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Abstract
This paper constructs a simple optimal monetary growth model in which an endogenous and variable rate of time preference provides a rational foundation for a Tobin-effect in a system where otherwise strong neoclassical assumptions (e.g., perfect foresight, an infinite planning horizon, and continuous marketclearing) are maintained. Changes in the proportional rate of growth of the nominal money supply affect both the rate of time preference (ñ) and the equilibrium capital—labour ratio. The impact effect of a fall in ñ (less impatience), and the induced capital accumulation that goes with it, drives the result. Proper transformation rules for two-state variable control problems and curvature and simulation results for the rate of time preference function are also established. The latter in particular provides a reasonable and easily understood foundation for simple systems in which the rate of time preference depends on an index of future consumption, and provides a counter-argument to well-known criticisms (e.g., Blanchard and Fischer (1989) and Barro and Sali-i-Martin (1995)) of Epstein—Uzawa rate of time preference functions. All results are obtained in an analytically simple way, using standard techniques.

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This paper constructs a simple optimal monetary growth model in which an endogenous and variable rate of time preference provides a rational foundation for a Tobin-effect in a system where otherwise strong neoclassical assumptions (e.g., perfect foresight, an infinite planning horizon, and continuous market-clearing) are maintained. Changes in the proportional rate of growth of the nominal money supply affect both the rate of time preference (ρ) and the equilibrium capital–labour ratio. The impact effect of a fall in ρ (less impatience), and the induced capital accumulation that goes with it, drives the result. Proper transformation rules for two-state variable control problems and curvature and simulation results for the rate of time preference function are also established. The latter in particular provides a reasonable and easily understood foundation for simple systems in which the rate of time preference depends on an index of future consumption, and provides a counter-argument to well-known criticisms (e.g., Blanchard and Fischer (1989) and Barro and Sali-i-Martin (1995)) of Epstein–Uzawa rate of time preference functions. All results are obtained in an analytically simple way, using standard techniques.

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1. Introduction

The issue of 'money neutrality' in monetary growth models has always attracted considerable attention. Early attempts, such as Sidrauski (1967a), generated 'Tobin-effects', or super-nonneutrality, where increases in the proportional rate of growth in the nominal stock of money (θ) result in larger steady-state capital-labour ratios (k). At the time the intuition seemed clear. In a world with only two assets, if the monetary authority drives down the rate of return on holding money, i.e., generates higher steady-state rates of inflation, individuals (with exogenous average propensities to save) are induced to reallocate their wealth in favor of an asset whose return (at least partially) offsets the effect of rising prices. In short, they are induced to hold greater stocks of capital.[1]

Unfortunately, such a straightforward effect was quickly seen to depend on the 'descriptive' or *ad hoc* nature of the savings function characteristic of a wide class of Keynesian models. Studies which instead based savings behaviour on explicit neoclassical intertemporal optimization, such as Sidrauski (1967b), recaptured super-neutral results.[2] Again, the intuition seemed clear, and especially so given the structure of standard optimal control models. In a world with infinitely-lived and optimizing agents the equilibrium capital stock was easily seen to have modified 'golden-rule' properties, clearly independent of the growth rate in the nominal supply of money. In other words, since a change in (θ) cannot affect either the rates (assumed positive and constant) of time preference (ρ) or population growth (g), technology alone determines the steady–state value of k. The presence of a inverse-monotonic function for the rate of return $r(k)$ and the steady-state condition $r(k) = \rho + g$ uniquely determines the value of k regardless of θ.

More recently, a number of papers have generated Tobin-effects within an optimizing framework by analyzing special cases where non-neutrality holds. The finite-horizon, overlapping generations models of Drazen (1976), Calvo's (1979) analysis of money in the production function, Brock's (1974) endogenous labour supply and Fischer's (1979b) characterization of out-of-steady state effects are notable examples.[3] However, none of these works explicitly addresses the importance of one of the key assumptions that drives super-neutrality in the standard optimizing framework, that of an exogenous rate of time preference. This paper does so. Following Uzawa (1968), it employs a variable rate of time preference function to form a connection between the monetary and real sectors of the economy, providing, so to speak, a 'rational' foundation for super-nonneutrality in a world where otherwise strict neoclassical assumptions (e.g., perfect foresight, an infinite planning horizon, continuous market-clearing) are maintained.[4]

[1] See Tobin (1965) and especially Fischer (1979a) for a complete treatment.

[2] See Brock (1974), Calvo (1979) and Fischer (1979b) as well.

[3] There are a class of endogenous monetary growth models that also generate non-neutral outcomes, where changes in π determine the rate of growth through endogenous productivity effects. De Gregorio (1993) and Gomme (1993) are good examples. All of this is ignored in the present paper, of course, since g is exogenous. The idea is to isolate the effects of a variable and endogenous rate of time preference in the simplest possible setting.

[4] There is a related point. Generally speaking, deterministic monetary growth models can be considered as special cases of (stochastic) rational expectations (RE) models with systematic monetary policy. However, rational expectations models are invariably constructed with super-neutral outcomes. Some time ago, Begg (1980:293), in particular, noted the conflict between (early) growth models with Tobin-effects and the RE approach, arguing that the "models analyzed in the rational expectations literature typically impose super-neutrality, whereas the literature on growth models with money emphasizes a mechanism which necessarily violates the super-neutrality condition." Nevertheless, Begg makes his case not from an optimizing framework, but with a 'descriptive' model where, as one might expect, non-neutrality depends simply on whether or not real money balances enter the consumption function as an argument, or whether the demand for money depends on the nominal rate of interest. Cf. Fischer (1979a) and Gale (1983:84–109).

In this context, the present paper re-affirms Begg's argument in a system with explicit intertemporal optimization. Indeed, in the model below it is easily seen that although steady-state utility falls with an increase in θ, level utility is still greater than in any comparable RE system which simply imposes super-neutrality with a constant rate of time preference and, as a result, generates

In simple terms the argument runs as follows. Standard monetary growth models employ utility functionals which are additive and discounted by a constant and positive value for ρ. The marginal rate of substitution between any two time-periods is thus independent of the utility from consumption at any other point in the time horizon. Such a structure breaks the link between long-run holdings of per-capita real money balances and k. Alternatively, by constructing a function for the rate of time preference so that it depends at each point in time on an index of current and future utility from consumption and real money balances, a connection between θ and k can be re-established. Across steady-state comparisons, an increase in θ results in an increase in the equilibrium capital-labour ratio and a fall in both the rate of return to capital and the rate of time preference.[5] A Tobin-effect is restored, and the impact effect of a fall in ρ (less impatience), and the induced capital accumulation that goes with it, drives the result.

This paper has an additional objective, specific to the construction of a wide-variety of models with variable rates of time preference. With money excluded, the Uzawa-type functional studied below takes the general form[6]

$$J(c) = \int_0^{\infty} u(c(t)) \exp\left\{ -\int_0^t \rho(u(c(t))d\tau \right\} dt \tag{1.1}$$

where the rate of time preference (ρ) depends on the utility derived from consumption ($c(t)$) along a given path, and where the value

$$\Delta(t) = \int_0^t \rho(u(c(t))d\tau \tag{1.2}$$

defines a utility discount factor applied at each time t.

There are two points of concern here. First, including Δ as an extra state variable clearly complicates the analysis, requiring (along with the usual state-variable constraint) now four dimensions to qualitatively characterize the four differential equations describing the co-state and state variables. To circumvent the problem, Uzawa (1968:491) also introduced a simple method for reducing the problem (dimensionally) to a single state variable by transforming the time scale from t to Δ, taking Δ as the independent variable and thus treating ρ as constant at each point on a given path. The problem, however, is that for this transformation to be valid the underlying system to be analyzed must be autonomous and except for the simplest control problems this is rarely the case. Non-autonomous transition equations (such as in the optimal monetary growth model developed below) imply that the correspondence between Δ

no change in the equilibrium capital-labour ratio.

[5] Uzawa (1968) and Epstein and Hynes (1983) are two closely related papers, developing similar systems and results. In a sense, the present paper simply elaborates on these two works. However, Uzawa's (1968) monetary growth model uses a transformation rule (as detailed in section 2 below), which generates considerable errors in first-order conditions and optimal paths, and although the functional form contained in Epstein and Hynes (1983) is actually less general than that of equation (1.1) below, the analysis in sections 4–7 on monetary growth is arguably more straightforward and complete. A good part of the motivation for the present paper is a desire to construct and explicitly solve an analytically simple model, using only standard techniques familiar to the theory of optimal monetary growth. Nothing in sections 4–7 below requires a knowledge of Volterra derivatives or the properties of weakly-separable utility functionals.

[6] Uzawa (1968) preferences belong to the general class of utility functionals used by Epstein (1983,1987a,1987b). See Buckholtz and Hartwick (1989), Epstein and Shi (1993), Nairay (1984) and Uzawa (1969) as well.

and t is no longer unique and thus Uzawa's transformation is not applicable. Section 2 of the paper shows this point clearly and indicates the required conditions for proper transformations of two-state variable problems of this form.[7]

The issue is doubly important since Uzawa's transformation has been used extensively in models of international trade and finance (e.g., Obstfeld (1981a,1981b,1982) and Engel and Kletzer (1989)), where the use of a variable rate of time preference nicely avoids some of the "disturbing implications" drawn from typical open-economy Ramsey models.[8] However, once again, given the non-autonomous nature of these systems, such a transformation is improper and as such generates errors in first-order conditions and resulting optimal paths that are incorrect.

Second, to ensure (saddlepoint) stability of the resulting dynamic system, the function $\rho = \rho(u(c(t))$ is usually restricted such that

$$\rho > 0 \qquad \rho' > 0 \qquad \rho'' > 0 \qquad \rho - \rho' u > 0 \tag{1.3}$$

where, to guarantee the existence of an optimal program, requires

$$(u')^2 \rho''(u) + u'' \rho'(u) < 0 \tag{1.4}$$

so that, roughly speaking, the concavity of $u(c)$ offsets the convexity of $\rho(u)$, and thus $\tilde{\rho}(c) = \rho[u(c)]$ is concave.[9] However, such restrictions have generated considerable criticism, particularly with regard to $\rho' > 0$, and regardless of whether c or u is the relevant argument in the functional form.[10] For example, Barro and Sala-i-Martin (1995:108–09) argue that ρ as a positive function of consumption $(c(t))$ is "unappealing" because it is "counterintuitive that people would raise their rates of time preference as their levels of consumption rise." Likewise, Blanchard and Fischer (1989:72–75) contend that Uzawa's rate of time preference function is "not particularly attractive as a description of preferences and is not recommended for general use" since the assumption $\rho' > 0$ is "difficult to defend *a priori*; indeed, we usually think of the rich who are more likely to be patient." As it goes, these criticisms are too harsh. Following Epstein (1987a), section 3 of the paper shows that what matters for the proper interpretation of the curvature of $\rho(u)$ or $\tilde{\rho}(c)$ is not only the level of consumption at time t but the path of consumption for all times subsequent to t. In the case where the growth in consumption increases, the tendency toward consumption smoothing, given convex preferences over $u(c)$, implies that more weight is given to present consumption. The same holds for globally constant consumption profiles with a path for consumption that is now everywhere higher. The result is intuitive and this is all that is meant by $\rho' > 0$ in this case. In effect, those households who know they will be 'more rich' in the future evaluate current consumption more highly.

[7] The issue is treated at length in Francis and Kompas (1998). Section 2 below draws on the results of this paper.

[8] For example, with ρ constant, it follows that all countries except the most patient one eventually become credit constrained. See Barro and Sali-i-Martin (1995:103–110) for a general discussion of the issue.

[9] Nairay (1984:285) was the first to introduce the restriction given by equation (1.4). For the most part in this paper, where there is no ambiguity, the condition $\rho' > 0$ will be taken as a representation implying that both $\rho'(u)$ and $\tilde{\rho}'(c) > 0$ hold.

[10] In fact, in the monetary growth model to follow, none of the reduced-form equations, the sign of the relevant coefficient matrix for the stability of the steady state, or the comparative dynamics exercise for variations in θ depend on the sign of $\rho''(u)$. See sections 6 and 7 and the appendix below. Nairay (1984) shows that the assumptions given by (1.3) correspond exactly to the discrete-time notions of 'impatience' and 'time perspective' contained in Koopmans, Diamond and Williamson (1964).

In addition, section 3 of the paper also shows that an exogenous decrease in the rate of time preference (where rates of time preference are variable over a given time horizon), results in larger steady-state values of wealth. A simulation makes this especially clear and thus confirms a Becker (1980) and Epstein (1987a) result that varying levels of impatience imply different final distributions of wealth and consumption across agents, with, as Blanchard and Fischer (1989:73) might suggest for this case, the patient being more wealthy in steady state. The result carries over naturally to the macro model and, indeed, such a plausible outcome in fact requires that $\rho' > 0$ holds as a necessary condition.

With sections 2 and 3 in mind, section 4 of the paper returns to the macro economy and sets out the optimal monetary growth model with variable rates of time preference. Section 5 derives conditions for individual optimization. The system is treated as an explicit two-state variable problem to avoid a transformation error, and in a way that is nicely applicable to any class of growth models (e.g., models of international trade and finance and life-cycle studies) which employ a rate of time preference as an additional state variable. Section 6 constructs the macro-dynamics for a perfect foresight equilibrium and section 7, finally, illustrates super-nonneutrality. Section 8 concludes. An appendix collects some technical details.

2. Transformation Errors and Rules

Consider a typical intertemporal (life-cycle) problem without money, where any given path assigns values of consumption $c(t) \geq 0$ at each time $t \geq 0$ over an infinite horizon. Let the rate of time preference be a variable that depends on an index of current and future consumption defined by equation (1.2). For an otherwise standard utility functional, the problem is to maximize (1.1) subject to

$$\dot{\Delta} = \rho(u(c)) \tag{2.1}$$

and

$$\dot{w}(t) = g(w(t), c(t), t) \tag{2.2}$$

for given initial conditions, $\Delta(0) = 0$ and $w(0) = w_0$. The state variable constraint for (say) wealth (w), or equation (2.2), is written in general and non-autonomous form. Equation (2.1) explicitly adds a second state variable, or Δ.

As mentioned, the idea of Uzawa's (1968:491) transformation is to reduce the dimension of the problem (to that of a single state variable) by transforming the time scale from t to Δ, so that the rate of time preference can be treated as a constant and the usual solution techniques can be applied. To follow this approach, note first that $dt = d\Delta/\rho(u)$ from (2.1). With equation (1.1) redefined, the problem now becomes one of maximizing

$$\tilde{J}(c) = \int_0^\infty \frac{u(c(t))}{\rho} e^{-\Delta} d\Delta \tag{2.3}$$

subject to

$$\frac{dw}{d\Delta} = \frac{g(w, c, t)}{\rho} \tag{2.4}$$

and $w(0) = w_0$, a single state variable.

To see the error in Uzawa's transformation (as applied to non-autonomous systems) most clearly, consider a specific example. Let preferences be given by $u(c(t)) =$

$c^{\frac{1}{2}}$ or $c = u^2$ and let equation (2.1) be represented by

$$\dot{\Delta} = \rho(u(c)) = \alpha + \beta u. \tag{2.5}$$

Define the state variable constraint as

$$\dot{w}(t) = rw + \nu(t) - c = rw + \nu(t) - u^2 \tag{2.6}$$

for r a given and exogenous rate of interest. The presence of $\nu(t)$ in (2.6) makes the system non-autonomous.[11]

Using Uzawa's transformation, maximize equation (2.3) subject to

$$\frac{dw}{d\Delta} = \frac{1}{\rho}(rw + \nu(t) - u^2) \tag{2.7}$$

to obtain the following first–order necessary conditions:

$$\frac{1}{\rho^2}\left[\rho e^{-\Delta} - \rho' e^{-\Delta}u - 2\rho u \lambda_1(\Delta) - \rho'\lambda_1(\Delta)(rw + \nu(t) - u^2\right] = 0 \tag{2.8}$$

$$\frac{d\lambda_1}{d\Delta} = \frac{-\lambda_1 r}{\rho}\ . \tag{2.9}$$

To convert the problem to current–values define $\phi_1(\Delta) \equiv e^{\Delta}\lambda_1$ and note that

$$\frac{d\phi_1}{d\Delta} = e^{\Delta}\frac{d\lambda_1}{d\Delta} + e^{\Delta}\lambda_1 = \phi_1 - \frac{r\phi_1}{\rho} \tag{2.10}$$

so that transforming the time scale back to t, using $d\Delta = \rho(u)dt$, implies that (2.8) and (2.9) now become

$$\rho(1 - 2\phi_1 u) - \rho'(u + \phi_1(rw + \nu(t) - u^2)) = 0 \tag{2.11}$$

and

$$\dot{\phi}_1(t) = \phi_1(\rho(u(c)) - r). \tag{2.12}$$

As a means of comparison, write (2.11) as a single second-order differential equation. Using (2.12), (2.5) and (2.6) and differentiating twice with respect to time gives

$$2\left[\ddot{u} + (\alpha + \beta u - 2r)\dot{u} - r(\alpha + \beta u - r)u\right] + \left[\frac{1}{\rho}\{(\rho - r)\beta\dot{\nu} + \beta\ddot{\nu}\} - \frac{\beta^2\dot{\nu}\dot{u}}{\rho^2}\right] = 0. \tag{2.13}$$

Now rewrite the problem explicitly in terms of two state variables, without Uzawa's transformation. Maximizing (1.1) subject to initial conditions and (2.5) and (2.6) gives

$$e^{-\Delta} - 2u\lambda_1 + \lambda_2\rho' = 0 \tag{2.14}$$

[11] In a monetary growth model with variable rates of time preference (as in Section 6 below), the value of $\nu(t)$ is represented by flow additions to the stock of real money balances (m), or θm, for θ the proportional rate of growth of the nominal supply of money. Net changes in real money balances or $(\theta - \pi)m$, for $\pi(t)$ the rate of inflation, imply that the transition equation is non-autonomous. Uzawa's transformation can thus not be used. In models of international trade and finance, $\nu(t)$ would stand for anticipated shocks, imported foreign goods or bond holdings. See Obstfeld (1982) and Engel and Kletzer (1989) for example.

$$\dot{\lambda}_1(t) = -\lambda_1 r \tag{2.15}$$

$$\dot{\lambda}_2 = ue^{-\Delta} \tag{2.16}$$

$$\dot{\Delta}(t) = \alpha + \beta u \tag{2.17}$$

and

$$\dot{w}(t) = rw + \nu(t) - u^2. \tag{2.18}$$

To eliminate the exponential, convert to current-value variables for $\phi_1(t) \equiv e^{\Delta}\lambda_1(t)$ and $\phi_2(t) \equiv e^{\Delta}\lambda_2(t)$. Since $\dot{\phi}_1(t) = e^{\Delta}\dot{\lambda}_1(t) + \rho\phi_1$ and $\dot{\phi}_2(t) = e^{\Delta}\dot{\lambda}_2(t) + \rho\phi_2$, first-order conditions now become

$$1 - 2\phi_1 u + \phi_2 \rho' = 0 \tag{2.19}$$

$$\dot{\phi}_1(t) = \phi_1(\rho(u(c)) - r) \tag{2.20}$$

and

$$\dot{\phi}_2(t) = \rho\phi_2 + u. \tag{2.21}$$

Finally, differentiating (2.19) twice with respect to time, substituting from (2.5), (2.20) and (2.21), gives

$$\ddot{u} + (\alpha + \beta u - 2r)\dot{u} - r(\alpha + \beta u - r)u = 0 \tag{2.22}$$

or the system as described (again) by a comparable single, second–order differential equation in u.

Comparing equations (2.13) and (2.22) makes the point. The two are equivalent only when $\dot{\nu}(t) = \ddot{\nu}(t) = 0$.[12] In other words, Uzawa's transformation is valid only when the problem is autonomous. When this condition does not hold, Uzawa's transformation generates errors in first-order conditions and resulting optimal paths. In effect, the transformation ignores the non-linearities caused by the explicit presence of time as a variable in the transition equation.

3. Variable Rates of Time Preference and Simulation Results

Consider again the life-cycle problem of maximizing (1.1) subject to (2.1) and the state-variable constraint

$$\dot{w} = rw - c \tag{3.1}$$

for $\Delta(0) = 0$ and $w(0) = w_0$.[13] From first-order conditions, the equation of motion for consumption is given by

$$\dot{c} = \frac{u_c(\rho - \rho' u)[(r - \rho) - \rho' u_c(rw - c)]}{\rho'' u_c^2 (rw - c) - u_{cc}(\rho - \rho' u)} \tag{3.2}$$

[12] It can be easily shown that this condition, as expected, amounts to a system where $dH/dt = 0$, for H the value of the relevant Hamiltonian. Since the problem is clearly autonomous, such a system can thus be properly transformed using Uzawa's technique. See Francis and Kompas (1998).

[13] See Nairay (1984), Obstfeld (1990) and for a full analysis of this problem, especially Kompas and Preston (1998). Uzawa's transformation is applicable in this case (see Francis and Kompas (1998) for the relevant proof). Note, too, that for $\rho = \bar{\rho}$ in equation (3.2) below that the standard result is obtained, or $\dot{c} = (u_c/u_{cc})(\rho - r)$.

and in steady-state $rw = c$ and $\rho[u(c)] = r$. A linear approximation to the steady state implies that

$$\begin{bmatrix} \dot{c} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} 0 & \rho' u_c/A \\ -1 & r \end{bmatrix} \begin{bmatrix} c - c^* \\ w - w^* \end{bmatrix} \tag{3.3}$$

for

$$A = u_{cc}/u_c - \rho'' u_c u/(\rho - \rho' u) < 0 \tag{3.4}$$

with the determinant of the above matrix given by $|D| = \rho' u_c/A < 0$, indicating a saddlepoint.

Given the value of $|D|$, restricting ρ' to be greater than zero is clearly necessary for saddlepoint stability (the steady state is locally unstable if $\rho' < 0$). But, as mentioned in the introduction above, this does not necessarily imply that the functional form $\rho = \rho(u(c(t))$ and the variation $\rho' > 0$ is "unappealing". In particular, it does not imply that the rate of time preference depends only on the level of current consumption. It also depends on the path of consumption for all periods subsequent to t. For a given household, in other words, the condition $\rho' > 0$ implies that an increase in the path of $c(t)$, formally, a Volterra derivative, results in an increase in the current rate of time preference. A greater weight is thus placed on present consumption, and in a way fully consistent with convex preferences (or the desire for consumption smoothing).

To get a picture of all of this, consider first the marginal rate of substitution as defined in terms of variations in the value function $J(c)$ in equation (1.1). Following Ryder and Heal (1973), let c^* be the reference path and $c(t) = c^*(t) + \mu(t)\Delta c$ the variational path, and define the value of forward consumption from T, or ${}_Tc(t)$ as

$$J({}_Tc(t)) = \int_T^\infty u(c(t)) \exp\left\{-\int_T^t \rho(u(c(\tau))d\tau\right\} dt. \tag{3.5}$$

For an arbitrary period of time, $\Delta\varepsilon$, and using the definition of Δ in equation (1.2) so that

$$\Delta(0, T - \Delta\varepsilon) = \int_0^{T-\Delta\varepsilon} \rho(u(c(t))d\tau \tag{3.6}$$

defined (for example) over a specific time interval $(0, T - \Delta\varepsilon)$, the value of the variational path is given by

$$J(c(t)) = \int_0^{T-\Delta\varepsilon} u(c(t))\Delta(0,t)dt + \Delta(0, T - \Delta\varepsilon)J({}_{T-\Delta\varepsilon}c(t)) \tag{3.7}$$

and comparing this to a comparable expression for $c^*(t)$ implies that

$$J(c^*(t)) - J(c(t)) = \Delta(0, T - \Delta\varepsilon)[J({}_{T-\Delta\varepsilon}c^*(t)) - J({}_{T-\Delta\varepsilon}c(t))]. \tag{3.8}$$

A Volterra derivative is defined when $\Delta\varepsilon$ and $\Delta c \to 0$. Now consider another variational path c^{**} which differs from $c(t)$ over two distinct intervals $(\tau_1 - \Delta\varepsilon, \tau_1)$ and $(\tau_2 - \Delta\varepsilon, \tau_2)$. Replace c^{**} with c^*, repeat equations (3.7) and (3.8) and let the total change in the value function be zero. In the limit, the marginal rate of substitution between consumption at dates $\tau_1$and τ_2 is

$$\chi(c; \tau_1, \tau_2) \equiv \frac{J'(c; \tau_1)}{J'(c; \tau_2)} \tag{3.9}$$

and truncating the original consumption path in equations (1.1) and (1.2) above, as in (3.5), while taking the time derivative of $J(c(t))$, obtains

$$\rho(t) = \left(\frac{\partial\Delta(\tau_1,\tau_2)}{\partial\tau_2}\right) \tag{3.10}$$

when $\tau_1 = \tau_2$. Equations (3.7)–(3.10) make it clear that both the marginal rate of substitution and the rate of time preference depend on the entire path of consumption. In fact, for a truncated path through consumption at T, the functions $\chi(c(T), J(_Tc))$ and $\rho(c(T), J(_Tc))$, separating current from future consumption, clearly depend both on the value of consumption at time T along this path, and the utility gained from the consumption path for all times subsequent to T.[14] It follows that the variation $\rho' > 0$ simply measures a path effect, applying to changes in both current and future consumption. An increase in the path of consumption (or with globally constant consumption profiles, a path that is now everywhere higher in the future), thus implies an increase in $\rho(T)$, so that current consumption is given more weight.[15]

This last result can be seen in a slightly different way. As Epstein (1987a) shows, the general value of $\rho(c(T), J(_Tc))$ obtained from equation (1.1), and comparable to (3.10), is

$$\rho(c,\varphi) = \frac{u(c)\rho'(c) - \rho(c)u_c(c)}{\varphi\rho'(c) - u_c(c)} \tag{3.11}$$

for $\varphi \equiv J(_tc)$, the utility from the path of consumption for times subsequent to t. The variation in ρ from a change in this path is

$$\rho_\varphi(c,\varphi) = \frac{\rho'(c)}{u_c - u(c)\rho'(c)/\rho(c)} \tag{3.12}$$

for $\rho_\varphi \equiv \partial\rho/\partial\varphi$. Given the restrictions in (1.3) above, the denominator of this expression is clearly positive, so that, truncating at T, if the value $\rho_\varphi(c(T),\varphi) > 0$, then, by equation (3.12), it must be the case that $\rho'(c(T)) > 0$. As above (see equations (3.7)–(3.10)), if there is an increase in the path passing through $c(T)$, and for all times subsequent to T, and this results in an increase in $\rho(J(_Tc))$, it must imply that $\rho(c(T))$ increases.

The only final point to note is that given such a path effect the value $\rho_\varphi(c(T),\varphi) > 0$, and the resulting increase in the rate of time preference at T, is consistent with a tendency for consumption smoothing, or convex preferences. With a logarithmic change in marginal utility, it is easily seen that

$$-\frac{d}{dT}\log J(_Tc) = \rho(c(T), J(_Tc)) - \frac{u_{cc}}{u_c}\dot{c} \tag{3.13}$$

[14] See equation (3.7) as $\Delta\varepsilon \to 0$ and Kompas (1999) for a complete exposition and relevant proofs.

[15] Along the lines of Uzawa (1991), Kompas (1999) also analyzes a system where intertemporal preferences are homothetic so that if a path $c(t) \succeq \tilde{c}(t)$, it follows that $\alpha c(t) \succeq \alpha\tilde{c}(t)$ for any given scalar $\alpha > 0$. For such a preference ordering, χ in equation (3.9) remains unchanged and hence the rate of time preference does not change when any given consumption path is multiplied by $\alpha > 0$. The function ρ simply depends on the growth in consumption. This assumption, however, is not applicable to the monetary growth model developed below since, in its classical form, all paths are restricted to be piece-wise continuous. In such a case, a change in θ results in a 'jump' in prices and real money balances at $t = 0$ and the resulting path effect on real money balances must imply a comparable 'impact effect' on the rate of time preference at $t = 0$. In short, the rate of time preference must also depend on the current level of real money balances and not just its growth (or path) through time. In a macro model with 'everywhere smooth' price-paths, the assumption of homothetic (intertemporal) preferences would thus imply that the controversy over the sign of ρ' at $t = 0$ could be ignored altogether.

so that with $u_c > 0$ and $u_{cc} < 0$, a $\dot{c} > 0$, or any increase in the growth of consumption, thus increases the current rate of time preference. For the case of constant consumption profiles, equation (3.13) reduces to the more familiar

$$\rho(c(T), J(_T c)) = r \tag{3.14}$$

for r a given rate of return. The difference, of course, between (3.14) and the standard additive model is that ρ is now a variable that depends the path of present and future consumption. If $\rho_\varphi(c(T), \varphi) > 0$, for $\varphi \equiv J(_T c)$, an increase in a globally constant consumption path implies, from equation (3.12), once again, that $\rho(c(T))$ increases, with a new equilibrium (depending on the context) that corresponds to a larger value of r.[16] As long as the argument runs in terms of consumption paths that are everywhere higher (or lower), whether constant or not, everything follows. The restriction $\rho' > 0$ simply means that if a household knows that it will be 'more rich' in the future, it evaluates consumption today more highly.

Finally, consider a comparative dynamics exercise. As mentioned in the introduction, the idea is to show that when $\rho' > 0$ a parametric fall in ρ is fully consistent with an increase in steady-state wealth.[17] To make the analysis especially clear let the rate of time preference function (for now) be affine[18]

$$\rho = \kappa u + \beta \tag{3.15}$$

for κ and β given constants, so that in steady state the following hold

$$\rho(u(c)) = \kappa u + \beta = r$$

$$rw = c \tag{3.16}$$

$$\beta u_c = \psi[\kappa u(c) + \beta] \tag{3.17}$$

for ψ the co-state variable on wealth. Given a change in κ, the relevant variational system is

$$\begin{bmatrix} \kappa u_c & 0 & 0 \\ 1 & -r & 0 \\ A & 0 & -\rho \end{bmatrix} \begin{bmatrix} dc^*/d\kappa \\ dw^*/d\kappa \\ d\psi^*/d\kappa \end{bmatrix} = \begin{bmatrix} -u(c) \\ 0 \\ \psi u(c) \end{bmatrix} \tag{3.18}$$

for $A = \beta u_{cc} - \psi \kappa u_c < 0$, with a determinant value $|D| = \kappa u_c r \rho > 0$. Solving (note that $\kappa > 0$ is necessary for stability) gives

$$\frac{dc^*}{d\kappa} = \frac{-u\rho r}{|D|} < 0 \tag{3.19}$$

[16] In fact, this is exactly the reverse case of what occurs in the macro model developed below. See equations (5.10), (7.7) and (7.8). Here, an increase in θ results in a steady-state fall in the rate of return (r) and since the path for real money balances ($m(t)$) and utility, or $u(c(t), m(t))$, is everywhere lower, the rate of time preference also falls in steady-state equilibrium.

[17] This will be particularly relevant in the macro model contained in sections 4–7 below where, in this case, an *induced* fall in ρ, given an increase in inflation, results in capital accumulation. The simulation result also applies to a model which forms a comparison across households (or countries) with varying rates of time preference.

[18] See Kompas and Preston (1998) for additional (technical) derivations and a comparative dynamics exercise in the sense of Oniki (1973). Nairay (1984) shows that the strict convexity of $\rho(u)$ implies that the function is approximately affine in u for large values of u. Nonetheless, the point of contention over the function $\rho = \rho(u)$ usual centers on $\rho' > 0$, which still holds in equation (3.15). The simulation exercise that follows below is less restrictive in that both ρ' and $\rho'' > 0$ as in (1.3).

$$\frac{dw^*}{d\kappa} = \frac{-u\rho}{|D|} < 0 \tag{3.20}$$

and

$$\frac{d\psi^*}{d\kappa} = \frac{-uru_{cc}\beta}{|D|} > 0. \tag{3.21}$$

An decrease in the rate of time preference results in an increase in steady-state wealth and consumption.

A simulation exercise confirms the result. Set $\beta = 0$ (without loss of generality) and let

$$\rho = \frac{\kappa}{c_0}u^2 \tag{3.22}$$

so that both ρ' and $\rho'' > 0$, for c_0 an arbitrary constant. Let $u(c) = c^{0.75}$, $r = 0.1$, $c_0 = 100$ and $w_0 = 1000$. Begin, at first, with a value of $\kappa = 0.002$. (In this numerical case the problem is bounded in that $\kappa < 0.01$.) After calculating the value of A in equation (3.4), the idea is to form a log-linear approximation around steady-state values for the system given by (3.3) in order to generate numerical solutions to the approximate nonlinear system. In so doing, the system now becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{c} \\ \dot{w} \end{bmatrix} + \begin{bmatrix} 0 & 0.08 \\ 1 & -0.1 \end{bmatrix} \begin{bmatrix} c \\ w \end{bmatrix} = \begin{bmatrix} 0.08w^* \\ c^* - 0.1w^* \end{bmatrix} \tag{3.23}$$

where the general solution contains two roots, $\gamma_1 = 0.14$ and $\gamma_2 = -0.06$, consistent with a saddlepoint. Ignoring the positive root, the general solution is

$$\begin{bmatrix} 0 & 0.08 \\ 1 & -0.1 \end{bmatrix} \begin{bmatrix} c \\ w \end{bmatrix} = \begin{bmatrix} 0.08w^* \\ c^* - 0.1w^* \end{bmatrix}. \tag{3.24}$$

A similar procedure is performed for values of $\kappa = 0.004$ and $\kappa = 0.006$ and the results for steady-state values of wealth (w), years, and speed of converge are summarized in Table 1.[19] Each calculated path can be shown to be increasing and monotone and that it converges to a higher steady-state value of w^* in finite time as κ decreases.

	$\kappa = 0.002$	$\kappa = 0.004$	$\kappa = 0.006$
Steady-state w^*	13000	8300	6400
Time-scale of convergence (years)	200	150	110
Speed of convergence	0.06	0.08	0.09

Table 1: Numerical results for w^* given a change in κ

4. A Simple Monetary Growth Model

The results of Sections 2 and 3 are now applied to a simple monetary growth model. Detailed assumptions are as set out in Sidrauski (1967b). Individuals are assumed identical, can hold either money or capital and production is neoclassical (time–subscripts are often dropped for convenience)

$$y = y(k); \qquad y' > 0, \qquad y'' < 0, \qquad y(0) = 0 \tag{4.1}$$

[19] For example, for the speed of convergence for $\kappa = 0.002$, the value of $w(t) = -12000e^{-0.06t} + 1300$.

for y real output per–capita. The instantaneous utility function is defined as $u = u(c(t), m(t))$ for c per–capita consumption and m per-capita real money balances, with $u_c, u_m > 0$, $u_{cc}, u_{mm} < 0$ and $J = u_{cc}u_{mm} - u_{mc}^2 > 0$. Restricting

$$J_1 = u_{mm} - (u_{cm}u_m)/u_c < 0 \qquad J_2 = (u_{cc}u_m)/u_c - u_{cm} < 0 \tag{4.2}$$

guarantees that both c and m are normal goods.

Money is introduced into the system through net lump-sum transfers where the individual's flow constraint is of the form

$$\dot{a} = y(k) + v - (\delta + g)k - (\pi + g)m - c \tag{4.3}$$

for a non–human wealth, v the value of net transfers (or θm in the aggregate economy), δ and g the exogenous rates of depreciation and population growth and π the rate of inflation. Unlike Sidrauski, assume 'myopic perfect foresight' so that actual and expected rates of change in the price level are equal.[20] The corresponding stock constraint is

$$a = k + m \tag{4.4}$$

for each t in the planning horizon.

Following Uzawa (1968), define $\Delta(t)$ to be the utility discount factor at t, where

$$\Delta(t) = \int_0^t \rho(u)d\tau \qquad \Delta(0) = 0 \tag{4.5}$$

for $\rho = \rho(u) = \rho[u(c, m)]$ the individual rate of (subjective) time preference, assumed now to be a function of c and m and over a given path. As discussed in sections 1 and 3 above, restrict $\rho(u)$ such that

$$\rho > 0, \qquad \rho' > 0, \qquad \rho'' > 0, \qquad \rho - \rho' u > 0. \tag{4.6}$$

The individual's problem amounts to finding the time–profiles (c_t, m_t) which maximize

$$L = \int_0^\infty u(c, m)e^{-\Delta(t)}dt \tag{4.7}$$

subject to constraints (4.3) and (4.4), $a(0) = a_0$ and

$$\dot{\Delta}(t) = \rho(u) = \rho(u(c, m)). \tag{4.8}$$

5. Individual Optimization

To avoid a transformation error (see section 2), define the current–value Hamiltonian as a two–state variable problem, or

$$\hat{H} = u(c, m) + \sigma[y(a - m) + v - (\delta + g)(a - m) - (\pi + g)m - c] + \phi\rho[u(c, m)] \tag{5.1}$$

for $\sigma(t) \equiv e^{\Delta}\lambda(t)$ and $\phi(t) \equiv e^{\Delta}\gamma(t)$, with $\lambda(t)$ and $\gamma(t)$ the co–state variables for the constraints given by equations (4.3) and (4.8). Since

$$\dot{\sigma}(t) = e^{\Delta}\dot{\lambda}(t) + e^{\Delta}\lambda(t)\dot{\Delta} = e^{\Delta}\dot{\lambda} + \rho\phi \tag{5.2}$$

[20] As is well-known, 'myopic perfect foresight' means that agents can calculate the right–hand time derivative of $p(t)$ over the entire planning horizon. Following Sargent and Wallace (1973), and given the 'classical' nature of this monetary growth model, $p(t)$ is assumed to be piece–wise continuous, with a finite number of jumps.

and

$$\dot{\phi}(t) = e^{\Delta}\dot{\gamma} + \rho\phi \tag{5.3}$$

first–order necessary conditions (assuming an interior solution and the appropriate transversality conditions) are

$$u_c = \frac{\sigma}{1+\phi\rho'} \tag{5.4}$$

$$u_m = \frac{\sigma}{1+\phi\rho'}(y' - \delta + \pi) \tag{5.5}$$

$$\dot{\sigma} = \sigma[\rho(u(c,m)) - r + g] \tag{5.6}$$

$$\dot{\phi} = \phi\rho[u(c,m)] + u(c,m) \tag{5.7}$$

$$\dot{a} = y(a-m) + v - (\delta - g)(a-m) - (\pi + g)m - c \tag{5.8}$$

for $r = y' - g$. Equations (5.4) and (5.5) given the usual static result for utility maximization, or

$$\frac{u_m}{u_c} = y' - \delta + \pi = r + \pi \tag{5.9}$$

and (5.6) implies that

$$\rho(u(c,m)) = r - g \tag{5.10}$$

when $\dot{\sigma} = 0$. Note that for the case in which $\rho(u) = \bar{\rho}$, all the standard first–order conditions for utility maximization with a constant rate of time preference clearly return.

6. Perfect Foresight Macro Dynamics

Assume that market equilibrium requires that the supply and demand for money be equal at each time t. For $v = \theta m$, a perfect foresight equilibrium is represented by a bounded time–profile for prices $p(t)$ and corresponding paths $(c_t, m_t, k_t, \sigma_t)$ that satisfy[21]

$$\dot{k} = y(k) - (\delta + g)k - c \tag{6.1}$$

Asset market (flow) equilibrium now becomes

$$\dot{a} = y(k) - (\delta + g)k + (\theta - \pi - g)m - c \tag{6.2}$$

where the change in per capita real money balances with respect to time is given by

$$\dot{m} = (\theta - \pi - g)m \tag{6.3}$$

so that the equilibrium rate of inflation is $\pi^* = \theta - g$.

The macro model consists of equations (5.4)–(5.6), (6.1) and (6.3). A linear approximation to the steady–state values $(c^*, m^*, k^*, \sigma^*)$ gives

$$\begin{bmatrix} \dot{c} \\ \dot{m} \\ \dot{k} \\ \dot{\sigma} \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 & -\rho/A_4 \\ mJ_2/u_c & -mJ_1/u_c & my'' & 0 \\ -1 & 0 & \rho & 0 \\ \sigma\rho' u_c & \sigma\rho' u_m & -\sigma y'' & 0 \end{bmatrix} \begin{bmatrix} c - c^* \\ m - m^* \\ k - k^* \\ \sigma - \sigma^* \end{bmatrix} \tag{6.4}$$

[21] A bounded and strictly positive value for $p(t)$ for all t guarantees (in an otherwise saddlepoint unstable system) that the economy always converges to a new steady-state in response to an exogenous shock. See Sargent and Wallace (1973), Burmeister (1980), and Kompas and Spotton (1989).

for

$$A_1 = 1/A_4[(A_5 m J_2/u_c) + u_{cc}(\rho - \rho' u) - \rho'' u u_c^2]$$
$$A_2 = 1/A_4[(-A_5 m J_1/u_c) + u_{cm}(\rho - \rho' u) - \rho'' u u_c u_m]$$
$$A_3 = 1/A_4[A_5 m y'' - \sigma y'']$$
$$A_4 = \sigma u_{cc}/u_c + [(\sigma - u_c)/\rho']\rho'' u_c$$
$$A_5 = -[(\sigma u_{cm}/u_c) + \{(\sigma - u_c)/\rho'\}\rho'' u_m]$$

where the determinant of the above coefficient matrix is negative, indicating a saddlepoint.[22]

7. The Effect of a Change in θ

Super–nonneutrality can finally be illustrated. For steady–state values, the following hold

$$\rho u_c - \rho' u u_c = \sigma(r - g) \tag{7.1}$$

$$\frac{u_m}{u_c} - \theta = r - g \tag{7.2}$$

$$y(k) - (\delta + g)k = c \tag{7.3}$$

$$\rho(u(c, m)) = r - g \tag{7.4}$$

and comparative steady states gives

$$\begin{bmatrix} A_6 & A_7 & \sigma y'' & -\rho \\ J_2 & -J_1/u_c & y'' & 0 \\ -1 & 0 & \rho & 0 \\ \rho' u_c & \rho' u_m & -y'' & 0 \end{bmatrix} \begin{bmatrix} dc^*/d\theta \\ dm^*/d\theta \\ dk^*/d\theta \\ d\sigma^*/d\theta \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \tag{7.5}$$

for

$$A_6 = u_{cc}(\rho - \rho' u) - \rho'' u u_c^2$$
$$A_7 = u_{cm}(\rho - \rho' u) - \rho'' u u_c u_m.$$

The determinant of the above matrix is

$$|D| = \rho[(J_1 y''/u_c) - \rho' u_m y'' - \rho\{(\rho' J_2 u_m/u_c) + \rho' J_1\}] > 0 \tag{7.6}$$

and solving gives

$$\frac{dm^*}{d\theta} = \frac{\rho(y'' - \rho\rho' u_c)}{|D|} < 0 \tag{7.7}$$

$$\frac{dk^*}{d\theta} = \frac{\rho\rho' u_m}{|D|} > 0 \tag{7.8}$$

An increase in θ decreases steady-state per capita real money balances (just as in the super-neutral case), but increases both the steady-state capital-labour ratio and consumption per capita.[23]

The intuition for this effect is generally the same as that contained in 'descriptive' monetary growth models, except that the rate of time preference, or more to the point,

[22] See the Appendix below.

[23] Note that for $\rho = \bar{\rho}$, $dc^*/d\theta$ and $dk^*/d\theta$ equal zero, as usual.

changes in the rate of time preference drive the super-nonneutral result. Moreover, both the 'impact' and steady-state effects are fully consistent with the comparative dynamics exercise (given change in the rate of time preference) described at the end section 3 above. An increase in θ lowers the rate of return on holding money balances as π increases. With perfect foresight and $\rho' > 0$, the fall in the path of $m(t)$ decreases utility and hence the rate of time preference at a given $k(0)$. With the resulting fall in ρ, individuals are thus induced to accumulate or hold larger steady-state stocks of capital, with a higher nominal rate of return $(r + \pi)$ partially offsetting the increase in π. With an increase in k the value of consumption (and thus ρ) also rises, but since real returns fall in steady-state $(y''(k) < 0)$, the steady-state rate of time preference, from equation (7.4), must still be lower than at $t = 0$.[24] In short, real and nominal returns move in opposite directions and a smaller value of ρ corresponds to a larger equilibrium value for k.

8. Conclusion

This paper has constructed a simple optimal monetary growth model where changes in the proportional rate of growth of the money supply affect both the rate of time preference and the equilibrium capital-labour ratio. A Tobin-effect is restored in a system where otherwise strong neoclassical assumptions (e.g., perfect foresight, an infinite planning horizon, and continuous market-clearing) are maintained. With a change in θ, increases in the rate of inflation result in a fall in the rate of time preference and induced capital accumulation. The steady-state value of k increases. In addition, a justification for curvature restrictions on the rate of time preference function has also been provided. The analysis demonstrates that ρ depends on the entire path of consumption and that $\rho' > 0$ measures a variation in a given path that is nicely consistent with the desire for consumption smoothing. In effect, $\rho' > 0$ implies those households who know they will be 'more rich' in the future evaluate current consumption more highly. A comparative dynamics exercise also shows that, with $\rho' > 0$, a decrease in the rate of time preference must imply a larger steady-state value of wealth. The effect is fully consistent with the increase in k (from an induced fall in ρ) obtained in the macro model.

Two further lines of research immediately come to mind. First, it would be helpful to have a precise specification of the transitional dynamics, detailed at the end of section 7 above, given by the effect of a change in θ on ρ and k for all time $t > 0$. Adapting the work of Fischer (1979) might be particularly helpful in this case since the constant relative risk aversion utility functions that he uses are known to be broadly consistent (see Epstein (1983)) with the curvature assumptions on $\rho(u)$ used in section 3 of this paper. Second, it may now be worthwhile to reconsider models of international trade and finance using variable rates of time preference, along the lines of Obstfeld (1981a,1981b). This is important since if the justification for the curvature of $\rho(u)$ in this paper is correct, such systems can be usefully constructed to avoid what Barro and Sala-i-Martin (1995:108) term the "disturbing implications" of standard open-economy (Ramsey) models. Unlike Obstfeld, however, the model must be explicitly written as a multi-state variable problem in order to avoid the error in

[24] Although it is clear that the steady-state value of $u(c(t), m(t))$ and ρ falls with an increase in θ, the value of $u(c(t), m(t))$ does not fall as much as in the super-neutral case since, with the fall in the rate of time preference, an increase in the equilibrium value of $k(t)$ results in higher values of $c(t)$. See equation (7.3).

Uzawa's transformation detailed in section 2 above

APPENDIX

The equation for $\dot{c}$ used in system (6.4) is derived from first–order conditions. Differentiate (5.9) with respect to time to obtain

$$u_{cc}\dot{c} + u_{cm}\dot{m} = \dot{\sigma}/(1+\phi\rho') - \sigma(\dot{\phi}\rho' + \dot{\rho}'\phi)/(1+\phi\rho')^2 \tag{A1}$$

where $\dot{\rho}' = \rho'' u_c \dot{c} + \rho'' u_m \dot{m}$. With equations (5.6) and (5.7),

$$\phi = (\sigma - u_c)/\rho' u_c \tag{A2}$$

so that substituting in (A1) gives

$$\dot{c} = 1/A_4[A_5\dot{m} + \sigma(\rho - y' + \delta + g) - \rho(\sigma - u_c) - \rho' u u_c] \tag{A3}$$

as given in Section 6.

The determinant of the coefficient matrix in system (6.4) is

$$|F| = \rho/A_4[(\sigma m J_1 y''/u_c) - \sigma\rho' m u_m y'' - \rho\{(\sigma m \rho' u_m J_2/u_c) + \sigma\rho' m J_1\}]. \tag{A4}$$

Since J_1, J_2, $y'' < 0$ and $\sigma, \rho, \rho', m, u_c, u_m > 0$, the sign of $|F| = sign\,\{A_4\}$. At a steady state $\phi = -u/\rho$, so that $u_c = \sigma/(1 - \rho' u/\rho)$ implying that $(\sigma - u_c)$ in A_4 is negative, indicating a saddlepoint. It can also be shown formally that the characteristic equation for this system has a unique negative root.

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