

# A double fibration transform for complex projective space

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*To Sigurdur Helgason on the occasion of his eighty-fifth birthday.*

ABSTRACT. We develop some theory of double fibration transforms where the cycle space is a smooth manifold and apply it to complex projective space.

## 1. Introduction

The classical *Penrose transform* is concerned with (anti)-self-dual 4-dimensional Riemannian manifolds. If  $M$  is such a manifold then, as shown in [1], there is a canonically defined 3-dimensional complex manifold  $Z$ , known as the *twistor space* of  $M$ , that fibres over  $M$

$$(1.1) \quad \tau : Z \rightarrow M$$

in the sense that  $\tau$  is a submersion with holomorphic fibres intrinsically isomorphic to  $\mathbb{C}\mathbb{P}_1$ . In fact, this construction depends only on the conformal structure on  $M$  and the Penrose transform then identifies the Dolbeault cohomology  $H^r(Z, \mathcal{O}(V))$  for the various natural holomorphic vector bundles  $V$  on  $Z$  with the cohomology of certain conformally invariant elliptic complexes of linear differential operators on  $M$ . Some typical examples are presented in [12, 16].

The two main examples of this construction are for  $M = S^4$ , the flat model of 4-dimensional conformal geometry, and for  $M = \mathbb{C}\mathbb{P}_2$  with its Fubini-Study metric. In both cases, the twistor space is a well-known complex manifold. For  $S^4$  it is  $\mathbb{C}\mathbb{P}_3$  and for  $\mathbb{C}\mathbb{P}_2$  it is the flag manifold

$$\mathbb{F}_{1,2}(\mathbb{C}^3) \equiv \{(L, P) \mid L \subset P \subset \mathbb{C}^3 \text{ with } \dim_{\mathbb{C}} L = 1, \dim_{\mathbb{C}} P = 2\}.$$

For  $\mathbb{C}\mathbb{P}_2$  the fibration is

$$(1.2) \quad \mathbb{F}_{1,2}(\mathbb{C}^3) \ni (L, P) \xrightarrow{\tau} L^\perp \cap P \in \mathbb{C}\mathbb{P}_2,$$

where the orthogonal complement  $L^\perp$  of  $L$  is taken with respect to a fixed Hermitian inner product on  $\mathbb{C}^3$ , namely the same inner product that induces the Fubini-Study metric on  $\mathbb{C}\mathbb{P}_2$  as a homogeneous space  $SU(3)/S(U(1) \times U(2))$ . The Penrose transform in this setting is carried out in detail in [7, 9].

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There are several options for generalising this twistor geometry of  $\mathbb{CP}_2$  to higher dimensions. Perhaps the most obvious is to take as twistor space the flag manifold  $\mathbb{F}_{1,2}(\mathbb{C}^{n+1})$  and define  $\tau : \mathbb{F}_{1,2}(\mathbb{C}^{n+1}) \rightarrow \mathbb{CP}_n$  by  $(L, P) \mapsto L^\perp \cap P$ . This is the option adopted in [10]. Perhaps a more balanced option is to take as twistor space the flag manifold

$$Z = \mathbb{F}_{1,n}(\mathbb{C}^{n+1}) \equiv \{(L, H) \mid L \subset H \subset \mathbb{C}^{n+1} \text{ with } \dim_{\mathbb{C}} L = 1, \dim_{\mathbb{C}} H = n\}$$

and consider the *double fibration*

$$(1.3) \quad \begin{array}{ccc} & X & \\ \eta \swarrow & & \searrow \tau \\ Z & & \mathbb{CP}_n \end{array}$$

where  $X \subset \mathbb{F}_{1,n}(\mathbb{C}^{n+1}) \times \mathbb{CP}_n$  is the *incidence variety* given by

$$(1.4) \quad X = \{(L, H, \ell) \mid \ell \subseteq L^\perp \cap H\}$$

and the fibrations  $\eta$  and  $\tau$  are the *forgetful mappings*,

$$\mathbb{F}_{1,n}(\mathbb{C}^{n+1}) \ni (L, H) \xleftarrow{\eta} (L, H, \ell) \xrightarrow{\tau} \ell \in \mathbb{CP}_n.$$

Of course, when  $n = 2$  the dimensions force  $\eta$  to be an isomorphism and this double fibration (1.3) reverts to the single fibration (1.2).

The aim of this article is to explain a transform on Dolbeault cohomology for double fibrations of this type and then execute the transform in this particular case. Then, since the Bott-Borel-Weil Theorem [4] computes the Dolbeault cohomology of  $Z = \mathbb{F}_{1,n}(\mathbb{C}^{n+1})$  with coefficients in any homogeneous vector bundle, we may draw conclusions concerning the cohomology of various elliptic complexes on  $\mathbb{CP}_n$ .

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## 2. The general transform

There is a better established double fibration transform defined for

$$\begin{array}{ccc} & X & \\ \mu \swarrow & & \searrow \nu \\ Z & & M \end{array}$$

in which all manifolds are complex and both  $\mu$  and  $\nu$  are holomorphic. Classical twistor theory, for example, is concerned with the holomorphic correspondence

$$\begin{array}{ccc} & \mathbb{F}_{1,2}(\mathbb{C}^4) & \\ \mu \swarrow & & \searrow \nu \\ \mathbb{CP}_3 & & \text{Gr}_2(\mathbb{C}^4). \end{array}$$

The Penrose transform in this setting is explained in [11] and generalised to arbitrary holomorphic correspondences between complex flag manifolds in [2]. Another vast generalisation is concerned with the holomorphic correspondences arising from the cycle spaces of general flag domains as in [15].

On the face of it, the double fibration (1.3) is of a different nature since  $\mathbb{CP}_n$  is only to be considered as a smooth manifold. In fact, a link will emerge with the complex correspondences and this will ease some of the computations involved. For the moment, however, let us develop some general machinery applicable to this

*smooth* setting. This machinery is a generalisation of the Penrose transform for a single fibration (1.1), which goes as follows. The only requirements on (1.1) are that  $\tau$  should be a smooth submersion from a complex manifold  $Z$  to a smooth manifold  $M$  and that the fibres of  $\tau$  should be compact complex submanifolds of  $Z$ .

Let us denote by  $\Lambda_Z^{0,q}$  the bundle of  $(0, q)$ -forms on  $Z$  and by  $\bar{\partial}_Z : \Lambda_Z^{0,q} \rightarrow \Lambda_Z^{0,q+1}$  the  $\bar{\partial}$ -operator on  $Z$  so that

$$H^r(Z, \mathcal{O}) \equiv H^r(\Gamma(Z, \Lambda_Z^{0,\bullet}), \bar{\partial}_Z)$$

is the Dolbeault cohomology of  $Z$ . The 1-forms along the fibres of  $\tau$ , defined by the short exact sequence

$$0 \rightarrow \tau^* \Lambda_M^1 \rightarrow \Lambda_Z^1 \rightarrow \Lambda_\tau^1 \rightarrow 0,$$

are decomposed as  $\Lambda_\tau^1 = \Lambda_\tau^{0,1} \oplus \Lambda_\tau^{1,0}$  by the complex structure on these fibres and the fact that this complex structure is acquired from that on  $Z$  implies that there is a commutative diagram

$$(2.1) \quad \begin{array}{ccccccc} 0 & \rightarrow & \Lambda^{0,0} & \xrightarrow{\bar{\partial}_\tau} & \Lambda_\tau^{0,1} & \xrightarrow{\bar{\partial}_\tau} & \Lambda_\tau^{0,2} & \xrightarrow{\bar{\partial}_\tau} & \dots \\ & & \parallel & & \uparrow & & \uparrow & & \\ 0 & \rightarrow & \Lambda^{0,0} & \xrightarrow{\bar{\partial}_Z} & \Lambda_Z^{0,1} & \xrightarrow{\bar{\partial}_Z} & \Lambda_Z^{0,2} & \xrightarrow{\bar{\partial}_Z} & \dots \end{array}$$

where the top row is the  $\bar{\partial}$ -complex along the fibres of  $\tau$ . Though the notation may seem bizarre at first, let us define the bundle  $\Lambda_\mu^{1,0}$  on  $Z$  by the short exact sequence

$$(2.2) \quad 0 \rightarrow \Lambda_\mu^{1,0} \rightarrow \Lambda_Z^{0,1} \rightarrow \Lambda_\tau^{0,1} \rightarrow 0.$$

Regarded as a filtration of  $\Lambda_Z^{0,1}$ , this short exact sequence induces filtrations on  $\Lambda_Z^{0,q}$  for all  $q$  and (2.1) implies that  $\bar{\partial}_Z$  is compatible with this filtration. An immediate consequence is that the bundle  $\Lambda_\mu^{1,0}$  acquires a holomorphic structure along the fibres of  $\tau$ . To see this by hand, one notes that the composition

$$\Lambda_\mu^{1,0} \rightarrow \Lambda_Z^{0,1} \xrightarrow{\bar{\partial}_Z} \Lambda_Z^{0,2} \rightarrow \Lambda_\tau^{0,2}$$

vanishes by dint of the definition (2.2) of  $\Lambda_\mu^{1,0}$  and the commutative diagram (2.1) whence the short exact sequence

$$0 \rightarrow \Lambda_\mu^{2,0} \rightarrow [\ker : \Lambda_Z^{0,2} \rightarrow \Lambda_\tau^{0,2}] \rightarrow \Lambda_\tau^{0,1} \otimes \Lambda_\mu^{1,0} \rightarrow 0$$

induced by (2.2) implies that  $\bar{\partial}_Z|_{\Lambda_\mu^{1,0}}$  induces an operator

$$\bar{\partial}_\tau : \Lambda_\mu^{1,0} \rightarrow \Lambda_\tau^{0,1} \otimes \Lambda_\mu^{1,0},$$

as required. To see this (and much more) by machinery, one employs the spectral sequence of the filtered complex  $\Lambda_Z^{0,\bullet}$ , arriving at the  $E_0$ -level

$$\begin{array}{ccccccc} & & \uparrow q & & & & \\ & & \Lambda_\tau^{0,3} & & & & \\ & & \uparrow \bar{\partial}_\tau & & & & \\ & & \Lambda_\tau^{0,2} & & \Lambda_\tau^{0,2} \otimes \Lambda_\mu^{1,0} & & \\ & & \uparrow \bar{\partial}_\tau & & \uparrow \bar{\partial}_\tau & & \\ & & \Lambda_\tau^{0,1} & & \Lambda_\tau^{0,1} \otimes \Lambda_\mu^{1,0} & & \Lambda_\tau^{0,1} \otimes \Lambda_\mu^{2,0} \\ & & \uparrow \bar{\partial}_\tau & & \uparrow \bar{\partial}_\tau & & \uparrow \bar{\partial}_\tau \\ \Lambda^{0,0} & \xrightarrow{\quad} & \Lambda_\mu^{1,0} & \xrightarrow{\quad} & \Lambda_\mu^{2,0} & \xrightarrow{\quad} & \Lambda_\mu^{3,0} \xrightarrow{p} \end{array}$$

and, in particular, the differential  $\Lambda_\mu^{1,0} \xrightarrow{\bar{\partial}_\tau} \Lambda_\tau^{0,1} \otimes \Lambda_\mu^{1,0}$ . This spectral sequence for  $\Gamma(Z, \Lambda_Z^{0,\bullet})$ , at the  $E_1$ -level, reads

$$\begin{array}{ccccccc}
& & \uparrow q & & & & \\
& & \Gamma(M, \tau_*^3 \Lambda_\mu^{0,0}) & & & & \\
& & \downarrow & & & & \\
& & \Gamma(M, \tau_*^2 \Lambda_\mu^{0,0}) & \longrightarrow & \Gamma(M, \tau_*^2 \Lambda_\mu^{1,0}) & & \\
& & \downarrow & & & & \\
& & \Gamma(M, \tau_*^1 \Lambda_\mu^{0,0}) & \longrightarrow & \Gamma(M, \tau_*^1 \Lambda_\mu^{1,0}) & \longrightarrow & \Gamma(M, \tau_*^1 \Lambda_\mu^{2,0}) \\
& & \downarrow & & & & \\
& & \Gamma(M, \tau_* \Lambda_\mu^{0,0}) & \longrightarrow & \Gamma(M, \tau_* \Lambda_\mu^{1,0}) & \longrightarrow & \Gamma(M, \tau_* \Lambda_\mu^{2,0}) & \longrightarrow & \Gamma(M, \tau_* \Lambda_\mu^{3,0}) & \xrightarrow{p}
\end{array}$$

where  $\tau_*^q \Lambda_\mu^{p,0}$  is the  $q^{\text{th}}$  direct image of the vector bundle  $\Lambda_\mu^{p,0}$  with respect to its holomorphic structure in the fibre directions. Note that, with the fibres of  $\tau$  being compact, these direct images generically define smooth vector bundles on  $M$  and certainly this will be the case when the fibration (1.1) is homogeneous. In any case, we have proved the following.

**THEOREM 2.1.** *Suppose that  $\tau : Z \rightarrow M$  is a submersion of smooth manifolds and that  $Z$  has a complex structure such that the fibres of  $\tau$  are compact complex submanifolds of  $Z$ . Then the bundle  $\Lambda_\mu^{1,0} \equiv \ker : \Lambda_Z^{0,1} \rightarrow \Lambda_\tau^{0,1}$  acquires a natural holomorphic structure along the fibres of  $\tau$  and there is a spectral sequence*

$$E_1^{p,q} = \Gamma(M, \tau_*^q \Lambda_\mu^{p,0}) \implies H^{p+q}(Z, \mathcal{O}).$$

This theorem only comes to life with examples in which it is possible to compute the direct images  $\tau_*^q \Lambda_\mu^{p,0}$ . There is also a coupled version of the spectral sequence

$$E_1^{p,q} = \Gamma(M, \tau_*^q \Lambda_\mu^{p,0}(V)) \implies H^{p+q}(Z, \mathcal{O}(V))$$

for any holomorphic vector bundle  $V$  on  $Z$ . The proof is easily modified but the added scope for interesting examples is significantly increased.

For the moment, however, let us continue with generalities, firstly by extending Theorem 2.1 to cover double fibrations of the form

$$(2.3) \quad \begin{array}{ccc} & X & \\ \eta \swarrow & & \searrow \tau \\ Z & & M \end{array}$$

(of which (1.2) is typical) where  $M$  is smooth and the fibres of  $\tau$  are identified by  $\eta$  as compact complex submanifolds of the complex manifold  $Z$ . To do this, let us define a bundle  $\Lambda_X^{0,1}$  on  $X$  by means of the short exact sequence

$$(2.4) \quad 0 \rightarrow \eta^* \Lambda_Z^{1,0} \rightarrow \Lambda_X^1 \rightarrow \Lambda_X^{0,1} \rightarrow 0,$$

where  $\Lambda_X^1$  is the bundle of complex-valued 1-forms on  $X$ . Geometrically, this pulls back the complex structure from  $Z$  to an *involutive structure* [3] on  $X$ . In particular, there is an induced complex of differential operators

$$0 \rightarrow \Lambda_X^{0,0} \xrightarrow{\bar{\partial}_X} \Lambda_X^{0,1} \xrightarrow{\bar{\partial}_X} \Lambda_X^{0,2} \xrightarrow{\bar{\partial}_X} \dots$$

Comparing (2.4) with the bundle  $\Lambda_\eta^1$  of 1-forms along the fibres of  $\eta$  defined by the short exact sequence

$$0 \rightarrow \eta^* \Lambda_Z^1 \rightarrow \Lambda_X^1 \rightarrow \Lambda_\eta^1 \rightarrow 0$$

we see that there is a short exact sequence

$$0 \rightarrow \eta^* \Lambda_Z^{0,1} \rightarrow \Lambda_X^{0,1} \rightarrow \Lambda_\eta^1 \rightarrow 0.$$

The complex  $\Gamma(X, \Lambda_X^{0,\bullet})$  thereby acquires a filtration, the spectral sequence for which reads at the  $E_0$ -level

$$(2.5) \quad \begin{array}{ccccc} & & \uparrow^q & & \\ & & \Gamma(X, \Lambda_\eta^3 \otimes \eta^* \Lambda_Z^{0,0}) & & \\ & & \uparrow^{d_\eta} & & \\ & & \Gamma(X, \Lambda_\eta^2 \otimes \eta^* \Lambda_Z^{0,0}) & & \Gamma(X, \Lambda_\eta^2 \otimes \eta^* \Lambda_Z^{0,1}) \\ & & \uparrow^{d_\eta} & & \uparrow^{d_\eta} \\ & & \Gamma(X, \Lambda_\eta^1 \otimes \eta^* \Lambda_Z^{0,0}) & & \Gamma(X, \Lambda_\eta^1 \otimes \eta^* \Lambda_Z^{0,1}) & & \Gamma(X, \Lambda_\eta^1 \otimes \eta^* \Lambda_Z^{0,2}) \\ & & \uparrow^{d_\eta} & & \uparrow^{d_\eta} & & \uparrow^{d_\eta} \\ & & \Gamma(X, \eta^* \Lambda_Z^{0,0}) & \longrightarrow & \Gamma(X, \eta^* \Lambda_Z^{0,1}) & \longrightarrow & \Gamma(X, \eta^* \Lambda_Z^{0,2}) & \longrightarrow & \Gamma(X, \eta^* \Lambda_Z^{0,3}) & \xrightarrow{p} \end{array}$$

where  $d_\eta : \Lambda_\eta^q \otimes \eta^* \Lambda^{0,p} \rightarrow \Lambda_\eta^{q+1} \otimes \eta^* \Lambda^{0,p}$  is the exterior derivative along the fibres of  $\eta$  coupled with the pullback bundle  $\eta^* \Lambda^{0,p}$ . Notice that such a coupling

$$d_\eta : \eta^* V \rightarrow \Lambda_\eta^1 \otimes \eta^* V \quad \text{and hence} \quad d_\eta : \Lambda_\eta^q \otimes \eta^* V \rightarrow \Lambda_\eta^{q+1} \otimes \eta^* V$$

is valid for any smooth vector bundle  $V$  on  $Z$  because the pullback  $\eta^* V$  may be defined by transition functions that are constant along the fibres, hence annihilated by  $d_\eta$ . When the fibres of  $\eta$  are contractible, this is exactly the setting in which Buchdahl's theorem [6] applies and we deduce the following.

PROPOSITION 2.2. *Suppose that the fibres of  $\eta : X \rightarrow Z$  are contractible. Then*

$$0 \rightarrow \Gamma(Z, V) \rightarrow \Gamma(X, \eta^* V) \xrightarrow{d_\eta} \Gamma(X, \Lambda_\eta^1 \otimes \eta^* V) \xrightarrow{d_\eta} \Gamma(X, \Lambda_\eta^2 \otimes \eta^* V) \xrightarrow{d_\eta} \dots$$

*is exact for any smooth vector bundle  $V$  on  $Z$ .*

In this case, our spectral sequence (2.5) collapses at the  $E_1$ -level and we have proved the following.

PROPOSITION 2.3. *Suppose that the fibres of  $\eta : X \rightarrow Z$  are contractible. Then*

$$H^r(Z, \mathcal{O}) \cong H^r(\Gamma(X, \Lambda_X^{0,\bullet}), \bar{\partial}_X) \quad \text{for all } r = 0, 1, 2, \dots$$

In fact, for the double fibration (1.3), the fibres of  $\eta$  are not contractible and in §3 we shall have to revisit the spectral sequence (2.5) to relate the Dolbeault cohomology  $H^r(Z, \mathcal{O})$  with the *involutive cohomology*  $H^r(\Gamma(X, \Lambda_X^{0,\bullet}), \bar{\partial}_X)$ .

Nevertheless, we may deal with the fibration  $\tau : X \rightarrow M$  exactly as in our proof of Theorem 2.1. Specifically, we define a bundle  $\Lambda_\mu^{1,0}$  on  $X$  by the exact sequence

$$(2.6) \quad 0 \rightarrow \Lambda_\mu^{1,0} \rightarrow \Lambda_X^{0,1} \rightarrow \Lambda_\tau^{0,1} \rightarrow 0$$

and employ the spectral sequence of the corresponding filtered complex  $\Lambda_X^{0,\bullet}$  to conclude that the following theorem holds.

THEOREM 2.4. *Suppose that*

$$(2.3) \quad \begin{array}{ccc} & X & \\ \eta \swarrow & & \searrow \tau \\ Z & & M \end{array}$$

*is a double fibration of smooth manifolds such that*

- $Z$  is a complex manifold,
- the fibres of  $\tau$  are embedded by  $\eta$  as compact complex submanifolds of  $Z$ .

Then the bundle  $\Lambda_\mu^{1,0}$ , defined as the middle cohomology of the complex

$$(2.7) \quad 0 \rightarrow \eta^* \Lambda_Z^{1,0} \rightarrow \Lambda_X^1 \rightarrow \Lambda_\tau^{0,1} \rightarrow 0,$$

acquires a natural holomorphic structure along the fibres of  $\tau$  and there is a spectral sequence

$$(2.8) \quad E_1^{p,q} = \Gamma(M, \tau_*^q \Lambda_\mu^{p,0}) \implies H^{p+q}(\Gamma(X, \Lambda_X^{0,\bullet}), \bar{\partial}_X).$$

COROLLARY 2.5. *If, in addition, the fibres of  $\eta$  are contractible, then*

$$E_1^{p,q} = \Gamma(M, \tau_*^q \Lambda_\mu^{p,0}) \implies H^{p+q}(Z, \mathcal{O}).$$

PROOF. Immediate from Proposition 2.3.  $\square$

In §4 we shall present an example for which the fibres of  $\eta$  are, indeed, contractible and to which Corollary 2.5 applies.

Before we continue, let us glance ahead to §3 in which the first thing we do is use (2.5) to deal with the topology along the fibres of  $\eta$  for the double fibration (1.3). Another thing we need in order to apply Theorem 2.4 is a computation of the direct images  $\tau_*^q \Lambda_\mu^{p,0}$  as homogeneous bundles on  $\mathbb{C}\mathbb{P}_n$ . This computation is best viewed in the light of a geometric interpretation of  $\Lambda_\mu^{1,0}$  as follows.

Suppose that  $M$  is a totally real submanifold of a complex manifold  $\mathbb{M}$  such that the double fibration (2.3) embeds as

$$(2.9) \quad \begin{array}{ccc} & X & \\ \eta \swarrow & & \searrow \tau \\ Z & & M \end{array} \hookrightarrow \begin{array}{ccc} & \mathbb{X} & \\ \mu \swarrow & & \searrow \nu \\ Z & & \mathbb{M}, \end{array}$$

where the ambient double fibration is in the holomorphic category and the fibres of  $\nu$  coincide with the fibres of  $\tau$  over  $M$ .

PROPOSITION 2.6. *Under these circumstances the bundle  $\Lambda_\mu^{1,0}$  of  $(1,0)$ -forms along the fibres of  $\mu$  coincides, when restricted to  $X \subset \mathbb{X}$ , with the bundle already denoted in the same way and defined as the middle cohomology of (2.7).*

PROOF. If we write

$$n = \dim_{\mathbb{C}} Z \quad m = \dim_{\mathbb{R}} M \quad s = \dim_{\mathbb{C}}(\text{fibres of } \tau),$$

then  $\dim_{\mathbb{R}} X = m + 2s$  and  $X$  has real codimension  $2(n - s)$  in  $Z \times M$ . This is the same as the real codimension of  $\mathbb{X}$  in  $Z \times \mathbb{M}$  and it follows that the complexified conormal bundle  $\mathcal{C}$  of  $X$  in  $Z \times M$  coincides with the restriction to  $X$  of the complexified conormal bundle of  $\mathbb{X}$  in  $Z \times \mathbb{M}$ . Hence it splits as  $\mathcal{C} = \mathcal{C}^{0,1} \oplus \mathcal{C}^{1,0}$  in line with the complex structure on the ambient double fibration. For any double fibration there is a basic commutative diagram with exact rows and columns, which

in the case of (2.3) looks as follows.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 & & \Lambda_\tau^1 & = & \Lambda_\tau^1 & & \\
 & & \uparrow & & \uparrow & & \\
 0 & \rightarrow & \eta^* \Lambda_Z^1 & \rightarrow & \Lambda_X^1 & \rightarrow & \Lambda_\eta^1 \rightarrow 0 \\
 & & \uparrow & & \uparrow & & \parallel \\
 0 & \rightarrow & \mathcal{C} & \rightarrow & \tau^* \Lambda_M^1 & \rightarrow & \Lambda_\eta^1 \rightarrow 0 \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

But we have just observed that the left hand column has the additional feature that it splits

$$\begin{array}{ccc}
 \Lambda_\tau^1 & = & \Lambda_\tau^{0,1} \oplus \Lambda_\tau^{1,0} \\
 \uparrow & & \uparrow \quad \uparrow \\
 \eta^* \Lambda_Z^1 & & \eta^* \Lambda_Z^{0,1} \oplus \eta^* \Lambda_Z^{1,0} \\
 \uparrow & & \uparrow \quad \uparrow \\
 \mathcal{C} & = & \mathcal{C}^{0,1} \oplus \mathcal{C}^{1,0}
 \end{array}$$

in line with the ambient complex structure. Hence we obtain the diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 & & \Lambda_\tau^{0,1} & = & \Lambda_\tau^{0,1} & & \\
 & & \uparrow & & \uparrow & & \\
 0 & \rightarrow & \eta^* \Lambda_Z^{0,1} & \rightarrow & \Lambda_X^1 / \eta^* \Lambda_Z^{1,0} & \rightarrow & \Lambda_\eta^1 \rightarrow 0 \\
 & & \uparrow & & \uparrow & & \parallel \\
 0 & \rightarrow & \mathcal{C}^{0,1} & \rightarrow & \tau^* \Lambda_M^1 / \mathcal{C}^{1,0} & \rightarrow & \Lambda_\eta^1 \rightarrow 0 \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

and it follows from (2.4) and (2.6) that  $\Lambda_\mu^{1,0} = \tau^* \Lambda_M^1 / \mathcal{C}^{1,0}$ . On the other hand, since  $M \hookrightarrow \mathbb{M}$  is totally real, we may identify  $\Lambda_M^1$  with  $\Lambda_{\mathbb{M}}^{1,0}$  along  $M$  and therefore  $\tau^* \Lambda_M^1$  with  $\nu^* \Lambda_{\mathbb{M}}^{1,0}$  along  $X \subset \mathbb{X}$ , at which point the basic diagram on  $\mathbb{X}$

$$(2.10) \quad \begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 & & \Lambda_\nu^{1,0} & = & \Lambda_\nu^{1,0} & & \\
 & & \uparrow & & \uparrow & & \\
 0 & \rightarrow & \mu^* \Lambda_Z^{1,0} & \rightarrow & \Lambda_{\mathbb{X}}^{1,0} & \rightarrow & \Lambda_\mu^{1,0} \rightarrow 0 \\
 & & \uparrow & & \uparrow & & \parallel \\
 0 & \rightarrow & \mathcal{C}^{1,0} & \rightarrow & \nu^* \Lambda_{\mathbb{M}}^{1,0} & \rightarrow & \Lambda_\mu^{1,0} \rightarrow 0 \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

for the ambient double fibration in the holomorphic setting finishes the proof.  $\square$

Finally, it is left to the reader also to check that the holomorphic structure for the bundle  $\Lambda_\mu^{1,0}$  on  $X$  along the fibres of  $\tau$  coincides with the standard holomorphic structure along the fibres of  $\mu$  for the bundle  $\Lambda_\mu^{1,0}$  on  $\mathbb{X}$  when restricted to  $X \hookrightarrow \mathbb{X}$ .

In summary, for a double fibration of the form (2.3), firstly we have a spectral sequence (2.5) that can be used to interpret Dolbeault cohomology on  $Z$  in terms

of involutive cohomology on  $X$ , secondly another spectral sequence (2.8) that can be used to interpret the involutive cohomology on  $X$  in terms of smooth data on  $M$  and, thirdly, in case that (2.3) complexifies as (2.9), a geometric interpretation of the bundles  $\Lambda_\mu^{p,0}$  occurring in this spectral sequence. In the following section, we shall see that this is just what we need to operate the transform onto complex projective space starting with the double fibration (1.3).

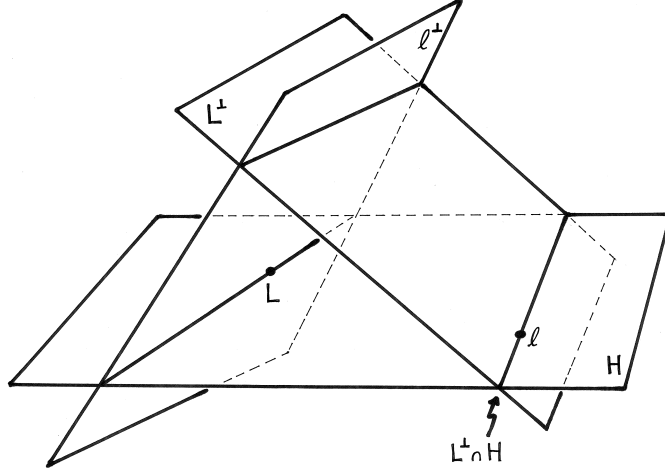
### 3. A particular transform

This section is entirely concerned with the double fibration (1.3), which will be dealt with mainly by means of Theorem 2.4. But, as foretold in §2, the first thing we should do is deal with the topology of the fibres of  $\eta$ .

PROPOSITION 3.1. *For the double fibration (1.3)*

- *the fibres of  $\eta$  are isomorphic to  $\mathbb{C}\mathbb{P}_{n-2}$  as smooth manifolds,*
- *the fibres of  $\tau$  are isomorphic to  $\mathbb{F}_{1,n-1}(\mathbb{C}^n)$  as complex manifolds.*

PROOF. It is useful to draw a picture in  $\mathbb{C}\mathbb{P}_n$  of the incidence variety (1.4) (although, of course, this is a picture over the reals in case  $n = 3$ ).



There are two points  $L$  and  $\ell$  and three hyperplanes  $L^\perp$ ,  $\ell^\perp$ , and  $H$ . Since  $L^\perp \cap H$  is the intersection of two hyperplanes in  $\mathbb{C}\mathbb{P}_n$  it is intrinsically  $\mathbb{C}\mathbb{P}_{n-2}$  and, since the mapping  $\eta$  from this configuration to  $\mathbb{F}_{1,n}(\mathbb{C}^{n+1})$  forgets everything but  $L \in H$ , we have identified its fibres with  $\mathbb{C}\mathbb{P}_{n-2}$ . On the other hand, if  $\ell$  is fixed, then the rest of the configuration may be constructed by choosing an arbitrary point  $L \in \ell^\perp$  and an arbitrary hyperplane in  $\ell^\perp$  passing through  $L$ , defining  $H$  as the join of this hyperplane with  $\ell$ .  $\square$

Examining this configuration also shows how the double fibration (1.3) may be naturally complexified to obtain (2.9). One simply allows the point  $\ell \in \mathbb{C}\mathbb{P}_n$  and the hyperplane  $\ell^\perp \in \mathbb{C}\mathbb{P}_n^*$  to become unrelated save for retaining that  $\ell \notin \ell^\perp$ . More precisely, let

$$\mathbb{M} \equiv \{(\ell, h) \in \mathbb{C}\mathbb{P}_n \times \mathbb{C}\mathbb{P}_n^* \mid \ell \notin h\} = \mathbb{F}_1(\mathbb{C}^{n+1}) \times \mathbb{F}_n(\mathbb{C}^{n+1}) \setminus \mathbb{F}_{1,n}(\mathbb{C}^{n+1})$$



with  $\mathbb{C}\mathbb{P}_n \equiv M \hookrightarrow M$  given by  $\ell \mapsto (\ell, \ell^\perp)$ , where the orthogonal complement is taken with respect to a fixed Hermitian inner product on  $\mathbb{C}^{n+1}$ . If we set

$$\mathbb{X} \equiv \{(L, H, \ell, h) \mid L \subset h \text{ and } \ell \subset H\}$$

then clearly this extends  $X$  in (1.4): the geometry is exactly the same except that  $\ell^\perp$  is replaced by the less constrained hyperplane  $h$ . An advantage of the complexified double fibration is that it is homogeneous under the action of  $\mathrm{GL}(n+1, \mathbb{C})$ :

$$(3.1) \quad \begin{array}{ccc} & \mathrm{GL}(n+1, \mathbb{C}) / \left\{ \begin{array}{c|ccc|c} * & 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & * & * & \cdots & * & * \\ \hline 0 & 0 & * & \cdots & * & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & * & \cdots & * & * \\ \hline 0 & 0 & 0 & \cdots & 0 & * \end{array} \right\} & \\ \mu \swarrow & & \searrow \nu \\ \mathrm{GL}(n+1, \mathbb{C}) / \left\{ \begin{array}{c|ccc|c} * & 0 & * & \cdots & * & * \\ \hline * & * & * & \cdots & * & * \\ \hline * & 0 & * & \cdots & * & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ * & 0 & * & \cdots & * & * \\ \hline 0 & 0 & 0 & \cdots & 0 & * \end{array} \right\} & \mathrm{GL}(n+1, \mathbb{C}) / \left\{ \begin{array}{c|ccc|c} * & 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & * & * & \cdots & * & * \\ \hline 0 & * & * & \cdots & * & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & * & * & \cdots & * & * \\ \hline 0 & * & * & \cdots & * & * \end{array} \right\} \end{array}$$

Before exploiting this homogeneity, however, there is an immediate consequence of Proposition 3.1, as follows.

PROPOSITION 3.2. *Concerning the double fibration (1.3), there are canonical isomorphisms*

$$H^r(\Gamma(X, \Lambda_X^{0,\bullet}), \bar{\partial}_X) = \mathbb{C} \quad \text{for } r = 0, 2, 4, 6, \dots, 2n - 4,$$

and the cohomology in other degrees vanishes.

PROOF. From Proposition 3.1 and the well-known de Rham cohomology of complex projective space [5], it follows from (2.5) that the  $E_1$ -level of this spectral sequence is isomorphic to

$$(3.2) \quad \begin{array}{ccccccc} & \uparrow q & & & & & \\ \Gamma(Z, \Lambda^{0,0}) & \longrightarrow & \Gamma(Z, \Lambda^{1,0}) & \longrightarrow & \cdots & & \\ \downarrow 0 & \longrightarrow & 0 & \longrightarrow & 0 & & \\ \Gamma(Z, \Lambda^{0,0}) & \longrightarrow & \Gamma(Z, \Lambda^{1,0}) & \longrightarrow & \Gamma(Z, \Lambda^{2,0}) & \longrightarrow & \cdots \\ \downarrow 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ \Gamma(Z, \Lambda^{0,0}) & \longrightarrow & \Gamma(Z, \Lambda^{1,0}) & \longrightarrow & \Gamma(Z, \Lambda^{2,0}) & \longrightarrow & \Gamma(Z, \Lambda^{3,0}) \xrightarrow{p} \end{array}$$

But, the fibres of  $\eta$  are not only isomorphic to  $\mathbb{C}\mathbb{P}_{n-2}$  as smooth manifolds but as Kähler manifolds—the fixed Hermitian inner product on  $\mathbb{C}^{n+1}$  endows each fibre with a canonical Kähler metric. In particular, the Kähler form and its exterior powers provide an explicit basis for the de Rham cohomology and therefore this identification of the  $E_1$ -level becomes canonical. Now, as a very special case of the

Bott-Borel-Weil Theorem [4], the cohomology of each row of (3.2) is concentrated in zeroth position where it is canonically identified with  $\mathbb{C}$ . As this spectral sequence converges to  $H^{p+q}(\Gamma(X, \Lambda_X^{0,\bullet}), \bar{\partial}_X)$ , the proof is complete.  $\square$

Now we come to the task of interpreting the spectral sequence (2.8). As already mentioned in §2 we shall use Proposition 2.6 and the complexified double fibration

$$\begin{array}{ccc} & X & \\ \eta \swarrow & & \searrow \tau \\ \mathbb{F}_{1,n}(\mathbb{C}^{n+1}) & & \mathbb{C}\mathbb{P}_n \end{array} \hookrightarrow \begin{array}{ccc} & \mathbb{X} & \\ \mu \swarrow & & \searrow \nu \\ \mathbb{F}_{1,n}(\mathbb{C}^{n+1}) & & \mathbb{M} = \{(\ell, h) \in \mathbb{C}\mathbb{P}_n \times \mathbb{C}\mathbb{P}_n^* \mid \ell \notin h\} \end{array}$$

to identify the direct images  $\tau_*^q \Lambda_\mu^{0,p}$ . This, in turn, will be facilitated by the fact that the complexification is  $\mathrm{GL}(n+1, \mathbb{C})$ -homogeneous as in (3.1).

For simplicity, we shall now restrict to the case  $n = 3$ , the general case being only notationally more awkward. Adapting the notation of [8], the irreducible homogeneous vector bundles on  $\mathbb{M}$  may be denoted

$$(a \parallel b, c, d) \quad \text{for integers } a, b, c, d \text{ with } b \leq c \leq d.$$

For example, the holomorphic cotangent bundle is

$$(3.3) \quad (-1 \parallel 0, 0, 1) \oplus (1 \parallel -1, 0, 0),$$

being the analytic continuation of the bundle  $\Lambda_M^1 = \Lambda_M^{0,1} \oplus \Lambda_M^{1,0}$  on  $M$ . Similarly, the irreducible homogeneous vector bundles on  $\mathbb{X}$  are necessarily line bundles and may be denoted

$$(a \parallel b \mid c \mid d) \quad \text{for arbitrary integers } a, b, c, d.$$

By carefully unravelling the meaning of these symbols in terms of weights, one can check that the bundle  $\Lambda_\mu^{1,0}$  is reducible and

$$(3.4) \quad \Lambda_\mu^{1,0} = \begin{array}{c} (-1 \parallel 0 \mid 0 \mid 1) + (-1 \parallel 0 \mid 1 \mid 0) \\ \oplus \\ (1 \parallel -1 \mid 0 \mid 0) + (1 \parallel 0 \mid -1 \mid 0) \end{array}$$

where  $(-1 \parallel 0 \mid 0 \mid 1) + (-1 \parallel 0 \mid 1 \mid 0)$ , for example, means that this is a rank 2 bundle with composition factors as indicated, equivalently that there is an exact sequence

$$0 \rightarrow (-1 \parallel 0 \mid 1 \mid 0) \rightarrow (-1 \parallel 0 \mid 0 \mid 1) + (-1 \parallel 0 \mid 1 \mid 0) \rightarrow (-1 \parallel 0 \mid 0 \mid 1) \rightarrow 0.$$

The procedure for computing direct images is explained in [8] and here we find

$$\nu_*(-1 \parallel 0 \mid 0 \mid 1) = (-1 \parallel 0, 0, 1) \quad \nu_*(1 \parallel -1 \mid 0 \mid 0) = (1 \parallel -1, 0, 0)$$

with all other direct images vanishing (e.g.  $(-1 \parallel 0 \mid 1 \mid 0)$  is singular along the fibres of  $\nu$ ). Bearing in mind that the fibres of  $\nu$  coincide with those of  $\tau$  over  $M$ , we have proved the following.

LEMMA 3.3. *For the double fibration (1.3) and  $\Lambda_\mu^{1,0}$  defined on  $X$  by the exact sequence (2.6), we have*

$$\tau_* \Lambda_\mu^{1,0} = \Lambda_M^1 \quad \text{and all higher direct images vanish.}$$

From (3.4) and the algorithms in [8] the higher forms are

$$\Lambda_\mu^{2,0} = (-2 \parallel 0 \mid 1 \mid 1) \oplus \left[ \begin{array}{c} (0 \parallel -1 \mid 0 \mid 1) + \\ \oplus \\ (0 \parallel 0 \mid 0 \mid 0) \end{array} \right] \oplus (2 \parallel -1 \mid -1 \mid 0)$$

and therefore

$$\nu_*\Lambda_\mu^{2,0} = (-2 \parallel 0, 1, 1) \oplus (0 \parallel -1, 0, 1) \oplus (0 \parallel 0, 0, 0) \oplus (2 \parallel -1, -1, 0) = \Lambda_M^2$$

with all higher direct images vanishing. Next,

$$\Lambda_\mu^{3,0} = \begin{array}{c} (-1 \parallel -1 \mid 1 \mid 1) + (-1 \parallel 0 \mid 0 \mid 1) \\ \oplus \\ (1 \parallel -1 \mid -1 \mid 1) + (1 \parallel -1 \mid 0 \mid 0) \end{array}$$

whence

$$\nu_*\Lambda_\mu^{3,0} = \begin{array}{ccc} (-1 \parallel -1, 1, 1) \oplus (-1 \parallel 0, 0, 1) & \Lambda_M^{1,2} \\ \oplus & = \oplus \\ (1 \parallel -1, -1, 1) \oplus (1 \parallel -1, 0, 0) & \Lambda_M^{2,1} \end{array}$$

with all higher direct images vanishing. Finally,

$$\Lambda_\mu^{4,0} = (0 \parallel -1 \mid 0 \mid 1) \implies \nu_*\Lambda_\mu^{4,0} = (0 \parallel -1, 0, 1) = \Lambda_{M,\perp}^{2,2},$$

where  $\Lambda_{M,\perp}^{2,2}$  denotes the  $(2, 2)$ -forms orthogonal to  $\kappa \wedge \kappa$  where  $\kappa$  is the Kähler form on  $M = \mathbb{C}\mathbb{P}_3$ . Again, the higher direct images vanish.

Feeding all this information into the spectral sequence of Theorem 2.4 causes it to collapse to an identification of the involutive cohomology  $H^r(\Gamma(X, \Lambda_X^{0,\bullet}), \bar{\partial}_X)$  as the global cohomology of the elliptic complex

$$(3.5) \quad 0 \rightarrow \Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \rightarrow \begin{array}{c} \Lambda^{1,2} \\ \oplus \\ \Lambda^{2,1} \end{array} \rightarrow \Lambda_{\perp}^{2,2} \rightarrow 0$$

on  $\mathbb{C}\mathbb{P}_3$  and from Proposition 3.2 we deduce the following.

**THEOREM 3.4.** *The complex (3.5) is exact on  $\mathbb{C}\mathbb{P}_3$  except at  $\Lambda^0$  and  $\Lambda^2$ , where its cohomology is canonically identified with  $\mathbb{C}$ .*

In fact, the Kähler form on  $\mathbb{C}\mathbb{P}_3$  generates the cohomology at  $\Lambda^2$ . It is interesting to compare (3.5) with the complex that emerges from the Penrose transform of  $H^r(\mathbb{F}_{1,2}(\mathbb{C}^4), \mathcal{O})$  under the submersion  $\mathbb{F}_{1,2}(\mathbb{C}^4) \rightarrow \mathbb{C}\mathbb{P}_3$  as computed in [10], namely

$$0 \rightarrow \Lambda^0 \xrightarrow{d} \Lambda^1 \rightarrow \begin{array}{c} \Lambda^{0,2} \\ \oplus \\ \Lambda_{\perp}^{1,1} \end{array} \rightarrow \Lambda_{\perp}^{1,2} \rightarrow 0,$$

which is exact except for the constants at  $\Lambda^0$ . The complex (3.5) is better balanced with respect to type, as one would expect.

As a simple variation on this theme, one can consider a similar transform for the Dolbeault cohomology of  $Z = \mathbb{F}_{1,3}(\mathbb{C}^4)$  but having coefficients in any complex homogeneous line bundle or, indeed, vector bundle on  $Z$ . Following the notation of [8], let us next consider the homogeneous line bundle  $(1 \mid 0, 0 \mid 0)$  on  $Z$ . The only additional difficulty that must be addressed is that  $\mathbb{F}_{1,3}(\mathbb{C}^4)$ , as it appears in (3.1), is not written in standard form. Specifically, we have

$$\mathrm{GL}(4, \mathbb{C}) / \left\{ \begin{bmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \right\} \quad \text{rather than} \quad \mathrm{GL}(4, \mathbb{C}) / \left\{ \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \right\}.$$

But these two realisations are equivalent under conjugation by

$$(3.6) \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and, as explained in [10, 13], the effect of this conjugation is that the formula for pulling back a homogeneous vector bundle from  $Z$  to  $\mathbb{X}$  includes the action of the Weyl group element represented by (3.6). Specifically,

$$\mu^*(a | b, c | d) = (b | a | c | d) + \dots$$

and, in particular,

$$(3.7) \quad \mu^*(1 | 0, 0 | 0) = (0 | 1 | 0 | 0).$$

This bundle on  $\mathbb{X}$  makes its effect felt in modifying the spectral sequence (2.8) as

$$E_1^{p,q} = \Gamma(M, \tau_*^q(\Lambda_\mu^{p,0} \otimes (0 | 1 | 0 | 0)|_X)) \implies H^{p+q}(\Gamma(X, \Lambda_X^{0,\bullet} \otimes (0 | 1 | 0 | 0)|_X), \bar{\partial}_X)$$

and also the spectral sequence (2.5) as applied in proving Proposition 3.2. In fact, since  $H^r(\mathbb{F}_{1,3}(\mathbb{C}^4), \mathcal{O}(1 | 0, 0 | 0)) = 0$  for all  $r$  (as a particular instance of the Bott-Borel-Weil Theorem [4]), following the proof of Proposition 3.2 demonstrates the following.

PROPOSITION 3.5. *Concerning the double fibration (1.3), we have*

$$H^r(\Gamma(X, \Lambda_X^{0,\bullet} \otimes (0 | 1 | 0 | 0)|_X), \bar{\partial}_X) = 0 \quad \forall r.$$

Therefore, the spectral sequence

$$E_1^{p,q} = \Gamma(M, \tau_*^q(\Lambda_\mu^{p,0} \otimes (0 | 1 | 0 | 0)|_X))$$

converges to zero. It remains to compute the bundles involved and for this we may proceed as before, instead computing

$$\nu_*^q(\Lambda_\mu^{p,0} \otimes (0 | 1 | 0 | 0)) \quad \text{for } \nu : \mathbb{X} \rightarrow \mathbb{M}$$

and then restricting to  $\mathbb{CP}_3 = M \hookrightarrow \mathbb{M}$ . This is a matter of combining (3.7) with (3.4) and applying the Bott-Borel-Weil Theorem as formulated in [8].

PROPOSITION 3.6. *The direct images  $\nu_*^q(\Lambda_\mu^{p,0} \otimes (0 | 1 | 0 | 0))$  vanish for  $q \geq 1$  and for  $q = 0$  are as follows*

$$(3.8) \quad \begin{array}{c|c|c|c|c} p=0 & p=1 & p=2 & p=3 & p=4 \\ \hline 0 & 0 & (-2 | 1, 1, 1) & (-1 | 0, 1, 1) \oplus (1 | 0, 0, 0) & (0 | 0, 0, 1) \end{array}.$$

PROOF. According to the Bott-Borel-Weil Theorem, some particular direct images are

$$\begin{aligned} \nu_*(a | b | c | d) &= (a | b, c, d) \text{ if } b \leq c \leq d \\ \nu_*^1(a | b | c | d) &= (a | b + 1, c - 1, d) \text{ if } b + 1 \leq c - 1 \leq d \\ \nu_*^1(a | b | c | d) &= (a | b, d + 1, c - 1) \text{ if } b \leq d + 1 \leq c - 1 \end{aligned}$$

and, in these cases, all other direct images vanish. Furthermore,  $(a | b | c | d)$  has all direct images vanishing if any two of  $b, c + 1, d + 2$  coincide. This will be sufficient for our purposes. In particular,

$$\nu_*^q(\Lambda_\mu^{0,0} \otimes (0 | 1 | 0 | 0)) = \nu_*^q(0 | 1 | 0 | 0) = 0 \quad \forall q.$$

Next,

$$\Lambda_\mu^{1,0} \otimes (0 \parallel 1 \mid 0 \mid 0) = \begin{array}{c} (-1 \parallel 1 \mid 0 \mid 1) + (-1 \parallel 1 \mid 1 \mid 0) \\ \oplus \\ (1 \parallel 0 \mid 0 \mid 0) + (1 \parallel 1 \mid -1 \mid 0) \end{array}$$

so

$$(3.9) \quad \nu_*^q(\Lambda_\mu^{1,0} \otimes (0 \parallel 1 \mid 0 \mid 0)) = \nu_*^q((1 \parallel 0 \mid 0 \mid 0) + (1 \parallel 1 \mid -1 \mid 0)).$$

This requires further work since we need to know the connecting homomorphism

$$(3.10) \quad \begin{array}{ccc} \nu_*(1 \parallel 0 \mid 0 \mid 0) & \rightarrow & \nu_*^1(1 \parallel 1 \mid -1 \mid 0) \\ \parallel & & \parallel \\ (1 \parallel 0, 0, 0) & \xrightarrow{?} & (1 \parallel 0, 0, 0) \end{array}$$

induced by this extension. For this, we consult again the diagram (2.10), finding that the bottom row in case of (3.1) with  $n = 3$  is

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{C}^{1,0} & \rightarrow & \nu^* \Lambda_{\mathbb{M}}^{1,0} & \rightarrow & \Lambda_\mu^{1,0} & \rightarrow & 0 \\ & & \parallel & & \parallel & & \parallel & & \\ 0 & \rightarrow & \begin{array}{c} (-1 \parallel 1 \mid 0 \mid 0) \\ \oplus \\ (1 \parallel 0 \mid 0 \mid -1) \end{array} & \rightarrow & \nu^* \left[ \begin{array}{c} (-1 \parallel 0, 0, 1) \\ \oplus \\ (1 \parallel -1, 0, 0) \end{array} \right] & \rightarrow & \begin{array}{c} (-1 \parallel 0 \mid 0 \mid 1) + (-1 \parallel 0 \mid 1 \mid 0) \\ \oplus \\ (1 \parallel -1 \mid 0 \mid 0) + (1 \parallel 0 \mid -1 \mid 0) \end{array} & \rightarrow & 0, \end{array}$$

which, in particular, yields the short exact sequence

$$0 \rightarrow (1 \parallel 0 \mid 0 \mid -1) \rightarrow \nu^*(1 \parallel -1, 0, 0) \rightarrow (1 \parallel -1 \mid 0 \mid 0) + (1 \parallel 0 \mid -1 \mid 0) \rightarrow 0$$

and, therefore, when tensored with  $(0 \parallel 1 \mid 0 \mid 0)$  the short exact sequence

$$0 \rightarrow (1 \parallel 1 \mid 0 \mid -1) \rightarrow \nu^*(1 \parallel -1, 0, 0) \otimes (0 \parallel 1 \mid 0 \mid 0) \rightarrow (1 \parallel 0 \mid 0 \mid 0) + (1 \parallel 1 \mid -1 \mid 0) \rightarrow 0$$

from which it follows that all the direct images (3.9) vanish (equivalently, that the connecting homomorphism (3.10) is an isomorphism, as one might expect).

Next, we should compute the direct images of  $\Lambda_\mu^{2,0} \otimes (0 \parallel 1 \mid 0 \mid 0)$ , i.e. of

$$(-2 \parallel 1 \mid 1 \mid 1) \oplus \left[ \begin{array}{c} (0 \parallel 1 \mid -1 \mid 1) \\ (0 \parallel 0 \mid 0 \mid 1) + \oplus + (0 \parallel 1 \mid 0 \mid 0) \\ (0 \parallel 0 \mid 1 \mid 0) \end{array} \right] \oplus (2 \parallel 0 \mid -1 \mid 0).$$

The induced connecting homomorphism  $\nu_*(0 \parallel 0 \mid 0 \mid 1) \rightarrow \nu_*^1(0 \parallel 1 \mid -1 \mid 1)$  is again an isomorphism by similar reasoning and only  $(-2 \parallel 1 \mid 1 \mid 1)$  contributes to the direct images, as claimed in (3.8).

Next,

$$\Lambda_\mu^{3,0} \otimes (0 \parallel 1 \mid 0 \mid 0) = \begin{array}{c} (-1 \parallel 0 \mid 1 \mid 1) + (-1 \parallel 1 \mid 0 \mid 1) \\ \oplus \\ (1 \parallel 0 \mid -1 \mid 1) + (1 \parallel 0 \mid 0 \mid 0) \end{array}$$

and, finally,

$$\Lambda_\mu^{4,0} \otimes (0 \parallel 1 \mid 0 \mid 0) = (0 \parallel 0 \mid 0 \mid 1)$$

from which the rest of (3.8) is immediate.  $\square$

Assembling these various computations yields the following.

**THEOREM 3.7.** *There is an elliptic and globally exact complex on  $\mathbb{C}P_3$*

$$(3.11) \quad \begin{array}{ccccccc} 0 & \rightarrow & (-2 \parallel 1, 1, 1) & \rightarrow & (-1 \parallel 0, 1, 1) & \rightarrow & (0 \parallel 0, 0, 1) & \rightarrow & 0. \\ & & & & \oplus & \nearrow & & & \\ & & & & (1 \parallel 0, 0, 0) & & & & \end{array}$$

PROOF. Everything is shown save for the following two observations. Firstly, there is no possible first order differential operator  $(-2 \parallel 1, 1, 1) \rightarrow (1 \parallel 0, 0, 0)$  since there is no possible  $SU(4)$ -invariant symbol. Indeed, from (3.3), we have

$$\Lambda_M^1 \otimes (-2 \parallel 1, 1, 1) = (-3 \parallel 1, 1, 2) \oplus (-1 \parallel 0, 1, 1).$$

Similar symbol considerations

$$\begin{aligned} \Lambda_M^1 \otimes (-1 \parallel 0, 1, 1) &= (-2 \parallel 0, 1, 2) \oplus (-2 \parallel 1, 1, 1) \oplus (0 \parallel -1, 1, 1) \oplus (0 \parallel 0, 0, 1) \\ \Lambda_M^1 \otimes (1 \parallel 0, 0, 0) &= (0 \parallel 0, 0, 1) \oplus (2 \parallel -1, 0, 0) \end{aligned}$$

also allow one to check that the complex is elliptic.  $\square$

Invariance under  $SU(4)$  identifies the operators explicitly. Specifically, if we denote by  $L$  the homogeneous line bundle  $L = (-2 \parallel 1, 1, 1)$ , then (3.11) becomes

$$\begin{array}{ccccccc} 0 & \rightarrow & L & \xrightarrow{\partial} & \Lambda_M^{1,0} \otimes L & \xrightarrow{\partial} & \Lambda_M^{2,0} \otimes L \rightarrow 0 \\ & & & & \oplus & & \downarrow \kappa \wedge \kappa \wedge \_ \\ & & & & \Lambda_M^{3,0} \otimes L & \xrightarrow{\bar{\partial}} & \Lambda_M^{3,1} \otimes L \end{array}$$

As a check, Theorem 3.7 says that  $L$  has no global anti-holomorphic sections and this is certainly true because its complex conjugate  $(2 \parallel -1, -1, -1)$  has no global holomorphic sections (it is the homogeneous holomorphic bundle  $(2 \mid -1, -1, -1)$  in the notation of [8], which has singular infinitesimal character).

Other homogeneous holomorphic bundles on the twistor space  $Z = \mathbb{F}_{1,n}(\mathbb{C}^{n+1})$  will give rise to other invariant complexes of differential operators on  $\mathbb{C}\mathbb{P}_n$ . The author suspects that the holomorphic tangent bundle  $\Theta$  will give rise to an especially interesting complex (since  $H^1(Z, \Theta)$  parameterises the infinitesimal deformations of  $Z$  as a complex manifold). Unfortunately, he has not yet been able to complete the calculation in this case.

#### 4. Another particular transform

This section is concerned with an instance of the holomorphic double fibration transform as formulated in general in [15]. Specifically, let us consider the complex flag manifold  $Z = \mathbb{F}_{1,n}(\mathbb{C}^{n+1})$  under the action of  $SU(n, 1)$ . There are three open orbits for this action, easily described in terms of geometry in  $\mathbb{C}\mathbb{P}_n$ . The orbits of  $SU(n, 1)$  acting on  $\mathbb{C}\mathbb{P}_n$  are the open ball  $B$ , its boundary, and the complement of its closure. As in §1 and §3, an element  $(L, H)$  in  $Z$  may be viewed as a point on a hyperplane in  $\mathbb{C}\mathbb{P}_n$ . The three open orbits are given by the following restrictions.

- the point  $L$  lies in the ball  $B$ ,
- the hyperplane  $H$  lies outside the ball  $B$ ,
- the point  $L$  lies outside  $B$  but the hyperplane  $H$  intersects  $B$ .

The set of hyperplanes lying outside  $B$  defines an open subset in the dual projective space  $\mathbb{C}\mathbb{P}_n^*$ , which we may identify with  $\bar{B}$ , i.e. the ball  $B$  with its conjugate complex structure.

Let us consider the third of the options above for the open orbits of  $SU(n, 1)$  acting on  $Z$  and call it  $D$ . By definition it is a *flag domain*. Following the notation of [15], its *cycle space*  $\mathcal{M}_D$  is  $B \times \bar{B}$  inside  $\mathbb{C}\mathbb{P}_n \times \mathbb{C}\mathbb{P}_n^*$  and the correspondence space  $\mathfrak{X}_D$  is exactly  $\nu^{-1}(\mathcal{M}_D)$  for the complexified correspondence of §3. Thus, we

have an open inclusion

$$\begin{array}{ccc}
 & \mathfrak{X}_D & \\
 \mu \swarrow & & \searrow \nu \\
 D & & \mathcal{M}_D
 \end{array}
 \xrightarrow{\text{open } \hookrightarrow}
 \begin{array}{ccc}
 & \mathbb{X} & \\
 \mu \swarrow & & \searrow \nu \\
 \mathbb{F}_{1,n}(\mathbb{C}^{n+1}) & & \mathbb{M}
 \end{array}$$

and, in particular, the fibres of  $\nu$  over  $\mathcal{M}_D$  coincide with the fibres of  $\nu : \mathbb{X} \rightarrow \mathbb{M}$  restricted to  $\mathcal{M}_D$ . The spectral sequence [2, 13] for the resulting double fibration transform starting with a holomorphic vector bundle  $E$  on  $D$  reads

$$(4.1) \quad E_1^{p,q} = \Gamma(\mathcal{M}_D, \nu_*^q(\Lambda_\mu^{p,0} \otimes \mu^* E)) \implies H^{p+q}(D, \mathcal{O}(E))$$

under the assumptions that  $\mathcal{M}_D$  is Stein and that  $\mu : \mathfrak{X}_D \rightarrow D$  has contractible fibres, both of which are true for any flag domain [15] and directly seen to be the case here.

Observe that the terms in this spectral sequence (when  $E$  is trivial) are almost the same as in (2.8). Certainly, the direct image bundles may be obtained by working on the homogeneous correspondence (3.1) and then restricting to  $\mathcal{M}_D$ . As our final example, let us carry this out for  $n = 3$  and for  $E$  being the canonical bundle on  $D$ . This is precisely the restriction to  $D$  of the homogeneous line bundle  $(3|0,0|-3)$  on  $Z$ . Following exactly the procedures of §3 we find the following homogeneous bundles for  $\Lambda_\mu^{p,0} \otimes \mu^*(3|0,0|-3) = \Lambda_\mu^{p,0} \otimes (0||3|0|-3)$ .

$p = 0$	$(0  3 0 -3)$
$p = 1$	$(-1  3 0 -2) + (-1  3 1 -3)$ $\oplus$ $(1  2 0 -3) + (1  3 -1 -3)$
$p = 2$	$(-2  3 1 -2) \oplus \left[ (0  2 0 -2) + \begin{array}{c} (0  3 -1 -2) \\ \oplus \\ (0  2 1 -3) \end{array} + (0  3 0 -3) \right] \oplus (2  2 -1 -3)$
$p = 3$	$(-1  2 1 -2) + (-1  3 0 -2)$ $\oplus$ $(1  2 -1 -2) + (1  2 0 -3)$
$p = 4$	$(0  2 0 -2)$

Using the Bott-Borel-Weil Theorem, as formulated in [8], we find that the only non-zero direct images  $\nu_*^q(\Lambda_\mu^{p,0} \otimes \mu^*(3|0,0|-3))$  are when  $q = 3$  as follows.

$p = 0$	$(0  -1,0,1)$
$p = 1$	$(-1  0,0,1) \oplus (-1  -1,1,1)$ $\oplus$ $(1  -1,0,0) \oplus (1  -1,-1,1)$
$p = 2$	$(-2  0,1,1) \oplus [(0  0,0,0) \oplus (0  -1,0,1)] \oplus (2  -1,-1,0)$
$p = 3$	$(-1  0,0,1)$ $\oplus$ $(1  -1,0,0)$
$p = 4$	$(0  0,0,0)$

We conclude, for example, from (4.1) that  $H^3(D, \mathcal{O}(3|0,0|-3))$  is realised on

$$\mathcal{M}_D = B \times \bar{B} \subset \mathbb{C}\mathbb{P}_3 \times \mathbb{C}\mathbb{P}_3^*$$

as the kernel of the holomorphic differential operator

$$\begin{array}{ccc}
 & \bar{\partial} & \\
 (0||-1,0,1) & \begin{array}{l} \nearrow \\ \searrow \end{array} & \begin{array}{l} (-1||0,0,1) \oplus (-1||-1,1,1) \\ \oplus \\ (1||-1,0,0) \oplus (1||-1,-1,1) \end{array}
 \end{array}$$

where  $\bar{\partial}$  (respectively  $\bar{\partial}$ ) denotes holomorphic differentiation in the direction of  $\mathbb{C}\mathbb{P}_3$  (respectively  $\mathbb{C}\mathbb{P}_3^*$ ) followed by projection to the indicated bundles:

$$\begin{aligned} (0 \parallel -1, 0, 1) &\longrightarrow \Lambda_{\mathbb{C}\mathbb{P}_3}^{1,0} \otimes (0 \parallel -1, 0, 1) \\ &= (1 \parallel -1, 0, 0) \otimes (0 \parallel -1, 0, 1) \\ &= (1 \parallel -2, 0, 1) \oplus (1 \parallel -1, -1, 1) \oplus (1 \parallel -1, 0, 0) \\ &\rightarrow (1 \parallel -1, -1, 1) \oplus (1 \parallel -1, 0, 0). \end{aligned}$$

It is interesting to note that the entire complex  $\nu_*^3(\Lambda_{\mu}^{\bullet,0} \otimes \mu^*(3|0,0|-3))$  on  $\mathbb{M}$  is the analytic continuation of the elliptic complex

$$\begin{array}{ccccccc} & & & & \Lambda^{1,3} & & \\ & & & & \oplus & & \\ & & & & \Lambda^{2,2} & & \\ & & & & \oplus & & \\ & & & & \Lambda^{3,1} & & \\ 0 & \rightarrow & \Lambda_{\perp}^{1,1} & \begin{array}{c} \nearrow \\ \searrow \end{array} & \Lambda^{1,2} & \begin{array}{c} \nearrow \\ \searrow \end{array} & \Lambda^{2,3} \\ & & & & \oplus & & \oplus \\ & & & & \Lambda^{2,1} & \begin{array}{c} \nearrow \\ \searrow \end{array} & \Lambda^{3,2} \\ & & & & \oplus & & \\ & & & & \Lambda^{3,1} & \begin{array}{c} \nearrow \\ \searrow \end{array} & \Lambda^{3,3} \rightarrow 0 \end{array}$$

on  $M = \mathbb{C}\mathbb{P}_3$  and that this complex is the formal adjoint of (3.5), exactly as predicted by duality [13, Theorem 4.1]. The transform described in this section is an example of the much more general theory developed in [14].

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