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Abstract

Uzawa (1968) first introduced a simple and appealing method for reducing problems with variable rates of time preference to single-state systems by transforming the time scale from t to Δ, a utility discount factor. This transformation has been used extensively, particularly in models of international trade and finance (e.g., Obstfeld, 1981a, 1981b, 1982, Engel and Kletzer, 1989, and Turnovsky, 1997), where the use of a variable rate of time preference avoids some of the "disturbing implications" drawn from typical open-economy Ramsey models. The purpose of this paper, however, is to show that Uzawa's transformation is valid only when the underlying system to be analyzed is autonomous. Unfortunately, except for the simplest control problems, this is rarely the case. In particular, systems with non-autonomous transition equations imply that the correspondence between Δ and t is no longer unique, and thus Uzawa's transformation is not applicable.

JEL Classification: C61, D91, F30

Keywords: optimal control, variable rates of time preference, Uzawa's transformation, life-cycle models.

1. Introduction

Uzawa (1968) first introduced a simple and appealing method for endogenizing the rate of time preference, apparently applicable to a broad class of control models, by defining

$$J(c) = \int_0^\infty u(c(t)) \exp\left\{-\int_0^t \rho(u(c(t))d\tau\right\} dt \tag{1.1}$$

for ρ a rate of time preference that depends on the utility derived from consumption ($c(t)$) along a given path, and where the value

$$\Delta(t) = \int_0^t \rho(u(c(t))d\tau \tag{1.2}$$

defines a utility discount factor applied at each time t. The analysis provided a clear generalization of classical growth and life-cycle models, where the rate of time preference is assumed to be constant and exogenous, and extensions of Uzawa's approach have generated considerable interest since.[1]

Nevertheless, including a second state variable (Δ), in addition to the usual state variable constraint, complicates the analysis, requiring now four dimensions to qualitatively characterize the properties of the four differential equations describing the co-state and state variables. Standard qualitative techniques are also, of course, no longer available. To simplify the problem, Uzawa (1968:491) also introduced a straightforward method for reducing the problem (dimensionally) to a single state variable by transforming the time scale from t to Δ, taking Δ as the independent variable and thus treating ρ as constant at each point on a given path. This transformation has been used extensively, particularly in models of international trade and finance (e.g., Obstfeld, 1981a, 1981b, 1982, Engel and Kletzer 1989, Turnovsky, 1997), where the use of a variable rate of time preference avoids some of the "disturbing implications" drawn from typical open-economy Ramsey models.[2]

The purpose of this paper, however, is to show that Uzawa's transformation is valid only when the underlying system to be analyzed is autonomous.[3] Unfortunately, except for the simplest control problems, this is rarely the case. In particular, systems with non-autonomous transition equations (e.g., ones where there are anticipated shocks, changes in money or bond holdings, or rates of inflation that are time dependent) imply that the correspondence between Δ and t is no longer unique, and thus Uzawa's transformation is not applicable.

Section 2 of the paper sets out the nature of Uzawa's transformation, as applied to a standard life-cycle problem with a variable rate of time preference. Section 3 shows that this procedure is invalid when the system is non-autonomous.

[1] Uzawa preferences belong to the general class of utility functionals developed by Epstein (1983,1987). See Epstein and Hynes (1983), Nairay (1984), Obstfeld and Rogoff (1996) and Uzawa (1969,1991) as well. Curvature restrictions on the function $\rho = \rho(u(c))$ have been notably criticized. See Turnovsky (2000:357), Barro and Sala-i-Martin (1995:109) and Blanchard and Fischer (1989:73) for example. Nothing in the argument to follow, however, requires any prior restrictions on the form of $\rho(u(c))$.

[2] For example, in standard models where $\rho = \bar{\rho}$, either all but the most 'patient' country follows a path where consumption goes to zero or residents are effectively constrained on international credit markets. See Barro and Sala-i-Martin (1995:103–110) for a general discussion of this issue.

[3] This point was first noticed in Kompas and Abdel-Razeq (1987), as cited in Obstfeld (1990:56), but remained relatively unclear.

Section 4 provides a specific example, and constructs a single comparable second-order differential equation to describe the various systems. This device makes it especially clear that if the transition equation contains terms that are non-autonomous, then Uzawa's transformation cannot be used. Section 5 concludes.

2. Uzawa's Transformation

Consider a typical intertemporal (life-cycle) problem where any given path assigns values of consumption $c(t) \geq 0$ at each time $t \geq 0$ over an infinite horizon. As noted in section 1, let the rate of time preference be a variable that depends on an index of current and future consumption and define

$$\Delta(t) = \int_0^t \rho(u(c))d\tau \tag{2.1}$$

as a utility discount factor at each time t. For an otherwise standard utility functional, the problem is to maximize

$$J(c) = \int_0^\infty u(c(t))e^{-\Delta(t)}dt \tag{2.2}$$

subject to

$$\dot{\Delta} = \rho(u(c)) \tag{2.3}$$

and

$$\dot{w}(t) = g(w(t), c(t), t) \tag{2.4}$$

for given initial conditions, $\Delta(0) = 0$ and $w(0) = w_0$. The state variable constraint for (say) wealth (or w), or equation (2.4), is written in general and non-autonomous form. Equation (2.3) explicitly adds a second state variable.

The idea of Uzawa's (1968:491) transformation, once again, is to reduce the dimension of the above problem by transforming the time scale from t to Δ, so that the rate of time preference can be treated as a constant and the usual solution techniques can be applied. To follow this approach, note first that $dt = d\Delta/\rho(u)$ from equation (2.3). With (2.1), the problem given by (2.2)–(2.4) now becomes

$$\tilde{J}(c) = \int_0^\infty \frac{u(c(t))}{\rho} e^{-\Delta} d\Delta \tag{2.5}$$

subject to

$$\frac{dw}{d\Delta} = \frac{g(w, c, t)}{\rho} \tag{2.6}$$

and $w(0) = w_0$, a single state variable. Define

$$\tilde{H}(c, w, \lambda_1(\Delta), \Delta) = \frac{u(c)}{\rho} e^{-\Delta} + \lambda_1(\Delta) \frac{g(w, c, t)}{\rho} \tag{2.7}$$

for $\lambda_1(\Delta)$ a co-state variable as a function of the new time-scale Δ. First-order necessary conditions are

$$\rho(u_c e^{-\Delta} + \lambda_1 g_c) + \rho'(u_c u(c) e^{-\Delta} - u_c \lambda_1 g(w, c, t)) = 0 \tag{2.8}$$

$$\frac{d\lambda_1}{d\Delta} = -\frac{\lambda_1 g_w}{\rho} \tag{2.9}$$

$$\frac{dw}{d\Delta} = \frac{g(w, c, t)}{\rho} \tag{2.10}$$

where rearranging equation (2.8) gives[4]

$$u_c = \frac{-\lambda_1 g_c}{e^{-\Delta} - \rho' \tilde{H}} \tag{2.11}$$

for $\tilde{H}$ as defined in (2.7). In principle, solving (2.11) gives the value $c^*(t)$ for each value of Δ along an optimal path.

3. The Transformation Error

However, for the case of a non-autonomous system there is an error in the transformation used above. To see this, solve the system given by equations (2.1)–(2.4) directly as a non-transformed, two-state variable problem, using

$$H(w, c, \Delta, \lambda_1, \lambda_2, t) = u(c) e^{-\Delta(t)} + \lambda_1 g(x, c, t) + \lambda_2 \rho(u(c)) \tag{3.1}$$

to obtain the following first-order condition

$$u_c e^{-\Delta} + \lambda_1 g_c + \lambda_2 \rho' u_c = 0 \tag{3.2}$$

for the optimal value of $c(t)$, or

$$u_c = \frac{-\lambda_1 g_c}{e^{-\Delta} + \lambda_2 \rho'}. \tag{3.3}$$

If it is the case that Uzawa's transformation of a non-autonomous system is correct, equations (2.11) and (3.3) should be equivalent. But this is true only when $\tilde{H} = -\lambda_2$ for all t, or when the two-state variable problem is autonomous since, when (2.11) and (3.3) are equal,

$$\frac{u(c)}{\rho} e^{-\Delta} + \lambda_1 \frac{g(w, c, t)}{\rho} + \lambda_2 = 0 \tag{3.4}$$

or

$$u(c) e^{-\Delta(t)} + \lambda_1 g(x, c, t) + \lambda_2 \rho(u(c)) = 0 \tag{3.5}$$

so that the value of H given in equation (3.1) equals zero. Thus, for the systems to be identical, the Hamiltonian for the non-autonomous two-state variable problem must be constant and zero along an optimal trajectory, which of course can never be the case. Uzawa's transformation is not valid here.

[4] All notation is conventional, so that $u_c \equiv \partial u/\partial c$, $g_w \equiv \partial g/\partial w$, $\rho' \equiv \partial \rho/\partial u$, and so on.

4. An Example

To get a clear understanding of the result in Section 3, consider a specific example. Let preferences be given by $u(c(t)) = c^{0.5}$ or $c = u^2$ and let equation (2.3) be represented by

$$\dot{\Delta} = \rho(u(c)) = \alpha + \beta u. \tag{4.1}$$

Define the state variable constraint as

$$\dot{w}(t) = rw + \nu(t) - c = rw + \nu(t) - u^2 \tag{4.2}$$

for r a given and exogenous rate of interest. The presence of $\nu(t)$ in (4.2) makes the system non-autonomous.[5] With equation (2.1), the problem is to maximize (2.2) subject to equations (4.1) and (4.2), and initial conditions, $\Delta(0) = 0$ and $w(0) = w_0$. Treated as a two-state variable problem (i.e., without invoking Uzawa's transformation) first-order necessary conditions for an optimal path are

$$e^{-\Delta} - 2u\lambda_1 + \lambda_2\rho' = 0 \tag{4.3}$$

$$\dot{\lambda}_1(t) = -\lambda_1 r \tag{4.4}$$

$$\dot{\lambda}_2 = ue^{-\Delta} \tag{4.5}$$

$$\dot{\Delta}(t) = \alpha + \beta u \tag{4.6}$$

and

$$\dot{w}(t) = rw + \nu(t) - u^2. \tag{4.7}$$

Convert to current-values by defining $\phi_1(t) \equiv e^{\Delta}\lambda_1(t)$ and $\phi_2(t) \equiv e^{\Delta}\lambda_2(t)$. Since $\dot{\phi}_1(t) = e^{\Delta}\dot{\lambda}_1(t) + \rho\phi_1$ and $\dot{\phi}_2(t) = e^{\Delta}\dot{\lambda}_2(t) + \rho\phi_2$, first-order conditions now become

$$1 - 2\phi_1 u + \phi_2\rho' = 0 \tag{4.8}$$

$$\dot{\phi}_1(t) = \phi_1(\rho(u(c)) - r) \tag{4.9}$$

and

$$\dot{\phi}_2(t) = \rho\phi_2 + u. \tag{4.10}$$

[5] In a monetary growth model with variable rates of time preference the value of $\nu(t)$ would be represented by flow additions to the stock of real money balances (m), or θm, for θ the proportional rate of growth of the nominal supply of money. Net changes in real money balances or $(\theta - \pi)m$, for $\pi(t)$ the rate of inflation, imply that the transition equation is non-autonomous. In models of international trade and finance, $\nu(t)$ would stand for anticipated shocks, imported foreign goods or bond holdings.

In order to compare this system to that using Uzawa's transformation, convert the system into a single differential equation. Differentiating (4.8) with respect to time, and substituting using (4.1), (4.9) and (4.10) gives

$$-2\phi_1(\rho - r)u - 2\phi_1\dot{u} + \rho'(\rho\phi_2 + u) = 0. \tag{4.11}$$

Using (4.1) and equation (4.8) to eliminate ϕ_2 implies that

$$-2\phi_1(\dot{u} - ru) + \rho' u - \rho = 0 \tag{4.12}$$

or

$$-2\phi_1(\dot{u} - ru) = \alpha \tag{4.13}$$

and differentiating a second time, using (4.2) and (4.9) to eliminate ϕ_1, gives

$$\ddot{u} + (\alpha + \beta u - 2r)\dot{u} - r(\alpha + \beta u - r)u = 0 \tag{4.14}$$

or the system as described by a single second–order differential equation in u.

Now, rewrite the problem using Uzawa's transformation and maximize equation (2.5) subject to

$$\frac{dw}{d\Delta} = \frac{1}{\rho}(rw + \nu(t) - u^2) \tag{4.15}$$

to obtain the following first-order necessary conditions:

$$\frac{1}{\rho^2}\left[\rho e^{-\Delta} - \rho' e^{-\Delta} u - 2\rho u \lambda_1(\Delta) - \rho' \lambda_1(\Delta)(rw + \nu(t) - u^2\right] = 0 \tag{4.16}$$

$$\frac{d\lambda_1}{d\Delta} = \frac{-\lambda_1 r}{\rho} \ . \tag{4.17}$$

Using current-values define $\phi_1(\Delta) \equiv e^{\Delta}\lambda_1$ and note that

$$\frac{d\phi_1}{d\Delta} = e^{\Delta}\frac{d\lambda_1}{d\Delta} + e^{\Delta}\lambda_1 = \phi_1 - \frac{r\phi_1}{\rho} \ . \tag{4.18}$$

Transforming the time scale back to t, using $d\Delta = \rho(u)dt$, implies that (4.17) and (4.18) now become

$$\rho(1 - 2\phi_1 u) - \rho'(u + \phi_1(rw + \nu(t) - u^2)) = 0 \tag{4.19}$$

and

$$\dot{\phi}_1(t) = \phi_1(\rho(u(c)) - r). \tag{4.20}$$

To convert the system to a single differential equation, use (4.1) so that equation (4.19) can be written as

$$2\alpha u + \beta u^2 + \beta(rw + \nu(t)) = \frac{\alpha}{\phi_1} \tag{4.21}$$

and differentiate with respect to time, substituting from (4.20), to obtain

$$2\alpha\dot{u} + 2\beta u\dot{u} + \beta r\dot{w} + \beta\dot{v} = -\frac{\alpha}{\phi_1}(\rho - r). \tag{4.22}$$

Using (4.7) and (4.23) and multiplying through by α/ϕ_1 gives

$$2\phi_1(\dot{u} - ru) + \frac{1}{\rho}\phi_1\beta\dot{\nu} = -\alpha \tag{4.23}$$

and, finally, differentiating again with respect to time, with (4.1) and (4.20), obtains

$$2\left[\ddot{u} + (\alpha + \beta u - 2r)\dot{u} - r(\alpha + \beta u - r)u\right] + \left[\frac{1}{\rho}\{(\rho - r)\beta\dot{\nu} + \beta\ddot{\nu}\} - \frac{\beta^2\dot{\nu}\dot{u}}{\rho^2}\right] = 0. \tag{4.24}$$

Compare equations (4.14) and (4.24). The two are equivalent only when $\dot{\nu}(t) = \ddot{\nu}(t) = 0$. In other words, Uzawa's transformation is valid only when the problem is autonomous.

5. Conclusion

This paper has demonstrated that Uzawa's transformation of optimal control problems with variable rates of time preference is a valid only when applied to a system that is autonomous. If non-autonomous terms in the transition equation matter, as is often the case in models of optimal monetary growth and international trade and finance, then the correspondence between Δ and t is no longer unique and Uzawa's transformation is not applicable. As such, the transformation overlooks the non-linearities caused by the explicit presence of time as a variable, causing errors in first-order conditions and resulting optimal paths.

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