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Gauge transformation between retarded and multipolar gauges

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Abstract. The gauge function, expressed in terms of the sources, required for a gauge transformation between the retarded electromagnetic gauge and the three-vector version of the multipolar gauge is obtained.

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The retarded solutions to the inhomogeneous wave equations for the electromagnetic scalar and vector potentials $\phi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$

$$\partial^2\mathbf{A}/\partial(ct)^2 - \nabla^2\mathbf{A} + \nabla(\nabla\cdot\mathbf{A} + c^{-2}\partial\phi/\partial t) = \mu_0\mathbf{J} \tag{1}$$

and

$$\nabla^2\phi + (\partial/\partial t)\nabla\cdot\mathbf{A} = -\rho/\varepsilon_0 \tag{2}$$

are

$$\phi_1(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r}', t')\delta\{t' - t + |\mathbf{r} - \mathbf{r}'|/c\}\mathrm{d}\mathbf{r}'\mathrm{d}t'}{|\mathbf{r} - \mathbf{r}'|} \tag{3}$$

and

$$\mathbf{A}_1(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}', t')\delta\{t' - t + |\mathbf{r} - \mathbf{r}'|/c\}\mathrm{d}\mathbf{r}'\mathrm{d}t'}{|\mathbf{r} - \mathbf{r}'|}, \tag{4}$$

where δ is the Dirac delta function and c is the velocity of light. They describe the potentials at position $\mathbf{r}$ and time t arising from charge and current densities ρ and $\mathbf{J}$ at position $\mathbf{r}'$ and time t' [1, 2] and satisfy the Lorentz gauge condition $\nabla\cdot\mathbf{A}_1 + c^{-2}\partial\phi_1/\partial t = 0$. The electromagnetic fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ are obtained from the potentials by the relations

$$\mathbf{B} = \nabla\times\mathbf{A} \quad \text{and} \quad \mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t, \tag{5}$$

where ∇ is the spatial gradient operator with respect to $\mathbf{r}$. In consequence, if the potentials are transformed to

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla\chi \quad \text{and} \quad \phi \to \phi' = \phi - \partial\chi/\partial t, \tag{6}$$

the electromagnetic fields are unchanged. The gauge function $\chi(\mathbf{r}, t)$ is required to satisfy the condition $\{(\partial/\partial i)(\partial/\partial j) - (\partial/\partial j)(\partial/\partial i)\}\chi = 0$, where i and j are any pair of the co-ordinates x, y, z and t. The principle of gauge invariance requires all observable quantities, such as the fields $\mathbf{E}$ and $\mathbf{B}$, to be independent of the gauge function [3,4].

In the retarded gauge above, the potentials, denoted by the subscripts 1 are described in terms of the charge and current source densities ρ and $\mathbf{J}$ at the retarded time $t' = t - |\mathbf{r} - \mathbf{r}'|/c$. Another gauge that is of interest in the theory of magnetism [5,6] and in semiclassical electrodynamics which describes the interaction of atoms with radiation [6] is the three-vector version of the multipolar gauge [5,7–9]. In this gauge the potentials, denoted by the subscript 2, are described in terms of the instantaneous but non-local values of the fields $\mathbf{E}$ and $\mathbf{B}$

$$\phi_2(\mathbf{r}, t) = -\mathbf{r} \cdot \int_0^1 \mathbf{E}(u\mathbf{r}, t)\mathrm{d}u \quad \text{and} \quad \mathbf{A}_2(\mathbf{r}, t) = -\mathbf{r} \times \int_0^1 \mathbf{B}(u\mathbf{r}, t)u\mathrm{d}u. \tag{7}$$

The multipolar gauge satisfies the condition $\mathbf{r} \cdot \mathbf{A}_2(\mathbf{r}, t) = 0$ and is obtained from a gauge function that is essentially given by $-\int_0^1 \mathbf{r} \cdot \mathbf{A}(u\mathbf{r}, t)\mathrm{d}u$ [9].

It is the purpose of this note to obtain the gauge function $\chi(\mathbf{r}, t)$ that effects a gauge transformation between these two important gauges by means of the relations $\mathbf{A}_2 = \mathbf{A}_1 + \nabla\chi$ and $\phi_2 = \phi_1 - \partial\chi/\partial t$. The procedure that is used is to get $\mathbf{B}_1(\mathbf{r}, t)$ and $\mathbf{E}_1(\mathbf{r}, t)$ from eqs (3)–(5) and substitute them in eq. (7) to get $\mathbf{A}_2(\mathbf{r}, t)$ and $\phi_2(\mathbf{r}, t)$. A gauge function $\chi(\mathbf{r}, t)$ is then found that relates the two sets of potentials.

First we get $\mathbf{B}_1$ by taking the curl of $\mathbf{A}_1$ with respect to $\mathbf{r}$. Noting that $\mathbf{J}$ is a function of $\mathbf{r}'$ but not of $\mathbf{r}$ this gives

$$\mathbf{B}_1(u\mathbf{r}, t) = -\frac{\mu_0}{4\pi} \int \mathrm{d}\mathbf{r}'\mathrm{d}t'\mathbf{J}(\mathbf{r}', t') \times \nabla_{u\mathbf{r}} h(u\mathbf{r}), \tag{8}$$

where $h(u\mathbf{r}) = \delta(t' - t + |u\mathbf{r} - \mathbf{r}'|)/c/|u\mathbf{r} - \mathbf{r}'|$ and the gradient is taken with respect to the parameter $u\mathbf{r}$. Hence

$$\mathbf{A}_2(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \mathrm{d}\mathbf{r}'\mathrm{d}t' \int_0^1 u\mathrm{d}u\, \mathbf{r} \times \{\mathbf{J}(\mathbf{r}', t') \times \nabla_{u\mathbf{r}} h(u\mathbf{r})\}. \tag{9}$$

The triple vector product may be expanded as

$$\begin{aligned} \mathbf{r} \times \{\mathbf{J}(\mathbf{r}', t') \times \nabla_{u\mathbf{r}} h(u\mathbf{r})\} &= \mathbf{J}(\mathbf{r}', t')\{\mathbf{r} \cdot \nabla_{u\mathbf{r}} h(u\mathbf{r})\} \\ &- \{\mathbf{J}(\mathbf{r}', t') \cdot \mathbf{r}\}\nabla_{u\mathbf{r}} h(u\mathbf{r}) \end{aligned} \tag{10}$$

and this gives rise to two terms in (9). When the relations between derivatives $u\nabla_{u\mathbf{r}} h(u\mathbf{r}) = \nabla h(u\mathbf{r})$ and $(\mathbf{r} \cdot \nabla)h(u\mathbf{r}) = u(\partial h(u\mathbf{r})/\partial u)$ derived in the appendix are used, where h is any function of $u\mathbf{r}$, the first term in the integrand becomes $u(\partial h/\partial u) = (\partial/\partial u)(uh) - h$ and so (9) is

$$\mathbf{A}_2(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \mathrm{d}\mathbf{r}'\mathrm{d}t'\mathbf{J}(\mathbf{r}',t') \times \int_0^1 \mathrm{d}u\{[\partial/\partial u\{uh(u\mathbf{r})\} - h(u\mathbf{r})] - \cdot\mathbf{r}\nabla_{\mathbf{r}}h(u\mathbf{r})\}, \tag{11}$$

where in the last term the vector dot product is between $\mathbf{J}$ and $\mathbf{r}$. The integral of u over the perfect differential can be carried out to give

$$\mathbf{A}_2(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \mathrm{d}\mathbf{r}'\mathrm{d}t'\mathbf{J}(\mathbf{r}',t')\left[h(\mathbf{r}) - \int_0^1 \mathrm{d}u\{h(u\mathbf{r}) + \cdot\mathbf{r}\nabla_{\mathbf{r}}h(u\mathbf{r})\}\right]. \tag{12}$$

The first term can be recognized to be $\mathbf{A}_1(\mathbf{r},t)$ of eq. (4) so,

$$\mathbf{A}_2(\mathbf{r},t) - \mathbf{A}_1(\mathbf{r},t) = -\frac{\mu_0}{4\pi}\int \mathrm{d}\mathbf{r}'\mathrm{d}t'\mathbf{J}(\mathbf{r}',t')\int_0^1 \mathrm{d}u\{h(u\mathbf{r}) + \cdot\mathbf{r}\nabla_{\mathbf{r}}h(u\mathbf{r})\}. \tag{13}$$

Next we calculate $\phi_2(\mathbf{r},t)$. From eq. (5), $\mathbf{E}$ is the sum of two parts, denoted by the superscripts a and b, which, from eq. (7), give rise to two terms in the potential. The first, involving the gradient, is

$$\phi_2^a(\mathbf{r},t) = \int_0^1 \mathrm{d}u(\mathbf{r}\cdot\nabla_{u\mathbf{r}})\phi_1(u\mathbf{r},t), \tag{14}$$

so using the results in the appendix, $\mathbf{r}\cdot\nabla_{u\mathbf{r}}h(\mathbf{q}) = \partial h/\partial u$ we obtain $\phi_2^a(\mathbf{r},t) - \phi_1(\mathbf{r},t) = -\phi_1(0,t)$. The other term is $\phi_2^b(\mathbf{r},t) = \mathbf{r}\cdot\int_0^1(\partial/\partial t)\mathbf{A}_1(u\mathbf{r},t)\mathrm{d}u$ and leads to

$$\phi_2(\mathbf{r},t) - \phi_1(\mathbf{r},t) = -\phi_1(0,t) - \frac{\mu_0}{4\pi}\int \mathrm{d}\mathbf{r}'\mathrm{d}t'\{\mathbf{r}\cdot\mathbf{J}(\mathbf{r}',t')\} \times \int_0^1 \mathrm{d}u\frac{\delta'\{t'-t+|u\mathbf{r}-\mathbf{r}'|/c\}}{|u\mathbf{r}-\mathbf{r}'|}, \tag{15}$$

where δ' is the derivative of the delta function with respect to its argument.

Consider now the scalar function $\chi(\mathbf{r},t) = f(\mathbf{r},t) + g(t)$ where

$$f(\mathbf{r},t) = -\frac{\mu_0}{4\pi}\int \mathrm{d}\mathbf{r}'\mathrm{d}t'\{\mathbf{r}\cdot\mathbf{J}(\mathbf{r}',t')\}\int_0^1 \mathrm{d}uh(u\mathbf{r}) \tag{16}$$

and

$$g(t) = -\frac{1}{4\pi\varepsilon_0}\int \mathrm{d}\mathbf{r}'\mathrm{d}t'\rho(\mathbf{r}',t')\theta\{t'-t+|\mathbf{r}'|/c\}/|\mathbf{r}'| \tag{17}$$

and $\theta(x)$ is the function which is 1 for $x > 0$ and zero otherwise; its derivative is the delta function. The gradient of χ, which is the gradient of f, is obtained by noting that $\nabla\{\mathbf{r}\cdot\mathbf{J}(\mathbf{r}',t')h(u\mathbf{r})\} = h(u\mathbf{r})\nabla\{\mathbf{r}\cdot\mathbf{J}(\mathbf{r}',t')\} + \{\mathbf{r}\cdot\mathbf{J}(\mathbf{r}',t')\}\nabla h(u\mathbf{r}) = h(u\mathbf{r})\mathbf{J}(\mathbf{r}',t') + \{\mathbf{r}\cdot\mathbf{J}(\mathbf{r}',t')\}\nabla h(u\mathbf{r})$ since $\nabla\{\mathbf{J}(\mathbf{r}',t')\cdot\mathbf{r}\} = \mathbf{J}(\mathbf{r}',t')$. Hence

$$\nabla\chi = -\frac{\mu_0}{4\pi}\int \mathrm{d}\mathbf{r}'\mathrm{d}t'\mathbf{J}(\mathbf{r}',t')\int_0^1 \mathrm{d}u\{h(u\mathbf{r}) + \cdot\mathbf{r}\nabla h(u\mathbf{r})\}. \tag{18}$$

The time derivative $\partial\chi/\partial t$ has two terms one coming from f and one from g.

$$\partial f/\partial t = \frac{\mu_0}{4\pi}\int \mathrm{d}\mathbf{r}'\mathrm{d}t'\{\mathbf{r}\cdot\mathbf{J}(\mathbf{r}',t')\}\int_0^1 \mathrm{d}u\frac{\delta'\{t'-t+|u\mathbf{r}-\mathbf{r}'|/c\}}{|u\mathbf{r}-\mathbf{r}'|} \tag{19}$$

and

$$\begin{aligned}\partial g/\partial t &= \frac{1}{4\pi\varepsilon_0}\int \mathrm{d}\mathbf{r}'\mathrm{d}t'\rho(\mathbf{r}',t')\delta\{t'-t+|\mathbf{r}'|/c\}/|\mathbf{r}'|\\ &= \phi_1(0,t).\end{aligned} \tag{20}$$

By comparing eqs (18) and (20) with (13) and (15) it can be seen that the gauge function $\chi = f + g$ is indeed able to transform the retarded gauge into the three-vector version of the multipolar gauge.

The integral over u in eq. (16) can be simplified. Using the standard relation

$$\delta[f(u)] = \Sigma_i\delta[u-u^i]/|\partial f/\partial u|_{u^i}, \tag{21}$$

where $f[u^i] = 0$, in this case with $f[u] = t' - t + |u\mathbf{r}-\mathbf{r}'|/c$, we obtain the roots u^i from $(u\mathbf{r}-\mathbf{r}')^2 - c^2(t-t')^2 = 0$ to be

$$u^{+-} = \frac{r'}{r}\left[\cos\varphi \pm \sqrt{c^2(t-t')^2/r'^2 - \sin^2\varphi}\right], \tag{22}$$

where r and r' are the lengths of the vectors $\mathbf{r}$ and $\mathbf{r}'$ and φ is the angle between them. From the relation $c^2(t-t')^2 = u^2r^2 + r'^2 - 2urr'\cos\varphi$ it follows that $c^2(t-t')^2 - r'^2\sin^2\varphi = (ur - r'\cos\varphi)^2 \geq 0$ so the square root is always real. Next, it is straightforward to show that

$$\frac{\partial f}{\partial u} = \frac{\mathbf{r}\cdot(u\mathbf{r}-\mathbf{r}')}{c|u\mathbf{r}-\mathbf{r}'|} \tag{23}$$

and that

$$\mathbf{r}\cdot(u\mathbf{r}-\mathbf{r}')|_{u^\pm} = \pm\sqrt{c^2(t-t')^2 - r'^2\sin^2\varphi}, \tag{24}$$

so

$$\int_0^1 \mathrm{d}u h(u\mathbf{r}) = \frac{c}{r\sqrt{c^2(t-t')^2 - r'^2\sin^2\varphi}}\int_0^1 \mathrm{d}u[\delta(u-u^+) + \delta(u-u^-)], \tag{25}$$

as the $|u\mathbf{r}-\mathbf{r}'|$ terms in the numerator and denominator cancel and where u^+ and u^- are given by eq. (22) so

$$f(\mathbf{r},t) = -\frac{\mu_0}{4\pi}\int \frac{\mathrm{d}\mathbf{r}'\mathrm{d}t'c\{\mathbf{r}\cdot\mathbf{J}(\mathbf{r}',t')\}}{r\sqrt{c^2(t-t')^2 - r'^2\sin^2\varphi}}\int_0^1 \mathrm{d}u[\delta(u-u^+) + \delta(u-u^-)]. \tag{26}$$

In carrying out the integrations over $\mathbf{r}'$ and t' in (26) the integral over u gives plus one when u^+ and u^- calculated from eq. (22) lie between zero and unity and zero otherwise.

Appendix

We show that $\mathbf{r} \cdot \nabla h(\mathbf{q}) = u(\partial h/\partial u)$ where ∇ is the gradient operator with respect to $\mathbf{r}$, $\mathbf{q} = u\mathbf{r}$ and h is any function of the vector $\mathbf{q}$. Noting that $\partial q^i/\partial x^j = u\delta_{ij}$ and $\partial q^i/\partial u = x^i$ we find that $\partial h/\partial x^i = u(\partial h/\partial q^i)$ so $\nabla h(u\mathbf{r}) = u\nabla_{u\mathbf{r}} h(u\mathbf{r})$. Also $\partial h/\partial u = \Sigma_i x^i(\partial h/\partial q^i)$ so $u\partial h(\mathbf{q})/\partial u = \Sigma_i u x^i \partial h/\partial q^i$. Next, $\mathbf{r} \cdot \nabla h(\mathbf{q}) = \Sigma_i x^i \partial/\partial x^i h(\mathbf{q}) = \Sigma_i u x^i \partial h/\partial q^i$ so it follows that $\mathbf{r} \cdot \nabla h(\mathbf{q}) = u(\partial h/\partial u)$ and $\mathbf{r} \cdot \nabla_{u\mathbf{r}} h(\mathbf{q}) = \partial h/\partial u$.

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