Hardy spaces of differential forms on Riemannian manifolds

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Abstract. Let M be a complete connected Riemannian manifold. Assuming that the Riemannian measure is doubling, we define Hardy spaces H^p of differential forms on M and give various characterizations of them, including an atomic decomposition. As a consequence, we derive the H^p-boundedness for Riesz transforms on M, generalizing previously known results. Further applications, in particular to H^∞ functional calculus and Hodge decomposition, are given.

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Contents

1 Introduction and main results

The study of Hardy spaces started in the 1910's and was closely related to Fourier series and complex analysis in one variable (see [50], Chapters 7 and 14). In the 1960's, an essential feature of the development of real analysis in several variables was the theory of real Hardy spaces $H^p(\mathbb{R}^n)$, and in particular $H^1(\mathbb{R}^n)$, which began with the paper of Stein and Weiss [46]. In this work, Hardy spaces were defined and studied by means of Riesz transforms and harmonic functions. The celebrated paper of Fefferman and Stein [26] provided many characterizations of Hardy spaces on $\mathbb{R}^n$, in particular in terms of suitable maximal functions. The dual space of $H^1(\mathbb{R}^n)$ was also identified as $BMO(\mathbb{R}^n)$. An important step was the atomic decomposition of $H^1(\mathbb{R}^n)$, due to Coifman (for $n = 1$, [14]) and to Latter (for $n \geq 2$, [31]). A detailed review and bibliography on these topics may be found in [43]. Hardy spaces have been generalized to various geometric settings. See for example the work of Strichartz [47] for compact manifolds with a characterization via pseudo-differential operators, and more generally, starting from the point of view of the atomic decomposition, the work of Coifman and Weiss [17] for spaces of homogeneous type, which are known to be a relevant setting for most tools in harmonic analysis such as Hardy-Littlewood maximal function, covering lemmata, Whitney decomposition, Calderón-Zygmund decomposition and singular integrals (see [17], [16], [22]).

The connection between Hardy spaces and area functionals will be most important to us thanks to the theory of tent spaces developed by Coifman, Meyer and Stein in [15]. Let us recall the main line of ideas. For suitable functions f on $\mathbb{R}^n$ and all $x \in \mathbb{R}^n$, define

$$\mathcal{S}f(x) = \left(\iint_{|y-x|<t} \left| t\sqrt{\Delta} e^{-t\sqrt{\Delta}} f(y) \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}$$

where $\Delta = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$. Hence, $e^{-t\sqrt{\Delta}}$ is nothing but the Poisson semigroup. It is proved in

[26] that, if $f \in H^1(\mathbb{R}^n)$, then $\mathcal{S}f \in L^1(\mathbb{R}^n)$ and

$$\|\mathcal{S}f\|_{L^1(\mathbb{R}^n)} \leq C \|f\|_{H^1(\mathbb{R}^n)}.$$

This fact exactly means that for $f \in H^1(\mathbb{R}^n)$ the function F defined by $F(t,x) = t\sqrt{\Delta}e^{-t\sqrt{\Delta}}f(x)$ belongs to the tent space $T^{1,2}(\mathbb{R}^n)$. Conversely, for any $F \in T^{1,2}(\mathbb{R}^n)$ and $F_t(x) = F(t,x)$, the function f given by

$$f = \int_0^{+\infty} t\sqrt{\Delta}e^{-t\sqrt{\Delta}} F_t \frac{dt}{t} \tag{1.1}$$

is in the Hardy space $H^1(\mathbb{R}^n)$ with appropriate estimate. The round trip is granted by the Calderón reproducing formula

$$f = 4\int_0^{+\infty} t\sqrt{\Delta}e^{-t\sqrt{\Delta}} t\sqrt{\Delta}e^{-t\sqrt{\Delta}} f \frac{dt}{t}.$$

It is proved in [15] that the tent space $T^{1,2}(\mathbb{R}^n)$ has an atomic decomposition. Applying this to the function $F(t,x) = t\sqrt{\Delta}e^{-t\sqrt{\Delta}}f(x)$ and plugging this into the reproducing formula, one obtains a decomposition of f, not as a sum of atoms but of so-called molecules (see [17]). Molecules do not have compact support but decay sufficiently fast that they can be used in place of atoms for many purposes. Had we changed the operator $t\sqrt{\Delta}e^{-t\sqrt{\Delta}}$ to an appropriate convolution operator with compactly supported kernel, then the same strategy would give an atomic decomposition of f. The tent space method can be used in different contexts, for instance to obtain an atomic decomposition for Hardy spaces defined by maximal functions involving second order elliptic operators, see [7]. See also [49] for a variant of this argument. Let us also mention that the duality for tent spaces provides an alternative proof of the $H^1 - BMO$ duality (see [26], [45]).

If one wants to replace functions by forms given some differential structure, then the first thing that changes is the mean value condition. A function f in the Hardy space $H^1(\mathbb{R}^n)$ has vanishing mean, that is $\int f = 0$. For general forms, the integral has no meaning. However, an appropriate atomic decomposition of the Hardy space of divergence free vector fields was proved in [28]. This space turned out to be a specific case of the Hardy spaces of exact differential forms on $\mathbb{R}^n$ defined by Lou and the second author in [32] via the tent spaces approach. There, atomic decompositions, duality results, among other things, were obtained. What replaces the mean value property in [32] is the fact that atoms are *exact* forms (see section 6).

One motivation for studying Hardy spaces of forms is the Riesz transforms. Indeed, the Riesz transforms $R_j = \partial_{x_j}(\sqrt{\Delta})^{-1}$ are well-known bounded operators on $H^1(\mathbb{R}^n)$ (and $L^p(\mathbb{R}^n)$, $1 < p < \infty$). However, the vector map $(R_1, R_2, ..., R_n) = \nabla(\sqrt{\Delta})^{-1}$ is geometrically meaningful as its target space is a space of gradient vector fields (in a generalized sense). This observation is valid in any Riemannian manifold. On such manifolds the understanding of the L^p boundedness property of Riesz transforms was proposed by Strichartz in [48]. What happens at $p = 1$ is interesting in itself and also part of this quest as it can give results for $p > 1$ by interpolation. As the "geometric" Riesz transform is form-valued, getting satisfactory H^1 boundedness statements for this operator requires the notion of Hardy spaces of differential forms. Our aim is, therefore, to develop an appropriate theory of Hardy spaces on Riemannian manifolds (whether or not compact) and to apply this to the Riesz transform (for us, only the "geometric" Riesz transform matters so that we drop the "s" in transform). This will indeed generalize the theory in [32] to a geometric context. In particular, their Hardy spaces will be our spaces $H^1_d(\Lambda^k T^*\mathbb{R}^n)$ for $0 \le k \le n$ (see Section 8.2.2). Also, our Hardy spaces are designed so that the Riesz transform is automatically bounded on them. Specializing to specific situations allows us to recover results obtained by the third author alone [38] or with M. Marias [33].

We now describe precisely our setting. Let M be a complete Riemannian manifold, ρ the geodesic distance and $d\mu$ the Riemannian measure. Complete means that any two points can be joined by a geodesic, thus M is connected. For all $x \in M$ and all $r > 0$, $B(x,r)$ stands for the open geodesic ball with center x and radius r, and its measure will be denoted $V(x,r)$.

For all $x \in M$, denote by ΛT_x^*M the complex exterior algebra over the cotangent space T_x^*M. Let $\Lambda T^*M = \oplus_{0\le k\le \dim\, M}\Lambda^k T^*M$ be the bundle over M whose fibre at each $x \in M$ is given by ΛT_x^*M, and let $L^2(\Lambda T^*M)$ be the space of square integrable sections of ΛT^*M. Denote by d the exterior differentiation. Recall that, for $0 \le k \le \dim\, M - 1$, d maps, for instance, $C_0^\infty(\Lambda^k T^*M)$ into $C_0^\infty(\Lambda^{k+1}T^*M)$ and that $d^2 = 0$. Denote also by d^* the adjoint of d on $L^2(\Lambda T^*M)$. Let $D = d + d^*$ be the Hodge-Dirac operator on $L^2(\Lambda T^*M)$, and $\Delta = D^2 = dd^* + d^*d$ the (Hodge-de Rham) Laplacian. The L^2 Hodge decomposition, valid on any complete Riemannian manifold, states that

$$L^2(\Lambda T^*M) = \overline{\mathcal{R}(d)} \oplus \overline{\mathcal{R}(d^*)} \oplus \mathcal{N}(\Delta),$$

where $\mathcal{R}(T)$ (resp. $\mathcal{N}(T)$) stands for the range (resp. the nullspace) of T, and the decomposition is orthogonal. See for instance [25], Theorem 24, p. 165, and [12].

In view of the previous discussions, we will start the approach of Hardy spaces via tent spaces. The first observation is that this theory can be developed in spaces of homogeneous type subject to an additional technical condition [39]. For us, it only means that we impose on M the doubling property: there exists $C > 0$ such that, for all $x \in M$ and all $r > 0$,

$$V(x,2r) \le CV(x,r). \tag{1.2}$$

A straightforward consequence of (1.2) is that there exist $C, \kappa > 0$ such that, for all $x \in M$, all $r > 0$ and all $\theta > 1$,

$$V(x,\theta r) \le C\theta^\kappa V(x,r). \tag{1.3}$$

The hypothesis (1.2) exactly means that M, equipped with its geodesic distance and its Riemannian measure, is a space of homogeneous type in the sense of Coifman and Weiss.

There is a wide class of manifolds on which (1.2) holds. First, it is true on Lie groups with polynomial volume growth (in particular on nilpotent Lie groups), and in this context the heat kernel on functions does satisfy Gaussian estimates, see [40]. In particular, (1.2) is true if M has nonnegative Ricci curvature thanks to the Bishop comparison theorem (see [11]).

Recall also that (1.2) remains valid if M is quasi-isometric to a manifold with nonnegative Ricci curvature, or is a cocompact covering manifold whose deck transformation group has polynomial growth, [20]. Contrary to the doubling property, the nonnegativity of the Ricci curvature is not stable under quasi-isometry.

The second observation is that the Euclidean proofs using tent spaces and formulæ such as (1.1) use pointwise bounds on kernels of the Poisson semigroup (or of appropriate convolution operators). Here, the only available operators are functions of Δ and this would require some knowledge, say, on the kernel $p_t(x,y)$ of the heat semigroup. If one deals with the Laplace-Beltrami operator on functions, "Gaussian" pointwise estimates may hold for the heat kernel p_t, but this depends on further geometric assumptions on M. For instance, when M non-compact, it is well-known (see [41], [29]) that M satisfies the doubling property and a scaled L^2 Poincaré inequality on balls if and only if p_t satisfies a "Gaussian" upper and lower estimate and is Hölder continuous. More precisely, there exist $C_1, c_1, C_2, c_2, \alpha > 0$ such that, for all $x, x', y, y' \in M$ and all $t > 0$,

$$\begin{aligned} &\frac{c_2}{V(x,\sqrt{t})} e^{-C_2 \frac{\rho^2(x,y)}{t}} \leq p_t(x,y) \leq \frac{C_1}{V(x,\sqrt{t})} e^{-c_1 \frac{\rho^2(x,y)}{t}}, \\ &|p_t(x,y) - p_t(x',y)| \leq C \left(\frac{\rho(x,x')}{\sqrt{t}} \right)^{\alpha}, \ |p_t(x,y) - p_t(x,y')| \leq C \left(\frac{\rho(y,y')}{\sqrt{t}} \right)^{\alpha}. \end{aligned} \tag{1.4}$$

Note that such a result concerns the heat kernel on functions, *i.e.* on 0-forms. For the heat kernel on 1-forms, the pointwise Gaussian domination holds for $|p_t(x,y)|$ if M has non negative Ricci curvature from the Weitzenböck formula (see [10] and also the recent work [21] for more and the references therein). Very little seems to be known about estimates for the heat kernel on general forms.

Hence, for our theory to be applicable, we have to forbid the use of Gaussian estimates similar to (1.4). Fortunately, there is a weaker notion of Gaussian decay, which holds on any complete Riemannian manifold, namely the notion of L^2 off-diagonal estimates, as introduced by Gaffney [27]. This notion has already proved to be a good substitute of Gaussian estimates for such questions as the Kato square root problem or L^p-bounds for Riesz transforms when dealing with elliptic operators (even in the Euclidean setting) for which Gaussian estimates do not hold (see [1, 4, 9] in the Euclidean setting, and [2] in a complete Riemannian manifold). We show in the present work that a theory of Hardy spaces of differential forms can be developed under such a notion.

The results of this work have been announced in the Note [5]. Let us state the main ones. We define in fact three classes of Hardy spaces of differential forms on manifolds satisfying (1.2). These definitions require some preliminary material, and we remain vague at this stage. The first class, denoted by $H^1(\Lambda T^*M)$, is the one defined via tent spaces. Actually, using fully the theory of tent spaces, we also define $H^p(\Lambda T^*M)$ for all $1 \leq p \leq +\infty$. The second class, $H^1_{mol}(\Lambda T^*M)$, is defined via "molecules" (see above). Our third class, $H^1_{max}(\Lambda T^*M)$, is defined in terms of an appropriate maximal function associated to the Hodge-de Rham Laplacian. Within each class, the Hardy spaces are Banach spaces with norms depending on some parameters. We show they are identical spaces with equivalence of norms. Eventually we prove that the three classes are the same. This can be summarized as follows:

Theorem 1.1 *Assume (1.2). Then,* $H^1(\Lambda T^*M) = H^1_{mol}(\Lambda T^*M) = H^1_{max}(\Lambda T^*M)$.

Let us mention that we do not use much of the differential structure to prove the first equality. As a matter of fact, it can be proved on a space of homogeneous type for an operator satisfying L^2 off-diagonal bounds and L^2 quadratic estimates. We leave this point to further works (see also Remark 1.4 below).

As a corollary of Theorem 1.1, we derive the following comparison between $H^p(\Lambda T^*M)$ and $L^p(\Lambda T^*M)$:

Corollary 1.2 *Assume (1.2).*

(*a*) *For all* $1 \leq p \leq 2$, $H^p(\Lambda T^*M) \subset L^p(\Lambda T^*M)$, *and more precisely,* $H^p(\Lambda T^*M) \subset \overline{\mathcal{R}(D) \cap L^p(\Lambda T^*M)}^{L^p(\Lambda T^*M)}$.

(*b*) *For* $2 \leq p < +\infty$, $\overline{\mathcal{R}(D) \cap L^p(\Lambda T^*M)}^{L^p(\Lambda T^*M)} \subset H^p(\Lambda T^*M)$.

Of course, it may or may not be that equalities hold for some/all $p \in (1, \infty) \setminus \{2\}$.

For our motivating operator, namely the Riesz transform $D\Delta^{-1/2}$ on M, we obtain a satisfactory answer.

Corollary 1.3 *Assume (1.2). Then, for all* $1 \leq p \leq +\infty$, $D\Delta^{-1/2}$ *is* $H^p(\Lambda T^*M)$ *bounded. Consequently, it is* $H^1(\Lambda T^*M) - L^1(\Lambda T^*M)$ *bounded.*

The plan of the paper is as follows. As a preliminary section (Section 2), we focus on the case of $H^2(\Lambda T^*M)$ and define what we mean by Riesz transform, because this case just requires well-known facts of the Hodge-de Rham theory of $L^2(\Lambda T^*M)$, and this motivates the foregoing technical tools needed to define and study $H^p(\Lambda T^*M)$ spaces for $p \neq 2$. Section 3 is devoted to the statement and the proof of the off-diagonal L^2 estimates for the Hodge-Dirac operator and the Hodge-de Rham Laplacian. In Section 4, we present tent spaces on M and establish the boundedness of some "projectors" on these spaces. Relying on this fact, we define Hardy spaces $H^p(\Lambda T^*M)$ for $1 \leq p \leq +\infty$ in Section 5 and state duality and interpolation results. We also establish the $H^p(\Lambda T^*M)$ boundedness of Riesz transforms (Corollary 1.3) and show more generally that there is a functional calculus on $H^p(\Lambda T^*M)$. Section 6 is devoted to the description of molecules and the identification of $H^1(\Lambda T^*M)$ with $H^1_{mol}(\Lambda T^*M)$. As a consequence, we obtain Corollary 1.2, which completes the proof of Corollary 1.3. In Section 7, we prove the maximal characterization of $H^1(\Lambda T^*M)$, which ends the proof of Theorem 1.1.

In Section 8, we give further examples and applications of the previous results. Namely, specializing to the case of 0-forms, we compare our Hardy spaces with the classical Hardy spaces for functions under suitable assumptions on M (such as Poincaré inequalities), generalizing known results about the Riesz transform. We also go further in the comparison of $H^p(\Lambda T^*M)$ with $L^p(\Lambda T^*M)$ assuming "Gaussian" estimates for the heat kernel, and recover well-known results about the L^p boundedness for the Riesz transform.

Remark 1.4 *During the preparation of this manuscript, we learnt that S. Hofmann and S. Mayboroda have been developing the theory of Hardy spaces associated with second order elliptic operators in divergence form in* $\mathbb{R}^n$ *[30]. This is an alternative generalisation of the usual theory, which is associated with the Laplacian on* $\mathbb{R}^N$*. Although there is much in common, such as the use of off-diagonal estimates, the results are different, and the proofs have been obtained independently.*

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Notation: If two quantities $A(f), B(f)$ depend on a function f ranging over a certain space L, $A(f) \sim B(f)$, for $f \in L$, means that there exist $c, C > 0$ such that $c\,A(f) \leq B(f) \leq C\,A(f)$, $\forall\, f \in L$.

2 The $H^2(\Lambda T^*M)$ space and the Riesz transform

Set $H^2(\Lambda T^*M) = \overline{\mathcal{R}(D)} = \overline{\{Du \in L^2(\Lambda T^*M); u \in L^2(\Lambda T^*M)\}}$ and note that

$$L^2(\Lambda T^*M) = \overline{\mathcal{R}(D)} \oplus \mathcal{N}(D) = H^2(\Lambda T^*M) \oplus \mathcal{N}(D).$$

It is an essential fact for the sequel that $H^2(\Lambda T^*M)$ can be described in terms of tent spaces and appropriate quadratic functionals, which we describe now. If $\theta \in \left(0, \frac{\pi}{2}\right)$, set

$$\begin{array}{rcl} \Sigma_{\theta+} & = & \{z \in \mathbb{C} \setminus \{0\}\,;\ |\arg z| \leq \theta\} \cup \{0\}\,, \\ \Sigma^0_{\theta+} & = & \{z \in \mathbb{C} \setminus \{0\}\,;\ |\arg z| < \theta\}\,, \\ \Sigma_{\theta} & = & \Sigma_{\theta+} \cup (-\Sigma_{\theta+})\,, \\ \Sigma^0_{\theta} & = & \Sigma^0_{\theta+} \cup \left(-\Sigma^0_{\theta+}\right). \end{array}$$

and denote by $H^\infty(\Sigma^0_\theta)$ the algebra of bounded holomorphic functions on Σ^0_θ. Given $\sigma, \tau > 0$, define $\Psi_{\sigma,\tau}(\Sigma^0_\theta)$ to be the set of holomorphic functions $\psi \in H^\infty(\Sigma^0_\theta)$ which satisfy

$$|\psi(z)| \leq C \inf\{|z|^\sigma, |z|^{-\tau}\}$$

for some $C > 0$ and all $z \in \Sigma^0_\theta$. Then let $\Psi(\Sigma^0_\theta) = \cup_{\sigma,\tau>0}\Psi_{\sigma,\tau}(\Sigma^0_\theta)$.

For example, if $\psi(z) = z^N(1 \pm iz)^{-\alpha}$ for integers N, α with $1 \leq N < \alpha$ then $\psi \in \Psi_{N,\alpha-N}(\Sigma_\theta^0)$, if $\psi(z) = z^N(1+z^2)^{-\beta}$ for integers N, β with $1 \leq N < 2\beta$ then $\psi \in \Psi_{N,2\beta-N}(\Sigma_\theta^0)$, and if $\psi(z) = z^N \exp(-z^2)$ for a non-negative integer N then $\psi \in \Psi_{N,\tau}(\Sigma_\theta^0)$ for all $\tau > 0$.

Define $\mathcal{H} = L^2\left((0,+\infty), L^2(\Lambda T^*M), \dfrac{dt}{t}\right)$, equipped with the norm

$$\|F\|_{\mathcal{H}} = \left(\int_0^{+\infty}\int_M |F(x,t)|^2\, dx \frac{dt}{t}\right)^{1/2},$$

with $|F(x,t)|^2 = \langle F(x,t), F(x,t)\rangle_x$, where $\langle .,.\rangle_x$ stands for the inner complex product in T_x^*M, and we drop the subscript x in the notation to simplify the exposition. Note also that, here and after, we write $dx, dy, \ldots$ instead of $d\mu(x), d\mu(y), \ldots$. If $F \in \mathcal{H}$ and $t > 0$, denote by F_t the map $x \mapsto F(x,t)$.

Given $\psi \in \Psi(\Sigma_\theta^0)$ for some $\theta > 0$, set $\psi_t(z) = \psi(tz)$ for all $t > 0$ and all $z \in \Sigma_\theta^0$ and define the operator $\mathcal{Q}_\psi : L^2(\Lambda T^*M) \to \mathcal{H}$ by

$$(\mathcal{Q}_\psi h)_t = \psi_t(D)h\ , \quad t > 0\ .$$

Since D is a self-adjoint operator on $L^2(\Lambda T^*M)$, it follows from the spectral theorem that $\mathcal{Q}_\psi$ is bounded, and indeed that

$$\|\mathcal{Q}_\psi f\|_{\mathcal{H}} \sim \|f\|_2$$

for all $f \in H^2(\Lambda T^*M)$. (Note that $\mathcal{Q}_\psi f = 0$ for all $f \in \mathcal{N}(D)$.) Also define the operator $\mathcal{S}_\psi : \mathcal{H} \to L^2(\Lambda T^*M)$ by

$$\mathcal{S}_\psi H = \int_0^{+\infty} \psi_t(D)H_t \frac{dt}{t} = \lim_{\varepsilon\to 0, N\to\infty} \int_\varepsilon^N \psi_t(D)H_t\frac{dt}{t}$$

where the limit is in the $L^2(\Lambda T^*M)$ strong topology. This operator is also bounded, as $\mathcal{S}_\psi = \mathcal{Q}_{\overline{\psi}}{}^*$ where $\overline{\psi}$ is defined by $\overline{\psi}(z) = \overline{\psi(\bar{z})}$.

If $\widetilde{\psi} \in \Psi(\Sigma_\theta^0)$ is chosen to satisfy $\int_0^\infty \psi(\pm t)\widetilde{\psi}(\pm t)\frac{dt}{t} = 1$ (*e.g.* by taking $\widetilde{\psi}(z) = \{\int_0^\infty |\psi(\pm t)|^2 \frac{dt}{t}\}^{-1}\overline{\psi}(z)$ when $z \in \Sigma_{\theta\pm}^0$), then the spectral theorem implies the following version of the Calderón reproducing theorem:

$$\mathcal{S}_{\widetilde{\psi}}\mathcal{Q}_\psi f = \mathcal{S}_\psi\mathcal{Q}_{\widetilde{\psi}} f = f$$

for all $f \in \mathcal{R}(D)$ and hence for all $f \in H^2(\Lambda T^*M)$. (Indeed $\mathcal{S}_\psi\mathcal{Q}_{\widetilde{\psi}}$ is the orthogonal projection of $L^2(\Lambda T^*M)$ onto $H^2(\Lambda T^*M)$.) It follows that $\mathcal{R}(\mathcal{S}_\psi) = H^2(\Lambda T^*M)$ and that

$$\|f\|_2 \sim \inf\{\|H\|_{\mathcal{H}}\,;\ f = S_\psi H\}$$

for all $f \in H^2(\Lambda T^*M)$.

Remark 2.1 *With a little more care we could take* $\widetilde{\psi} \in \Psi_{\sigma,\tau}(\Sigma_\theta^0)$ *for any given* σ, τ*. This fact will be used in Section 5.1.*

Thus, we have two descriptions of $H^2(\Lambda T^*M)$ in terms of quadratic functionals, involving the $\mathcal{H}$ space (which is nothing but the tent space $T^{2,2}(\Lambda T^*M)$ of Section 4 below) and independent of the choice of the function ψ.

Let $\Delta = D^2$. Note that $\mathcal{N}(\Delta) = \mathcal{N}(D)$ and, hence, D and Δ are one-one operators on $H^2(\Lambda T^*M)$. Observe also that replacing D by Δ and $\Psi(\Sigma_\theta^0)$ by $\Psi(\Sigma_{\theta+}^0)$ would lead exactly to the similar descriptions of the Hardy space $H^2(\Lambda T^*M)$ in terms of functions of Δ only.

We define the Riesz transform on M as the bounded operator $D\Delta^{-1/2} \colon H^2(\Lambda T^*M) \to H^2(\Lambda T^*M)$.

Set $H^2_d(\Lambda T^*M) = \overline{\mathcal{R}(d)}$ and $H^2_{d^*}(\Lambda T^*M) = \overline{\mathcal{R}(d^*)}$, so that by the Hodge decomposition

$$H^2(\Lambda T^*M) = H^2_d(\Lambda T^*M) \oplus H^2_{d^*}(\Lambda T^*M), \tag{2.1}$$

and the sum is orthogonal. The orthogonal projections are given by dD^{-1} and d^*D^{-1}.

The Riesz transform $D\Delta^{-1/2}$ splits naturally as the sum of $d\Delta^{-1/2}$ and $d^*\Delta^{-1/2}$, which we call the Hodge-Riesz transforms. As

$$d\Delta^{-1/2} = (dD^{-1})(D\Delta^{-1/2}) \quad \text{and} \quad d^*\Delta^{-1/2} = (d^*D^{-1})(D\Delta^{-1/2}),$$

they extend to bounded operators on $H^2(\Lambda T^*M)$. One further checks that $d\Delta^{-1/2}$ is bounded and invertible from $H^2_{d^*}(\Lambda T^*M)$ to $H^2_d(\Lambda T^*M)$, that $d^*\Delta^{-1/2}$ is bounded and invertible from $H^2_d(\Lambda T^*M)$ to $H^2_{d^*}(\Lambda T^*M)$, and that they are inverse to one another.

3 Off-diagonal L^2-estimates for Hodge-Dirac and Hodge-Laplace operators

Throughout this section, M is an arbitrary complete Riemannian manifold (we stress the fact that M is not assumed to satisfy the doubling property (1.2)). We collect and prove all the off-diagonal L^2-estimates which will be used in the sequel for the Hodge-Dirac operator and the Hodge-de Rham Laplacian (and also for d and d^*). We will make use of the following terminology:

Definition 3.1 *Let $A \subset \mathbb{C}$ be a non-empty set, $(T_z)_{z\in A}$ be a family of $L^2(\Lambda T^*M)$-bounded operators, $N \geq 0$ and $C > 0$. Say that $(T_z)_{z\in A}$ satisfies $OD_z(N)$ estimates with constant C if, for all disjoint closed subsets $E, F \subset M$ and all $z \in A$,*

$$\|M_{\chi_F} T_z M_{\chi_E}\|_{2,2} \leq C \inf\left(1, \left(\frac{|z|}{\rho(E,F)}\right)^N\right), \tag{3.1}$$

where, for any $G \subset M$, χ_G denotes the characteristic function of G and, for any bounded function η on M, M_η stands for the multiplication by η.

In this definition and in the sequel, if E and F are any subsets of M, $\rho(E,F)$ is the infimum of $\rho(x,y)$ for all $x \in E$ and all $y \in F$. Moreover, if T is a bounded linear operator from $L^p(\Lambda T^*M)$ to $L^q(\Lambda T^*M)$, its functional norm is denoted by $\|T\|_{q,p}$.

Remark 3.2 *We remark that if* $(T_z)_{z\in A}$ *satisfies* $OD_z(N)$ *estimates, and* $0 \le N_1 \le N$*, then* $(T_z)_{z\in A}$ *satisfies* $OD_z(N_1)$ *estimates.*

The off-diagonal estimates to be used in the sequel will be presented in four lemmata.

Lemma 3.3 *Let* N *and* α *be nonnegative integers with* $0 \le N \le \alpha$ *and* $\mu \in \left(0, \frac{\pi}{2}\right)$*. Then, for all integers* $N' \ge 0$, $\left((zD)^N (I + izD)^{-\alpha}\right)_{z\in\Sigma_{\frac{\pi}{2}-\mu}}$ *satisfies* $OD_z(N')$ *estimates with constants only depending on* μ, N, N' *and* α*.*

Remark 3.4 *Note that, with the same notations, if* $\alpha \ge N+1$, $(zd(zD)^N(I+izD)^{-\alpha})_{z\in\Sigma_{\frac{\pi}{2}-\mu}}$ *and* $(zd^*(zD)^N(I+izD)^{-\alpha})_{z\in\Sigma_{\frac{\pi}{2}-\mu}}$ *satisfy* $OD_z(N')$ *estimates with constants only depending on* μ, N, N', α*. However, these estimates will not be used in the sequel.*

Lemma 3.5 *Let* k, N *and* α *be nonnegative integers with* $0 \le N \le \alpha$ *and* $\mu \in \left(0, \frac{\pi}{2}\right)$*. Then, for all* $\tau \in \Sigma_{\frac{\pi}{2}-\mu}$, $\left((I + i\tau D)^{-k}(zD)^N(I + izD)^{-\alpha}\right)_{z\in\Sigma_{\frac{\pi}{2}-\mu}}$ *satisfies* $OD_z(N)$ *estimates with constants only depending on* μ, N, k, α *and* β *(and, in particular, uniform in* τ*).*

Since the operator D is self-adjoint in $L^2(\Lambda T^*M)$, one may define the L^2-bounded operator $f(D)$ for any $f \in H^\infty(\Sigma^0_\theta)$. If $f \in \Psi(\Sigma^0_\theta)$, then $f(D)$ can be computed with the Cauchy formula:

$$f(D) = \frac{1}{2i\pi}\int_\gamma (\zeta I - D)^{-1} f(\zeta) d\zeta, \tag{3.2}$$

where γ is made of two rays $re^{\pm i\beta}$, $r \ge 0$ and $\beta < \theta$, described counterclockwise (see [35], [8, Section 0.1]). Moreover, for every $f \in H^\infty(\Sigma^0_\theta)$ there is a uniformly bounded sequence of functions $f_n \in \Psi(\Sigma^0_\theta)$ which converges to f uniformly on compact sets, and then $f(D)\psi(D) = \lim f_n(D)\psi(D)$ in the strong operator topology for all $\psi \in \Psi(\Sigma^0_\theta)$.

Lemma 3.6 *Let* N *be a positive integer and* $\mu \in \left(0, \frac{\pi}{2}\right)$*.*

(*a*) *If* $\left(g_{(t)}\right)_{t>0}$ *is a uniformly bounded family of functions in* $H^\infty(\Sigma^0_{\frac{\pi}{2}-\mu})$ *and* α *is an integer such that* $\alpha \ge N+1$*, then* $(g_{(t)}(D)(tD)^N(I \pm itD)^{-\alpha})_{t>0}$ *satisfies* $OD_t(N-1)$ *estimates with constant bounded by* $C \sup\limits_{t>0} \left\|g_{(t)}\right\|_\infty$*.*

(*b*) *If* $f \in H^\infty(\Sigma^0_{\frac{\pi}{2}-\mu})$ *and* $\psi \in \Psi_{N,1}(\Sigma^0_{\pi/2-\mu})$*, then* $(f(D)\psi_t(D))_{t>0}$ *satisfies* $OD_t(N-1)$ *estimates with constants bounded by* $C\left\|f\right\|_\infty$*.*

In what follows, we set $h_{a,b}(u) = \inf\left(u^a, u^{-b}\right)$, where $a, b, u > 0$. Recall that if $\psi \in \Psi(\Sigma^0_\theta)$ and $t > 0$, then ψ_t is defined by $\psi_t(z) = \psi(tz)$.

Lemma 3.7 *Let* $\psi \in \Psi_{N_1,\alpha_1}(\Sigma^0_{\pi/2-\mu})$ *and* $\widetilde{\psi} \in \Psi_{N_2,\alpha_2}(\Sigma^0_{\pi/2-\mu})$ *where* $\alpha_1, \alpha_2, N_1, N_2$ *are positive integers and* $\mu \in \left(0, \frac{\pi}{2}\right)$*, and suppose that* a, b *are nonnegative integers satisfying* $a \le \min\{N_1, \alpha_2 - 1\}$*,* $b \le \min\{N_2, \alpha_1 - 1\}$*. Then, there exists* $C > 0$ *such that, for all* $f \in H^\infty(\Sigma^0_\mu)$*, there exists, for all* $s, t > 0$*, an operator* $T_{s,t}$ *with the following properties:*

(*i*) $\psi_s(D)f(D)\widetilde{\psi}_t(D) = h_{a,b}\left(\frac{s}{t}\right)T_{s,t}$;

(*ii*) $(T_{s,t})_{t\geq s}$ *satisfies* $OD_t(N_2+a-1)$ *estimates uniformly in* $s>0$;

(*iii*) $(T_{s,t})_{s\geq t}$ *satisfies* $OD_s(N_1+b-1)$ *estimates uniformly in* $t>0$.

Proof of Lemma 3.3: The proof is exactly as the one of Proposition 5.2 in [9]. □

Proof of Lemma 3.5: We use the notation

$$[T,S]=TS-ST$$

for the commutator of two operators T and S. The proof is done by induction on k and relies on a commutator argument, as in [9], Proposition 5.2.

For $k=0$, the conclusion is given by Lemma 3.3. Let $k\geq 1$ and assume that, for all integers $0\leq N\leq\alpha$, $\left((I+i\tau D)^{-(k-1)}(zD)^N(I+izD)^{-\alpha}\right)_{z\in\Sigma_{\frac{\pi}{2}-\mu}}$ satisfies $OD_z(N)$ estimates uniformly in τ. To establish that $\left((I+i\tau D)^{-k}(zD)^N(I+izD)^{-\alpha}\right)_{z\in\Sigma_{\frac{\pi}{2}-\mu}}$ satisfies $OD_z(N)$ estimates uniformly in τ whenever $0\leq N\leq\alpha$, we argue by induction on N. The case when $N=0$ is obvious. Let $1\leq N\leq\alpha$ and assume that $\left((I+i\tau D)^{-k}(zD)^{(N-1)}(I+izD)^{-\alpha}\right)_{z\in\Sigma_{\frac{\pi}{2}-\mu}}$ satisfies $OD_z(N-1)$ estimates uniformly in τ. We intend to show that $\left((I+i\tau D)^{-k}(zD)^N(I+izD)^{-\alpha}\right)_{z\in\Sigma_{\frac{\pi}{2}-\mu}}$ satisfies $OD_z(N)$ estimates uniformly in τ. Let E,F be two disjoint closed subsets of M, χ the characteristic function of E and η a Lipschitz function on M equal to 1 on F, to 0 on E and satisfying

$$\left\|\left|\nabla\eta\right|\right\|_\infty\leq C\rho(E,F)^{-1},\rho(\text{supp }\eta,E)\sim\rho(E,F).$$

Our conclusion reduces to proving that

$$\left\|M_\eta\left((I+i\tau D)^{-k}(zD)^N(I+izD)^{-\alpha}\right)M_\chi\right\|_{2,2}\leq C\left(\frac{|z|}{\rho(E,F)}\right)^N \tag{3.3}$$

where we recall that M_η and M_χ denote the multiplication by η and χ respectively. But, because of the supports of χ and η, the left hand side of (3.3) is equal to the $\|.\|_{2,2}$ norm of

$$\begin{aligned} &\left[M_\eta,\left((I+i\tau D)^{-k}(zD)^N(I+izD)^{-\alpha}\right)\right]M_\chi\\ =\ &(I+i\tau D)^{-1}\left[M_\eta,\left((I+i\tau D)^{-(k-1)}(zD)^N(I+izD)^{-\alpha}\right)\right]M_\chi\\ &+\left[M_\eta,(I+i\tau D)^{-1}\right]\left((I+i\tau D)^{-(k-1)}(zD)^N(I+izD)^{-\alpha}\right)M_\chi. \end{aligned} \tag{3.4}$$

By the induction assumption, the $\|.\|_{2,2}$ norm of the first term is bounded by

$$\left\|\left[M_\eta,\left((I+i\tau D)^{-(k-1)}(zD)^N(I+izD)^{-\alpha}\right)\right]M_\chi\right\|_{2,2}\leq C\left(\frac{|z|}{\rho(E,F)}\right)^N.$$

The second term in (3.4) is equal to

$$(I+i\tau D)^{-1}z\left[D,M_\eta\right](i\tau D)(I+i\tau D)^{-k}(zD)^{N-1}(I+izD)^{-\alpha}M_\chi$$

and its $\|.\|_{2,2}$ norm is therefore bounded by

$$\begin{aligned}
&\left\|(I+i\tau D)^{-1}z\,[D,M_\eta]\,(I+i\tau D)^{-(k-1)}(zD)^{N-1}(I+izD)^{-\alpha}M_\chi\right\|_{2,2} \quad + \\
&\left\|(I+i\tau D)^{-1}z\,[D,M_\eta]\,(I+i\tau D)^{-k}(zD)^{N-1}(I+izD)^{-\alpha}M_\chi\right\|_{2,2} \\
&\leq |z|\,\||\nabla\eta|\|_\infty \left(\frac{|z|}{\rho(E,F)}\right)^{N-1} \\
&\leq C\left(\frac{|z|}{\rho(E,F)}\right)^{N},
\end{aligned}$$

where the penultimate inequality follows from the induction assumptions and the formula

$$D(\eta b) = \eta Db + d\eta\wedge b - d\eta\vee b, \tag{3.5}$$

where

$$\langle\alpha\vee\beta,\gamma\rangle := \langle\beta,\alpha\wedge\gamma\rangle.$$

This concludes the proof of (3.3), and therefore of Lemma 3.5. □

Proof of Lemma 3.6: We begin with assertion (a). First note that for $f\in\Psi(\Sigma^0_{\pi/2-\mu})$ and $0<r<R<\infty$, then

$$\left|\frac{1}{2i\pi}\int_{\zeta\in\gamma;r\leq|\zeta|\leq R} f(\zeta)\tfrac{1}{\zeta}\,d\zeta\right| \leq 2||f||_\infty\ .$$

To see this, apply Cauchy's theorem to change to an integral over four arcs. This fact is used to handle the second last term in the following expression.

$$\begin{aligned}
g_{(t)}(D)(tD)^N(I+itD)^{-\alpha} &= \frac{1}{2i\pi}\int_\gamma g_{(t)}(\zeta)(\zeta I-D)^{-1}(tD)^N(I+itD)^{-\alpha}\,d\zeta \\
&= \frac{t}{2i\pi}\int_{\zeta\in\gamma;|\zeta|<1/t} g_{(t)}(\zeta)\tfrac{1}{\zeta}D(I-\tfrac{1}{\zeta}D)^{-1}(tD)^{N-1}(I+itD)^{-\alpha}\,d\zeta \\
&\quad+\lim_{R\to\infty}\frac{1}{2i\pi}\int_{\zeta\in\gamma;1/t\leq|\zeta|\leq R} g_{(t)}(\zeta)\tfrac{1}{\zeta}\,(tD)^N(I+itD)^{-\alpha}\,d\zeta \\
&\quad+\lim_{R\to\infty}\frac{1}{2i\pi}\int_{\zeta\in\gamma;1/t\leq|\zeta|\leq R} g_{(t)}(\zeta)\tfrac{1}{t\zeta^2}(I-\tfrac{1}{\zeta}D)^{-1}(tD)^{N+1}(I+itD)^{-\alpha}\,d\zeta
\end{aligned}$$

Apply Lemmas 3.3 and 3.5 to see that each term satisfies $OD_t(N-1)$ estimates. A limiting argument gives the result for a family $(g_{(t)})_{t>0}$ uniformly bounded in $H^\infty(\Sigma^0_{\pi/2-\mu})$.

To prove assertion (b) in Lemma 3.6, apply assertion (a) with $g_t(z)=f(z)\psi_t(z)(tz)^{-N}(1+itz)^{(N+1)}$. □

Proof of Lemma 3.7: If $s\leq t$, write

$$\psi_s(D)f(D)\widetilde{\psi}_t(D) = \left(\frac{s}{t}\right)^a(sD)^{-a}\psi_s(D)f(D)(tD)^a\widetilde{\psi}_t(D) = \left(\frac{s}{t}\right)^a T_{s,t}$$

where

$$T_{s,t} = f_{(s)}(D)\widetilde{\widetilde{\psi}}_t(D)\ .$$

with $f_{(s)}(z) = (sz)^{-a}\psi(sz)f(z)$ and $\widetilde{\widetilde{\psi}}(z) = z^a\widetilde{\psi}(z)$. Now $f_{(s)} \in H^\infty(\Sigma^0_{\pi/2-\mu})$ with $\left\|f_{(s)}\right\|_\infty \leq C_1 \|f\|_\infty$, and $\widetilde{\widetilde{\psi}} \in \Psi_{N_2+a,\alpha_2-a}$, so Lemma 3.6 ensures that $T_{s,t}$ satisfies $OD_t(N_2+a-1)$ estimates with a constant not exceeding $C\|f\|_\infty$. The part $t \leq s$ is proved in a similar way. □

We remark that, since $\Delta = D^2$, Lemmata 3.3, 3.5, 3.6 and 3.7 imply similar off-diagonal estimates when $(tD)^N(I+itD)^{-\alpha}$ is replaced by $(t^2\Delta)^N(I+t^2\Delta)^{-\alpha}$ for appropriate N and α. Furthermore, we can strengthen these to "Gaffney" type estimates for the heat semigroup.

Lemma 3.8 *For all $N \geq 0$, there exists $C, \alpha > 0$ such that, for all disjoint closed subsets $E, F \subset M$ and all $t > 0$,*

$$\left\|M_{\chi_F}(t^2\Delta)^N e^{-t^2\Delta}M_{\chi_E}\right\|_{2,2} + \left\|M_{\chi_F}tD(t^2\Delta)^N e^{-t^2\Delta}M_{\chi_E}\right\|_{2,2} \leq Ce^{-\alpha\frac{\rho^2(E,F)}{t^2}}.$$

In particular, $((t^2\Delta)^N e^{-t^2\Delta})_{t>0}$ and $(tD(t^2\Delta)^N e^{-t^2\Delta})_{t>0}$ satisfy $OD_t(N')$ estimates for any integer $N' \geq 0$.

Proof: The proof of the estimate for the first term is analogous to [23] (this kind of estimate originated in Gaffney's work [27]) and [24], Lemma 7, whereas the second term can be estimated by the same method as in [2], estimate (3.1) p. 930. □

Observe that the same argument yields

$$\left\|M_{\chi_F}tde^{-t^2\Delta}M_{\chi_E}\right\|_{2,2} + \left\|M_{\chi_F}td^*e^{-t^2\Delta}M_{\chi_E}\right\|_{2,2} \leq Ce^{-\alpha\frac{\rho^2(E,F)}{t^2}}.$$

4 Tent spaces on M

4.1 Definition, atomic decomposition and duality for tent spaces

We first present tent spaces on M, following [15]. For all $x \in M$ and $\alpha > 0$, the cone of aperture α and vertex x is the set

$$\Gamma_\alpha(x) = \{(y,t) \in M \times (0,+\infty)\, ;\ y \in B(x,\alpha t)\}.$$

When $\alpha = 1$, $\Gamma_\alpha(x)$ will simply be denoted by $\Gamma(x)$. For any closed set $F \subset M$, let $\mathcal{R}(F)$ be the union of all cones with aperture 1 and vertices in F. Finally, if $O \subset M$ is an open set and $F = M \setminus O$, the tent over O, denoted by $T(O)$, is the complement of $\mathcal{R}(F)$ in $M \times (0,+\infty)$.

Let $F = (F_t)_{t>0}$ be a family of measurable sections of ΛT^*M. Write $F(y,t) := F_t(y)$ for all $y \in M$ and all $t > 0$ and assume that F is measurable on $M \times (0,+\infty)$. Define then, for all $x \in M$,

$$\mathcal{S}F(x) = \left(\iint_{\Gamma(x)} |F(y,t)|^2 \frac{dy}{V(x,t)}\frac{dt}{t}\right)^{1/2},$$

and, if $1 \leq p < +\infty$, say that $F \in T^{p,2}(\Lambda T^*M)$ if

$$\|F\|_{T^{p,2}(\Lambda T^*M)} := \|\mathcal{S}F\|_{L^p(M)} < +\infty.$$

Remark 4.1 *Assume that (1.2) holds. If $\alpha > 0$ and if we define, for all $x \in M$,*

$$\mathcal{S}_\alpha F(x) = \left(\iint_{\Gamma_\alpha(x)} |F(y,t)|^2 \, \frac{dy}{V(x,t)} \frac{dt}{t} \right)^{1/2},$$

then $\|F\|_{T^{p,2}(\Lambda T^*M)} \sim \|\mathcal{S}_\alpha F\|_{L^p(M)}$ for all $1 \leq p < +\infty$ (see [15]).

In order to ensure duality results for tent spaces, we do not define $T^{\infty,2}(\Lambda T^*M)$ in the same way. For any family $(F_t)_{t>0}$ of measurable sections of ΛT^*M and all $x \in M$, define

$$\text{\c{F}}(x) = \sup_{B \ni x} \left(\frac{1}{V(B)} \iint_{T(B)} |F(y,t)|^2 \, dy \frac{dt}{t} \right)^{1/2},$$

where the supremum is taken over all open balls B containing x, and say that $F \in T^{\infty,2}(\Lambda T^*M)$ if $\|F\|_{T^{\infty,2}(\Lambda T^*M)} := \|\text{\c{F}}\|_\infty < +\infty$.

We first state a density result for tent spaces which does not require (1.2)

Proposition 4.2 *When $1 \leq p < \infty$, then $T^{p,2}(\Lambda T^*M) \cap T^{2,2}(\Lambda T^*M)$ is dense in $T^{p,2}(\Lambda T^*M)$.*

Proof: Set

$$\mathcal{E} = \left\{ F \in T^{2,2}(\Lambda T^*M);\ F \text{ is bounded and has compact support in } M \times (0,+\infty) \right\},$$

which is obviously contained in $T^{p,2}(\Lambda T^*M) \cap T^{2,2}(\Lambda T^*M)$. Fix any point x_0 in M and, for all $n \geq 1$, define $\chi_n = \chi_{B(x_0,n) \times \left(\frac{1}{n}, n\right)}$. Then, it is easy to check that, for all $F \in T^{p,2}(\Lambda T^*M)$, if

$$F_n = \chi_n \chi_{\{(x,t) \in M \times (0,+\infty);\ |F(x,t)| < n\}} F$$

for all $n \geq 1$, then $F_n \in \mathcal{E}$ and $F_n \to F$ in $T^{p,2}(\Lambda T^*M)$. □

Remark 4.3 *The same argument shows that, if $F \in T^{p,2}(\Lambda T^*M) \cap T^{2,2}(\Lambda T^*M)$, then $\chi_n F \to F$ both in $T^{p,2}(\Lambda T^*M)$ and in $T^{2,2}(\Lambda T^*M)$.*

If we assume furthermore property (1.2), duality, atomic decomposition and interpolation results hold for tent spaces as in the Euclidean case. The proofs are analogous to the corresponding ones in [15], and we will therefore not write them down (see however [39] for the atomic decomposition for tent spaces on spaces of homogeneous type). Let us just mention that, apart from property (1.2), these proofs rely on the existence of $\alpha > 0$ such that, for all $r > 0$ and all $x, y \in M$ satisfying $\rho(x,y) < r$,

$$\mu(B(x,r) \cap B(y,r)) \geq \alpha V(x,r).$$

This last assertion follows from the definition of the geodesic distance on M, the completeness of M and the doubling property.

The duality for tent spaces is as follows:

Theorem 4.4 *Assume (1.2). Then:*

(*a*) *There exists* $C>0$ *such that, for all* $F\in T^{1,2}(\Lambda T^*M)$ *and all* $G\in T^{\infty,2}(\Lambda T^*M)$,

$$\iint_{M\times(0,+\infty)} |F(x,t)|\,|G(x,t)|\,dx\frac{dt}{t} \le C\int_M \mathcal{S}F(x)\mathcal{G}(x)dx.$$

(*b*) *The pairing* $\langle F,G\rangle \mapsto \iint_{M\times(0,+\infty)}\langle F(x,t),G(x,t)\rangle dx\frac{dt}{t}$ *realizes* $T^{\infty,2}(\Lambda T^*M)$ *as equivalent with the dual of* $T^{1,2}(\Lambda T^*M)$ *and* $T^{p',2}(\Lambda T^*M)$ *as equivalent with the dual of* $T^{p,2}(\Lambda T^*M)$ *if* $1<p<+\infty$ *and* $1/p+1/p'=1$.

In assertion (*b*) and in the sequel, $\langle .,.\rangle$ denotes the complex inner product in ΛT^*M.

The $T^{p,2}(\Lambda T^*M)$ spaces interpolate by the complex interpolation method:

Theorem 4.5 *Assume (1.2). Let* $1\le p_0<p<p_1\le+\infty$ *and* $\theta\in(0,1)$ *such that* $1/p=(1-\theta)/p_0+\theta/p_1$. *Then* $[T^{p_0,2}(\Lambda T^*M),T^{p_1,2}(\Lambda T^*M)]_\theta = T^{p,2}(\Lambda T^*M)$.

The $T^{1,2}(\Lambda T^*M)$ space admits an atomic decomposition. An atom is a function $A\in L^2\left((0,+\infty),L^2(\Lambda T^*M),dt/t\right)$ supported in $T(B)$ for some ball $B\subset M$ and satisfying

$$\iint_{T(B)} |A(x,t)|^2\,dx\frac{dt}{t}\le\frac{1}{V(B)}.$$

An atom belongs to $T^{1,2}(\Lambda T^*M)$ with a norm controlled by a constant only depending on M. It turns out that every $F\in T^{1,2}(\Lambda T^*M)$ has an atomic decomposition (see [39]):

Theorem 4.6 *Assume (1.2). There exists* $C>0$ *such that every* $F\in T^{1,2}(\Lambda T^*M)$ *can be written as* $F=\sum_j\lambda_jA_j$, *where the* A_j*'s are atoms and* $\sum_{j\ge0}|\lambda_j|\le C\left\Vert F\right\Vert_{T^{1,2}(\Lambda T^*M)}$.

Remark 4.7 *It is plain to see that, in the definition of an atom, up to changing the constant in Theorem 4.6, the tent* $T(B)$ *over the ball* B *can be replaced by the Carleson box*

$$\mathcal{B}(B)=B\times[0,r(B)]$$

where $r(B)$ *is the radius of* B.

We end up this section by a technical lemma for later use:

Lemma 4.8 *Assume (1.2).*

(*a*) *Let* $1\le p<+\infty$. *If* $(H_n)_{n\ge1}$ *is any sequence in* $T^{p,2}(\Lambda T^*M)$ *which converges to* $H\in T^{p,2}(\Lambda T^*M)$, *there exists an increasing map* $\varphi:\mathbb{N}^*\to\mathbb{N}^*$ *such that* $H_{\varphi(n)}(y,t)\to H(y,t)$ *for almost every* $(y,t)\in M\times(0,+\infty)$.

(*b*) *Let* $(H_n)_{n\ge1}$ *be a sequence in* $T^{1,2}(\Lambda T^*M)\cap T^{2,2}(\Lambda T^*M)$ *which converges to* H *in* $T^{1,2}(\Lambda T^*M)$ *and to* G *in* $T^{2,2}(\Lambda T^*M)$. *Then,* $H=G$.

Proof: For assertion (*a*), since, for all $j\ge1$, $\mathcal{S}_j(H_n-H)\to0$ in $L^p(\Lambda T^*M)$ (see Remark 4.1), a diagonal argument shows that, up to a subsequence, $\mathcal{S}_j(H_n-H)(x)\to0$ for all $j\ge1$ and almost every $x\in M$. Fix then $x\in M$ such that $\mathcal{S}_j(H_n-H)(x)\to0$ for all $j\ge1$. Thanks to a diagonal argument again, one has $|(H_n-H)(y,t)|\to0$ for almost every $(y,t)\in\Gamma_j(x)$, up to a subsequence, which gives the conclusion. Assertion (*b*) is an immediate consequence of assertion (*a*). □

4.2 The main estimate

Recall from Section 2 that $\mathcal{H} = L^2\left((0,+\infty), L^2(\Lambda T^*M), \frac{dt}{t}\right)$, equipped with the norm

$$\|F\|_{\mathcal{H}} = \left(\int_0^{+\infty}\int_M |F(x,t)|^2\,dx\frac{dt}{t}\right)^{1/2}.$$

It is easy to see that $\mathcal{H} = T^{2,2}(\Lambda T^*M)$ with equivalent norms.

The main result of the present section is the following theorem, which will play a crucial role in our definition of Hardy spaces via tent spaces on M, and in establishing the functional calculus for Hardy spaces.

Theorem 4.9 *Assume that M is a complete connected Riemannian manifold which satisfies the doubling property (1.2). Define κ as in (1.3), and let $\beta = \left[\frac{\kappa}{2}\right]+1$ (the smallest integer larger than $\frac{\kappa}{2}$) and $\theta \in \left(0,\frac{\pi}{2}\right)$. For given $\psi, \widetilde{\psi} \in \Psi(\Sigma_\theta^0)$ and $f \in H^\infty(\Sigma_\theta^0)$, define the bounded operator $Q_f : \mathcal{H} \to \mathcal{H}$ to be $Q_f = \mathcal{Q}_\psi f(D)\mathcal{S}_{\widetilde{\psi}}$, i.e.,*

$$Q_f(F)_s = \int_0^{+\infty} \psi_s(D) f(D)\widetilde{\psi}_t(D)F_t\frac{dt}{t}$$

for all $F \in \mathcal{H}$ and all $s > 0$. Suppose either
(a) $1 \leq p < 2$ and $\psi \in \Psi_{1,\beta+1}(\Sigma_\theta^0)$, $\widetilde{\psi} \in \Psi_{\beta,2}(\Sigma_\theta^0)$; or
(b) $2 < p \leq \infty$, and $\psi \in \Psi_{\beta,2}(\Sigma_\theta^0)$, $\widetilde{\psi} \in \Psi_{1,\beta+1}(\Sigma_\theta^0)$.

Then Q_f extends to a $T^{p,2}(\Lambda T^*M)$-bounded map, and, for all $F \in T^{p,2}(\Lambda T^*M)$,

$$\|Q_f(F)\|_{T^{p,2}(\Lambda T^*M)} \leq C_p \|f\|_\infty \|F\|_{T^{p,2}(\Lambda T^*M)}, \tag{4.1}$$

where $C_p > 0$ only depends on the constant in (1.2), κ, θ, p, ψ and $\widetilde{\psi}$.

Remark 4.10 *In the case when $\int_0^\infty \psi(\pm t)\widetilde{\psi}(\pm t)\frac{dt}{t} = 1$ and hence $\mathcal{S}_{\widetilde{\psi}}\mathcal{Q}_\psi h = h$ for all $h \in \mathcal{R}(D)$, then $Q_f Q_g = Q_{fg}$. In particular $\mathcal{P}_{\{\psi,\widetilde{\psi}\}} := Q_1$ is a bounded projection on $T^{p,2}(\Lambda T^*M)$ when $1 \leq p \leq \infty$. In order that these operators be the same for all p, choose $\psi = \widetilde{\psi} \in \Psi_{1,\beta+1}(\Sigma_\theta^0) \cap \Psi_{\beta,2}(\Sigma_\theta^0)$ and set $\mathcal{P}_\psi := \mathcal{P}_{\{\psi,\widetilde{\psi}\}} = \mathcal{Q}_\psi\mathcal{S}_\psi$. In this case we see that the spaces $\mathcal{P}_\psi T^{p,2}(\Lambda T^*M)$ interpolate by the complex method for $1 \leq p \leq \infty$.*

Proof of Theorem 4.9: This proof will be divided in several steps.

Step 1: The boundedness of Q_f in $T^{2,2}(\Lambda T^*M) = \mathcal{H}$ follows immediately from the results in Section 2.

Step 2: An inequality for $T^{1,2}(\Lambda T^*M)$ atoms. We now assume that $\psi \in \Psi_{1,\beta+1}(\Sigma_\theta^0)$ and $\widetilde{\psi} \in \Psi_{\beta,2}(\Sigma_\theta^0)$. Let us prove that, for any atom $A \in T^{1,2}(\Lambda T^*M)$,

$$\|Q_f(A)\|_{T^{1,2}(\Lambda T^*M)} \leq C. \tag{4.2}$$

Let A be an atom in $T^{1,2}(\Lambda T^*M)$. There exists a ball $B\subset M$ such that A is supported in $T(B)$ and

$$\iint_{T(B)} |A(x,t)|^2\,dx\frac{dt}{t}\leq V^{-1}(B).$$

Set $\widetilde{A}=Q_f(A)$, $\widetilde{A_1}=\widetilde{A}\chi_{T(4B)}$ and, for all $k\geq 2$, $\widetilde{A_k}=\widetilde{A}\chi_{T(2^{k+1}B)\setminus T(2^kB)}$, so that $\widetilde{A}=\sum\limits_{k\geq 1}\widetilde{A_k}$ (actually, we should truncate $\widetilde{A}$ by imposing $\delta\leq s\leq R$, obtain bounds independent of δ and R, and then let δ go to 0 and R to $+\infty$; we ignore this point and argue directly without this truncation, to simplify the notation).

We need to show that, for some $\varepsilon>0$ and $C>0$ independent of k,A and f, $\frac{2^{k\varepsilon}}{C}\widetilde{A_k}$ is a $T^{1,2}$ atom, which will prove that $\widetilde{A}\in T^{1,2}(\Lambda T^*M)$ with a controlled norm, by the atomic decomposition of $T^{1,2}(\Lambda T^*M)$. Since $\widetilde{A_k}$ is supported in $T\left(2^{k+1}B\right)$, it is enough to check that, for all $k\geq 1$,

$$\iint \left|\widetilde{A_k}(x,s)\right|^2 dx\frac{ds}{s}\leq \frac{C^2}{V(2^{k+1}B)}2^{-2k\varepsilon}. \tag{4.3}$$

For $k=1$, using the $T^{2,2}(\Lambda T^*M)$ boundedness of Q_f, the fact that $\|A\|_{\mathcal{H}}\leq V(B)^{-1/2}$ and the doubling property, one obtains

$$\left\|\widetilde{A_1}\right\|_{\mathcal{H}}\leq C\,\|A\|_{\mathcal{H}}\leq CV^{-1/2}(B)\leq C'V^{-1/2}(4B).$$

Fix now $k\geq 2$, and suppose $0<\delta<\beta-\kappa/2$. Applying Lemma 3.7 with $a=1,b=\beta,N_1=1,N_2=\beta,\alpha_1=\beta+1,\alpha_2=2$ and the fact that A is supported in $T(B)$, write

$$\widetilde{A}_s=Q(A)_s=\int_0^{+\infty}\psi_s(D)f(D)\widetilde{\psi}_t(D)A_t\frac{dt}{t}=\int_0^r h_{1,\beta}\left(\frac{s}{t}\right)T_{s,t}A_t\frac{dt}{t},$$

where r is the radius of B. The Cauchy-Schwarz inequality yields

$$\begin{aligned}\left|\widetilde{A_{k}}_s\right|^2 &\leq \left(\int_0^{+\infty} h_{1,\delta}\left(\frac{s}{t}\right)\frac{dt}{t}\right)\left(\int_0^r h_{1,2\beta-\delta}\left(\frac{s}{t}\right)|T_{s,t}A_t|^2\frac{dt}{t}\right)\\ &\leq C\int_0^r h_{1,2\beta-\delta}\left(\frac{s}{t}\right)|T_{s,t}A_t|^2\frac{dt}{t}.\end{aligned}$$

Since $\widetilde{A_k}$ is supported in $T\left(2^{k+1}B\right)\setminus T\left(2^kB\right)$, one may assume that $0<s<2^{k+1}r$. Moreover, if $s<2^{k-1}r$ and if (x,s) belongs to $T(2^{k+1}B)\setminus T(2^kB)$, then $x\in 2^{k+1}B\setminus 2^{k-1}B$, so that

$$\begin{aligned}\iint\left|\widetilde{A_k}(x,s)\right|^2 dx\frac{ds}{s} &\leq C\int_0^{2^{k-1}r}\int_0^r h_{1,2\beta-\delta}\left(\frac{s}{t}\right)\left\|\chi_{2^{k+1}B\setminus 2^{k-1}B}T_{s,t}A_t\right\|^2_{L^2(\Lambda T^*M)}\frac{dt}{t}\frac{ds}{s}\\ &+ C\int_{2^{k-1}r}^{2^{k+1}r}\int_0^r h_{1,2\beta-\delta}\left(\frac{s}{t}\right)\|T_{s,t}A_t\|^2_{L^2(\Lambda T^*M)}\frac{dt}{t}\frac{ds}{s}\ .\end{aligned} \tag{4.4}$$

Thanks to (1.3), the last integral in (4.4) is bounded by

$$\begin{aligned}C\int_{2^{k-1}r}^{2^{k+1}r}\int_0^r\left(\frac{t}{s}\right)^{2\beta-\delta}\|A_t\|^2_{L^2(\Lambda T^*M)}\frac{dt}{t}\frac{ds}{s} &\leq C\int_0^r\left(\frac{t}{2^kr}\right)^{2\beta-\delta}\|A_t\|^2_{L^2(\Lambda T^*M)}\frac{dt}{t}\\ &\leq C2^{-k(2\beta-\delta)}V^{-1}(B)\\ &\leq C2^{-k(2\beta-\delta-\kappa)}V^{-1}(2^{k+1}B),\end{aligned}$$

where we now need the fact that $2\beta-\delta-\kappa>0$. Moreover, Lemma 3.7 yields that $(T_{s,t})_{s\geq t}$ satisfies $OD_s(\beta)$ estimates, and hence $OD_s(\beta-\delta)$ estimates by Remark 3.2. The first integral on the right hand side of (4.4) is therefore dominated by

$$
\begin{aligned}
&C\int_0^r \|A_t\|^2_{L^2(\Lambda T^*M)}\left(\int_0^t \left(\frac{s}{t}\right)\left(\frac{t}{2^kr}\right)^{2\beta}\frac{ds}{s}+\int_t^{2^{k-1}r}\left(\frac{t}{s}\right)^{2\beta-\delta}\left(\frac{s}{2^kr}\right)^{2\beta-2\delta}\frac{ds}{s}\right)\frac{dt}{t} \leq\\
&C\int_0^r \|A_t\|^2_{L^2(\Lambda T^*M)}\left(\left(\frac{t}{2^kr}\right)^{2\beta}+\left(\frac{t}{2^kr}\right)^{2\beta-2\delta}\right)\frac{dt}{t} \leq\\
&C2^{-k(2\beta-2\delta)}\int_0^r \|A_t\|^2_{L^2(\Lambda T^*M)}\frac{dt}{t}\leq C2^{-k(2\beta-2\delta)}V^{-1}(B)\leq C2^{-k(2\beta-2\delta-\kappa)}V^{-1}(2^{k+1}B)\ ,
\end{aligned}
$$

using Lemma 3.7. We now need the fact that $2\beta-2\delta-\kappa>0$ to complete the proof of (4.2).

Step 3: conclusion of the proof when $p=1$. Consider again ψ and $\widetilde{\psi}$ as in assertion (a) of Theorem 4.9. Observe first that the extension of Q_f to a $T^{1,2}(\Lambda T^*M)$-bounded operator does not follow at once from (4.2). Indeed, up to this point, Q_f is only defined on $T^{2,2}(\Lambda T^*M)$, and our task is to define it properly on $T^{1,2}(\Lambda T^*M)$. One way to do this could be to observe that, by Theorem 4.6, any element $F\in T^{1,2}(\Lambda T^*M)$ has an atomic decomposition $F=\sum_j \lambda_j A_j$, and to define $Q_f(F)=\sum_j \lambda_j Q_f(A_j)$ (which converges in $T^{1,2}(\Lambda T^*M)$), but we should then check that this definition does not depend on the decomposition of F (which is not unique). Here, we argue differently. Since, by Proposition 4.2, $T^{1,2}(\Lambda T^*M)\cap T^{2,2}(\Lambda T^*M)$ is dense in $T^{1,2}(\Lambda T^*M)$, it is enough to show that there exists $C>0$ such that, for all $F\in T^{1,2}(\Lambda T^*M)\cap T^{2,2}(\Lambda T^*M)$, (4.1) holds for F with $p=1$.

Consider such an F, and write $F=\sum_j \lambda_j A_j$ where $\sum_j |\lambda_j|\sim \|F\|_{T^{1,2}(\Lambda T^*M)}$ and, for each $j\geq 1$, A_j is a $T^{1,2}(\Lambda T^*M)$ atom supported in $B_j\times[0,r_j]$ (r_j denotes the radius of B_j). By Remark 4.3, if $x_0\in M$ and $\chi_n=\chi_{B(x_0,n)\times\left(\frac{1}{n},n\right)}$, then $F_n:=\chi_n F$ converges to F both in $T^{1,2}(\Lambda T^*M)$ and in $T^{2,2}(\Lambda T^*M)$. For all $n\geq 1$, F_n has an atomic decomposition in $T^{1,2}(\Lambda T^*M)$:

$$
F_n=\sum_j \lambda_j(\chi_n A_j), \tag{4.5}
$$

where, for each j, $\chi_n A_j$ is a $T^{1,2}(\Lambda T^*M)$ atom. In particular, The series in (4.5) clearly converges in $T^{1,2}(\Lambda T^*M)$, but we claim that it also converges in $T^{2,2}(\Lambda T^*M)$. This relies on the following observation:

Fact 4.11 *For all $n\geq 1$, there exists $\kappa_n>0$ such that, for all $j\geq 1$, if $V(B_j)\leq\kappa_n$, then* $\left(B(x_0,n)\times\left(\frac{1}{n},n\right)\right)\cap(B_j\times[0,r_j])=\emptyset$.

Proof of the fact: We claim that

$$
\kappa_n=\frac{V(x_0,n)}{C(1+4n^2)^\kappa},
$$

where C and κ appear in (1.3), does the job. Indeed, assume now that $V(B_j)\leq\kappa_n$. If $B(x_0,n)\cap B_j=\emptyset$, there is nothing to do. Otherwise, let $y\in B(x_0,n)\cap B_j$, and write

$B_j = B(x_j, r_j)$. The doubling property yields

$$\begin{array}{rcl} V(x_0,n) & \leq & V(x_j, n+d(x_0,y)+d(y,x_j)) \\ & \leq & V(x_j, 2n+r_j) \\ & \leq & CV(B_j)\left(1+\dfrac{2n}{r_j}\right)^{\kappa}. \end{array}$$

Since $V(B_j) \leq \kappa_n$, it follows at once that $r_j \leq \frac{1}{2n}$, which obviously implies the desired conclusion. □

This fact easily implies that the series in (4.5) converges in $T^{2,2}(\Lambda T^*M)$. Indeed, we can drop in this series all the j's such that $V(B_j) \leq \kappa_n$, and, if $V(B_j) > \kappa_n$,

$$\|\chi_n A_j\|_{T^{2,2}(\Lambda T^*M)} \leq V(B_j)^{-1/2} \leq \kappa_n^{-1/2},$$

which proves the convergence (remember that $\sum |\lambda_j| < +\infty$).

As a consequence,

$$Q_f(F_n) = \sum_j \lambda_j Q_f(\chi_n A_j), \tag{4.6}$$

and this series converges in $T^{2,2}(\Lambda T^*M)$. But, since $\|Q_f(\chi_n A_j)\|_{T^{1,2}(\Lambda T^*M)} \leq C$ for all $j \geq 1$, the series in the right-hand side of (4.6) also converges in $T^{1,2}(\Lambda T^*M)$ to some $G \in T^{1,2}(\Lambda T^*M)$, and, according to Lemma 4.8, $G = Q_f(F_n)$. Therefore,

$$\|Q_f(F_n)\|_{T^{1,2}(\Lambda T^*M)} \leq \sum_j |\lambda_j| \, \|Q_f(\chi_n A_j)\|_{T^{1,2}(\Lambda T^*M)} \leq C \, \|F\|_{T^{1,2}(\Lambda T^*M)}. \tag{4.7}$$

Let us now prove that $(Q_f(F_n))_{n\geq 1}$ is a Cauchy sequence in $T^{1,2}(\Lambda T^*M)$. From (4.6), one has

$$Q_f(F_n - F_m) = \sum_j \lambda_j Q_f((\chi_n - \chi_m)A_j),$$

where the series converges in $T^{1,2}(\Lambda T^*M)$. Thus,

$$\|Q_f(F_n - F_m)\|_{T^{1,2}(\Lambda T^*M)} \leq C \sum_j |\lambda_j| \, \|Q_f((\chi_n - \chi_m)A_j)\|_{T^{1,2}(\Lambda T^*M)}.$$

Fix now $\varepsilon > 0$. There exists $J \geq 2$ such that $\sum\limits_{j \geq J} |\lambda_j| < \varepsilon$. Moreover, for each $1 \leq j \leq J-1$, there exists $N_j \geq 1$ such that, for all $n \geq N_j$, $\chi_n A_j = A_j$. Therefore, there exists $N \geq 1$ such that, for all $n, m \geq N$ and all $1 \leq j \leq J-1$, $(\chi_n - \chi_m)A_j = 0$. As a consequence,

$$\|Q_f(F_n - F_m)\|_{T^{1,2}(\Lambda T^*M)} \leq C\varepsilon$$

for all $n, m \geq N$. Since $(Q_f(F_n))_{n\geq 1}$ is a Cauchy sequence in $T^{1,2}(\Lambda T^*M)$, $Q_f(F_n) \to U$ in $T^{1,2}(\Lambda T^*M)$ for some $U \in T^{1,2}(\Lambda T^*M)$. Moreover, since $F_n \to F$ in $T^{2,2}(\Lambda T^*M)$, $Q_f(F_n) \to Q_f(F)$ in $T^{2,2}(\Lambda T^*M)$, and a new application of Lemma 4.8 yields $Q_f(F) = U$. It follows that $Q_f(F_n) \to Q_f(F)$ in $T^{1,2}(\Lambda T^*M)$. Letting n go to $+\infty$ in (4.7) gives the desired result, which ends up the proof of the case $p = 1$.

Step 4: End of the proof. Using the interpolation results for tent spaces (Theorem 4.5), we obtain the conclusion of Theorem 4.9 for all $1 \leq p \leq 2$. Finally, the duality for tent spaces (Theorem 4.4) also yields this conclusion for $p \geq 2$ (note that the assumptions on ψ and $\widetilde{\psi}$ have been switched), which ends the proof of Theorem 4.9. □

From now on, we constantly assume the doubling property (1.2)**.**

5 Definition of Hardy spaces and first results

5.1 Definition and first properties of Hardy spaces

We are now able to give the definition of the $H^p(\Lambda T^*M)$ space for all $1 \leq p \leq +\infty$, $p \neq 2$, by means of quadratic functionals (as was done for $H^2(\Lambda T^*M)$ in Section 2) and tent spaces. Theorem 4.9 tells us that we have to distinguish between the cases $1 \leq p < 2$ and $2 < p \leq +\infty$.

Given $\psi \in \Psi(\Sigma_\theta^0)$ for some $\theta > 0$, set $\psi_t(z) = \psi(tz)$ for all $t > 0$ and all $z \in \Sigma_\theta^0$. Recall that the operator $\mathcal{S}_\psi : T^{2,2}(\Lambda T^*M) \longrightarrow L^2(\Lambda T^*M)$ is defined by

$$\mathcal{S}_\psi H = \int_0^{+\infty} \psi_t(D) H_t \frac{dt}{t}$$

and $\mathcal{Q}_\psi : L^2(\Lambda T^*M) \longrightarrow T^{2,2}(\Lambda T^*M)$ by

$$\left(\mathcal{Q}_\psi h\right)_t = \psi_t(D) h$$

for all $h \in L^2(\Lambda T^*M)$ and all $t > 0$.

Definition 5.1 *For each $\psi \in \Psi(\Sigma_\theta^0)$, define $E^p_{D,\psi}(\Lambda T^*M) = \mathcal{S}_\psi(T^{p,2}(\Lambda T^*M) \cap T^{2,2}(\Lambda T^*M))$ with semi-norm*

$$\|h\|_{H^p_{D,\psi}(\Lambda T^*M)} = \inf\{\|H\|_{T^{p,2}(\Lambda T^*M)} \, ; H \in T^{p,2}(\Lambda T^*M) \cap T^{2,2}(\Lambda T^*M), \mathcal{S}_\psi H = h\} \ .$$

The case when $1 \leq p < 2$: Recall that $\beta = \left[\frac{\kappa}{2}\right] + 1$ (the smallest integer larger than $\frac{\kappa}{2}$). It turns out that, provided $\psi \in \Psi_{\beta,2}(\Sigma_\theta^0)$, $E^p_{D,\psi}(\Lambda T^*M)$ is actually independent from the choice of ψ, and can be described by means of the operators $\mathcal{Q}_{\widetilde{\widetilde{\psi}}}$ if $\widetilde{\widetilde{\psi}} \in \Psi_{1,\beta+1}(\Sigma_\theta^0)$ (see Section 2) :

Lemma 5.2 *If $\psi, \widetilde{\psi} \in \Psi_{\beta,2}(\Sigma_\theta^0)$ and $\widetilde{\widetilde{\psi}} \in \Psi_{1,\beta+1}(\Sigma_\theta^0)$, then*

$$E^p_{D,\psi}(\Lambda T^*M) = E^p_{D,\widetilde{\psi}}(\Lambda T^*M) = \{h \in H^2(\Lambda T^*M) \ ; \ \|\mathcal{Q}_{\widetilde{\widetilde{\psi}}} h\|_{T^{p,2}(\Lambda T^*M)} < \infty\}$$

with norm

$$\|h\|_{H^p_{D,\psi}(\Lambda T^*M)} \sim \|h\|_{H^p_{D,\widetilde{\psi}}(\Lambda T^*M)} \sim \|\mathcal{Q}_{\widetilde{\widetilde{\psi}}} h\|_{T^{p,2}(\Lambda T^*M)} \ .$$

Proof: Fix $\psi, \widetilde{\psi}$ and $\widetilde{\widetilde{\psi}}$ as in Lemma 5.2. Observe first that, if $\varphi \in \Psi_{1,\beta+1}(\Sigma_\theta^0)$, then Theorem 4.9 tells us that $\mathcal{Q}_\varphi \mathcal{S}_\psi$ extends to a bounded operator in $T^{p,2}(\Lambda T^*M)$.
(a) First, let $h \in E^p_{D,\psi}(\Lambda T^*M)$ and $\varphi \in \Psi_{1,\beta+1}(\Sigma_\theta^0)$. There exists $H \in T^{p,2}(\Lambda T^*M)$ with $\|H\|_{T^{p,2}(\Lambda T^*M)} \leq 2\|h\|_{H^p_{D,\psi}(\Lambda T^*M)}$ such that $h = \mathcal{S}_\psi H$, so that $h \in H^2(\Lambda T^*M)$ and, because of our observation, $\|\mathcal{Q}_\varphi h\|_{T^{p,2}(\Lambda T^*M)} \leq C\|H\|_{T^{p,2}(\Lambda T^*M)} \leq C\|h\|_{H^p_{D,\psi}(\Lambda T^*M)}$. In particular, $E^p_{D,\psi}(\Lambda T^*M) \subset \{h \in H^2(\Lambda T^*M)\ ;\ \|\mathcal{Q}_{\widetilde{\widetilde{\psi}}} h\|_{T^{p,2}(\Lambda T^*M)} < \infty\}$.
(b) Assume now that $h \in H^2(\Lambda T^*M)$ and $\mathcal{Q}_{\widetilde{\widetilde{\psi}}} h \in T^{p,2}(\Lambda T^*M)$. We claim that there exists $\zeta \in \Psi_{\beta,2}(\Sigma_\theta^0)$ such that $h \in E^p_{D,\zeta}(\Lambda T^*M)$. Indeed, by Remark 2.1, there exists $\zeta \in \Psi_{\beta,2}(\Sigma_\theta^0)$ such that $\mathcal{S}_\zeta \mathcal{Q}_{\widetilde{\widetilde{\psi}}} = \mathrm{Id}$ on $H^2(\Lambda T^*M)$. Therefore, if $H = Q_{\widetilde{\widetilde{\psi}}} h$, one has $h = \mathcal{S}_\zeta H$, which shows that $\|h\|_{H^p_{D,\zeta}(\Lambda T^*M)} \leq \left\|Q_{\widetilde{\widetilde{\psi}}} h\right\|_{T^{p,2}(\Lambda T^*M)}$.
(c) We check now that, if ζ is as in step (b) and $h \in E^p_{D,\zeta}(\Lambda T^*M)$, then $h \in E^p_{D,\psi}(\Lambda T^*M)$. Indeed, thanks to Remark 2.1 again, there exists $\varphi \in \Psi_{1,\beta+1}(\Sigma_\theta^0)$ such that $\mathcal{S}_\psi \mathcal{Q}_\varphi = \mathrm{Id}$ on $H^2(\Lambda T^*M)$. According to (a), $\|\mathcal{Q}_\varphi h\|_{T^{p,2}(\Lambda T^*M)} \leq C\|h\|_{H^p_{D,\zeta}(\Lambda T^*M)}$. Since $h \in H^2(\Lambda T^*M)$, one has $h = \mathcal{S}_\psi \mathcal{Q}_\varphi h$, which shows that $\|h\|_{H^p_{D,\psi}(\Lambda T^*M)} \leq C\|h\|_{H^p_{D,\zeta}(\Lambda T^*M)}$.
(d) It remains to be shown that $\|h\|_{H^p_{D,\psi}(\Lambda T^*M)}$ is a norm rather than a seminorm on $E^p_{D,\psi}(\Lambda T^*M)$. Let $h \in E^p_{D,\psi}(\Lambda T^*M)$ with $\|h\|_{H^p_{D,\psi}(\Lambda T^*M)} = \|\mathcal{Q}_{\widetilde{\widetilde{\psi}}} h\|_{T^{p,2}(\Lambda T^*M)} = 0$. Then $h \in H^2(\Lambda T^*M) \cap \mathcal{N}(\mathcal{Q}_{\widetilde{\widetilde{\psi}}}) = \overline{\mathcal{R}(D)} \cap \mathcal{N}(D) = \{0\}$. *i.e.* $h = 0$ as required. □

Remark 5.3 *It follows that these spaces and maps are independent of $\theta \in (0, \frac{\pi}{2})$ too.*

Proposition 5.4 *With the notation of Lemma 5.2, $\{h \in \mathcal{R}(D)\ ;\ \|\mathcal{Q}_{\widetilde{\widetilde{\psi}}} h\|_{T^{p,2}(\Lambda T^*M)} < \infty\}$ is dense in $E^p_{D,\psi}(\Lambda T^*M)$ for all $\widetilde{\widetilde{\psi}} \in \Psi_{1,\beta+1}(\Sigma_\theta^0)$.*

Proof: Fix $\widetilde{\widetilde{\psi}} \in \Psi_{1,\beta+1}(\Sigma_\theta^0)$ and choose $\widetilde{\psi} \in \Psi_{\beta,2}(\Sigma_\theta^0)$ such that $\mathcal{S}_{\widetilde{\psi}} \mathcal{Q}_{\widetilde{\widetilde{\psi}}} h = h$ for all $h \in H^2(\Lambda T^*M)$. For a given $h \in E^p_{D,\psi}(\Lambda T^*M)$, set $H = \mathcal{Q}_{\widetilde{\widetilde{\psi}}} h \in T^{p,2}(\Lambda T^*M) \cap T^{2,2}(\Lambda T^*M)$, and define, for each natural number N, $H_N \in T^{p,2}(\Lambda T^*M) \cap T^{2,2}(\Lambda T^*M)$ by $H_N(x,t) = H(x,t)\chi_{[\frac{1}{N},N]}(t)$. It is not difficult to show that $H_N \to H$ in $T^{p,2}(\Lambda T^*M)$, and so $h_N := \mathcal{S}_{\widetilde{\psi}} H_N \to h$ in $E^p_{D,\psi}(\Lambda T^*M) = E^p_{D,\widetilde{\psi}}(\Lambda T^*M)$.

It remains to be shown that $h_N \in \mathcal{R}(D)$. This holds because

$$h_N = \int_{\frac{1}{N}}^N \widetilde{\psi}_t(D) H_t \frac{dt}{t} = D \int_{\frac{1}{N}}^N \phi(tD) H_t \, dt$$

where $\phi \in H^\infty(S^0_\mu)$ is defined by $\phi(z) = \frac{1}{z}\widetilde{\psi}(z)$. □

We are now in a position to define the *Hardy spaces* associated with D.

Definition 5.5 *Suppose $1 \leq p < 2$. Define $H^p_D(\Lambda T^*M)$ to be the completion of $E^p_{D,\psi}(\Lambda T^*M)$ under any of the equivalent norms $\|h\|_{H^p_{D,\psi}(\Lambda T^*M)}$ with $\psi \in \Psi_{\beta,2}(\Sigma_\theta^0)$, which we write as just $\|h\|_{H^p_D(\Lambda T^*M)}$.*

In particular,

$$H_D^p(\Lambda T^*M) = \overline{\{h \in \mathcal{R}(D)\ ;\ \|\mathcal{Q}_\psi h\|_{T^{p,2}(\Lambda T^*M)} < \infty\}}$$

under the norm $\|\mathcal{Q}_\psi h\|_{T^{p,2}(\Lambda T^*M)}$ for any $\psi \in \Psi_{1,\beta+1}(\Sigma_\theta^0)$. For example,

$$\begin{aligned} \|h\|_{H_D^p(\Lambda T^*M)} &\sim \left\| tDe^{-t\sqrt{\Delta}} h \right\|_{T^{p,2}(\Lambda T^*M)} \\ &\sim \left\| t^2 \Delta e^{-t^2\Delta} h \right\|_{T^{p,2}(\Lambda T^*M)} \\ &\sim \left\| tD(I + t^2\Delta)^{-N} h \right\|_{T^{p,2}(\Lambda T^*M)} \end{aligned}$$

where $N \geq \frac{\beta}{2} + 1$.

The case when $2 < p < \infty$: The same procedure works, but with the roles of $\Psi_{\beta,2}(\Sigma_\theta^0)$ and $\Psi_{1,\beta+1}(\Sigma_\theta^0)$ interchanged.

Definition 5.6 *Suppose $2 < p < \infty$. Define $H_D^p(\Lambda T^*M)$ to be the completion of $E_{D,\psi}^p(\Lambda T^*M)$ under any of the equivalent norms $\|h\|_{H_{D,\psi}^p(\Lambda T^*M)}$ with $\psi \in \Psi_{1,\beta+1}(\Sigma_\theta^0)$, which we write as just $\|h\|_{H_D^p(\Lambda T^*M)}$.*

In particular,

$$H_D^p(\Lambda T^*M) = \overline{\{h \in \mathcal{R}(D)\ ;\ \|\mathcal{Q}_\psi h\|_{T^{p,2}(\Lambda T^*M)} < \infty\}}$$

under the norm $\|\mathcal{Q}_\psi h\|_{T^{p,2}(\Lambda T^*M)}$ for any $\psi \in \Psi_{\beta,2}(\Sigma_\theta^0)$. For example,

$$\begin{aligned} \|h\|_{H_D^p(\Lambda T^*M)} &\sim \left\| (tD)^\beta e^{-t\sqrt{\Delta}} h \right\|_{T^{p,2}(\Lambda T^*M)} \\ &\sim \left\| (t^2 \Delta)^M e^{-t^2\Delta} h \right\|_{T^{p,2}(\Lambda T^*M)} \\ &\sim \left\| (tD)^\beta (I + t^2\Delta)^{-N} h \right\|_{T^{p,2}(\Lambda T^*M)} \end{aligned}$$

where $M \geq \frac{\beta}{2}$ and $N \geq \frac{\beta}{2} + 1$.

Suppose that the function ψ used in any of the above norms is an even function. Then $\psi_t(D) = \widetilde{\psi}(t^2\Delta)$, where $\widetilde{\psi} \in \Psi(\Sigma_{2\theta+}^0)$. We thus see that we have defined Hardy spaces $H_\Delta^p(\Lambda T^*M)$ corresponding to the Laplacian Δ, and that they are the same as the spaces $H_D^p(\Lambda T^*M)$. From now on, for all $1 \leq p < +\infty$, the $H_D^p(\Lambda T^*M)$ space, which coincides with $H_\Delta^p(\Lambda T^*M)$, will be denoted by $H^p(\Lambda T^*M)$.

We define $H^\infty(\Lambda T^*M)$ in a different way. This definition relies on the following lemma:

Lemma 5.7 *Let $\psi \in \Psi_{1,\beta+1}(\Sigma_\theta^0)$.*

(a) Let $G \in T^{\infty,2}(\Lambda T^*M)$. Then the map T_G, initially defined on $E_D^1(\Lambda T^*M)$ by $T_G(f) = \iint \langle (\mathcal{Q}_\psi f)_t (x), G(x,t) \rangle dx \frac{dt}{t}$, extends in a unique way to a bounded linear functional on $H^1(\Lambda T^*M)$, denoted again by T_G.

(*b*) *Conversely, if* U *is a bounded linear functional on* $H^1(\Lambda T^*M)$*, there exists* $G \in T^{\infty,2}(\Lambda T^*M)$ *such that* $U = T_G$.

The proof is an immediate consequence of assertion (*b*) in Theorem 4.4 and the definition of $E^1_{D,\psi}(\Lambda T^*M)$. We define $H^\infty(\Lambda T^*M)$ as the dual space of $H^1(\Lambda T^*M)$, equipped with the usual dual norm. Observe that, by Lemma 5.7, one has

$$\|U\|_{H^\infty(\Lambda T^*M)} \sim \inf \left\{ \|G\|_{T^{\infty,2}(\Lambda T^*M)} \, ; \; U = T_G \right\}.$$

Theorems 4.4 and 4.5 yield duality and interpolation results for Hardy spaces:

Theorem 5.8 *The pairing* $\langle g, h\rangle \mapsto \int_M \langle g(x), h(x)\rangle dx$ *realizes* $H^{p'}(\Lambda T^*M)$ *as equivalent with the dual of* $H^p(\Lambda T^*M)$ *if* $1 < p < +\infty$ *and* $1/p + 1/p' = 1$*. Moreover, by definition, the dual of* $H^1(\Lambda T^*M)$ *is* $H^\infty(\Lambda T^*M)$.

Theorem 5.9 *Let* $1 \le p_0 < p < p_1 \le +\infty$ *and* $\theta \in (0,1)$ *such that* $1/p = (1-\theta)/p_0 + \theta/p_1$*. Then* $[H^{p_0}(\Lambda T^*M), H^{p_1}(\Lambda T^*M)]_\theta = H^p(\Lambda T^*M)$.

Proof: The spaces $\mathcal{P}_\psi T^{p,2}(\Lambda T^*M)$ defined in Remark 4.10 interpolate by the complex method for $1 \le p \le \infty$, where we have taken $\psi \in \Psi_{1,\beta+1}(\Sigma^0_\theta) \cap \Psi_{\beta,2}(\Sigma^0_\theta)$ with $\int_0^\infty \psi(\pm t)\psi(\pm t)\frac{dt}{t} = 1$ and defined the projection $\mathcal{P}_\psi := \mathcal{Q}_\psi \mathcal{S}_\psi$. It is straightforward to see that the map $\mathcal{Q}_\psi$ extends to an isomorphism from $\mathcal{P}_\psi T^{p,2}(\Lambda T^*M)$ to $H^p(\Lambda T^*M)$ with inverse $\mathcal{S}_\psi$ for each p, and that these maps coincide for different values of p. The result follows. □

Remark 5.10 *Since* $H^\infty(\Lambda T^*M)$ *is the dual space of* $H^1(\Lambda T^*M)$*, it turns out that* $H^\infty(\Lambda T^*M)$ *is actually a BMO-type space. Recall that* $BMO(\mathbb{R}^n)$ *is the dual space of* $H^1(\mathbb{R}^n)$ *([26]) and that similar duality results have been established for other kinds of Hardy spaces, in particular in [32] for Hardy spaces of exact differential forms in* $\mathbb{R}^n$*. To keep homogeneous notations and simplify our previous and foregoing statements about Hardy spaces, we write* $H^\infty(\Lambda T^*M)$ *instead of* $BMO(\Lambda T^*M)$.

5.2 Riesz transform and Functional calculus

We are now ready to prove the first part of Corollary 1.3, namely

Theorem 5.11 *For all* $1 \le p \le +\infty$*, the Riesz transform* $D\Delta^{-1/2}$*, initially defined on* $\mathcal{R}(\Delta)$*, extends to a* $H^p(\Lambda T^*M)$*-bounded operator. More precisely, one has* $\left\|D\Delta^{-1/2}h\right\|_{H^p(\Lambda T^*M)} \sim \|h\|_{H^p(\Lambda T^*M)}$.

Proof: The case when $p = 2$ is in Section 2. Consider now the case when $1 \le p < +\infty$ and $p \ne 2$. Choose $\psi \in \Psi_{1,\beta+1}(\Sigma^0_\theta)$ when $1 \le p < 2$, and $\psi \in \Psi_{\beta,2}(\Sigma^0_\theta)$ when $2 < p < \infty$. In either case the holomorphic function $\widetilde{\psi}$ defined by $\widetilde{\psi}(z) = \mathrm{sgn}(\mathrm{Re}\, z)\psi(z)$ belongs to the same space, and moreover $\widetilde{\psi}(D) = D\Delta^{-1/2}\psi(D)$. Hence, by Lemma 5.2,

$$\left\|D\Delta^{-1/2}h\right\|_{H^p(\Lambda T^*M)} \sim \left\|D\Delta^{-1/2}\psi(D)h\right\|_{T^{p,2}(\Lambda T^*M)} = \left\|\widetilde{\psi}(D)h\right\|_{T^{p,2}(\Lambda T^*M)} \sim \|h\|_{H^p(\Lambda T^*M)}$$

for all $h \in H^p(\Lambda T^*M)$. The case $p = +\infty$ follows from the case $p = 1$ by duality. $\square$

Similar estimates actually give the following more general result on the holomorphic functional calculus of D:

Theorem 5.12 *For all $1 \le p \le +\infty$, $f(D)$ is $H^p(\Lambda T^*M)$-bounded for all $f \in H^\infty(\Sigma_\theta^0)$ with $\|f(D)h\|_{H^p(\Lambda T^*M)} \le C \|f\|_\infty \|h\|_{H^p(\Lambda T^*M)}$.*

When $1 \le p < \infty$, this estimate follows from Theorem 4.9 and the definitions of $H^p(\Lambda T^*M)$. When $p = \infty$, use duality.

Let us finish this section by discussing the boundedness of the Hodge-Riesz transforms. Let $1 \le p \le +\infty$, and denote by n the dimension of M. First, the splitting $\Lambda T^*M = \oplus_{0\le k\le n}\Lambda^k T^*M$ allows us to define naturally $H^p(\Lambda^k T^*M)$ for all $0 \le k \le n$ (first for $1 \le p < +\infty$, then for $p = +\infty$ by duality), and one has, if $f = (f_0, \dots, f_n) \in \Lambda T^*M$,

$$\|f\|_{H^p(\Lambda T^*M)} \sim \sum_{k=0}^{n} \|f_k\|_{H^p(\Lambda^k T^*M)} . \tag{5.1}$$

To see this when $1 \le p < \infty$, recall that $H^p(\Lambda^k T^*M) = H^p_\Delta(\Lambda^k T^*M)$, and note that Δ preserves the decomposition into k-forms. Specializing Theorem 5.11 to k forms implies that, for all $0 \le k \le n-1$, $d\Delta^{-1/2}$ is $H^p(\Lambda^k T^*M) - H^p(\Lambda^{k+1}T^*M)$ bounded, and that, for all $1 \le k \le n$, $d^*\Delta^{-1/2}$ is $H^p(\Lambda^k T^*M) - H^p(\Lambda^{k-1}T^*M)$ bounded. Using (5.1) we have obtained:

Theorem 5.13 *For all $1 \le p \le +\infty$, $d\Delta^{-1/2}$ and $d^*\Delta^{-1/2}$ are both $H^p(\Lambda T^*M)$ bounded.*

5.3 The Hodge decomposition for $H^p(\Lambda T^*M)$

We can define other Hardy spaces, associated to the operators d and d^*, which leads us to a Hodge decomposition for $H^p(\Lambda T^*M)$. Recall from Section 2 that

$$H^2(\Lambda T^*M) = H^2_d(\Lambda T^*M) \oplus H^2_{d^*}(\Lambda T^*M),$$

where $H^2_d(\Lambda T^*M) = \overline{\mathcal{R}(d)}$ and $H^2_{d^*}(\Lambda T^*M) = \overline{\mathcal{R}(d^*)}$ and that the orthogonal projections are given by dD^{-1} and d^*D^{-1}.

For $1 \le p < +\infty$ and $p \ne 2$, set

$$H^p_d(\Lambda T^*M) = \overline{\mathcal{R}(d) \cap H^p(\Lambda T^*M)},\ H^p_{d^*}(\Lambda T^*M) = \overline{\mathcal{R}(d^*) \cap H^p(\Lambda T^*M)}$$

where the closure is taken in the $H^p(\Lambda T^*M)$ topology. We have the following Hodge decomposition for $H^p(\Lambda T^*M)$:

Theorem 5.14 *For all $1 \le p < +\infty$, one has $H^p(\Lambda T^*M) = H^p_d(\Lambda T^*M) \oplus H^p_{d^*}(\Lambda T^*M)$, and the sum is topological.*

Proof: The orthogonal projection dD^{-1} from $H^2(\Lambda T^*M)$ to $H^2_d(\Lambda T^*M)$ defines a bounded operator from $H^p(\Lambda T^*M)$ to $H^p_d(\Lambda T^*M)$. Indeed, $dD^{-1} = d\Delta^{-1/2}D\Delta^{-1/2}$, $D\Delta^{-1/2}$ is $H^p(\Lambda T^*M)$ bounded by Theorem 5.11 and $d\Delta^{-1/2}$ is $H^p(\Lambda T^*M) - H^p_d(\Lambda T^*M)$ bounded by Theorem 5.13. Similarly, d^*D^{-1} is a bounded operator from $H^p(\Lambda T^*M)$ to $H^p_{d^*}(\Lambda T^*M)$. Since, for all $f \in H^p(\Lambda T^*M)$, one has $f = dD^{-1}f + d^*D^{-1}f$, the theorem is proved. □

Note that, for all $1 \leq p < +\infty$, $H^p_d(\Lambda T^*M)$ and $H^p_{d^*}(\Lambda T^*M)$ can also be described by means of tent spaces in the same way as $H^p(\Lambda T^*M)$. More precisely, if $\psi \in \Psi_{1,\tau}(\Sigma^0_\theta)$ for some $\tau, \theta > 0$, define $\phi \in H^\infty(\Sigma^0_\theta)$ by $\phi(z) = \frac{1}{z}\psi(z)$ and then define, for $H \in T^{2,2}(\Lambda T^*M)$,

$$\mathcal{S}_{d,\psi}(H) = \int_0^{+\infty} td\phi_t(D)H_t\frac{dt}{t} \text{ and } \mathcal{S}_{d^*,\psi}(H) = \int_0^{+\infty} td^*\phi_t(D)H_t\frac{dt}{t},$$

and, for all $h \in L^2(\Lambda T^*M)$, define

$$(\mathcal{Q}_{d,\psi}h)_t = td\phi_t(D)h \text{ and } (\mathcal{Q}_{d^*,\psi}h)_t = td^*\phi_t(D)h.$$

Then, for $1 \leq p < +\infty$, replacing $\mathcal{S}_{\widetilde{\psi}}$ by $\mathcal{S}_{d,\widetilde{\psi}}$ (resp. by $\mathcal{S}_{d^*,\widetilde{\psi}}$) and $\mathcal{Q}_\psi$ by $\mathcal{Q}_{d^*,\psi}$ (resp. $\mathcal{Q}_{d,\psi}$) in Section 5.1, one obtains a characterization of $H^p_d(\Lambda T^*M)$ (resp. $H^p_{d^*}(\Lambda T^*M)$) by means of $\mathcal{S}_{d,\widetilde{\psi}}$ and $\mathcal{Q}_{d^*,\psi}$ (resp. $\mathcal{S}_{d^*,\widetilde{\psi}}$ and $\mathcal{Q}_{d,\psi}$), provided that $\psi \in \Psi_{1,\beta+1}(\Sigma^0_\theta)$ and $\widetilde{\psi} \in \Psi_{\beta,2}(\Sigma^0_\theta)$ if $1 \leq p < 2$, and $\psi \in \Psi_{\beta,2}(\Sigma^0_\theta)$ and $\widetilde{\psi} \in \Psi_{1,\beta+1}(\Sigma^0_\theta)$ if $2 < p < \infty$.

It is plain to observe that, if $1 < p < +\infty$, the dual of $H^p_d(\Lambda T^*M)$ is isomorphic to $H^{p'}_d(\Lambda T^*M)$, where $1/p + 1/p' = 1$. We define $H^\infty_d(\Lambda T^*M)$ (resp. $H^\infty_{d^*}(\Lambda T^*M)$) as the dual space of $H^1_d(\Lambda T^*M)$ (resp. $H^1_{d^*}(\Lambda T^*M)$). Lemma 5.7 provides another description of $H^\infty_d(\Lambda T^*M)$ and $H^\infty_{d^*}(\Lambda T^*M)$. Namely, fix $\widetilde{\psi} \in \Psi_{1,\beta+1}(\Sigma^0_\theta)$. For all $G \in T^{\infty,2}(\Lambda T^*M)$ and all $f \in H^1(\Lambda T^*M) \cap \mathcal{R}(d)$, define

$$T_{d^*,G}(f) = \iint \langle (\mathcal{Q}_{d^*,\psi}f)_t(x), G(x,t)\rangle dx\frac{dt}{t}.$$

Then, $T_{d^*,G}$ is a bounded linear functional on $H^1_d(\Lambda T^*M)$, and conversely, any bounded linear functional U on $H^1_d(\Lambda T^*M)$ is equal to $T_{d^*,G}$ for some $G \in T^{\infty,2}(\Lambda T^*M)$. Furthermore, $\|T_{d^*,G}\| \sim \|G\|_{T^{\infty,2}(\Lambda T^*M)}$. The description of $(H^1_{d^*}(\Lambda T^*M))'$ is similar.

As a consequence we have a more precise statement for the Hodge-Riesz transforms.

Theorem 5.15 *Let $1 \leq p \leq +\infty$. Then $d\Delta^{-1/2}$ extends to a continuous isomorphism from $H^p_{d^*}(\Lambda T^*M)$ onto $H^p_d(\Lambda T^*M)$ and $d^*\Delta^{-1/2}$ to a continuous isomorphism from $H^p_d(\Lambda T^*M)$ onto $H^p_{d^*}(\Lambda T^*M)$. These operators are inverse to one another.*

Proof: Assume first that $1 \leq p < +\infty$. Theorem 5.13 shows that $d\Delta^{-1/2}$ extends to a $H^p(\Lambda T^*M)$-bounded linear map, and the very definition of $H^p_d(\Lambda T^*M)$ therefore ensures that it is $H^p_{d^*}(\Lambda T^*M) - H^p_d(\Lambda T^*M)$ bounded. Similarly, $d^*\Delta^{-1/2}$ extends to a $H^p_d(\Lambda T^*M) - H^p_{d^*}(\Lambda T^*M)$ bounded map. Next, for $f \in H^p_d(\Lambda T^*M) \cap \mathcal{R}(d)$, $(d\Delta^{-1/2})(d^*\Delta^{-1/2})f = dd^*\Delta^{-1}f = f$ since $d^*df = 0$, which shows that $d\Delta^{-1/2}$ is onto $H^p_d(\Lambda T^*M)$ and $d^*\Delta^{-1/2}$

is one-to-one from $H^p_d(\Lambda T^*M)$. Symmetrically, $d^*\Delta^{-1/2}$ is onto $H^p_{d^*}(\Lambda T^*M)$ and $d\Delta^{-1/2}$ is one-to-one from $H^p_{d^*}(\Lambda T^*M)$.

Finally, the conclusion for $p=+\infty$ follows from the case $p=1$ by duality. The proof is straightforward and relies on the fact that $d^*\Delta^{-1/2}=\Delta^{-1/2}d^*$ on a dense subspace of $H^1_d(\Lambda T^*M)$. □

To finich, let us specialize the above to k-forms. For $1\leq p\leq +\infty$, one can also naturally define $H^p_d(\Lambda^k T^*M)$ for all $1\leq k\leq n$ and $H^p_{d^*}(\Lambda^k T^*M)$ for all $0\leq k\leq n-1$ (first for $1\leq p<+\infty$, then using duality for $p=+\infty$), and Theorem 5.15 shows the following result:

Theorem 5.16 *Let $1\leq p\leq +\infty$.*

(a) For all $0\leq k\leq n-1$, $d\Delta^{-1/2}$ is a continuous isomorphism from $H^p_{d^*}(\Lambda^k T^*M)$ onto $H^p_d(\Lambda^{k+1}T^*M)$.

(b) For all $1\leq k\leq n$, $d^*\Delta^{-1/2}$ is a continuous isomorphism from $H^p_d(\Lambda^k T^*M)$ onto $H^p_{d^*}(\Lambda^{k-1}T^*M)$.

6 The decomposition into molecules

As recalled in the introduction, an essential feature of the classical $H^1(\mathbb{R}^n)$ space is that every function in $H^1(\mathbb{R}^n)$ admits an atomic decomposition. Recall that an atom in $H^1(\mathbb{R}^n)$ is a measurable function $a\in L^2(\mathbb{R}^n)$, supported in a ball B, with zero integral and satisfying $\left\Vert a\right\Vert_2\leq |B|^{-1/2}$. The Coifman-Latter theorem says that an integrable function f belongs to $H^1(\mathbb{R}^n)$ if and only if it can be written as

$$f=\sum_{k\geq 1}\lambda_k a_k$$

where $\sum_k |\lambda_k|<+\infty$ and the a_k's are atoms. Moreover, $\left\Vert f\right\Vert_{H^1(\mathbb{R}^n)}$ is comparable with the infimum of $\sum |\lambda_k|$ over all such decompositions.

In [32], Lou and the second author establish an atomic decomposition for $H^1_d(\mathbb{R}^n,\Lambda^k)$ for all $1\leq k\leq n$. In this context, an atom is a form $a\in L^2(\mathbb{R}^n,\Lambda^k)$ such that there exists $b\in L^2(\mathbb{R}^n,\Lambda^{k-1})$ supported in a ball $B\subset\mathbb{R}^n$ with radius r, $a=db$ and $\left\Vert a\right\Vert_2+r^{-1}\left\Vert b\right\Vert_2\leq |B|^{-1/2}$. Note that the cancellation condition (in the case of functions) is replaced by the fact that an atom is the image of some other form under d (that is, a is exact), which implies in particular that $da=0$ whenever a in an atom. The proof relies on a classical result due to Necas ([37], Lemma 7.1, Chapter 3) and on ([42], Theorem 3.3.3, Chapter 3).

In the present section, we prove a "molecular" decomposition for $H^1(\Lambda T^*M)$ inspired by the Coifman-Weiss terminology (see the introduction). In our context, we do not know how to get atoms with compact support. Roughly speaking, a "molecule" is a form f in $L^2(\Lambda T^*M)$ which is the image under D^N of some $g\in L^2(\Lambda T^*M)$, with L^2 decay for f and g, and for some integer N large enough.

To be more precise, we adopt the following terminology. Fix $C > 0$. If $B \subset M$ is a ball with radius r and if $(\chi_k)_{k\geq 0}$ is a sequence of nonnegative C^∞ functions on M with bounded support, say that $(\chi_k)_{k\geq 0}$ is adapted to B if χ_0 is supported in $4B$, χ_k is supported in $2^{k+2}B \setminus 2^{k-1}B$ for all $k \geq 1$,

$$\sum_{k\geq 0} \chi_k = 1 \text{ on } M \text{ and } \left\| |\nabla \chi_k| \right\|_\infty \leq \frac{C}{2^k r}, \tag{6.1}$$

where $C > 0$ only depends on M. Note that, when $C > 0$ is large enough, there exist sequences adapted to any fixed ball.

Let N be a positive integer. If $a \in L^2(\Lambda T^*M)$, a is called an N-molecule if and only if there exists a ball $B \subset M$ with radius r, $b \in L^2(\Lambda T^*M)$ such that $a = D^N b$, and a sequence $(\chi_k)_{k\geq 0}$ adapted to B such that, for all $k \geq 0$,

$$\|\chi_k a\|_{L^2(\Lambda T^*M)} \leq 2^{-k} V^{-1/2}(2^k B) \text{ and } \|\chi_k b\|_{L^2(\Lambda T^*M)} \leq 2^{-k} r^N V^{-1/2}(2^k B). \tag{6.2}$$

Note that the first set of estimates in (6.2) imply

$$\|a\|_{L^2(\Lambda T^*M)} \leq 2V^{-1/2}(B) \text{ and } \|b\|_{L^2(\Lambda T^*M)} \leq 2r^N V^{-1/2}(B)\ . \tag{6.3}$$

Thus $a \in \mathcal{R}(D) \subset H^2(\Lambda T^*M)$. Furthermore, any N-molecule a belongs to $L^1(\Lambda T^*M)$ and one has

$$\|a\|_{L^1(\Lambda T^*M)} \leq 2C \tag{6.4}$$

where C is the constant in (1.2).

Definition 6.1 *Say that a section f belongs to $H^1_{mol,N}(\Lambda T^*M)$ if there exists a sequence $(\lambda_j)_{j\geq 1} \in l^1$ and a sequence of N-molecules $(a_j)_{j\geq 1}$ such that*

$$f = \sum_{j\geq 1} \lambda_j a_j,$$

and define $\|f\|_{H^1_{mol,N}(\Lambda T^*M)}$ as the infimum of $\sum |\lambda_j|$ over all such decompositions.

It is plain to see that $H^1_{mol,N}(\Lambda T^*M)$ is a Banach space. We prove in this section the following

Theorem 6.2 *Assume (1.2) and let κ is given by (1.3). Then, for integers $N > \frac{\kappa}{2} + 1$, $H^1_{mol,N}(\Lambda T^*M) = H^1(\Lambda T^*M)$. As a consequence, $H^1_{mol,N}(\Lambda T^*M)$ is independent of N provided that $N > \frac{\kappa}{2} + 1$.*

Corollary 6.3 *(a) For $1 \leq p \leq 2$, $H^p(\Lambda T^*M) \subset \overline{\mathcal{R}(D) \cap L^p(\Lambda T^*M)}^{L^p(\Lambda T^*M)}$.*

(b) For $2 \leq p < +\infty$, $\overline{\mathcal{R}(D) \cap L^p(\Lambda T^*M)}^{L^p(\Lambda T^*M)} \subset H^p(\Lambda T^*M)$.

Proof of Corollary 6.3: For assertion (a), the inclusion $H^1(\Lambda T^*M) \subset L^1(\Lambda T^*M)$ is an immediate consequence of Theorem 6.2 and of (6.4). Since $H^2(\Lambda T^*M) \subset L^2(\Lambda T^*M)$, we obtain by interpolation (Theorem 5.9) that $H^p(\Lambda T^*M) \subset L^p(\Lambda T^*M)$. Therefore

$$\begin{aligned} H^p(\Lambda T^*M) &= \overline{\mathcal{R}(D) \cap H^p(\Lambda T^*M)}^{H^p(\Lambda T^*M)} \\ &\subset \overline{\mathcal{R}(D) \cap L^p(\Lambda T^*M)}^{L^p(\Lambda T^*M)}. \end{aligned}$$

For assertion (b), observe first that, for all $1 \leq p' \leq 2$, there exists $C > 0$ such that, for all $G \in T^{p',2}(\Lambda T^*M) \cap T^{2,2}(\Lambda T^*M)$,

$$\|\mathcal{S}_\psi G\|_{L^{p'}(\Lambda T^*M)} \leq C \|G\|_{T^{p',2}(\Lambda T^*M)} \tag{6.5}$$

where $\psi \in \Psi_{\beta,2}(\Sigma_\theta^0)$ with $\beta = \left[\frac{\kappa}{2}\right] + 1$. Indeed, $\mathcal{S}_\psi G \in H^{p'}(\Lambda T^*M)$, and therefore belongs to $L^{p'}(\Lambda T^*M)$ by assertion (a).

Let $p \geq 2$ and $f \in \mathcal{R}(D) \cap L^p(\Lambda T^*M)$. For all $G \in T^{p',2}(\Lambda T^*M) \cap T^{2,2}(\Lambda T^*M)$ where $1/p + 1/p' = 1$, since $\mathcal{Q}_\psi^* = \mathcal{S}_{\overline{\psi}}$ (see Section 2), one obtains, using (6.5),

$$\begin{aligned} \left| \iint \langle (\mathcal{Q}_\psi f)_t(x), G(x,t) \rangle dx \frac{dt}{t} \right| &= \left| \int_M \langle f(x), \mathcal{S}_{\overline{\psi}} G(x) \rangle dx \right| \\ &\leq \|f\|_{L^p(\Lambda T^*M)} \left\| \mathcal{S}_{\overline{\psi}} G \right\|_{L^{p'}(\Lambda T^*M)} \\ &\leq C \|f\|_{L^p(\Lambda T^*M)} \|G\|_{T^{p',2}(\Lambda T^*M)}, \end{aligned}$$

which shows that $\|\mathcal{Q}_\psi f\|_{T^{p,2}(\Lambda T^*M)} \leq C \|f\|_{L^p(\Lambda T^*M)}$ (remember that, by Proposition 4.2, $T^{p',2}(\Lambda T^*M) \cap T^{2,2}(\Lambda T^*M)$ is dense in $T^{p',2}(\Lambda T^*M)$), therefore $\|f\|_{H^p(\Lambda T^*M)} \leq C \|f\|_{L^p(\Lambda T^*M)}$. Next, if $f \in \overline{\mathcal{R}(D) \cap L^p(\Lambda T^*M)}^{L^p(\Lambda T^*M)}$, there exists a sequence $(f_j)_{j \geq 1} \in \mathcal{R}(D) \cap L^p(\Lambda T^*M)$ which converges to f in the $L^p(\Lambda T^*M)$ norm, therefore in the $H^p(\Lambda T^*M)$ norm, which shows that $f \in H^p(\Lambda T^*M)$. □

Remark 6.4 *Note that the inclusion $H^1(\Lambda T^*M) \subset L^1(\Lambda T^*M)$ did not seem to be an immediate consequence of the definition of $H^1(\Lambda T^*M)$.*

Remark 6.5 *What assertion (b) in Corollary 6.3 tells us is that, for all $\psi \in \Psi_{\beta,2}(\Sigma_0^\theta)$, for all $2 \leq p < +\infty$, there exists $C > 0$ such that, for all $f \in \mathcal{R}(D) \cap L^p(\Lambda T^*M)$, $\mathcal{Q}_\psi f \in T^{p,2}(\Lambda T^*M)$ and*

$$\|\mathcal{Q}_\psi f\|_{T^{p,2}(\Lambda T^*M)} \leq C \|f\|_{L^p(\Lambda T^*M)}. \tag{6.6}$$

In the Euclidean case and with the Laplacian on functions, this inequality is nothing but the well-known L^p-boundedness of the so-called Lusin area integral (for $p \geq 2$, it follows directly from the L^p-boundedness of the vertical quadratic g function and the L^p boundedness of the Hardy-Littlewood maximal function, see for instance [44], p. 91). In the context of spaces of homogeneous type, the L^p boundedness (for all $1 < p < +\infty$) of the area integral associated to an operator L was proved in [3] under the following assumptions: L is the generator of a holomorphic semigroup acting on L^2, the kernel of which satisfies Gaussian upper bounds, and L has a bounded holomorphic calculus on L^2. Note that, in the framework of the present paper, we do not require any Gaussian upper estimate for the heat kernel of the Hodge Laplacian to obtain (6.6) for $p \geq 2$.

As a consequence of Corollary 6.3 and Theorem 5.11, we obtain the last part of Corollary 1.3:

Corollary 6.6 *Assume (1.2). Then $D\Delta^{-1/2}$ is $H^1(\Lambda T^*M) - L^1(\Lambda T^*M)$ bounded.*

In Section 8 below, this theorem will be compared with previously known results for the Riesz transform on manifolds.

The proof of Theorem 6.2 will be divided into two subsections, each corresponding to one inclusion.

6.1 $H^1(\Lambda T^*M) \subset H^1_{mol,N}(\Lambda T^*M)$ for all $N \geq 1$.

Fix $N \geq 1$ and set $\beta = \left[\frac{\kappa}{2}\right] + 1$ as usual. Choose $M \geq \max\{\beta, N\}$ and define $\psi(z) = z^M(1+iz)^{-M-2} \in \Psi_{M,2}(\Sigma^0_+)$ and let $\phi(z) = z^{M-N}(1+iz)^{-M-2}$ so that $\psi(z) = z^N\phi(z)$. It is enough to prove that $E^1_{D,\psi}(\Lambda T^*M) \subset H^1_{mol,N}(\Lambda T^*M)$, which means that, if

$$f = \int_0^{+\infty} \psi_t(D)F_t \frac{dt}{t}$$

with $F \in T^{1,2}(\Lambda T^*M) \cap T^{2,2}(\Lambda T^*M)$, then $f \in H^1_{mol,N}(\Lambda T^*M)$. According to the atomic decomposition for tent spaces (Theorem 4.6), one may assume that $F = A$ is a $T^{1,2}(\Lambda T^*M)$-atom, supported in $T(B)$ where B is a ball in M with radius r. Let

$$g = \int_0^{+\infty} t^N\phi_t(D)A_t \frac{dt}{t}$$

so that $f = D^N g$, and let $(\chi_k)_{k\geq 0}$ be a sequence adapted to B. We claim that, up to a multiplicative constant, f is a N-molecule, which gives the desired conclusion. To show this, we just have to establish the following L^2 estimates for f and g:

Lemma 6.7 *There exists $C > 0$ only depending on M such that, for all $k \geq 0$,*

$$\|\chi_k f\|_{L^2(\Lambda T^*M)} \leq C2^{-k}V^{-1/2}(2^kB),\ \|\chi_k g\|_{L^2(\Lambda T^*M)} \leq Cr^N 2^{-k}V^{-1/2}(2^kB).$$

Proof: We first deal with the estimates for f. First, since D is self-adjoint, one has

$$\|f\|^2_{L^2(\Lambda T^*M)} \leq C\int_M \int_0^{+\infty} |A(x,t)|^2\, dx\frac{dt}{t} \leq CV(B)^{-1}.$$

This shows that $\|\chi_0 f\|_{L^2(\Lambda T^*M)} \leq \|f\|_2 \leq CV(B)^{-1/2}$.

Fix now $k \geq 1$ and $m \geq \dfrac{\kappa}{2} + 1$. Lemma 3.3 and the fact that A is supported in $T(B)$ yield

$$\begin{aligned}
\|\chi_k f\|_{L^2(\Lambda T^*M)} &\leq \int_0^r \left\|\chi_{2^{k+1}B\setminus 2^{k-1}B}\psi_t(D)A_t\right\|_{L^2(\Lambda T^*M)} \frac{dt}{t} \\
&\leq C\int_0^r \left(\frac{t}{2^kr}\right)^m \|A_t\|_{L^2(\Lambda T^*M)} \frac{dt}{t} \\
&\leq C\left(\int_0^r \left(\frac{t}{2^kr}\right)^{2m} \frac{dt}{t}\right)^{1/2} \left(\int_0^r \|A_t\|^2_{L^2(\Lambda T^*M)} \frac{dt}{t}\right)^{1/2} \\
&\leq C(2^kr)^{-m}r^mV(B)^{-1/2} \\
&\leq C2^{-k\left(m-\frac{\kappa}{2}\right)}V^{-1/2}(2^kB).
\end{aligned}$$

We now turn to the estimates on g. First,

$$\begin{aligned}
\|g\|_{L^2(\Lambda T^*M)} &\leq \int_0^r t^N \|A_t\|_{L^2(\Lambda T^*M)} \frac{dt}{t} \\
&\leq \left(\int_0^r t^{2N}\frac{dt}{t}\right)^{1/2} \left(\int_0^r \|A_t\|^2_{L^2(\Lambda T^*M)} \frac{dt}{t}\right)^{1/2} \\
&\leq Cr^N V(B)^{-1/2},
\end{aligned}$$

which shows that $\|\chi_0 g\|_{L^2(\Lambda T^*M)} \leq \|g\|_2 \leq Cr^N V(B)^{-1/2}$.

Fix now $k \geq 1$ and $m \geq \frac{\kappa}{2} + 1$. Lemma 3.3 yields

$$\begin{aligned}
\|\chi_k g\|_{L^2(\Lambda T^*M)} &\leq \int_0^r t^N \left\|\chi_{2^{k+1}B\setminus 2^{k-1}B}\phi_t(D)A_t\right\|_{L^2(\Lambda T^*M)} \frac{dt}{t} \\
&\leq C\int_0^r t^N \left(\frac{t}{2^k r}\right)^m \|A_t\|_{L^2(\Lambda T^*M)} \frac{dt}{t} \\
&\leq C(2^k r)^{-m} \left(\int_0^r t^{2N+2m}\frac{dt}{t}\right)^{1/2} \left(\int_0^r \|A_t\|^2_{L^2(\Lambda T^*M)} \frac{dt}{t}\right)^{1/2} \\
&\leq C2^{-km} r^N V^{-1/2}(B) \\
&\leq Cr^N V^{-1/2}(2^k B) 2^{-k\left(m-\frac{\kappa}{2}\right)}.
\end{aligned}$$

This completes the proof of Lemma 6.7, and provides the desired inclusion. □

6.2 $H^1_{mol,N}(\Lambda T^*M) \subset H^1(\Lambda T^*M)$ for all $N > \frac{\kappa}{2} + 1$.

For the converse inclusion, it is enough to prove that there exists $C > 0$ such that, for every N- molecule f in $H^1_{mol,N}(\Lambda T^*M)$, $f \in H^1(\Lambda T^*M)$ with $\|f\|_{H^1(\Lambda T^*M)} \leq C$.

Let f be such a N-molecule. Since $f \in \mathcal{R}(D)$, according to Lemma 5.2, it suffices to show that, if $F(x,t) = (\mathcal{Q}_\psi f)_t(x)$ with $\psi(z) = z(1+iz)^{-\gamma-2} \in \Psi_{1,\beta+1}(\Sigma^0_\theta)$ for some $\gamma > N-1$, then

$$\|F\|_{T^{1,2}(\Lambda T^*M)} \leq C, \tag{6.7}$$

There exists a ball B, a section $g \in L^2(\Lambda T^*M)$ and a sequence $(\chi_k)_{k\geq 0}$ adapted to B such that $f = D^N g$ and (6.2) holds. Define

$$\eta_0 = \chi_{2B\times(0,2r)}$$

and, for all $k \geq 1$,

$$\eta_k = \chi_{(2^{k+1}B\setminus 2^k B)\times(0,r)},\ \eta'_k = \chi_{(2^{k+1}B\setminus 2^k B)\times\left(r,2^{k+1}r\right)},\ \eta''_k = \chi_{2^k B\times\left(2^k r,2^{k+1}r\right)},$$

where these functions χ_S are the (un-smoothed) characteristic functions of $S \subset M \times (0,\infty)$. Write

$$F = \eta_0 F + \sum_{k\geq 1}\eta_k F + \sum_{k\geq 1}\eta'_k F + \sum_{k\geq 1}\eta''_k F.$$

The estimate (6.7) will be an immediate consequence of the following

Lemma 6.8 *(a) For each $k \geq 0$, $\|\eta_k F\|_{T^{1,2}(\Lambda T^*M)} \leq C2^{-k}$.*

(b) For each $k \geq 1$, $\|\eta'_k F\|_{T^{1,2}(\Lambda T^*M)} \leq C2^{-k}$.

(c) For each $k \geq 1$, $\|\eta''_k F\|_{T^{1,2}(\Lambda T^*M)} \leq C2^{-k}$.

Proof:

Assertion (a)**:** Since $\eta_k F$ is supported in the box $\mathcal{B}(2^{k+1}B)$ (see Remark 4.7), we just have to prove that its $T^{2,2}(\Lambda T^*M)$ norm is controlled by $C2^{-k}V^{-1/2}(2^kB)$ (recall that the $T^{2,2}(\Lambda T^*M)$ norm is equivalent to the norm in $\mathcal{H}$, see Section 4.2), which will prove that $\frac{1}{C}2^k\eta_k F$ is an atom in $T^{1,2}(\Lambda T^*M)$. First, by the spectral theorem, one has

$$\begin{aligned}
\|\eta_0 F\|^2_{T^{2,2}(\Lambda T^*M)} &\leq \|F\|^2_{T^{2,2}(\Lambda T^*M)} \\
&\leq C\int_0^{+\infty} \|\psi_t(D)f\|^2_{L^2(\Lambda T^*M)}\frac{dt}{t} \\
&\leq C\,\|f\|^2_{L^2(\Lambda T^*M)} \\
&\leq CV(B)^{-1}.
\end{aligned}$$

Fix now $k \geq 1$. One has

$$\begin{aligned}
\|\eta_k F\|_{T^{2,2}(\Lambda T^*M)} &\leq \sum_{l\geq 0}\left\|\chi_{(2^{k+1}B\setminus 2^kB)\times(0,r)}\psi_t(D)(\chi_l f)\right\|_{T^{2,2}(\Lambda T^*M)} \\
&:= \sum_{l\geq 0} I_l.
\end{aligned}$$

Assume that $0 \leq l \leq k-2$. Then, using (1.3), Lemma 3.3, (6.2) for f and the fact that $\rho\left(\text{supp }\chi_l f, 2^{k+1}B\setminus 2^kB\right) \geq c(2^k - 2^l)r$ and choosing $m \geq \frac{\kappa}{2}+1$,

$$\begin{aligned}
I_l^2 &= \int_0^r\left(\int_{2^{k+1}B\setminus 2^kB}|\psi_t(D)(\chi_l f)(x)|^2\,dx\right)\frac{dt}{t} \\
&\leq C\int_0^r\left(\frac{t}{(2^k-2^l)r}\right)^{2m}\|\chi_l f\|^2_{L^2(\Lambda T^*M)}\frac{dt}{t} \\
&\leq C2^{k(\kappa-2m)}2^{-l(\kappa+2)}V^{-1}(2^kB).
\end{aligned}$$

It follows that

$$\sum_{l=0}^{k-2}\left\|\chi_{(2^{k+1}B\setminus 2^kB)\times(0,r)}\psi_t(D)(\chi_l f)\right\|_{T^{2,2}(\Lambda T^*M)} \leq C2^{-k}V^{-1/2}(2^kB).$$

Assume now that $k-2 \leq l \leq k+2$. Then, by the spectral theorem,

$$\begin{aligned}
I_l^2 &\leq \int_0^{+\infty}\|\psi_t(D)(\chi_l f)\|^2_{L^2(\Lambda T^*M)}\frac{dt}{t} \\
&\leq C\,\|\chi_l f\|^2_{L^2(\Lambda T^*M)} \\
&\leq C2^{-2k}V^{-1}(2^kB).
\end{aligned}$$

Assume finally that $l \geq k+3$. Then, using Lemma 3.3 and $\rho\left(\text{supp } \chi_l f, 2^{k+1}B \setminus 2^k B\right) \geq c2^l r$, and choosing $m \geq \frac{\kappa}{2}+1$,

$$\begin{aligned} I_l^2 &\leq \int_0^r \left(\frac{t}{2^l r}\right)^{2m} \|\chi_l f\|^2_{L^2(\Lambda T^*M)} \frac{dt}{t} \\ &\leq C2^{-2lm}2^{-2l}V^{-1}(2^k B). \end{aligned}$$

As a consequence,

$$\sum_{l=k+3}^{\infty} \left\|\chi_{(2^{k+1}B\setminus 2^k B)\times(0,r)}\psi_t(D)(\chi_l f)\right\|_{T^{2,2}(\Lambda T^*M)} \leq C2^{-k}V^{-1/2}(2^k B).$$

This ends the proof of assertion (a) in Lemma 6.8.

Assertion (b)**:** Similarly, we now estimate

$$\begin{aligned} \|\eta_k' F\|_{T^{2,2}(\Lambda T^*M)} &\leq \sum_{l\geq 0} \left\|\chi_{(2^{k+1}B\setminus 2^k B)\times(r,2^{k+1}r)}\psi_t(D)D^N(\chi_l g)\right\|_{T^{2,2}(\Lambda T^*M)} \\ &:= \sum_{l\geq 0} J_l. \end{aligned}$$

Define now $\widetilde{\psi}(z) = z^N\psi(z) \in \Psi_{N+1,\gamma+1-N}(\Sigma_\theta^0)$, so that

$$J_l^2 = \int_r^{2^{k+1}r} \left\|\chi_{2^{k+1}B\setminus 2^k B}\widetilde{\psi}_t(D)(\chi_l g)\right\|^2_{L^2(\Lambda T^*M)} \frac{dt}{t^{2N+1}}.$$

Assume first that $0 \leq l \leq k-2$. Then, (6.2) applied to g, Lemma 3.3 and the support conditions on χ_l yield

$$\begin{aligned} J_l^2 &\leq C\int_r^{2^{k+1}r} \left(\frac{t}{(2^k-2^l)r}\right)^{2m} \|\chi_l g\|^2_{L^2(\Lambda T^*M)} \frac{dt}{t^{2N+1}} \\ &\leq C2^{-l(\kappa+2)}2^{-k(2m-\kappa)}V^{-1}(2^k B), \end{aligned}$$

if m is chosen so that $\kappa+2 \leq 2m < 2N$, which is possible since $N > \frac{\kappa}{2}+1$. This yields

$$\sum_{l=0}^{k-2} \left\|\chi_{(2^{k+1}B\setminus 2^k B)\times(r,2^{k+1}r)}\psi_t(D)D^N(\chi_l g)\right\|_{L^2(\Lambda T^*M)} \leq C2^{-k}V^{-1/2}(2^k B).$$

Assume now that $k-1 \leq l \leq k+1$. Then one has

$$J_l^2 \leq C2^{-2k}V^{-1}(2^k B).$$

Assume finally that $l \geq k+2$. Then, using Lemma 3.3 and the support conditions again,

$$\begin{aligned} J_l^2 &\leq \int_r^{2^{k+1}r} \left(\frac{t}{2^l r}\right)^{2m} \|\chi_l g\|^2_{L^2(\Lambda T^*M)} \frac{dt}{t^{2N+1}} \\ &\leq C2^{-2(m+1)l}V^{-1}(2^k B), \end{aligned}$$

provided that $m < N$. Thus,

$$\sum_{l=k+2}^{+\infty} \left\| \chi_{(2^{k+1}B \setminus 2^k B)\times(r,2^{k+1}r)} \psi_t(D) D^N(\chi_l g) \right\|_{T^{2,2}(\Lambda T^*M)} \le C 2^{-k} V^{-1/2}(2^k B)$$

and assertion (b) is proved.

Assertion (c)**:** Finally, as in assertions (a) and (b), we have to estimate

$$\begin{aligned} \|\eta_k'' F\|_{T^{2,2}(\Lambda T^*M)} &\le \sum_{l\ge 0} \left\| \chi_{2^k B \times (2^k r, 2^{k+1} r)} \psi_t(D) D^N(\chi_l g) \right\|_{T^{2,2}(\Lambda T^*M)} \\ &:= \sum_{l \ge 0} K_l. \end{aligned}$$

Similarly, one has

$$K_l^2 = \int_{2^k r}^{2^{k+1} r} \left\| \chi_{2^k B} \widetilde{\psi}_t(D)(\chi_l g) \right\|^2_{L^2(\Lambda T^*M)} \frac{dt}{t^{2N+1}}.$$

Assume first that $0 \le l \le k$. Then, one obtains

$$\begin{aligned} K_l^2 &\le C \int_{2^k r}^{2^{k+1} r} \frac{dt}{t^{2N+1}} \|\chi_l g\|^2_{L^2(\Lambda T^*M)} \\ &\le C 2^{-(\kappa+2)l} V^{-1}(2^k B) 2^{k(\kappa - 2N)}. \end{aligned}$$

Since $2N \ge \kappa + 1$, it follows that

$$\sum_{l=0}^{k} \left\| \chi_{2^k B \times (2^k r, 2^{k+1} r)} \psi_t(D) D^N(\chi_l g) \right\|_{T^{2,2}(\Lambda T^*M)} \le C 2^{-k} V^{-1/2}(2^k B).$$

Assume now that $l \ge k+1$. Then, using Lemma 3.3 once more,

$$\begin{aligned} K_l^2 &\le C \int_{2^k r}^{2^{k+1} r} \left(\frac{t}{2^l r} \right)^{2m} \frac{dt}{t^{2N+1}} \|\chi_l g\|^2_{L^2(\Lambda T^*M)} \\ &\le C 2^{-(2m+2)l} 2^{k(2m-2N)} V^{-1}(2^k B), \end{aligned}$$

provided that $m < N$. It follows that

$$\sum_{l \ge k+1} \left\| \chi_{2^k B \times (2^k r, 2^{k+1} r)} \psi_t(D) D^N(\chi_l g) \right\|_{T^{2,2}(\Lambda T^*M)} \le C 2^{-k} V^{-1/2}(2^k B).$$

Assertion (c) is therefore proved. □

Remark 6.9 *Let $N > \frac{1}{2}(\frac{\kappa}{2} + 1)$ be an integer. Say that $a \in L^2(\Lambda T^*M)$ is an N-molecule for Δ if there exists $b \in L^2(\Lambda T^*M)$ such that $a = \Delta^N b$, a ball $B \subset M$ with radius r and a sequence $(\chi_k)_{k \ge 0}$ adapted to B such that, for all $k \ge 0$,*

$$\|\chi_k a\|_2 \le 2^{-k} V^{-1/2}(2^k B) \text{ and } \|\chi_k b\|_2 \le 2^{-k} r^{2N} V^{-1/2}(2^k B). \qquad (6.8)$$

Moreover, define $H^1_{\Delta,mol,N}$ as the space of all sections f such that $f = \sum_{j\ge 1} \lambda_j a_j$, where $\sum_j |\lambda_j| < +\infty$ and the a_j's are N-molecules for Δ, and equip $H^1_{\Delta,mol,N}$ with the usual norm. Then, using Theorem 6.2, it is plain to see that, for all $N > \frac{1}{2}(\frac{\kappa}{2}+1)$, $H^1_{\Delta,mol,N} = H^1(\Lambda T^*M)$.

Finally, we also have a decomposition into molecules for $H^1_d(\Lambda T^*M)$ and $H^1_{d^*}(\Lambda T^*M)$. Let $N > \frac{\kappa}{2}+1$. An N-molecule for d is a section $a \in L^2(\Lambda T^*M)$, such that there exists a ball $B \subset M$ with radius r, $b \in L^2(\Lambda T^*M)$ with $a = dD^{N-1}b$ and a sequence $(\chi_k)_{k\geq 0}$ adapted to B such that, for all $k \geq 0$,

$$\|\chi_k a\|_{L^2(\Lambda T^*M)} \leq 2^{-k}V^{-1/2}(2^kB) \text{ and } \|\chi_k b\|_{L^2(\Lambda T^*M)} \leq 2^{-k}r^N V^{-1/2}(2^kB). \tag{6.9}$$

Then, $f \in H^1_d(\Lambda T^*M)$ if and only if $f = \sum_j \lambda_j a_j$ where the a_j's are atoms in $H^1_d(\Lambda T^*M)$ and $\sum |\lambda_j| < +\infty$. The proof is analogous and uses the characterization of $H^1_d(\Lambda T^*M)$ by means of $\mathcal{S}_{d,\psi}$ and $\mathcal{Q}_{d^*,\psi}$ given in Section 5.3. One obtains a similar decomposition for $H^1_{d^*}(\Lambda T^*M)$, defining an N-molecule for d^* similarly to an N-molecule for d.

Remark 6.10 *It turns out that, under some Gaussian upper estimates for the heat kernel of the Hodge-de Rham Laplacian, we can take $N=1$ in Theorem 6.2 and other similar results. We will come back to this in Section 8.2.2.*

7 The maximal characterization

In this section, we provide a characterization of $H^1(\Lambda T^*M)$ in terms of maximal functions. Recall that, for classical Hardy spaces of functions in the Euclidean case, such maximal functions are defined, for instance, in the following way: if $\int_{\mathbb{R}^n} |f(y)|\,(1+|y|^2)^{-(n+1)/2} < +\infty$ and $x \in \mathbb{R}^n$, define

$$f^*(x) = \sup_{|y-x|<t} \left| e^{-t\sqrt{\Delta}} f(y) \right|.$$

Then, a possible characterization of $H^1(\mathbb{R}^n)$ is the following one: $f \in H^1(\mathbb{R}^n)$ if and only if $f^* \in L^1(\mathbb{R}^n)$.

In the present context, such a definition has to be adapted. Let us explain the main lines before coming to the details. First, the lack of pointwise estimates forces us to replace the value at (y,t) by an L^2 average on a ball centered at (y,t). Secondly, the Poisson semigroup (on forms) $e^{-t\sqrt{\Delta}}$ only satisfies $OD_t(1)$ estimates in general, which is unsufficient to carry out the argument in [26] or its adaptation in [6]. Hence, we abandon in the maximal function the Poisson semigroup in favor of the heat semigroup. Thirdly, the good-λ argument of [26] or [6] with the heat semigroup produces then uncontrolled error terms involving the time derivatives due to the parabolic nature of the equation associated with. The trick is to modify the maximal function to incorporate the errors in the very definition of the maximal function (see the function $\tilde{f}^*_{\alpha,c}$ below) so that they are under control in the argument.

In the sequel, if $x \in M$ and $0 < r < t$, $B((x,t),r) = B(x,r) \times (t-r,t+r)$. For all $x \in M$ and all $\alpha > 0$, recall that

$$\Gamma_\alpha(x) = \{(y,t) \in M \times (0,+\infty);\ y \in B(x,\alpha t)\}.$$

Let $0 < \alpha$. Fix $c > 0$ such that, for all $x \in M$, whenever $(y,t) \in \Gamma_\alpha(x)$, $B((y,t),ct) \subset \Gamma_{2\alpha}(x)$. Elementary geometry shows that $c \leq \alpha/(1+2\alpha)$ works. For $f \in L^2(\Lambda T^*M)$ and all $x \in M$, define

$$f^*_{\alpha,c}(x) = \sup_{(y,t)\in\Gamma_\alpha(x)} \left(\frac{1}{tV(y,t)} \iint_{B((y,t),ct)} \left| e^{-s^2\Delta} f(z) \right|^2 dz ds \right)^{1/2}$$

Define $H^1_{max}(\Lambda T^*M)$ as the completion of $\{f \in \mathcal{R}(D); \left\|f^*_{\alpha,c}\right\|_{L^1(M)} < \infty\}$ for that norm and set

$$\|f\|_{H^1_{max}(\Lambda T^*M)} = \left\|f^*_{\alpha,c}\right\|_{L^1(M)}.$$

The norm depends a priori on α, c. However, the doubling condition (1.2) allows us to compare them. For fixed α, the pointwise bound $f^*_{\alpha,c} \le C(1+c/c')^{\kappa/2} f^*_{2\alpha,c'}$ holds if $0 < c \le \alpha/(1+2\alpha)$ and $0 < c' \le 2\alpha/(1+4\alpha)$. Next, if $0 < \alpha \le \beta$ and $c \le \alpha/(1+2\alpha)$ then $f^*_{\alpha,c} \le f^*_{\beta,c}$ while if $0 < \beta < \alpha$ and $c \le \beta/(1+2\beta)$ (hence $c \le \alpha/(1+2\alpha)$), we have $\|f^*_{\alpha,c}\|_1 \le C(\beta/\alpha)^\kappa \|f^*_{\beta,c}\|_1$ by a variant of the Fefferman-Stein argument in [26] which is skipped. Hence, the space $H^1_{max}(\Lambda T^*M)$ is independent from the choice of α, c. Notice also that, in the definition of $f^*_{\alpha,c}$, because of the doubling property again, replacing $V(y,t)$ by $V(y,ct)$ yields an equivalent norm.

The following characterization holds as part of Theorem 1.1

Theorem 7.1 *Assume (1.2). Then* $H^1(\Lambda T^*M) = H^1_{max}(\Lambda T^*M)$.

Remark 7.2 *The average in s in the definition of $f^*_{\alpha,c}$ is useful only in the proof of the inclusion $H^1_{max}(\Lambda T^*M) \subset H^1(\Lambda T^*M)$. Equivalent norms occur without the average is s in the definition.*

As a consequence of this result and Corollary 6.6, we have

Corollary 7.3 *Assume (1.2). Then $D\Delta^{-1/2}$ is $H^1_{max}(\Lambda T^*M) - L^1(\Lambda T^*M)$ bounded.*

For the proof of the theorem, we introduce an auxiliary space $\widetilde{H}^1_{max}(\Lambda T^*M)$ and show the following chain of inclusions: $H^1(\Lambda T^*M) \subset H^1_{\max}(\Lambda T^*M) = \widetilde{H}^1_{max}(\Lambda T^*M) \subset H^1(\Lambda T^*M)$. This space is built as $H^1_{max}(\Lambda T^*M)$ with $f^*_{\alpha,c}$ changed to

$$\tilde{f}^*_{\alpha,c}(x) = \sup_{(y,t)\in\Gamma_\alpha(x)} \left(\frac{1}{tV(y,t)} \iint_{B((y,t),ct)} \left| e^{-s^2\Delta} f(z)\right|^2 + \left| s\frac{\partial}{\partial s} e^{-s^2\Delta} f(z)\right|^2 dzds \right)^{1/2}. \quad (7.1)$$

7.1 Proof of $H^1(\Lambda T^*M) \subset H^1_{max}(\Lambda T^*M)$

Fix $\alpha = 1/2$ and $0 < c \le 1/4$ and set $f^* = f^*_{1/2,c}$. In view of Theorem 6.2, it is enough to show that any N-molecule for Δ (for suitable N) in $H^1(\Lambda T^*M)$ belongs to $H^1_{max}(\Lambda T^*M)$. We denote by $\mathcal{M}$ the usual Hardy-Littlewood maximal function:

$$\mathcal{M} f(x) = \sup_{B \ni x} \frac{1}{V(B)} \int_B |f(y)|\, dy,$$

where the supremum is taken over all the balls $B \subset M$ containing x. Here is our first technical lemma:

Lemma 7.4 *Assume that $(T_t)_{t>0}$ satisfies $OD_t(N)$ estimates with $N > \kappa/2$. Then, there exists $C > 0$ such that, for all $f \in L^2_{loc}(\Lambda T^*M)$, all $x \in M$ and all $(y,t) \in \Gamma_\alpha(x)$,*

$$\frac{1}{tV(y,ct)} \iint_{B((y,t),ct)} |T_s f(z)|^2\, dzds \le C\mathcal{M}(|f|^2)(x).$$

Proof: Decompose $f=\sum_k f_k$, with $f_0=\chi_{B(y,2ct)}f$ and $f_k=\chi_{B(y,2^{k+1}ct)\setminus B(y,2^kct)}f$ for all $k\geq 1$ (where χ_A stands for the characteristic function of A). For $k=0$, the L^2-boundedness of T_s and the fact that $s\sim t$ and $V(x,(2c+1)t)\sim V(y,ct)$ yield

$$\frac{1}{tV(y,ct)}\iint_{B((y,t),ct)}|T_sf_0(z)|^2\,dzds\leq C\mathcal{M}(|f|^2)(x).$$

For $k\geq 1$, the estimate $OD_t(N)$ and the fact that $s\sim t$ and $V(x,(2^{k+1}c+1)t)\leq C2^{k\kappa}V(y,ct)$ give us

$$\begin{aligned}\frac{1}{tV(y,ct)}\iint_{B((y,t),ct)}|T_sf_k(z)|^2\,dzds &\leq \frac{C}{V(y,ct)}\left(\frac{1}{2^k}\right)^{2N}\int|f_k(z)|^2\,dz\\ &\leq \frac{C2^{k\kappa}}{2^{2kN}}\mathcal{M}(|f|^2)(x).\end{aligned}$$

Since $2N>\kappa$, one can sum up these estimates by the Minkowski inequality. □

We prove now that $H^1(\Lambda T^*M)\subset H^1_{max}(\Lambda T^*M)$. Let $a=\Delta^{N_0}b\in\mathcal{R}(D)$ be a N_0-molecule for Δ in $H^1(\Lambda T^*M)$, for some $N_0\geq\frac{\kappa}{2}+1$, $r>0$ the radius of the ball B associated with a, $(\chi_j)_{j\geq 0}$ a sequence adapted to B such that (6.8) holds (see Remark 6.9) . For each $j\geq 0$, set $a_j=\chi_ja$. First, the Kolmogorov inequality ([36], p. 250), Lemma 7.4, the maximal theorem and the doubling property (1.2) show that

$$\int_{2B}a^*(x)dx\leq CV(2B)^{1/2}\left\|\mathcal{M}\left(|a|^2\right)\right\|_{1,\infty}^{1/2}\leq CV(B)^{1/2}\left\|a\right\|_2\leq C.$$

We next show that, for some $\delta>0$ only depending on doubling constants and all $k\geq 1$,

$$\int_{2^{k+1}B\setminus 2^kB}a^*(x)dx\leq C2^{-k\delta}. \tag{7.2}$$

Fix $k\geq 1$ and write $a^*\leq a^*_{low}+a^*_{medium}+a^*_{high}$, where a^*_{low} (resp. a^*_{medium},a^*_{high}) correspond to the constraint $t<r$ (resp. $r\leq t<2^{k-1}r$, $t\geq 2^{k-1}r$) in the supremum defining a^*. We first deal with a^*_{low}. According to the definition of a_j we have $a^*_{low}\leq\sum_{j\geq 0}a^*_{j,low}$. Fix $(y,t)\in\Gamma_\alpha(x)$, $t<r$, $x\in 2^{k+1}B\setminus 2^kB$. If $j\leq k-2$, so that $\rho(B(y,ct),\text{supp }a_j)\sim 2^kr$, then, using the fact that $s\sim t$ and off-diagonal estimates for $e^{-s^2\Delta}$ (Lemma 3.8) and arguing as in Lemma 7.4 (using the fact that $t<r$), one obtains, provided that $2N>\kappa$,

$$\begin{aligned}\frac{1}{t}\frac{1}{V(y,ct)}\iint_{B(y,t),ct)}\left|e^{-s^2\Delta}a_j(z)\right|^2dzds &\leq C\left(\frac{t}{2^kr}\right)^{2N-\kappa}\mathcal{M}\big(\left|a_j\right|^2\big)(x)\\ &\leq \frac{C}{2^{k(2N-\kappa)}}\mathcal{M}\big(\left|a_j\right|^2\big)(x).\end{aligned}$$

If $j\geq k+2$, so that $\rho(B(y,ct),\text{supp }a_j)\sim 2^jr$, then, one has similarly

$$\frac{1}{t}\frac{1}{V(y,ct)}\iint_{B(y,t),ct)}\left|e^{-s^2\Delta}a_j(z)\right|^2dzds\leq\frac{C}{2^{j(2N-\kappa)}}\mathcal{M}\big(\left|a_j\right|^2\big)(x).$$

Setting

$$c_{j,k} = \begin{cases} 2^{-k(N-\kappa/2)} & \text{if } j \leq k-2, \\ 1 & \text{if } k-1 \leq j \leq k+1, \\ 2^{-j(N-\kappa/2)} & \text{if } j \geq k+2, \end{cases}$$

one therefore has, using the Kolmogorov inequality and (6.2) again,

$$\begin{aligned} \int_{2^{k+1}B \setminus 2^k B} a^*_{low} &\leq \sum_{j\geq 0} \int_{2^{k+1}B\setminus 2^k B} a^*_{j,low} \\ &\leq C \sum_{j \geq 0} c_{j,k} V^{1/2}(2^{k+1}B) 2^{-j} V^{-1/2}(2^j B) \\ &\leq C \sum_{j\geq 0} c_{j,k} \sup\left(1, 2^{(k-j)\kappa/2}\right) 2^{-j} \\ &\leq C 2^{-k\delta} \end{aligned}$$

if $N > \kappa + \delta$.

To estimate a^*_{medium} on $2^{k+1}B \setminus 2^k B$, write $a = \Delta^{N_0} b$ and $b = \sum_{j\geq 0} b_j$ where $b_j = \chi_j b$. Let $(y,t) \in \Gamma_\alpha(x)$, $x \in 2^{k+1}B \setminus 2^k B$ and $r \leq t < 2^{k-1}r$. Then, $\rho(B(y,ct), \text{supp } b_j) \sim 2^k r$ if $j \leq k-2$ and $\rho(B(y,ct), \text{supp } b_j) \sim 2^j r$ if $j \geq k+2$. Hence, arguing as before, using off-diagonal estimates for $(s^2\Delta)^N e^{-s^2\Delta}$ (see Lemma 3.8 again) and $s \sim t$, one has, if $2N > \kappa$,

$$\frac{1}{t}\frac{1}{V(y,ct)} \iint_{B((y,t),ct)} \left|\Delta^{N_0} e^{-s^2\Delta} b_j(z)\right|^2 dz ds \leq \begin{cases} \frac{1}{t^{4N_0}} \left(\frac{t}{2^k r}\right)^{2N-\kappa} \mathcal{M}\left(|b_j|^2\right)(x) & \text{if } j \leq k-2, \\ \frac{1}{t^{4N_0}} \mathcal{M}\left(|b_j|^2\right)(x) & \text{if } |j-k| \leq 1, \\ \frac{1}{t^{4N_0}} \left(\frac{t}{2^j r}\right)^{2N-\kappa} \mathcal{M}\left(|b_j|^2\right)(x) & \text{if } j \geq k+2, \end{cases}$$

and, if we choose N such that $N < \frac{\kappa}{2} + 2N_0$, one obtains

$$\frac{1}{t}\frac{1}{V(y,ct)} \iint_{B((y,t),ct)} \left|\Delta^{N_0} e^{-s^2\Delta} b_j(z)\right|^2 dz ds \leq \frac{1}{r^{4N_0}} c^2_{j,k} \mathcal{M}\left(|b_j|^2\right)(x)$$

where $c_{j,k}$ was defined above. Thus, provided that $N > \kappa$,

$$\int_{2^{k+1}B\setminus 2^k B} a^*_{medium} \leq C 2^{-k\delta}$$

for some $\delta > 0$. Note that this choice of N is possible since $N_0 \geq \frac{\kappa}{2} + 1$.

It remains to look at a^*_{high}. Let $x \in 2^{k+1}B \setminus 2^k B$, $(y,t) \in \Gamma_\alpha(x)$ and $t > 2^{k-1}r$. Using $a = \Delta^{N_0} b$ and the L^2-boundedness of $(s^2\Delta)^{N_0} e^{-s^2\Delta}$, one obtains

$$\begin{aligned} \frac{1}{t}\frac{1}{V(y,ct)} \iint_{B((y,t),ct)} \left|\Delta^{N_0} e^{-s^2\Delta} b(z)\right|^2 dz ds &\leq \frac{C}{t^{4N_0}} \int |b|^2 \frac{1}{V(y,ct)} \\ &\leq C 2^{-k(4N_0-\kappa)} \frac{1}{V^2(2^{k+1}B)}. \end{aligned}$$

It follows that

$$\int_{2^{k+1}B\setminus 2^k B} a^*_{high} \le C 2^{-k(2N_0-\kappa/2)}.$$

Finally, (7.2) is proved since $N_0 \ge \kappa/2+1$. Thus, $\int_M a^* \le C$, which ends the proof of the inclusion $H^1(\Lambda T^*M) \subset H^1_{max}(\Lambda T^*M)$. □

7.2 $H^1_{max}(\Lambda T^*M) = \widetilde{H}^1_{max}(\Lambda T^*M)$

As $f^*_{\alpha,c} \le \tilde{f}^*_{\alpha,c}$, it follows that $H^1_{max}(\Lambda T^*M) \supset \widetilde{H}^1_{max}(\Lambda T^*M)$. We turn to the opposite inclusion. For that argument we fix $\alpha = 1$ and $c \le 1/12$ and write f^*, $\tilde{f}^*$. The term in $\tilde{f}^*$ coming from $e^{-s^2\Delta}f(z)$ is immediately controlled by f^*. Next, fix $x \in M$ and $(y,t) \in \Gamma_1(x)$. Let $(z,s) \in B((y,t),ct)$. Write

$$s\partial_s e^{-s^2\Delta}f(z) = -2s^2\Delta e^{-s^2\Delta}f(z) = (-2s^2\Delta e^{-(\sqrt{3}s/2)^2\Delta})e^{-(s/2)^2\Delta}f(z)$$

and observe that $-2s^2\Delta e^{-(\sqrt{3}s/2)^2\Delta}$ satisfies $OD_s(N)$ for any N (see Lemma 3.8). Since $s\sim t$,

$$\left(\frac{1}{tV(y,t)}\iint_{B((y,t),ct)}\left|s\partial_s e^{-s^2\Delta}f(z)\right|^2 dzds\right)^{1/2}$$

is controlled by

$$C\left(\frac{1}{tV(y,t)}\int_{t-ct}^{t+ct}\int_{B(y,2ct)}\left|e^{-(s/2)^2\Delta}f(z)\right|^2 dzds\right)^{1/2}$$
$$+C\sum_{k=1}^{\infty}2^{-kN}\left(\frac{1}{tV(y,t)}\int_{t-ct}^{t+ct}\int_{B(y,2^{k+1}ct)\setminus B(y,2^kct)}\left|e^{-(s/2)^2\Delta}f(z)\right|^2 dzds\right)^{1/2}.$$

Change $s/2$ to s. This first term is controlled by $f^*_{2,4c}(x)$. For the kth term in the series, one covers $B(y,2^{k+1}ct)$ by balls $B(y_j,2ct)$ with $y_j \in B(y,2^{k+1}ct)$ and $B(y_j,ct)$ pairwise disjoint. By doubling, the balls $B(y_j,2ct)$ have bounded overlap. Observe also that each point $(y_j,t/2)$ belongs to $\Gamma_{2+2^{k+2}c}(x)$. Hence, the kth term is bounded by

$$\begin{aligned} C2^{-kN}\left(\sum_j \frac{V(y_j,2t)}{V(y,t)}\right)^{1/2} f^*_{2+2^{k+2}c,4c}(x) &\le C\left(\frac{V(y,2^{k+1}ct+2t)}{V(y,t)}\right)^{1/2} f^*_{2+2^{k+2}c,4c}(x) \\ &\le C2^{k\kappa/2}f^*_{2+2^{k+2}c,4c}(x). \end{aligned}$$

Hence, using the comparisons between the $\|f^*_{\alpha,c}\|_1$ norms, we obtain

$$\begin{aligned} \|\tilde{f}^*\|_1 &\le \|f^*\|_1 + C\|f^*_{2,4c}\|_1 + C\sum_{k=1}^{\infty}2^{-k(N-\kappa/2)}\|f^*_{2+2^{k+2}c,4c}\|_1. \\ &\le \|f^*\|_1(1+C+C\sum_{k=1}^{\infty}2^{-k(N-\kappa/2)}2^{k\kappa}) \end{aligned}$$

so that if $N > 3\kappa/2$, we obtain $\|\tilde{f}^*\|_1 \le C\|f^*\|_1$. □

7.3 $\widetilde{H}^1_{max}(\Lambda T^*M) \subset H^1(\Lambda T^*M)$

We fix some notation. If $f \in L^2(\Lambda T^*M)$, the section $u(y,t) = e^{-t^2\Delta} f(y)$ for all $y \in M$ and all $t > 0$ satisfies the equation

$$D^2 u = \Delta u = -\frac{1}{2t}\frac{\partial u}{\partial t}. \tag{7.3}$$

For all $\alpha > 0$, $0 \le \varepsilon < R \le +\infty$, $x \in M$ set the truncated cone

$$\Gamma_\alpha^{\varepsilon,R}(x) = \{(y,t) \in M \times (\varepsilon, R);\ y \in B(x,\alpha t)\} = \{(y,t) \in \Gamma_\alpha(x); \varepsilon < t < R\}.$$

and for all $f \in L^2(\Lambda T^*M)$,

$$S_\alpha^{\varepsilon,R} f(x) = \left(\iint_{\Gamma_\alpha^{\varepsilon,R}(x)} \frac{|tDu(y,t)|^2}{V(y,t)} dy \frac{dt}{t} \right)^{1/2}$$

where $|Du(y,t)|^2 = \langle Du(y,t), Du(y,t)\rangle$ (remember that $\langle .,.\rangle$ denotes the inner product in ΛT^*M). Note that, contrary to the definition of $\mathcal{S}F$ given in Section 4, we divide here by $V(y,t)$ instead of $V(x,t)$. By (1.2), this amounts to the same, since, when $(y,t) \in \Gamma_\alpha(x)$, $d(y,x) \le \alpha t$, and it turns out that $V(y,t)$ is more handy here. For our purpose, it is enough to show that, for all $f \in L^2(\Lambda T^*M) \cap \widetilde{H}^1_{max}(\Lambda T^*M)$ and all $0 < \varepsilon < R$,

$$\left\| S_\alpha^{\varepsilon,R} f \right\|_1 \le C \|f\|_{\widetilde{H}^1_{max}(\Lambda T^*M)}. \tag{7.4}$$

Indeed, if furthermore $f \in \mathcal{R}(D) \cap \widetilde{H}^1_{max}(\Lambda T^*M)$, letting ε go to 0 and R to $+\infty$, this means that $(y,t) \mapsto tDe^{-t^2\Delta} f(y) \in T^{1,2}(\Lambda T^*M)$ and, since

$$f = a \int_0^{+\infty} (tD)^{2N_1+1}(I + t^2D^2)^{-\alpha_1} tDe^{-t^2D^2} f \frac{dt}{t}$$

for suitable integers N_1, α_1 and constant a (we use $\Delta = D^2$), this yields $f \in H^1(\Lambda T^*M)$ by definition of the Hardy space.

The proof of (7.4) is inspired by the one of Proposition 7 in [6] where the Poisson semigroup is changed to the heat semigroup. We first need the following inequality (see Lemma 8 in [6]):

Lemma 7.5 *There exists $C > 0$ such that, for all $f \in L^2(\Lambda T^*M)$, all $0 < \varepsilon < R < +\infty$ and all $x \in M$,*

$$S_{1/20}^{\varepsilon,R}(x) \le C\,(1 + \ln(R/\varepsilon))\, \widetilde{f}^*_{1,c}(x).$$

Proof: Fix any $0 < c < 1/3$. Let $x \in M$. One can cover $\Gamma_{1/20}^{\varepsilon,R}(x)$ by balls in $M \times (0,+\infty)$ in the following way: for each $l \in \mathbb{Z}$, let $(B(x_{j,l}, \frac{c}{2}\tau^l))_{j\in\mathbb{Z}}$ be a covering of M by balls (where $\tau = \frac{1+c/2}{1-c/2}$) so that the balls $B\left(x_{j,l}, \frac{c}{4}\tau^l\right)$ are pairwise disjoint. Let $K_{j,l} = B(x_{j,l}, \frac{c}{2}\tau^l) \times \left[\tau^l - \frac{c}{2}\tau^l, \tau^l + \frac{c}{2}\tau^l\right]$ and $\widetilde{K_{j,l}} = B(x_{j,l}, c\tau^l) \times \left[\tau^l - c\tau^l, \tau^l + c\tau^l\right] = B((x_{j,l}, \tau^l), c\tau^l)$. Since, for

$(y,t)\in K_{j,l}$, if $K_{j,l}\cap\Gamma^{\varepsilon,R}_{1/20}(x)\neq\emptyset$, one has $t\sim\tau^l$ and $V(y,t)\sim V(x_{j,l},\tau^l)$, we obtain

$$\begin{aligned} S^{\varepsilon,R}_{1/20}f(x)^2 &\leq \sum_{l,j;\ K_{j,l}\cap\Gamma^{\varepsilon,R}_{1/20}(x)\neq\emptyset}\iint_{K_{j,l}}\frac{\left|tDu(y,t)\right|^2}{V(y,t)}dy\frac{dt}{t}\\ &\leq C\sum_{l,j;\ K_{j,l}\cap\Gamma^{\varepsilon,R}_{1/20}(x)\neq\emptyset}\frac{\tau^l}{V(x_{j,l},\tau^l)}\iint_{K_{j,l}}\left|Du(y,t)\right|^2dydt. \end{aligned}$$

At this stage, we need the following parabolic Caccioppoli inequality recalling that $u(y,t)=e^{-t^2\Delta}f(y)$: for some constant $C>0$ only depending on M, but not on j, l,

$$\iint_{K_{j,l}}\left|Du(y,t)\right|^2dydt\leq C\tau^{-2l}\iint_{\widetilde{K}_{j,l}}\left|u(y,t)\right|^2dydt+C\tau^{-2l}\iint_{\widetilde{K}_{j,l}}\left|t\partial_tu(y,t)\right|^2dydt. \quad (7.5)$$

The proof of this inequality is classical and will therefore be skipped (see for instance [8], Chapter 1). Since, by the choice of c, $x_{j,l}\in\Gamma^{\varepsilon,R}_1(x)$ whenever $K_{j,l}\cap\Gamma^{\varepsilon,R}_{1/20}(x)\neq\emptyset$, it follows that

$$\begin{aligned} S^{\varepsilon,R}_{1/20}f(x)^2 &\leq C\sum_{l,j;\ K_{j,l}\cap\Gamma^{\varepsilon,R}_{1/20}(x)\neq\emptyset}\frac{1}{\tau^lV(x_{j,l},\tau^l)}\iint_{\widetilde{K}_{j,l}}\left(\left|u(y,t)\right|^2+\left|t\partial_tu(y,t)\right|^2\right)dydt\\ &\leq C\sharp\left\{(l,j);\ K_{j,l}\cap\Gamma^{\varepsilon,R}_{1/20}(x)\neq\emptyset\right\}\tilde{f}^*_{1,c}(x)^2. \end{aligned}$$

For fixed $l\in\mathbb{Z}$, the bounded overlap property of the balls $B(x_{j,l},c\tau^l)$ implies that the number of j's such that $K_{j,l}\cap\Gamma^{\varepsilon,R}_{1/20}(x)\neq\emptyset$ is uniformly bounded with respect to l. Now, if $l\in\mathbb{Z}$ is such that $K_{j,l}\cap\Gamma^{\varepsilon,R}_{1/20}(x)\neq\emptyset$, then one has $(1+c)^{-1}\varepsilon<\tau^l\leq R(1-c)^{-1}$, which yields the desired conclusion. □

Proof of (7.4): it suffices to establish the following "good λ" inequality:

Lemma 7.6 *There exists $C>0$ such that, for all $0\leq\varepsilon<R\leq+\infty$, all $f\in L^2(\Lambda T^*M)$, all $0<\gamma<1$ and all $\lambda>0$,*

$$\mu\left(\left\{x\in M;\ S^{\varepsilon,R}_{1/20}f(x)>2\lambda,\ \tilde{f}^*(x)\leq\gamma\lambda\right\}\right)\leq C\gamma^2\mu\left(\left\{x\in M;\ S^{\varepsilon,R}_{1/2}f(x)>\lambda\right\}\right) \quad (7.6)$$

where $\tilde{f}^*=\tilde{f}^*_{1,c}$ is as in (7.1) with $0<c<1/3$ to be chosen in the proof.

Indeed, assume that Lemma 7.6 is proved. Then, if $f\in L^2(\Lambda T^*M)\cap\tilde{H}^1_{max}(\Lambda T^*M)$, integrating (7.6) with respect to λ and using Remark 4.1 yield

$$\left\|S^{\varepsilon,R}_{1/20}f\right\|_1\leq C\gamma^{-1}\left\|\tilde{f}^*\right\|_1+C\gamma^2\left\|S^{\varepsilon,R}_{1/2}f\right\|_1\leq C\gamma^{-1}\left\|\tilde{f}^*\right\|_1+C'\gamma^2\left\|S^{\varepsilon,R}_{1/20}f\right\|_1.$$

Since $f\in\widetilde{H}^1_{max}(\Lambda T^*M)$, Lemma 7.5 ensures that $\left\|S^{\varepsilon,R}_{1/20}f\right\|_1<+\infty$, and (7.4) follows at once if γ is chosen small enough.

Proof of Lemma 7.6: Assume first that M is unbounded, which, by (1.2), implies that $\mu(M) = +\infty$ (see [34]). Let $O = \left\{x \in M; S_{1/2}^{\varepsilon,R} f(x) > \lambda\right\}$. Observe that, since $f \in L^2(\Lambda T^*M)$, $S_{1/2}^{\varepsilon,R} f \in L^2(\Lambda T^*M)$, whence $\mu(O) < +\infty$, therefore $O \neq M$. Morever, O is open since the map $x \mapsto S_{1/2}^{\varepsilon,R} f(x)$ is continuous. Let $(B_k)_{k\geq 1}$ be a Whitney decomposition of O, so that $2B_k \subset O$ and $4B_k \cap (M \setminus O) \neq \emptyset$[1] for all $k \geq 1$. For all $k \geq 1$, define

$$E_k = \left\{x \in B_k;\ S_{1/20}^{\varepsilon,R} f(x) > 2\lambda,\ \tilde{f}^*(x) \leq \gamma\lambda\right\}.$$

Because of the bounded overlap property of the B_k's, and since $\left\{S_{1/20}^{\varepsilon,R} f > 2\lambda\right\} \subset \left\{S_{1/2}^{\varepsilon,R} f > \lambda\right\}$, it is enough to prove that

$$\mu(E_k) \leq C\gamma^2 \mu(B_k). \tag{7.7}$$

Observe first that, if $\varepsilon \geq 20r(B_k)$ (where $r(B_k)$ is the radius of B_k), then $E_k = \emptyset$. Indeed, there exists $x_k \in 4B_k$ such that $S_{1/2}^{\varepsilon,R} f(x_k) \leq \lambda$. Let now $x \in B_k$ and $(y,t) \in \Gamma_{1/20}^{\varepsilon,R}(x)$. Then,

$$d(x_k, y) \leq d(x_k, x) + d(x, y) \leq 5r(B_k) + \frac{t}{20} \leq \frac{5\varepsilon}{20} + \frac{t}{20} \leq \frac{6t}{20} < \frac{t}{2},$$

so that $(y,t) \in \Gamma_{1/2}^{\varepsilon,R}(x_k)$. As a consequence, $S_{1/20}^{\varepsilon,R} f(x) \leq S_{1/2}^{\varepsilon,R} f(x_k) \leq \lambda$. We may therefore assume that $\varepsilon < 20r(B_k)$. Since one has $S_{1/20}^{20r(B_k),R} f(x) \leq \lambda$ by similar arguments, we deduce that

$$E_k \subset \widetilde{E_k} = \left\{x \in B_k \cap F;\ S_{1/20}^{\varepsilon,20r(B_k)} f(x) > \lambda\right\}$$

where

$$F = \left\{x \in M\ ;\ \tilde{f}^*(x) \leq \gamma\lambda\right\}$$

(note that F is closed). By Tchebycheff inequality,

$$\begin{aligned} \mu\left(\widetilde{E_k}\right) &\leq \frac{1}{\lambda^2} \int_{B_k \cap F} \left|S_{1/20}^{\varepsilon,20r(B_k)} f(x)\right|^2 dx \\ &= \frac{1}{\lambda^2} \int_{x \in B_k \cap F} \left(\iint_{\varepsilon < t < 20r(B_k),\ y \in B(x,t/20)} \frac{|tDu(y,t)|^2}{V(y,t)} dy \frac{dt}{t} \right) dx \\ &\leq \frac{C}{\lambda^2} \iint_{\Omega_k^\varepsilon} t^2 \left|Du(y,t)\right|^2 dy \frac{dt}{t}, \end{aligned}$$

where Ω_k^ε is the region in $M \times (0, +\infty)$ defined by the following conditions:

$$\varepsilon < t < 20r(B_k),\ \psi(y) < t/20$$

with

$$\psi(y) = \rho(y, B_k \cap F).$$

[1] To be correct $4B_k$ should be $c_1 B_k$ where c_1 depends on the doubling property. To avoid too many constants, we set $c_1 = 4$ to fix ideas.

Note that $\Omega_k^\varepsilon \subset \widetilde{\Omega_k}$, where the region $\widetilde{\Omega_k}$ is defined by $0 < \psi(y) < t$. By definition of F, one has

$$\frac{1}{tV(y,t)} \iint_{B((y,t),ct)} |u(s,z)|^2 + |s\partial_s u(z,s)|^2 \, dzds \leq \gamma^2\lambda^2 \tag{7.8}$$

for all $(y,t) \in \widetilde{\Omega_k}$. To avoid the use of surface measure on $\partial\widetilde{\Omega_k}$, let us introduce

$$\zeta(y,t) = \eta^2\left(\frac{\psi(y)}{t}\right)\chi_1^2\left(\frac{t}{\varepsilon}\right)\chi_2^2\left(\frac{t}{20r(B_k)}\right),$$

where η, χ_1 and χ_2 are nonnegative C^∞ functions on $\mathbb{R}$, η is supported in $\left[0, \frac{1}{10}\right]$ and is equal to 1 on $\left[0, \frac{1}{20}\right]$, χ_1 is supported in $\left[\frac{9}{10}, +\infty\right[$ and is equal to 1 on $[1, +\infty[$, and χ_2 is supported in $\left[0, \frac{11}{10}\right]$ and is equal to 1 on $[0, 1]$. One therefore has

$$\mu\left(\widetilde{E_k}\right) \leq \frac{1}{\lambda^2} \iint \zeta(y,t) \, |Du(y,t)|^2 \, dydt := \frac{1}{\lambda^2} I.$$

The integral is over $M \times (0, \infty)$. An integration by parts in space using $D^* = D$ yields

$$\begin{aligned} I &= \Re \iint \langle D\left(\zeta(y,t)Du(y,t)\right), u(y,t)\rangle t dydt \\ &= \Re \iint \langle D\zeta(y,t)Du(y,t), u(y,t)\rangle t dydt \\ &+ \Re \iint \langle \zeta(y,t)D^2u(y,t), u(y,t)\rangle t dydt. \end{aligned}$$

In the first integral in the right hand side $D\zeta Du$ is the Clifford product (its exact expression is not relevant as we merely use $|D\zeta Du| \leq |D\zeta||Du|$). In the second, we use (7.3) and since ζ is real-valued, $\Re\langle \zeta D^2 u, u\rangle = -\frac{1}{4t}\zeta\partial_t |u|^2$. Then integration by parts in t gives us

$$\begin{aligned} I &= \Re \iint t\langle D\zeta(y,t)Du(y,t), u(y,t)\rangle dydt \\ &+ \tfrac{1}{4} \iint \partial_t\zeta(y,t) \, |u(y,t)|^2 \, dydt \\ &:= I_1 + I_2. \end{aligned}$$

Estimates and support considerations: Observe that, because of the support conditions on η, χ_1 and χ_2 and since ψ is a Lipschitz function with Lipschitz constant 1,

$$|D\zeta(y,t)| + |\partial_t\zeta(y,t)| \leq \frac{C}{t} \tag{7.9}$$

independently of ε and k. Now let us look more closely at the supports of $D\zeta$ and $\partial_t\zeta$. Examination shows that they are both supported in the region G_k^ε of $M \times (0, \infty)$ defined by $0 < \psi(y) < \frac{t}{10}$, $\frac{9\varepsilon}{10} < t < 22r(B_k)$ and

$$\frac{t}{20} < \psi(y) < \frac{t}{10} \quad \text{or} \quad \frac{9\varepsilon}{10} < t < \varepsilon \quad \text{or} \quad 20r(B_k) < t < 22r(B_k).$$

Observe that $G_k^\varepsilon \subset \widetilde{\Omega_k}$. Consider again the balls $K_{j,l}$ introduced above. It is possible to choose c small enough such that, for all k, j, l, ε, if $K_{j,l} \cap G_k^\varepsilon \neq \emptyset$, then $\widetilde{K_{j,l}} \subset \widetilde{\Omega_k}$. Thus, (7.8) yields

$$\iint_{\widetilde{K_{j,l}}} |u(z,s)|^2 + |s\partial_s u(z,s)|^2 \, dzds \leq \gamma^2\lambda^2\tau^l V(x_{j,l},\tau^l) \leq C\gamma^2\lambda^2\tau^l V(x_{j,l},c\tau^l)$$

where we used the doubling property in the last inequality.

Estimate of I_2: Using the considerations above and $t \sim \tau^l$ on $K_{j,l}$

$$\begin{aligned}
|I_2| &\leq C \iint_{G_k^\varepsilon} |u(y,t)|^2 \, dy\frac{dt}{t} \\
&= C \sum_{l,j;\ K_{j,l}\cap G_k^\varepsilon \neq \emptyset} \iint_{K_{j,l}} |u(y,t)|^2 \, dy\frac{dt}{t} \\
&\leq C \sum_{l,j;\ K_{j,l}\cap G_k^\varepsilon \neq \emptyset} \tau^{-l} \iint_{K_{j,l}} |u(y,t)|^2 \, dydt \\
&\leq C\gamma^2\lambda^2 \sum_{l,j;\ K_{j,l}\cap G_k^\varepsilon \neq \emptyset} V(x_{j,l},c\tau^l) \\
&\leq C\gamma^2\lambda^2 \iint_{\widetilde{G_k^\varepsilon}} dy\frac{dt}{t},
\end{aligned}$$

where $\widetilde{G_k^\varepsilon}$ is the region defined by $0 < \psi(y) < 5t$, $\frac{9\varepsilon}{100} < t < 100r(B_k)$ and

$$\frac{t}{40} < \psi(y) < 5t \ \text{ or } \ \frac{9\varepsilon}{20} < t < 10\varepsilon \ \text{ or } \ 2r(B_k) < t < 100r(B_k).$$

The last inequality is due to the bounded overlap property of the balls $B(x_{j,l},c\tau^l)$ for each $l \in \mathbb{Z}$ and $t \sim \tau^l$ on each of them. Thus,

$$|I_2| \leq C\gamma^2\lambda^2\mu(H_k^\varepsilon),$$

where $H_k^\varepsilon = \left\{ y \in M;\ \exists t > 0,\ (y,t) \in \widetilde{G_k^\varepsilon} \right\}$. It remains to observe that $H_k^\varepsilon \subset 221B_k$. Indeed, if $y \in H_k^\varepsilon$ and $t > 0$ is such that $(y,t) \in \widetilde{G_k^\varepsilon}$, one has $\psi(y) < 5t$, so that there exists $z \in B_k \cap F$ such that $\rho(y,z) < 5t < 220r(B_k)$. Thus, $y \in 221B_k$. Using the doubling property, we have therefore obtained

$$|I_2| \leq C\gamma^2\lambda^2 V(B_k). \ \square$$

Estimate of I_1: Using the same notation, one has

$$\begin{aligned}
|I_1| &\leq C \iint_{G_k^\varepsilon} |Du(y,t)|\,|u(y,t)| \, dydt \\
&\leq C \sum_{l,j;\ K_{j,l}\cap G_k^\varepsilon \neq \emptyset} \iint_{K_{j,l}} |Du(y,t)|\,|u(y,t)| \, dydt \\
&\leq C \sum_{l,j;\ K_{j,l}\cap G_k^\varepsilon \neq \emptyset} \left(\iint_{K_{j,l}} |Du(y,t)|^2 \, dydt \right)^{1/2} \left(\iint_{K_{j,l}} |u(y,t)|^2 \, dydt \right)^{1/2}.
\end{aligned}$$

The Caccioppoli inequality (7.5) yields

$$\begin{aligned}
|I_1| &\leq C \sum_{l,j;\ K_{j,l}\cap G_k^\varepsilon\neq\emptyset} \tau^{-l} \iint_{\widetilde{K_{j,l}}} |u(y,t)|^2\, dydt + \tau^{-l} \iint_{\widetilde{K_{j,l}}} |t\partial_t u(y,t)|^2\, dydt \\
&\leq C\gamma^2\lambda^2 \sum_{l,j;\ K_{j,l}\cap G_k^\varepsilon\neq\emptyset} V(x_{j,l}, c\tau^l),
\end{aligned}$$

and the same computations as before yield

$$|I_1| \leq C\gamma^2\lambda^2 V(B_k). \ \square$$

Finally, (7.7) holds and Lemma 7.6 is proved when M is unbounded.

When M is bounded, call δ the diameter of M. We claim that there exists a constant $C > 0$ such that, for all $R \geq 20\delta$ and all $x \in M$,

$$S_{1/20}^{20\delta,R} f(x) \leq C\tilde{f}^*(x). \tag{7.10}$$

Assume that (7.10) is proved. It is enough to prove Lemma 7.6 for γ small, say $\gamma \leq 1/C$, where C is the constant in (7.10). In this case, (7.10) ensures that, if $S_{1/20}^{20\delta,R} f(x) > \lambda$, then $\tilde{f}^*(x) > \gamma\lambda$, so that it remains to establish that

$$\mu\left(\left\{x \in M;\ S_{1/20}^{\varepsilon,20\delta} f(x) > \lambda,\ \tilde{f}^*(x) \leq \gamma\lambda\right\}\right) \leq C\gamma^2 \mu\left(\left\{x \in M;\ S_{1/2}^{\varepsilon,R} f(x) > \lambda\right\}\right). \tag{7.11}$$

If $O = \left\{x \in M;\ S_{1/2}^{\varepsilon,R} f(x) > \lambda\right\}$ is a proper subset of M, argue as before, using the Whitney decomposition. If $O = M$, then O is a ball itself, and we argue directly, without the Whitney decomposition, replacing $\mu\left(\widetilde{E_k}\right)$ by the left-hand side of (7.11).

It remains to prove (7.10). First, if $t \geq 20\delta$, $B(x, t/20) = M$ and $V(y,t) = \mu(M)$ for any $y \in M$, so that

$$S_{1/20}^{20\delta,R} f(x)^2 = \mu(M)^{-1} \int_M \int_{20\delta}^R |tDu(y,t)|^2\, dy \frac{dt}{t}.$$

Next, computations similar to the estimates of I above yield, for all $t > 0$,

$$\int_M |tDu(y,t)|^2\, dy = -\frac{t}{4} \int_M \partial_t\, |u(y,t)|^2\, dy,$$

so that, integrating by parts with respect to t, we obtain

$$S_{1/20}^{20\delta,R} f(x)^2 \leq \frac{1}{4\mu(M)} \int_M |u(y,20\delta)|^2\, dy.$$

But, for any $s \leq 20\delta$, the semigroup contraction property shows that

$$\int_M |u(y,20\delta)|^2\, dy \leq \int_M |u(y,s)|^2\, dy.$$

It follows that

$$S_{1/20}^{20\delta,R}f(x)^2 \leq \frac{1}{8\delta\mu(M)}\int_M\int_{\delta(1/c-1)}^{\delta(1/c+1)}|u(y,s)|^2\,dyds.$$

Noticing that $M = B(x,\delta) = B(x,c(\delta/c))$, one concludes, by definition of the maximal function, that

$$S_{1/20}^{20\delta,R}f(x)^2 \leq \frac{1}{8c}\tilde{f}^*(x)^2,$$

which is (7.10). The proof of Lemma 7.6 is now complete. □

Remark 7.7 *The same proof shows that (7.4) holds if $|Du(y,t)|^2$ is replaced by the sum $|du(y,t)|^2+|d^*u(y,t)|^2$ in the definition of $S_\alpha^{\varepsilon,R}$ (the sum is important). This is a stronger fact, since $|Du(y,t)|^2 \leq 2\,|du(y,t)|^2+2\,|d^*u(y,t)|^2$ (but observe that, if one restricts to k-forms for fixed $0\leq k\leq \dim M$, the two versions are equal). We could then conclude using the $H_d^1(\Lambda T^*M)$ and $H_{d^*}^1(\Lambda T^*M)$ spaces.*

8 Further examples and applications

8.1 The Coifman–Weiss Hardy space

In this section, we focus on the case of functions, *i.e.* 0-forms. Assuming that M satisfies (1.2), we may compare $H_{d^*}^1(\Lambda^0T^*M) = H_D^1(\Lambda^0T^*M) = H_\Delta^1(\Lambda^0T^*M)$ with the Coifman-Weiss Hardy space, *i.e.* the H^1 space defined in the general context of a space of homogeneous type in [17].

We first recall what this space is. A (Coifman-Weiss) atom is a function $a\in L^2(M)$ supported in a ball $B\subset M$ and satisfying

$$\int_M a(x)dx = 0 \text{ and } \left\|a\right\|_2 \leq V(B)^{-1/2}.$$

A complex-valued function f on M belongs to $H_{CW}^1(M)$ if and only if it can be written as

$$f=\sum_{k\geq 1}\lambda_k a_k$$

where $\sum_k|\lambda_k|<+\infty$ and the a_k's are Coifman-Weiss atoms. Define

$$\left\|f\right\|_{H_{CW}^1(M)} = \inf\sum_{k\geq 1}|\lambda_k|\,,$$

where the infimum is taken over all such decompositions of f. Equipped with this norm, $H_{CW}^1(M)$ is a Banach space. The link between this Coifman-Weiss space and the $H_{d^*}^1(\Lambda^0T^*M)$ space is as follows:

Theorem 8.1 *Assume (1.2). Then $H_{d^*}^1(\Lambda^0T^*M)\subset H_{CW}^1(M)$.*

Proof: Recall that a (Coifman-Weiss) molecule is a function $f \in L^1(M) \cap L^2(M)$ such that

$$\int_M f(x)dx = 0$$

and there exist $x_0 \in M$ and $\varepsilon > 0$ with

$$\left(\int_M |f(x)|^2 \, dx\right) \left(\int_M |f(x)|^2 \, m(x,x_0)^{1+\varepsilon} dx\right)^{\frac{1}{\varepsilon}} \leq 1, \tag{8.1}$$

where $m(x, x_0)$ is the infimum of the measures of the balls both containing x and x_0. It is shown in [17] (Theorem C, p. 594) that such a molecule belongs to $H^1_{CW}(M)$ with a norm only depending on the constant in (1.2) and ε. Note that condition (8.1) is satisfied if

$$\left(\int_M |f(x)|^2 \, dx\right) \left(\int_M |f(x)|^2 \, V(x_0, d(x,x_0))^{1+\varepsilon} dx\right)^{\frac{1}{\varepsilon}} \leq 1.$$

Let $f \in \mathcal{R}(D) \cap H^1_{d^*}(\Lambda^0 T^* M)$ and $F \in T^{1,2}(\Lambda^0 T^* M)$ with $\|F\|_{T^{1,2}(\Lambda^0 T^* M)} \sim \|f\|_{H^1_{d^*}(\Lambda^0 T^* M)}$ and

$$f = \int_0^{+\infty} (tD)^N (I + itD)^{-\alpha} F_t \frac{dt}{t}$$

with $N > \kappa/2 + 1$ and $\alpha = N + 2$. Since F has an atomic decomposition in $T^{1,2}(\Lambda^0 T^* M)$, it is enough to show that, whenever A is a (scalar-valued) atom in $T^{1,2}(\Lambda^0 T^* M)$ supported in $T(B)$ for some ball $B \subset M$,

$$a = \int_0^{+\infty} (tD)^N (I + itD)^{-\alpha} A_t \frac{dt}{t}$$

belongs to $H^1_{CW}(M)$ and satisfies

$$\|a\|_{H^1_{CW}(M)} \leq C. \tag{8.2}$$

To that purpose, it suffices to check that, up to a multiplicative constant, a is a Coifman-Weiss molecule. First, since $a \in \mathcal{R}(D)$ and a is a function, it is clear that a has zero integral. Furthermore, the spectral theorem shows that

$$\|a\|_2 \leq C V^{-1/2}(B).$$

Moreover, if $B = B(x_0, r)$ and if $\varepsilon > 0$,

$$\begin{aligned}
\int_M |a(x)|^2 \, V^{1+\varepsilon}(x_0, d(x,x_0)) dx &= \int_{2B} |a(x)|^2 \, V^{1+\varepsilon}(x_0, d(x,x_0)) dx \\
&+ \sum_{k \geq 1} \int_{2^{k+1}B \setminus 2^k B} |a(x)|^2 \, V^{1+\varepsilon}(x_0, d(x,x_0)) dx \\
&= A_0 + \sum_{k \geq 1} A_k.
\end{aligned}$$

On the one hand, by the doubling property,

$$A_0 \leq CV^{1+\varepsilon}(B) \left\| a \right\|_2^2 \leq CV(B)^{\varepsilon}.$$

On the other hand, if $k \geq 1$, using Lemma 3.3 and choosing N' such that $N' > \kappa(1+\varepsilon)/2$, one has

$$\begin{aligned} A_k^{1/2} &\leq V^{(1+\varepsilon)/2}(2^{k+1}B) \int_0^r \left\| (tD)^N (I+itD)^{-\alpha} A_t \right\|_{L^2(2^{k+1}B \setminus 2^k B)} \frac{dt}{t} \\ &\leq V^{(1+\varepsilon)/2}(2^{k+1}B) \int_0^r \left(\frac{t}{2^k r} \right)^{N'} \left\| A_t \right\|_2 \frac{dt}{t} \\ &\leq 2^{k(\kappa(1+\varepsilon)/2 - N')} V(B)^{\varepsilon/2}. \end{aligned}$$

Finally,

$$\int_M \left| a(x) \right|^2 V^{1+\varepsilon}(x_0, d(x,x_0)) dx \leq CV^{\varepsilon}(B),$$

which ends the proof of (8.2), therefore of Theorem 8.1. □

We will focus on the converse inclusion in Theorem 8.1 in the following section.

8.2 Hardy spaces and Gaussian estimates

In this section, we give further results about $H^p(\Lambda T^*M)$ spaces assuming some "Gaussian" upper bounds for the heat kernel of the Hodge-de Rham Laplacian on M. Denote by n the dimension of M. For each $0 \leq k \leq n$, let p_t^k be the kernel of $e^{-t\Delta_k}$, where Δ_k is the Hodge-de Rham Laplacian restricted to k-forms. Say that (G_k) holds if there exist $C, c > 0$ such that, for all $t > 0$ and all $x, y \in M$,

$$\left| p_t^k(x,y) \right| \leq \frac{C}{V(x, \sqrt{t})} e^{-cd^2(x,y)/t}. \tag{8.3}$$

Say that (G) holds if (G_k) holds for all $0 \leq k \leq n$. See the introduction for comments on the validity of $(G_{(k)})$ when $k \geq 1$.

8.2.1 The Coifman-Weiss Hardy space and Gaussian estimates

Under the assumptions of Theorem 8.1, and even if one assumes furthermore that (G_0) holds, the inclusion $H^1_{d^*}(\Lambda^0 T^*M) \subset H^1_{CW}(M)$, proved in Theorem 8.1, is strict in general. This can be seen by considering the example where M is the union of two copies of $\mathbb{R}^n$ ($n \geq 2$) glued smoothly together by a cylinder. First, on this manifold, (1.2) and (8.3) clearly hold (see [18]). Moreover, Theorem 5.16 asserts that the Riesz transform $d\Delta^{-1/2}$ is $H^1_{d^*}(\Lambda^0 T^*M) - L^1(\Lambda^1 T^*M)$ bounded on this manifold. But, as was kindly explained to us by A. Hassell, it is possible to prove, using arguments analogous to those contained in [13], that the Riesz transform is not $H^1_{CW}(M) - L^1(M)$ bounded (while it is shown in [13] that the Riesz transform is $L^p(M)$-bounded for all $1 < p < n$).

However, under a stronger assumption on M, the spaces $H^1_{CW}(M)$ and $H^1_{d^*}(\Lambda^0 T^*M)$ do coincide. Say that M satisfies an L^2 Poincaré inequality on balls if there exists $C>0$ such that, for any ball $B\subset M$ and any function $f\in C^\infty(2B)$,

$$\int_B |f(x)-f_B|^2\,dx \leq Cr^2\int_{2B}|\nabla f(x)|^2\,dx, \tag{8.4}$$

where f_B denotes the mean-value of f on B and r the radius of B. Then we have:

Theorem 8.2 *Assume (1.2) and (8.4). Then $H^1_{d^*}(\Lambda^0T^*M)=H^1_{CW}(M)$.*

Indeed, as recalled in the introduction, these assumptions on M imply that p_t satisfies the estimates (1.4), and these estimates, in turn, easily imply that any atom in $H^1_{CW}(M)$ belongs to $H^1_{max}(\Lambda^0T^*M)$ with a controlled norm. See, for instance, [6]. □

As a consequence of Theorem 8.2 and of Theorem 5.16, we recover the following result, already obtained in [38]:

Corollary 8.3 *Assume (1.2) and (8.4). Then, the Riesz transform on functions $d\Delta^{-1/2}$ is $H^1_{CW}(M)-L^1(M)$ bounded.*

Moreover, under the assumptions of Corollary 8.3, some kind of H^1-boundedness result for the Riesz transform had been proved by M. Marias and the third author in [33]. Namely, if u is a harmonic function on M (in the sense that $\Delta u=0$ in M) with a growth at most linear (which means that $|u(x)|\leq C(1+d(x_0,x))$ for some $x_0\in M$), the operator $R_uf=du\cdot d\Delta^{-1/2}f$ is $H^1_{CW}(M)$-bounded (here and after in this section, $\cdot$ stands for the real scalar product on 1-forms). Actually, we can also recover this result using the Hardy spaces defined in the present paper. Indeed, since, by Theorem 5.16, $d\Delta^{-1/2}$ is $H^1_{d^*}(\Lambda^0T^*M)-H^1_d(\Lambda^1T^*M)$ bounded, it suffices to prove that the map $g\mapsto du\cdot g$ is $H^1_d(\Lambda^1T^*M)-H^1_{d^*}(\Lambda^0T^*M)$ bounded.

To that purpose, because of the decomposition into molecules for $H^1_d(\Lambda^1T^*M)$, one may assume that $g=a$ is a 1-molecule for d^* in $H^1_d(\Lambda^1T^*M)$, see Section 6.1. Namely, one has $a=db$ where $b\in L^2(\Lambda^0T^*M)$ and there exists a ball B and a sequence $(\chi_k)_{k\geq 0}$ adapted to B such that, for each $k\geq 0$,

$$\|\chi_k a\|_2\leq 2^{-k}V^{-1/2}(2^kB) \text{ and } \|\chi_k b\|_2\leq r2^{-k}V^{-1/2}(2^kB).$$

But it is plain to see that, up to a constant, $du\cdot a$ is a 1-molecule for d^* in $H^1_{d^*}(\Lambda^0T^*M)$. Indeed, since du is bounded on M, one has, for each $k\geq 0$,

$$\|\chi_k du\cdot a\|_2\leq C\|\chi_k a\|_2\leq C2^{-k}V^{-1/2}(2^kB).$$

Moreover, since $\Delta u=0$ on M, one has

$$du\cdot a=du\cdot db=-d^*(bdu)+b\Delta u=-d^*(bdu),$$

and, for each $k\geq 0$,

$$\|\chi_k bdu\|_2\leq C\|\chi_k b\|_2\leq Cr2^{-k}V^{-1/2}(2^kB).$$

This ends the proof. □

8.2.2 The decomposition into molecules and Gaussian estimates

We state here an improved version of Theorem 6.2, assuming furthermore some Gaussian upper estimates:

Theorem 8.4 *Assume (1.2).*

(a) *If* (G) *holds, then* $H^1(\Lambda T^*M) = H^1_{mol,1}(\Lambda T^*M)$.

(b) *If* $1 \leq k \leq n$ *and* (G_{k-1}) *holds, then* $H^1_d(\Lambda^k T^*M) = H^1_{d,mol,1}(\Lambda^k T^*M)$.

(c) *If* $0 \leq k \leq n-1$ *and* (G_{k+1}) *holds, then* $H^1_{d^*}(\Lambda^k T^*M) = H^1_{d^*,mol,1}(\Lambda^k T^*M)$.

This theorem roughly says that, assuming Gaussian estimates, any section of $H^1(\Lambda T^*M)$ can be decomposed by means of 1-molecules instead of N-molecules for $N > \frac{\kappa}{2}+1$. Observe that, in assertion (c), if $M = \mathbb{R}^n$, the conclusion for $k=0$ is nothing but the usual atomic decomposition for functions in $H^1(\mathbb{R}^n)$.

Proof: We just give a sketch, which follows the same lines as the one of Theorem 6.2, focusing on assertion (a). The inclusion $H^1(\Lambda T^*M) \subset H^1_{mol,1}(\Lambda T^*M)$ was proved in Section 6.1 and does not require Gaussian estimates. As for the converse inclusion, consider a molecule $f = Dg$ where f and g satisfy (6.2), and define $F(x,t) = tDe^{-t^2\Delta}f(x)$. We argue exactly as in Section 6.2 for $\eta_k F$. For $\eta'_k F$, we use the fact that, if $0 \leq l \leq k-2$, $y \in 2^{k+1}B \setminus 2^k B$ and $r < t < 2^{k+1}r$, assumption (G) yields

$$\begin{aligned} \left| tDe^{-t^2\Delta} tD(\chi_l g)(y) \right| &= \left| t^2\Delta e^{-t^2\Delta}(\chi_l g)(y) \right| \\ &\leq \frac{C}{V(y,t)} e^{-c\frac{2^{2k}r^2}{t^2}} \int_{2^{l+1}B} |\chi_l g(z)|\, dz \\ &\leq \frac{C}{V(y,t)} r 2^{-l} e^{-c\frac{2^{2k}r^2}{t^2}}. \end{aligned}$$

Using this estimate, one concludes for $\eta'_k F$ in the same way as in Section 6.2. The other terms in the proof of Theorem 6.2 are dealt with in a similar way. This kind of argument can easily be transposed for assertions (b) and (c). □

An observation related to Theorem 8.4 is that the elements of $H^1_{mol,1}(\Lambda T^*M)$ actually have an **atomic** decomposition. More precisely, a section $a \in L^2(\Lambda T^*M)$ is called an atom if there exist a ball $B \subset M$ with radius r and a section $b \in L^2(\Lambda T^*M)$ such that b is supported in B, $a = Db$ and

$$\|a\|_{L^2(\Lambda T^*M)} \leq V^{-1/2}(B) \text{ and } \|b\|_{L^2(\Lambda T^*M)} \leq rV^{-1/2}(B). \tag{8.5}$$

Say that a section f of ΛT^*M belongs to $H^1_{at}(\Lambda T^*M)$ if and only if there exist a sequence $(\lambda_j)_{j\geq 1} \in l^1$ and a sequence $(a_j)_{j\geq 1}$ of atoms such that $f = \sum_j \lambda_j a_j$, and equip $H^1_{at}(\Lambda T^*M)$ with the usual norm. We claim that $H^1_{mol,1}(\Lambda T^*M) = H^1_{at}(\Lambda T^*M)$. Indeed, an atom is clearly a 1-molecule up to a multiplicative constant. Conversely, let $a = Db$ be a 1-molecule, B a ball and $(\chi_k)_{k\geq 0}$ a sequence of $C^\infty(M)$ functions adapted to B, such that (6.2) holds with $N = 1$. Notice that, for some universal constant $C' > 0$, one has

$$\left\| \chi_{2^{k+2}B \setminus 2^{k-1}B} b \right\|_{L^2(\Lambda T^*M)} \leq C' r 2^{-k} V^{-1/2}(2^{k+2}B)$$

(this fact is a consequence of the support properties of the χ_j's). Define now $C'' = \max(CC', 1)$ where $C > 0$ is the constant in (6.1). For all $k \geq 0$, set

$$b_k = \frac{2^{k-1}}{C''}\chi_k b \text{ and } a_k = Db_k.$$

It is obvious that b_k is supported in $2^{k+2}B$ and that $\|b_k\|_{L^2(\Lambda T^*M)} \leq 2^{k+2}rV^{-1/2}(2^{k+2}B)$. Moreover, by (3.5),

$$a_k = \frac{2^{k-1}}{C''}\left(\chi_k Db + d\chi_k \wedge b - d\chi_k \vee b\right) = \frac{2^{k-1}}{C''}\left(\chi_k a + d\chi_k \wedge b - d\chi_k \vee b\right), \tag{8.6}$$

which implies $\|a_k\|_{L^2(\Lambda T^*M)} \leq \frac{2^{k-1}}{C''}\left(2^{-k}V^{-1/2}(2^{k+2}B) + \frac{C}{2^k r}C'r2^{-k}V^{-1/2}(2^{k+2}B)\right) \leq V^{-1/2}(2^{k+2}B)$. Thus, for each $k \geq 0$, a_k is an atom. Moreover, since $\sum_k \chi_k = 1$, one has

$$\sum_k d\chi_k \wedge b = \sum_k d\chi_k \vee b = 0,$$

and (8.6) therefore yields

$$a = \sum_{k\geq 0} \frac{C''}{2^{k-1}} a_k,$$

which shows that $a \in H^1_{at}(\Lambda T^*M)$.

One can similarly define $H^1_{d,at}(\Lambda T^*M)$, and $H^1_{d,at}(\Lambda T^*M) = H^1_{d,mol,1}(\Lambda T^*M)$ holds by an analoguous argument. As a corollary of this fact and Theorem 8.4, we get that, when $M = \mathbb{R}^n$, $H^1_d(\Lambda^k T^*\mathbb{R}^n)$ coincides with the $\mathcal{H}^1_d(\mathbb{R}^n, \Lambda^k)$ space introduced in [32], as was claimed in the introduction.

8.2.3 H^p spaces and L^p spaces

It turns out that, assuming that (G) holds (which is the case of $M = \mathbb{R}^n$), one can compare precisely $H^p(\Lambda T^*M)$ and $L^p(\Lambda T^*M)$ for $1 < p \leq 2$:

Theorem 8.5 *Assume (1.2). Let $1 < p < 2$.*

- (a) *Assume (G). Then, $H^p(\Lambda T^*M) = \overline{\mathcal{R}(D) \cap L^p(\Lambda T^*M)}^{L^p(\Lambda T^*M)}$, $H^p_d(\Lambda T^*M) = \overline{\mathcal{R}(d) \cap L^p(\Lambda T^*M)}^{L^p(\Lambda T^*M)}$ and $H^p_{d^*}(\Lambda T^*M) = \overline{\mathcal{R}(d^*) \cap L^p(\Lambda T^*M)}^{L^p(\Lambda T^*M)}$*
- (b) *Assume (G_k) for some $0 \leq k \leq n$. Then $H^p(\Lambda^k T^*M) = \overline{\mathcal{R}(D) \cap L^p(\Lambda^k T^*M)}^{L^p(\Lambda^k T^*M)}$, and the corresponding equalities for $H^p_d(\Lambda^k T^*M)$ and $H^p_{d^*}(\Lambda^k T^*M)$ also hold.*

Proof: For assertion (a), the inclusion $H^p(\Lambda T^*M) \subset \overline{\mathcal{R}(D) \cap L^p(\Lambda T^*M)}^{L^p(\Lambda T^*M)}$ has already been proved (Corollary 6.3) and does not require assumption (G). Conversely, it is enough to deal with $f \in \mathcal{R}(D) \cap L^p(\Lambda T^*M)$. Theorem 6 in [3] ensures that $\mathcal{Q}_\psi f \in T^{p,2}(\Lambda T^*M)$, where $\psi \in \Psi_{1,\beta+1}(\Sigma^\theta_0)$ with $\beta = \left[\frac{\kappa}{2}\right] + 1$ (this is where we use Gaussian estimates). Now, for suitable $\widetilde{\psi} \in \Psi_{\beta,2}(\Sigma^\theta_0)$, one has $f = \mathcal{S}_{\widetilde{\psi}}\mathcal{Q}_\psi f$ since $f \in \mathcal{R}(D)$, which shows that $f \in H^p(\Lambda T^*M)$. The other equalities, as well as assertion (b), have a similar proof. □

Remark 8.6 *It is not known whether equality* $H^p(\Lambda^0 T^*M) = \overline{\mathcal{R}(D) \cap L^p(\Lambda^0 T^*M)}^{L^p(\Lambda^0 T^*M)}$ *holds for* $1 < p < 2$ *in general (i.e. without assuming Gaussian estimates for the heat kernel).*

Remark 8.7 *What happens in Theorem 8.5 for* $p \geq 2$ *? We proved in Corollary 6.3 that* $\overline{\mathcal{R}(D) \cap L^p(\Lambda T^*M)}^{L^p(\Lambda T^*M)} \subset H^p(\Lambda T^*M)$ *for* $2 \leq p < +\infty$. *The converse inclusion cannot be true in general. Indeed, assume that* $\overline{\mathcal{R}(D) \cap L^p(\Lambda T^*M)}^{L^p(\Lambda T^*M)} = H^p(\Lambda T^*M)$. *Then, if* $f \in L^p(M) \cap \mathcal{R}(D)$ *is a function, one has* $f \in H^p_{d^*}(\Lambda^0 T^*M)$ *and Theorem 5.15 shows that* $d\Delta^{-1/2} f \in H^p_d(\Lambda^0 T^*M)$. *Our assumption therefore implies* $d\Delta^{-1/2} f \in L^p(M)$. *In other words, the Riesz transform on functions* $d\Delta^{-1/2}$ *is* L^p*-bounded, which is false in general for* $p > 2$, *even if Gaussian upper estimates for the heat kernel hold ([18]), and even if the* L^2 *Poincaré inequality for balls is true (see [19]).*

As a corollary of Theorem 8.5 and of Theorem 5.16, we obtain:

Corollary 8.8 *Assume (1.2) and* (G_0). *Then, for all* $1 < p \leq 2$, *the Riesz transform on functions* $d\Delta^{-1/2}$ *is* $L^p(\Lambda^0 T^*M) - L^p(\Lambda^1 T^*M)$ *bounded.*

Proof: It suffices to consider $f \in \mathcal{R}(d^*) \cap L^p(\Lambda T^*M)$. Then, Theorem 8.5 ensures that $f \in H^p_{d^*}(\Lambda^0 T^*M)$, and Theorem 5.16 yields that $d\Delta^{-1/2} f \in H^p_d(\Lambda^1 T^*M) \subset L^p(\Lambda^1 T^*M)$. □

Note that Corollary 8.8 is not new and was originally proved in [18]. Actually, in [18], the weak $(1,1)$ boundedness of $d\Delta^{-1/2}$ is established, and the L^p boundedness for $1 < p \leq 2$ follows at once by interpolation with the L^2 boundedness. Our approach by Hardy spaces does not allow us to recover the weak $(1,1)$ boundedness for $d\Delta^{-1/2}$.

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