mathcal{D}^*, n]\eta\rangle - \langle \mathcal{D}^*\xi, n\eta\rangle + \langle [\mathcal{D}^*, n]\xi, \eta\rangle = \langle [\xi], [\eta]\rangle_n . \qquad \square
\end{aligned}$$

Definition 9.8. Let n be a Clifford normal for the relative spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. Assume that $\rho_\partial(a) \in B(\mathcal{H}_n)$ for all $a \in \mathcal{A}/\mathcal{J}$, so that Lemma 9.5 implies that ρ_∂ extends to a $*$-homomorphism $\rho_\partial : A/J \to B(\mathcal{H}_n)$. We consider the cases that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is even and odd separately.

Suppose first $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is even. Then the boundary Hilbert space $\mathcal{H}_n$ inherits the $\mathbb{Z}_2$-grading of $\mathcal{H}$, and we use the operator n_∂ to define a representation of $A/J\widehat{\otimes}\mathbb{C}\mathrm{l}_1$ on $\mathcal{H}_n$. The representation of $\mathbb{C}\mathrm{l}_1$ provides the appropriate shift in KK-degree for the boundary map, since $K^1(A/J) = K^0(A/J\widehat{\otimes}\mathbb{C}\mathrm{l}_1)$. Define a representation of $\mathbb{C}\mathrm{l}_1$ on $\mathcal{H}_\partial$ by $c \mapsto -in_\partial$, where c is the self-adjoint unitary generator of $\mathbb{C}\mathrm{l}_1$. This is a well-defined representation by Lemma 9.7. Condition 4) of Definition 9.1 ensures that $[n_\partial, [a]]_\pm = 0$ for all $a \in A/J$, and hence $a\widehat{\otimes}z \mapsto \rho_\partial(a)z$ is a representation of $A/J\widehat{\otimes}\mathbb{C}\mathrm{l}_1$ on the boundary Hilbert space $\mathcal{H}_n$.

Suppose now that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is odd. Lemma 9.7 shows that $\gamma := -in_\partial$ is a grading operator on $\mathcal{H}_n$; i.e. $\gamma = \gamma^*$ and $\gamma^2 = 1$, and so we define a $\mathbb{Z}_2$-grading on $\mathcal{H}_n$ by $\mathcal{H}_n^j = \{\xi \in \mathcal{H}_n : \gamma\xi = (-1)^j\xi\}$.

9.2 The boundary operator

Let n be a Clifford normal for the relative spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. We construct an unbounded operator $\mathcal{D}_n$ on the boundary Hilbert space $\mathcal{H}_n$, so that $(\mathcal{A}/\mathcal{J}\widehat{\otimes}\mathbb{C}\mathrm{l}_1, \mathcal{H}_n, \mathcal{D}_n)$ is a candidate for a spectral triple (assuming ρ_∂ maps $\mathcal{A}/\mathcal{J}$ into $B(\mathcal{H}_n)$). We need an additional assumption in order to construct $\mathcal{D}_n$, however.

Definition 9.9. Let n be a Clifford normal for the relative spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. We define the spaces

$$\mathcal{H}_n^2 := \{\xi \in \mathrm{dom}(n) : \mathcal{D}^*(n\xi), \mathcal{D}^*\xi \in \mathrm{dom}(n)\}, \qquad \mathcal{H}_{n,0}^2 := \mathcal{H}_n^2 \cap \mathrm{dom}(\mathcal{D}).$$

Let

$$\mathrm{dom}(\mathcal{D}_n) = \{[\xi] : \xi \in \mathcal{H}_n^2\} \subset \mathrm{dom}(n)/\,\mathrm{dom}(\mathcal{D}),$$

and, assuming that $[\mathcal{D}^*, n] \cdot \mathcal{H}_{n,0}^2 \subset \mathrm{dom}(\mathcal{D})$, define $\mathcal{D}_n : \mathrm{dom}(\mathcal{D}_n) \to \mathrm{dom}(n)/\,\mathrm{dom}(\mathcal{D})$ by

$$\mathcal{D}_n[\xi] := \left[\frac{1}{2} n[\mathcal{D}^*, n]\xi\right].$$

We call the operator $\mathcal{D}_n$ the **boundary operator**. Since $(n^2+1) \cdot \mathrm{dom}(n) \subset \mathrm{dom}(\mathcal{D})$, $\mathcal{D}_n$ anticommutes with n_∂, and so $\mathcal{D}_n$ is an odd operator for $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ either even or odd.

Remark. Not every representative of $[\xi]$ is necessarily in $\mathcal{H}_n^2$, so such a representative must be chosen when defining $\mathcal{D}_n[\xi]$. However, the assumption that $[\mathcal{D}^*, n]$ maps $\mathcal{H}_{n,0}^2$ into $\mathrm{dom}(\mathcal{D})$ means that $\mathcal{D}_n$ does not depend on the particular choice of representative.

The next result shows that there are several equivalent statements to the assumption $[\mathcal{D}^*, n] \cdot \mathcal{H}_{n,0}^2 \subset \mathrm{dom}(\mathcal{D})$. Note that $\{\cdot, \cdot\}$ denotes the anticommutator $\{a, b\} = ab + ba$.

Proposition 9.10. *Let n be a Clifford normal for the relative spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, and assume that $n \cdot \mathcal{H}_n^2 \subset \mathcal{H}_n^2$. Then the following are equivalent.*

1) $[\mathcal{D}^*, n] \cdot \mathcal{H}_{n,0}^2 \subset \mathrm{dom}(\mathcal{D})$.

2) For all $\xi \in \mathcal{H}_{n,0}^2$, $\eta \in \mathcal{H}_n^2$,

$$\langle \{\mathcal{D}^*, [\mathcal{D}^*, n]\}\xi, \eta\rangle = \langle \xi, \{\mathcal{D}^*, [\mathcal{D}^*, n]\}\eta\rangle\,.$$

3) For all $\xi \in \mathcal{H}_{n,0}^2$, $\eta \in \mathcal{H}_n^2$,

$$\langle [\mathcal{D}^*, [\mathcal{D}^*, n]]\xi, \eta\rangle = -\,\langle \xi, [\mathcal{D}^*, [\mathcal{D}^*, n]]\eta\rangle\,.$$

4) For all $\xi \in \mathcal{H}_{n,0}^2$, $\eta \in \mathcal{H}_n^2$,

$$\langle \mathcal{D}^*[\mathcal{D}^*, n]\xi, \eta\rangle = \langle \xi, [\mathcal{D}^*, n]\mathcal{D}^*\eta\rangle\,.$$

Proof. We will show that 2) $\Leftrightarrow$ 3), 3) $\Leftrightarrow$ 4) and 4) $\Leftrightarrow$ 1). Let $\xi \in \mathcal{H}_{n,0}^2$ and $\eta \in \mathcal{H}_n^2$.

Then

$$\begin{aligned}
&\langle [\mathcal{D}^*,[\mathcal{D}^*,n]]\xi,\eta\rangle = \langle \mathcal{D}^*[\mathcal{D}^*,n]\xi,\eta\rangle - \langle [\mathcal{D}^*,n]\mathcal{D}\xi,\eta\rangle \\
&= \langle \mathcal{D}^*[\mathcal{D}^*,n]\xi,\eta\rangle - \langle \mathcal{D}\xi,[\mathcal{D}^*,n]\eta\rangle = \langle \mathcal{D}^*[\mathcal{D}^*,n]\xi,\eta\rangle - \langle \xi,\mathcal{D}^*[\mathcal{D}^*,n]\eta\rangle \\
&= \langle \{\mathcal{D}^*,[\mathcal{D}^*,n]\}\xi,\eta\rangle - \langle [\mathcal{D}^*,n]\mathcal{D}\xi,\eta\rangle - \langle \xi,\mathcal{D}^*[\mathcal{D}^*,n]\eta\rangle \\
&= \langle \{\mathcal{D}^*,[\mathcal{D}^*,n]\}\xi,\eta\rangle - 2\langle \xi,\mathcal{D}^*[\mathcal{D}^*,n]\eta\rangle ,
\end{aligned}$$

so if 2) is true, then

$$\langle [\mathcal{D}^*,[\mathcal{D}^*,n]]\xi,\eta\rangle = \langle \xi,\{\mathcal{D}^*,[\mathcal{D}^*,n]\}\eta\rangle - 2\langle \xi,\mathcal{D}^*[\mathcal{D}^*,n]\eta\rangle = -\langle \xi,[\mathcal{D}^*,[\mathcal{D}^*,n]]\eta\rangle ,$$

and so 3) is true. If on the other hand 3) is true, then

$$\begin{aligned}
\langle \xi,\{\mathcal{D}^*,[\mathcal{D}^*,n]\}\eta\rangle &= \langle [\mathcal{D}^*,[\mathcal{D}^*,n]]\xi,\eta\rangle + 2\langle \xi,\mathcal{D}^*[\mathcal{D}^*,n]\eta\rangle \\
&= -\langle \xi,[\mathcal{D}^*,[\mathcal{D}^*,n]]\eta\rangle + 2\langle \xi,\mathcal{D}^*[\mathcal{D}^*,n]\eta\rangle = \langle \xi,\{\mathcal{D}^*,[\mathcal{D}^*,n]\}\eta\rangle
\end{aligned}$$

and so 2) is true. We have

$$\begin{aligned}
\langle [\mathcal{D}^*,[\mathcal{D}^*,n]]\xi,\eta\rangle &= \langle \mathcal{D}^*[\mathcal{D}^*,n]\xi,\eta\rangle - \langle \xi,\mathcal{D}^*[\mathcal{D}^*,n]\eta\rangle \\
&= \langle \mathcal{D}^*[\mathcal{D}^*,n]\xi,\eta\rangle - \langle \xi,[\mathcal{D}^*,n]\mathcal{D}^*\eta\rangle - \langle \xi,[\mathcal{D}^*,[\mathcal{D}^*,n]]\eta\rangle ,
\end{aligned}$$

so 3) is true if and only if 4) is true. Suppose that 1) is true. Then

$$\langle \mathcal{D}^*[\mathcal{D}^*,n]\xi,\eta\rangle = \langle \mathcal{D}[\mathcal{D}^*,n]\xi,\eta\rangle = \langle [\mathcal{D}^*,n]\xi,\mathcal{D}^*\eta\rangle = \langle \xi,[\mathcal{D}^*,n]\mathcal{D}^*\eta\rangle ,$$

and so 4) is true. Now suppose that 4) is true, and let $\xi \in \mathcal{H}^2_{n,0}$, $\psi \in \ker(\mathcal{D}^* - i) \oplus \ker(\mathcal{D}^* + i)$. We will show that $[\mathcal{D}^*,n]\xi$ and ψ are orthogonal in the graph inner product of $\mathrm{dom}(\mathcal{D}^*)$. Since $\mathrm{dom}(\mathcal{D}^*) = \mathrm{dom}(\mathcal{D}) \oplus \ker(\mathcal{D}^* - i) \oplus \ker(\mathcal{D}^* + i)$ as a graph orthogonal sum, the graph orthogonality of $[\mathcal{D}^*,n]\xi$ and ψ for any $\psi \in \ker(\mathcal{D}^* - i) \oplus \ker(\mathcal{D}^* + i)$ shows that $[\mathcal{D}^*,n]\xi \in \mathrm{dom}(\mathcal{D})$ and hence that 1) is true. So assuming 4), we have

$$\begin{aligned}
&\langle [\mathcal{D}^*,n]\xi,\psi\rangle_{\mathrm{dom}(\mathcal{D}^*)} = \langle [\mathcal{D}^*,n]\xi,\psi\rangle + \langle \mathcal{D}^*[\mathcal{D}^*,n]\xi,\mathcal{D}^*\psi\rangle \\
&= \langle [\mathcal{D}^*,n]\xi,\psi\rangle + \left\langle \xi,[\mathcal{D}^*,n](\mathcal{D}^*)^2\psi\right\rangle = \langle [\mathcal{D}^*,n]\xi,\psi\rangle - \langle \xi,[\mathcal{D}^*,n]\psi\rangle = 0. \qquad \square
\end{aligned}$$

Proposition 9.11. *Let n be a Clifford normal for the relative spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ such that $n \cdot \mathrm{dom}((\mathcal{D}^*)^2) \subset \mathrm{dom}((\mathcal{D}^*)^2)$. The boundary operator $\mathcal{D}_n$ is well-defined and symmetric with respect to the boundary inner product $\langle\cdot,\cdot\rangle_n$ if and only if $\{\mathcal{D}^*,[\mathcal{D}^*,n]\}$ is a symmetric operator on $\mathcal{H}$ with the domain $\mathcal{H}^2_n$.*

Proof. If $\{\mathcal{D}^*,[\mathcal{D}^*,n]\}$ is symmetric on the domain $\mathcal{H}^2_n$, then $[\mathcal{D}^*,n] \cdot \mathcal{H}^2_{n,0} \subset \mathrm{dom}(\mathcal{D})$ by Proposition 9.10 and hence $\mathcal{D}_n$ is well-defined.

Let $\xi, \eta \in \mathcal{H}_n^2$. We compute the two terms of $\langle \mathcal{D}_n[\xi], [\eta]\rangle_n - \langle [\xi], \mathcal{D}_n[\eta]\rangle_n$ separately. We have

$$\begin{aligned}
&\langle \mathcal{D}_n[\xi], [\eta]\rangle_n = \frac{1}{2}\langle n[\mathcal{D}^*, n]\xi, \mathcal{D}^* n\eta\rangle - \frac{1}{2}\langle \mathcal{D}^* n[\mathcal{D}^*, n]\xi, n\eta\rangle \\
&= -\frac{1}{2}\langle [\mathcal{D}^*, n]\xi, n\mathcal{D}^* n\xi\rangle - \frac{1}{2}\langle n\mathcal{D}^*[\mathcal{D}^*, n]\xi, n\eta\rangle - \frac{1}{2}\langle [\mathcal{D}^*, n][\mathcal{D}^*, n]\xi, n\eta\rangle \\
&= -\frac{1}{2}\left\langle [\mathcal{D}^*, n]\xi, \mathcal{D}^* n^2\xi\right\rangle + \frac{1}{2}\langle [\mathcal{D}^*, n]\xi, [\mathcal{D}^*, n]n\xi\rangle + \frac{1}{2}\left\langle \mathcal{D}^*[\mathcal{D}^*, n]\xi, n^2\eta\right\rangle \\
&\quad - \frac{1}{2}\langle [\mathcal{D}^*, n][\mathcal{D}^*, n]\xi, n\eta\rangle \\
&= \frac{1}{2}\langle [\mathcal{D}^*, n]\xi, \mathcal{D}^*\eta\rangle - \frac{1}{2}\left\langle [\mathcal{D}^*, n]\xi, \mathcal{D}(n^2+1)\eta\right\rangle - \frac{1}{2}\langle \mathcal{D}^*[\mathcal{D}^*, n]\xi, \eta\rangle \\
&\quad + \frac{1}{2}\left\langle \mathcal{D}^*[\mathcal{D}^*, n]\xi, (n^2+1)\eta\right\rangle = \frac{1}{2}\langle \xi, [\mathcal{D}^*, n]\mathcal{D}^*\eta\rangle - \frac{1}{2}\langle \mathcal{D}^*[\mathcal{D}^*, n]\xi, \eta\rangle .
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\langle [\xi], \mathcal{D}_n[\eta]\rangle_n = \frac{1}{2}\left\langle \xi, \mathcal{D}^* n^2[\mathcal{D}^*, n]\eta\right\rangle - \frac{1}{2}\left\langle \mathcal{D}^*\xi, n^2[\mathcal{D}^*, n]\eta\right\rangle \\
&= -\frac{1}{2}\langle \xi, \mathcal{D}^*[\mathcal{D}^*, n]\eta\rangle + \frac{1}{2}\left\langle \xi, \mathcal{D}(n^2+1)[\mathcal{D}^*, n]\eta\right\rangle + \frac{1}{2}\langle \mathcal{D}^*\xi, [\mathcal{D}^*, n]\eta\rangle \\
&\quad - \frac{1}{2}\left\langle \mathcal{D}^*\xi, (n^2+1)[\mathcal{D}^*, n]\eta\right\rangle \\
&= \frac{1}{2}\langle [\mathcal{D}^*, n]\mathcal{D}^*\xi, \eta\rangle - \frac{1}{2}\langle \xi, \mathcal{D}^*[\mathcal{D}^*, n]\eta\rangle .
\end{aligned}$$

Combining the expressions for these two terms, we have

$$\langle \mathcal{D}_n[\xi], [\eta]\rangle_n - \langle [\xi], \mathcal{D}_n[\eta]\rangle_n = \frac{1}{2}\langle \xi, \{\mathcal{D}^*, [\mathcal{D}^*, n]\}\eta\rangle - \frac{1}{2}\langle \{\mathcal{D}^*, [\mathcal{D}^*, n]\}\xi, \eta\rangle ,$$

which establishes that $\mathcal{D}_n$ is symmetric on the image of $\mathcal{H}_n^2$ in $\mathcal{H}_n$ if and only if $\{\mathcal{D}^*, [\mathcal{D}^*, n]\}$ is symmetric on $\mathcal{H}_n^2$. □

Remark. It is unknown whether, under the existing assumptions, $\mathcal{D}_n$ is self-adjoint, $\mathcal{D}_n$ has bounded commutators with $\mathcal{A}/\mathcal{J}$ or $\mathcal{D}_n$ has (relatively) compact resolvent.

9.3 The doubling construction in the unital case

Recall that A is a $\mathbb{Z}_2$-graded separable C^*-algebra, $J \lhd A$ is an ideal and $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a relative spectral triple for $J \lhd A$. We now additionally require that $(\mathcal{A}, \mathcal{H}, D)$ is *even*. We also fix a Clifford normal n for $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.

In this section, we will use the Clifford normal n to construct the "doubled" spectral triple $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{H}}, \widetilde{\mathcal{D}})$ in the case that A is unital and represented non-degenerately on $\mathcal{H}$.

The doubled spectral triple is a spectral triple for the pullback C^*-algebra

$$\widetilde{A} := \{(a,b) \in A \oplus A : a - b \in J\}.$$

This construction mimics the doubling construction on a manifold with boundary, [7, Ch. 9].

Let $\widetilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$. We equip $\widetilde{\mathcal{H}}$ with the $\mathbb{Z}_2$-grading $\widetilde{\mathcal{H}}^j = \mathcal{H}^j \oplus \mathcal{H}^{j+1}$, $j \in \mathbb{Z}_2$. The pullback algebra $\widetilde{A}$ is represented on $\widetilde{\mathcal{H}}$ by $(a,b) \cdot (\xi, \eta) = (a\xi, b\eta)$.

Definition 9.12. Recall that n is the Clifford normal for the even relative spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, and let γ be the grading operator on $\mathcal{H}$ (i.e. γ is $(-1)^j$ on $\mathcal{H}^j$). Define an operator $\widetilde{\mathcal{D}}$ on $\widetilde{\mathcal{H}}$ on the domain

$$\mathrm{dom}(\widetilde{\mathcal{D}}) = \{(\xi, \eta) \in \mathrm{dom}(n) \oplus \mathrm{dom}(n) : \eta - n\gamma\xi \in \mathrm{dom}(\mathcal{D})\}$$

by $\widetilde{\mathcal{D}}(\xi, \eta) = (\mathcal{D}^*\xi, \mathcal{D}^*\eta)$.

Proposition 9.13. *The operator $\widetilde{\mathcal{D}}$ is self-adjoint on $\widetilde{\mathcal{H}}$.*

Proof. We first show that $\widetilde{\mathcal{D}}$ is symmetric, and then show that $\mathrm{dom}(\widetilde{\mathcal{D}}^*) \subset \mathrm{dom}(\widetilde{\mathcal{D}})$ which establishes that $\widetilde{\mathcal{D}}$ is self-adjoint. Let $(\xi, n\gamma\xi + \varphi), (\xi', n\gamma\xi' + \varphi') \in \mathrm{dom}(\widetilde{\mathcal{D}})$, where $\xi, \xi' \in \mathrm{dom}(n)$ and $\varphi, \varphi' \in \mathrm{dom}(\mathcal{D})$. Then

$$\begin{aligned}
&\left\langle \widetilde{\mathcal{D}}(\xi, n\gamma\xi + \varphi), (\xi', n\gamma\xi' + \varphi') \right\rangle \\
&= \left\langle \mathcal{D}^*\xi, \xi' \right\rangle + \left\langle \mathcal{D}^* n\gamma\xi, n\gamma\xi' \right\rangle + \left\langle \mathcal{D}\varphi, n\gamma\xi' \right\rangle + \left\langle \mathcal{D}^* n\gamma\xi, \varphi' \right\rangle \\
&= \left\langle \mathcal{D}^*\xi, \xi' \right\rangle + \left\langle [\mathcal{D}^*, n]\gamma\xi, n\gamma\xi' \right\rangle + \left\langle n\mathcal{D}^*\gamma\xi, n\gamma\xi' \right\rangle + \left\langle \varphi, \mathcal{D}^* n\gamma\xi' \right\rangle + \left\langle n\gamma\xi, \mathcal{D}\varphi' \right\rangle \\
&= \left\langle \mathcal{D}^*\xi, \xi' \right\rangle + \left\langle \gamma\xi, [\mathcal{D}^*, n] n\gamma\xi' \right\rangle + \left\langle \mathcal{D}^*\xi, n^2\xi' \right\rangle + \left\langle \varphi, \mathcal{D}^* n\gamma\xi' \right\rangle + \left\langle n\gamma\xi, \mathcal{D}\varphi' \right\rangle \\
&\qquad \text{(since } [\mathcal{D}^*, n] \text{ is symmetric)} \\
&= \left\langle \xi, \mathcal{D}^*\xi' \right\rangle + \left\langle \gamma\xi, [\mathcal{D}^*, n] n\gamma\xi' \right\rangle + \left\langle \xi, \mathcal{D}^* n^2\xi' \right\rangle + \left\langle \varphi, \mathcal{D}^* n\gamma\xi' \right\rangle + \left\langle n\gamma\xi, \mathcal{D}\varphi' \right\rangle \\
&\qquad \text{(since } (n^2+1) \cdot \mathrm{dom}(n) \subset \mathrm{dom}(\mathcal{D})) \\
&= \left\langle (\xi, n\gamma\xi + \varphi), \widetilde{\mathcal{D}}(\xi', n\gamma\xi' + \varphi') \right\rangle,
\end{aligned}$$

after some rearranging, which shows that $\widetilde{\mathcal{D}}$ is symmetric.

We now show that $\mathrm{dom}(\widetilde{\mathcal{D}}^*) \subset \mathrm{dom}(\widetilde{\mathcal{D}})$. Let $(\eta, \zeta) \in \mathrm{dom}(\widetilde{\mathcal{D}}^*)$, which means that there exists $(\rho, \sigma) \in \widetilde{\mathcal{H}}$ such that for all $(\xi, n\gamma\xi + \varphi) \in \mathrm{dom}(\widetilde{\mathcal{D}})$, with $\varphi \in \mathrm{dom}(\mathcal{D})$, we have

$$\left\langle \widetilde{\mathcal{D}}(\xi, n\gamma\xi + \varphi), (\eta, \zeta) \right\rangle = \langle (\xi, n\gamma\xi + \varphi), (\rho, \sigma) \rangle. \tag{9.2}$$

Since $\widetilde{\mathcal{D}}$ is an extension of $\mathcal{D} \oplus \mathcal{D}$, the adjoint $\widetilde{\mathcal{D}}^*$ is a restriction of $\mathcal{D}^* \oplus \mathcal{D}^*$, and so

$(\rho, \sigma) = (\mathcal{D}^*\eta, \mathcal{D}^*\zeta)$. Rearranging Equation (9.2), we have

$$-\langle \mathcal{D}^*\xi, \eta\rangle + \langle \xi, \mathcal{D}^*\eta\rangle = \langle \mathcal{D}^* n\xi, \gamma\zeta\rangle - \langle n\xi, \mathcal{D}^*\gamma\zeta\rangle$$

for all $\xi \in \mathrm{dom}(n)$, which by Condition 7) of Definition 9.1 implies that $\gamma\zeta + n\eta \in \mathrm{dom}(\mathcal{D})$. Applying the grading operator γ yields $\zeta - n\gamma\eta \in \mathrm{dom}(\mathcal{D})$ and hence $(\eta, \zeta) \in \mathrm{dom}(\widetilde{\mathcal{D}})$, and thus we have established that $\widetilde{\mathcal{D}}$ is self-adjoint. □

Proposition 9.14. *Recall that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is an even relative spectral triple for $J \lhd A$. Let*

$$\widetilde{\mathcal{A}} := \{(a, b) \in \mathcal{A} : a - b \in J\},$$

and let $\widetilde{\mathcal{D}}$ be as above. The $\mathbb{Z}_2$-graded commutators $[\widetilde{\mathcal{D}}, \widetilde{a}]_\pm$ are defined and bounded on $\mathrm{dom}(\widetilde{\mathcal{D}})$ for all $\widetilde{a} \in \widetilde{\mathcal{A}}$.

Proof. Let $(\xi, n\gamma\xi + \varphi) \in \mathrm{dom}(\widetilde{\mathcal{D}})$, where $\varphi \in \mathrm{dom}(\mathcal{D})$, and let $(a, a + j) \in \widetilde{\mathcal{A}}$, where $j \in \mathcal{J} = J \cap \mathcal{A}$. Then

$$(a\xi, an\gamma\xi + jn\gamma\xi + (a + j)\varphi) = (a\xi, n\gamma a\xi + [a, n]_\pm\gamma\xi + jn\gamma\xi + (a + j)\varphi)$$

which is in $\mathrm{dom}(\widetilde{\mathcal{D}})$ since $[a, n] \cdot \mathrm{dom}(n) \subset \mathrm{dom}(\mathcal{D})$, which is Condition 4) of Definition 9.1. The boundedness of the commutators follows from the fact that $[\mathcal{D}^*, a]$ is bounded for all $a \in \mathcal{A}$. □

The following result will be used to show that $\widetilde{\mathcal{D}}$ has compact resolvent in the case that $(1 + \mathcal{D}^*\mathcal{D})^{-1/2}$ is compact.

Proposition 9.15. *Let T be a closed symmetric operator on a separable Hilbert space $\mathcal{H}$ such that $(1+T^*T)^{-1/2}$ is compact, and let $T \subset T_e \subset T^*$ be a closed extension of T. Then $(1 + T_e^*T_e)^{-1/2}$ is compact if and only if $\ker(T_e)$ is finite dimensional.*

Proof. Let V be the phase of T, which is the partial isometry with initial space $\ker(T)^\perp$ and final space $\ker(T^*)^\perp$ defined by $T = V|T|$, [39, Thm. VIII.32]. Then $TT^* = V|T|^2V^* = VT^*TV^*$, which implies that

$$(1 - P_{\ker(T^*)})(1 + TT^*)^{-1/2} = VV^*(1 + TT^*)^{-1/2} = V(1 + T^*T)^{-1/2}V^*,$$

and hence $(1 - P_{\ker(T^*)})(1 + TT^*)^{-1/2}$ is compact.

For a closed operator S on $\mathcal{H}$, let $\iota_S : \mathrm{dom}(S) \hookrightarrow \mathcal{H}$ be the inclusion, where we equip $\mathrm{dom}(S)$ with the graph inner product. Since $(1 + S^*S)^{-1/2} : \mathcal{H} \to \mathrm{dom}(S)$ is unitary, Lemma C.2, (where $\mathrm{dom}(S)$ is equipped with the graph inner product), $(1 + S^*S)^{-1/2}$ is compact as an operator on $\mathcal{H}$ if and only if ι_S is compact.

We can write

$$\iota_{T^*} = \iota_{T^*} P_{\ker(T^*)} + \iota_{T^*}(1 - P_{\ker(T^*)}),$$

where the second term is compact since $(1 - P_{\ker(T^*)})(1 + TT^*)^{-1/2}$ is compact.

Let T_e be a closed operator with $T \subset T_e \subset T^*$. Then

$$\begin{aligned}\iota_{T_e} &= \iota_{T^*} i_{\mathrm{dom}(T_e)} = \iota_{T^*} P_{\ker(T^*)} i_{\mathrm{dom}(T_e)} + \iota_{T^*}(1 - P_{\ker(T^*)}) i_{\mathrm{dom}(T_e)} \\ &= \iota_{T^*} P_{\ker(T_e)} + \text{compact operator},\end{aligned}$$

where $i_{\mathrm{dom}(T_e)} \in B(\mathrm{dom}(T_e) \to \mathrm{dom}(T^*))$ is the inclusion of $\mathrm{dom}(T_e)$ in $\mathrm{dom}(T^*)$. Since the graph inner product and $\mathcal{H}$-inner product agree on $\ker(T^*)$, $\iota_{T^*} P_{\ker(T_e)}$ is compact if and only if $P_{\ker(T_e)}$ is compact. Therefore ι_{T_e} is compact if and only if $\ker(T_e)$ is finite dimensional. Using the fact that ι_{T_e} is compact if and only if $(1 + T_e^* T_e)^{-1/2} \in K(\mathcal{H})$ completes the proof. □

Lemma 9.16. *With $\widetilde{\mathcal{D}}$ as above,* $\ker(\widetilde{\mathcal{D}}) = \ker(\mathcal{D}) \oplus \ker(\mathcal{D})$.

Proof. Let $(\xi, n\gamma\xi + \varphi) \in \ker(\widetilde{\mathcal{D}})$, where $\varphi \in \mathrm{dom}(\mathcal{D})$, so $\xi, n\gamma\xi + \varphi \in \ker(\mathcal{D}^*)$. Since $\{\mathcal{D}^*, \gamma\} = 0$, γ preserves $\ker(\mathcal{D}^*)$, and so $-\gamma(n\gamma\xi + \varphi) = n\xi - \gamma\varphi \in \ker(\mathcal{D}^*)$. Hence

$$0 = \langle \xi, \mathcal{D}^*(n\xi - \gamma\varphi)\rangle - \langle \mathcal{D}^*\xi, n\xi - \gamma\varphi\rangle = \langle \xi, \mathcal{D}^* n\xi\rangle - \langle \mathcal{D}^*\xi, n\xi\rangle = \langle [\xi], [\xi]\rangle_n$$

since $\gamma\varphi \in \mathrm{dom}(\mathcal{D})$ and so $\langle \xi, \mathcal{D}\gamma\varphi\rangle = \langle \mathcal{D}^*\xi, \gamma\varphi\rangle$. The definiteness of $\langle \cdot, \cdot\rangle_n$ implies that $[\xi] = 0$; i.e. $\xi \in \mathrm{dom}(\mathcal{D})$. This in turn implies that $n\gamma\xi + \varphi \in \mathrm{dom}(\mathcal{D})$, and hence $(\xi, n\gamma\xi + \varphi) \in \ker(\mathcal{D}) \oplus \ker(\mathcal{D})$. □

Remark. Let $\mathcal{D}_{\min}$ be the minimal extension of a Dirac operator $\mathcal{D}$ on a (Clifford module over a) compact manifold with boundary, as in Example 8.2. Then $\ker(\mathcal{D}_{\min}) = \{0\}$, [7, Cor. 8.3]. The above result corresponds to the doubled operator $\widetilde{\mathcal{D}}$ being invertible in this case, [7, Thm. 9.1].

By combining Propositions 9.13, 9.14, 9.15 and Lemma 9.16, and using the fact that if A is unital and represented non-degenerately on $\mathcal{H}$, then $(1 + \mathcal{D}^*\mathcal{D})^{-1/2}$ is compact, we arrive at the main result of this section.

Theorem 9.17. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a relative spectral triple for an ideal J in a $\mathbb{Z}_2$-graded, separable, unital C^*-algebra A which is represented non-degenerately on $\mathcal{H}$, and let n be a Clifford normal for $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. Then the triple $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{H}}, \widetilde{\mathcal{D}})$ is a spectral triple for the pullback algebra $\widetilde{A} = \{(a, b) \in A \oplus A : a - b \in J\}$.*

Remark. It is an open question as to whether Theorem 9.17 holds in the non-unital case. If we assumed that $a(1 - P_{\ker(\mathcal{D}^*)})(1 + \mathcal{D}\mathcal{D}^*)^{-1/2}$ is compact for all $a \in \mathcal{A}$, this assumption could be used in place of Proposition 9.15 in order to prove that Theorem

9.17 holds for non-unital C^*-algebras. It is worth noting that this same assumption can also be used to generalise Proposition 8.12 to non-unital algebras, as noted in the remark following Proposition 8.12.

Appendix A

Odd cycles and circle factorisation

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be an odd $\mathbb{T}$-equivariant spectral triple for a trivially $\mathbb{Z}_2$-graded unital $\mathbb{T}$-algebra A, with A represented non-degenerately on $\mathcal{H}$, and $\mathcal{A}$ compatible with the action of $\mathbb{T}$. In order to factorise $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, we require a character $\ell \in \mathbb{Z}$ and a Clifford representation $\eta : \Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))^{\mathbb{T}} \cong \mathbb{C}\mathrm{l}_1 \to B(\mathcal{H})$ satisfying the conditions of Definition 3.9. We associate to $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ an even equivariant spectral triple $(\mathcal{A}\widehat{\otimes}\mathbb{C}\mathrm{l}_1, \mathcal{H}\widehat{\otimes}\mathbb{C}^2, \mathcal{D}\widehat{\otimes}\omega)$, as in §1.2.2, so that the Clifford representation $\widetilde{\eta} : \Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))^{\mathbb{T}} \to B(\mathcal{H}\widehat{\otimes}\mathbb{C}^2)$ associated to η defines the right-hand module, which is the even equivariant spectral triple $(\mathcal{A}^{\mathbb{T}}\widehat{\otimes}\mathbb{C}\mathrm{l}_1\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))^{\mathbb{T}}, \mathcal{H}_\ell\widehat{\otimes}\mathbb{C}^2, \mathcal{D}_\ell\widehat{\otimes}\omega)$, and hence a class $x \in KK^2_{\mathbb{T}}(A^{\mathbb{T}}, \mathbb{C})$.

Since $KK^2_{\mathbb{T}}(A^{\mathbb{T}}, \mathbb{C}) \cong KK_{\mathbb{T}}(A^{\mathbb{T}}, \mathbb{C})$, the class x is also represented by some even spectral triple for $A^{\mathbb{T}}$. So one would like to avoid having to associate an even spectral triple to the odd spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, and simply use $\gamma := i\eta(c(dt))$ as a grading operator on $\mathcal{H}_\ell$, so that $(\mathcal{A}^{\mathbb{T}}, \mathcal{H}_\ell, \mathcal{D}_\ell)$ is an even spectral triple for $A^{\mathbb{T}}$. Using $i\eta(c(dt))$ as a grading operator on $\mathcal{H}_\ell$ is the approach taken in [8, §6]. The difficulty one encounters is that $\mathcal{D}_\ell$ need not anticommute with $\eta(c(dt))$, so $\mathcal{D}_\ell$ need not be odd. In [8, §6] this failure to be odd is solved by subtracting the even part of $\mathcal{D}_\ell$ from $\mathcal{D}_\ell$, so that what remains is odd. This subtraction is also how the "horizontal Dirac operator" is constructed in [15, 16] (if their definition is repaired so that their equivalent of $\eta(c(dt))$ preserves the domain of $\mathcal{D}$).

In this appendix we show that the seemingly *ad hoc* method of subtracting the even part of $\mathcal{D}_\ell$ to obtain a spectral triple is in fact completely canonical. We give a brief outline of the canonical procedure here. The canonical isomorphism $KK^2_{\mathbb{T}}(A^{\mathbb{T}}, \mathbb{C}) = KK_{\mathbb{T}}(A^{\mathbb{T}}\widehat{\otimes}\mathbb{C}\mathrm{l}_1\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))^{\mathbb{T}}, \mathbb{C}) \to KK_{\mathbb{T}}(A^{\mathbb{T}}, \mathbb{C})$ is

$$y \mapsto [(A^{\mathbb{T}}, (A^{\mathbb{T}}\widehat{\otimes}(\mathbb{C}^2)^*)_{A^{\mathbb{T}}\widehat{\otimes}\mathbb{C}\mathrm{l}_1\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))^{\mathbb{T}}}, 0)]\widehat{\otimes}_{A^{\mathbb{T}}\widehat{\otimes}\mathbb{C}\mathrm{l}_1\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))^{\mathbb{T}}} y, \tag{A.1}$$

where $\mathbb{C}^2$ is the Morita equivalence between $\mathbb{C}\mathrm{l}_1\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))^{\mathbb{T}} \cong M_2(\mathbb{C})$ and $\mathbb{C}$, and $(\mathbb{C}^2)^*$ is the conjugate module. We use a connection (Definition 5.7) to take the Kasparov product (A.1) with the class $x \in KK^2_{\mathbb{T}}(A^{\mathbb{T}}, \mathbb{C})$ of the even equivariant spectral triple. Normally the choice of a connection is not canonical, but we will show below that in this case there is a unique *odd* connection (so that the operator one obtains is odd) facilitating the Kasparov product. We will show that the resulting even spectral triple for $A^{\mathbb{T}}$ is unitarily equivalent to $(\mathcal{A}^{\mathbb{T}}, \mathcal{H}_\ell, (\mathcal{D}_\ell)^1)$, where $(\mathcal{D}_\ell)^1$ denotes the odd part of $\mathcal{D}_\ell$ and $\mathcal{H}_\ell$ is $\mathbb{Z}_2$-graded by

$$\mathcal{H}_\ell^j = \{\xi \in \mathcal{H}_\ell : i\eta(c(dt))\xi = (-1)^j\xi\}.$$

Since all steps the the procedure to obtain the even spectral triple $(\mathcal{A}^{\mathbb{T}}, \mathcal{H}_\ell, (\mathcal{D}_\ell)^1)$ are canonical, we see that simply subtracting the non-odd parts of $\mathcal{D}_\ell$ is the correct thing to do.

A.1 The isomorphism $KK^2_{\mathbb{T}}(A^{\mathbb{T}}, \mathbb{C}) \to KK_{\mathbb{T}}(A^{\mathbb{T}}, \mathbb{C})$

Recall that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is an odd $\mathbb{T}$-equivariant spectral triple for a trivially $\mathbb{Z}_2$-graded unital $\mathbb{T}$-algebra A, and (ℓ, η) satisfy the conditions of Definition 3.9. We will use the Morita equivalence between $\mathbb{C}\mathrm{l}_1\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))^{\mathbb{T}}$ and $\mathbb{C}$ to obtain an even spectral triple for $A^{\mathbb{T}}$, which represents the class of $(\mathcal{A}^{\mathbb{T}}\widehat{\otimes}\mathbb{C}\mathrm{l}_1\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))^{\mathbb{T}}, \mathcal{H}_\ell\widehat{\otimes}\mathbb{C}^2, \mathcal{D}_\ell\widehat{\otimes}\omega)$ under the isomorphism $KK_{\mathbb{T}}(A^{\mathbb{T}}\widehat{\otimes}\mathbb{C}\mathrm{l}_1\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))^{\mathbb{T}}, \mathbb{C}) \cong KK_{\mathbb{T}}(A^{\mathbb{T}}, \mathbb{C})$.

We first write down an isomorphism between $\mathbb{C}\mathrm{l}_1\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))^{\mathbb{T}}$ and $M_2(\mathbb{C})$, which is defined on the generators of $\mathbb{C}\mathrm{l}_1\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))^{\mathbb{T}}$ by

$$1\widehat{\otimes}ic(dt) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad c\widehat{\otimes}1 \mapsto \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

This isomorphism represents $\mathbb{C}\mathrm{l}_1\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))^{\mathbb{T}}$ on $\mathbb{C}^2$, which is $\mathbb{Z}_2$-graded by

$$(\mathbb{C}^2)^j = \left\{v \in \mathbb{C}^2 : \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) v = (-1)^j v\right\},$$

so that $\mathbb{C}^2$ is a $\mathbb{Z}_2$-graded $\mathbb{C}\mathrm{l}_1\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))^{\mathbb{T}}$-$\mathbb{C}$ Morita equivalence bimodule. The conjugate module $(\mathbb{C}^2)^*$ (Definition 1.9) is a $\mathbb{C}$-$\mathbb{C}\mathrm{l}_1\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))^{\mathbb{T}}$ Morita equivalence bimodule. Hence we have the equivariant Kasparov module $\left(A^{\mathbb{T}}, (A^{\mathbb{T}}\widehat{\otimes}(\mathbb{C}^2)^*)_{A^{\mathbb{T}}\widehat{\otimes}\mathbb{C}\mathrm{l}_1\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))^{\mathbb{T}}}, 0\right)$, where all actions of $\mathbb{T}$ are trivial, which defines a class $KK_{\mathbb{T}}(A^{\mathbb{T}}, A^{\mathbb{T}}\widehat{\otimes}\mathbb{C}\mathrm{l}_1\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))^{\mathbb{T}})$, which we denote by z. Taking the Kasparov product with the class z implements the canonical isomorphism $KK^2_{\mathbb{T}}(A^{\mathbb{T}}, \mathbb{C}) \to KK_{\mathbb{T}}(A^{\mathbb{T}}, \mathbb{C})$. We use a connection as in §5.3 to construct an even equivariant spectral triple for $A^{\mathbb{T}}$ which represents the Kasparov product with the class of $(\mathcal{A}^{\mathbb{T}}\widehat{\otimes}\mathbb{C}\mathrm{l}_1\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))^{\mathbb{T}}, \mathcal{H}_\ell\widehat{\otimes}\mathbb{C}^2, \mathcal{D}_\ell\widehat{\otimes}\omega)$, and hence the corre-

sponding class in $KK_{\mathbb{T}}(A^{\mathbb{T}},\mathbb{C})$.

Proposition A.1. *There is a unique odd $\mathcal{D}_\ell$-connection*

$$\nabla : \mathcal{A}^{\mathbb{T}}\widehat{\otimes}(\mathbb{C}^2)^* \to (A^{\mathbb{T}}\widehat{\otimes}(\mathbb{C}^2)^*)\widehat{\otimes}_{A^{\mathbb{T}}\widehat{\otimes}\mathbb{C}\mathrm{l}_1\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))^{\mathbb{T}}} B(\mathcal{H}_\ell\widehat{\otimes}\mathbb{C}^2).$$

Proof. We first show that if ∇ exists, then is unique. Suppose ∇_1 and ∇_2 are two odd $\mathcal{D}_\ell$-connections. The graded Leibniz rule (5.3) implies that the difference $\nabla_1 - \nabla_2$ is $\mathcal{A}^{\mathbb{T}}\widehat{\otimes}\mathbb{C}\mathrm{l}_1\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))^{\mathbb{T}}$-linear. Since the Hilbert module $\mathcal{A}^{\mathbb{T}}\widehat{\otimes}(\mathbb{C}^2)^*$ is generated by $1\widehat{\otimes}\overline{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}$, $\nabla_1 - \nabla_2$ is determined by $(\nabla_1 - \nabla_2)(1\widehat{\otimes}\overline{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)})$. Since $\nabla_1 - \nabla_2$ is odd,

$$(\nabla_1 - \nabla_2)(1\widehat{\otimes}\overline{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}) = (1\widehat{\otimes}\overline{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)})\widehat{\otimes}F$$

for some odd $F \in B(\mathcal{H}_\ell\widehat{\otimes}\mathbb{C}^2)$, which we can write $F = F'\widehat{\otimes}\left(\begin{smallmatrix}0 & -i\\ i & 0\end{smallmatrix}\right) + F''\widehat{\otimes}\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$ for some $F', F'' \in B(\mathcal{H}_\ell)$. The $\mathcal{A}^{\mathbb{T}}\widehat{\otimes}\mathbb{C}\mathrm{l}_1\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))^{\mathbb{T}}$-linearity of $\nabla_1 - \nabla_2$ means that F commutes with $1\widehat{\otimes}\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, which implies that $F' = F'' = 0$, so $\nabla_1 - \nabla_2 = 0$.

We will now construct ∇. We can write any $a\widehat{\otimes}\overline{\left(\begin{smallmatrix}v_1\\v_2\end{smallmatrix}\right)} \in A^{\mathbb{T}}\widehat{\otimes}(\mathbb{C}^2)^*$ as

$$a\widehat{\otimes}\overline{\begin{pmatrix}v_1\\v_2\end{pmatrix}} = 1\widehat{\otimes}\overline{\begin{pmatrix}1\\0\end{pmatrix}}\cdot\frac{1}{2}\big(\overline{v_1}a\widehat{\otimes}1\widehat{\otimes}1 - i\overline{v_1}a\widehat{\otimes}c\widehat{\otimes}ic(dt) + \overline{v_2}a\widehat{\otimes}1\widehat{\otimes}ic(dt) - i\overline{v_2}a\widehat{\otimes}c\widehat{\otimes}1\big). \tag{A.2}$$

Recall that $\mathbb{C}\mathrm{l}_1\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))^{\mathbb{T}}$ is represented on $\mathcal{H}\widehat{\otimes}\mathbb{C}^2$ by

$$1\widehat{\otimes}ic(dt) \mapsto i\eta(c(dt)))\widehat{\otimes}\omega, \quad c\widehat{\otimes}1 \mapsto 1\widehat{\otimes}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

where $\omega = \left(\begin{smallmatrix}0 & -i\\ i & 0\end{smallmatrix}\right)$. For convenience we also write $1 = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$, $c = \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$, and $\gamma = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$. Then Equation (A.2) leads us to define

$$\begin{aligned}&\nabla\left(a\widehat{\otimes}\overline{\begin{pmatrix}v_1\\v_2\end{pmatrix}}\right)\\ &:= 1\widehat{\otimes}\overline{\begin{pmatrix}1\\0\end{pmatrix}}\widehat{\otimes}\frac{1}{2}[\mathcal{D}_\ell\widehat{\otimes}\omega, \overline{v_1}a\widehat{\otimes}1 + i\overline{v_1}a\eta(c(dt))\widehat{\otimes}\gamma + i\overline{v_2}a\eta(c(dt))\widehat{\otimes}\omega - i\overline{v_2}a\widehat{\otimes}c]_{\pm}.\end{aligned}$$

By construction, ∇ is odd and obeys the $\mathbb{Z}_2$-graded Leibniz rule (5.3), and it is equivariant since all the actions of $\mathbb{T}$ are trivial. (The action of $\mathbb{T}$ on $\mathcal{H}_\ell$ is not necessarily trivial, but since it is multiplication by the character χ_ℓ the induced action on $B(\mathcal{H}_\ell\widehat{\otimes}\mathbb{C}^2)$ given by conjugation by χ_ℓ is trivial.) $\square$

Lemma A.2. *Recall that $\mathcal{D}_\ell$ is the restriction of $\mathcal{D}$ to the character space $\mathcal{H}_\ell$, and let ∇ be the unique odd $\mathcal{D}_\ell$-connection of Proposition A.1. Let $P^\pm = \frac{1\pm i\eta(c(dt))}{2} \in B(\mathcal{H}_\ell)$. Then the operator $1\widehat{\otimes}_\nabla(\mathcal{D}_\ell\widehat{\otimes}\omega)$ on the Hilbert space $(A^{\mathbb{T}}\widehat{\otimes}(\mathbb{C}^2)^*)\widehat{\otimes}_{A^{\mathbb{T}}\widehat{\otimes}\mathbb{C}\mathrm{l}_1\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))}(\mathcal{H}_\ell\widehat{\otimes}\mathbb{C}^2)$ can be expressed as*

$$\begin{aligned}
&\left(1\widehat{\otimes}_\nabla(\mathcal{D}_\ell\widehat{\otimes}\omega)\right)\left(a\widehat{\otimes}\overline{\begin{pmatrix}v_1\\v_2\end{pmatrix}}\widehat{\otimes}\xi\widehat{\otimes}\begin{pmatrix}w_1\\w_2\end{pmatrix}\right)\\
&= 1\widehat{\otimes}\overline{\begin{pmatrix}1\\0\end{pmatrix}}\widehat{\otimes}\mathcal{D}_\ell(aP^+\xi)\widehat{\otimes}\begin{pmatrix}0\\i\overline{v_1}w_1+\overline{v_2}w_2\end{pmatrix} + 1\widehat{\otimes}\overline{\begin{pmatrix}1\\0\end{pmatrix}}\widehat{\otimes}\mathcal{D}_\ell(aP^-\xi)\widehat{\otimes}\begin{pmatrix}-i\overline{v_1}w_2-\overline{v_2}w_1\\0\end{pmatrix}.
\end{aligned}$$

Proof. By the definition of $1\widehat{\otimes}_\nabla(\mathcal{D}_\ell\widehat{\otimes}\omega)$ (Definition 5.7), we compute

$$\begin{aligned}
&\left(1\widehat{\otimes}_\nabla(\mathcal{D}_\ell\widehat{\otimes}\omega)\right)\left(a\widehat{\otimes}\overline{\begin{pmatrix}v_1\\v_2\end{pmatrix}}\widehat{\otimes}\xi\widehat{\otimes}\begin{pmatrix}w_1\\w_2\end{pmatrix}\right) = a\widehat{\otimes}\overline{\begin{pmatrix}v_1\\-v_2\end{pmatrix}}\widehat{\otimes}\mathcal{D}_\ell\xi\widehat{\otimes}\omega\begin{pmatrix}w_1\\w_2\end{pmatrix}\\
&+ 1\widehat{\otimes}\overline{\begin{pmatrix}1\\0\end{pmatrix}}\widehat{\otimes}\frac{1}{2}[\mathcal{D}_\ell\widehat{\otimes}\omega, \overline{v_1}a\widehat{\otimes}1 + \overline{v_1}ai\eta(c(dt))\widehat{\otimes}\gamma + \overline{v_2}ai\eta(c(dt))\widehat{\otimes}\omega\\
&\qquad\qquad - i\overline{v_2}a\widehat{\otimes}c]_\pm\left(\xi\widehat{\otimes}\begin{pmatrix}w_1\\w_2\end{pmatrix}\right)\\
&= a\widehat{\otimes}\overline{\begin{pmatrix}v_1\\-v_2\end{pmatrix}}\widehat{\otimes}\mathcal{D}_\ell\xi\widehat{\otimes}\begin{pmatrix}-iw_2\\iw_1\end{pmatrix} + \frac{1}{2}\widehat{\otimes}\overline{\begin{pmatrix}1\\0\end{pmatrix}}\widehat{\otimes}\mathcal{D}_\ell a\xi\widehat{\otimes}\begin{pmatrix}-i\overline{v_1}w_2\\i\overline{v_1}w_1\end{pmatrix}\\
&\quad - \frac{1}{2}a\widehat{\otimes}\overline{\begin{pmatrix}v_1\\0\end{pmatrix}}\widehat{\otimes}\mathcal{D}_\ell\xi\widehat{\otimes}\begin{pmatrix}-iw_2\\iw_1\end{pmatrix} + \frac{1}{2}\widehat{\otimes}\overline{\begin{pmatrix}1\\0\end{pmatrix}}\widehat{\otimes}\mathcal{D}_\ell ai\eta(c(dt))\xi\widehat{\otimes}\begin{pmatrix}i\overline{v_1}w_2\\i\overline{v_1}w_1\end{pmatrix}\\
&\quad - \frac{1}{2}a\widehat{\otimes}\overline{\begin{pmatrix}v_1\\0\end{pmatrix}}\widehat{\otimes}\mathcal{D}_\ell\xi\widehat{\otimes}\begin{pmatrix}-iw_2\\iw_1\end{pmatrix} + \frac{1}{2}\widehat{\otimes}\overline{\begin{pmatrix}1\\0\end{pmatrix}}\widehat{\otimes}\mathcal{D}_\ell ai\eta(c(dt))\xi\widehat{\otimes}\begin{pmatrix}\overline{v_2}w_1\\\overline{v_2}w_2\end{pmatrix}\\
&\quad + \frac{1}{2}a\widehat{\otimes}\overline{\begin{pmatrix}0\\v_2\end{pmatrix}}\widehat{\otimes}\mathcal{D}_\ell\xi\widehat{\otimes}\begin{pmatrix}-iw_2\\iw_1\end{pmatrix} + \frac{1}{2}\widehat{\otimes}\overline{\begin{pmatrix}1\\0\end{pmatrix}}\widehat{\otimes}\mathcal{D}_\ell a\xi\widehat{\otimes}\begin{pmatrix}-\overline{v_2}w_1\\\overline{v_2}w_2\end{pmatrix}\\
&\quad + \frac{1}{2}a\widehat{\otimes}\overline{\begin{pmatrix}0\\v_2\end{pmatrix}}\widehat{\otimes}\mathcal{D}_\ell\xi\widehat{\otimes}\begin{pmatrix}-iw_2\\iw_1\end{pmatrix}\\
&= 1\widehat{\otimes}\overline{\begin{pmatrix}1\\0\end{pmatrix}}\widehat{\otimes}\mathcal{D}_\ell a\frac{1+i\eta(c(dt))}{2}\xi\widehat{\otimes}\begin{pmatrix}0\\i\overline{v_1}w_1+\overline{v_2}w_2\end{pmatrix}\\
&\quad + 1\widehat{\otimes}\overline{\begin{pmatrix}1\\0\end{pmatrix}}\widehat{\otimes}\mathcal{D}_\ell a\frac{1-i\eta(c(dt))}{2}\xi\widehat{\otimes}\begin{pmatrix}-i\overline{v_1}w_2-\overline{v_2}w_1\\0\end{pmatrix},
\end{aligned}$$

after making some cancellations. □

A.2 The unitary equivalence of spectral triples

Recall that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is an odd $\mathbb{T}$-equivariant spectral triple for a unital, trivially $\mathbb{Z}_2$-graded C^*-algebra A, and $\ell \in \mathbb{Z}$ and $\eta : \Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))^{\mathbb{T}} \to B(\mathcal{H})$ are as in Definition 3.9.

Lemma A.3. *The map*

$$\Phi : (A^{\mathbb{T}} \widehat{\otimes} (\mathbb{C}^2)^*) \widehat{\otimes}_{A^{\mathbb{T}} \widehat{\otimes} \mathbb{C}\mathrm{l}_1 \widehat{\otimes} \Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))^{\mathbb{T}}} (\mathcal{H}_\ell \widehat{\otimes} \mathbb{C}^2) \to \mathcal{H}_\ell$$

given by (where $P^{\pm} = \frac{1 \pm i\eta(c(dt))}{2}$*)*

$$\Phi\left(a \widehat{\otimes} \overline{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}} \widehat{\otimes} \xi \widehat{\otimes} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) = (\overline{v_1} w_1 - i\overline{v_2} w_2) a P^+ \xi + (-i\overline{v_1} w_2 - \overline{v_2} w_1) a P^- \xi$$

is a $\mathbb{Z}_2$*-graded, equivariant,* $A^{\mathbb{T}}$*-linear unitary isomorphism, where* $\mathcal{H}_\ell$ *is* $\mathbb{Z}_2$*-graded by* $i\eta(c(dt))$*. Its inverse is*

$$\Phi^{-1}(\xi) = 1 \widehat{\otimes} \overline{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \widehat{\otimes} P^+ \xi \widehat{\otimes} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \widehat{\otimes} \overline{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \widehat{\otimes} P^- \xi \widehat{\otimes} \begin{pmatrix} 0 \\ i \end{pmatrix}.$$

Proof. That Φ is $\mathbb{Z}_2$-graded, equivariant and $A^{\mathbb{T}}$-linear is clear. It is easy to check that $\Phi \circ \Phi^{-1} = 1$, so Φ is surjective. Showing that Φ is unitary is a simple calculation, using the fact that P^+ and P^- are mutually orthogonal projections. □

Proposition A.4. *The isomorphism* Φ *is a unitary equivalence between the triple* $(\mathcal{A}^{\mathbb{T}}, (A^{\mathbb{T}} \widehat{\otimes} (\mathbb{C}^2)^*) \widehat{\otimes}_{A^{\mathbb{T}} \widehat{\otimes} \mathbb{C}\mathrm{l}_1 \widehat{\otimes} \Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))^{\mathbb{T}}} (\mathcal{H}_\ell \widehat{\otimes} \mathbb{C}^2), 1 \widehat{\otimes}_\nabla (\mathcal{D}_\ell \widehat{\otimes} \omega))$ *and the even spectral triple* $(\mathcal{A}^{\mathbb{T}}, \mathcal{H}_\ell, P^+ \mathcal{D}_\ell P^- + P^- \mathcal{D}_\ell P^+)$*, where* $\mathcal{H}_\ell$ *is* $\mathbb{Z}_2$*-graded by* $i\eta(c(dt))$*, and* $P^{\pm} = \frac{1 \pm i\eta(c(dt))}{2}$*. In particular,* $(\mathcal{A}^{\mathbb{T}}, (A^{\mathbb{T}} \widehat{\otimes} (\mathbb{C}^2)^*) \widehat{\otimes}_{A^{\mathbb{T}} \widehat{\otimes} \mathbb{C}\mathrm{l}_1 \widehat{\otimes} \Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))^{\mathbb{T}}} (\mathcal{H}_\ell \widehat{\otimes} \mathbb{C}^2), 1 \widehat{\otimes}_\nabla (\mathcal{D}_\ell \widehat{\otimes} \omega))$ *is an even spectral triple.*

Proof. Using Lemma A.2, we compute

$$\begin{aligned}
&\Phi \circ 1 \widehat{\otimes}_\nabla (\mathcal{D}_\ell \widehat{\otimes} \omega) \circ \Phi^{-1}(\xi) \\
&= \Phi\left(1 \widehat{\otimes} \overline{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \widehat{\otimes} \mathcal{D}_\ell P^+ \xi \widehat{\otimes} \begin{pmatrix} 0 \\ i \end{pmatrix} + 1 \widehat{\otimes} \overline{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \widehat{\otimes} \mathcal{D}_\ell P^- \xi \widehat{\otimes} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \\
&= P^- \mathcal{D}_\ell P^+ \xi + P^+ \mathcal{D}_\ell P^- \xi,
\end{aligned}$$

which shows that the triples are unitarily equivalent. Since

$$P^+ \mathcal{D}_\ell P^- + P^- D_\ell P^+ = \mathcal{D}_\ell + \frac{1}{2} \eta(c(dt)) \{\mathcal{D}_\ell, \eta(c(dt))\}$$

is a perturbation of $\mathcal{D}_\ell$ by a bounded self-adjoint operator, it follows that the triple $(\mathcal{A}^{\mathbb{T}}, \mathcal{H}_\ell, P^+\mathcal{D}_\ell P^- + P^-\mathcal{D}_\ell P^+)$ is an even equivariant spectral triple. The unitary equivalence via Φ proves that $(\mathcal{A}^{\mathbb{T}}, (A^{\mathbb{T}}\widehat{\otimes}(\mathbb{C}^2)^*)\widehat{\otimes}_{A^{\mathbb{T}}\widehat{\otimes}\mathbb{C}\mathrm{l}_1\widehat{\otimes}\Gamma(\mathbb{C}\mathrm{l}(\mathbb{T}))^{\mathbb{T}}}(\mathcal{H}_\ell\widehat{\otimes}\mathbb{C}^2), 1\widehat{\otimes}_\nabla(\mathcal{D}_\ell\widehat{\otimes}\omega))$ is also an even equivariant spectral triple. $\square$

Appendix B

A (counter)example: the unit disc

In this chapter we develop the example of the spin Dirac operator on the unit disc. This provides a counterexample to claims appearing in [5, pp. 164–165] and [49] that weaker definitions of spectral triple yield K-homology classes. It also gives us a concrete example of a relative spectral triple, and helps to determine the limits of the definition of a relative spectral triple. The work in this chapter is joint work with Bram Mesland and Adam Rennie, appearing in [18].

B.1 The Dirac operator on the unit disc

Let $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ be the closed unit disc in $\mathbb{R}^2$, and let $\mathring{\mathbb{D}}$ be its interior. Since $\mathring{\mathbb{D}}$ is flat and trivial, its spin Dirac operator is easy to describe. The spinor bundle $\mathbf{\$}$ is a trivial Hermitian vector bundle of rank 2, so we can take $\mathbf{\$} = \mathring{\mathbb{D}} \times \mathbb{C}^2$. Choosing Pauli matrices $c(dx) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $c(dy) = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$, the Dirac operator is

$$\mathcal{D} = c(dx)\frac{\partial}{\partial x} + c(dy)\frac{\partial}{\partial y} = \begin{pmatrix} 0 & -\partial_x + i\partial_y \\ \partial_x + i\partial_y & 0 \end{pmatrix},$$

defined on $\mathrm{dom}(\mathcal{D}) = C_c^\infty(\mathring{\mathbb{D}}) \otimes \mathbb{C}^2 \subset L^2(\mathbb{D}) \otimes \mathbb{C}^2$. This is a densely-defined symmetric operator on the Hilbert space $L^2(\mathbb{D}) \otimes \mathbb{C}^2$. The spinor bundle is $\mathbb{Z}_2$-graded by the grading operator $\gamma = ic(dx)c(dy) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$; observe that $\mathcal{D}$ is odd with respect to this grading.

On the disc it is more convenient to work with the polar coordinates $(r, \theta) \in (0, 1) \times (0, 2\pi)$ defined by $x = r\cos\theta$, $y = r\sin\theta$. In these coordinates, the Dirac operator is

$$\mathcal{D} = \begin{pmatrix} 0 & e^{-i\theta}(-\partial_r + ir^{-1}\partial_\theta) \\ e^{i\theta}(\partial_r + ir^{-1}\partial_\theta) & 0 \end{pmatrix}. \tag{B.1}$$

Let $\mathcal{D}_{\min}$ be the closure of $\mathcal{D}$, which is called the **minimal extension** of $\mathcal{D}$ (i.e. the smallest closed extension). The **maximal extension** of $\mathcal{D}$ is $\mathcal{D}_{\max} = \mathcal{D}_{\min}^* = \mathcal{D}^*$. This

extension can be described using distributions. The symmetric operator $\mathcal{D}$ induces a dual operator

$$\mathcal{D}^\dagger : (C_c^\infty(\mathring{\mathbb{D}}) \otimes \mathbb{C}^2)^\dagger \to (C_c^\infty(\mathring{\mathbb{D}}) \otimes \mathbb{C}^2)^\dagger,$$

on the space of distributions $(C_c^\infty(\mathring{\mathbb{D}}) \otimes \mathbb{C}^2)^\dagger$, uniquely determined by the formula

$$\langle \mathcal{D}^\dagger \phi, f\rangle := \langle \phi, \mathcal{D} f\rangle, \qquad \phi \in (C_c^\infty(\mathring{\mathbb{D}}) \otimes \mathbb{C}^2)^\dagger,\ f \in C_c^\infty(\mathring{\mathbb{D}}) \otimes \mathbb{C}^2.$$

A similar formula embeds $L^2(\mathbb{D}) \otimes \mathbb{C}^2$ into the space of distributions. Using these identifications, the domain of $\mathcal{D}_{\max}$ is given by

$$\mathrm{dom}(\mathcal{D}_{\max}) = \{f \in L^2(\mathbb{D}) \otimes \mathbb{C}^2 : \mathcal{D}^\dagger f \in L^2(\mathbb{D}) \otimes \mathbb{C}^2\}.$$

The domain of $\mathcal{D}_{\min}$ is

$$\mathrm{dom}(\mathcal{D}_{\min}) = \{f \in H^1(\mathbb{D}) \otimes \mathbb{C}^2 : f|_{\partial\mathbb{D}} = 0\},$$

where $H^1(\mathbb{D})$ is the first Sobolev space.

Remark. It is not immediately obvious that $f|_{\partial\mathbb{D}}$ is well-defined for $f \in H^1(\mathbb{D})$, and indeed $f|_{\partial\mathbb{D}}$ makes no sense for $f \in L^2(\mathbb{D})$ since the boundary $\partial\mathbb{D}$ is a set of measure zero. The trace theorem however states that if $\overline{M}$ is a Riemannian manifold with boundary and $s > 1/2$, then $f|_{\partial M}$ is a well-defined element of $H^{s-1/2}(\partial M)$ for all $f \in H^s(\overline{M})$, [7, Cor. 11.6].

We claim that $(C^\infty(\mathbb{D}), L^2(\mathbb{D}) \otimes \mathbb{C}^2, \mathcal{D}_{\min})$ is an even relative spectral triple for $C_0(\mathring{\mathbb{D}}) \lhd C(\mathbb{D})$. Using the above characterisations of the domains of $\mathcal{D}_{\min}$ and $\mathcal{D}_{\max}$, it is straightforward to check that Conditions 1) and 2) of Definition 8.1 are satisfied. Condition 4) is satisfied since $C(\mathbb{D})$ is unital and represented non-degenerately on $L^2(\mathbb{D}) \otimes \mathbb{C}^2$. It remains to check Condition 3), the compact resolvent condition.

Lemma B.1. *The operator* $(1 + \mathcal{D}_{min}^* \mathcal{D}_{min})^{-1/2}$ *is compact.*

Proof. The eigenvectors of $\mathcal{D}_{\min}^* \mathcal{D}_{\min}$ are

$$\left\{ \begin{pmatrix} J_n(r\alpha_{n,k})e^{in\theta} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ J_n(r\alpha_{n,k})e^{in\theta} \end{pmatrix} : n \in \mathbb{Z}, k = 1, 2, \dots \right\},$$

where $\alpha_{n,k}$ denotes the kth positive root of the Bessel function J_n. We claim that these eigenvectors are complete for $L^2(\mathbb{D}) \otimes \mathbb{C}^2$. With the measure $r dr d\theta$, we can take $\mathbb{D} = [0,1] \times S^1/\sim$, where $\sim$ is the identification $(0,z) \sim (0,1)$ for $z \in S^1$. It is well known that $\{e^{in\theta}\}_{n=-\infty}^{\infty}$ is complete for $L^2(S^1)$ and $\{J_n(r\alpha_{n,k}) : k \geq 1\}$ is complete for $L^2([0,1], r\,dr)$ for all $n \in \mathbb{Z}$, [6], proving the claim.

The eigenvalues corresponding to these eigenvectors are

$$\mathcal{D}_{\min}^*\mathcal{D}_{\min}\begin{pmatrix} J_n(r\alpha_{n,k})e^{in\theta} \\ 0 \end{pmatrix} = \alpha_{n,k}^2\begin{pmatrix} J_n(r\alpha_{n,k})e^{in\theta} \\ 0 \end{pmatrix},$$
$$\mathcal{D}_{\min}^*\mathcal{D}_{\min}\begin{pmatrix} 0 \\ J_n(r\alpha_{n,k})e^{in\theta} \end{pmatrix} = \alpha_{n,k}^2\begin{pmatrix} 0 \\ J_n(r\alpha_{n,k})e^{in\theta} \end{pmatrix},$$

so the eigenvalues of $\mathcal{D}_{\min}^*\mathcal{D}_{\min}$ are $\{\alpha_{n,k}^2\}_{n=0,k=1}^{\infty}$. Each of these eigenvalues has multiplicity 4. Since $\alpha_{n,k} \to \infty$ as $n, k \to \infty$, it follows that $(1 + \mathcal{D}_{\min}^*\mathcal{D}_{\min})^{-1/2}$ is compact. □

We have proved the claim that $(C^\infty(\mathbb{D}), L^2(\mathbb{D})\otimes\mathbb{C}^2, \mathcal{D}_{\min})$ is an even relative spectral triple for $C_0(\mathring{\mathbb{D}}) \lhd C(\mathbb{D})$.

B.2 A counterexample

It is asserted in [5, pp. 164–165] that the definition of an unbounded Kasparov module $(\mathcal{A}, E_B, \mathcal{D})$ (Definition 1.18) may be weakened, by replacing Condition 2) by

2') for all $a \in \mathcal{A}$, there is a subspace $X \subset \mathrm{dom}(\mathcal{D})$ which is dense in E such that $a \cdot X \subset \mathrm{dom}(\mathcal{D})$ and $[\mathcal{D}, \rho(a)]_\pm$ is bounded on X,

and that this weaker definition yields a KK-class since $(\rho, E_B, \mathcal{D}(1 + \mathcal{D}^2)^{-1/2})$ is a bounded Kasparov module. This assertion is false, as has been noted in [23, Sect. 4], and a self-adjoint extension of the Dirac operator on the disc furnishes us with a concrete counterexample. The Dirac operator on the disc also provides a counterexample to the claim [49, Thms. 1.2, 1.3, 6.2] that a Fredholm module can be obtained from any self-adjoint extension of a symmetric operator $\mathcal{D}$ satisfying certain spectral triple-like conditions, [49, Defns. 1.1 and 6.3].

Before proceeding to the counterexample, we examine Condition 2) of Definition 1.18 more closely. It is asserted in [9, p. 686] that Condition 2) may be weakened by allowing a to map a core for $\mathcal{D}$ into the domain of $\mathcal{D}$ for all $a \in \mathcal{A}$. (A core is a graph norm dense subspace of $\mathrm{dom}(\mathcal{D})$.) In fact this condition turns out to be equivalent to Condition 2). We show this equivalence using the following proposition, which generalises [23, Lemma 2.1].

Proposition B.2. *Let $\mathcal{D} : \mathrm{dom}(\mathcal{D}) \subset E \to E$ be a closed operator on a Hilbert module E, let $X \subset \mathrm{dom}(\mathcal{D})$ be a core for $\mathcal{D}$, and let $a \in \mathrm{End}(E)$ satisfy*

(1) $a \cdot X \subset \mathrm{dom}(\mathcal{D})$, and

(2) $[\mathcal{D}, a] : X \to E$ is bounded on X.

Then $a \cdot \mathrm{dom}(\mathcal{D}) \subset \mathrm{dom}(\mathcal{D})$ *so that* $[\mathcal{D}, a] : \mathrm{dom}(\mathcal{D}) \to E$ *is well-defined. If moreover there is a subspace* $Y \subset \mathrm{dom}(\mathcal{D}^*)$ *such that* Y *is dense in* E *and* $a^* \cdot Y \subset \mathrm{dom}(\mathcal{D}^*)$*, then* $[\mathcal{D}, a] : \mathrm{dom}(\mathcal{D}) \to E$ *is bounded.*

Proof. Since X is a core for $\mathcal{D}$, it is dense in $\mathrm{dom}(\mathcal{D})$ in the graph norm. Let $x \in \mathrm{dom}(\mathcal{D})$, and choose a sequence $(x_n)_{n=1}^{\infty} \subset X$ such that $x_n \to x$ in the graph norm, which is equivalent to $x_n \to x$ and $\mathcal{D}x_n \to \mathcal{D}x$ in the usual norm. Since a is continuous, $ax_n \to ax$, and $(\mathcal{D}ax_n)_{n=1}^{\infty}$ is Cauchy in the usual norm since

$$\begin{aligned}\|\mathcal{D}ax_n - \mathcal{D}ax_m\| &= \|a\mathcal{D}x_n - a\mathcal{D}x_m + [\mathcal{D}, a]x_n - [\mathcal{D}, a]x_m\| \\ &\leq \|a\|\|\mathcal{D}x_n - \mathcal{D}x_m\| + \|[\mathcal{D}, a]\|\|x_n - x_m\| \to 0.\end{aligned}$$

Hence $(ax_n)_{n=1}^{\infty}$ is Cauchy in the graph norm, and since $\mathcal{D}$ is closed, there is some $y \in \mathrm{dom}(\mathcal{D})$ such that $ax_n \to y$ in the graph norm. This graph convergence implies that $ax_n \to y$ in the usual norm, and since $ax_n \to ax$ in the usual norm we see that $y = ax$. Hence $ax \in \mathrm{dom}(\mathcal{D})$.

Now suppose that $Y \subset \mathrm{dom}(\mathcal{D}^*)$, $a^* \cdot Y \subset \mathrm{dom}(\mathcal{D}^*)$. To show that the commutator $[\mathcal{D}, a] : \mathrm{dom}(\mathcal{D}) \to E$ is bounded, it is enough to show that $[\mathcal{D}, a]$ is closeable, since then $\overline{[\mathcal{D}, a]} \supset \overline{[\mathcal{D}, a]|_X}$ which is everywhere defined and bounded. Let $\xi \in \mathrm{dom}(\mathcal{D})$ and $\eta \in Y$. Then

$$\begin{aligned}\langle [\mathcal{D}, a]\xi, \eta\rangle &= \langle a\xi, \mathcal{D}^*\eta\rangle - \langle \mathcal{D}\xi, a^*\eta\rangle = \langle \xi, a^*\mathcal{D}^*\eta\rangle - \langle \xi, \mathcal{D}^*a^*\eta\rangle \\ &= \langle \xi, -[\mathcal{D}^*, a^*]\eta\rangle .\end{aligned}$$

Hence $\mathrm{dom}([\mathcal{D}, a])^* \supset Y$. Since $[\mathcal{D}, a]$ is closeable if and only if $([\mathcal{D}, a])^*$ is densely defined, if Y is dense in E then $[\mathcal{D}, a]$ is closeable and thus bounded. □

Corollary B.3. *Condition 2) of Definition 1.18 is equivalent to*

ii) for all $a \in \mathcal{A}$ *there exists a core* X *for* $\mathcal{D}$ *such that* $a \cdot X \subset \mathrm{dom}(\mathcal{D})$*, and such that* $[\mathcal{D}, a] : X \to E$ *is bounded on* X*.*

To construct the counterexample, we consider APS-type extensions of the Dirac operator $\mathcal{D}$ on $\mathring{\mathbb{D}}$ arising from the projections $P_N : L^2(S^1) \to L^2(S^1)$, $N \in \mathbb{Z}$, defined by

$$P_N\left(\sum_{k\in\mathbb{Z}} c_k e^{ik\theta}\right) = \sum_{k\geq N} c_k e^{ik\theta}, \qquad \sum_{k\in\mathbb{Z}} c_k e^{ik\theta} \in L^2(S^1).$$

We use P_N to define self-adjoint extensions by setting

$$\begin{aligned}\operatorname{dom}(\mathcal{D}_{P_N}) &:= \left\{ \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \operatorname{dom}(\mathcal{D}_{\max}) : \ P_N(\xi_1|_{\partial\mathbb{D}}) = 0, \ (1 - P_{N+1})(\xi_2|_{\partial\mathbb{D}}) = 0 \right\} \\ \mathcal{D}_{P_N}\xi &:= \mathcal{D}_{\max}\xi, \qquad \text{for } \xi \in \operatorname{dom}(\mathcal{D}_{P_N}).\end{aligned}$$

(One can check directly using integration by parts that these are in fact self-adjoint.)

Remark. The self-adjoint extensions $\mathcal{D}_{P_N}$ *do* define even spectral triples for the C^*-algebra of continuous functions constant on the boundary $\partial\mathbb{D}$, since smooth functions which are constant on the boundary preserve the domain, and $(1+\mathcal{D}_{P_N}^2)^{-1/2}$ is compact. Each extension $\mathcal{D}_{P_N}$ defines a different K-homology class. This is easy, and not new: see [3, Appendix A]. We can determine that the classes are distinct using the **index map**, which for any unital separable C^*-algebra is the pullback $\iota^* : K^0(A) \to K^0(\mathbb{C}) \cong \mathbb{Z}$ of the unital inclusion $\iota : \mathbb{C} \hookrightarrow A$. The index of the class of a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is $\dim\ker(\mathcal{D}^0) - \dim\ker(\mathcal{D}^1) \in \mathbb{Z}$ provided A is represented non-degenerately on $\mathcal{H}$, where $\mathcal{D} = \left(\begin{smallmatrix} 0 & \mathcal{D}^1 \\ \mathcal{D}^0 & 0 \end{smallmatrix}\right)$ with respect to the $\mathbb{Z}_2$-grading $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$. In this case, we compute

$$\dim\ker(\mathcal{D}^0_{P_N}) - \dim(\ker \mathcal{D}^1_{P_N}) = N.$$

The reason is that

$$\ker(\mathcal{D}_{\max}) = \overline{\operatorname{span}} \left\{ \begin{pmatrix} r^n e^{in\theta} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ r^n e^{-in\theta} \end{pmatrix} : n = 0, 1, 2, \dots \right\},$$

and so

$$\ker((\mathcal{D}_{P_N})^0) = \begin{cases} \{0\} & N \le 0 \\ \overline{\operatorname{span}}\{r^n e^{in\theta} : 0 \le n < N\} & N > 0, \end{cases}$$

while

$$\ker((\mathcal{D}^0_{P_N}) = \begin{cases} \{0\} & N > -1 \\ \overline{\operatorname{span}}\{r^n e^{-in\theta} : 0 \le n \le -N-1\} & N \le -1. \end{cases}$$

We focus on the extension $\mathcal{D}_{P_0}$, which we abbreviate as $\mathcal{D}_P$. For $k \ge 1$, let $\alpha_{n,k}$ denote the kth positive zero of the Bessel function J_n. Then the eigenvectors of $\mathcal{D}_P^2$ are

$$\begin{aligned}&\left\{ \begin{pmatrix} J_n(r\alpha_{n-1,k})e^{-in\theta} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ J_n(r\alpha_{n-1,k})e^{in\theta} \end{pmatrix} \right\}_{n,k=1}^{\infty}, \\ &\left\{ \begin{pmatrix} J_n(r\alpha_{n,k})e^{in\theta} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ J_n(r\alpha_{n,k})e^{-in\theta} \end{pmatrix} \right\}_{n=0,k=1}^{\infty}.\end{aligned} \tag{B.2}$$

Lemma B.4. *The eigenvectors* (B.2) *of* $\mathcal{D}_P^2$ *span* $L^2(\mathbb{D}, \mathbb{C}^2)$*. The corresponding set of eigenvalues is* $\{\alpha_{n,k}^2\}_{n=0,k=1}^{\infty}$ *each of multiplicity 4, and hence* $(1+\mathcal{D}_P^2)^{-1/2}$ *is compact.*

Proof. Proceeding as in the proof of Lemma B.1, we take $\mathbb{D} = [0,1] \times S^1/\sim$, where $\sim$ is the identification $(0,z) \sim (0,1)$ for $z \in S^1$. Since $\{e^{in\theta}\}_{n=-\infty}^{\infty}$ is complete for $L^2(S^1)$, it is enough to show that

(a) $\{r \mapsto J_n(r\alpha_{n-1,k})\}_{k=1}^{\infty}$ spans $L^2([0,1], r\,dr)$ for all $n = 1, 2, \ldots$, and

(b) $\{r \mapsto J_n(r\alpha_{n,k})\}_{k=1}^{\infty}$ spans $L^2([0,1], r\,dr)$ for all $n = 0, 1, 2, \ldots$.

Statement (a) is true by [6, Thm. 6], and (b) is true by [6, Thm. 2]. Hence the eigenfunctions above are the entire set of eigenfunctions, and the set of eigenvalues is $\{\alpha_{n,k}^2\}_{n=0,k=1}^{\infty}$. Each of these eigenvalues has multiplicity 4. Since $\alpha_{n,k} \to \infty$ as $n, k \to \infty$, $(1+\mathcal{D}_P^2)^{-1/2}$ is compact. □

Since $[\mathcal{D}_P, a]$ is defined and bounded on the dense subspace $\mathrm{dom}(\mathcal{D}_{\min}) \subset L^2(\mathbb{D}) \otimes \mathbb{C}^2$ for all $a \in C^\infty(\mathbb{D})$, $(C^\infty(\mathbb{D}), L^2(\mathbb{D}) \otimes \mathbb{C}^2, \mathcal{D}_P)$ satisfies the weaker definition of spectral triple of [5, Defn. 17.11.1], which replaces Condition 2) of Definition 1.18 by Condition 2'). We show however that the bounded transform $F := \mathcal{D}_P(1+\mathcal{D}_P^2)^{-1/2}$ does not define a Fredholm module for $C(\mathbb{D})$, since $[F, re^{-i\theta}]$ is not compact. (Note that $re^{-i\theta}$ does not preserve the domain of $\mathcal{D}_P$.)

To simplify calculations involving the commutator $[F, re^{-i\theta}]$, we include the following elementary Lemma.

Lemma B.5. *Let* $\mathcal{D}$ *be a self-adjoint operator on the Hilbert space* $\mathcal{H}$*, and suppose that* $(1+\mathcal{D}^2)^{-1/2}$ *is compact. Then with* $F = \mathcal{D}(1+\mathcal{D}^2)^{-1/2}$*,* $P_+ = \chi_{[0,\infty)}(\mathcal{D})$*,* $P_- = 1 - P_+$*, and* $A \subset B(\mathcal{H})$ *a* $*$*-algebra, the operator* $[F, a]$ *is compact for all* $a \in A$ *if and only if* $P_+ a P_-$ *is compact for all* $a \in A$*.*

Proof. Let

$$\mathrm{Ph}(\mathcal{D}) = P_+ - P_-,$$

which is a compact perturbation of $F = \mathcal{D}(1+\mathcal{D}^2)^{-1/2}$, so for $a \in A$, the commutator $[F, a]$ is compact if and only if $[\mathrm{Ph}(\mathcal{D}), a]$ is compact. Since $P_+ + P_- = 1$, we see that

$$[\mathrm{Ph}(\mathcal{D}), a] = (P_+ + P_-)[\mathrm{Ph}(\mathcal{D}), a](P_+ + P_-) = 2P_+aP_- - 2P_-aP_+,$$

so that $[\mathrm{Ph}(\mathcal{D}), a]$ is compact if and only if $P_+aP_- - P_-aP_+$ is compact. If $P_+aP_- - P_-aP_+$ is compact, then so are

$$P_+(P_+aP_- - P_-aP_+) = P_+aP_- \quad \text{and} \quad -P_-(P_+aP_- - P_-aP_+) = P_-aP_+,$$

so $[F, a]$ is compact if and only if P_+aP_- and P_-aP_+ are compact. Since $(P_+aP_-)^* = P_-a^*P_+$, we have $[F, a]$ is compact for all $a \in A$ if and only if P_+aP_- is compact for all $a \in A$. □

To facilitate our computations we now describe an orthonormal eigenbasis for $\mathcal{D}_P$.

Proposition B.6. *The vectors*

$$|1,n,k,\pm\rangle = \frac{1}{\sqrt{2\pi}J_n(\alpha_{n-1,k})}\begin{pmatrix} J_n(r\alpha_{n-1,k})e^{-in\theta} \\ \pm J_{n-1}(r\alpha_{n-1,k})e^{-i(n-1)\theta}\end{pmatrix},$$
$$|2,n,k,\pm\rangle = \frac{1}{\sqrt{2\pi}J_n(\alpha_{n-1,k})}\begin{pmatrix} J_{n-1}(r\alpha_{n-1,k})e^{i(n-1)\theta} \\ \mp J_n(r\alpha_{n-1,k})e^{in\theta}\end{pmatrix},$$

$n,k=1,2,\ldots$ form a normalised complete set of eigenvectors for $\mathcal{D}_P$. The corresponding set of eigenvalues is given by $\mathcal{D}_P\,|j,n,k,\pm\rangle = \pm\alpha_{n-1,k}\,|j,n,k,\pm\rangle$.

Proof. From Lemma B.4 it is straightforward to show that the eigenvectors and eigenvalues of $\mathcal{D}_P$ are

$$\mathcal{D}_P\begin{pmatrix} J_n(r\alpha_{n-1,k})e^{-in\theta} \\ \pm J_{n-1}(r\alpha_{n-1,k})e^{-i(n-1)\theta}\end{pmatrix} = \pm\alpha_{n-1,k}\begin{pmatrix} J_n(r\alpha_{n-1,k})e^{-in\theta} \\ \pm J_{n-1}(r\alpha_{n-1,k})e^{-i(n-1)\theta}\end{pmatrix},$$
$$\mathcal{D}_P\begin{pmatrix} J_{n-1}(r\alpha_{n-1,k})e^{i(n-1)\theta} \\ \mp J_n(r\alpha_{n-1,k})e^{in\theta}\end{pmatrix} = \pm\alpha_{n-1,k}\begin{pmatrix} J_{n-1}(r\alpha_{n-1,k})e^{i(n-1)\theta} \\ \mp J_n(r\alpha_{n-1,k})e^{in\theta}\end{pmatrix},$$

for $n,k=1,2,\ldots$. Note that these eigenvectors are complete for $L^2(\mathbb{D},\mathbb{C}^2)$ since we can recover our spanning set (B.2) from linear combinations of these.

To normalise these eigenvectors, we use the following standard integrals, [51]:

$$\begin{aligned}
&\left\langle \begin{pmatrix} J_n(r\alpha_{n-1,k})e^{-in\theta} \\ \pm J_{n-1}(r\alpha_{n-1,k})e^{-i(n-1)\theta}\end{pmatrix}, \begin{pmatrix} J_n(r\alpha_{n-1,k})e^{-in\theta} \\ \pm J_{n-1}(r\alpha_{n-1,k})e^{-i(n-1)\theta}\end{pmatrix}\right\rangle \\
&= \int_0^{2\pi}\int_0^1 (J_n^2(r\alpha_{n-1,k}) + J_{n-1}^2(r\alpha_{n-1,k}))r\,dr\,d\theta \\
&= \pi\left(J_n^2(\alpha_{n-1,k}) + J_n^2(\alpha_{n-1,k})\right) = 2\pi J_n^2(\alpha_{n-1,k}), \quad \text{and similarly} \\
&\left\langle \begin{pmatrix} J_{n-1}(r\alpha_{n-1,k})e^{i(n-1)\theta} \\ \pm J_n(r\alpha_{n-1,k})e^{in\theta}\end{pmatrix}, \begin{pmatrix} J_{n-1}(r\alpha_{n-1,k})e^{i(n-1)\theta} \\ \pm J_n(r\alpha_{n-1,k})e^{in\theta}\end{pmatrix}\right\rangle = 2\pi J_n^2(\alpha_{n-1,k}). \qquad \square
\end{aligned}$$

Our aim is to show that $[F, re^{-i\theta}]$ is not compact. Let P_+ be the non-negative spectral projection associated to $\mathcal{D}_P$, and let $P_- = 1 - P_+$. By Lemma B.5, we need only show that $P_+re^{-i\theta}P_-$ is not compact. To do this, we will construct a bounded sequence of vectors $(\xi_n)_{n=1}^\infty$, with the property that $P_+re^{-i\theta}P_-$ maps $(\xi_n)_{n=1}^\infty$ to a sequence with no convergent subsequences. In order to find the sequence $(\xi_n)_{n=1}^\infty$, we first derive an explicit formula for $P_+re^{-i\theta}P_-$, using the eigenbasis expression

$$P_+re^{-i\theta}P_- = \sum_{i,j=1}^{2}\sum_{n,m,k,\ell=1}^{\infty} |i,n,k,+\rangle\,\langle i,n,k,+|\,re^{-i\theta}\,|j,m,\ell,-\rangle\,\langle j,m,\ell,-|\,. \tag{B.3}$$

Lemma B.7. *Let $P_+ = \chi_{[0,\infty)}(\mathcal{D}_P)$ and $P_- = 1 - P_+$ be the spectral projections of $\mathcal{D}_P$. The operator $P_+ r e^{-i\theta} P_-$ can be expressed as*

$$\begin{aligned} P_+ r e^{-i\theta} P_- = &\sum_{m,k,\ell=1}^{\infty} \frac{2\alpha_{m,k}}{(\alpha_{m,k} - \alpha_{m-1,\ell})(\alpha_{m,k} + \alpha_{m-1,\ell})^2} \left|1, m+1, k, +\right\rangle \left\langle 1, m, \ell, -\right| \\ &+ \sum_{n,k,\ell=1}^{\infty} \frac{2\alpha_{n,\ell}}{(\alpha_{n-1,k} - \alpha_{n,\ell})(\alpha_{n,\ell} + \alpha_{n-1,k})^2} \left|2, n, k, +\right\rangle \left\langle 2, n+1, \ell, -\right| \\ &+ \sum_{k \neq \ell} \frac{1}{\alpha_{0,k} + \alpha_{0,\ell}} \left|1, 1, k, +\right\rangle \left\langle 2, 1, \ell, -\right| + \sum_{k=1}^{\infty} \frac{1}{\alpha_{0,k}} \left|1, 1, k, +\right\rangle \left\langle 2, 1, k, -\right|. \end{aligned}$$

Proof. We first compute the operators $\left\langle i, n, k, +\right| r e^{-i\theta} \left|j, m, \ell, -\right\rangle$ for $i, j = 1, 2$. Using integration by parts and standard recursion relations and identities for the Bessel functions and their derivatives, [51], we find:

1. Case $i = j = 1$:

$$\begin{aligned} &\left\langle 1, n, k, +\right| r e^{-i\theta} \left|1, m, \ell, -\right\rangle \\ &= \frac{1}{2\pi J_n(\alpha_{n-1,k}) J_m(\alpha_{m-1,\ell})} \int_0^{2\pi} \int_0^1 r^2 e^{i(n-m-1)\theta} \big(J_n(r\alpha_{n-1,k}) J_m(r\alpha_{m-1,\ell}) \\ &\qquad - J_{n-1}(r\alpha_{n-1,k}) J_{m-1}(r\alpha_{m-1,\ell}) \big) \, dr \, d\theta \\ &= \frac{\delta_{n,m+1}}{J_{m+1}(\alpha_{m,k}) J_m(\alpha_{m-1,\ell})} \int_0^1 r^2 J_{m+1}(r\alpha_{m,k}) J_m(r\alpha_{m-1,\ell}) \\ &\qquad - r^2 J_m(r\alpha_{m,k}) J_{m-1}(r\alpha_{m-1,\ell}) \, dr \\ &= \frac{2\alpha_{m,k} \delta_{n,m+1}}{(\alpha_{m,k} - \alpha_{m-1,\ell})(\alpha_{m,k} + \alpha_{m-1,\ell})^2}; \end{aligned}$$

2. Case $i = 1, j = 2$:

$$\begin{aligned} &\left\langle 1, n, k, +\right| r e^{-i\theta} \left|2, m, \ell, -\right\rangle \\ &= \frac{1}{2\pi J_n(\alpha_{n-1,k}) J_m(\alpha_{m-1,\ell})} \int_0^{2\pi} \int_0^1 e^{i(m+n-2)\theta} \big(J_n(r\alpha_{n-1,k}) J_{m-1}(r\alpha_{m-1,\ell}) \\ &\qquad + J_{n-1}(r\alpha_{n-1,k}) J_m(r\alpha_{m-1,\ell}) \big) r^2 \, dr \, d\theta \\ &= \frac{\delta_{n,1} \delta_{m,1}}{J_1(\alpha_{0,k}) J_1(\alpha_{0,\ell})} \int_0^1 r^2 \big(J_1(r\alpha_{0,k}) J_0(r\alpha_{0,\ell}) + J_0(r\alpha_{0,k}) J_1(r\alpha_{0,\ell}) \big) \, dr \\ &= \begin{cases} \frac{1}{\alpha_{0,k} + \alpha_{0,\ell}} & n = m = 1 \text{ and } k \neq \ell \\ \frac{1}{\alpha_{0,k}} & n = m = 1 \text{ and } k = \ell \\ 0 & \text{otherwise;} \end{cases} \end{aligned}$$

3. Case $i = 2, j = 1$:

$$\begin{aligned}
&\langle 2, n, k, +| \, re^{-i\theta} \, |1, m, \ell, -\rangle \\
&= \frac{1}{2\pi J_n(\alpha_{n-1,k}) J_m(\alpha_{m-1,\ell})} \int_0^{2\pi}\int_0^1 e^{-i(n+m)\theta} \big(J_{n-1}(r\alpha_{n-1,k}) J_m(r\alpha_{m-1,\ell}) \\
&\qquad\qquad + J_n(r\alpha_{n-1,k}) J_{m-1}(r\alpha_{m-1,k\ell}) \big) r^2 \, dr \, d\theta \\
&= 0;
\end{aligned}$$

4. Case $i = j = 2$:

$$\begin{aligned}
&\langle 2, n, k, +| \, re^{-i\theta} \, |2, m, \ell, -\rangle \\
&= \frac{1}{2\pi J_n(\alpha_{n-1,k}) J_m(\alpha_{m-1,\ell})} \int_0^{2\pi}\int_0^1 \big(J_{n-1}(r\alpha_{n-1,k}) J_{m-1}(r\alpha_{m-1,\ell}) \\
&\qquad\qquad - J_n(r\alpha_{n-1,k}) J_m(r\alpha_{m-1,\ell}) \big) r^2 e^{i(m-n-1)\theta} \, dr \, d\theta \\
&= \frac{\delta_{m,n+1}}{J_n(\alpha_{n-1,k}) J_{n+1}(\alpha_{n,\ell})} \int_0^1 \big(J_{n-1}(r\alpha_{n-1,k}) J_n(r\alpha_{n,\ell}) \\
&\qquad\qquad - J_n(r\alpha_{n-1,k}) J_{n+1}(r\alpha_{n,\ell}) \big) r^2 \, dr \\
&= \frac{2\alpha_{n,\ell}\delta_{m,n+1}}{(\alpha_{n-1,k} - \alpha_{n,\ell})(\alpha_{n,\ell} + \alpha_{n-1,k})^2}.
\end{aligned}$$

The result is now obtained by using these cases in combination with (B.3). □

For convenience we write

$$|\ell, -\rangle := |2, 1, \ell, -\rangle \,, \qquad |k, +\rangle := |1, 1, k, +\rangle \,,$$

and define the sequence

$$\xi_n := \sum_{\ell=1}^{\infty} \frac{\sqrt{n}}{n+\ell} \, |\ell, -\rangle \,, \quad n = 1, 2, \ldots.$$

Lemma B.8. *The sequence* $(\xi_n)_{n=1}^{\infty}$ *is bounded.*

Proof. We have

$$\|\xi_n\|^2 = n \sum_{\ell=1}^{\infty} \frac{1}{(n+\ell)^2} = n\psi^{(1)}(n+1),$$

where $\psi^{(m)}(x) = (d^{m+1}/dx^{m+1})(\log(\Gamma))(x)$ is the polygamma function of order m. As $n \to \infty$, $(n+1)\psi^{(1)}(n+1) \to 1$, so $\|\xi_n\|^2 \to 1$. □

To simplify the computations, we subtract the operator

$$K := \sum_{k=1}^{\infty} \frac{1}{2\alpha_{0,k}} \left|1,1,k,+\right\rangle \left\langle 2,1,k,-\right|$$

from $P_+ re^{-i\theta} P_-$, since K is obviously compact, and define a sequence

$$\zeta_n := (P_+ re^{-i\theta} P_- - K)\xi_n.$$

Our purpose is to show that $(\zeta_n)_{n=1}^{\infty}$ has no convergent subsequence. To this end we investigate its limiting behaviour.

Lemma B.9.

$$\liminf_{n\to\infty} \|\zeta_n\| \geq \frac{1}{2\pi}.$$

Proof. We have

$$\zeta_n = \sum_{k,\ell=1}^{\infty} \frac{\sqrt{n}}{(n+\ell)(\alpha_{0,k}+\alpha_{0,\ell})} \left|k,+\right\rangle.$$

It is proved in [31, Lemma 1] that for all $\ell \geq 1$,

$$\pi(\ell - 1/4) < \alpha_{0,\ell} < \pi(\ell - 1/8), \tag{B.4}$$

yielding the inequality

$$\frac{\sqrt{n}}{(n+\ell)(\alpha_{0,k}+\alpha_{0,\ell})} > \frac{\sqrt{n}}{(n+\ell)(\alpha_{0,k}+\pi(\ell-1/8))}.$$

This inequality allows us to estimate the coefficients of ζ_n via

$$\begin{aligned}
&\sum_{\ell=1}^{\infty} \frac{\sqrt{n}}{(n+\ell)(\alpha_{0,k}+\alpha_{0,\ell})} \geq \sum_{\ell=1}^{\infty} \frac{\sqrt{n}}{(n+\ell)(\alpha_{0,k}+\pi(\ell-1/8))} \\
&= \frac{\sqrt{n}}{\pi(n-\alpha_{0,k}/\pi+1/8)} \sum_{\ell=1}^{\infty} \left(\frac{1}{\ell+\alpha_{0,k}/\pi-1/8} - \frac{1}{n+\ell} \right) \\
&= \frac{\sqrt{n}}{\pi(n-\alpha_{0,k}/\pi+1/8)} \sum_{\ell=1}^{\infty} \left(\frac{1}{\ell+\alpha_{0,k}/\pi-1/8} - \frac{1}{\ell} + \frac{1}{\ell} - \frac{1}{\ell+n} \right) \\
&= \frac{\sqrt{n}}{\pi(n-\alpha_{0,k}/\pi+1/8)} \left(-\psi^{(0)}(\alpha_{0,k}/\pi+7/8) + \psi^{(0)}(n+1) \right) \\
&= \frac{\sqrt{n}}{\pi} \frac{\psi^{(0)}(n+1) - \psi^{(0)}(\alpha_{0,k}/\pi+7/8)}{n-\alpha_{0,k}/\pi+1/8}
\end{aligned}$$

which allows us to bound $\|\zeta_n\|$ by

$$\begin{aligned}\|\zeta_n\|^2 &\geq \frac{n}{\pi^2}\sum_{k=1}^{\infty}\left(\frac{\psi^{(0)}(n+1)-\psi^{(0)}(\alpha_{0,k}/\pi+7/8)}{n-\alpha_{0,k}/\pi+1/8}\right)^2\\ &\geq \frac{n}{\pi^2}\sum_{k=1}^{n}\left(\frac{\psi^{(0)}(n+1)-\psi^{(0)}(\alpha_{0,k}/\pi+7/8)}{n-\alpha_{0,k}/\pi+1/8}\right)^2. \end{aligned} \tag{B.5}$$

Now, $\alpha_{0,k}/\pi \in (k-1/4, k-1/8)$ by Equation (B.4), and $\psi^{(0)}$ increases monotonically on $(0,\infty)$, so for $k \leq n$ we have

$$\begin{aligned}0 \leq \psi^{(0)}(n+1)-\psi^{(0)}(k+1) &< \psi^{(0)}(n+1)-\psi^{(0)}(k+3/4)\\ &< \psi^{(0)}(n+1)-\psi^{(0)}(\alpha_{0,k}/\pi+7/8).\end{aligned}$$

For $k \leq n$,

$$\psi^{(0)}(n+1)-\psi^{(0)}(k+1) = \sum_{j=0}^{n-k-1}\frac{1}{k+j+1},$$

and so

$$0 \leq \sum_{j=0}^{n-k-1}\frac{1}{k+j+1} < \psi^{(0)}(n+1)-\psi^{(0)}(\alpha_{0,k}/\pi+7/8).$$

For $k \leq n$ we also have

$$0 < n-\alpha_{0,k}/\pi+1/8 < n-k+3/8,$$

allowing us to obtain the estimate

$$\begin{aligned}&\sum_{k=1}^{n}\left(\frac{\psi^{(0)}(n+1)-\psi^{(0)}(\alpha_{0,k}/\pi+7/8)}{n-\alpha_{0,k}/\pi+1/8}\right)^2\\ &> \sum_{k=1}^{n}\frac{1}{(n-k+3/8)^2}\left(\sum_{j=0}^{n-k-1}\frac{1}{k+j+1}\right)^2\\ &\geq \sum_{k=1}^{n}\frac{1}{(n-k+3/8)^2}\cdot\left(\frac{n-k}{k+(n-k-1)+1}\right)^2 = \sum_{k=1}^{n}\frac{(n-k)^2}{(n-k+3/8)^2}\frac{1}{n^2}\\ &\geq \sum_{k=1}^{n}\frac{(n-k)^2}{(n-k+1)^2}\frac{1}{n^2} = \frac{1}{n^2}\sum_{j=2}^{n}\frac{(j-1)^2}{j^2} \geq \frac{1}{n^2}\frac{n-1}{4}.\end{aligned} \tag{B.6}$$

Thus combining Equations (B.5) and (B.6) yields

$$\|\zeta_n\|^2 \geq \frac{n}{\pi^2}\sum_{k=1}^{n}\left(\frac{\psi^{(0)}(n+1)-\psi^{(0)}(\alpha_{0,k}/\pi+7/8)}{n-\alpha_{0,k}/\pi+1/8}\right)^2 \geq \frac{n-1}{4n\pi^2}.$$

As $n \to \infty$,

$$\liminf_{n\to\infty} \|\zeta_n\|^2 \geq \frac{1}{4\pi^2} \qquad \square$$

Next we analyse the possible limits of convergent subsequences of ζ_n.

Lemma B.10. *If $(\zeta_n)_{n=1}^\infty$ has a norm convergent subsequence $(\zeta_{n_j})_{j=1}^\infty$, then $\zeta_{n_j} \to 0$.*

Proof. We show that $\lim_{n\to\infty} \langle \zeta_n \,|k,+\rangle = 0$ for all $k = 1, 2, \ldots$, which shows that if $\zeta_{n_j} \to \zeta$, then $\zeta = 0$. We have

$$\langle \zeta_n \,|\, k, +\rangle = \sum_{\ell=1}^{\infty} \frac{\sqrt{n}}{(n+\ell)(\alpha_{0,k} + \alpha_{0,\ell})}$$

Since $\alpha_{0,k} \in (\pi k - \pi/4, \pi k - \pi/8)$ by Equation (B.4), $(\alpha_{0,k} + \alpha_{0,\ell})^{-1} < (\pi(k+\ell-1/2))^{-1}$. Hence

$$\begin{aligned} 0 \leq \langle \zeta_n | k, +\rangle &\leq \frac{\sqrt{n}}{\pi} \sum_{\ell=1}^{\infty} \frac{1}{(n+\ell)(k+\ell-1/2)} \\ &= \frac{\sqrt{n}}{\pi(n-k+1/2)} \sum_{\ell=1}^{\infty} \left(\frac{1}{k+\ell-1/2} - \frac{1}{n+\ell} \right) \\ &= \frac{\sqrt{n}}{\pi(n-k+1/2)} \left(\psi^{(0)}(n+1) - \psi^{(0)}(k+1/2) \right). \end{aligned}$$

As $n \to \infty$, $\psi^{(0)}(n) \sim \ln(n)$. Hence

$$\begin{aligned} &\lim_{n\to\infty} \frac{\sqrt{n}}{\pi(n-k+1/2)} \left(\psi^{(0)}(n+1) - \psi^{(0)}(k+1/2) \right) \\ &= \lim_{n\to\infty} \left(\frac{\sqrt{n}(\ln(n+1) - \psi^{(0)}(k+1/2))}{\pi(n-k+1/2)} \right) = 0. \end{aligned}$$

Hence $\lim_{n\to\infty} \langle \zeta_n \,|\, k, +\rangle = 0$. $\square$

Corollary B.11. *The sequence $(\zeta_n)_{n=1}^\infty$ has no norm convergent subsequences.*

Proof. If ζ_n had a convergent subsequence $(\zeta_{n_j})_{j=1}^\infty$, then $\zeta_{n_j} \to 0$ by Lemma B.10. But by Lemma B.9, $\|\zeta_{n_j}\| \not\to 0$, which is a contradiction. $\square$

Corollary B.12. *The operator $P_+ r e^{-i\theta} P_-$ is not compact.*

Proof. By Lemma B.8, $(\xi_n)_{n=1}^\infty$ is bounded, but $((P_+ r e^{-i\theta} P_- - K)\xi_n)_{n=1}^\infty$ contains no convergent subsequence. As $P_+ r e^{-i\theta} P_-$ differs from $P_+ r e^{-i\theta} P_- - K$ by a compact operator, $P_+ r e^{-i\theta} P_-$ is not compact. $\square$

In summary we have shown the following:

Proposition B.13. *The self-adjoint extension $\mathcal{D}_P$ of the Dirac operator $\mathcal{D}$ has compact resolvent, and for all $a \in C^\infty(\mathbb{D})$, the commutators $[\mathcal{D}_P, a]$ are defined on* $\mathrm{dom}(\mathcal{D})$*, and are bounded on this dense subset. The commutator $[F, re^{-i\theta}]$ of the bounded transform $F := \mathcal{D}_P(1+\mathcal{D}_P^2)^{-\frac{1}{2}}$ with $re^{-i\theta} \in C(\mathbb{D})$ is not a compact operator. Therefore the triple $(\rho, L^2(\mathbb{D}) \otimes \mathbb{C}^2, F)$ (where ρ is the representation) is not a Fredholm module for $C(\mathbb{D})$.*

Appendix C

Green's formula and the boundary map

Given a relative spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ for an ideal J in a separable $\mathbb{Z}_2$-graded C^*-algebra A, one would like to construct a spectral triple for A/J (the boundary spectral triple) which represents the class $\partial[(\mathcal{A}, \mathcal{H}, \mathcal{D})]$, where $\partial : K^*(J \lhd A) \to K^{*+1}(A/J)$ is the boundary map (when this boundary map makes sense). In this chapter we detail an unsuccessful attempt to construct the boundary spectral triple motivated by Green's formula.

Let $\mathcal{D}$ be a Dirac operator on a Clifford module S over a compact Riemannian manifold with boundary $\overline{M}$. We recall Green's formula (9.1). For sections $\xi, \eta \in \Gamma^\infty(S)$,

$$\langle \mathcal{D}\xi, \eta \rangle_{L^2(S)} - \langle \xi, \mathcal{D}\eta \rangle_{L^2(S)} = \langle \xi, c(n)\eta \rangle_{L^2(S|_{\partial M})}$$

where $n \in \Gamma^\infty(T^*\overline{M}|_{\partial M})$ is the inward unit normal and c denotes Clifford multiplication. So the boundary Hilbert space is captured by the failure of $\mathcal{D}$ to be symmetric.

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a relative spectral triple for an ideal J in a $\mathbb{Z}_2$-graded separable C^*-algebra A. Motivated by the left-hand side of Green's formula, we examine the form

$$\langle \mathcal{D}^*\xi, \eta \rangle - \langle \xi, \mathcal{D}^*\eta \rangle .$$

Let $(\phi_k)_{k=1}^\infty$ be an approximate identity for $\mathcal{J} = \mathcal{A} \cap J$ of homogeneous degree zero. If J is represented non-degenerately on $\mathcal{H}$, then $\phi_k \to 1$ in the strong operator topology.

Using $\phi_k \cdot \mathrm{dom}(\mathcal{D}^*) \subset \mathrm{dom}(\mathcal{D})$, we see that

$$\begin{aligned}\langle \mathcal{D}^*\xi, \eta\rangle - \langle \xi, \mathcal{D}^*\eta\rangle &= \lim_{k\to\infty} \big(\langle \mathcal{D}^*\xi, \phi_k\eta\rangle - \langle \xi, \phi_k\mathcal{D}^*\eta\rangle \big) \\ &= \lim_{k\to\infty} \big(\langle \xi, \mathcal{D}\phi_k\eta\rangle - \langle \xi, \phi_k\mathcal{D}^*\eta\rangle \big) \\ &= \lim_{k\to\infty} \langle \xi, [\mathcal{D}^*, \phi_k]_\pm \eta\rangle \,.\end{aligned}$$

Our first attempt at constructing the boundary spectral triple is examining the limit of $[\mathcal{D}^*, \phi_k]_\pm$ as $k \to \infty$ in a suitable sense.

Definition C.1. Let $T : \mathrm{dom}(T) \subset \mathcal{H} \to \mathcal{H}$ be a closed operator on a separable Hilbert space $\mathcal{H}$. The **graph inner product** on $\mathrm{dom}(T)$ is

$$\langle \xi, \eta\rangle_{\mathrm{dom}(T)} = \langle \xi, \eta\rangle + \langle T\xi, T\eta\rangle \,.$$

With the graph inner product, $\mathrm{dom}(T)$ is a Hilbert space.

Lemma C.2. *Let $T : \mathrm{dom}(T) \subset \mathcal{H} \to \mathcal{H}$ be a closed operator on a separable Hilbert space $\mathcal{H}$. Then the graph inner product is given by*

$$\langle \xi, \eta\rangle_{\mathrm{dom}(T)} = \Big\langle (1+T^*T)^{1/2}\xi, (1+T^*T)^{1/2}\eta \Big\rangle \,.$$

Proof. We claim that $(1+T^*T)^{-1/2} : \mathcal{H} \to \mathrm{dom}(T)$ is unitary when $\mathrm{dom}(T)$ is equipped with the graph inner product. We have

$$\begin{aligned}&\Big\langle (1+T^*T)^{-1/2}\xi, (1+T^*T)^{-1/2}\eta \Big\rangle_{\mathrm{dom}(T)} \\ &= \Big\langle (1+T^*T)^{-1/2}\xi, (1+T^*T)^{-1/2}\eta \Big\rangle \\ &\quad + \Big\langle T(1+T^*T)^{-1/2}\xi, T(1+T^*T)^{-1/2}\eta \Big\rangle \\ &= \big\langle (1+T^*T)^{-1}\xi, \eta\big\rangle + \Big\langle T^*(1+TT^*)^{-1/2}T(1+T^*T)^{-1/2}\xi, \eta \Big\rangle \\ &= \big\langle (1+T^*T)^{-1}\xi, \eta\big\rangle + \big\langle T^*T(1+T^*T)^{-1}\xi, \eta\big\rangle = \langle \xi, \eta\rangle \,.\end{aligned}$$

Thus $(1+T^*T)^{-1/2} : \mathcal{H} \to \mathrm{dom}(T)$ is unitary, and therefore so is its inverse $(1+T^*T)^{1/2} : \mathrm{dom}(T) \to \mathcal{H}$. □

Proposition C.3. *Let A be a separable $\mathbb{Z}_2$-graded C^*-algebra, and let $J \lhd A$ be an ideal. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a relative spectral triple for $J \lhd A$ such that J is represented non-degenerately on $\mathcal{H}$, and let $(\phi_k)_{k=1}^\infty$ be an approximate identity for $\mathcal{J}$. The sequence $((1+\mathcal{D}\mathcal{D}^*)^{-1}[\mathcal{D}^*, \phi_k])_{k=1}^\infty$ converges in the weak operator topology on $\mathrm{dom}(\mathcal{D}^*)$ with the graph inner product to an anti-self-adjoint partial isometry $n \in B(\mathrm{dom}(\mathcal{D}^*))$, which is*

given by

$$n\xi = \begin{cases} 0 & \textit{if } \xi \in \mathrm{dom}(\mathcal{D}) \\ -i\xi & \textit{if } \xi \in \ker(\mathcal{D}^* - i) \\ i\xi & \textit{if } \xi \in \ker(\mathcal{D}^* + i). \end{cases}$$

We can also express the operator n as

$$n = \mathcal{D}(1 + \mathcal{D}^*\mathcal{D})^{-1} - (1 + \mathcal{D}\mathcal{D}^*)^{-1}\mathcal{D}^*.$$

Proof. Let $\xi, \eta \in \mathrm{dom}(\mathcal{D}^*)$. Then

$$\begin{aligned} &\left\langle \xi, (1 + \mathcal{D}\mathcal{D}^*)^{-1}[\mathcal{D}^*, \phi_k]\eta \right\rangle_{\mathrm{dom}(\mathcal{D}^*)} = \left\langle (1 + \mathcal{D}\mathcal{D}^*)^{1/2}\xi, (1 + \mathcal{D}\mathcal{D}^*)^{-1/2}[\mathcal{D}^*, \phi_k]\eta \right\rangle \\ &= \langle \xi, \mathcal{D}\phi_k\eta \rangle - \langle \xi, \phi_k\mathcal{D}^*\eta \rangle = \langle \mathcal{D}^*\xi, \phi_k\eta \rangle - \langle \xi, \phi_k\mathcal{D}^*\eta \rangle \\ &\to \langle \mathcal{D}^*\xi, \eta \rangle - \langle \xi, \mathcal{D}^*\eta \rangle \text{ as } k \to \infty, \end{aligned}$$

which proves the weak convergence of $((1 + \mathcal{D}\mathcal{D}^*)^{-1}[\mathcal{D}^*, \phi_k])_{k=1}^{\infty}$. Let $\xi = \xi' + \xi^+ + \xi^-$ and $\eta = \eta' + \eta^+ + \eta^-$, where $\xi', \eta' \in \mathrm{dom}(\mathcal{D})$ and $\xi^\pm, \eta^\pm \in \ker(\mathcal{D}^* \mp i)$, using the graph inner product orthogonal decomposition $\mathrm{dom}(\mathcal{D}^*) = \mathrm{dom}(\mathcal{D}) \oplus \ker(\mathcal{D}^* - i) \oplus \ker(\mathcal{D}^* + i)$, [40, p. 138]. Then

$$\begin{aligned} \langle \mathcal{D}^*\xi, \eta \rangle - \langle \xi, \mathcal{D}^*\eta \rangle &= \left\langle \mathcal{D}\xi', \eta \right\rangle - \left\langle \xi', \mathcal{D}^*\eta \right\rangle + \left\langle \mathcal{D}^*(\xi^+ + \xi^-), \eta' \right\rangle - \left\langle \xi^+ + \xi^-, \mathcal{D}\eta' \right\rangle \\ &\quad + \left\langle i\xi^+ - i\xi^-, \eta^+ + \eta^- \right\rangle - \left\langle \xi^+ + \xi^-, i\eta^+ - i\eta^- \right\rangle \\ &= -i \left\langle \xi^+, \eta^+ \right\rangle - i \left\langle \xi^+, \eta^- \right\rangle + i \left\langle \xi^-, \eta^+ \right\rangle + i \left\langle \xi^-, \eta^- \right\rangle \\ &\quad - i \left\langle \xi^+, \eta^+ \right\rangle + i \left\langle \xi^+, \eta^- \right\rangle - i \left\langle \xi^-, \eta^+ \right\rangle + i \left\langle \xi^-, \eta^- \right\rangle = -2i \left\langle \xi^+, \eta^+ \right\rangle + 2i \left\langle \xi^-, \eta^- \right\rangle. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\langle \xi, n\eta \rangle_{\mathrm{dom}(\mathcal{D}^*)} = \left\langle \xi^+, -i\eta^+ \right\rangle_{\mathrm{dom}(\mathcal{D}^*)} + \left\langle \xi^-, i\eta^- \right\rangle_{\mathrm{dom}(\mathcal{D}^*)} \\ &= -i \left\langle \xi^+, \eta^+ \right\rangle - i \left\langle \mathcal{D}^*\xi^+, \mathcal{D}^*\eta^+ \right\rangle + i \left\langle \xi^-, \eta^- \right\rangle + i \left\langle \mathcal{D}^*\xi^-, \mathcal{D}^*\eta^- \right\rangle \\ &= -2i \left\langle \xi^+, \eta^+ \right\rangle + 2i \left\langle \xi^-, \eta^- \right\rangle. \end{aligned}$$

This establishes the first claim. To prove the second claim, we compute

$$\begin{aligned} &\mathcal{D}(1 + \mathcal{D}^*\mathcal{D})^{-1}\xi' - (1 + \mathcal{D}\mathcal{D}^*)^{-1}\mathcal{D}^*\xi \\ &= (1 + \mathcal{D}\mathcal{D}^*)^{-1}\mathcal{D}\xi - (1 + \mathcal{D}\mathcal{D}^*)^{-1}\mathcal{D}\xi' - i\mathcal{D}(1 + \mathcal{D}^*\mathcal{D})^{-1}\mathcal{D}^*\xi^+ \\ &\quad - i(1 + \mathcal{D}\mathcal{D}^*)^{-1}\xi^+ + i\mathcal{D}(1 + \mathcal{D}^*\mathcal{D})^{-1}\mathcal{D}^*\xi^- + i(1 + \mathcal{D}\mathcal{D}^*)^{-1}\xi^- \\ &= -i\mathcal{D}\mathcal{D}^*(1 + \mathcal{D}\mathcal{D}^*)^{-1}\xi^+ - i(1 + \mathcal{D}\mathcal{D}^*)^{-1}\xi^+ + i\mathcal{D}\mathcal{D}^*(1 + \mathcal{D}\mathcal{D}^*)^{-1}\xi^- \\ &\quad + i(1 + \mathcal{D}\mathcal{D}^*)^{-1}\xi^- = -i\xi^+ + i\xi^-. \end{aligned}$$

□

If we think of $n = \mathcal{D}(1+\mathcal{D}^*\mathcal{D})^{-1} - (1+\mathcal{D}\mathcal{D}^*)^{-1}\mathcal{D}^*$ as the inward normal on the boundary, as in Green's formula (9.1), then this leads us to the following definition.

Definition C.4. Let A be a separable $\mathbb{Z}_2$-graded C^*-algebra, let $J \lhd A$ be an ideal, and let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a relative spectral triple for $J \lhd A$. Define an inner product on $\operatorname{dom}(\mathcal{D}^*)/\operatorname{dom}(\mathcal{D})$ by

$$\langle [\xi], [\eta] \rangle_\partial := \langle P\xi, P\eta \rangle_{\operatorname{dom}(\mathcal{D}^*)},$$

where $P \in B(\operatorname{dom}(\mathcal{D}^*))$ is the orthogonal projection onto $\ker(\mathcal{D}^* - i) \oplus \ker(\mathcal{D}^* + i)$. We denote by $\mathcal{H}_\partial$ the Hilbert space $\operatorname{dom}(\mathcal{D}^*)/\operatorname{dom}(\mathcal{D})$ with the inner product $\langle \cdot, \cdot \rangle_\partial$.

For $n = \mathcal{D}(1+\mathcal{D}^*\mathcal{D})^{-1} - (1+\mathcal{D}\mathcal{D}^*)^{-1}\mathcal{D}^*$, we then have an analogue to Green's formula (9.1),

$$\langle \mathcal{D}^*\xi, \eta \rangle - \langle \xi, \mathcal{D}^*\eta \rangle = \langle \xi, n\eta \rangle_\partial .$$

At first glance this looks promising, since we even have a $\mathbb{Z}_2$-grading by $-in$ (in the case that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is odd) or a representation of $\mathbb{C}\mathrm{l}_1$ by $c \mapsto -in$ (in the case that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is even). This grading or Clifford representation provides the appropriate change of KK-dimension going from $K^*(J \lhd A)$ to $K^{*+1}(A/J)$. However, this construction fails since the algebra A/J is not represented on $\mathcal{H}_\partial$ (at least not by the obvious map).

The obvious candidate for a representation of $\mathcal{A}/\mathcal{J} \subset A/J$ on $\mathcal{H}_\partial$ is the map $[a] \cdot [\xi] := [a\xi]$, which is a well-defined linear map since $\mathcal{J}$ maps $\operatorname{dom}(\mathcal{D}^*)$ into $\operatorname{dom}(\mathcal{D})$. Since $\mathcal{A}$ preserves $\operatorname{dom}(\mathcal{D})$, it is multiplicative, so all that remains is to check whether it is involutive.

Proposition C.5. *Let A be a separable $\mathbb{Z}_2$-graded C^*-algebra, let $J \lhd A$ be an ideal, and let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a relative spectral triple for $J \lhd A$. The map $\mathcal{A}/\mathcal{J} \to B(\mathcal{H}_\partial)$ given by $[a] \cdot [\xi] : [a\xi]$ is involutive if and only if $\langle \xi^+, [\mathcal{D}^*, a]_\pm \xi^- \rangle = 0$ for all $\xi^\pm \in \ker(\mathcal{D}^* \mp i)$, $a \in \mathcal{A}$.*

Proof. Let $\xi^\pm, \eta^\pm \in \ker(\mathcal{D}^* \mp i)$, and let $a \in \mathcal{A}$ be of homogeneous degree. Then by using $\mathcal{D}^*\xi^\pm = \pm i\xi^\pm$ and some rearranging, we can compute

$$\begin{aligned}
&\left\langle a(\xi^+ + \xi^-), \eta^+ + \eta^- \right\rangle_{\operatorname{dom}(\mathcal{D}^*)} - \left\langle \xi^+ + \xi^-, a^*(\eta^+ + \eta^-) \right\rangle_{\operatorname{dom}(\mathcal{D}^*)} \\
&= \left\langle a(\xi^+ + \xi^-), \eta^+ + \eta^- \right\rangle + \left\langle \mathcal{D}^* a(\xi^+ + \xi^-), \mathcal{D}^*(\eta^+ + \eta^-) \right\rangle \\
&\quad - \left\langle \xi^+ + \xi^-, a^*(\eta^+ + \eta^-) \right\rangle - \left\langle \mathcal{D}^*(\xi^+ + \xi^-), \mathcal{D}^* a^*(\eta^+ + \eta^-) \right\rangle \\
&= -i \left\langle \xi^+, [\mathcal{D}^*, a^*]_\pm \eta^+ \right\rangle + i \left\langle \xi^+, [\mathcal{D}^*, a^*]_\pm \eta^- \right\rangle - i \left\langle \xi^-, [\mathcal{D}^*, a^*]_\pm \eta^+ \right\rangle \\
&\quad + i \left\langle \xi^-, [\mathcal{D}^*, a^*]_\pm \eta^- \right\rangle + i \left\langle \xi^+, [\mathcal{D}^*, a^*]_\pm \eta^+ \right\rangle + i \left\langle \xi^+, [\mathcal{D}^*, a^*]_\pm \eta^- \right\rangle \\
&\quad - i \left\langle \xi^-, [\mathcal{D}^*, a^*]_\pm \eta^+ \right\rangle - i \left\langle \xi^-, [\mathcal{D}^*, a^*]_\pm \eta^- \right\rangle \\
&= 2i \left\langle \xi^+, [\mathcal{D}^*, a^*]_\pm \eta^- \right\rangle - 2i \left\langle \xi^-, [\mathcal{D}^*, a^*]_\pm \eta^+ \right\rangle .
\end{aligned}$$

Hence the map $\mathcal{A}/\mathcal{J} \to B(\mathcal{H}_\partial)$ is involutive if and only if

$$2i\left\langle \xi^+, [\mathcal{D}^*, a^*]_\pm \eta^- \right\rangle - 2i\left\langle \xi^-, [\mathcal{D}^*, a^*]_\pm \eta^+ \right\rangle = 0$$

for all $\xi^\pm, \eta^\pm \in \ker(\mathcal{D}^* \mp i)$ and $a \in \mathcal{A}$, or equivalently

$$\left\langle \xi^+, [\mathcal{D}^*, a]_\pm \xi^- \right\rangle = 0$$

for all $\xi^\pm \in \ker(\mathcal{D}^* \mp i)$, $a \in \mathcal{A}$. □

Unfortunately, the map $\mathcal{A}/\mathcal{J} \to B(\mathcal{H}_\partial)$ fails to be involutive even in simple examples.

Example C.6 ([39, pp. 257–529])**.** Let $A = C([0,1])$, $J = C_0((0,1))$ and take the odd relative spectral triple $(C^\infty([0,1]), L^2([0,1]), \mathcal{D})$, where

$$\mathrm{dom}(\mathcal{D}) = \{f \in AC([0,1]) : f' \in L^2([0,1]), f(0) = f(1) = 0\}, \quad \mathcal{D} = -i\frac{d}{dx}.$$

Here AC denotes the absolutely continuous functions, which are functions of the form $f(x) = f(0) + \int_0^x h(y)\,dy$ for some $h \in L^1([0,1])$. Then $\ker(\mathcal{D}^* \mp i) = \mathrm{span}\{e^{\mp x}\}$. Let $a = e^{-2x} \in \mathcal{A}$. Then

$$\left\langle e^{-x}, [\mathcal{D}^*, a] e^x \right\rangle = 2i \int_0^1 e^{-x} e^{-2x} e^x \, dx \neq 0.$$

Example C.7. Let $\mathbb{D} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ be the closed unit disc and let $\mathring{\mathbb{D}}$ be its interior. Let $A = C(\mathbb{D})$, let $J = C_0(\mathring{\mathbb{D}})$ and take the even relative spectral triple $(C^\infty(\mathbb{D}), L^2(\mathbb{D}) \oplus L^2(\mathbb{D}), \mathcal{D})$, where

$$\mathrm{dom}(\mathcal{D}) = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{H} : f, g \in H^1(\mathbb{D}), f|_{\partial\mathbb{D}} = g|_{\partial\mathbb{D}} = 0 \right\},$$
$$\mathcal{D} = \begin{pmatrix} 0 & e^{-i\theta}(-\partial_r + ir^{-1}\partial_\theta) \\ e^{i\theta}(\partial_r + ir^{-1}\partial_\theta) & 0 \end{pmatrix}.$$

Here H^1 denotes the first Sobolev space. Then

$$\ker(\mathcal{D}^* \mp i) \supset \overline{\mathrm{span}} \left\{ \begin{pmatrix} \pm i e^{in\theta} I_n(r) \\ e^{i(n+1)\theta} I_{n+1}(r) \end{pmatrix}, \begin{pmatrix} \pm i e^{-i(n+1)\theta} I_{n+1}(r) \\ e^{-in\theta} I_n(r) \end{pmatrix} : n \in \mathbb{N} \right\},$$

where the I_n are modified Bessel functions of the first kind. Let $a = r^2 e^{2i\theta} \in \mathcal{A}$. Then

$$[\mathcal{D}^*, a] = \begin{pmatrix} 0 & -4re^{i\theta} \\ 0 & 0 \end{pmatrix},$$

and

$$\left\langle \begin{pmatrix} ie^{3i\theta} I_3(r) \\ e^{4i\theta} I_4(r) \end{pmatrix}, [\mathcal{D}^*, a] \begin{pmatrix} -ie^{i\theta} I_1(r) \\ e^{2i\theta} I_2(r) \end{pmatrix} \right\rangle = \int_0^{2\pi} \int_0^1 4ir I_3(r) I_2(r) r \, dr d\theta$$
$$\approx 0.009i \neq 0.$$

The non-vanishing of the integral can also be deduced from the fact that $r^2 I_2(r) I_3(r)$ is strictly positive on $(0, 1)$.

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