Invariant Distances and Metrics in Complex Analysis

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Constructing a distance that is invariant under a given class of mappings is one of the fundamental tools for the geometric approach in mathematics. The idea goes back to Klein and even to Riemann. In this article we will consider distances invariant under biholomorphic mappings of complex manifolds. There will be many such distances. A number of these will come from functions on the tangent spaces, in the way that a Riemannian metric on a manifold yields a distance on the manifold.

Following Riemann's lead, we think of a suitable function on the tangent spaces as a way to measure "lengths" of tangent vectors, and we can use it to define lengths of curves and ultimately a distance on the manifold. We do not insist that this function be related to an inner product. If the complex manifold is M and its tangent spaces are denoted $T_p(M)$, then we shall work with any nonnegative function $\mathfrak{f}(p, v)$, for $v \in T_p(M)$, such that $\mathfrak{f}(p, v)$ suitably respects scalar multiplication (real or complex as appropriate) in v. In most cases the function will be continuous in (p, v), but sometimes we allow the continuity to be slightly relaxed. Motivated by the Riemannian case, we shall refer to this function as a *metric* if $\mathfrak{f}(p, v)$ vanishes only for $v = 0$, or a *pseudometric* in general. We do not assume that $\mathfrak{f}(p, v)$ satisfies the triangle inequality in v if p is fixed; if the triangle inequality is in fact satisfied, the function will be called a *norm* or *pseudonorm*.

If the "length" function comes from an inner product, then the metric is said to be *Riemannian* as usual. If that inner product is the real part of a Hermitian inner product, then we call the metric *Hermitian*. If the Hermitian inner product behaves almost like the Euclidean metric (in the sense that they agree to order two), then we call the metric *Kählerian*. If, on the other hand, there is no inner product present at all and only the notion of vector length is defined, then we call the metric *Finslerian*.

The way that the word "Hermitian" is used here may be unfamiliar to some readers, and we offer a note of clarification by considering the simple example of $\mathbb{C}^n$. If we think of $\mathbb{C}^n$ as a complex space, then it is natural to use the inner product, for $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, given by

$$\langle z, w \rangle = \sum_{j=1}^{n} z_j \overline{w}_j .$$

This is the inner product that we call the (standard) *Hermitian* inner product on $\mathbb{C}^n$. Sometimes, however, we wish to think of $\mathbb{C}^n$ as just $\mathbb{R}^{2n}$. So we write $z = (z_1, \ldots, z_n) \approx (x_1, y_1, \ldots, x_n, y_n)$ and $w = (w_1, \ldots, w_n) \approx (u_1, v_1, \ldots, u_n, v_n)$, where we have made the natural identifications $z_j = x_j + iy_j \approx (x_j, y_j)$ and $w_j = u_j + iv_j \approx (u_j, v_j)$. Then a natural inner product to use is

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$$z \cdot w = \sum_{\ell=1}^{n} x_j u_j + y_j v_j .$$

Notice that this *real inner product* is simply the real part of the Hermitian inner product introduced a moment ago.

All manifolds in this article will be complex. For a complex manifold, the real tangent space at a point may be regarded canonically as a complex vector space of half the dimension. We will regard it that way throughout. The differential of a holomorphic mapping is then complex linear.

We intend in this article to demonstrate that different distances are suitable for different applications. In five different contexts, we shall try to indicate what some of these may be.

The first result in this general line of inquiry is due to Poincaré. On the unit disc $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane, he constructed the distance

$$\rho(z, w) := \frac{1}{2} \ln \frac{1 + \left| \frac{z-w}{1-\bar{z}w} \right|}{1 - \left| \frac{z-w}{1-\bar{z}w} \right|}, \tag{1}$$

which is invariant under conformal transformations of Δ. Poincaré used this distance in his model for the Lobachevskii geometry in Δ.

Consider the Hermitian metric on Δ given in traditional notation by

$$\mathcal{P}(z) := \frac{dz\, d\bar{z}}{(1 - |z|^2)^2}.$$

Then, for a vector v tangent to Δ at a point z, one can calculate the length of v as

$$\mathfrak{p}(z, v) := \frac{|v|}{1 - |z|^2} \tag{2}$$

(here $|\cdot|$ is just Euclidean length). It turns out that $\mathfrak{p}$ is an infinitesimal form of the distance ρ. This means that, for given points $p, q \in \Delta$, the distance $\rho(p, q)$ can be obtained from $\mathfrak{p}$ as follows: for any smooth curve γ (parametrized by the interval $[0, 1]$) that joins p and q, we define its length $|\gamma|_{\mathfrak{p}}$ as

$$|\gamma|_{\mathfrak{p}} := \int_0^1 \mathfrak{p}(\gamma(t), \gamma'(t))\, dt\, ;$$

then we take the infimum of the lengths of all such curves to determine the distance from p to q. Thus the distance ρ on Δ can be recovered by measuring the lengths of tangent vectors in the metric $\mathfrak{p}$. This relation will be of importance for us below when we look at various generalizations of ρ and $\mathfrak{p}$.

Buried inside the Poincaré distance is another distance. Define

$$\tilde{\rho}(z, w) = \left| \frac{z - w}{1 - \bar{z}w} \right| .$$

Then it can be verified that $\tilde{\rho}$ is a distance function. The classical name for $\tilde{\rho}$ is the *pseudohyperbolic* distance (see [Gar])—not to be confused with the "pseudodistances" that will be discussed in the sequel. If ϕ is any holomorphic function from the unit disc to itself, then

$$\tilde{\rho}(\phi(z), \phi(w)) \le \tilde{\rho}(z, w). \tag{3}$$

This inequality comes from the Schwarz-Pick lemma, which in turn is just the conformally invariant version of the classical Schwarz lemma.

So holomorphic functions are distance nonincreasing in the metric $\tilde{\rho}$. If ϕ happens to be a *holomorphic automorphism* of the disc (i.e., $\phi : \Delta \to \Delta$ is one-to-one and onto, as well as holomorphic), then inequality (3) applies both to ϕ and to ϕ^{-1}, so that we obtain

$$\tilde{\rho}(\phi(z), \phi(w)) = \tilde{\rho}(z, w).$$

If the pseudohyperbolic distance has the nice invariance properties that we want, then why bother with the Poincaré metric, which seems to involve an additional level of (computational) complexity? Let us try to answer this question as follows. The group of holomorphic automorphisms of the unit disc consists of certain Möbius transformations. These act transitively on the disc: if $z, w \in \Delta$, then there is an automorphism ϕ such that $\phi(z) = w$. In fact, more is true: if ξ, ζ are tangent directions at z, w respectively, then there is an automorphism ϕ such that $\phi(z) = w$ and the tangent direction ξ is taken to a scalar multiple of the tangent direction ζ. From this observation it follows that there is, up to multiplication by a scalar, just one Riemannian metric on the disc that is invariant under the automorphism group. And of course that metric is the Poincaré metric.

So how does the pseudohyperbolic distance fit in? It is plainly *not* a constant multiple of the Poincaré distance. The answer is that $\tilde{\rho}$ *does not come from a Riemannian metric*: it is in fact impossible to find a way to measure the lengths of vectors tangent to Δ to obtain $\tilde{\rho}$ in the same way as ρ was obtained from $\mathfrak{p}$. Nevertheless, the pseudohyperbolic distance meshes naturally with the Schwarz-Pick lemma and is therefore a cornerstone of classical geometric function theory (see [Gar]).[1]

In what follows we will give five separate examples of generalizations of the Poincaré distance and metric and provide one application of each. Among a large number of potential illustrative examples we concentrate on just a few results, mostly those relevant to the study of the group Aut(M) of biholomorphic automorphisms of a given

[1] *Specialists are fond of the pseudohyperbolic distance also because it meshes well with Nevanlinna-Pick interpolation, which is the backbone of the celebrated corona problem.*

complex manifold M (all manifolds throughout the paper are assumed to be connected). Here

$$\mathrm{Aut}(M) = \left\{ \phi : M \to M \;\middle|\; \begin{array}{l} \phi \text{ is holomorphic,} \\ \text{one-to-one and onto} \end{array} \right\}.$$

Observe that $\mathrm{Aut}(M)$, equipped with the binary operation of composition of mappings, is a group (the inverse of a holomorphic, one-to-one mapping is automatically holomorphic). We topologize $\mathrm{Aut}(M)$ with the compact-open topology, which turns $\mathrm{Aut}(M)$ into a topological group. For some classes of manifolds the group $\mathrm{Aut}(M)$ can be given the additional structure of a Lie group; one such class can be defined via an invariant distance (see Theorem 2 below).

The Carathéodory Pseudodistance

The "pseudodistance" that now bears his name was introduced by Carathéodory in 1926. Let M be a complex manifold and $p, q \in M$. Set

$$C_M(p,q) := \sup_f \rho(f(p), f(q)), \tag{4}$$

where the supremum is taken over all holomorphic mappings from M into the unit disc Δ and ρ is the Poincaré distance (1). In effect, C_M is the pullback to M of the distance ρ on Δ (Figure 1). It turns out that C_M is a *pseudodistance*; i.e., it satisfies

(i) $C_M(p,q) \geq 0$,
(ii) $C_M(p,p) = 0$,
(iii) $C_M(p,q) = C_M(q,p)$,
(iv) $C_M(p,q) \leq C_M(p,r) + C_M(r,q)$,

for all $p, q, r \in M$. It is in general not a distance; that is to say, it may happen that, for $p \neq q$, one has $C_M(p,q) = 0$. For example, $C_{\mathbb{C}^n} \equiv 0$. On the other hand, it follows from the Schwarz-Pick lemma that $C_\Delta = \rho$. It can be shown that the Carathéodory pseudodistance C_M is invariant under biholomorphic mappings and, moreover, is *nonincreasing* under holomorphic mappings. Specifically, if $f : M_1 \to M_2$ is a holomorphic mapping between complex manifolds M_1 and M_2, then

$$C_{M_2}(f(p), f(q)) \leq C_{M_1}(p,q),$$

for all $p, q \in M_1$. It turns out that C_M is the least pseudodistance among all pseudodistances on M that are nonincreasing under holomorphic mappings from M to Δ, where distances on Δ are measured in the Poincaré distance.

A pseudodistance d is said to be *inner* if $d(p,q)$ equals $d^I(p,q)$, which is the infimum of the lengths of all curves connecting p to q. Here the length of a curve $\gamma : [0,1] \to M$ is understood as the supremum of $\sum_{i=1}^k d(\gamma(t_{i-1}), \gamma(t_i))$ over all partitions $0 = t_0 < t_1 < \ldots < t_k = 1$ of the interval $[0,1]$.

The Carathéodory pseudodistance C_M is not in general inner. Although C_M is continuous as a mapping from $M \times M$ into $\mathbb{R}$, it does not in general induce the topology of M, even if it is in fact a distance.

For a point $p \in M$ and a tangent vector $v \in T_p(M)$ to M at p, one can also define the Finslerian pseudometric

$$\mathfrak{c}_M(p,v) := \sup_f \mathfrak{p}(f(p), df(p)v),$$

where the supremum is taken over the same set of mappings f as in (4) and $\mathfrak{p}$ is defined in (2). The function $\mathfrak{c}_M$ is a pseudonorm on the tangent bundle $T(M)$ to M and is *not* in general given by a Hermitian pseudometric. It is an infinitesimal form of the inner pseudodistance C_M^I induced by C_M and is continuous on $T(M)$. The pseudometric $\mathfrak{c}_M$ is nonincreasing under holomorphic mappings. This last fact generalizes the usual Schwarz-Pick lemma for the unit disc Δ. Indeed, for the unit disc Δ, one has (cf. (2))

$$\mathfrak{c}_\Delta(z,v) = \mathfrak{p}(z,v) = \frac{|v|}{1-|z|^2},$$

and the property that $\mathfrak{c}_\Delta$ does not increase under a holomorphic mapping $\phi : \Delta \to \Delta$ means precisely that

$$\frac{|\phi'(z)|}{1-|\phi(z)|^2} \leq \frac{1}{1-|z|^2},$$

which is the infinitesimal version of the Schwarz-Pick lemma.

Both C_M and $\mathfrak{c}_M$ are directed to the study of holomorphic mappings between manifolds rather than to the geometric properties of manifolds. Applications of C_M and $\mathfrak{c}_M$ to the understanding of holomorphic mappings are numerous, and we mention just one of them.

A complex manifold M is called *C-hyperbolic* if, for its universal cover M', $C_{M'}^I$ is a genuine distance. Examples of C-hyperbolic manifolds are bounded domains in complex space $\mathbb{C}^n$. Also, a compact quotient of a bounded domain D by a properly discontinuous group acting freely on D is C-hyperbolic.

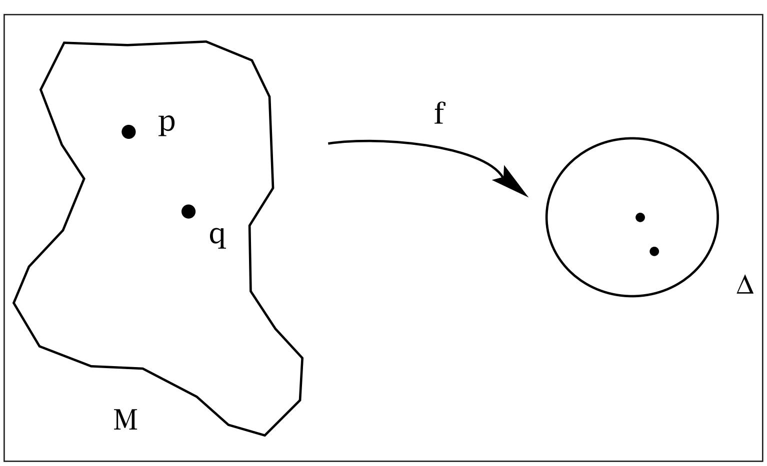

Figure 1. Constructing the Carathéodory pseudodistance.

Theorem 1. *There exist only finitely many holomorphic mappings from a compact complex manifold M_1 onto a C-hyperbolic compact complex manifold M_2. In particular, the group of biholomorphic automorphisms of a compact C-hyperbolic manifold is finite.*

Sketch of the proof of Theorem 1 (Urata). Let $\mathrm{Hol}(M_1, M_2)$ denote the set of all holomorphic mappings from M_1 into M_2. It is known that $\mathrm{Hol}(M_1, M_2)$, equipped with the compact-open topology, admits a complex structure (think of the graphs of the mappings as forming a sort of Teichmüller space). Using the C-hyperbolicity of M_2, one can show that $\mathrm{Hol}(M_1, M_2)$ is compact.

Further, let $S \subset \mathrm{Hol}(M_1, M_2)$ be the set of all holomorphic surjections in $\mathrm{Hol}(M_1, M_2)$. The set S is a complex subvariety of $\mathrm{Hol}(M_1, M_2)$ because S is given by the equation $f(M_1) = M_2$; i.e.,

$$S = \{f \in \mathrm{Hol}(M_1, M_2) \mid f(M_1) = M_2\},$$

and hence it is also compact. The core of the proof is to show that $\dim S = 0$ (which is far too technical to be discussed here). Since S is compact, this gives that S is in fact finite. □

The Kobayashi Pseudodistance

The pseudodistance named after Kobayashi was introduced by him in 1967. Let M be a complex manifold and $p, q \in M$. A *chain of discs* from p to q is a collection of points $p = p_0, p_1, \ldots, p_k = q$ of M; pairs of points $a_1, b_1, \ldots, a_k, b_k$ of Δ; and holomorphic mappings $f_1, \ldots f_k$ from Δ to M such that $f_j(a_j) = p_{j-1}$ and $f_j(b_j) = p_j$ for all j (Figure 2). The *length* $l(\alpha)$ of a chain α is the sum $\rho(a_1, b_1) + \cdots + \rho(a_k, b_k)$, where ρ is the Poincaré distance (1). The Kobayashi pseudodistance is then defined to be

$$K_M(p, q) := \inf_{\alpha} l(\alpha), \tag{5}$$

where the infimum is taken over all chains of discs from p to q.

The function K_M is indeed a pseudodistance, but not a distance in general (e.g., $K_{\mathbb{C}^n} \equiv 0$); it coincides with the Poincaré distance (1) on Δ and is nonincreasing under holomorphic mappings. The Kobayashi pseudodistance K_M is the greatest pseudodistance among all pseudodistances on M that do not increase under holomorphic mappings from Δ to M, where distances on Δ are measured in the Poincaré distance. For any complex manifold M, one has $C_M(p, q) \le K_M(p, q)$.

The Kobayashi pseudodistance is continuous as a mapping from $M \times M$ to $\mathbb{R}$ and, unlike the Carathéodory pseudodistance, is always inner. In particular, if K_M is a distance on M, then it induces the topology of M. A manifold for which K_M is a genuine distance is called (Kobayashi) *hyperbolic*. Any C-hyperbolic manifold, any bounded domain in complex space in particular, is hyperbolic. The property "hyperbolic" is preserved under biholomorphic mappings. There are unbounded domains that are not biholomorphically equivalent with bounded domains and are nevertheless hyperbolic. For example, one can show that the domain

$$\Omega := \Big\{(z_1, z_2) \in \mathbb{C}^2 \;\Big|\; |z_1| < 1, |z_2| < 1/(1 - |z_1|)\Big\}$$

is hyperbolic but is not biholomorphically equivalent to any bounded domain in $\mathbb{C}^2$, as any bounded holomorphic function on Ω is independent of z_2. This last fact also implies that Ω is not C-hyperbolic.

An infinitesimal form of the Kobayashi pseudodistance is given by the Finslerian pseudometric

$$\mathfrak{k}_M(p, v) := \inf_{f,z,u} \mathfrak{p}(z, u),$$

where the infimum is taken over all holomorphic mappings f from Δ into M, points $z \in \Delta$, and tangent vectors $u \in T_z(\Delta)$ such that $f(z) = p$ and $df(z)u = v$. In general $\mathfrak{k}_M$ is not even a pseudonorm as $\mathfrak{c}_M$ is (i.e., $\mathfrak{k}_M$ does not in general satisfy the triangle inequality). Similar to $\mathfrak{c}_M$, the infinitesimal pseudometric $\mathfrak{k}_M$ does not increase under holomorphic mappings.

As with the Carathéodory pseudodistance, K_M and $\mathfrak{k}_M$ are primarily used for studying holomorphic mappings between manifolds rather than a manifold's intrinsic geometry. The Kobayashi pseudodistance now has probably many more applications than any other invariant pseudodistance, and we are not even going to try to mention all the areas in complex analysis where it is used. We will give just one example of such an application below.

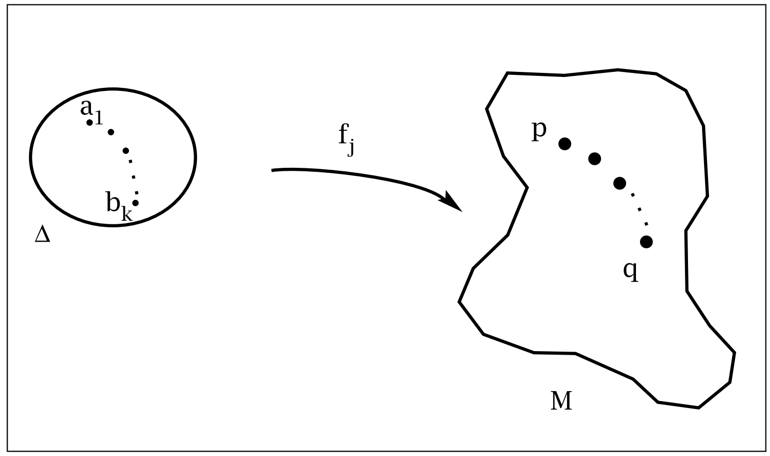

Figure 2. Constructing the Kobayashi pseudodistance.

Theorem 2. *Let M be a hyperbolic complex manifold of complex dimension n. Then the following holds:*

(i) The group Aut(M) *can be given the structure of a real Lie group whose topology agrees with the compact-open topology, and for every* $p \in M$*, the stabilizer* $I_p(M) := \{f \in \mathrm{Aut}(M) \mid f(p) = p\}$ *of* p *in* Aut(M) *is compact.*

(ii) If $n \geq 2$ *and* $\dim \mathrm{Aut}(M) \geq n^2 + 3$*, then* M *is biholomorphically equivalent to the unit ball in* $\mathbb{B}^n \subset \mathbb{C}^n$.

The proof of Theorem 2 involves some interesting arguments of counting dimension, together with some rather subtle Lie group theory. We give an indication of the ideas.

Sketch of the proof of Theorem 2.[2] Since K_M is an inner distance on M, it induces the topology of M and thus turns M into a locally compact metric space. By a classical result of van Dantzig and van der Waerden, the group $G(M)$ of all isometries of M is locally compact and the stabilizer $\mathrm{St}_p(M) := \{f \in G(M) \mid f(p) = p\}$ of p in $G(M)$ is compact for every $p \in M$ with respect to the compact-open topology. Since Aut(M) and $I_p(M)$ are closed in $G(M)$ and $\mathrm{St}_p(M)$ respectively, it follows that Aut(M) is locally compact and $I_p(M)$ is compact. Further, by a theorem of Bochner and Montgomery, a locally compact group of differentiable transformations of a manifold is a Lie transformation group, and (i) is proved.

For (ii) we need the following theorem due to Greene-Krantz (1985) and Bland-Duchamp-Kalka (1987).

Theorem 3. *Let* M *be a complex manifold of complex dimension* n *and* p *a point in* M*. Suppose that there exists a compact group* $K \subset I_p(M)$ *such that, for every pair of nonzero vectors* $v, u \in T_p(M)$*, there exists* $\phi \in K$ *and* $\mu \in \mathbb{C}$ *such that* $d\phi(p)u = \mu v$*. Then* M *is biholomorphically equivalent to either the unit ball* $\mathbb{B}^n$ *or the complex space* $\mathbb{C}^n$ *or the complex projective space* $\mathbb{CP}^n$.

We will now proceed with the proof of part (ii) of Theorem 2. Since the complex dimension of M is n, it follows that the real dimension of any orbit of the action of Aut(M) on M does not exceed $2n$, and therefore we have $\dim I_p(M) \geq n^2 - 2n + 3$ for every $p \in M$. Consider the isotropy representation $\alpha_p : I_p(M) \to GL(T_p(M), \mathbb{C})$:

$$\alpha_p(\phi) := d\phi(p), \qquad \phi \in I_p(M).$$

It can be shown that α_p is faithful (one-to-one). Since $I_p(M)$ is compact by (i) of Theorem 2, there is a positive definite Hermitian form h_p on $T_p(M)$ such that $\alpha_p(I_p(M)) \subset U_{h_p}(n)$, where $U_{h_p}(n)$ is the group of complex-linear transformations of $T_p(M)$ preserving the form h_p. We choose a basis in $T_p(M)$ such that h_p in this basis is given by the identity matrix. Since α_p is faithful, we see that $\alpha(I_p(M))$ is a compact subgroup of $U(n)$ of dimension $\geq n^2 - 2n + 3$.

To wrap up the proof, it is necessary to apply some fairly technical results from Lie theory to see that a connected closed subgroup of $U(n)$ of dimension $\geq n^2 - 2n + 3$ must in fact be either $U(n)$ or $SU(n)$ (this holds for $n \neq 4$; for $n = 4$ there are additional possibilities). Each of $U(n)$ and $SU(n)$ acts transitively on the unit sphere $S^{2n-1} \subset \mathbb{C}^n$. Then, assuming $n \neq 4$, we can fix $p \in M$ and apply Theorem 3 with $K = \alpha_p(I_p(M))$ to conclude that M is biholomorphically equivalent to either $\mathbb{B}^n$ or $\mathbb{C}^n$ or $\mathbb{CP}^n$. Since M is hyperbolic, it in fact has to be equivalent to $\mathbb{B}^n$ ($\mathbb{C}^n$ and $\mathbb{CP}^n$ are not hyperbolic). For $n = 4$ an additional technical argument is required. □

[2] *The proof of (i) follows the book [Kob1], whereas (ii) is part of our new result (to appear) on characterization of hyperbolic manifolds by the dimensions of their automorphism groups.*

The Bergman Pseudometric

The Bergman pseudometric is a Hermitian pseudometric on complex manifolds introduced by Bergman in 1922 for one variable and in 1933 for several variables.

We first describe Bergman's construction for M a bounded domain in $\mathbb{C}^n$. Consider $L^2(M)$ with respect to Lebesgue measure. The subspace of $L^2(M)$ of holomorphic functions is closed, and we let $\{f_j\}_{j=1}^{\infty}$ be an orthonormal basis. Then the *Bergman kernel function* is given by

$$b_M(z, \overline{w}) = \sum_{j=1}^{\infty} f_j(z)\overline{f_j(w)}.$$

It is independent of the choice of orthonormal basis. For $z = w$, we have $b_M(z, \overline{z}) > 0$.

For a general M of complex dimension n, let $\omega_1, \omega_2, \ldots$ be a complete orthonormal basis in the space of square-integrable holomorphic n-forms on M. Then the differential form

$$B_M(z, \overline{w}) := \sum_{j=1}^{\infty} \omega_j(z) \wedge \overline{\omega_j(w)},$$

for $z, w \in M$, is independent of the basis and is called the *Bergman kernel form*. In a local coordinate system $z_1, \ldots, z_n$ in M one can write every $\omega_j(z)$ as $\omega_j(z) = f_j(z)dz_1 \wedge \cdots \wedge dz_n$ where $f_j(z)$ is a locally defined holomorphic function, and therefore B_M can be written locally as

$$B_M(z, \overline{w}) = b_M(z, \overline{w})dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{w_1} \wedge \cdots \wedge d\overline{w_n},$$

where $b_M(z, \overline{w}) := \sum_{j=1}^{\infty} f_j(z)\overline{f_j(w)}$.

Assume now that $b_M(z, \overline{z}) > 0$ for every $z \in M$, and define the Bergman pseudometric as follows:

$$\mathcal{B}_M(z) := \frac{1}{2} \sum_{m,k=1}^{n} \frac{\partial^2 \log b_M(z, \overline{z})}{\partial z_m \, \partial \overline{z_k}} \, dz_m \, d\overline{z_k}.$$

It turns out that $\mathcal{B}_M$ is independent of the coordinate system, is positive semidefinite, biholomorphically invariant, smooth, and Kählerian. For the unit disc Δ we have $\mathcal{B}_\Delta = \mathcal{P}$. However, it is *not* true in general that $\mathcal{B}_M$ is nonincreasing under holomorphic mappings. Those manifolds for which $\mathcal{B}_M$ is well defined and positive definite (and thus is a Hermitian metric) are of particular interest, although they are relatively few. Bounded domains in complex space are examples of such manifolds, and for them the Bergman metric has been studied most extensively. If D is a bounded domain in $\mathbb{C}^n$, then one has $\mathfrak{c}_D^2(p, v) \leq 4 \langle v, v \rangle_{\mathcal{B}_D(p)}$, but no general relation between $\mathfrak{k}_D$ and $\mathcal{B}_D$ is known (in fact, a famous example of Diederich and Fornæss suggests that there is no such relation).

Unlike $\mathfrak{c}_D$ and $\mathfrak{k}_D$, the Bergman pseudometric $\mathcal{B}_D$ is Hermitian and even Kählerian, so one would expect that it could be used to obtain differential-geometric information about the domain D. Below we give an example of one such application.

For any invariant metric on a domain $D \subset \mathbb{C}^n$, an important characteristic is its boundary behavior, in particular, the boundary behavior of its curvature tensor. The best-studied case is that of bounded *strongly pseudoconvex* domains with C^∞ smooth boundary. Strong pseudoconvexity means that the boundary of the domain can be locally made strongly convex by a biholomorphic change of coordinates. For such domains C. Fefferman (1974) found a remarkable asymptotic formula for the Bergman kernel form. He used it to show that suitable geodesics of the Bergman metric approach the boundary of the domain in a "pseudotransverse" manner, and therefore that biholomorphic mappings of strongly pseudoconvex domains extend smoothly to the boundaries.

Using Fefferman's asymptotic formula, Klembeck showed—for a bounded strongly pseudoconvex domain D with C^∞ smooth boundary—that the holomorphic sectional curvature of $\mathcal{B}_D$ near the boundary of D approaches the negative constant $-2/(n+1)$; that number is the holomorphic sectional curvature of the Bergman metric of the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$. This fact is essential for the following characterization of the unit ball $\mathbb{B}^n$, which is entirely different from that in Theorem 2. We sketch the proof because it is readily appreciated and the techniques have been quite influential.

Theorem 4 *(Bun Wong, Rosay). Let $D \subset \mathbb{C}^n$ be a bounded, strongly pseudoconvex domain with C^∞ smooth boundary, and suppose that the automorphism group* $\mathrm{Aut}(D)$ *of D is noncompact in the compact-open topology. Then D is biholomorphically equivalent to $\mathbb{B}^n$.*

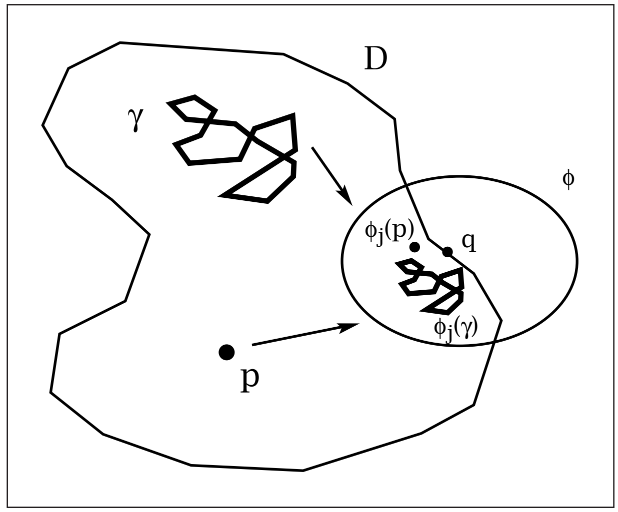

Figure 3. Proof of Theorem 4.

Proof of Theorem 4 (Klembeck). For any bounded domain $D \subset \mathbb{C}^n$, the noncompactness of $\mathrm{Aut}(D)$ is equivalent to the existence of a sequence $\{\phi_j\} \subset \mathrm{Aut}(M)$ such that $\phi_j \to \phi$ as $j \to \infty$, uniformly on compact subsets of D, with $\phi : D \to \mathbb{C}^n$ holomorphic and $\phi \notin \mathrm{Aut}(D)$. Clearly, ϕ maps D into $\overline{D}$. Moreover, by a classical theorem of H. Cartan, ϕ in fact maps D into ∂D, the boundary of D. The set $\phi(D)$ is then a complex variety in ∂D.

If D is strongly pseudoconvex, then ∂D does not contain any nontrivial complex variety (as follows from a maximum modulus principle-type argument), and hence ϕ is a constant mapping into a point $q \in \partial D$. Therefore, for every point $p \in D$ one has $\phi_j(p) \to q$ as $j \to \infty$ (Figure 3). Further, we know that near q the holomorphic sectional curvature of $\mathcal{B}_D$ approaches $-2/(n+1)$. Since the Bergman metric is invariant under each of the automorphisms ϕ_j, this implies that the holomorphic sectional curvature of $\mathcal{B}_D$ is constant and equal to $-2/(n+1)$.

We will now show that D is simply connected. Let γ be a closed curve in D. Since ϕ_j converges to ϕ uniformly on compact subsets of D, we see that the curve $\phi_j(\gamma)$ for j large enough sits in a prescribed neighborhood of the point q, and this neighborhood can be chosen (by the boundary smoothness) to be simply connected. Therefore, $\phi_j(\gamma)$ is homotopic to zero, and hence so is γ, thus proving that D is simply connected.

In summary, D is a simply connected Kähler manifold with negative constant holomorphic sectional curvature. It now follows from standard results of differential geometry (see [KoN]) that D is biholomorphically equivalent to $\mathbb{B}^n$. □

The boundary behavior of the Bergman metric on more general domains is an interesting subject in its own right; it is very important, for example, in problems of extendability of holomorphic mappings between domains to mappings between their boundaries. The literature on this topic is vast, and we shall not discuss the results here (see, e.g., [Kra]).

The Kähler-Einstein Metric

A Kähler-Einstein metric on a complex manifold is a Hermitian metric for which the Ricci tensor coincides up to multiplication by a real constant with the metric tensor. Thanks to Cheng-Yau (1980) and Mok-Yau (1983), such a metric is known to exist, for example, on any domain $D \subset \mathbb{C}^n$ that is bounded and *pseudoconvex* (i.e., can be exhausted by an increasing union of strongly pseudoconvex domains). It is given by

$$\mathcal{E}_D(z) := \sum_{m,k=1}^{n} \frac{\partial^2 u(z)}{\partial z_m \, \partial \overline{z_k}} \, dz_m \, d\overline{z_k},$$

where u is a solution to the boundary value problem

$$\det\left(\frac{\partial^2 u}{\partial z_m \, \partial \overline{z_k}}\right) = e^{2u} \quad \text{on } D, \qquad u = \infty \quad \text{on } \partial D. \tag{6}$$

The solution u is required to be *strongly plurisubharmonic*; that is, the matrix $(\partial^2 u / \partial z_m \, \partial \overline{z_k})$ is required to be positive definite. The metric $\mathcal{E}_D$ is Kählerian, complete, and biholomorphically invariant. For the unit disc Δ one has $\mathcal{E}_\Delta = \mathcal{P}$. It is, however, not true in general that $\mathcal{E}_M$ does not increase under holomorphic mappings.

For a bounded strongly pseudoconvex domain D with C^∞ smooth boundary, it is known from the work of Cheng-Yau that the holomorphic sectional curvature of $\mathcal{E}_D$ near the boundary of D approaches the negative constant $-2/(n+1)$. Therefore, $\mathcal{E}_D$ can be used in the proof of Theorem 4 above instead of $\mathcal{B}_D$. In some cases the boundary behavior of $\mathcal{E}_D$ is easier to determine than that of $\mathcal{B}_D$, since $\mathcal{E}_D$ is found from a solution to the boundary value problem (6) and therefore can be studied by methods of partial differential equations.

More Pseudometrics and Pseudodistances

In 1993 Hung-Hsi Wu proposed a way to define a whole new family of biholomorphically invariant pseudometrics. The actual definitions are rather complicated, and we mention only the following existence theorem related to one of these pseudometrics (see [Wu] for more detail).

Theorem 5. *It is possible to construct on every complex manifold M an upper semicontinuous Hermitian (not just Finslerian!) pseudometric $\mathcal{W}_M$ such that:*

(i) $\mathcal{W}_\Delta = \mathcal{P}$.
(ii) *If $f : M_1 \to M_2$ is a holomorphic mapping and $\dim M_1 = n$, then $f^* \mathcal{W}_{M_2} \le \sqrt{n}\, \mathcal{W}_{M_1}$.*
(iii) *If $f : M_1 \to M_2$ is a biholomorphic mapping, then $f^* \mathcal{W}_{M_2} = \mathcal{W}_{M_1}$.*
(iv) *If M is hyperbolic, then $\mathcal{W}_M$ is an upper semicontinuous Hermitian metric.*

The Wu pseudometric $\mathcal{W}_M$ in fact combines some of the most attractive features of the Finsler pseudometrics (Kobayashi and Carathéodory) and the Hermitian pseudometrics (Bergman and Kähler-Einstein) discussed earlier. For the Wu pseudometric $\mathcal{W}_M$ is (essentially) nonincreasing under holomorphic mappings, yet it is Hermitian. One of the first triumphs of the Wu pseudometrics is the following theorem of Wu:

Theorem 6. *A compact C-hyperbolic manifold is a projective variety.*

It is also conjectured that a similar result will be true for general Kobayashi hyperbolic manifolds.

The pseudometric $\mathcal{W}_M$ becomes a Hermitian metric on hyperbolic manifolds. This fact once again indicates the importance of the concept of hyperbolicity. It turns out that the hyperbolicity of a manifold can be understood by using a pseudometric introduced by Sibony in 1981 via plurisubharmonic functions (see [For, pp. 357–72] for details). Klimek in 1985 and Azukawa in 1986 also used plurisubharmonic functions to define an invariant pseudodistance and an invariant pseudometric different from the one of Sibony. More information about this subject can be found in [Kob2].

In fact there are other invariant pseudodistances and pseudometrics, too numerous to mention here explicitly, that have been created for various special purposes in complex analysis. We mention particularly the pseudometric of Hahn and the Lempert function. The monograph [JaP] is an excellent source of information about the panorama of ideas that is available.

Concluding Remarks

In this short article we have discussed some ideas in the geometric approach to complex analysis—in particular, ideas that use invariant distances and metrics. We would like to stress once again that such distances and metrics have many applications, of which we have indicated just a few. We hope, however, that we have been able to show at least to some extent the beauty of this active and rapidly developing subject. The geometric approach has given rise to powerful new versions of function theory on manifolds, and the problems currently under study should occupy researchers for many years to come. For more information we refer the reader to the monographs and survey articles listed in the reference section below.

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